Calculus

DEMTH137

Edited by
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LOVELY PROFESSIONAL UNIVERSITY

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CONTENT

Unit 1: Sequence of Real Numbers

CONTENTS

Objectives

Students will be able to

- cite some properties of real numbers.
- discuss the sequence of real numbers.
- explain the concept of convergence of a sequence.
- validate the bounded and monotonic sequences

Introduction

You have the idea about the number system since you began your journey of learning mathematics. In this unit we will learn about some more properties of real numbers, the concept of sequence of real numbers and their properties of being bounded and of convergence.

1.1 Real numbers

All around us, the one thing which is constant is the change. The study of change is called CALCULUS. It has applications in almost all the fields of science and social science. For understanding the change, one needs to have the concept of measurement. We need to measure the time, distance, heat, force, intelligence etc. in order to adjudge the change happening in them.

One way to judge the change in any situation is by finding the average rate of change over a range. If the change is to be judged at a point, then instantaneous rate of change is required. To understand the instantaneous rate of change, one needs to understand the concept of limit first. Moreover, to understand anything and everything one needs to understand the number system first.

The advent of Calculus is accredited to two mathematicians. One is British mathematician Sir Issac Newton and other is German Mathematician Gottfried Leibnitz.

1.2 Number System

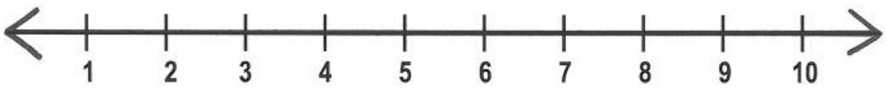
We know that since ancient times, the pebbles were used by the shepherds etc. to count their sheep and this amounts to the counting numbers, which are now known as natural numbers.

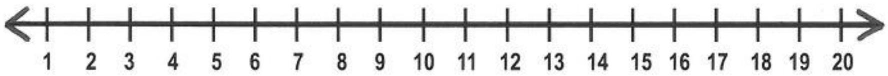
With the invention of zero, natural numbers {1, 2, 3,…} along with {0} are known as whole numbers. Then the idea of negative numbers got evolved and the set {…, -2, -1, 0, 1, 2,…} was named as the set of integers.

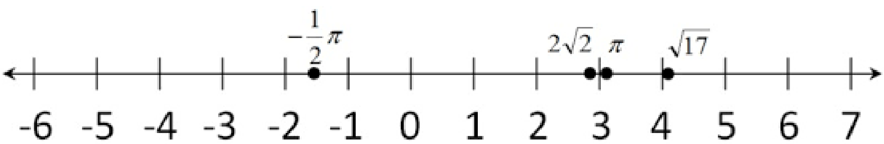
Then the idea of rational numbers came up where a rational number takes the form p/q where q is non-zero number and p and q are integers. The numbers which are not rational were called irrational numbers.

The rational and irrational numbers taken together are called real numbers. Therefore all the terminating, nonterminating-repeating, nonterminating-nonrepeating decimals are the real numbers.

All the real numbers can be plotted on the real line.







Closed and Open interval

Let a and b be two given numbers such that $a < b$. Then, the set $\{x: a \le x \le b\}$ is called a closed interval and is denoted by $[a, b]$; the set $\{x: a < x < b\}$ is called a closed interval and is denoted by (a, b) and the set $\{x: a \le x < b\}$ or $\{x: a < x \le b\}$ is called a closed interval and is denoted by $[a, b)$ or $(a, b]$ respectively.

Absolute function

Let x be a real number then there exist three possibilities, then it can be more than, less than or equal to zero. The modulus value or absolute value of x is defined as

$$|x| = \begin{cases} x \ if \ x \ge 0 \\ -x \ if \ x < 0 \end{cases} \quad \forall\, x \in \mathbb{R}$$

Moreover, we can write $|x| = \max\{x, -x\}$.

Also $|x - a| > l \Rightarrow \mathrm{x - a > l}$ or $\mathrm{x - a < -l}$ and

$|x - a| < l \Rightarrow -\mathrm{l} < \mathrm{x - a} < \mathrm{l}$

1.3 Sequence of real number

Sequence is a kind of ordered list of numbers where a pattern can be seen. For example, 3, 6, 9, . . . is a list in which the first term is 3, second is 6 and so on. We can easily tell that the fourth term will be 12 and so on. This example here is of an infinite sequence.

Technically, a function whose domain is a set of natural numbers and range is a subset of real numbers, is called a real sequence (and just sequence in this course). Since the domain of all sequences is the set of natural numbers, therefore a sequence is completely determined if

$f(n)\ \forall\ n \in \boldsymbol{N}$ is known. The sequence f is denoted as $< f_n >$ *or* $\{f_n\}$ mostly.

$\left\{\frac{1}{n}\right\}$, $\{(-1)^n\}$ etc.

Range of sequence is the set of all distinct terms of a sequence. It can be infinite or finite. The first sequence in the example is an infinite one and the second one is finite.

1.4 Bounded and unbounded sequence

A sequence is said to be bounded if and only if its range is bounded.

A sequence $\{f_n\}$ is ***bounded above*** if there exists a real number K such that

$$f_n \le K \quad \forall\ n \in \boldsymbol{N}$$

A sequence $\{f_n\}$ is ***bounded below*** if there exists a real number k such that

$$f_n \ge k \quad \forall\ n \in \boldsymbol{N}$$

A sequence is said to be ***bounded*** if it is bounded above as well as below. Hence a sequence $\{f_n\}$ is said to be bounded if there exist two real numbers k and K such that

$$k \le f_n \le K \quad \forall\ n \in \boldsymbol{N}$$

We call the sequence to be unbounded if it is not bounded.

Prove that the sequence {n} is not convergent and is not bounded.

Let p be a real number. Then, the neighbourhood $\left(p-\frac{1}{4}, p+\frac{1}{4}\right)$ will contain at most one term of the sequence $\{n\}$. Therefore, in this neighbourhood, we cannot find infinitely many terms of the sequence. Hence the sequence is not convergent.

On the other hand, we can see that all the elements of the sequence are more than 1 but we cannot find any upper bound i.e. any number so that all the elements of the sequence are below that number. Therefore, the sequence is just bounded below and not bounded above or simply the sequence is not bounded.

1.5 Limit of a sequence

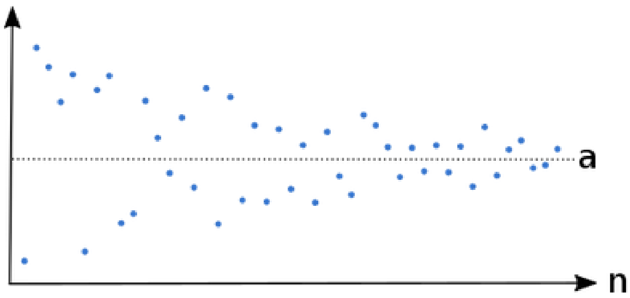
This concept of approach is of utmost importance to understand the calculus. A number a is called the limit of an infinite sequence $\{f_n\}$ if for any positive number ϵ we can find a positive number $\boldsymbol{N}$ depending on ϵ such that

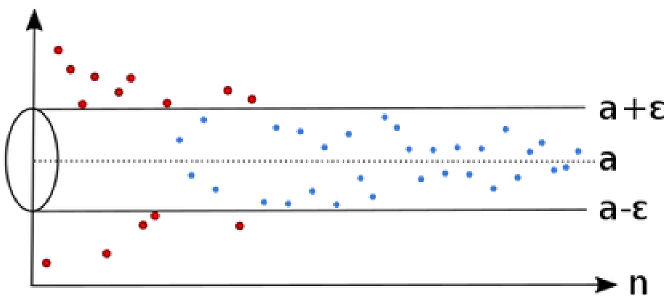
$$|f_n - a| < \epsilon \ \forall \text{ integers } n > \boldsymbol{N}$$

That is in any neighbourhood of a , f_n belongs to the same neighbourhood for infinite values of n. Consider the sequence $\{n\}$. If you consider any neighbourhood of any number a, it will not contain an infinite number of elements of the sequence. On the other hand any neighbourhood of any number a , for the sequence $\left\{\frac{1}{n}\right\}$ will contain infinitely many elements of the sequence.

1.6 Convergence of sequences

A sequence is said to be convergent if its limit exists and is unique. In other words, a sequence $\{f_n\}$ has a limit a if the successive terms get closer and closer to a.





Theorems on limits of sequence

If $\lim_{n\to\infty} f_n = A$ and $\lim_{n\to\infty} g_n = B$, then

(i) $\lim_{n\to\infty} (f_n + g_n) = A + B$

(ii) $\lim_{n\to\infty} (f_n - g_n) = A - B$

(iii) $\lim_{n\to\infty} f_n . g_n = A . B$

(iv) $\lim_{n\to\infty} \left(\frac{f_n}{g_n}\right) = \frac{A}{B}$ provided $B \neq 0$

If B=0, A≠0 then the limit does not exist and if B=0, A=0 then the limit may or may not exist.

(v) $\lim_{n\to\infty} f_n^p = A^p$, $p\epsilon \boldsymbol{R}$

(vi) $\lim_{n\to\infty} p^{f_n} = p^A$, $p\epsilon \boldsymbol{R}$

1.7 Infinite series

Let $f_1, f_2, f_3,$ . . . be a sequence. Create a new sequence by taking the sum of the terms taken one, two, three… at a time. Let $S_1, S_2, S_3,$ … be such that

$$S_1 = f_1$$
$$S_2 = f_1 + f_2$$
$$S_3 = f_1 + f_2 + f_3$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$S_n = f_1 + f_2 + f_3 + \cdots + f_n$$

Here S_n is called the n^{th} partial sum. The sequence $S_1, S_2, S_3, \ldots$ is symbolized by

$$f_1 + f_2 + f_3 + \cdots = \sum_{n=1}^{\infty} f_n$$

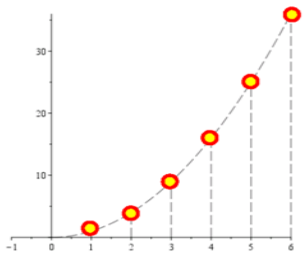
which is called an infinite series. If $\lim_{n\to\infty} S_n = S$ exists, the series is called convergent and S is called its sum, otherwise the series is called divergent.

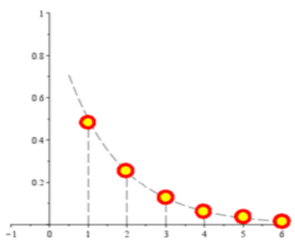
For instance, the geometric progression (G. P.) $a + ar + ar^2 + \cdots . + ar^{n-1} + \cdots$ is a series which converges to sum $\frac{a}{(1-r)}$ provided $|r| < 1$ and diverges if $|r| \geq 1$.

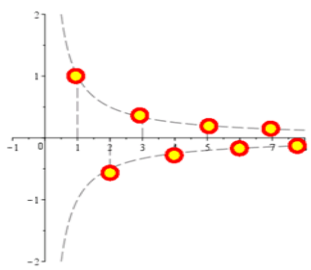
1.8 Monotonic sequences

A sequence $\{f_n\}$ is said to be monotonically increasing if $f_{n+1} \geq f_n \ \forall \ n \in \boldsymbol{N}$ and monotonically decreasing if $f_{n+1} \leq f_n \ \forall \ n \in \boldsymbol{N}$**.** For example, the sequence $\{n\}$ monotonically increasing and the sequence $\left\{\frac{1}{n}\right\}$ is monotonically decreasing. The sequence which is either monotonically increasing or decreasing is called the monotonic sequence.

Can you recognize what kind of sequence is being depicted by the following graphs?







Prove that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is (i) monotonically increasing (ii) is bounded and (iii) tends to limit $\left\{\frac{2}{3}\right\}$.

Let $f_n = \frac{2n-7}{3n+2}$

then $f_{n+1} = \frac{2n-5}{3n+5}$

And $f_{n+1} - f_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2} = \frac{25}{(3n+5)(3n+2)} > 0 \; \forall \, n \in \boldsymbol{N}$

This shows that the given sequence is monotonically increasing. Now if we write few terms of the sequence $\{f_n\}$, we can observe that all the terms are more than or equal to -1.

i.e. $f_n \geq -1 \; \forall \, n \in \boldsymbol{N}$

Moreover $1 - f_n = \frac{n+9}{3n+2} > 0 \;\; \forall \, n \in \boldsymbol{N}$

This implies $f_n < 1 \;\; \forall \, n \in \boldsymbol{N}$

And $-1 \leq f_n < 1 \;\; \forall \, n \in \boldsymbol{N}$, therefore the sequence is bounded.

Now the limit can be found as follows,

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2-\frac{7}{n}}{3+\frac{2}{n}} = \frac{2}{3}$$

Summary

In this chapter we have seen the concept of the sequence, its convergence, boundedness, and monotonicity.

Key words

Real Numbers, Sequence, Series, Convergence, Bounded function, Monotonic function

Self Assessment

1. The interval [3,99) is a

   (a) closed interval

   (b) open interval

   (c) semi closed or semi open interval

(d) semi open interval

2. $|x|$ can be written as

   (a) x
   (b) $-x$
   (c) $\min\{-x, x\}$
   (d) $\max\{-x, x\}$

3. The solution of $|x - 5| < 3$ can be written as

   (a) [2,8]
   (b) (2,8)
   (c) (2,8]
   (d) (-2,8)

4. The solution of $|x - 5| \geq 3$ can be written as

   (a) $(-\infty, 2]U[8, \infty)$
   (b) [2,8]
   (c) $(-\infty, 2)U(8, \infty)$
   (d) None of these

5. Which of the following is an infinite sequence?

   (a) The prime numbers between 2 to 2000
   (b) The set of even numbers
   (c) The set of odd numbers between 3 to 30
   (d) First ten multiples of seven

6. Function f is bounded if its range f(A) is a __________ subset.

7. The supremum of a set is its least upper bound.

True

False

8. The infimum of a set is its greatest upper bound.

True

False

9. The sequence $\{n\}$ is

   (a) not convergent
   (b) not bounded

(c) convergent and bounded

(d) neither convergent nor bounded

10. A sequence is said to be convergent

(a) if its limit exists

(b) if it is unique

(c) if its limit exists and is unique

(d) None of these

11. If $\lim_{n\to\infty} f_n = A$ and $\lim_{n\to\infty} g_n = B$, then

(a) $\lim_{n\to\infty} (f_n + g_n) = A + B$

(b) $\lim_{n\to\infty} (f_n - g_n) = -A - B$

(c) $\lim_{n\to\infty} f_n . g_n = A + B$

(d) $\lim_{n\to\infty} f_n . g_n = A/B$

12. The sequence $\left\{\frac{2n-7}{3n+2}\right\}$ converges to

(a) 2/3

(b) -1

(c) 1

(d) none of these

13. A sequence $\{f_n\}$ is said to be monotonically increasing if

(a) $f_{n+1} \leq f_n \ \forall \ n \in N$

(b) $f_{n+1} < f_n \ \forall \ n \in N$

(c) $f_{n+1} > f_n \ \forall \ n \in N$

(d) $f_{n+1} \geq f_n \ \forall \ n \in N$

14. A sequence $\{f_n\}$ is said to be strictly monotonically increasing if

(a) $f_{n+1} \leq f_n \ \forall \ n \in N$

(b) $f_{n+1} < f_n \ \forall \ n \in N$

(c) $f_{n+1} > f_n \ \forall \ n \in N$

(d) $f_{n+1} \geq f_n \ \forall \ n \in N$

15. The sequence $\{n\}$ is

(a) convergent

(b) monotonic

(c) convergent and monotonic

(d) neither convergent nor monotonic

Answers:

1	c	6	bounded	11	a
2	d	7	True	12	a
3	b	8	True	13	d
4	a	9	d	14	c
5	b	10	c	15	B

Review Questions

1. What is the solution of $|x-1|<5$?
2. Check whether the sequence $\{(-n)^n\}$ is bounded or not?
3. What is the solution of $|x-2|<6$?
4. What is the solution of $|x-1|>5$?
5. Prove that $|ab|=|a||b|$.
6. State true or false: $\{x: |x-3|<4\}=\{x: -1<x<7\}$
7. Evaluate $\lim_{n\to\infty} \log_5 \frac{n}{\log_9(n)}$
8. Check if the sequence $\{(-2)^n\}$ is bounded or not?
9. Check if the sequence $\left\{\frac{2}{n}\right\}$ is bounded or not?
10. Check if the sequence $\{n^2\}$ is bounded or not?
11. Write a short note on boundedness of a sequence.
12. Check the monotonicity of the sequence $\left\{\frac{2}{n}\right\}$.
13. Check the monotonicity of the sequence $\left\{\frac{2n}{n+1}\right\}$.
14. Write a short note on the monotonicity of a sequence.
15. Write a short note on the convergence of a sequence.

Further/Suggested Readings

1. George B. Thomas Jr., Joel Hass, Christopher Heil & Maurice D. Weir (2018). *Thomas*' *Calculus* (14th edition). Pearson Education.
2. Howard Anton, I. Bivens & Stephan Davis (2016). *Calculus* (10th edition). Wiley India.
3. http://mathonline.wikidot.com/calculus

Unit 02: Definite integral as a limit of sum

CONTENTS

Objectives

Students will -

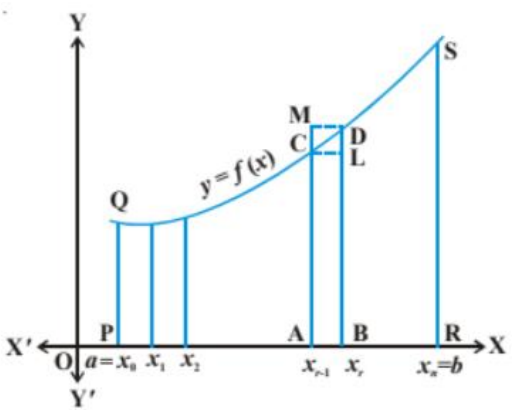
- learn the concept of integral as a limit of sum.
- learn about the hyperbolic functions
- be able to integrate irrational algebraic functions.
- be able to integrate the transcendental functions

Introduction

You have learnt the rules of differentiation and integration at the senior secondary level. The definite integral of a function gives you the area under the curve of that function between the specified limits. In this unit we will look into the detail as to what is the integral as a limit of sum. We will evaluate some definite integrals with this ab initio method. The functions can be classified as algebraic and transcendental functions. The polynomial functions, rational functions are the algebraic functions and exponential, logarithmic, trigonometric, inverse trigonometric functions, hyperbolic functions, inverse hyperbolic functions are the examples of the transcendental functions. We will solve some problems on integration of irrational as well as transcendental functions in this unit.

2.1 Integral as a limit of sum

Consider a continuous function defined on a closed interval [a, b], where all the values of the function are non-negative. The area bound between the curve, the point $x = a$ and $x = b$ and the x-axis is the definite integral $\int_a^b f(x)dx$ of any such continuous function f.



Notes

First idea is to divide the interval $[a, b]$ onto n equal sub-intervals as:

$a = x_0, x_1, x_2, \dots, x_{n-2}, x_{n-1}, x_n = b$. Now we can see that the length of each sub interval must be $\frac{b-a}{n}$ and let it be denoted by h. Therefore we can write the points of the partition as follows:

$$x_0 = a,$$

$$x_1 = a + h$$

$$x_2 = a + 2h$$

.

.

.

$$x_n = a + nh = b$$

Clearly as $\to \infty$, $h \to 0$. In the above figure The region PRSQP is the sum of all the n sub-regions, where each sub-region is defined on sub-interval $[x_{r-1}, x_r], r = 1, 2, 3, \dots n$. Observe region ABDM. Area of the rectangle (ABLC) < Area of the region (ABDCA) < Area of the rectangle (ABDM).

As $h \to 0$ all these areas become almost equal to each other. Hence, we have

$$s_n = h[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] = h \sum_{r=0}^{n-1} f(x_r)$$

and

$$S_n = h[f(x_1) + f(x_2) + \cdots + f(x_n)] = h \sum_{r=1}^{n} f(x_r)$$

Here s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r], r = 1, 2, 3, \dots n$ respectively.

Therefore we can write

$s_n <$ area of the region $< S_n$

As $n \to \infty$, these strips become narrower. Further it is assumed that the limiting value of s_n and S_n are the same in both cases and the common limiting value is the required area under the curve.

Symbolically we can write

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n$ =area (PRSQP) $=\int_a^b f(x)dx$

This area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For convenience, we shall take the rectangles having height equal to that of the curve at the left-hand-edge of each sub- interval. Hence, we can write

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a) + f(a+h) + \cdots + f(a + (n-1)h)]$$

$$\int_a^b f(x)dx = \frac{(b-a) \lim_{n\to\infty} [f(a) + f(a+h) + \cdots + f(a + (n-1)h)]}{n}$$

where $h = \frac{b-a}{n} \to 0 \; as \; n \to \infty$. This equation is the definition of definite integral as the limit of a sum.

Evaluate $\int_2^5 x^2 \, dx$ as a limit of sum.

Here $f(x) = x^2$

$$a = 2, b = 5, h = \frac{b-a}{n} = \frac{3}{n}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a+h) + f(a+2h) + \cdots + f(a+nh)]$$

$$\int_2^5 x^2 dx = \lim_{n\to\infty} \frac{3}{n}[(2+h)^2 + (2+2h)^2 + \cdots + (2+nh)^2]$$

$$= \lim_{n\to\infty} \frac{3}{n}[n\cdot 2^2 + h^2(1^2 + 2^2 + \cdots + n^2) + 4h(1 + 2 + \cdots n)]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n\cdot 2^2 + h^2\left(\frac{n(n+1)(2n+1)}{6}\right) + 4h\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n\cdot 2^2 + \left(\frac{3}{n}\right)^2\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{3}{n}\right)\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n\cdot 2^2 + \left(\frac{3}{n}\right)^2\left(\frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right) + 4\left(\frac{3}{n}\right)\left(\frac{n^2\left(1+\frac{1}{n}\right)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\left[3.2^2 + 3(3)^2\left(\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right) + 3(4)(3)\left(\frac{\left(1+\frac{1}{n}\right)}{2}\right)\right]$$

$$= 12 + 9 + 18 = 39$$

Thus the limit of the sum or in other words the area under the curve x^2 between lines $x = 2$ and $x = 5$ and above the x-axis is 39 square units.

Evaluate $\int_a^b \cos x\, dx$ as the limit of a sum.

Here $f(x) = \cos x$

By definition of integral as a limit of sum we can write

$$\int_a^b \cos x\, dx = \lim_{n\to\infty} h[\cos(a+h) + \cos(a+2h) + \cos(a+3h) + \cdots + \cos(a+nh)]$$

Let $S = \cos(a+h) + \cos(a+2h) + \cos(a+3h) + \cdots + \cos(a+nh)$

To calculate this sum, we must multiply both sides $2\sin\left(\frac{h}{2}\right)$. We get,

$$2\sin\left(\frac{h}{2}\right) S = 2\sin\left(\frac{h}{2}\right)(\cos(a+h) + \cos(a+2h) + \cos(a+3h) + \cdots + \cos(a+nh))$$

$$= \sin\left(a+\tfrac{3}{2}h\right) - \sin\left(a+\tfrac{1}{2}h\right) + \sin\left(a+\tfrac{5}{2}h\right) - \sin\left(a+\tfrac{3}{2}h\right) + \ldots + \sin\left(a+\tfrac{2n+1}{2}h\right) - \sin\left(a+\tfrac{2n-1}{2}h\right)$$

$$= \sin\left(a+\tfrac{2n+1}{2}h\right) - \sin\left(a+\tfrac{1}{2}h\right)$$

$$= \sin\left(b+\tfrac{1}{2}h\right) - \sin\left(a+\tfrac{1}{2}h\right) \quad \text{as } nh = b - a$$

Thus $\int_a^b \cos x\, dx = \lim_{n\to\infty} h \frac{\sin\left(b+\frac{1}{2}h\right) - \sin\left(a+\frac{1}{2}h\right)}{2\sin\left(\frac{h}{2}\right)}$

$$= \sin b - \sin a$$

In the senior secondary level, you have learnt how to integrate the algebraic rational functions and the trigonometric functions. In the next section we will see some cases of the integration of irrational algebraic function.

Evaluate $\int_a^b \frac{1}{\sqrt{x}} dx$ as limit of sum.

Notes

2.2 Integration of irrational algebraic functions

Certain types of integrals containing irrational expressions can be reduced to integrals of rational functions by making appropriate substitutions. These substitutions are done with an intention to convert the irrational function into a rational one.

> To integrate a function that contains only one irrational expression of the form $x^{\frac{m}{n}}$ we make the substitution for $x^{\frac{1}{n}}$.

An expression of the form $\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}$ can be integrated by using the substitution for $\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}$, where a, b, c, d are real numbers. These substitutions reduce the integrals rational functions in the transformed variable.

The integrals containing radicals of the form $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ can be evaluated with the help of trigonometric and hyperbolic substitutions.

> Consider the integral of $\int \frac{x}{(a+bx)^{\frac{1}{3}}} dx$

The denominator is having an irrational function and if we substitute $a + bx = y^3$ then the given integrand can be written as a rational function in y. Let us see the steps.

If we substitute $a + bx = y^3$

Then $b\, dx = 3y^2 dy$

And $\int \frac{x}{(a+bx)^{\frac{1}{3}}} dx = \int \left(\frac{y^3 - a}{b}\right)\left(\frac{3y^2}{yb}\right) dy$

$= \frac{1}{b^2} \int (y^4 - ay) dy$

$= \frac{3}{b^2}\left[\frac{y^5}{5} - \frac{ay^2}{2}\right]$+C

$= \frac{3}{10b^2} y^2(2y^3 - 5a) + C$

where C is the constant of integration.

> Evaluate $\int \left(\frac{\sqrt{x+9}}{x}\right) dx$

Put $\sqrt{x+9} = u$

$$dx = 2\,u\,du$$

Then

$\int \left(\frac{\sqrt{x+9}}{x}\right) dx = \int \left(\frac{u}{u^2-9}\right) 2u du = 2\int \frac{u^2}{u^2-9} du$

$= 2\int \frac{u^2 - 9 + 9}{u^2 - 9}\, du$

$= 2\int du + 2\int \frac{9}{u^2 - 9} du$

$= 2\,u + 2.9.\frac{1}{6} \ln \left|\frac{u-3}{u+3}\right| + C$

$= 2\sqrt{x+9} + 3 \ln |(\sqrt{x+9} - 3)/(\sqrt{x+9} + 3)\ + C$

where C is the constant of integration.

> Evaluate $\int (5x - 1)^{\frac{1}{3}} dx$

To solve this integral, substitute $(5x - 1)^{\frac{1}{3}} = u$

$(5x-1)=u^3$
$5\,dx = 3\,u^2 du$

Then

$\int (5x-1)^{\frac{1}{3}}dx = \int u \frac{3u^2}{5} du$

$= \frac{3}{20}u^4 + C$

$= \frac{3}{20}(5x-1)^{\frac{4}{3}} + C$

2.3 Integration of transcendental functions

A number that is not the root of any integer polynomial is termed as a transcendental number. And on the same lines a function that cannot be written using roots and the arithmetic found in polynomials is known as a transcendental function.

e.g. Exponential function

Logarithmic function

Trigonometric function

Inverse trigonometric function

Hyperbolic function

Inverse hyperbolic functions etc.

You are familiar with all the above mentioned functions except the hyperbolic function. So here is a short introduction to this family of transcendental functions.

Hyperbolic functions are the functions defined in terms of the exponential functions as follows.

$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x = \frac{e^x + e^{-x}}{2}$
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$	$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Just like with trigonometric functions, there are identities related to the hyperbolic functions.

$\cosh^2 x - \sinh^2 x = 1;$

$1 - \tanh^2 x = \operatorname{sech}^2 x;$

$\coth^2 x - 1 = \operatorname{csch}^2 x;$

$\sinh 2x = 2\sinh x \cosh x;$

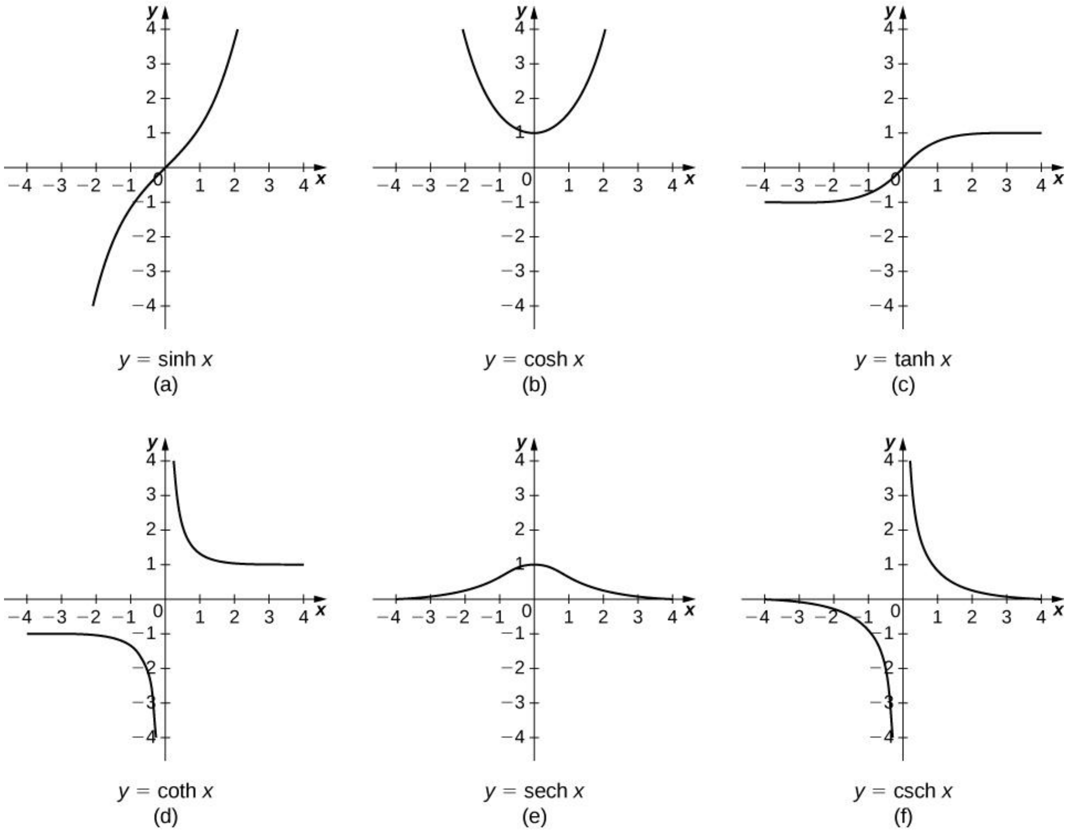
$\cosh 2x = \cosh^2 x + \sinh^2 x.$

Rules of differentiation and integration are as follows. Here the prime symbol is for denoting the first derivative.

Notes

$(\sinh x)' = \cosh x$	$\int \cosh x dx = \sinh x + C$
$(\cosh x)' = \sinh x$	$\int \sinh x dx = \cosh x + C$
$(\tanh x)' = \text{sech}^2 x$	$\int \text{sech}^2 x dx = \tanh x + C$
$(\coth x)' = -\text{csch}^2 x$	$\int \text{csch}^2 x dx = -\coth x + C$
$(\text{sech}\, x)' = -\text{sech}\, x \tanh x$	$\int \text{sech}\, x \tanh x dx = -\text{sech}\, x + C$
$(\text{csch}\, x)' = -\text{csch}\, x \coth x$	$\int \text{csch}\, x \coth x dx = -\text{csch}\, x + C$

The graphs of the hyperbolic functions are shown below and it can be seen that with appropriate range restrictions, they all have inverses (same as the case with the inverse trigonometric functions).



Hyperbolic Function	Inverse Hyperbolic Function	Domain	Range
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$-\infty \le x \le \infty$	$(-\infty, \infty)$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \ge 1$	$[0, \infty)$
$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$	$\lvert x\rvert < 1$	$(-\infty, \infty)$
$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$	$\operatorname{csch}^{-1} x = \sinh^{-1}\left(\dfrac{1}{x}\right)$ $= \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$x \ne 0$	$(-\infty, \infty)$
$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$	$\operatorname{sech}^{-1} x = \coth^{-1}\left(\dfrac{1}{x}\right)$ $= \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$	$0 < x \le 1$	
$\coth x = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$\coth^{-1} x = \tanh^{-1}\left(\dfrac{1}{x}\right)$ $= \dfrac{1}{2}\ln\dfrac{x+1}{x-1}$	$\lvert x\rvert > 1$	

The range in case of $\sec h^{-1} x$ is $[0, \infty)$ and for $\coth^{-1} x$ is $(-\infty, 0)U(0,\ \infty)$. The derivative of the inverse hyperbolic functions are as follows:

$\dfrac{d}{dx}\sinh^{-1} u = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$	$u \in \mathbb{R}$	$\dfrac{d}{dx}\operatorname{csch}^{-1} u = \dfrac{-1}{\lvert u\rvert\sqrt{1+u^2}}\dfrac{du}{dx}$	$u \ne 0$
$\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$	$u > 1$	$\dfrac{d}{dx}\operatorname{sech}^{-1} u = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$	$0 < u < 1$
$\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1-u^2}\dfrac{du}{dx}$	$\lvert u\rvert < 1$	$\dfrac{d}{dx}\coth^{-1} u = \dfrac{1}{1-u^2}\dfrac{du}{dx}$	$\lvert u\rvert > 1$

The integral as an anti-derivative can be written easily from the above rules of differentiation.

$\displaystyle\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1} x + C$ $= \ln\left(x + \sqrt{x^2+1}\right) + C$	$\displaystyle\int \frac{1}{x\sqrt{1+x^2}}dx = -\operatorname{csch}^{-1} x + C, \quad x \ne 0$
$\displaystyle\int \frac{1}{x^2-1}dx = \cosh^{-1} x + C, \quad x > 1$	$\displaystyle\int \frac{1}{x\sqrt{1-x^2}}dx = -\operatorname{sech}^{-1} x + C, \quad 0 < \lvert x\rvert < 1$
$\displaystyle\int \frac{1}{1-x^2}dx = \tanh^{-1} x + C, \quad \lvert x\rvert < 1$	$\displaystyle\int \frac{1}{1-x^2}dx = \coth^{-1} x + C, \quad \lvert x\rvert > 1$

Notes

Thus you got a brief idea about the calculus of hyperbolic functions and inverse hyperbolic functions. For more information you can check the following link.

https://math.libretexts.org/Courses/Monroe_Community_College/MTH_211_Calculus_II/Chapter_6%3A_Applications_of_Integration/6.9%3A_Calculus_of_the_Hyperbolic_Functions

Now with the knowledge of the transcendental functions, we can look into the examples related to the integration of the transcendental functions.

Evaluate $\int \frac{xe^x}{(x+1)^2}dx$

$$\int \frac{xe^x}{(x+1)^2}dx = \int \frac{(x+1-1)e^x}{(x+1)^2}dx$$

$$= \int \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)e^x dx$$

$$= \frac{e^x}{x+1} + C \qquad \text{(Using integration by parts on the first term)}$$

Evaluate $\int \sin(\log x)\, dx$

Put $\log x = t$

i.e. $\frac{1}{x}dx = dt$

$dx = e^t dt$

Therefore $\int \sin(\log x)\, dx = \int e^t \text{sin}t\, dt$

$$= -e^{-t}\cos t + e^t \sin t - \int e^t \text{sin}t\ \ dt$$

Clubbing the integral term, we get

$2\int e^t \sin t\, dt = e^t(\sin t - \cos t) + C$

$$\int e^t \sin t\, dt = \frac{e^t}{2}(\sin t - \cos t) + \frac{C}{2}$$

$$\int \sin(\log x)\, dx = \frac{1}{2}x(\sin(\log x) - \cos(logx)) + \boldsymbol{C}$$

Calculate $I = \int e^{ax}\cos(bx+c)dx$

$$I = \frac{e^{ax}}{a}\cos(bx+c) + \int \left(\frac{b}{a}\right)\sin\ (bx+c)\, e^{ax}dx$$

$$I = \frac{e^{ax}}{a}\cos(bx+c) + \left(\frac{b}{a^2}\right)\sin\ (bx+c)\, e^{ax} - \frac{b^2}{a^2}\int cos(bx+c)\, e^{ax}dx \quad \text{(integrating by parts)}$$

$$\left(1+\frac{b^2}{a^2}\right)I = \frac{e^{ax}}{a^2}(a\cos(bx+c) + b\sin(bx+c))$$

$$I = \frac{e^{ax}}{a^2+b^2}(\ a\cos(bx+c) + b\sin(bx+c))$$

Integrate $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ with respect to x.

Let $I = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Notes

Put $x = \cos\theta$

$dx = -\sin\theta \, d\theta$

Therefore $I = \tan^{-1}\sqrt{[(1-\cos\theta)/(1+\cos\theta)]}$

$= \tan^{-1}\tan\left(\frac{\theta}{2}\right)$

$= \frac{\theta}{2}$

Now $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{\theta}{2}(-\sin\theta)d\theta$

$= -\frac{1}{2}(-\theta\cos\theta + sin\theta)$

$= -\frac{1}{2}(-x\cos^{-1}x + \sqrt{1-x^2}) + C$

Evaluate $\int \frac{sinhx}{1+coshx}dx$

For this kind of integral we can go for a suitable substitution.

Let $1 + coshx = u$

It gives $sinhx\,dx = du$

$\int \frac{sinhx}{1+coshx}dx = \int \frac{du}{u} = \ln|u| + C = \ln\,|1+\cosh x| + C$

Evaluate $I = \int \cos x/\surd(1+\sin^2 x)\,dx$

Put $\sin x = u$

$\cos x \;\; dx = du$

$I = \int \frac{1}{\sqrt{1+u^2}}du$ (Refer to the rules of differentiation (integration) of the inverse hyperbolic functions)

$= \sinh^{-1}u + C$

$= \sinh^{-1}\sin x + C$

Evaluate the integral $\int \frac{1}{\sqrt{4x^2-1}}dx$

Let us substitute $2x = u$

$2dx = du$ and this results in

$$\int \frac{1}{\sqrt{4x^2-1}}dx = \int \frac{1}{2\sqrt{u^2-1}}du$$
$$= \frac{1}{2}\cosh^{-1}u + C$$
$$= \frac{1}{2}\cosh^{-1}(2x) + C$$

Summary

In this chapter we have seen how a definite integral can be calculated as a limit of sum. This is the ab initio way to calculate the area under a curve $y = f(x)$ bounded by the two vertical lines $x = a, x = b$ and the x-axis.

Key words

Real Numbers, Sequence, Series, Convergence, Bounded function, Monotonic function

Notes

Review Questions

Evaluate the following:

1. $\int \sinh^3 x \cosh x\, dx$
2. $\int \operatorname{sech}^2 3x\, dx$
3. $\frac{d}{dx}(\cosh^{-1} 3x)$
4. $\int \frac{1}{\sqrt{1-e^{2x}}} dx$
5. $\int \sqrt{e^x + 1} dx$
6. $\int \frac{x^2-2}{x+1}\, dx$
7. $\int \frac{1-x}{1+x} \frac{dx}{x}$
8. $\int \sqrt{secx - 1}\, dx$
9. $\int \frac{dx}{5+4cosx}$
10. $\int cosec^5 x\, dx$
11. $\int \sinh^3 x\ \ dx$
12. $\int x \sinh x\, dx$
13. Evaluate the definite integral $\int_2^3 x^3 dx$ as limit of sum.
14. Evaluate the definite integral $\int_a^b \sinh\ x\, dx$ as limit of sum.
15. Evaluate the definite integral $\int_0^{\pi/2} \cos\ x\, dx$ as limit of sum.

Further/Suggested Readings

George B. Thomas Jr., Joel Hass, Christopher Heil & Maurice D. Weir (2018). *Thomas' Calculus* (14th edition). Pearson Education.

Howard Anton, I. Bivens & Stephan Davis (2016). *Calculus* (10th edition). Wiley India.

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Unit 03: Integration by using reduction formula

CONTENTS

Objectives

Students will

- learn about an alternate method to solve the integrals.
- be able to solve the problems employing reduction formula.
- prove various properties of definite integrals.
- be able to solve definite integrals by using its properties.

Introduction

By now, you are well aware of the methods of substitution, method of integration by parts and the method in which we decompose the given integrand in the sum of integrands with known integrals. There is one more technique and that is called integration by successive reduction or integration using reduction formula.

3.1 Reduction formula

Any formula which expresses an integral in terms of another integral of the same type but of lesser degree or order is called a reduction formula. The successive application of the reduction formula enables us to express the integral of the general member of the class of functions in terms of that of the simplest member of the class. Mostly we obtain the reduction formula by using integration by parts. We can understand the method by the following examples.

Establish a reduction formula for $\int x^n e^{ax} dx$.

Here the integrand is $x^n e^{ax}$. This is a general form of the function. In the reduction formula we seek a relation of the given integral with another integral having the same form but involving $n-1$ or $n-2$ etc. And that can be accomplished by integrating the given integral by parts.

Let $I_n = \int x^n e^{ax} dx$

Integrating by parts, we get

$$I_n = \frac{x^n e^{ax}}{a} - \int n \frac{x^{n-1} e^{ax}}{a} dx$$

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Here you can see that I_{n-1} is having the same type as that of given integral except that the $'n'$ has reduced to $'n-1'$. Hence you can see the justification for the name 'reduction formula' also.

Now if we need to calculate $\int x^3 e^{ax} dx,$ we know what to do!

In the reduction formula $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$ we will get I_3 in terms of I_2, I_2 in terms of I_1 and I_1in terms of I_0. The last integral in the recursion I_0 is so easy to calculate. Then backward substitution leads to the required integral.

Let's see how it goes.

$I_3 = \frac{x^3 e^{ax}}{a} - \frac{3}{a} I_2$ -----(1)

$I_2 = \frac{x^2 e^{ax}}{a} - \frac{2}{a} I_1$ -----(2)

$I_1 = \frac{x^1 e^{ax}}{a} - \frac{1}{a} I_0$ -----(3)

Solving the right side of (3).

$$I_1 = \frac{x^1 e^{ax}}{a} - \frac{1}{a} \frac{e^{ax}}{a}$$

$$I_2 = \frac{x^2 e^{ax}}{a} - \frac{2}{a} \left(\frac{x^1 e^{ax}}{a} - \frac{1}{a} \frac{e^{ax}}{a} \right)$$

$$I_3 = \frac{x^3 e^{ax}}{a} - \frac{3}{a} \left(\frac{x^2 e^{ax}}{a} - \frac{2}{a} \left(\frac{x^1 e^{ax}}{a} - \frac{1}{a} \frac{e^{ax}}{a} \right) \right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x e^{ax}}{a^3} - 6 \frac{e^{ax}}{a^4}$$

which is the required solution.

Establish a reduction formula for $\int x^m \text{sin} nx \, dx$.

In this integral, there are two parameters m and n. In order to have a reduction formula, let us integrate by parts taking x^m as the first function and $\sin nx$ as the second function.

$$\int x^m \text{sin} nx \, dx = -\frac{x^m cosnx}{n} + \frac{m}{n} \int x^{m-1} \cos nx dx$$

Again integrating by parts,

$$\int x^m \text{sin} nx \, dx = -\frac{x^m cosnx}{n} + \frac{m x^{m-1} sinnx}{n^2} - \frac{m(m-1)}{n^2} \int x^{m-2} \sin nx dx$$

Therefore here we got the relation in two integrals of the same type but the one on the right side is of lower degree. The left side integral can be written as $I_{m,n}$ and the right-side integral which is reduced version can be written as $I_{m-2,n}$.

Establish a reduction formula for $\int x^n e^{-x} dx$ and deduce that $\int_0^\infty x^n e^{-x} dx = n!$ where n is any natural number.

$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx$$

Or

$I_n = -x^n e^{-x} + n I_{n-1}$ is the required reduction formula. Now for the deduction, consider the integral,

$$\int_0^t x^n e^{-x} dx = |-x^n e^{-x}|_0^t + n \int_0^t x^{n-1} e^{-x} dx$$

$$= -t^n e^{-t} + n \int_0^t x^{n-1} e^{-x} dx$$

The first term on the right-hand side tends to zero as $t \to \infty$.

Therefore, $\int_0^\infty x^n e^{-x} dx = \int_0^\infty x^{n-1} e^{-x} dx$

Or $I_n = nI_{n-1}$

$I_{n-1} = (n-1)I_{n-2}$

$I_{n-2} = (n-2)I_{n-3}$

$I_{n-3} = (n-3)I_{n-4}$

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$I_1 = 1. I_0 = 1$ (The integral I_0 can easily be calculated to be 1.)

By back substitution we get

$I_n = n(n-1)(n-2) \dots 3.2.1 = n!$

Establish the reduction formula for $\int \frac{x^n}{(logx)^m} dx$

In this integral, there are two parameters, and the idea is to get a relation of the given integral with another integral of same type but reduced parameter(s) any one or both.

To integrate the given function, let us write the integrand in the following way:

$$I_{m,n} = \int x^{n+1} \left[\frac{1}{(logx)^m} \cdot \frac{1}{x}\right] dx$$

Now the integration by parts can be applied taking the term in bracket as the second function. Therefore,

$$I_{m,n} = x^{n+1} \frac{(\log x)^{-m+1}}{-m+1} - \int (n+1)x^n \frac{(\log x)^{-m+1}}{-m+1} dx$$

$$I_{m,n} = x^{n+1} \frac{(\log x)^{-m+1}}{-m+1} + \frac{n+1}{m-1} \int \frac{x^n}{(\log x)^{m-1}} dx$$

Or $I_{m,n} = x^{n+1} \frac{(\log x)^{-m+1}}{-m+1} + \frac{n+1}{m-1} I_{m-1,n}$ is the required reduction formula.

Evaluate $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ where n is a positive integer.

The reduction formula for the integral is given as,

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

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$I_3 = \frac{2}{3} I_1$ if n is odd

$I_2 = \frac{1}{2} I_0$ if n is even

Therefore it can be compiled as

$$I_n = \begin{cases} \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}\dots\dfrac{2}{3}I_1 & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}\dots\dfrac{1}{2}I_0 & \text{if } n \text{ is even} \end{cases}$$

Now $I_1 = \int_0^{\frac{\pi}{2}} sinx\, dx = 1$ and

$$I_0 = \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2}$$

Thus the above integral can finally be written as

$$I_n = \begin{cases} \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}\dots\dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}\dots\dfrac{1}{2}.\dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

Obtain a reduction formula for $\int \sin^p x \cos^q x\, dx$.

$$\int \sin^p x \cos^q x\, dx = \int \sin^{p-1} x(\sin x \cos^q x)dx$$

$$= \sin^{p-1} x\left(-\frac{\cos^{q+1}x}{q+1}\right) - \int (p-1)\sin^{p-2} x.\left(-\frac{\cos^{q+1}x}{q+1}\right)\cos x\, dx$$

$$= \left(-\frac{\sin^{p-1}\text{x}\cos^{q+1}x}{q+1}\right) + \frac{p-1}{q+1}\int \sin^{p-2} x.\cos^q x \cos^2 x\, dx$$

Now here we can use the trigonometric identity $cos^2 x = 1 - \sin^2 x$ in the last term.

$$\left(1+\frac{p-1}{q+1}\right)\int \sin^p x \cos^q x\, dx = -(\sin^{p-1} xcos^{q+1}x)/(q+1) + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx$$

$$\int \sin^p x \cos^q x\, dx = (-\sin^{p-1} x \cos^{q+1} x)/(p+q) + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx$$

is the required reduction formula.

Construct the reduction formula for $\int \frac{x^n}{\sqrt{ax^2+bx+c}}dx$ where $n \in \boldsymbol{N}$

Here the integrand is an irrational function and it can be re written by involving the derivative of the term in the denominator as follows:

$$x^n = \frac{2ax+b-b}{2a}x^{n-1}$$

$$I_n = \frac{1}{2a}\int \frac{(2ax+b)x^{n-1}}{\sqrt{ax^2+bx+c}}dx - \frac{b}{2a}\int \frac{x^{n-1}}{\sqrt{ax^2+bx+c}}dx$$

$$I_n = \frac{1}{a}x^{n-1}\sqrt{ax^2+bx+c} - (n-1)\, I_n - \frac{b(n-1)}{a}I_{n-1} - \frac{c(n-1)}{a}I_{n-2} - \frac{b}{2a}I_{n-1}$$

$I_n = \frac{1}{na}x^{n-1}\sqrt{ax^2+bx+c} - \frac{b(2n-1)}{2na}I_{n-1} - \frac{c(n-1)}{na}I_{n-2}$ is the required reduction formula.

Let us evaluate $\int \frac{x^3}{\sqrt{x^2-2x+2}}dx$. You can see that this integral is a particular case of the integral whose reduction formula is

$$\int \frac{x^n}{\sqrt{ax^2+bx+c}}dx = \frac{1}{na}x^{n-1}\sqrt{ax^2+bx+c} - \frac{b(2n-1)}{2na}\int \frac{x^{n-1}}{\sqrt{ax^2+bx+c}}dx - \frac{c(n-1)}{na}\int \frac{x^{n-2}}{\sqrt{ax^2+bx+c}}dx$$

The above reduction formula can be written as:

$$\int \frac{x^3}{\sqrt{x^2-2x+2}}dx = \frac{1}{3}x^2\sqrt{x^2-2x+2} + \frac{5}{3}\int \frac{x^2}{\sqrt{x^2-2x+2}}dx - \frac{4}{3}\int \frac{x^1}{\sqrt{x^2-2x+2}}dx \text{ -----(1)}$$

Also $\int \frac{x^2}{\sqrt{x^2-2x+2}}dx = \frac{1}{2}\sqrt{x^2-2x+2} + \frac{3}{2}\int \frac{x^1}{\sqrt{x^2-2x+2}}dx - \int \frac{1}{\sqrt{x^2-2x+2}}dx$ -----(2)

Let us work on the two integrals on the right hand side of the equation (2).

$\int \frac{x}{\sqrt{x^2-2x+2}}dx = \frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+2}}dx + \frac{1}{2}\int \frac{2}{\sqrt{(x-1)^2+1}}dx = (x^2-2x+2)^{1/2} + \sinh^{-1}(x-1)$ -----(3)

$\int \frac{1}{\sqrt{x^2-2x+2}}dx = \sinh^{-1}(x-1)$ -----(4)

Substituting (3) and (4) in (2), we get

$$\int \frac{x^2}{\sqrt{x^2-2x+2}}dx = \frac{1}{2}\sqrt{x^2-2x+2} + \frac{3}{2}(x^2-2x+2)^{\frac{1}{2}} + \frac{1}{2}\sinh^{-1}(x-1)$$

And the back substitution in equation (1) will give

$$\int \frac{x^3}{\sqrt{x^2-2x+2}}dx$$

$$= \frac{1}{3}x^2\sqrt{x^2-2x+2} + \frac{5}{3}\left(\frac{1}{2}\sqrt{x^2-2x+2} + \frac{3}{2}(x^2-2x+2)^{\frac{1}{2}} + \frac{1}{2}\sinh^{-1}(x-1)\right)$$
$$-\frac{4}{3}((x^2-2x+2)^{1/2} + \sinh^{-1}(x-1))$$

$$= \frac{(2x^2+12)}{6} - \frac{1}{2}\sinh^{-1}(x-1)$$

Form a reduction formula for $\int \frac{sinnx}{sinx}dx$

The parameter n can be seen in the numerator of the integrand. Here integration by parts may not help in getting a reduced form of the same type of integral. So we can use trigonometric identity ($\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$) as follows:

$$\sin nx - \sin(n-2)x = 2\cos(n-1)x\sin x$$

$$\sin nx = \sin(n-2)x + 2\cos(n-1)x\sin x$$

Now we can rewrite the given integral as

$$\int \frac{sinnx}{sinx}dx = \int \frac{\sin(n-2)x + 2\cos(n-1)x\sin x}{sinx}dx$$

$$\int \frac{sinnx}{sinx}dx = \int \frac{\sin(n-2)x}{sinx}dx + \int 2\cos(n-1)x\,dx$$

$$\int \frac{sinnx}{sinx}dx = \int \frac{\sin(n-2)x}{sinx}dx + \frac{2\sin(n-1)x}{n-1}$$

which is the required reduction formula connecting the given integral with its reduced version.

3.2 Properties of definite integral

Before studying the properties of definite integral, let us recapitulate some basics about them.

Definite Integral

The definite integral is an integral of the form $\int_a^b f(x)\ dx$. This integral is read as the integral from *a* to *b* of $f(x)\ dx$. The numbers *a* and *b* are said to be the limits of integration. For our problems, a is less than *b*. Definite Integrals are evaluated using the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function for $a \le x \le b$ and $F(x)$ be an anti-derivative of $f(x)$. Then $\int_a^b f(x)\ dx = [F(x)]|_a^b = F(b) - F(a)$.

If $f(x) \ge 0$ for $a \le x \le b$. Then

Definite Integral: $\int_a^b f(x)\ dx = \begin{matrix}\text{Area Between } f(x) \text{ and the} \\ x \text{ axis for } a \le x \le b\end{matrix}$

The Second Fundamental Theorem of Calculus

If *f* is continuous on an open interval, *I* containing *a*, then for every *x* in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x)$$

Chain Rule of Differentiation

$$\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$$

Suppose we let $F = f(g(x))$ and let $u = g(x)$. Then $F = f(u)$ and hence

$\frac{dF}{du} = f'(u)$ and $\frac{du}{dx} = g'(x)$

For $F = f(g(x))$ and $u = g(x)$ we can use these ideas to rewrite the chain rule as follows:

$$F'(x) = \frac{dF}{dx} = \frac{d}{dx}(f(g(x)) = f'(g(x))g'(x) = f'(u)\frac{du}{dx} = \frac{dF}{du}\frac{du}{dx}$$

This gives another way to write the chain rule, which is as follows:

Chain Rule: Alternative Form

If we want to differentiate the composition$F(x) = f(g(x))$, we set $u = g(x)$ and compute the following: $F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx}$

For example, suppose we want to differentiate$F(x) = (x^2 + 7)^{10}$

Then by taking, $u = x^2 + 7$, we have $F = u^{10}$

Hence, $\frac{dF}{du} = 10u^9$ and $\frac{du}{dx} = 2x$.

Therefore, for $F(x) = (x^2 + 7)^{10}$

$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = 10u^9(2x) = 10(x^2 + 7)^9(2) = 20x(x^2 + 7)^9$.

This way of expressing the chain rule can be useful when using the Second Fundamental Theorem of Calculus. Suppose $F(x) = \int_a^{g(x)} f(t)\,dt$, then taking $u = g(x)$ gives $F = \int_a^u f(t)\,dt$

Then $F'(x) = \frac{dF}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left(\int_a^u f(t)\,dt\right) \cdot \frac{du}{dx} = f(u) \cdot \frac{du}{dx} = f(g(x)) \cdot g'(x)$

Properties

1. $\int_a^b \phi(x)dx = \int_a^b \phi(t)dt$

Let $\int \phi(x)dx = F(x) + c_1$

and $\int \phi(t)dt = F(t) + c_2$

Therefore, $\int_a^b \phi(x)dx = [F(x) + c_1]_a^b = F(b) - F(a)$

Similarly, $\int_a^b \phi(t)dt = F(b) - F(a)$

This property explains the dummy nature of the variable of integration in a definite integral.

2. $\int_a^b \phi(x)dx = \int_a^c \phi(x)dx + \int_c^b \phi(x)$

Let $\int \phi(x)dx = F(x) + c_1$

Then the RHS= $[F(x) + c_1]_a^c + [F(x) + c_1]_c^b$

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$

$$= \int_a^b \phi(x)dx = LHS$$

3. $\int_a^b \phi(x)dx = -\int_b^a \phi(x)dx$

Let $\int \phi(x)dx = F(x) + c_1$

Consider the RHS = $-\int_b^a \phi(x)dx$

$= -[F(x) + c_1]_b^a$

$= [F(a) + c_1 - F(b) - c_1$

$= F(b) - F(a) = \int_a^b \phi(x)dx = LHS$

4. $\int_0^a \phi(x)dx = \int_0^a \phi(a-x)dx$

Letting $a - x = t$

or $x = a - t$

or $dx = -dt$

Now the RHS= $\int_0^a \phi(a-x)dx = -\int_a^0 \phi(t)dt = \int_0^a \phi(t)dt = \int_0^a \phi(x)dx$

5. $\int_0^{2a} \phi(x)dx = \begin{cases} 2\int_0^a \phi(x)dx & \text{if } \phi(2a-x) = \phi(x) \\ 0 & \text{if } \phi(2a-x) = -\phi(x) \end{cases}$

Consider $\int_a^{2a} \phi(x)dx$

Put $2a - x = t$ This implies $-dx = dt$. Therefore,

$$\int_a^{2a} \phi(x)dx = -\int_a^0 \phi(2a-t)dt$$

$$= \int_0^a \phi(2a-t)dt$$

$$= \int_0^a \phi(2a-x)dx$$

Now $\int_0^{2a} \phi(x)dx = \int_0^a \phi(x)dx + \int_a^{2a} \phi(x)dx$

$$= \int_0^a \phi(x)dx + \int_0^a \phi(2a-x)dx$$

The second integrand on the RHS can be $\phi(x)$ or $-\phi(x)$ and then accordingly it will yield $2\int_0^a \phi(x)dx$ or a zero. Hence the property.

6. $\int_{-a}^a \phi(x)dx = \begin{cases} 0 & \text{if } \phi(-x) = -\phi(x) \\ 2\int_0^a \phi(x)dx & \text{if } \phi(-x) = \phi(x) \end{cases}$

Consider $\int_{-a}^0 \phi(x)dx$

Put $x = -t, dx = -dt$

Then $\int_{-a}^0 \phi(x)dx = -\int_a^0 \phi(-t)dt = \int_0^a \phi(-x)dx$

Therefore, $\int_{-a}^a \phi(x)dx = \int_{-a}^0 \phi(x)dx + \int_0^a \phi(x)dx$

$$= \begin{cases} 0 & \text{if } \phi(-x) = -\phi(x) \\ 2\int_0^a \phi(x)dx & \text{if } \phi(-x) = \phi(x) \end{cases}$$

Now let us evaluate a few integrals using the properties.

Evaluate $\int_0^{\pi/2} \log(1 + tan\theta)\, d\theta$

Let $I = \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - tan\theta}{1 + tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1 + tan\theta}\right) d\theta$$

$$= \frac{\pi}{4}\log 2 - \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta$$

Or $I = \frac{\pi}{8}\log 2$

Evaluate $\int_0^{\pi} x \sin^6 x \cos^4 x\, dx$

Let $I = \int_0^{\pi} x \sin^6 x \cos^4 x\, dx$

$$= \int_0^{\pi} (\pi - x)\sin^6(\pi - x)\cos^4(\pi - x)dx$$

$$= \int_0^{\pi} (\pi)\sin^6(x)\cos^4(x)dx - \int_0^{\pi} (x)\sin^6(x)\cos^4(x)dx$$

$$2I = 2\pi \int_0^{\pi/2} \sin^6 x \cos^4 x\, dx$$

$$I = \frac{2\pi}{2}\frac{5.3.1.3.1}{10.8.6.4.2}\frac{\pi}{2}$$

$$I = \frac{3\pi^2}{512}$$

Summary

In this chapter we have seen one more technique of solving the integral by writing a recursion formula. Many general integrals can be solved by this method including their particular cases. We have also seen the proofs of the properties of the definite integral.

Key words

Definite integral, reduction formula, properties of definite integral

Self-Assessment

1. $\int_2^3 x^3 dx$ is equal to

   (a) 65 (b) 65/4 (c) ¼ (d) 63/4

2. If $m \neq n$, then $\int_0^{\pi} \cos mx \cos nx\, dx$ is

   (a) 0 (b) $\frac{\pi}{2}$ (c) π (d) 2π

3. $\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$ is

   (a) 5/15 (b) 6/15 (c) 7/15 (d) 8/15

4. $\int_0^{\frac{\pi}{2}} \sin^6 x\, dx$ is

   (a) 5/32 (b) $5\pi/32$ (c) 5/16 (d) $5\pi/16$

5. $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$ is

   (a) 16/35 (b) 6/15 (c) 17/15 (d) 8/35

6. $\int_0^{\frac{\pi}{2}} \cos^8 x\, dx$ is

   (a) 5/32 (b) $35\pi/256$ (c) 35/256 (d) $5\pi/16$

7. $\int_0^{\frac{\pi}{4}} \cos^6 2x \; dx$ is
   (a) $5/64$ (b) $5\pi/64$ (c) $35/256$ (d) $\pi/16$

8. $\int_0^1 x^2(1-x^2)^{\frac{3}{2}} dx$ is
   (a) $\frac{\pi}{32}$ (b) $\frac{\pi}{16}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

9. $\int_0^{\pi/2} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}} dx$ is
   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{8}$ (d) None of these

10. $\int_0^{\pi} xsin^6xcos^4x \; dx$ is
    (a) $3\pi^2$ (b) $\frac{3\pi^2}{51}$ (c) $\frac{3\pi^2}{512}$ (d) $\frac{\pi^2}{512}$

11. $\int_0^{\frac{\pi}{2}} \log \sin x \; dx$ is
    (a) $-\frac{\pi}{2}\log 2$ (b) $\frac{\pi}{2}\log 2$ (c) $-\frac{3\pi}{2}\log 2$ (d) $-\frac{\pi}{2}$

12. $\int_0^{\pi} \log\,(1+\cos x) dx$ is
    (a) $-\pi \log 2$ (b) $\pi \log\; 1/2$ (c) Both (a) and (b) (d) None of these

13. $\int_0^{\pi/2} \sin^5 x \cos^6 x \; dx$ is
    (a) $8/693$ (b) $8/69$ (c) $8/6$ (d) None of these

14. $\int_0^{\pi/2} \sin^6 x \cos^8 x \; dx$ is
    (a) $\frac{5}{4096}$ (b) $\frac{5\pi}{4096}$ (c) $\frac{\pi}{4096}$ (d) None of these

15. $\int_0^{\pi/2} \cos 2x \cos^3 x \; dx$ is
    (a) $\frac{8}{15}$ (b) $\frac{2}{15}$ (c) $\frac{2}{5}$ (d) $\frac{5}{2}$

Answers:

1	b	2	a	3	d	4	b	5	a
6	b	7	b	8	a	9	b	10	c
11	a	12	c	13	a	14	b	15	c

Review Questions

1. Construct the reduction formula for $\int \sin^p x \cos^q x \; dx$ where p, q are positive integers.
2. Evaluate the definite integral $\int_0^{\pi/2} \sin^p x \cos^q x \; dx$ where p, q are positive integers.
3. Using the properties prove that $\int_0^{\pi} (x \sin x)/(1+\cos^2 x) \; dx = \pi^2/4$.
4. Using the properties prove that $\int_0^{\pi/2} \sin^2 x/(\sin x + cos\, x) \;\; dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$
5. Evaluate $\int_0^{\pi/2} x \cot x \; dx$ using the properties of definite integral.

Further/Suggested Readings

George B. Thomas Jr., Joel Hass, Christopher Heil & Maurice D. Weir (2018). Thomas' Calculus (14th edition). Pearson Education.

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https://www.intmath.com/integration/4-definite-integral.php

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Unit 04: Limit of a Real Valued Function

CONTENTS

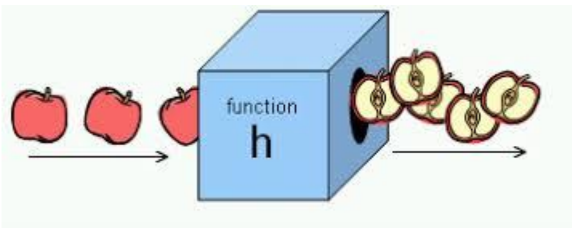
Objectives

Students will

- learn about the definition of real valued function
- learn about the limit of the function
- understand the epsilon delta definition of limit
- learn about infinite limit and limit at infinity

Introduction

In this unit we will understand one of the most crucial and fundamental concept of calculus. But before that let us have an idea about the function. You can consider a function as a kind of rule where you give in some input and get a specific output. The input is decided as everything that keeps the function well defined. The technical name for such input is the domain. Let us see what a function is! First of all there must be two non-empty sets A and B, then a function f from A to B is denoted as $f: A \rightarrow B$ and is defined as a function if for all the values in set A, there corresponds a unique value in set B. Set A is called the domain, B is called the codomain and $f(A)$ is called the range.

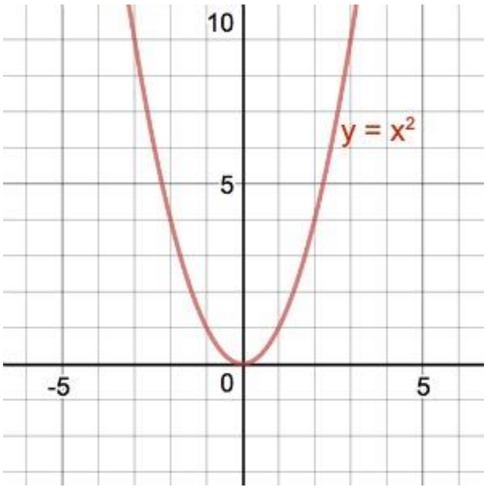


4.1 Real valued function

The function can be expressed as a set, a formula, a table or as a graph. If the range of the function is a set of real numbers, then it is called a real valued function.

For example: $\{(2,4),(3,9),(4,16),(5,25)\}$ is a function where domain is the set $\{2,3,4,5\}$ and range is the set $\{4,9,16,25\}$. Clearly, the domain and range are discrete in this case and are subsets of real numbers. In other form $y = x^2$ is a function. The input can be any real number as any real number will keep the function well defined and with any real input, the output is always going to be a non-negative real number. Therefore the range is set of non-negative real number. The same function can be represented in the form of the following graph:

Notes



We can show the first representation of function in the form of a table also, which is as follows:

x	y
2	4
3	9
4	16
5	25

Of course here the domain and range are discrete and finite subsets of real numbers.

Now let us consider a function $f(x) = \frac{x^2-9}{x-3}$. It is quite clear that the function is not well defined at $x = 3$, and is good for all other real numbers. Therefore the domain is set of all real numbers except 3. If you input these values of the domain, the function can give any real number as the output, except 6.

Now we know what happens at $x = 3$, can we just observe the behaviour of the function as x goes closer to 3 from all possible directions? See the following table and see the pattern of $f(x)$ as x goes close to 3 from left as well as from the right direction.

x	f(x)
2.9	5.9
2.99	5.99
2.999	5.999
3.01	6.01
3.001	6.001
3.0001	6.0001

So, you can observe that as x is approaching to 3, the $f(x)$ is approaching to 6. Then we say that the limit of the function exists at $x = 3$.

If the function does not approach to the same value from both directions, then we say that the limit of the function does not exist.

Here it must be mentioned that the word 'close' that has been used to define the concept of limit is not a crisp word. It can mean something to me and entirely different thing to you. This closeness has to be quantized in order to have a crisp definition of limit and that has been achieved through the epsilon delta definition.

4.2 Epsilon delta definition of the limit

A function is said to tend to limit l as x tends to c, if $\forall \epsilon > 0$ however small, $\exists\ \delta > 0$, such that

$$|f(x) - l| < \epsilon \text{ whenever } 0 < |x - c| < \delta$$

or

$$f(x) \in (l - \epsilon, l + \epsilon)\ \forall\ x \in (c - \delta, c)\ U\ (c, c + \delta)$$

The quantity ϵ is how close you would like $f(x)$ to be to its limit l; the quantity δ

is how close you have to choose x to c to achieve this.

1. Prove that $\lim_{x\to -1} 4x + 1 = -3$

To prove that $\lim_{x\to c} f(x) = l$ you can assume that someone has given you some small positive value of ϵ and you need to find a positive value of δ for which $|f(x) - l| < \epsilon$ whenever $0 < |x - c| < \delta$ holds. This δ surely depends on ϵ.

Here we want to find the δ such that whenever $|x + 1| < \delta$, $|4x + 1 + 3| < \epsilon$ for a predefined ϵ.

If we work out on the epsilon inequality, we can see $|4(x + 1)| < \epsilon$. That is $|x + 1| < \epsilon/4$. Now we can do a smart work here. If we consider $\epsilon/4$ as δ, we are done. Thus for a given ϵ and $\delta = \frac{\epsilon}{4}$, we have $|4x + 1 + 3| < \epsilon$ whenever $|x + 1| < \epsilon/4$.

2. Prove that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Let $\epsilon > 0$ be given. Then we would like to find a $\delta > 0$, such that

$$|f(x) - 0| < \epsilon \quad \text{whenever} \quad |x - 0| < \delta$$

Now

$$|f(x) - 0| = \left|x\sin\left(\frac{1}{x}\right)\right|$$
$$= |x|\left|\sin\left(\frac{1}{x}\right)\right|$$
$$\leq |x|$$

Now choosing $\delta = \epsilon$ we can see that

$$|f(x) - 0| < \epsilon \text{ whenever } |x| < \epsilon$$

$\therefore \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

4.3 Some results on limits

The precise definition of the limit is not so easy to use, and we won't use it very often in this course. Instead, there are a number of properties that limits have, which allow you to compute them without having to use the epsilon delta definition. Let us see some of the properties of limit in the form of following results.

Let f and g are two functions such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. Then,

$$\lim_{x\to a}(f + g)(x) = \lim_{x\to a}\big(f(x) + g(x)\big) = l + m$$

Notes

Here you can see that the limit of sum of two functions is the sum of their limits. In simple words, you can see that as $f(x)$ is getting close to l and $g(x)$ is getting close to m as $x \to a$, then $f(x)+g(x)$ will go close to $l+m$ only. Though it can be proved by definition, but here we will resort to common sense only. Similarly we can see some more properties:

$$\lim_{x\to a}(f-g)(x) = \lim_{x\to a}\big(f(x)-g(x)\big) = l-m$$

The limit of difference of two functions is the difference in the respective limit of the individual functions.

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x)g(x) = l\,m$$

The limit of the product of two functions is the product of the respective limits of the individual functions.

$$\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x)/g(x) = l\,/m \qquad (m \neq 0)$$

The limit of the quotient of two functions is the quotient of the limits of respective functions.

3. Evaluate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}$$

4. Evaluate $\lim_{x\to 0} \frac{e^{tanx}-e^x}{\tan x - x}$

$$\lim_{x\to 0} \frac{e^{tanx}-e^x}{\tan x-x} = \lim_{x\to 0} \frac{e^x(e^{tanx-x}-1)}{\tan x-x} = e^0.1 = 1$$

5. Evaluate $\lim_{x\to 0}\left(\frac{1^x+2^x+\cdots+n^x}{n}\right)^{a/x}$

Note that $\lim_{x\to 0}\ \big(1+f(x)\big)^x = e^{\lim_{x\to 0} f(x).x}$

Now we can modify the given function, so that it takes the form of the left hand side of the above result.

Then, $\lim_{x\to 0}\left(\frac{1^x+2^x+\cdots+n^x}{n}\right)^{a/x} = \lim_{x\to 0}\left(1+\frac{1^x+2^x+\cdots+n^x}{n}-1\right)^{a/x}$

$$= \lim_{x\to 0}\left(1+\frac{1^x+2^x+\cdots+n^x-n}{n}\right)^{a/x}$$

$$= e^{\lim_{x\to 0}\left(\frac{1^x-1+2^x-1+\cdots+n^x-1}{n}\right)^{a/x}}$$

$$= e^{\lim_{x\to 0}\left(\frac{xlog1+xlog2+\cdots+xlogn}{n}\right)^{\frac{a}{x}}}$$

$$= e^{\lim_{x\to 0}\left(\frac{xlog\, n!}{n}\right)^{\frac{a}{x}}}$$

$$= e^{\lim_{x\to 0}\frac{axlog\, n!}{xn}}$$

$$= n!^{\frac{a}{n}}$$

4.4 Limit at infinity and infinite limits

Here we will learn when do we say that a function is approaching to infinity as x is approaching to any number, as $x \to \infty$, when a function has a finite limit and what is the behaviour of the function as x approaches to positive/negative infinity.

$$\lim_{x\to c}\ f(x) = \infty$$

A function f is said to tend to ∞ as x tends to c, if for any $G > 0$, however large, there corresponds a $\delta > 0$ such that

$$\forall\, x \in (c-\delta)U(c, c+\delta)\,, \qquad f(x) > G$$

$$\lim_{x \to c} f(x) = -\infty$$

A function f is said to tend to $-\infty$ as x tends to c, if for any $G > 0$, however large, there corresponds a $\delta > 0$ such that

$$\forall\, x \in (c-\delta)U(c, c+\delta)\,, \qquad f(x) < -G$$

$$\lim_{x \to \infty} f(x) = l$$

A function f is said to tend to l as x tends to ∞, if for any given $\epsilon > 0$, there corresponds a $G > 0$ such that

$$|f(x) - l| < \epsilon\,, \forall\, x > G$$

$$\lim_{x \to -\infty} f(x) = l$$

A function f is said to tend to l as x tends to $-\infty$, if for any given $\epsilon > 0$, there corresponds a $G > 0$ such that

$$|f(x) - l| < \epsilon\,, \forall\, x < -G$$

$$\lim_{x \to \infty} f(x) = \infty$$

A function f is said to tend to ∞ as x tends to ∞, if for any given $\epsilon > 0$, there corresponds a $G > 0$ such that

$$f(x) > \epsilon\,, \forall\, x > G$$

$$\lim_{x \to -\infty} f(x) = \infty$$

A function f is said to tend to ∞ as x tends to $-\infty$, if for any given $\epsilon > 0$, there corresponds a $G > 0$ such that

$$f(x) > \epsilon\,, \forall\, x < -G$$

$$\lim_{x \to \infty} f(x) = -\infty$$

A function f is said to tend to $-\infty$ as x tends to ∞, if for any given $\epsilon > 0$, there corresponds a $G > 0$ such that

$$f(x) < \epsilon\,, \forall\, x > G$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

A function f is said to tend to $-\infty$ as x tends to $-\infty$, if for any given $\epsilon > 0$, there corresponds a $G > 0$ such that

$$f(x) < \epsilon\,, \forall\, x < -G$$

Let us see some problems now!

1. Prove that $\lim_{x \to 0+} \frac{1}{x} = \infty$, $\lim_{x \to 0-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Let $\frac{1}{x} = y$

Consider the case when $x > 0$. This implies that $y > 0$.

Let $G > 0$ be any number, then

$\frac{1}{x} > G$ if $0 < x < \frac{1}{G}$

This implies that $\lim_{x\to 0+} \frac{1}{x} = \infty$

Now consider the case when $x < 0$. This implies that $y < 0$.

Let $G > 0$ be any number, then

$\frac{1}{x} < -G$ if $-\frac{1}{G} < x < 0$

This implies $\lim_{x\to 0-} \frac{1}{x} = -\infty$

Therefore the left hand limit is different from the right hand limit as x is approaching to zero and $\lim_{x\to 0} \frac{1}{x}$ does not exist.

2. For $f(x) = \frac{2x+1}{x-3}$ show that $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$

Let $\epsilon > 0$ be given

Now $\left|\frac{2x+1}{x-3} - 2\right| = \frac{7}{|x-3|} < \epsilon$ for $x > \frac{7}{\epsilon} + 3$

$\therefore \lim_{x\to\infty} f(x) = 2$

Again $\left|\frac{2x+1}{x-3} - 2\right| = \frac{7}{|x-3|} < \epsilon$ for $x < -\frac{7}{\epsilon} + 3$

$\therefore \lim_{x\to-\infty} f(x) = 2$

If you look at the definitions of limits at infinity, you can find a positive number G, in both cases, which fulfills the required criterion.

3. Evaluate $\lim_{x\to\infty}(\sqrt{x^2+x+1} - \sqrt{x^2+1})$

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right)$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}+\frac{1}{x}+1} + \sqrt{\frac{1}{x^2}+1}} = \frac{1}{2}$$

Prove by epsilon delta definition that $\lim_{x\to 2}(3x-4) = 2$.

L'Hopital Rule

We have got a pretty good idea that for a quotient of two functions, such that the individual limit of numerator is non-zero and of denominator is zero, then the overall limit of the quotient function does not exist. In case the individual limit of numerator is zero and that of the denominator is non-zero, then the overall limit of the quotient function is zero. The third case needs a special attention. If the individual limit of both numerator as well as the denominator is zero, then this is called one of the indeterminate forms and there are chances to get its value by using L'Hopital's rule.

Consider $\lim_{x\to 0} \frac{\sin x}{x}$. Here the individual limit of the numerator is zero and of the denominator is also zero. This is $\frac{0}{0}$ form. To evaluate this, we will have the following rule (for derivation you can see the link 4 in the last section of the chapter).

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a.

Notes

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$ assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Therefore using the above result for $\lim_{x\to 0}\frac{\sin x}{x}$ as it is a $\frac{0}{0}$ form. We can write the given limit as $\lim_{x\to 0}\frac{\cos x}{1}$. And the limit can be evaluated to be one.

There are mainly seven indeterminate forms and we try to convert them in $\frac{0}{0}$ or in $\frac{\infty}{\infty}$ form first if they are not so. And the by the above mentioned formula the limit can be evaluated. Let us see one more question for better clarity.

Let us evaluate $\lim_{x\to 0}(x\log\ x)$. Clearly it is $0\cdot\infty$ form. So first of all we will rewrite the given function in the desirable $(0/0\ or\ \infty/\infty)$ form.

$$
\begin{aligned}
\lim_{x\to 0}(x\log\ x). &= \lim_{x\to 0}\frac{\log x}{1/x}\\
&= \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0}(-x)\\
&= 0
\end{aligned}
$$

Now let us find the $\lim_{x\to 0}(\cot\ x)^{1/\log\ x}$. Clearly this is an indeterminate form of type ∞^0. First of all we will rewrite the given function into the required form. Here you can see the function as a power of another function. So logarithm can simplify the system.

$$
\begin{aligned}
\log y &= \frac{1}{\log x}\log(\cot x)\\
\Rightarrow \lim_{x\to 0}\log\ y &= \lim_{x\to 0}\frac{\log\ \cot\ x}{\log x}\\
&= \lim_{x\to 0}\frac{-\text{cosec}^2 x/\cot x}{\frac{1}{x}}\\
&= \lim_{x\to 0}\frac{-x}{\sin x}\cdot\frac{1}{\cos x}\\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \log\lim_{x\to 0} y = -1\\
&\Rightarrow \lim_{x\to 0} y = e^{-1} = \frac{1}{e}
\end{aligned}
$$

Summary

We learnt about the concept of going close to a number from all possible directions. Here we are dealing with real numbers only so there are only two directions i.e. left and right. The limit of a function was defined more rigorously using the epsilon delta definition. Since the epsilon delta definition is tedious to apply, so some properties that can be proved by basic definition come handy to evaluate the limits of various composite functions. Moreover we learnt the concept of the limit at infinity and of infinite limits in eight different cases.

Key words

limit, epsilon-delta definition of limit, limit at infinity, infinite limits

Review Questions

1. When do you say that a function $f(x)$ is approaching to infinity as $x \to \infty$?
2. The L'Hopital rule is given as

Notes

(a) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

(b) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$

(c) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

(d) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} [\frac{f(x)}{g(x)}]$

3. Evaluate $\lim\limits_{x \to \frac{\pi}{2}} \frac{\tan 5x}{\tan x}$
4. $\lim\limits_{x \to 0} \frac{\sin x}{x}$ is
   (a) 0 (b) 1 (c) -1 (d) undefined
5. $\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n$ is given as
   (a) 1 (b) 2 (c) 3 (d) e
6. Show that $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$

Further/Suggested Readings

George B. Thomas Jr., Joel Hass, Christopher Heil & Maurice D. Weir (2018). *Thomas' Calculus (*14th edition). Pearson Education.

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Unit 05: Continuity of a real valued function

CONTENTS

Objectives

After studying this unit Students will

- learn about the concept of continuity and its geometric interpretation.
- explore some properties of the continuous functions
- learn various types of discontinuity
- be able to differentiate in continuity and uniform continuity

Introduction

With the word 'continuity', the first thing that comes to our mind is the ability to draw a graph without lifting the pen.In this unit we will learn when can we tag a function as a continuous function and if a function is continuous then what more can we know about the function. You will learn about the various types of discontinuities and the concept of uniform continuity also.

5.1 Concept of continuity

Consider a function $f(x)$.As the independent variable x will change, somehow $f(x)$ will also change. The idea of continuity is that if a small change is happening in x then a small change must happen in $f(x)$ i.e. the change in $f(x)$ should not be sudden for a small change occurring in x. Now here the word 'small' is not defined in a complete sense. My idea of small can differ from your idea of small. Thus a more precise epsilon delta definition is there to address this issue.

Continuity of a function at a point in an interval

A function f is said to be continuous at a point c, if to any $\epsilon > 0$, there corresponds a number $\delta > 0$ such that

$$|f(c+h) - f(c)| < \epsilon$$

for all values of h, such that $|h| < \delta$.

In a different manner, a function $f(x)$ is said to be continuous at c, if $\exists$ an interval $(c-\delta, c+\delta)$around c, such that for all $x \in (c-\delta, c+\delta)$, we have

$$f(c) - \epsilon < f(x) < f(c) + \epsilon$$

A function $f(x)$ is continuous at an interior point c if and only if $\lim_{x\to c} f(x) = f(c)$

Show that $f(x) = 3x + 1$ is continuous at $x = 1$.

Here $f(1) = 4$

$f(x) - f(1) = 3x - 3 = 3(x - 1)$

Let $\epsilon > 0$ be given, we will show that $|f(x) - f(1)| < \epsilon$ for some $\delta > 0$ such that $|x - 1| < \delta$.

Let $x > 1$

It implies $3(x - 1) > 0$

Now $|f(x) - f(1)| = 3(x - 1) < \epsilon$

i.e. if $x - 1 < \frac{\epsilon}{3}$

i.e. if $x < 1 + \frac{\epsilon}{3}$ ------ (1)

Now let $x < 1$

It implies $3(x - 1) < 0$

Now $|f(x) - f(1)| = 3(x - 1) < \epsilon$

i.e. if $1 - x < \frac{\epsilon}{3}$

i.e. if $x > 1 - \frac{\epsilon}{3}$ -----(2)

From (1) and (2) we can say that

$|f(x) - f(1)| < \epsilon$ if $1 - \frac{\epsilon}{3} < x < 1 + \frac{\epsilon}{3}$

i.e. if $-\frac{\epsilon}{3} < x - 1 < \frac{\epsilon}{3}$

In other words we have found a δ such that $-\delta < x - 1 < \delta$ or $|x - 1| < \delta$

Therefore f is continuous at 1.

Prove that $f(x) = sinx$ is continuous at any point c in the domain.

Let $\epsilon > 0$ be any number. Consider

$|f(x) - f(c)| = |\sin x - \sin c|$

$= |2\, cos\dfrac{x + c}{2}\ \ sin\dfrac{x - c}{2}|$

$= 2\left|\cos\frac{x+c}{2}\right|\left|sin\frac{x-c}{2}\right|$----- (1)

Now $\left|\sin\frac{x-c}{2}\right| \le \left|\frac{x-c}{2}\right|$ and $|\cos\frac{x+c}{2}| \le 1\ \forall\ x$ and c.

Now (1) can be written as

$|\sin x - \sin c| \le 2\left|\dfrac{x - c}{2}\right| = |x - c|$

Therefore $|\sin x - \sin c| < \epsilon$ when $|x - c| < \epsilon = \delta$

Thus there exists an interval around c, such that $\forall\ x \in (c - \delta, c + \delta)$,

$|\sin x - \sin c| < \epsilon$

Therefore $f(x) = \sin x$ is continuous at c.

Examine $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$

When $x \ne 1$ the given function can be written as $y = x + 1$.

Let $\epsilon > 0$ be any number, however small.

Let $x > 1$ then $y > 2$

$|y-2| = y-2 = x+1-2 = x-1 < \epsilon \text{ if } x < 1+\epsilon$

Therefore there exists an interval $(1, 1+\epsilon)$ such that $|y-2| < \epsilon$

$\therefore \lim y = 2$ as x tends to 1^+.

Let $x < 1$ then $y < 2$

$|y-2| = 2-y = 2-x-1 = 1-x < \epsilon \text{ if } x > 1-\epsilon$

Therefore there exists an interval $(1-\epsilon, 1)$ such that $|y-2| < \epsilon$

$\therefore \lim y = 2$ as x tends to 1^-.

Thus from both directions $|y-2| < \epsilon$ whenever $|x-1| < \epsilon$

$\therefore \lim y = 2$ as $x \to 1$.

5.2 Properties of continuous functions

A function is said to be continuous if it is continuous at every point of its domain. Check the domain of the function and apply the definition of continuity at the suspicious point.

Let us check the function $f(x) = \sin^2 x$ for continuity.

Cleary the set of real numbers is the domain of the function and to check if the function is continuous or not, we need to check that the function should be continuous on each point of its domain. Let us consider an arbitrary real number c and any $\epsilon > 0$. Then

$|f(x) - f(c)| = |\sin^2 x - \sin^2 c| = |\sin(x+c)||\sin(x-c)|$

$\leq |\sin(x-c)| \leq |x-c|$

If $|x-c| < \epsilon = \delta$ then $|f(x) - f(c)| < \epsilon$

Therefore by definition $\sin^2 x$ is continuous for $x = c \forall\, x$ as c is any number.

Let us now see a piecewise function for its continuity. Let the function be

$$f(x) = \begin{cases} x & \text{when } 0 \leq x < \frac{1}{2} \\ 1 & \text{when } x = \frac{1}{2} \\ 1-x & \text{when } \frac{1}{2} < x < 1 \end{cases}$$

Here the function is a polynomial function or a constant function in the piecewise domains.

The problem of discontinuity can occur at $\frac{1}{2}$. So let us work out on the left hand limit, the right hand limit and the value of the function at $\frac{1}{2}$.

The left hand limit is $\lim_{x \to \frac{1}{2}^-} x = \frac{1}{2}$

The right hand limit is $\lim_{x \to \frac{1}{2}^+} x = \frac{1}{2}$

The value of the function at $\frac{1}{2}$, $f\left(\frac{1}{2}\right) = 1$

The limit of the function is existing but is not equal to the value of the function, so the function is not continuous at $x = \frac{1}{2}$. The point $x = \frac{1}{2}$ is very much in the domain of the function, so we can say that the function is not continuous.

Theorems on continuous functions

Suppose f and g are two functions defined in a neighbourhood of the point a. Then, if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are well-defined, we have the following:

(1) $\lim_{x \to a}(f(x) + g(x))$ is defined, and equals the sum of the values $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$.

(2) $\lim_{x \to a}(f(x) - g(x))$ is defined, and equals $\lim_{x \to a} f(x) - \lim_{x \to a} g(x)$.

(3) $\lim_{x \to a} f(x)g(x)$ is defined, and equals the product $\lim_{x \to a} f(x) \lim_{x \to a} g(x)$.

The scalar multiples result basically states that if $\lim_{x\to a} f(x)$exists, and for any real number α

$$\lim_{x\to a} \alpha f(x) = \alpha \lim_{x\to a} f(x)$$

(4) $\lim_{x\to a} f(x)$and$\lim_{x\to a} g(x)$ exist and if $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

The notions of sum, difference, product and quotient of functions can be rewritten as:

$\lim_{x\to a} (f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$\lim_{x\to a} (f-g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

$\lim_{x\to a} (f.g)(x) = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$

$\lim_{x\to a} (f/g)(x) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$provided$g(a) \neq 0$

You are much familiar with some elementary functions such as the constant function, identity function, rational function, trigonometric functions, inverse trigonometric functions , exponential functions and the logarithmic functions.

The domain of continuity of a function is same as the domain of the definition of function.

All the above mentioned elementary functions are continuous in their domain.

For the composed functions, we need to check the continuity every time. Some properties of continuous functions can be stated as follows.

1. If $f(x)$ is continuous at c and $f(c) \neq 0$ the there exists an open interval $(c-\delta, c+\delta)$ around c such that $f(x)$has the sign of $f(c)$ for every x in this interval.
2. If f is continuous in a closed interval $[a,b]$ and $f(a), f(b)$ are of opposite signs, then $f(x)$ is zero for atleast one $x \in [a,b]$.
3. If f is continuous in $[a,b]$, then there exist points c and d in the interval $[a,b]$ where f assumes its greatest and least values M and m that is,
   $f(c) = M$and$f(d) = m$.

5.3 Intermediate value theorem

This theorem applies to the continuous functions. Using this theorem you can prove the solvability of the algebraic and transcendental equations.

For example, $\sin x + x^5 = 0$ is a transcendental equation and we can use the intermediate theorem to know whether it is solvable or not.

INTERMEDIATE VALUE THEOREM: Let f be a continuous function on the closed interval $[a,b]$. Assume that m is a number (y-value) between $f(a)$and$f(b)$. Then there is at least one number c(x- value) in the interval $[a,b]$which satisfies $f(c) = m$.

Assume that a function f is a continuous and$m = 0$. Then the conditions $f(a) < 0$ and$f(b) > 0$ would lead to the conclusionthat the equation $f(x) = 0$ is solvable for x, i.e., $f(c) = 0$.

Intermediate Value Theorem guarantees the existence of a solution, but not what the solution is.

Steps to solve a problem:

1. Define a function $y = f(x)$.
2. Establish that f is continuous.
3. Choose an interval $[a,b]$.
4. Define a number (y-value) m.
5. Establish that there exists a value c, in $[a,b]$ such that $f(c) = m$.

Now let us use the intermediate value theorem to prove that the equation is solvable on the given interval in the following examples.

$3x^5 - 4x^2 = 3$ on$[0,2]$

Let $f(x) = 3x^5 - 4x^2 - 3$

$f(x)$is continuous for every x as it is a polynomial function.

Now here $f(0) = -3, f(2) = 77$

Let $m = 3$ as $f(0) < m < f(2)$

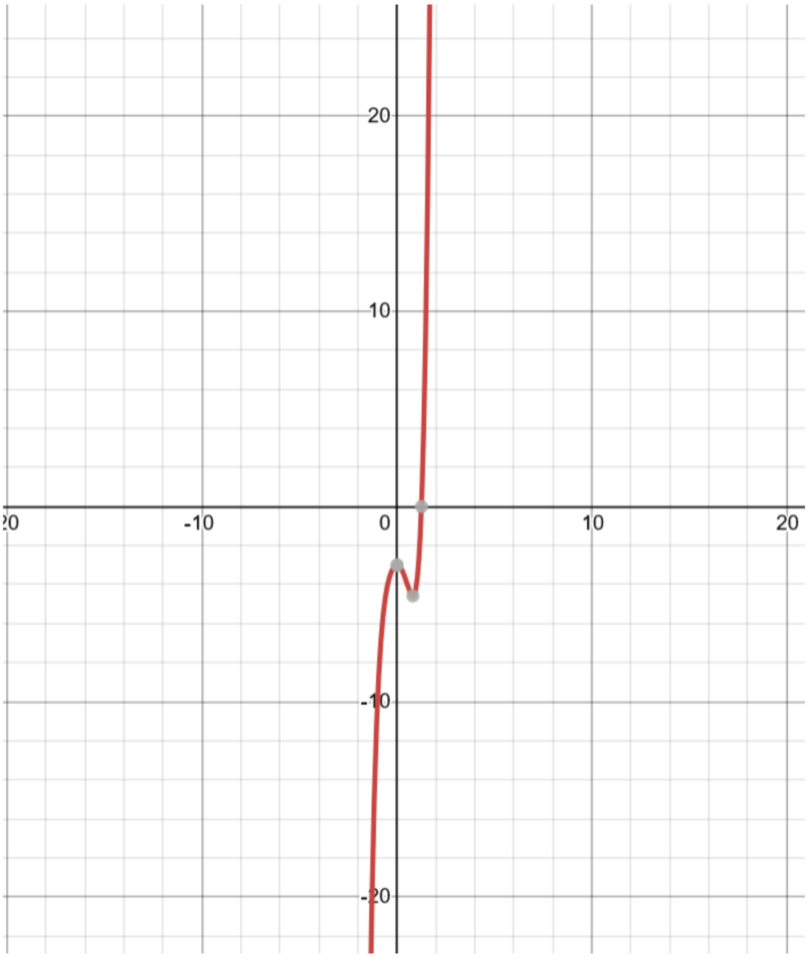
By intermediate value theorem, we can conclude that there exists $c \in [0,2]$ such that $f(c) = m$

i.e.$3c^5 - 4c^2 + 3 = 3$

or$c^2(3c^3 - 4) = 0$

or$c = 0,0,\left(\frac{4}{3}\right)^{1/3}$

Here all the values of c are lying in the given interval. In fact if only one value lies in the interval, that itself is sufficient to say that the equation is solvable. If we draw the function on xy −plane, the results are quite obvious.



$x^2 - 4x^3 + 1 = x - 7$

Let $f(x) = x^2 - 4x^3 + 1 - x + 7 = -4x^3 + x^2 - x + 8$

Here $f(x)$ is continuous for all x.

Now in this problem the interval is not given as in the previous example. Therefore by hit and trial we can look for two values x such that the value of the function at those values are of opposite signs.

$$f(0) = 8$$

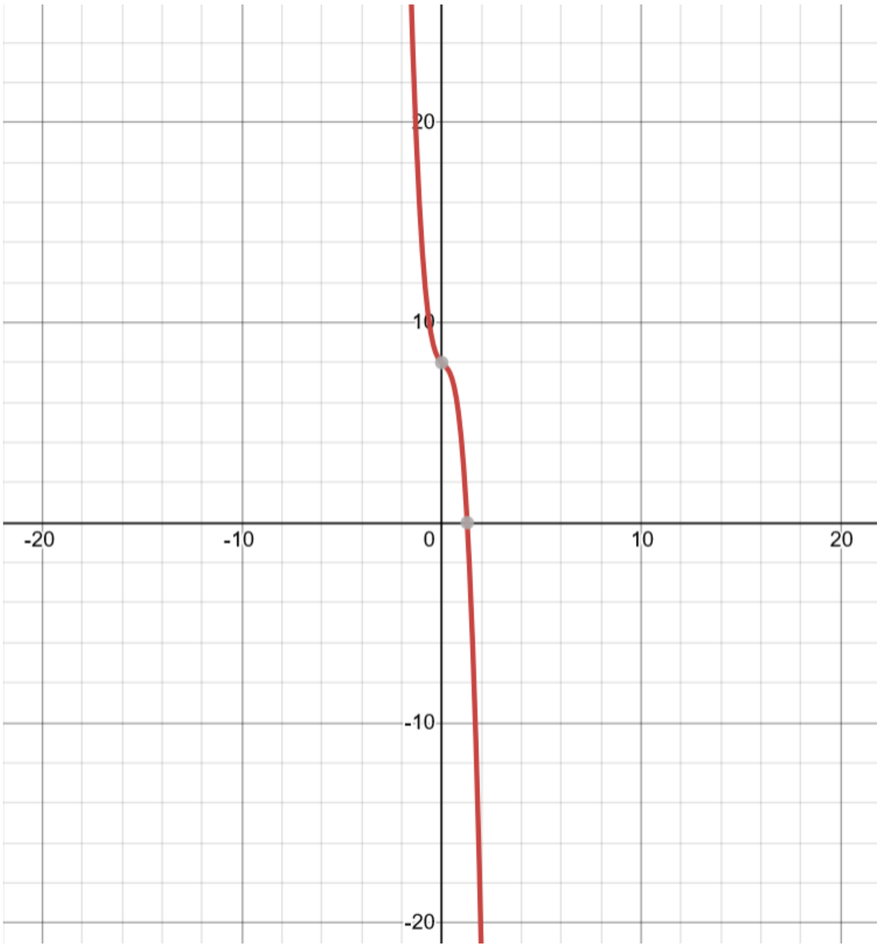
$$f(2) = -22$$

Let $m = 0$ (any number between -22 and 8 can be chosen as m)

Clearly all the assumptions of intermediate value theorem are met. Therefor there exists a $c \in [0,2]$ such that $f(c) = m$

i.e. $-4c^3 + c^2 - c + 8 = 0$ and this equation is solvable.

If we check it by actually drawing the graph, we can easily see that the intermediate value theorem is getting satisfied in the said interval.



$x^3 + 2 = sinx$

Now this equation is a transcendental one. and no interval is given.

Let $f(x) = x^3 + 2 - \sin x$

The function $f(x)$ is the sum of continuous functions so it is a continuous function for all x.

$$f(0) = 2$$

$$f(-\pi) = -29$$

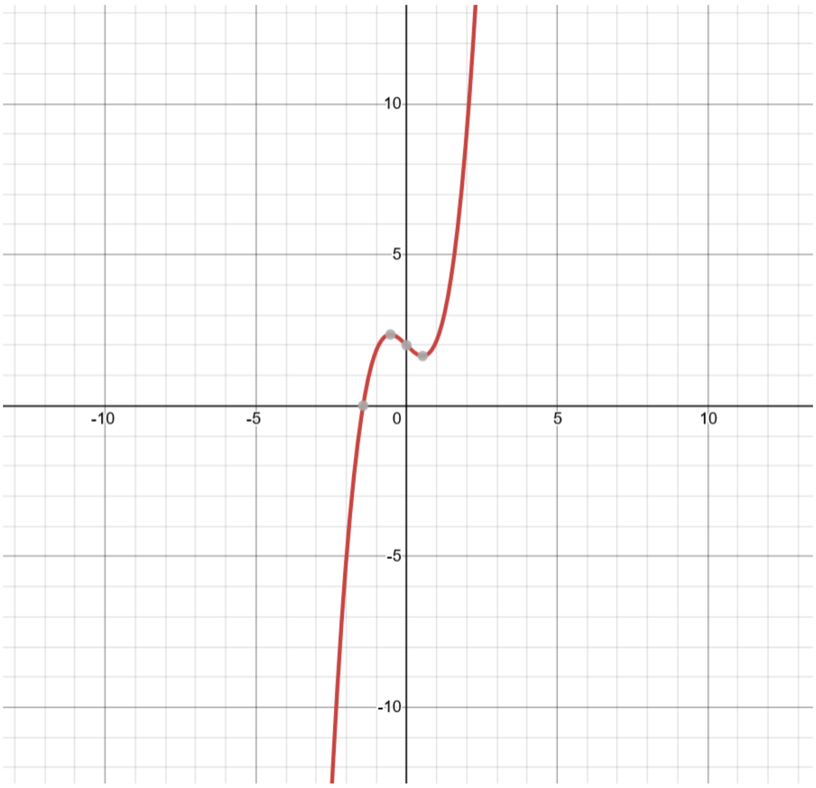
Choosing m such that $-29 < m < 2$.

Let $m = 0$

$$\therefore \exists\, c \in [-\pi, 0] \text{ such that } f(c) = m$$

i.e.$c^3 + 2 - \sin c = 0$

Therefore the equation is solvable. And we can verify this by actually plotting the graph which is as follows:



5.4 Geometric interpretation of continuity

Continuity at a point c can be defined for a function f on an open interval containing c. We may say that f is continuous at c if $f(x)$ tends to $f(c)$ as x tends to c. Or in plane words, the function f is continuous if the difference in x and c is small, the difference in $f(x)$ and $f(c)$ will also be small. That cannot be abrupt.

In simple words,

(i) Function f will be continuous at $x = c$ if there is no break in the graph of the function at the point $(c, f(c))$.

(ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

5.5 Types of discontinuity

Discontinuity of a function

A function is said to be discontinuous at a point of its domain if it is not continuous at that point. Moreover, that point is called the point of discontinuity of the function.

Two possibilities:

1. The limit of the functions exists as x tends to c but is different from the value of the function at c.
2. The limit of the function does not exist at c.

On this basis, we can classify the discontinuities as follows:

1. Removable discontinuity
2. Jump discontinuity (Discontinuity of the first kind)
3. Discontinuity of the first kind from the left
4. Discontinuity of the first kind from the right
5. Discontinuity of the second kind (Non removable or essential discontinuity)
6. Discontinuity of the second kind from the left

7. Discontinuity of the second kind from the right

Removable discontinuity

If $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$ (which may or may not exist), then that discontinuity is called removable because we can redefine the function so that the function becomes continuous at point c.

i.e.$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \neq f(c)$

For instance consider the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Clearly the limit of the function exists and is 1. But the value of the function at $x = 0$ is not 1.

Thus the function has a removable discontinuity at $x = 0$.

It means that we can redefine the function to remove this discontinuity by writing

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Jump discontinuity

This discontinuity is also called discontinuity of first kind. If $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist but are not equal, then we get a jump discontinuity.

i.e.$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

For instance $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

Clearly the left hand limit is -1 and right hand limit is +1 and you can see a jump right there near zero, hence the name 'jump discontinuity'.

Discontinuity of the first kind from the left

If $\lim_{x\to c^-} f(x) \neq f(c) = \lim\limits_{x\to c^+} f(x)$

Discontinuity of the first kind from the right

If $\lim_{x\to c^-} f(x) = f(c) \neq \lim\limits_{x\to c^+} f(x)$

Discontinuity of the second kind (Non removable or essential discontinuity)

If neither $\lim_{x\to c^-} f(x)$ nor $\lim_{x\to c^+} f(x)$ exists.

Discontinuity of the second kind from the left

If $\lim_{x\to c^-} f(x)$ does not exist.

Discontinuity of the second kind from the right

If $\lim_{x\to c^+} f(x)$ does not exist.

Consider a function $f(x) = 5^{\frac{x}{1-x^2}}$

Clearly the function is not defined at 1 and -1. But we can see how the function will behave as x approaches to 1 from both possible directions.

The left hand limit = $\lim_{x\to 1^-} 5^{\frac{x}{1-x^2}}$

Put $x = 1 - h$. $h \to 0$ as $x \to 1^-$

So the function can now be written as

$$\lim_{x\to 1^-} 5^{\frac{x}{1-x^2}} = \lim_{h\to 0} 5^{\frac{1-h}{1-(1-h)^2}}$$

$= \lim\limits_{h\to 0} 5^{\frac{1-h}{h(2-h)}}$ (Here $\frac{1-h}{2-h}$ tends to $\frac{1}{2}$ as h tends to zero.)

$$= \lim_{h\to 0} 5^{\frac{1}{2h}}$$

$= \infty$

The right hand limit is $\lim_{x \to 1^+} 5^{\frac{x}{1-x^2}}$ and

$$\lim_{x \to 1^+} 5^{\frac{x}{1-x^2}} = \lim_{h \to 0} 5^{\frac{1+h}{1-(1+h)^2}}$$

$= \lim_{h \to 0} 5^{\frac{1+h}{-2h-h^2}}$

$= \lim_{h \to 0} 5^{-\frac{1}{2h}} = 0$

Similarly at the point $x = -1$, the left hand limit is

$\lim_{x \to -1^-} 5^{\frac{x}{1-x^2}} = \lim_{h \to 0} 5^{\frac{-1-h}{1-(-1-h)^2}} = \infty$

And the right hand limit is

$\lim_{x \to -1^+} 5^{\frac{x}{1-x^2}} = \lim_{h \to 0} 5^{\frac{-1+h}{1-(-1+h)^2}} = 0$

Therefore in this example the point -1 is a discontinuity of the second kind from the left and 1 is also a discontinuity of the second kind from the left.

5.6 <u>Uniform continuity</u>

To understand uniform continuity let us recapitulate that a function $f$is said to be continuous on $\boldsymbol{R}$if for every $c \in \boldsymbol{R}$and for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x \in \boldsymbol{R}$with $|x - c| < \delta$ we have $|f(x) - f(c)| < \epsilon$. Here δ can depend upon ϵ and c.

On the other hand a function f is said to be uniformly continuous on $\boldsymbol{R}$if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in \boldsymbol{R}$with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. Here δ can depend upon ϵ.

The uniform continuity is a global concept. Here you have one single rectangle for the whole domain whereas the continuity of a function at a point is a local concept and the size of the rectangle will highly depend on the value of c.

We will see two theorems without proof to have a better idea of uniform continuity.

Theorem 1: Every uniformly continuous function on an interval is continuous on that interval but the converse is not true.

Theorem 2: If a function is continuous on a closed interval, then it is uniformly continuous on that closed interval.

Is the function $f(x) = \frac{x}{x+1}$ uniformly continuous for $x \in [0,2]$?

Let x, y be two arbitrary points in [0,2].Then$x \geq 0, y \geq 0$

or$x + 1 \geq 1, y + 1 \geq 1$

or$(x + 1)(y + 1) \geq 1$

or$\frac{1}{(x+1)(y+1)} \leq 1$

Now $|f(x) - f(y)| = \left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{|x-y|}{(x+1)(y+1)} \leq |x - y|$

Let $\epsilon > 0$ be given. Choosing $\delta = \epsilon$ we get

$|f(x) - f(y)| < \epsilon$when ever$|x - y| < \delta$ for every $x, y \in [0,2]$

The same problem can be solved by using the second theorem also. The only problem point for the function is $\{-1\}$ and that is not in the domain. So all conditions are getting fulfilled of Theorem 2 and that implies that the given function is uniformly continuous in [0,2].

Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ where $a > 0$, but not uniformly continuous on $[0, \infty)$.

Let $x, y \geq a > 0$ be two arbitrary numbers in $[0, \infty)$.

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right|$$

$$= \left|\frac{1}{x} - \frac{1}{y}\right| \left|\frac{1}{x} + \frac{1}{y}\right|$$

$$\leq \frac{2}{a}\left|\frac{y-x}{xy}\right|$$

$$\leq \frac{2}{a^3}|x - y|$$

Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon a^3}{2}$. Then,

$|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$ for all $x, y \geq a$.

Therefore f is uniformly continuous on $[a, \infty)$.

To show that f is not uniformly continuous on $[0, \infty)$ let us take two numbers in the interval $[0, \infty)$ as follows:

$x_1 = \frac{1}{\sqrt{n}}$ and $x_2 = \frac{1}{\sqrt{n+1}}$ be two numbers.

Now $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right|$

$$= |n - (n + 1)| = 1$$

And $|x_1 - x_2| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right|$

$$= \frac{1}{\sqrt{n}\sqrt{n+1}\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$< \frac{1}{\sqrt{n}.2\sqrt{n}} = \frac{1}{2n} = \delta$ (say)

Let $\epsilon = \frac{1}{2}$ and δ be any positive number such that $n > \frac{1}{2\delta}$ or $\frac{1}{2n} < \delta$.

Therefore $|f(x_1) - f(x_2)| > \epsilon$ when $|x_1 - x_2| < \delta$

$\therefore f$ is not uniformly continuous on $[0, \infty)$.

Summary

In this unit we have learnt about the technical definition of a continuous function and its various properties.

- A function $f(x)$ is said to be continuous at c, if $\exists$ an interval $(c - \delta, c + \delta)$ around c, such that for all $x \in (c - \delta, c + \delta)$, we have

$$f(c) - \epsilon < f(x) < f(c) + \epsilon$$

- For two continuous functions, the following properties hold.

$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$\lim_{x \to a} (f - g)(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

$\lim_{x \to a} (f.g)(x) = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$

$\lim_{x \to a} (f/g)(x) = \lim_{x \to a} f(x)/\lim_{x \to a} g(x)$ provided $g(a) \neq 0$

- A function is said to be discontinuous at a point of its domain if it is not continuous at that point. Moreover, that point is called the point of discontinuity of the function.
- We can classify the discontinuities mainly as:

1. Removable discontinuity
2. Jump discontinuity
3. Non removable or essential discontinuity

- The intermediate value theorem states that if a continuous function attains two values, it must also attain all values in between these two values.
- A function f is said to be uniformly continuous on $\boldsymbol{R}$if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in \boldsymbol{R}$with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. Here δ can depend upon ϵ.
- Every uniformly continuous function on an interval is continuous on that interval but the converse is not true.
- If a function is continuous on a closed interval, then it is uniformly continuous on that closed interval.

Key Words

continuity, discontinuity, intermediate value theorem, uniform continuity

Self Assessment

1. Which of the following is a continuous function?
   (a) Constant function
   (b) Polynomial function
   (c) Sine function
   (d) All of the above
2. To verify that any equation is solvable or not, which theorem must be used?
   (a) Mean value theorem
   (b) Rolle's theorem
   (c) Intermediate value theorem
   (d) None of these
3. If a function is continuous, it is definitely uniformly continuous.(True/False)
4. If you can redefine a function so that it becomes continuous, what kind of discontinuity are you tackling?
   (a) Jump
   (b) Removable
   (c) Discontinuity of first kind
   (d) Discontinuity of second kind
5. $\lim_{x \to 2} \frac{x-2}{x^2-4}$ is
   (a) 1
   (b) 4
   (c) $\frac{1}{4}$
   (d) None of these
6. Which of these functions is not uniformly continuous on (0,1)?
   (a) x^2
   (b) $\boldsymbol{\frac{1}{x^2}}$
   (c) $\sin x$
   (d) $\frac{sinx}{x}$
7. Find $\lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$ for $b > 0$.
   (a) 0
   (b) ∞
   (c) a
   (d) $\frac{1}{2\sqrt{a}}$
8. Which of the following is not a continuous function?
   (a) $[\boldsymbol{x}]$

(b) $|x|$

(c) x^2

(d) $\frac{1}{x}, x \neq 0$

9. If a function f is continuous for all real numbers and if $f(x) = \frac{x^2-4}{x+2}$ when $x \neq -2$, then $f(-2)$ is equal to

(a) -4

(b) -2

(c) -1

(d) 0

10. Which of the following functions are continuous at $x = 1$

I. $\ln x$

II. e^x

III. $\ln(e^x - 1)$

(a) I only

(b) I and II only

(c) II and III only

(d) I, II and III

11. The function $f(x) = \frac{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}{e^{\frac{1}{x}}-e^{-\frac{1}{x}}},\ x \neq 0, f(0) = 1$ has $x = 0$ as a

(a) removable discontinuity

(b) jump discontinuity

(c) discontinuity of second kind from left

(d) discontinuity of second kind from right

12. Discontinuity of second kind happens when

(a) the left hand limit does not exist

(b) the right hand limit does not exist

(c) both the left hand and right hand limits do not exist

(d) neither the left hand nor right hand limits exist

13. Discontinuity of the second kind is also known as

(a) removable discontinuity

(b) essential discontinuity

(c) jump discontinuity

(d) non-essential discontinuity

14. Jump discontinuity is a

(a) discontinuity of the first kind

(b) discontinuity of the first kind from left

(c) discontinuity of the first kind from right

(d) discontinuity of the second kind

15. Which of the following is/are true?

I. Every uniformly continuous function on an interval is continuous on that interval and conversely.

II. If a function is continuous on an open interval, then it is uniformly continuous on that closed interval.

(a) Only I is true

(b) Only II is true

(c) Both I and II are true

(d) None is true

Answers for Self Assessment

1.	D	2.	C	3.	False	4.	B	5.	C

6.	B	7.	D	8.	A	9.	A	10.	D
11.	B	12.	D	13.	B	14.	A	15.	D

Review Questions

1. Prove that a constant function is a continuous function.
2. State the intermediate value theorem with an example.
3. Show that the function $f(x) = 6x - 5$ is continuous at $x = 0$.
4. Discuss the continuity of the following function:
$$f(x) = \begin{cases} 3x - 5, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$
5. Determine the values of A and B so that the following function is continuous for all values of x.
$$f(x) = \begin{cases} Ax - B, & \text{if } x \leq -1 \\ 2x^2 + 3Ax + B, & \text{if -1} < x \leq 1 \end{cases}$$
and $f(x) = 4$, if $x > 1$
6. Verify if the equation $x^3 = \cos x - 2$ is solvable or not?
7. Examine the continuity of $f(x) = \begin{cases} \frac{|x-5|}{x-5}, & \text{if } x \neq 5 \\ 1, & \text{if } x = 5 \end{cases}$ and discuss in case of any discontinuity.
8. Check if the function $\frac{x}{x+2}$ is uniformly continuous on [0,2].
9. Examine the continuity of $f(x) = \begin{cases} \frac{|x-5|}{x-5}, & \text{if } x > 5 \\ 1, & \text{if } x \leq 5 \end{cases}$
10. Discuss the kind ofdiscontinuity for the following function:
$$f(x) = \begin{cases} 3x - 5, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$

Further Readings

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Unit 06: Differentiability of a Real Valued Function

CONTENTS

Objectives

Students will

- learn about the concept of differentiability
- explore the geometric interpretation of differentiability
- understand the relation in differentiability and continuity
- analyze the connection of differentiability and monotonicity

Introduction

After understanding the concept of limit, we can move on to know what is meant by differentiability of a real valued function. As we know, calculus is all about the study of the change. The general idea that comes to mind is the average rate of change. We want to say how fast we are, and to know that, we go for finding the average speed, so if you want to compare two persons, you just see their average speed or average velocity. Similarly, if you want to see where the bend is sharper, you would like to see the curvature. To explain the phenomena in more detail, we would like to know the velocity 'at a particular point' or the curvature 'at a particular point'. Here we are basically interested in the local change or the instantaneous rate of change. The average rate of change is kind of global phenomenon, we are telling something for the whole period of time or for the whole domain in general. In the first case, we need to understand something called differentiability which is an instantaneous phenomenon or alocal phenomenon.

6.1 Derivability and derivative

Consider a function $f: A \to B$. Let $c \in A$ be any point in the domain A and B is a set of real numbers. For $h > 0$, $c + h$ and $c - h$ lie on the right and left of c respectively. The value of the function at c and $c + h$ are respectively $f(c)$and $f(c + h)$. Thus we can say that change in x is h and change in $f(x)$ is $f(c + h) - f(c)$.

The average rate of change of the function w.r.t. the independent variable x is $\frac{f(c+h)-f(c)}{h}$. As the value of h approaches to zero, the expression $\frac{f(c+h)-f(c)}{h}$ may tend to a limit. The limit if exists, is called the derivative of the function f at point c, and is denoted by $f'(c)$and

$$f'(c) = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h}$$

If the derivative of the function takes a finite value, the function is called finitely derivable at c.

The left hand derivative is given as

$$f'(c^-) = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}$$

and the right hand derivative can be written as

$$f'(c^+) = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}$$

The function f is derivable if both the left and right hand derivatives exist and are equal.

Prove that the function $f(x) = x^2$ is derivable at $x = 1$.

Here the point of interest is $c = 1$, so we check the functional value at 1 and $1 + h$.

$$f(1) = 1$$

$$f(1+h) = 1 + 2h + h^2$$

To check the derivability, we need to check if the limit $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists or not!

Now $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} 2 + h = 2$

Thus the limit exists and hence the function is derivable at $x = 1$.

Check the differentiability of the modulus function at the zero.

Let the function be $f(x) = |x|$.

The domain of the function is the set of all real numbers. Here the point of interest is $c = 0$. So we will check if the limit

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$exists or not.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Since the modulus function is appearing in the function, whose limit is to be calculated, we need to apply the definition of modulus function which is as follows:

$$|h| = \begin{cases} h, & h > 0 \\ -h, & h < 0 \end{cases}$$

The left hand derivative is given as

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = -1$$

And the right hand derivative is given as

$$f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = +1$$

Clearly the limit does not exist at $x = 0$. Therefor $f'(0)$ does not exist or we can say that the modulus function is not differentiable at $x = 0$.

Find the derivative of $f(x) = \sqrt{x}$.

The domain of the function is the set of all non-negative real numbers. So we need to check the differentiability of the function for all $x > 0$ and at $x = 0$.

Let $x > 0$. We can write

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \frac{\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\sqrt{x+h} + \sqrt{x}\right)}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Therefore $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{1}{2\sqrt{x}}$provided $x > 0$.

To check the differentiability at $x = 0$, we can examine that

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\sqrt{h}}{h} = \lim_{h\to 0}\frac{1}{\sqrt{h}}$$

Clearly, $\lim_{h\to 0^+}\frac{1}{\sqrt{h}} \to \infty$ and $\lim_{h\to 0^-}\frac{1}{\sqrt{h}}$ is not defined.

So we can say that $f'(x) = \frac{1}{2\sqrt{x}}\quad \forall\, x \in (0,\infty)$

Check if the function $f(x) = x|x|$ is derivable at the origin.

Since there is modulus function involved in the definition of the function, we can simplify it first.

It can be written as $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

The point of interest is the origin. So let's find the left hand and right hand derivatives at origin.

$$f'(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{h\to 0^-}\frac{-x^2}{x} = \lim_{h\to 0^-} -x = 0$$

$$f'(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{h\to 0^+}\frac{x^2}{x} = \lim_{h\to 0^+} x = 0$$

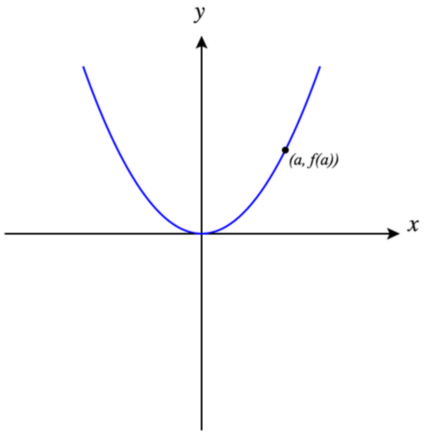
Therefore $f'(0^-) = f'(0^+)$ and it implies that f is derivable at the origin.

6.2 Geometrical interpretation of differentiability

Recall that if $y = f(x)$, then, for any real number Δx,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

is the average rate of change of y with respect to x over the interval $[x, x+\Delta x]$. Now if the graph of y is a straight line, that is, if $y = mx + b$ for some real numbers $m$and b, then $\frac{\Delta y}{\Delta x} = m$, the slope of the line. In fact, a straight line is characterized by the fact $\frac{\Delta y}{\Delta x}$is the same for any values of $x$and Δx. Moreover, $\frac{\Delta y}{\Delta x}$remains the same when Δx is infinitesimal; that is, the derivative of y with respect to x is the slope of the line. For other differentiable functions f, the value of $\frac{\Delta y}{\Delta x}$ depends upon both $x$and Δx. However, for infinitesimal values of Δx, the shadow of $\frac{\Delta y}{\Delta x}$, that is, the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$, depends on x alone. Hence it is reasonable to think of $\frac{\mathrm{d}y}{\mathrm{d}x}$ as the slope of the curve $y = f(x)$at a point$x$. Whereas the slope of a straight line is constant from point to point, for other differentiable functions the value of the slope of the curve will vary from point to point. If f is differentiable at a point a, we call the line with slope $f'(a)$passing through $(a, f(a))$the tangent line to the graph of $f$at $(a, f(a))$.



That is, the tangent line to the graph of $y = f(x)$ at $x = a$ is the line with equation

$$y = f'(a)\,(x - a) + f(a)$$

Hence a tangent line to the graph of a function f is a line through a point on the graph of f whose slope is equal to the slope of the graph at that point.

Kinematic interpretation of differentiability

The motion of a particle along a straight line can be written as $s = f(t)$, where s is the distance of the particle at point P from a fixed point of reference on the line,at time t. Let after some time Δt, the body covers a distance $s + \Delta s$ and is at point Q now. Clearly $\frac{\Delta s}{\Delta t}$ represents the average velocity for this interval of time and it approximates the actual velocity at P. Clearly as this interval of time gets smaller and smaller, the approximate value of the velocity gets better and better to the actual value.

i.e. $\lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta s}{\Delta t}\right) = \frac{ds}{dt}$= velocity (v) at time t

Similarly the instantaneous rate of change of velocity is interpreted as acceleration.

$\lim_{\Delta t \to 0} \frac{v(t+\Delta t)-v(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t}\right) = \frac{dv}{dt}$= acceleration (a) at time t

The third and fourth derivatives of distance w.r.t. time are called jerk and jounce.

$\lim_{\Delta t \to 0} \frac{a(t+\Delta t)-a(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta a}{\Delta t}\right) = \frac{da}{dt}$= jerk (j) at time t

$\lim_{\Delta t \to 0} \frac{j(t+\Delta t)-j(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta j}{\Delta t}\right) = \frac{dj}{dt}$= jounce (J) at time t

You can see and feel all these changes physically in the real world. But that will not be the case with all the functions other than the distance function.

The derivative of various functions have been developed through the ab-initio definition. Two examples are given.

Derivative of the function $f(x) = k$ where k is a given number:

$$y = k$$

$$y + \Delta y = k$$

$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0 \text{for every } x \in \text{Set of real numbers}$$

Derivative of the function $f(x) = x^n$ where n is a natural number:

$$y = x^n$$

$$y + \Delta y = (x + \Delta x)^n$$

$$= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = nx^{n-1}$$

Once the basic understanding is there, we can use the formulae of differentiation directly. You have studied and used them already still the lists of various formulae are presented below for your revision.

Summary of Rules of Differentiation

Derivative of Polynomial Function

Constant Function:	$y = c$	$y' = 0$
Linear Function:	$y = cx$	$y' = c$
Power Function:	$y = x^n$	$y' = nx^{n-1}$
Constant Multiple Function:	$y = cf(x)$	$y' = cf'(x)$
Sum of Functions:	$y = f(x) + g(x)$	$y' = f'(x) + g'(x)$
Difference of Functions:	$y = f(x) - g(x)$	$y' = f'(x) - g'(x)$
Product of Functions:	$y = f(x)g(x)$	$y' = f'(x)g(x) + f(x)g'(x)$
Quotient of Functions:	$y = \frac{f(x)}{g(x)}$	$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
Composite Function: (The chain rule)	$y = f(g(x))$	$y' = \frac{dy}{dg}g'(x)$
Inverse Function:	$x = f^{-1}(y) \Rightarrow y = f(x)$	$\frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}$

Derivative of Trigonometric Function

Sine Function:	$y = \sin x$	$y' = \cos x$
Cosine Function:	$y = \cos x$	$y' = -\sin x$
Tangent Function:	$y = \tan x$	$y' = \sec^2 x$
Cotangent Function:	$y = \cot x$	$y' = -\csc^2 x$
Secant Function:	$y = \sec x$	$y' = \sec x \tan x$
Cosecant Function:	$y = \csc x$	$y' = -\csc x \cot x$

Derivative of Exponential and Logarithmic Function

Natural Exponential Function:	$y = e^x$	$y' = e^x$
Exponential Function:	$y = a^x$	$y' = a^x \log_e a$
Natural Logarithmic Function:	$y = \log_e x$	$y' = \frac{1}{x}$
Logarithmic Function:	$y = \log_a x$	$y' = \frac{1}{x \log_e a}$

Derivative of Inverse Trigonometric Function

Inverse Sine Function:	$y=\sin^{-1}x \quad y'=\dfrac{1}{\sqrt{1-x^2}}$	$\begin{cases} x: -1<x<1 \\ y: -\pi/2<y<\pi/2 \end{cases}$
Inverse Cosine Function:	$y=\cos^{-1}x \quad y'=\dfrac{-1}{\sqrt{1-x^2}}$	$\begin{cases} x: -1<x<1 \\ y: \pi>y>0 \end{cases}$
Inverse Tangent Function:	$y=\tan^{-1}x \quad y'=\dfrac{1}{1+x^2}$	$\begin{cases} x: -\infty<x<+\infty \\ y: -\pi/2<y<\pi/2 \end{cases}$
Inverse Cotangent Function:	$y=\cot^{-1}x \quad y'=\dfrac{-1}{1+x^2}$	$\begin{cases} \begin{cases} x: -\infty<x<+\infty \\ y: \pi>y>0 \end{cases} \quad or \\ \begin{cases} x: (-\infty<x<0)\cup(0<x<+\infty) \\ y: (0>y>-\pi/2)\cup(\pi/2>y>0) \end{cases} \end{cases}$
Inverse Secant Function:	$y=\sec^{-1}x \quad y'=\dfrac{1}{x\sqrt{x^2-1}}$	$\begin{cases} x: (-\infty<x<-1)\cup(1<x<+\infty) \\ y: (\pi/2<y<\pi)\cup(0<y<\pi/2) \end{cases}$
Inverse Cosecant Function:	$y=\csc^{-1}x \quad y'=\dfrac{-1}{x\sqrt{x^2-1}}$	$\begin{cases} x: (-\infty<x<-1)\cup(1<x<+\infty) \\ y: (0>y>-\pi/2)\cup(\pi/2>y>0) \end{cases}$

Derivative of Hyperbolic Function

Hyperbolic Sine Function:	$y=\sinh x$	$y'=\cosh x$
Hyperbolic Cosine Function:	$y=\cosh x$	$y'=\sinh x$
Hyperbolic Tangent Function:	$y=\tanh x$	$y'=\operatorname{sech}^2 x$
Hyperbolic Cotangent Function:	$y=\coth x$	$y'=-\operatorname{csch}^2 x$
Hyperbolic Secant Function:	$y=\operatorname{sech} x$	$y'=-\operatorname{sech} x\tanh x$
Hyperbolic Cosecant Function:	$y=\operatorname{csch} x$	$y'=-\operatorname{csch} x\coth x$

Derivative of Inverse Hyperbolic Function

Inverse Hyperbolic Sine Function:	$y=\sinh^{-1}x \quad y'=\dfrac{1}{\sqrt{1+x^2}}$	$\begin{cases} x: -\infty<x<+\infty \\ y: -\infty<x<+\infty \end{cases}$
Inverse Hyperbolic Cosine Function:	$y=\cosh^{-1}x \quad y'=\pm\dfrac{1}{\sqrt{x^2-1}}$	$\begin{cases} \begin{cases} x: 1<x<+\infty \\ y: 0<y<+\infty \end{cases} \\ \begin{cases} x: 1<x<+\infty \\ y: 0>y>-\infty \end{cases} \end{cases}$
Inverse Hyperbolic Tangent Function:	$y=\tanh^{-1}x \quad y'=\dfrac{1}{1-x^2}$	$\begin{cases} x: -1<x<1 \\ y: -\infty<y<+\infty \end{cases}$
Inverse Hyperbolic Cotangent Function:	$y=\coth^{-1}x \quad y'=\dfrac{-1}{x^2-1}$	$\begin{cases} x: (-\infty<x<-1)\cup(1<x<+\infty) \\ y: (0>y>-\infty)\cup(+\infty>y>0) \end{cases}$
Inverse Hyperbolic Secant Function:	$y=\operatorname{sech}^{-1}x \quad y'=\mp\dfrac{1}{x\sqrt{1-x^2}}$	$\begin{cases} \begin{cases} x: 0<x<1 \\ y: +\infty>y>0 \end{cases} \quad or \\ \begin{cases} x: 0<x<1 \\ y: 0>y>-\infty \end{cases} \end{cases}$
Inverse Hyperbolic Cosecant Function:	$y=\operatorname{csch}^{-1}x \quad y'=-\dfrac{1}{\lvert x\rvert\sqrt{x^2+1}}$	$\begin{cases} x: (-\infty<x<0)\cup(0<x<+\infty) \\ y: (0>y>-\infty)\cup(+\infty>y>0) \end{cases}$

6.3 Relation between differentiability and continuity

In a layman language, if you want to check the differentiability of a function at a point, just by looking at its graph, you must zoom the figure at that point (in your mind, if not possible otherwise!) If you see a straight line at that point, the function is differentiable and otherwise it is non-differentiable. So corners or pointy edges in a graph imply non differentiable nature of the function while smooth curve suggests a differentiable function.

Similarly the concept of continuity in a domain relates to the graph without any kink or cut or break in that domain.

Theorem: If *f* is finitely derivable at *c*, then *f* is also continuous at *c*.

Let f be a finitely derivable function at c, so that the expression $\frac{f\,(c+\mathrm{h})-f(c)}{h} \to$ a finite limit as $h \to 0$.

We can write

$$f(c+\mathrm{h})-f(c)=\frac{f(c+\mathrm{h})-f(c)}{h}h$$

$$\lim_{h\to 0} f(c+\mathrm{h})-f(c)=\lim_{h\to 0}\frac{f(c+\mathrm{h})-f(c)}{h}\lim_{h\to 0} h$$

$$=f'(c)*0=0$$

$$\therefore \lim_{h\to 0} f(c+\mathrm{h})-f(c)=0$$

$$\lim_{h\to 0} f(c+\mathrm{h})=f(c)$$

Alternatively,

$$\lim_{x\to c} f(x)=f(c)$$

It implies that f is continuous at $x=c$.

The converse of this theorem is not necessarily true. One stark example is the absolute function. The function $f(x)=|x|$ is continuous on its domain but is not differentiable at the point 0 which is a part of the domain.

Examine $f(x)=\begin{cases}\frac{x\left(e^{-\frac{1}{x}}-e^{\frac{1}{x}}\right)}{e^{-\frac{1}{x}}+e^{\frac{1}{x}}} & x\neq 0\\ 0 & x=0\end{cases}$ for the continuity and differentiability at origin.

The right hand limit is $\lim_{x\to 0^+}\frac{x\left(e^{-\frac{1}{x}}-e^{\frac{1}{x}}\right)}{e^{-\frac{1}{x}}+e^{\frac{1}{x}}}$

$$=\lim_{x\to 0^+}\frac{x\left(e^{-\frac{2}{x}}-1\right)}{e^{-\frac{2}{x}}+1}=0$$

The left hand limit is $\lim_{x\to 0^-}\frac{x\left(e^{-\frac{1}{x}}-e^{\frac{1}{x}}\right)}{e^{-\frac{1}{x}}+e^{\frac{1}{x}}}$

$$=\lim_{x\to 0^-}\frac{x\left(1-e^{\frac{2}{x}}\right)}{1+e^{\frac{2}{x}}}=0$$

The value of the function at 0 is also zero.

Therefore the given function is continuous at origin.

The right hand derivative is $f'(0^+)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}-e^{\frac{1}{x}}}{e^{-\frac{1}{x}}+e^{\frac{1}{x}}}$

$$=\lim_{x\to 0^+}\frac{e^{-\frac{2}{x}}-1}{e^{-\frac{2}{x}}+1}=-1$$

The left hand derivative is $f'(0^-)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{e^{-\frac{1}{x}}-e^{\frac{1}{x}}}{e^{-\frac{1}{x}}+e^{\frac{1}{x}}}$

$$=\lim_{x\to 0^-}\frac{1-e^{\frac{2}{x}}}{1+e^{\frac{2}{x}}}=1$$

Therefore the function is not derivable at the origin.

6.4 Differentiability and monotonicity

Monotonicity gives an idea about the behaviour of the function. A function is said to be monotonic function if it is either increasing or decreasing in its entire domain. For example,

$f(x)=2x+3$has the set of all real numbers as its domain and the function is monotonically increasing on the entire domain.

$g(x) = -x^3$also has the set of all real numbers as its domain and the function is monotonically decreasing on the entire domain.

We can recall that for an increasing function $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ and for a decreasing function $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. The functions which are increasing as well as decreasing in their domain are known as non-monotonic functions. For example the absolute function, the sine function etc.

Monotonicity of a function at a point in its domain

A function is monotonically increasing at $x = a$ if $f(a+h) > f(a)$and $f(a-h) < f(a)$ for small $h > 0$. From the first expression, we can write

$$\frac{f(a+h) - f(a)}{h} > \frac{0}{h}$$

$$\frac{f(a+h) - f(a)}{h} > 0$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} > 0$$

$$f'(a) > 0$$

A function is monotonically decreasing at $x = a$ if $f(a+h) < f(a)$ and $f(a-h) > f(a)$ for small $h > 0$. From the first expression, we can write

$$\frac{f(a+h) - f(a)}{h} < \frac{0}{h}$$

$$\frac{f(a+h) - f(a)}{h} < 0$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} < 0$$

$$f'(a) < 0$$

Thus we can see a relation in differentiability and monotonicity. If a function is monotonically increasing at $x = a$, its first derivative at $x = a$ has to be positive and if a function is monotonically decreasing at $x = a$, its first derivative at $x = a$ has to be negative.

Monotonicity in an interval

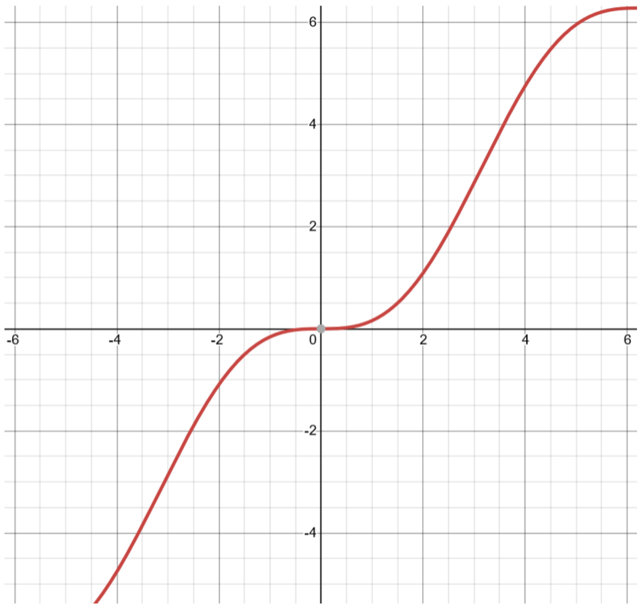
For an increasing function in some interval if $\Delta x > 0 \Leftrightarrow \Delta y > 0$or $\Delta x < 0 \Leftrightarrow \Delta y < 0$, then the function is said to be strictly monotonically increasing in that interval.

i.e. if $\frac{dy}{dx} > 0$ in some interval then y is said to be a strictly increasing function in that interval. Similarly, if $\frac{dy}{dx} < 0$ in some interval then y is said to be a strictly decreasing function in that interval.

if$\frac{dy}{dx} \geq 0$ in some interval then y is said to be a increasing function in that interval. Similarly, if $\frac{dy}{dx} \leq 0$ in some interval then y is said to be a decreasing function in that interval.

Prove that $f(x) = x - \sin x$ is an increasing function.

Let us see how the function looks like!



$$f(x) = x - \sin x$$

$$f'(x) = 1 - \cos x$$

$$f''(x) \geq 0 \quad \forall\, x \in \boldsymbol{R}$$

Therefore $f(x)$ is monotonically increasing $\forall\, x \in \boldsymbol{R}$

Greatest and least value of a function

We can discuss the greatest and least value of a function with specific conditions under the following three cases:

CaseI.$y = f(x)$ is strictly increasing in$[a, b]$, then $f(a)$ is the least value and $f(b)$ is the greatest value of the function.

Case II.$y = f(x)$is strictly decreasing in $[a, b]$, then $f(b)$ is the least value and $f(a)$ is the greatest value of the function.

Case III.$y = f(x)$is non-monotonic in $[a, b]$ and is continuous, then the greatest and least value of $f(x)$ in $[a, b]$ are those values where $f'(x) = 0$ or it does not exist or at the extreme values.

Find the interval in which the function $f(x) = 2x^2 - \ln\,|x|$ is (i) decreasing (ii) increasing.

$$f(x) = 2x^2 - \ln|x|$$

$$f'(x) = 4x \quad -\frac{1}{\mathrm{x}} = \frac{4x^2 - 1}{x}$$

Domain of the function is$(0, \infty)$. Therefore the denominator of $f'(x)$ is always positive and numerator has all the power to decide.

For $f(x)$ to be decreasing

$$f'(x) \leq 0$$

$$4x^2 - 1 \leq 0$$

$$x^2 \leq \frac{1}{4}$$

$$|x| \leq \frac{1}{2}$$

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$

$$x \in \left(0, \frac{1}{2}\right]$$

For $f(x)$ to be increasing

$$f'(x) \geq 0$$

$$4x^2 - 1 \geq 0$$

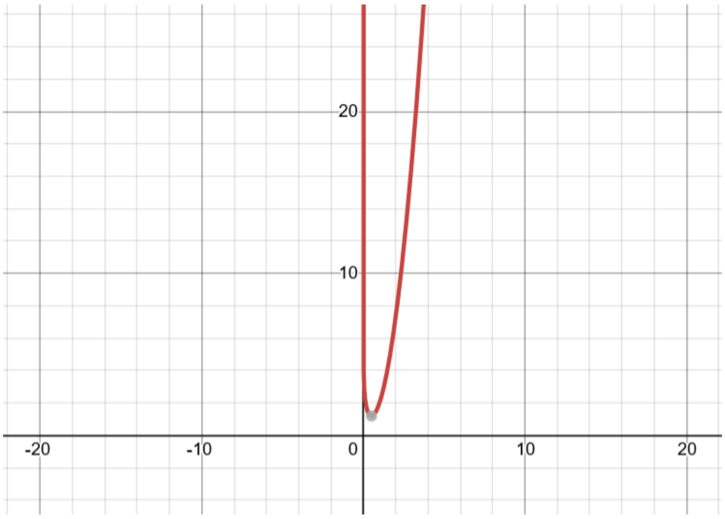
$$x^2 \geq \frac{1}{4}$$

$$|x| \geq \frac{1}{2}$$

$$x \geq \frac{1}{2} \text{ or } x \leq -\frac{1}{2}$$

$$x \in \left[\frac{1}{2}, \infty\right)$$

We can verify this by drawing the graph of the function and can observe the following:



- From $\frac{1}{2}$ onwards the function is increasing
- In $\left(0, \frac{1}{2}\right]$ the function is decreasing.
- At $\frac{1}{2}$ the function is having a minimum value.

Summary

In this unit we have learnt the basic definition of differentiability and its geometric and kinematic interpretations. We have learnt some results related to differentiability with continuity and monotonicity. The following are the main point:

- To check the derivability, we need to check if the limit $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists or not.
- If f is finitely derivable at c, then f is also continuous at c.
- If a function is monotonically increasing at $x = a$, its first derivative at $x = a$ has to be positive and if a function is monotonically decreasing at $x = a$, its first derivative at $x = a$ has to be negative.

Key Words

- Differentiability
- Derivability
- Differentiability and continuity
- Differentiability and monotonicity

Self Assessment

1. Which of the following does not lead to the idea of differentiability?

A. instantaneous rate of change

B. average rate of change
C. slope of the function
D. local rate of change

2. A function f is said to be derivable in $[a, b]$ if
A. f is finitely derivable at every point of $[a, b]$
B. f is infinitely derivable at every point of $[a, b]$
C. f is finitely derivable at some points of $[a, b]$
D. f is infinitely derivable at some points of $[a, b$

3. The derivative of the function $\sqrt{x+2}$ is
A. $\frac{1}{x+2}$
B. $\frac{1}{2(x+2)}$
C. $\frac{1}{2\sqrt{x+2}}$
D. $\frac{1}{2\sqrt{x}}$

4. A function f is said to be derivable at point c if
A. $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ exists
B. $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ exists
C. $\lim_{h\to 0} \frac{f(c+h)+f(c)}{h}$ exists
D. $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists

5. The derivative of $cosec\ x$ w.r.t. x is
A. $cosecx \cot x$
B. $-cosecx \cot x$
C. $-cosec^2x \cot x$
D. $-cosecx \cot^2 x$

6. The derivative of $log_a x$ w.r.t. x is
A. $\frac{1}{x}$
B. $\frac{1}{xlog_a x}$
C. $\frac{1}{xlog_e a}$
D. none of these

7. The derivative of $sin^{-1}x$ w.r.t. x is
A. $\frac{1}{1-x^2}$
B. $\frac{1}{\sqrt{1-x^2}}$
C. $\frac{1}{1+x^2}$
D. $\frac{1}{\sqrt{1+x^2}}$

8. The derivative of $cot^{-1}x$ w.r.t. x is
A. $\frac{1}{1-x^2}$
B. $\frac{1}{\sqrt{1-x^2}}$
C. $\frac{1}{1+x^2}$

D. none of these

9. The derivative of $tanh^{-1}x$ w.r.t. x is
A. $\frac{1}{1-x^2}$
B. $\frac{1}{\sqrt{1-x^2}}$
C. $\frac{1}{1+x^2}$
D. none of these

10. The derivative of $cosh\ x$ w.r.t. x is
A. $\sinh x$
B. $-\sinh x$
C. $-\sinh^2 x$
D. $\sinh^2 x$

11. Which of the following function is continuous and not differentiable in its domain?
A. x^2
B. $\sqrt{x}$
C. $|x|$
D. $\frac{1}{x}$

12. The function $x|x|$ is
A. derivable at origin
B. continuous at origin
C. both derivable and continuous at origin
D. none of these

13. If f is continuous at c, then f is also finitely derivable at c.
A. True
B. False

14. A function is said to be monotonic function if
A. it is increasing in its entire domain
B. it is decreasing in its entire domain
C. it is either increasing or decreasing in its entire domain
D. none of these

15. Which of the following suggests that the function is strictly decreasing?
A. $x_1 < x_2$implies $f(x_1) < f(x_2)$
B. $x_1 < x_2$implies $f(x_1) > f(x_2)$
C. $x_1 < x_2$implies $f(x_1) \le f(x_2)$
D. $x_1 < x_2$implies $f(x_1) \ge f(x_2)$

16. Which of the following suggests that the function is decreasing?
A. $x_1 < x_2$implies $f(x_1) < f(x_2)$
B. $x_1 < x_2$implies $f(x_1) > f(x_2)$
C. $x_1 < x_2$implies $f(x_1) \le f(x_2)$
D. $x_1 < x_2$implies $f(x_1) \ge f(x_2)$

17. The functions which are increasing as well as decreasing in their domain are known as

A. increasing functions
B. decreasing functions
C. monotonic functions
D. non monotonic functions

18. The function $2x^2 - log\ x$ is decreasing in the interval

A. $\left[-\frac{1}{2}, \frac{1}{2}\right]$
B. $\left[0, \frac{1}{2}\right]$
C. $\left(0, \frac{1}{2}\right)$
D. $\left(0, \frac{1}{2}\right]$

Answers for Self Assessment

1.	B	2.	A	3.	C	4.	B	5.	D
6.	C	7.	B	8.	D	9.	A	10.	A
11.	C	12.	C	13.	B	14.	C	15.	B
16.	D	17.	D	18.	D				

Review Questions

1. Find the derivative of $f(x) = 2ax + b$ using first principle.
2. Find the derivative of $f(x) = \frac{1}{x^2+3}$ using first principle.
3. Discuss the differentiability of the function $(x) = |x-2| + |x| + |x+2|$.
4. Find the slope of the tangents to the parabola $y = x^2$ at points (2, 4) and (-1, 1).
5. Find the interval in which the function $f(x) = 3x^2 - \ln\ |x|$ is (i) decreasing (ii) increasing.
6. Find the interval in which the function $f(x) = \log x + x$ is (i) decreasing (ii) increasing.
7. Examine $f(x) = \begin{cases} \frac{x\left(e^{-\frac{1}{x}} - e^{\frac{1}{x}}\right)}{e^{-\frac{1}{x}} + e^{\frac{1}{x}}} & x \neq 0 \\ 1 & x = 0 \end{cases}$ for the continuity and differentiability at origin.
8. Find the interval in which the function $f(x) = x - \cos x$ is (i) decreasing (ii) increasing.
9. Find the derivative of hyperbolic sine function using ab initio method.
10. Design a function which is increasing on some part of the domain and decreasing on other. Then discuss the differentiability and continuity of that function.

Further Readings

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Unit 7: Differentiability of a Real Valued Function

CONTENTS

Objectives

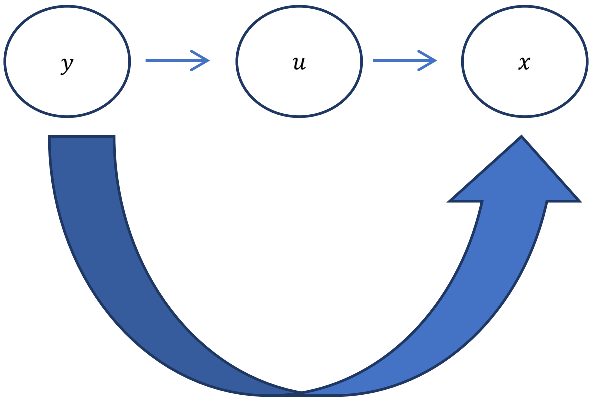
Students will

- learn about the derivative of the function of function
- explore the property of a differentiable function called Darboux's theorem
- learn to apply the Rolle's theorem

Introduction

If the function is made up of functions called composed functions or a composite function, then what to do in case, if we are interested in the derivative of a composed function! This question will be answered in this unit. The derivative of function of function is popularly known as the chain rule of differentiation.

Let f and ϕ be two derivable functions such that $y = f(u)$and $u = \phi(x)$. Clearly you can see that y is a function of u and u is a function of x and ultimately y is a function of x.



The range of ϕ must be a subset of the domain of f, then only we would be able to write $y = f(\phi(x))$ which is also called the composite function. Moreover we know that

$$(f o \phi)(x) = f(\phi(x))$$

7.1 Chain Rule of Differentiation

The chain rule is a rule for differentiating compositions of functions.

$$\frac{d}{dx}(f(\phi(x)) = \frac{d}{d(\phi(x))}f(\phi(x)).\frac{d}{dx}\phi(x)$$

Or it can be put simply as

$$\frac{d}{dx}(f(\phi(x)) = f'(\phi(x))\phi'(x)$$

However, we rarely use this formal approach when applying the chain rule to specific problems. Instead, we take an intuitive approach. For example, it is sometimes easier to think of the functions f and ϕ as ``layers'' of a problem. Function $f$ is the ``outer layer'' and function ϕ is the ``inner layer.'' Thus, the chain rule tells us to first differentiate the outer layer, leaving the inner layer unchanged (the term $f'(\phi(x))$, then differentiate the inner layer (the term $\phi'(x)$) .

The **chain rule** provides us a technique for finding the derivative of composite functions, with the number of functions that make up the composition determining how many differentiation steps are necessary. For example, if a composite function $f(x)$ is defined as

$$f(x) = (goh)(x) = g(h(x))$$

Then $f'(x) = g'(h(x).h'(x)$

If the function is defined as $f(x) = (gohok)(x) = g(h(k(x)))$

Then $f'(x) = g'(h(k(x)).h'(k(x)).k'(x)$

Let us consider f and ϕ be two derivable functions such that $y = f(u)$and $u = \phi(x)$. Let

Δxbe the change in x

Δube the change in u

Δybe the change in y.

Then we can write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}.\frac{\Delta u}{\Delta x}$$

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta u}.\frac{\Delta u}{\Delta x}\right)$$

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}.\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let us understand the rule better with the following examples:

Find the derivative of the function $\sqrt{1+x^2}$.

You can see that the function is a composition of the polynomial function and the square root function. So we can write this as

$u = 1 + x^2, y = \sqrt{u}$; then$y = \sqrt{1+x^2}$

Now $\frac{du}{dx} = 2x$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}}$$

Therefore, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{x}{\sqrt{1+x^2}}$.

We can look at the same problem by layers' point of view. The square root is the outer layer , it has to be dealt with first and then the polynomial as the inner layer will be considered. We can write it as

$$\frac{d}{dx}\left(\sqrt{1+x^2}\right) = \frac{d\sqrt{1+x^2}}{d(1+x^2)} \cdot \frac{d(1+x^2)}{dx}$$

$$= \frac{1}{2}(1+x^2)^{-\frac{1}{2}} . 2x$$

$$= \frac{x}{\sqrt{1+x^2}}$$

Find the derivative of the function $\sqrt{\frac{1+x}{1-x}}$.

Let $\frac{1+x}{1-x} = u$, then $y = \sqrt{u} = u^{\frac{1}{2}}$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}.\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{2}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}} . \frac{2}{(1-x)^2} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{\frac{3}{2}}}$$

Find the derivative of the function $\log(\cosh x)$.

$$\frac{dy}{dx} = \frac{d\log(\cosh x)}{d(coshx)} . \frac{d(coshx)}{dx}$$

$$= \frac{1}{coshx} . sinhx = \tanh x$$

Find the derivative of the function $\sinh^{-1} x$.

Let $y = \sinh^{-1} x$

$$\Rightarrow x = \sinh y$$

$$\frac{dx}{dy} = \cosh y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh \quad y}$$

$$= \pm \frac{1}{\sqrt{(1+\sinh^2 y)}}$$

$$= \pm \frac{1}{\sqrt{1+x^2}}$$

The sign of the radical must be same as that of $\cosh y$.

Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

Find the derivative of the function $e^{sinh^{-1}x}$.

Let $y = \sinh^{-1} x$

$$\frac{dy}{dx} = \frac{d\left(e^{sinh^{-1}x}\right)}{d(\sinh^{-1} x)} . \frac{d(\sinh^{-1} x)}{dx}$$

$$= e^{\sinh^{-1} x} . \frac{1}{\sqrt{1+x^2}}$$

7.2 Extreme Value Theorem

An major application of critical points or saddle points is in determining possible maximum and minimum values of a function on certain intervals. The **Extreme Value Theorem** guarantees both a maximum and minimum value for a function under certain conditions. It states the following:

If a function *f(x)* is continuous on a closed interval [*a, b*], then *f(x)* has both a maximum and minimum value on [*a, b*].

The steps for applying the extreme value theorem are as follows:

1. Establish that the function is continuous on the closed interval.
2. Determine all critical points in the given interval.
3. Evaluate the function at these critical points and at the endpoints of the interval.
4. Look for the largest and the smallest values of the function.

The largest function value from the previous step is the maximum value, and the smallest function value is the minimum value of the function on the given interval.

Find the maximum and minimum values of $f(x) = \sin x + \cos x$ on $[0, 2\pi]$.

The function is continuous on $[0, 2\pi]$.

$$f'(x) = \cos x - \sin x$$

The critical points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$. The function values at the end pointsof the given interval are $f(0) = 1$and $f(2\pi) = 1$.

Thus, we can see that the maximum value of the function is $\sqrt{2}$ and the minimum value is $-\sqrt{2}$.

Find the maximum and minimum values of $f(x) = x^4 - 3x^3 - 1$ on $[-2, 2]$.

The function is a polynomial, therefore is continuous on $[-2, 2]$.

Its derivative is given as $f'(x) = 4x^3 - 9x^2$

For critical points, put $f'(x) = 0$

or$4x^3 - 9x^2 = 0$

$$\Rightarrow x = 0, \frac{9}{4}.$$

Clearly $x = \frac{9}{4}$ does not belong to the interval $[-2, 2]$. The only critical point occurs at $x = 0$. which is $(0, -1)$.

The function values at the endpoints of the interval are $f(2)=-9$ and $f(-2)=39$; therefore, the maximum function value is 39 at $x = -2$, and the minimum function value is -9 at $x = 2$.

Note the importance of the closed interval in determining the values to consider for critical points.

7.3 Darboux's Theorem

If *f* is differentiable on the closed interval [*a, b*] and *r* is any number between *f* ′(*a*) and *f*′ (*b*), then there exists a number *c* in the open interval (*a, b*) such that *f*′ (*c*) = *r.*

Proof

Consider the function

$$h(x) = f(x) - \big(f(b) + r(x - b)\big)$$

Because $f(x)$ is differentiable, it is definitely continuous.

$f(b) + r(x - b)$ is also continuous and differentiable.

$\therefore$ $h(x)$is continuous and differentiable on $[a, b]$.

By the extreme value theorem, there exists $c \in (a, b)$ where $h(x)$ has an extreme value . At this point $h'(c) = 0$

We have $h'(x) = f'(x) - r$

So $f'(c) - r = 0$

$$\Rightarrow f'(c) = r$$

Thus if f is differentiable on the closed interval $[a, b]$ and r is any number between $f'(a)$ and $f'(b)$, then there exists a number c in the open interval (a, b) such that $f'(c) = r$.

Jean Gaston Darboux was a French mathematician who lived from 1842 to 1917. Of his several important theorems the one we just studied says that the derivative of a function has the Intermediate Value Theorem property – that is, the derivative takes on all the values between the values of the derivative at the endpoints of the interval under consideration.

Another interesting aspect of Darboux's Theorem is that there is no requirement that the derivative $f'(x)$ be continuous!

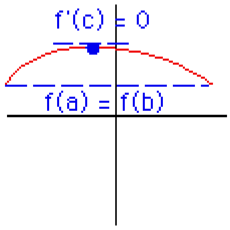
The common example of such a function is $f(x) = x^2 \sin\frac{1}{x}$ when $x \neq 0$ and $f(x) = 0$ when $x = 0$

With $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x},\ x \neq 0$

This function is differentiable and hence continuous. There is an oscillating discontinuity at the origin. The derivative is not continuous at the origin. Yet, every interval containing the origin as an interior point meets the conditions of Darboux's Theorem, so the derivative while not being continuous has the intermediate value property.

7.4 Rolle's Theorem

If f(x) is continuous an [a,b] and differentiable on (a,b) and if f(a) = f(b) then there is some c in the interval (a,b) such that f '(c) = 0.



Proof

In the statement of Rolle's theorem, f(x) is a continuous function on the closed interval [a,b]. Hence by the Intermediate Value Theorem it achieves a maximum and a minimum on [a,b]. Either one of these occurs at a point c with a < c < b,

Since f(x) is differentiable on (a,b) and c is an extremum we then conclude that f '(c) = 0.

or both the maximum and minimum occur at endpoints.

Since f(a) = f(b), this means that the function is never larger or smaller than f(a). In other words, the function f(x) is constant on the interval [a,b] and its derivative is therefore 0 at every point in (a,b).

Hence proved

Geometric interpretation

There is a point **C** on the interval (a,b)where the tangent to the graph of the function is parallel to the x-axis.

This property was known in the 12th century in ancient India. The outstanding Indian astronomer and mathematician Bhaskara II mentioned it in his writings.

For instance, consider $f(x) = |x|$ (where $|x|$ is the absolute value of x on the closed interval [-1,1]. This function does not have derivative at= 0. Though f(x) is continuous on the closed interval [-1,1] there is no point inside the interval $(-1,1)$ at which the derivative is equal to zero. The Rolle's Theorem fails here because $f(x)$ is not differentiable over the whole interval $(-1,1)$.

Physical interpretation

Rolle's Theorem has a clear physical meaning. Suppose that a body moves along a straight line, and after a certain period of time returns to the starting point. Then, in this period of time there is a moment, in which the instantaneous velocity of the body is equal to zero.

Verify the Rolle's Theorem for $f(x) = x^2$ in $[-1,1]$

The function is a polynomial, therefore it is continuous in $[-1,1]$.

The function is differentiable in $(-1,1)$. (You can verify it by ab initio definition or but just checking that its derivative by the usual rules of differential calculus, exists in $(-1,1)$.

And thirdly $f(1) = f(-1)$

All conditions are getting fulfilled therefore in the interval$(-1,1)$, there must exist at least a point such that the derivative of the function at that point is zero.

i.e.$f'(c) = 0$

$$\Rightarrow\ 2c = 0$$

$$\Rightarrow\ c = 0$$

And $0 \in (-1,1)$

Thus the Rolle's Theorem gets verified.

Verify Rolle's theorem for $f(x) = x(x+3)e^{-\frac{x}{2}}$in $[-3,0]$.

The function is a product of continuous functions, therefore it is continuous in $[-3,0]$.

The function is differentiable in $(-3,0)$. (solve for the derivative of the function and check if it exists in $(-3,0)$, It will be!)

$$f(-3) = f(0)$$

Therefore there will exist a point c , such that$f'(c) = 0$.

Or $c^2 - c - 6 = 0$

Or $c = -2, 3$

Thus we got at least a point $-2 \in (-3,0)$.

Thus Rolle's theorem gets verified for the given function in the given interval.

Summary

This unit is an extension of the differentiability to a function of a function. We have understood and learnt the formulae of the derivatives of elementary functions already.

- In order to differentiate a composite function, of course those formulae will not be applicable directly. We need to use the chain rule.
- $\frac{d}{dx}(f(\phi(x)) = f'(\phi(x))\phi'(x)$
- A significant result for a differentiable function on a closed interval, given as,
  If *f* is differentiable on the closed interval [*a, b*] and *r* is any number between *f* '(*a*) and *f* ' (*b*), then there exists a number *c* in the open interval (*a, b*) such that *f* ' (*c*) = *r.*
- Rolle's theorem states that 'If f(x) is continuous an [a,b] and differentiable on (a,b) and if f(a) = f(b) then there is some c in the interval (a,b) such that f '(c) = 0.'

Key Words

chain rule, derivative of a function of function, extreme value theorem, Darboux's theorem, Rolle's theorem

Self Assessment

1. The derivative of the function $\sqrt{2+x^2}$ is

A. $\frac{x}{2\sqrt{2+x^2}}$

B. $\frac{x}{\sqrt{2+2x^2}}$

C. $\frac{2x}{\sqrt{2+x^2}}$

D. $\frac{x}{\sqrt{2+x^2}}$

2. $\frac{d}{dx}\left(sin(cosx)\right) =$
   A. $\sin x \cos(\cos x)$
   B. $\cos(\sin x)$
   C. $-\sin x \cos(\cos x)$
   D. $\sin x \cos(\sin x)$

3. If *f* is differentiable on the closed interval [*a*, *b*] and *r* is any number between *f* ′ (*a*) and *f* ′(*b*), then there exists a number *c* in the open interval (*a*, *b*) such that *f*′(*c*) = *r*. This statement is of

A. intermediate value theorem
B. mean value theorem
C. Rolle's theorem
D. Darboux's theorem

4. $\frac{e^{sinh^{-1}x}}{\sqrt{1+x^2}}$ is the derivative of the function

A. x$\frac{e^{sinh^{-1}x}}{\sqrt{1+x^2}}$
B. $\frac{e^{sinh^{-1}x}}{\sqrt{1+x^2}}$
C. $e^{sinh^{-1}x}$ ans
D. $e^{cosh^{-1}x}$

5. The derivative of $log(cosh\, x)$ w.r.t. x is

A. $\log(\sin x)$
B. $\log(\sinh x)$
C. $cosech\, x$
D. $\tanh x$

6. $\frac{d}{dx}e^{sinx} =$

A. $e^{sinx}\cos x$
B. $\cos(\sin x)$
C. $-e^{sinx}\cos x$
D. $\sin x \cos(\sin x)$

7. If a function f is

I. continuous on $[a, b]$
II. derivable on (a, b)
III. $f(a) = f(b)$
IV. then there exists one value $c \in (a, b)$ such that $f'(c) = 0$

Which of the following are correct for f to satisfy the Rolle's Theorem?

A. I, II and III
B. I, II and IV
C. II, III and IV
D. III and IV

8. The function $f(x) = x^2$ in $[-1, 2]$ satisfies the Rolles's theorem.

A. True
B. False

9. The function $f(x) = x(x+3)e^{-\frac{x}{2}}$ in $[-1, 1]$ satisfies the Rolle's Theorem.

A. True
B. False

10. For all the second degree polynomials $y = ax^2 + bx + k$, it is seen that the Rolles' point is at $c = 0$. Also the value of k is zero. Then what is the value of b?

A. 0
B. 1
C. -1
D. 56

11. If f is continuous function on the closed interval $[a, b]$, and N is a number between $f(a)$ and $f(b)$, then there is $c \in [a, b]$such that $f(c) = N$ is:
A. The Intermediate Value Theorem
B. The Mean Value Theorem
C. Rolle's Theorem
D. The Extreme Value Theorem

12. According to Rolle's theorem, for a differentiable function f(x), if the start point f(a) and the end point f(b) equal 0 then:
A. Rolle's Theorem does not apply.
B. Somewhere between f(a) and f(b) the instantaneous rate of change must be 0.
C. Somewhere between f(a) and f(b) the function must equal 0.
D. The function is flat.

Answer for Self Assessment

1.	D	2.	C	3.	D	4.	C	5.	D
6.	A	7.	A	8.	B	9.	B	10.	A
11.	A	12.	B						

Review Questions

1. Find the derivative of the function $f(x) = \sin\left(\sqrt{x^2 - 5}\right)$
2. Find the derivative of the function $f(x) = \frac{\sqrt{1-x}}{1+x}$
3. Find the derivative of the function $f(x) = \sin h\left(\sqrt{x^2 + 5}\right)$
4. Find the derivative of the function $f(x) = \log\left(\cosh e^x\right)$
5. Find the derivative of the function $f(x) = 2^{(\cosh e^x)}$
6. Find the derivative of the function $f(x) = \tan^{-1} x$
7. Design a function which satisfies the Darboux's theorem in certain interval.
8. State and prove the Rolle's theorem.
9. Learn more about Michel Rolle!
10. Compare the Rolles' theorem with the Darboux's theorem.
11. Discuss the applicability of Rolle's theorem to the function

$$f(x) = \begin{cases} x^2 + 1, & 0 \leq x \leq 1 \\ 3 - x, & 1 < x \leq 2 \end{cases}$$

Further Reading

George B. Thomas Jr., Joel Hass, Christopher Heil & Maurice D. Weir (2018). *Thomas' Calculus* (14th edition). Pearson Education.

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https://www.math24.net/rolles-theorem

Unit 08: Mean Value Theorems

CONTENTS

Objectives

Students will

- learn about the properties of a differentiable function
- understand the basics of Lagrange's mean value theorem
- be able to use Cauchy's mean value theorem
- be able to interpret the mean value theorems geometrically

Introduction

If a function is appropriately differentiable and continuous then it can lead to much more information about the nature and behavior of the function. In this chapter we will learn the more general form of the Rolle's Theorem and then the general form of the Lagrange mean value theorem with their physical interpretations.

8.1 Lagrange's Mean Value Theorem

This theorem is also called the First Mean Value Theorem and allows to express the increment of a function on an interval through the value of the derivative at an intermediate point of the segment. Lagrange's mean value theorem (MVT) states that if a function$f(x)$ is continuous on a closed interval$[a, b]$and differentiable on the open interval(a, b), then there is at least one point$x = c$ on this interval, such that$\frac{f(b)-f(a)}{b-a} = f'(c)$.

Proof

Define a new function

$$\phi(x) = f(x) + Ax, \qquad x \in [a, b]$$

We choose a number A such that the condition $\phi(a) = \phi(b)$ is satisfied. Then

$$f(a) + Aa = f(b) + Ab$$

$$A = -\frac{f(b) - f(a)}{b - a}$$

As a result, we have

$$\phi(x) = f(x) + \left(-\frac{f(b)-f(a)}{b-a}\right)x$$

The function $\phi(x)$ is continuous on the closed interval$[a,b]$, differentiable on the open interval (a,b) and takes equal values at the endpoints of the interval. Therefore, it satisfies all the conditions of the Rolle's Theorem. Then there is at least a point c in the interval (a,b) such that

$$\phi'(c) = 0$$

It follows that

$$f'(c) + A = 0$$

or

$$f'(c) = -A$$

or

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Joseph Louis Lagrange, the greatest mathematician of the eighteenth century, was born at Turin on January 25, 1736, and died at Paris on April 10, 1813. His father, who had charge of the Sardinian military chest, was of good social position and wealthy, but before his son grew up he had lost most of his property in speculations, and young Lagrange had to rely for his position on his own abilities. He was educated at the college of Turin, but it was not until he was seventeen that he showed any taste for mathematics - his interest in the subject being first excited by a memoir by Halley across which he came by accident. Alone and unaided he threw himself into mathematical studies; at the end of a year's incessant toil he was already an accomplished mathematician, and was made a lecturer in the artillery school.

8.2 Alternate form of Lagrange's Mean Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a, a+h]$and differentiable on the open interval $(a, a+h)$, then there is at least one $\theta \in (0,1)$ such that $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$.

Let $b - a = h =$ length of the interval $[a,b]$

Therefore $[a,b]$ can be written as $[a, a+h]$

Also $a < c < a + h$

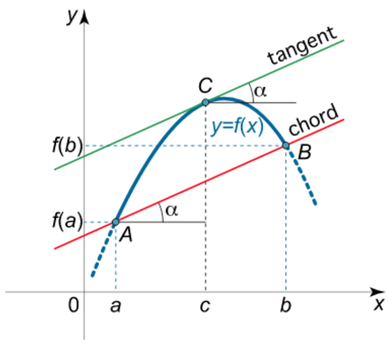
Therefore c can be written as $a + \theta h$ where $\theta \in (0,1)$.

So, the expression

$$f'(c) = -\frac{f(b)-f(a)}{b-a}$$

becomes

$$f(a+h) - f(a) = h\,f'(a+\theta h)$$



or

$$f(a+h) = f(a) + h\,f'(a+\theta h), \quad \theta \in (0,1)$$

8.3 Geometric and Physical Interpretation of Lagrange's Mean Value Theorem

Geometric interpretation

Lagrange's mean value theorem has a simple geometrical meaning. The chord passing through the points of the graph corresponding to the ends of the segment a and b has the slope equal to

$$\frac{f(b)-f(a)}{b-a}$$

Then there is a point $x = c$inside the interval$[a, b]$ where the tangent to the graph is parallel to the chord.

Physical interpretation

The mean value theorem has also a clear physical interpretation. If we assume that$f(t)$ represents the position of a body moving along a line, depending on the time$t$, then the ratio $\frac{f(b)-f(a)}{b-a}$is the average velocity of the body in the period of time$b - a$. Since$f'(t)$is the instantaneous velocity, this theorem means that there exists a moment of time c at which the instantaneous velocity is equal to the average velocity.

Lagrange's mean value theorem has many applications in mathematical analysis, computational mathematics and other fields. Let us further note two remarkable corollaries.

Corollary I

In a particular case when the values of the function$f(x)$ at the endpoints of the segment$[a, b]$are equal, i.e.$f(a) = f(b)$the mean value theorem implies that there is a point$c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$$

and that is the Rolle's theorem, which can hence be considered as a special case of Lagrange's mean value theorem.

Corollary II

If the derivative $f'(x)$ is zero at all points of the interval$[a, b]$ then the function$f(x)$ is constant on this interval.

For any two pointsx_1 and x_2 in the interval$[a, b]$, there exists a point $c \in (a, b)$such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$$

And this results in

$$f(x_1) = f(x_2)$$

If $f(x) = (x-1)(x-2)(x-3),\ x \in [0,4]$, find c such that the average rate of change of $f(x)$ is equal to the derivative of f(x) at c.

Here,

$$f(0) = -6$$

$$f(4) = 6$$

So, $\frac{f(4)-f(0)}{4-0} = 3$

Also $f'(x) = 3x^2 - 12\,x + 11$

According to the statement,

$$\frac{f(4)-f(0)}{4-0} = f'(c)$$

$$3c^2 - 12c + 11 - 3 = 0$$

$$\Rightarrow c = \frac{6 \pm 2\sqrt{3}}{8}$$

Verify the mean value theorem for $f(x) = x^3$in $[a, b]$

The function is a polynomial, therefore continuous in $[a, b]$

$f'(x) = 3x^2$exists in (a,b), therefore the function is differentiable in (a,b).

Since the function is satisfying the requirements of Lagrange's mean value theorem, there must exist a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 = \frac{b^3 - a^3}{b - a}$$

$$b^2 + ab + a^2 = 3c^2$$

$$c^2 = \frac{b^2 + ab + a^2}{3}$$

$$c = \pm\sqrt{\frac{b^2 + ab + a^2}{3}}$$

Prove that for any quadratic function $px^2 + qx + r$, the value of θ in Lagrange's theorem is always $\frac{1}{2}$, for any p, q, r, a, h.

Let $f(x) = px^2 + qx + r, x \in [a, a + h]$

Clearly, the given function is continuous in $[a, a + h]$

and derivable in $(a, a + h)$

Therefore, there exists a θ in (0,1) such that

$$f(a + h) - f(a) = hf'(a + \theta h)$$

Substituting and simplifying,

$$ph^2 = 2p\theta h^2$$

$$\Rightarrow \theta = \frac{1}{2} \in (0,1)$$

Hence proved

8.4 Cauchy's Mean Value Theorem

If two functions $f(x)$ and $F(x)$ are continuous on an interval $[a, b]$, differentiable on (a, b) and F'(x) is non zero for all $x \in (a, b)$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)}$$

This theorem is also known as generalized Lagrange's mean value theorem as it can be seen as a special case for $F(x) = x$.

Proof

Here $F(b) - F(a) \neq 0$

Define a new function

$$\phi(x) = f(x) + A\,F(x), \qquad x \in [a, b]$$

We choose a number A such that the condition $\phi(a) = \phi(b)$ is satisfied. Then

$$f(a) + AF(a) = f(b) + AF(b)$$

$$A = -\frac{f(b) - f(a)}{F(b) - F(a)}$$

As a result, we have

$$\phi(x) = f(x) + \left(-\frac{f(b) - f(a)}{F(b) - F(a)}\right) F(x)$$

The function $\phi(x)$ is continuous on the closed interval$[a, b]$, differentiable on the open interval (a, b) and takes equal values at the endpoints of the interval. Therefore, it satisfies all the conditions of the Rolle's Theorem. Then there is at least a point c in the interval (a, b) such that

$$\phi'(c) = 0$$

It follows that

$$f'(c) + AF'(c) = 0$$

or

$$f'(c) = -AF'(c)$$

or

$$\frac{f'(c)}{F'(c)} = \frac{f(b) - f(a)}{F(b) - F(a)}, \qquad F'(c) \neq 0$$

Hence the result.

Augustin-Louis Cauchy was one of the greatest mathematicians during the nineteenth century. In fact, there are sixteen concepts and theorems named after him, more than any other mathematician. His life began in Paris, France on August 21, 1789, and ended at Sceaux, France on May 22, 1857. His father, Luois-Francois, and his mother, Marie-Madeleine Desestre, provided him and his siblings a comfortable life.

Cauchy was exposed to famous scientists as a child. The Cauchy family once had Laplace and Berthollet as neighbors, and his father even knew Lagrange. In fact, Lagrange had foreseen Augustin's scientific greatness when he was a child by warning his father to not show him any mathematical text before he was seventeen years old.

8.5 Alternate form of Lagrange's Mean Value Theorem

We will see the following result without proof.

If two functions $f(x)$ and $F(x)$are continuous on a closed interval $[a, a+h]$and differentiable on the open interval $(a, a+h)$, then there is at least one $\theta \in (0,1)$ such that $\frac{f(a+h)-f(a)}{F(a+h)-F(a)} = \frac{f'(a+\theta h)}{F'(a+\theta h)}$.

If in the Cauchy's mean value theorem $f(x) = e^x$ and $F(x) = e^{-x}$, show that c is the arithmetic mean between a and b.

Here $f'(x) = e^x$

$$F'(x) = -e^{-x}$$

By Cauchy's mean value theorem,

$$\frac{f'(c)}{F'(c)} = \frac{f(b) - f(a)}{F(b) - F(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$-e^{2c} = -e^{a+b}$$

This implies

$$c = \frac{a+b}{2}$$

Show that $\frac{sin\alpha - sin\beta}{cos\beta - cos\alpha} = \cot\theta, \quad 0 < \alpha < \theta < \beta < \frac{\pi}{2}$

Looking at the result to be proved, you can see that the left hand side is a ratio of difference of two functions and Cauchy's theorem can be used to derive this, provided these two functions satisfy the requirements of Cauchy's mean value theorem.

Let $f(x) = \sin x$

$$F(x) = \cos x \; x \in [\alpha, \beta]$$

$$f'(x) = \cos x$$

$$F'(x) = -\sin x$$

By Cauchy's mean value theorem we can write,

$$\frac{f(\beta) - f(\alpha)}{F(\beta) - F(\alpha)} = \frac{f'(\theta)}{F'(\theta)}$$

$$\frac{sin\beta - sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-sin\theta}$$

$$\frac{sin\beta - sin\alpha}{\cos\beta - \cos\alpha} = -\cot\theta$$

or

$$\frac{sin\alpha - sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$$

Hence the result

Check the validity of Cauchy's mean value theorem for the functions $f(x) = x^4$ and $g(x) = x^2$on the interval[1,2].

Here

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

$$g(x) = x^2$$

$$g'(x) = 2x$$

Both functions are satisfying all the criteria of continuity and differentiability, therefore we can write

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{4c^3}{2c} = \frac{b^4 - a^4}{b^2 - a^2}$$

$$2c^2 = a^2 + b^2$$

Here $a = 1, b = 2$

Therefore, $c = \pm\sqrt{2.5}$ and $\sqrt{2.5} \in (1,2)$

Therefore having found at least a value in (1,2) such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$, we can say that the Cauchy' theorem is valid for the given functions.

Summary

This unit is an extension of the Rolle's Theorem. Its generalized form can be seen as Lagrange's mean value theorem, which further can be generalized as the Cauchy's mean value theorem.

- Lagrange's mean value theorem (MVT) states that if a function $f(x)$ is continuous on a closed interval $[a, b]$and differentiable on the open interval (a, b), then there is at least one point $x = c$ on this interval, such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

- If two functions $f(x)$ and $F(x)$ are continuous on an interval $[a, b]$, differentiable on (a, b) and F′(x) is non zero for all $x \in (a, b)$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)}$$

Key Words

Lagrange's mean value theorem, Cauchy's mean value theorem

Self Assessment

1. The Lagrange's mean value theorem is valid for the function $f(x) = \frac{x-1}{x-3}$ on the interval$[4, 5]$.

A. True

B. False

2. All points C satisfying the conditions of the MVT for the function $f(x) = x^3 - x$in the interval $[-2, 1]$ are

A. -1

B. 1, -1

C. 0

D. 1

3. For any quadratic function$px^2 + qx + r$, the value of θ in Lagrange's theorem, for any p, q, r, a, h is

A. less than ½

B. greater than ½

C. always ½

D. can take any value

4. For any quadratic function$3x^2 + 2x + 1$, the value of θ in Lagrange's theorem is

A. 0

B. 1

C. 0.5

D. 1.5

5. Cauchy's Mean Value Theorem can be reduced to Lagrange's Mean Value theorem.

A. True

B. False

6. Which of the following is not a necessary condition for Cauchy's Mean Value Theorem?

A. The functions, f(x) and g(x) be continuous in [a, b]

B. The derivative of g'(x) be equal to 0

C. The functions f(x) and g(x) be derivable in (a, b)

D. There exists a value c ∈ (a, b) such that, $(f(b)–f(a)) / (g(b)–g(a)) = f'(c) / g'(c)$

7. Cauchy's Mean Value Theorem is also known as 'Extended Mean Value Theorem'.

A. True

B. False

8. The Mean Value Theorem was stated and proved by _______

A. Leonhard Euler

B. Govindasvami

C. Michel Rolle

D. Augustin Louis Cauchy

9. The value of c which satisfies the Mean Value Theorem for the function $f(x) = x^2 + 2x + 1$ on [1,2] is

A. -5/2

B. -5/2

C. 7/2

D. -7/2

10. What is the value of c which lies in [1, 2] for the function f(x)= 4x and g(x)= $3x^2$?

A. 1

B. 1.5

C. 2

D. 2.5

Answer for Self Assessment

1.	A	2.	A	3.	C	4.	C	5.	A
6.	B	7.	A	8.	D	9.	D	10.	B

Review Questions

1. State and prove the Lagrange's mean value theorem.
2. State and prove the Cauchy's mean value theorem.
3. Check the validity of Cauchy's MVT for the functions f(x)= 4x and g(x)= $3x^2$
4. Check the validity of Lagrange's MVT for the function $f(x) = x^2 + 2x + 1$ on [1,2].
5. Check the validity of Lagrange's MVT for the function $f(x) = x^2 + 2x + 1$ on [-1,2].
6. Explain how the Lagrange's MVT is a special case of Cauchy's MVT.
7. Discuss the Lagrange's MVT in the interval $[a, a + h]$.
8. Discuss the Cauchy's MVT in the interval $[a, a + h]$.

Further Reading

George B. Thomas Jr., Joel Hass, Christopher Heil& Maurice D. Weir (2018). *Thomas' Calculus* (14th edition). Pearson Education.

Howard Anton, I. Bivens& Stephan Davis (2016).*Calculus* (10th edition).Wiley India.

https://www.math24.net/cauchys-mean-value-theorem

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https://tutorial.math.lamar.edu/classes/calci/MeanValueTheorem.aspx

Unit 09: Higher Order Derivatives

CONTENTS

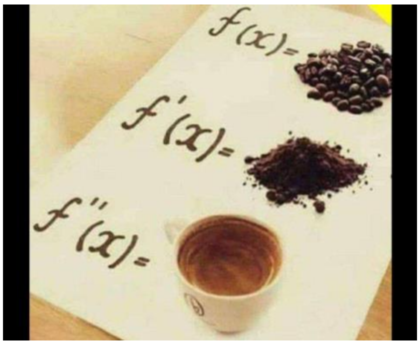
Objectives

Students will

- be able to find the n^{th} derivative of elementary functions
- be able to find the n^{th} derivative of the derived functions
- be able to calculate the n^{th} derivative of the product of two functions using the Leibnitz theorem

Introduction

By now, we have a good idea about what is differentiation, we know the technique how to differentiate a function, we have derived quite a few rules of the derivatives for some functions also. Continuing the stride, we now look into how the higher derivatives can be found out in a general manner. We will discuss how to find the nth derivative for some specific functions in this chapter. We will see the process of differentiating a given function successively n times, which is known as successive differentiation and the results that you get are called successive derivatives.



The adjacent image gives an idea of the higher derivatives. At first there are coffee beans, then some changes happen and you see the change in the beans with respect to time (x say), it is the coffee powder and then some more changes happening to this and you are getting the coffee.

So, we can say that $f'(x)$ is the first derivative and $f''(x)$ is the second derivative. So this second derivative has come up by differentiating the first derivative. So, this is the successive differentiation, you can further differentiate to get the third derivative and so, on. This successive differentiation is very much important for scientific and engineering applications.

9.1 Successive derivatives

The process of differentiating a given function successively n times are called successive differentiation and the results of such differentiation are called successive derivatives.

Let the function be $y = f(x)$.

Differentiating it once we get $\frac{dy}{dx} = f'(x)$

Differentiating it twice we get $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2} = f''(x)$

Differentiating it thrice we get $\frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{d^3y}{dx^3} = f'''(x)$

and so on

For instance, $f(x) = x^5 + \sin x + e^{2x}$

$$f'(x) = 5x^4 + \cos x + 2e^{2x}$$
$$f''(x) = 20x^3 - \sin x + 4e^{2x}$$
$$f'''(x) = 60x^2 - \cos x + 8\,e^{2x}$$

and so on.

9.2 The n^{th}derivative for $y = e^{ax}$

We will calculate the n^{th} derivative by generalizing first few derivatives.

$$y_1 = a\,e^{ax}$$
$$y_2 = a^2e^{ax}$$
$$y_3 = a^3e^{ax}$$
$$\vdots$$
$$y_n = a^ne^{ax}$$

9.3 Then^{th}derivative for $y = (ax + b)^m$ Where m is a Positive Integer More than n

We will calculate the n^{th} derivative by generalizing first few derivatives.

$$y_1 = ma(ax + b)^{m-1}$$
$$y_2 = m(m - 1)(ax + b)^{m-2}a^2$$
$$y_3 = m(m - 1)(m - 2)(ax + b)^{m-3}a^3$$
$$\vdots$$
$$y_n = m(m - 1)(m - 2) \dots (m - n + 1)(ax + b)^{m-n}a^n$$

The n^{th} derivative can further be written as

$$y_n = \frac{m!}{(m - n)!}(ax + b)^{m-n}a^n$$

9.4 The n^{th}derivative for $y = log(ax + b)$

We will calculate the n^{th} derivative by generalizing first few derivatives.

$$y_1 = a(ax + b)^{-1}$$
$$y_2 = -a^2(ax + b)^{-2}$$
$$y_3 = 2!\,a^3(ax + b)^{-3}$$
$$y_4 = -3!\,a^4(ax + b)^{-4}$$
$$\vdots$$
$$y_n = (-1)^{n-1}a^n(ax + b)^{-n}$$

9.5 The n^{th}derivative for $y = sin(ax + b)$

We will calculate the n^{th} derivative by generalizing first few derivatives.

$$y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$
$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

Similarly

$$y_3 = \quad a^3\sin\left(ax + b + \frac{3\pi}{2}\right)$$
$$\vdots$$
$$y_n = a^n\sin\left(ax + b + \frac{n\pi}{2}\right)$$

9.6 The n^{th}derivative for $y = cos(ax + b)$

We will calculate the n^{th} derivative by generalizing first few derivatives.

$$y_1 = -a\sin(ax + b) = a\cos\left(ax + b + \frac{\pi}{2}\right)$$
$$y_2 = -a^2\sin\left(ax + b + \frac{\pi}{2}\right) = a^2\cos\left(ax + b + \frac{2\pi}{2}\right)$$

Similarly

$$y_3 = \quad a^3\cos\left(ax + b + \frac{3\pi}{2}\right)$$
$$\vdots$$
$$y_n = a^n\cos\left(ax + b + \frac{n\pi}{2}\right)$$

9.7 The n^{th}derivative for $y = e^{ax}sin(bx + c)$

$$y_1 = a\,e^{ax}\sin(bx + c) + b\,e^{ax}\cos(bx + c)$$
$$= e^{ax}(a\sin(bx + c) + b\cos(bx + c))$$

Here substituting $a = r\cos\theta, b = r\sin\theta$ in the above expression.

$$y_1 = e^{ax}r\,(\cos\theta\sin(bx + c) + \sin\theta\cos(bx + c))$$
$$= r\,e^{ax}\sin(bx + c + \theta)$$

Similarly

$$y_2 = r^2e^{ax}\sin\,(bx + c + 2\theta)$$
$$\vdots$$
$$y_n = r^ne^{ax}\sin(bx + c + n\theta)$$

where $r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$

Therefore, $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + c + ntan^{-1}\left(\frac{b}{a}\right)\right)$

9.8 The n^{th}derivative for $y = e^{ax}cos(bx + c)$

$$y_1 = -b\, e^{ax}\sin(bx+c) + a\, e^{ax}\cos(bx+c)$$
$$= e^{ax}(-b\sin(bx+c) + a\cos(bx+c))$$

Here substituting $a = r\cos\theta, b = r\sin\theta$ in the above expression.

$$y_1 = e^{ax} r\,(-\sin\theta\sin(bx+c) + \cos\theta\cos(bx+c))$$
$$= r\, e^{ax}\cos(bx+c+\theta)$$

Similarly

$$y_2 = r^2 e^{ax}\cos\,(bx+c+2\theta)$$
$$\vdots$$
$$y_n = r^n e^{ax}\cos(bx+c+n\theta)$$

where $r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$

Therefore, $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + c + ntan^{-1}\left(\frac{b}{a}\right)\right)$

9.9 Determination of n^{th} Derivative of the Rational Functions

To calculate the n^{th} derivative of a rational function, we can decompose it into partial fractions. We may use the De Moivre's theorem also if the situation demands.

Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$.

Now here the given function is a composite function. We can work out on the function to write it as an elementary function whose n^{th} derivative is known.

Let $y = \frac{1}{1-5x+6x^2}$

$$= \frac{1}{(1-3x)(1-2x)}$$
$$= \frac{3}{1-3x} - \frac{2}{1-2x}$$

Now these two expressions are of the form $\frac{1}{ax+b}$. Working out on the n^{th} derivative of this function we get $\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

Using this general formula, we can write

$$y_n = n!\left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Find the n^{th} derivative of $\sin 6x\cos 4x$

Using the trigonometric identities the given function can be written as $y = \sin 6x\cos 4x = \frac{1}{2}(\sin 10x + \cos 2x)$

Applying the direct result of the sine and cosine functions, we get

$$y_n = \frac{1}{2}\left(10^n\sin\left(10x + \frac{n\pi}{2}\right) + 2^n\cos\left(2x + \frac{n\pi}{2}\right)\right)$$

If $y = x + \tan x$, show that $\cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$

We can find the first and second derivatives of y and on substituting them in the left hand side, you can easily get the result.

9.10 Leibnitz Theorem

If u and v are functions of x such that their n^{th} derivativesexist, then the n^{th} derivative of their product is given by

$$(uv)_n = u_n v + {}^n_1C\, u_{n-1}v_1 + {}^n_2C\, u_{n-2}v_2 + \cdots + {}^n_rC\, u_{n-r}v_r + \cdots + uv_n$$

whereu_r and v_r represent the r^{th} derivatives of u and v respectively.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2}{dx^2}(uv) = \frac{d}{dx}(uv_1 + vu_1) = u_2v + 2u_1v_1 + uv_2$$

and continuing in the same manner, the n^{th} derivative can be obtained as

$$\frac{d^n}{dx^n}(uv) = (uv)_n = u_n v + {}^n_1C\, u_{n-1}v_1 + {}^n_2C\, u_{n-2}v_2 + \cdots + {}^n_rC\, u_{n-r}v_r + \cdots + uv_n$$

Let us see some examples to understand the theorem better!

Find the n^{th} derivative of $x \log x$.

Here we can see the given function as a product of two functions and Leibnitz theorem can be applied.

Let $u = \log x$

Then $u_1 = \frac{1}{x}$

$$u_2 = -\frac{1}{x^2}$$

$$u_3 = \frac{2}{x^3}$$

$$u_4 = -\frac{2.3}{x^4}$$

$$\vdots$$

$$u_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Let $v = x$

$$v_1 = 1$$

$$v_2 = 0$$

$$v_3 = 0$$

$$\vdots$$

$$v_n = 0$$

By Leibnitz theorem,

$$(uv)_n = u_n v + {}^n_1C\, u_{n-1}v_1 + {}^n_2C\, u_{n-2}v_2 + \cdots + {}^n_rC\, u_{n-r}v_r + \cdots + uv_n$$

$$(xlogx)_n = \frac{(-1)^{n-1}(n-1)!}{x^{n-1}} + \frac{n(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$= \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

Find the n^{th} derivative of $x^2e^{3x}sin4x$

Let $u = e^{3x}sin4x$ and $v = x^2$

By Leibnitz theorem,

$$(uv)_n = u_n v + {}^n_1C\, u_{n-1}v_1 + {}^n_2C\, u_{n-2}v_2 + \cdots + {}^n_rC\, u_{n-r}v_r + \cdots + uv_n$$

$$(x^2 e^{3x} sin4x)_n = e^{3x}5^n \sin\left(4x + ntan^{-1}\left(\frac{4}{3}\right)\right)x^2 + n\, e^{3x}5^{n-1}\sin(4x + (n-1)\tan^{-1}\frac{4}{3})(2x) + \frac{n(n-1)}{2}e^{3x}5^{n-2}\sin(4x + (n-2)\tan^{-1}\frac{4}{3})(2)$$

The right hand side expression can be simplified for a concise form.

If $y = \tan^{-1}x$, show that $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$.

Here $y = \tan^{-1}x$

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)y_2 + 2xy_1 = 0$$

Differentiating 'n' times w.r.t. x, we get

$$(1+x^2)y_{n+2} + 2x\,n\,y_{n+1} + n(n-1)y_n + 2(x\,y_{n+1} + n\,y_n) = 0$$

$$(1+x^2)y_{n+2} + 2x(n+1)y_{n+1} + n(n+1)y_n = 0$$

which is the required expression to be proved. Now in order to deduce the second part, let us put $x = 0$ in the expressions of y, y_1, y_2 and y_{n+2}, we get

$$y(0) = 0$$

$$y_1(0) = 1$$

$$y_2(0) = 0$$

$$y_{n+2}(0) = -n(n+1)y_n(0)$$

From this recursion formula, higher derivatives can be obtained.

$$y_3(0) = -1.2.\,y_1(0) = -2.1 = -2!$$

$$y_4(0) = -2.3.\,y_2(0) = 0$$

$$y_5(0) = -3.4.\,y_3(0) = -3.4.(-2) = 4!$$

$$y_6(0) = -4.5.\,y_4(0) = 0$$

$$y_7(0) = -5.6.\,y_5(0) = -5.6.4! = -6!$$

$$\vdots$$

$$y_{2n+1}(0) = (-1)^n(2n)! \text{ and } y_{2n}(0) = 0$$

This expression shows that all the even derivatives of the given function are zero at $x = 0$ and the odd derivatives are given by $y_{2n+1}(0) = (-1)^n(2n)!$

Summary

This chapter is about the higher derivative of a function. We also learnt about how to find the nth derivative of the product of two functions.

- The process of differentiating a given function successively n times are called successive differentiation and the results of such differentiation are called successive derivatives.
- If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

  $$(uv)_n = u_n v + {}^n_1C\, u_{n-1}v_1 + {}^n_2C\, u_{n-2}v_2 + \cdots + {}^n_rC\, u_{n-r}v_r + \cdots + uv_n$$

  whereu_r and v_r represent the r^{th} derivatives of u and v respectively.

Keywords

Successive differentiation, higher derivatives, Leibnitz theorem

Self Assessment

1. The derivative of the function $\sqrt{2+x^2}$ is
A. $\frac{x}{2\sqrt{2+x^2}}$
B. $\frac{x}{\sqrt{2+2x^2}}$
C. $\frac{2x}{\sqrt{2+x^2}}$
D. $\frac{x}{\sqrt{2+x^2}}$

2. $\frac{d}{dx}\left(sin(cosx)\right) =$
A. $\sin x \cos(\cos x)$
B. $\cos(\sin x)$
C. $-\sin x \cos(\cos x)$
D. $\sin x \cos(\sin x)$

3. If *f* is differentiable on the closed interval [*a*, *b*] and *r* is any number between $f'(a)$ and $f'(b)$, then there exists a number *c* in the open interval (*a*, *b*) such that $f'(c) = r$. This statement is of
A. intermediate value theorem
B. mean value theorem
C. Rolle's theorem
D. Darboux's theorem

4. $\frac{e^{sinh^{-1}x}}{\sqrt{1+x^2}}$ is the derivative of the function
A. $x\frac{e^{sinh^{-1}x}}{\sqrt{1+x^2}}$
B. $\frac{e^{sinh^{-1}x}}{\sqrt{1+x^2}}$
C. $e^{sinh^{-1}x}$ ans
D. $e^{cosh^{-1}x}$

5. The derivative of $log(cosh\ x)$ w.r.t. x is
A. $\log(\sin x)$
B. $\log(\sinh x)$
C. $cosech\ x$
D. $\tanh x$

6. $\frac{d}{dx}e^{sinx} =$
A. $e^{sinx}\cos x$
B. $\cos(\sin x)$
C. $-e^{sinx}\cos x$
D. $\sin x \cos(\sin x)$

7. The Leibnitz theorem is about
A. the n^{th} derivative of the sum of two functions
B. the n^{th} derivative of the difference of two functions
C. the n^{th} derivative of the quotient of two functions
D. the n^{th} derivative of the product of two functions

8. Which of the following is correct?

A. $(uv)_n = u_n v + {}^n_1C u_{n-1} v_1 + {}^n_2C u_{n-2} v_2 + {}^n_3C u_{n-3} v_3 + \dots + {}^n_rC u_{n-r} v_r + \dots + u \;\; v_n$

B. $(uv)_n = u \;\; v + {}^n_1C u_{n-1} v_1 + {}^n_2C u_{n-2} v_2 + {}^n_3C u_{n-3} v_3 + \dots + {}^n_rC u_{n-r} v_r + \dots + u \;\; v_n$

C. $(uv)_n = u_n v + {}^n_1C u_{n-1} v_1 + {}^n_2C u_{n-2} v_2 + {}^n_3C u_{n-3} v_3 + \dots + {}^n_rC u_{n-r} v_r + \dots + u \;\; v$

D. $(uv)_n = u \;\; v + {}^n_1C u_{n-1} v_1 + {}^n_2C u_{n-2} v_2 + {}^n_3C u_{n-3} v_3 + \dots + {}^n_rC u_{n-r} v_r + \dots + v\, u$

9. Let $\boldsymbol{f(x) = \frac{sinx}{1+x^2}}$. The first derivative of $\boldsymbol{f(x)}$ at $\boldsymbol{x = 0}$ is given by

A. 1

B. 0

C. -1

D. 2

10. The number of terms in the n^{th} derivative of $x^2e^{3x}sin\ 4x$ are

A. 1

B. 2

C. 3

D. 4

11. For $y = tan^{-1}x, (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$, Then $y_3(0)$ is

A. 0

B. 1

C. 2

D. -2

12. $\frac{d}{dx}e^{-sinx} =$

A. $-e^{-sinx}\cos x$

B. $\cos(\sin x)$

C. $-e^{sinx}\cos x$

D. $\sin x\cos(\sin x)$

Answer for Self Assessment

1. D	2. C	3. D	4. C	5. D
6. A	7. D	8. A	9. A	10. C
11. D	12. A			

Review Questions

1. Find the first three derivatives of the following expressions w.r.t. x

(i) $\frac{x^2+a}{x+a}$

(ii) $8x^4 + 3.8x^3 - \frac{2}{3}x^2 + x - 7$

2. If a body move according to the law

$$s = 12 - 4.5t + 6.2t^2$$

find its velocity and acceleration when t=4 seconds, s being in feet. Is the acceleration the same for all values of t?

3. Find the $\boldsymbol{n^{th}}$derivative for $\boldsymbol{y} = \boldsymbol{e^{ax}}\cos(\boldsymbol{bx} + \boldsymbol{c})$
4. Find the $\boldsymbol{n^{th}}$derivative for $\boldsymbol{y} = \boldsymbol{e^{ax}}\sin(\boldsymbol{bx} - \boldsymbol{c})$
5. Find the $\boldsymbol{n^{th}}$derivative for $\boldsymbol{y} = \tan^{-1}\frac{1+x}{1-x}$
6. Find the $\boldsymbol{n^{th}}$derivative for $\boldsymbol{y} = \frac{1}{\mathrm{a}^2 - x^2}$
7. State and prove Leibnitz theorem.
8. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = nI_{n-1} + (n-1)$
9. Find the value of $\boldsymbol{n^{th}}$derivative for $\boldsymbol{y} = \mathrm{e}^{\mathrm{m}\sin^{-1}x}$for $x = 0$.
10. If $\boldsymbol{y} = \mathrm{e}^{\mathrm{a}\sin^{-1}x}$ prove that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$$

Further Reading

George B. Thomas Jr., Joel Hass, Christopher Heil& Maurice D. Weir (2018). *Thomas' Calculus (*14th edition). Pearson Education.

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Unit 10: Maclaurin's and Taylor's Theorems

CONTENTS

Objectives

Students will be able to

- expand the functions using Maclaurin's theorem
- expand the functions using Taylor's theorem
- apply the Taylor's theorem in finite form with Lagrange form of remainder
- apply the Taylor's theorem in finite form with Cauchy form of remainder

Introduction

In calculus, Taylor's theorem gives us a polynomial which approximates the function in terms of the derivatives of the function. Since the derivatives are usually easy to compute, these polynomials are also easy to compute.

A simple example of Taylor's theorem is the approximation of the exponential function e^x near $x = 0$. In other words, the exponential function can be approximated by an infinite polynomial given as follows

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{\mathrm{x}^{\mathrm{n}}}{n!} + \cdots$$

For a derivable function$f$, we can say that f' exists in certain neighborhood of point c and this further implies that f is defined and is continuous in a neighborhood of c.

Similarly, if f' has derivative at c has the same meaning as f has a second derivative at c. And this further implies that f' is continuous at c.

In general if $f^{n-1}(x)$ exists in the neighborhood of c, then the derivative of $f^{n-1}(x)$ at c, if exists, is called the n^{th} derivative of f at c and is written as $f^{(n)}(c)$.

10.1 Generalized Mean Value Theorem- Taylor's Theorem

If $n \geq 0$ is an integer and f is a function which is n times continuously differentiable on the closed interval$[a, x]$ and $n + 1$ times differentiable on the open interval(a, x), then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$$

Here, $n!$ denotes the factorial of , and $R_n(x)$ is a remainder term, denoting the difference between the Taylor polynomial of degree n and the original function. The remainder term $R_n(x)$depends on $x$and is small if $x$is close enough to a. There are several expressions available for it.

We can state the theorem in the following form also.

If a function f is such that

(i) the $(n-1)^{th}$ derivative f^{n-1} is continuous in $[a, a+h]$,

(ii) the n^{th} derivative f^n exists in $(a, a+h)$ and

(iii) p is a given positive integer

Then there exists at least one $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^n\frac{f^n(a)}{n!} + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h) \qquad \text{---- (1)}$$

Condition (i) assures that $f, f', f'', \dots, f^{n-1}$ are continuous in $[a, a+h]$.

Let a function $\phi(x)$ be defined by

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$$

---- (2)

where$A$ is to be determined such that

$$\phi(a) = \phi(a+h)$$

Therefore,

$\phi(x)$is a continuous in $[a, a+h]$, derivable in $(a, a+h)$ and $\phi(a) = \phi(a+h)$. So, Rolle's theorem suggests that there must exist at least a $\theta \in (0,1)$ such that

$$\phi'(a+\theta h) = 0$$

Now $\phi'(x) = f'(x) + (a+h-x)f''(x) - f'(x) + \frac{(a+h-x)^2}{2!}f'''(x) - (a+h-x)f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - \frac{(n-1)(a+h-x)^{n-2}}{(n-1)!}f^{n-1}(x) - pA(a+h-x)^{p-1}$

$$\phi'(x) = 0 \Rightarrow \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) = pA(a+h-x)^{p-1}$$

$$\phi'(a+\theta h) = 0 \Rightarrow \frac{(a+h-a-\theta h)^{n-1}}{(n-1)!}f^n(a+\theta h) = pA(a+h-a-\theta h)^{p-1}$$

$$\Rightarrow \frac{(h-\theta h)^{n-1}}{(n-1)!}f^n(a+\theta h) = pA(h-\theta h)^{p-1}$$

$$\Rightarrow \quad A = \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)!\,p}f^n(a+\theta h), \quad 1-\theta \neq 0, \qquad h \neq 0$$

Substituting A in the expression (2), we get the required result.

Corollary

Let x be a point of the interval $[a, a+h]$. Let f satisfies the conditions of Taylor's theorem in $[a, a+h]$, thus it satisfies the condition for $[a, x]$ also.

Writing $a+h$ as x or h as $x-a$ in the expression (1), we get

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + \cdots + (x-a)^n\frac{f^n(a)}{n!} + \frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n(a+\theta(x-a))$$

---- (3)

where$0 < \theta < 1$ and the expression (3) holds for all $x \in [a, a+h]$.

10.2 Maclaurin's Theorem

Substituting $a = 0$ in (3) i.e. for all $x \in [0, h]$

$f(x) = f(0) + (x)\frac{f'(0)}{1!} + (x)^2\frac{f''(0)}{2!} + \cdots + (x)^{n-1}\frac{f^{n-1}(0)}{(n-1)!} + \frac{(x)^n(1-\theta)^{n-p}}{(n-1)!p}f^n(\theta x)$ ---- (4)

which holds when

(i) f^{n-1} is continuous in $[0, h]$
(ii) f^n exists in $(0, h)$ and
(iii) $p$is a given positive integer.

Show that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$$

Here$f(x) = e^x$

$f^{n-1}(x)$is continuous in $[0, h]$

$f^n(x)$exists in $(0, h)$

Let $p = n$ in (4). Then,

$$f'(x) = e^x f'(0) = 1$$
$$f''(x) = e^x f''(0) = 1$$
$$f'''(x) = e^x f'''(0) = 1$$
$$f^{n-1}(x) = e^x f^{n-1}(0) = 1$$
$$f^n(x) = e^x f^n(\theta x) = \theta x$$

Therefore, fromthe expression (4), we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$$

Hence the proof.

10.3 Taylor's Theorem in Finite form with Lagrange form of Remainder

From the previous section we know, Taylor's theorem states that, if a function f is such that

(i) the $(n-1)^{th}$ derivative f^{n-1} is continuous in $[a, a+h]$,
(ii) the n^{th} derivative f^n exists in $(a, a+h)$ and
(iii) p is a given positive integer
Then there exists at least one $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^n\frac{f^n(a)}{n!} + \frac{h^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n(a+\theta h)$$

The term $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h)$ is known as the remainder after n terms, better known as Taylor's remainder R_n after n terms due to Schlomilch and Roche. In this expression if we substitute $p = n$

$R_n = \frac{h^n}{n!}f^n\,(a+\theta h)$is the remainder after n terms due to Lagrange.

Therefore the Taylor's theorem with Lagrange's form of remainder is given as,

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^{n-1}\frac{f^{n-1}(a)}{(n-1)!} + \frac{h^n}{n!}f^n(a+\theta h)$$

or

$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + \cdots + (x-a)^{n-1}\frac{f^{n-1}(a)}{(n-1)!} + \frac{(x-a)^n}{n!}f^n(a+\theta(x-a))$
----- (5)

The expression of $f(x)$ in (5) will fail for those values of x for which

(i) $f(x)$or any of its differential coefficient becomes infinite.
(ii) $f(x)$ or any of its differential coefficients is discontinuous and

(iii) $\lim_{n\to\infty} R_n \neq 0$ i.e. $\lim_{n\to\infty} \frac{h^n}{n!} f^n(a+\theta h) \neq 0$

Similarly the expansion of $f(x)$ by Maclaurin's theorem is not valid for the values of x for which

(i) $f(0)$ or any of $f'(0),\ f''(0), \ldots$ is not finite

(ii) $f(x)$ or any of its derivatives is discontinuous as x passes through zero and

(iii) $\lim_{n\to\infty} R_n \neq 0$ i.e. $\lim_{n\to\infty} \frac{x^n}{n!} f^n(\theta x) \neq 0$

10.4 Maclaurin's Power Series for a Given Function

Let a function f possesses continuous derivatives of all orders in the interval $[0, x]$, so that we have

$$f(x) = f(0) + xf'(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

whereR_n is the Lagrange form of remainder.

Therefore $f(x) = f(0) + xf'(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$ ---- (6)

is valid for all values of n for which $\lim_{n\to\infty} R_n = 0$. The expression (6) is called Maclaurin's infinite series for the expansion of $f(x)$ as power series.

Consider the function $f(x) = e^x$

$$f^n(x) = e^x \ \forall\, x \in \boldsymbol{R}$$

The Lagrange's form of remainder after n terms is

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x} \text{where } \ 0 < \theta < 1$$

Now consider the case if $x > 0$

$$\theta x < x \Rightarrow e^{\theta x} < e^x$$

And if $x < 0$

Then $-x > 0$

Therefore $\theta > 0$

$$\Rightarrow -\theta x > 0$$

$$\Rightarrow e^{-\theta} \ > e^0$$

$$\Rightarrow e^{\theta x} < 1$$

Assuming that for all x

$$\lim_{n\to\infty} \frac{x^n}{n!} \ = 0 \quad \text{(The proof is given after the next example)}$$

Therefore $R^n \to 0$ as $n \to \infty \ \ \forall\, x \in \boldsymbol{R}$

$\therefore e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$is valid for all $x \in \boldsymbol{R}$

Consider the function $f(x) = \sin x$

$$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right) \ \ \forall\, x \in \boldsymbol{R}$$

Lagrange's form of remainder

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

$$|R_n| = \left|\frac{x^n}{n!}\right| \left|\sin\left(\theta x + \frac{n\pi}{2}\right)\right| \leq \frac{x^n}{n!}$$

and$R_n \to 0$ as $n \to \infty\ \forall\ x \in \boldsymbol{R}$

Thus

$$\forall\ x \in \boldsymbol{R}$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

We can prove $\lim_{n\to\infty} \frac{x^n}{n!} = 0\ \forall\ x \in \boldsymbol{R}$as follows:

Let $a_n = \frac{x^n}{n!} \forall\ x \in \boldsymbol{R}\ and\ n \in \boldsymbol{N}$

If $x = 0 \quad \lim_{n\to\infty} a_n = 0$

If $x > 0$ then for $\in \boldsymbol{N}$, $a_n > 0$

For sufficiently large n (say $n \geq x$)

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!} = \frac{x}{n+1} a_n < a_n$$

This implies that after certain $n, a_{n+1} < a_n$

Since a bounded monotonically decreasing sequence of real numbers must have a limit,

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

$$= \lim_{n\to\infty} \frac{x}{n+1} \lim_{n\to\infty} a_n$$

$$\Rightarrow a = 0$$

If $x < 0$, we introduce $a(-1)^n$ factor

i.e. $\frac{x^n}{n!} = \frac{(-1)^n x^n}{n!}$ where $(-1)^n$ is bounded and $\frac{x^n}{n!}$ tends to zero.

$\{b_n\}$is bounded and $a_n \to 0$, then $\lim_{n\to\infty} a_n b_n = 0$

Therefore $\lim_{n\to\infty} \frac{x^n}{n!} = 0\ \forall\ x \in \boldsymbol{R}$

Hence proved

10.5 Taylor's Theorem in Finite form with Cauchy forms of Remainder

The Taylor's theorem states that, if a function f is such that

(i) the $(n-1)^{th}$ derivative f^{n-1} is continuous in $[a, a+h]$,
(ii) the n^{th} derivative f^n exists in $(a, a+h)$ and
(iii) p is a given positive integer

Then there exists at least one $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^n\frac{f^n(a)}{n!} + \frac{h^n(1-\theta)^{n-p}}{(n-1)!\,p} f^n(a+\theta h)$$

The term $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^n(a+\theta h)$ is known as the remainder after n terms, better known as Taylor's remainder R_n after n terms due to Schlomilch and Roche. In this expression if we substitute $p = 1$

$R_n = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!!} f^n\ (a+\theta h)$is the remainder after n terms due to Cauchy.

Therefore the Taylor's theorem with Cauchy's form of remainder is given as,

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^{n-1}\frac{f^{n-1}(a)}{(n-1)!} + \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!!} f^n\ (a+\theta h)$$

or

$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + \cdots + (x-a)^{n-1}\frac{f^{n-1}(a)}{(n-1)!} + \frac{(x-a)^{n-1}(1-\theta)^{n-1}}{(n-1)!!} f^n\ (a+\theta(x-a))$ ----- (6)

Expansion of $(1+x)^m, m \in R$

$(1+x)^m$ possesses continuous derivatives of every order when $1+x>0$ i.e. $x>-1$. Also

$$f^n(x) = m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n}$$

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1}\, m(m-1)\dots(m-n+1)(1+\theta x)^{m-n}$$

$$= x^n \frac{m(m-1)\dots(m-n+1)}{(n-1)!}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1}$$

Let $|x|<1$

$$\Rightarrow -1 < x < 1$$

Now $-1<x$

$$\Rightarrow -\theta < \theta x$$

$$\Rightarrow 1-\theta < 1+\theta x$$

$$\Rightarrow \frac{1-\theta}{1+\theta x} < 1$$

$$\Rightarrow 0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1$$

Let $m-1>0$, we have

$$0<\theta<1$$

$$\Rightarrow \theta x < x$$

$$\Rightarrow \theta x + 1 < x + 1$$

Moreover $x>-1 \Rightarrow x<1$

$$\Rightarrow \theta x + 1 < 2$$

Therefore

$$0 < \theta x + 1 < 2$$

$$\Rightarrow\ 0 < (\theta x + 1)^{m-1} < 2^{m-1} < 2^m$$

Let $m-1<0$, we have

$$\theta x > -|x|$$

$$\Rightarrow \theta x + 1 > 1 - |x|$$

$$\Rightarrow (\theta x + 1)^{m-1} \le (1-|x|)^{m-1}$$

We know, $\lim_{n\to\infty} \quad \frac{m(m-1)\dots(m-n+1)x^n}{(n-1)!} = 0$

$$\therefore R_n \to 0 \text{ as } n \to \infty \text{ if} |x| < 1$$

$\therefore (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$ when $-1<x<1$

Expansion of $\log(1+x)$

$(1+x)^m$ possesses continuous derivatives of every order when $1+x>0$ i.e. $x>-1$. Also

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Taking Cauchy's form of remainder

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= (-1)^{n-1}\, x^n \frac{1}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

Let $|x| < 1$

$$\Rightarrow -1 < x < 1$$

$$\Rightarrow -\theta < \theta\, x < \theta$$

$$\Rightarrow 1 - \theta < 1 + \theta\, x < 1 + \theta$$

$$\Rightarrow 0 < \frac{1-\theta}{1+\theta x} < 1$$

$$\Rightarrow \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1$$

Also we have

$$\theta x > -|x|$$

$$\Rightarrow 1 + \theta x > 1 - |x|$$

$$\Rightarrow \frac{1}{1+\theta x} < \frac{1}{1-|x|}$$

Therefore, for all n

$$|R_n| < |x|^n \frac{1}{1-|x|} \to 0 \text{ as } n \to \infty$$

Therefore, when $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^n x^{n-1}}{n-1} + \cdots$$

By taking the Lagrange's form of remainder we may show that the infinite series expansion is valid for $x = 1$ also.

For the formal expansion of a function, we will follow the following steps:

Calculate the n^{th} derivative of the function

Check the $\lim_{n\to\infty} R_n$

If $\lim_{n\to\infty} R_n$ vanishes, then the function can be expressed as a power series

If $f(x)$ can be expressed as an infinite Maclaurin's series, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

Can $f(x) = \begin{cases} e^{\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ be expanded by Maclaurin's theorem?

In this problem we need to check all the conditions first regarding the function's continuity anf differentiability on its domain.

At the point $x = 0$, we can observe the limit of the function as

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\frac{1}{x}} = \lim_{x\to 0}\left(1 + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + \cdots\right) = \infty$$

As the limit of the given function is not defined at $x = 0$, the continuity of the function can not be established, so the given function can not be expanded by Maclaurin's theorem

Can $f(x) = \sqrt{x}$ be expanded by Maclaurin's theorem?

Clearly the function is a continuous one on its domain. Let us check the differentiability also.

We have

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$$

$$\vdots$$

Clearly $f'(0), f''(0), \ldots$ do not exist.

Use Maclaurin's theorem to expand $y = \log(1 + e^x)$ ------ (1)

$$\Rightarrow e^y = 1 + e^x$$

Differentiating both sides w.r.t.x

$e^y y_1 = e^x$ ---- (2)

Differentiating again both sides w.r.t.x

$e^y y_2 + e^y y_1^2 = e^x$ ---- (3)

Differentiating again both sides w.r.t.x

$e^y y_3 + 3y_1 y_2 e^y + e^y y_1^3 = e^x$ ---- (4)

Differentiating again both sides w.r.t.x

$e^y y_4 + y_3 e^y y_1 + 3\left(y_1 y_2 e^y y_1 + e^y (y_1 y_3 + y_2^2)\right) + e^y 3y_1^2 y_2 + y_1^3 e^y y_1 = e^x$ ---- (5)

Put $x = 0$ in (1), (2), (3), (4) and (5)

(1) $\Rightarrow (y)_0 = \log 2$

(2) $\Rightarrow (y_1)_0 = \frac{1}{e^{log2}} = \frac{1}{2}$

(3) $\Rightarrow e^{log2} y_2 + e^{log2}\left(\frac{1}{2}\right)^2 = 1$

$$(y_2)_0 = \frac{1}{4}$$

(4) $\Rightarrow 2y_3 + 3\frac{1}{2}\frac{1}{4}2 + 2\frac{1}{8} = 1$

$$2y_3 + \frac{3}{4} + \frac{1}{4} = 1$$

$$(y_3)_0 = 0$$

(5) $\Rightarrow (y_4)_0 = -\frac{1}{8}$

$\therefore$ By Maclaurin's theorem

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots$$

$$\log(1 + e^x) = log2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \cdots$$

Can $f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ be expanded by Maclaurin's theorem?

Let $f(x) = e^{-\frac{1}{x^2}}, x \neq 0$

Let us look into the differentiability of the function at $x = 0$

$$\lim_{h\to 0^+} f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}}{h}$$

Substituting $\frac{1}{h} = \theta$, we can write

$$\lim_{h\to 0} f'(0) = \lim_{\theta\to\infty}\frac{e^{-\theta^2}}{1/\theta}$$

$$= \lim_{\theta\to\infty}\frac{\theta}{e^{\theta^2}}$$

$$= \lim_{\theta\to\infty}\frac{1}{2\theta e^{\theta^2}} = 0$$

$$\lim_{h\to 0^-} f'(0) = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}}{-h} = 0$$

$$\therefore f'(0) = 0$$

Also, $f'(x) = \frac{2}{x^2}e^{-\frac{1}{x^2}}, x \neq 0$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{2}{x^3}e^{-\frac{1}{x^2}}$$

Substituting $\frac{1}{x} = t$

$$\lim_{x\to 0} f'(x) = \lim_{t\to\infty} \frac{2t^3}{e^{t^2}} = 0 = f'(0)$$

Therefore the function is continuous at $x = 0$.

If we find the higher derivative of $f(x)$for $x \neq 0$, we will get $e^{-\frac{1}{x^2}}$ multiplied by a polynomial in $\frac{1}{x}$.

Therefore, higher derivatives of $f(x)$will be zero at $x = 0$.

So, the function possesses continuous derivatives for every value of x.

By Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$$

$$\text{and so}e^{-\frac{1}{x^2}} = 0 + x.0 + \frac{x^2}{2!}.0 + \cdots \frac{x^{n-1}}{(n-1)!}.0 + R_n$$

i.e.$R_n = e^{-\frac{1}{x^2}}$

$R_n$does not approach to zero as n approaches to infinity.

Therefore, $f(x)$ can not be expanded by Maclaurin's theorem

Summary

In this unit, we learnt about the finite form of Taylor's and Maclaurin's theorem.

- The Taylor's theorem states that, if a function f is such that

(i) the $(n-1)^{th}$ derivative f^{n-1} is continuous in $[a, a+h]$,

(ii) the n^{th} derivative f^n exists in $(a, a+h)$ and

(iii) p is a given positive integer

Then there exists at least one $\theta \in (0,1)$ such that

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + \cdots + (x-a)^n\frac{f^n(a)}{n!} + \frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n(a+\theta(x-a))$$

- The Taylor's theorem with Lagrange's form of remainder is given as,

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^{n-1}\frac{f^{n-1}(a)}{(n-1)!} + \frac{h^n}{n!}f^n(a+\theta h)$$

- The Taylor's theorem with Cauchy's form of remainder is given as,

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + \cdots + h^{n-1}\frac{f^{n-1}(a)}{(n-1)!} + \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!!}f^n\,(a+\theta h)$$

- For the Maclaurin's theorem with Lagrange and Cauchy's form of remainder, substitute$a = 0$ and $h = x$ in the above expressions.

Keywords

Taylor's theorem, Maclaurin's theorem, Lagrange's form of remainder, Cauchy's form of remainder, Taylor's series, Maclaurin's series

Self Assessment

1. If a function f is derivable then which of the following is true?

A. f is defined

B. f is defined and is continuous in a neighborhood of a point c

C. f is defined and is uniformly continuous in a neighborhood of a point c

D. none of these

2. Expansion of function f(x) is?

A. $f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0)$

B. $1 + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0)$

C. $f(0) - \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) - \cdots + \frac{x^n}{n!}f^n(0)$

D. $f(1) + \frac{x}{1!}f'(1) + \frac{x^2}{2!}f''(1) + \cdots + \frac{x^n}{n!}f^n(1)$

3. The necessary condition for the Maclaurin expansion to be true for function f(x) is that
A. f(x) should be continuous
B. f(x) should be differentiable
C. f(x) should exist at every point
D. f(x) should be continuous and differentiable

4. The expansion of f(a+h) is
A. $f(a) - \frac{h}{1!}f'(\mathrm{a}) + \frac{h^2}{2!}f''(\mathrm{a}) - \cdots + \frac{(-h)^n}{n!}f^n(\mathrm{a})$
B. $hf(a) + \frac{h}{1!}f'(\mathrm{a}) + \frac{h^2}{2!}f''(\mathrm{a}) + \cdots + \frac{h^n}{n!}f^n(\mathrm{a})$
C. $f(h) + \frac{a}{1!}f'(\mathrm{h}) + \frac{a^2}{2!}f''(h) + \cdots + \frac{a^n}{n!}f^n(\mathrm{h})$
D. $f(a) + \frac{h}{1!}f'(\mathrm{a}) + \frac{h^2}{2!}f''(\mathrm{a}) + \cdots + \frac{h^n}{n!}f^n(\mathrm{a})$

5. The expansion of e^{sinx} is
A. $1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^4}{8} + \cdots$
B. $1 - \frac{x}{1} + \frac{x^2}{2} + \frac{x^4}{8} + \cdots$
C. $1 + \frac{x}{1} - \frac{x^2}{2} + \frac{x^4}{8} + \cdots$
D. $1 + \frac{x}{1} + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

6. The $(n+1)^{th}$ term in the generalized mean value theorem or the Taylor theorem for the function $f(a+h)$ is
A. $\frac{h^n(1+\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h)$
B. $\frac{h^n(1-\theta)^{n-p}}{(n+1)!p}f^n(a+\theta h)$
C. $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h)$
D. $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a-\theta h)$

7. In the expression $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h)$ which of the following is true for the θ value?
A. $\theta \in [0,1]$
B. $\theta \in (0,1)$
C. θ can take any value
D. $\theta > 0$

8. The expression $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h)$ is known as
A. Remainder term
B. Remainder after n terms
C. Remainder after n+1 terms
D. Remainder after n-1 terms

9. $\frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a+\theta h)$ is due to
A. Schlomilch
B. Lagrange
C. Schlomilch and Roche
D. Cauchy

10. $\frac{h^n}{n\,!}f^n(a+\theta h)$ is the remainder after n terms due to
A. Schlomilch
B. Lagrange
C. Schlomilch and Roche
D. Cauchy

11. The Taylor's theorem with Lagrange's form of remainder for a function $f(x)$ will fail for those values of x for which
I. $f(x)$ or any of its differential coefficients becomes infinite
II. $f(x)$ or any of its differential coefficients becomes discontinuous
III. The remainder term is non-zero as $n \to \infty$

A. Only I is true
B. Only II is true
C. II and III are true
D. I, II and III are true

12. The expression $\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$ is known as
A. Remainder term
B. Remainder after n terms
C. Remainder after n+1 terms
D. Remainder after n-1 terms

13. $\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$ is due to
A. Schlomilch
B. Lagrange
C. Schlomilch and Roche
D. Cauchy

14. The function $f(x) = \begin{cases} e^{\frac{1}{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ can be expanded by Maclaurin's theorem.
A. True
B. False

15. The function $f(x) = \sqrt{x}$ can be expanded by Maclaurin's theorem.
A. True
B. False

Answer for Self Assessment

1.	B	2.	A	3.	D	4.	D	5.	D
6.	C	7.	B	8.	B	9.	C	10.	B
11.	D	12.	B	13.	D	14.	B	15.	B

Review Questions

1. Expand $\cos x$ by Maclaurin's series.
2. Expand $\log(1+x)$ by Maclaurin's theorem.
3. Expand $\log(x+a)$in the powers of x by Taylor's theorem.
4. Expand $\log\sin x$ in powers of $(x-2)$.
5. Expand $\sin^{-1}(x+h)$ in powers of x till the power of x^3.
6. Expand $\tan^{-1} x$ in the powers of $\left(x-\frac{\pi}{4}\right)$.
7. Differentiate in the Taylor's theorem with the Lagrange's and Cauchy's form of remainder.
8. Differentiate in the Maclaurin's theorem with the Lagrange's and Cauchy's form of remainder.
9. By Maclaurin's theorem, find first three non vanishing terms in the expansion of $\frac{e^x}{1+e^x}$.
10. Expand $e^x \cos x$ in the form of a power series.

Further Reading

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Unit 11: Maxima and Minima of a Function

CONTENTS

Objectives

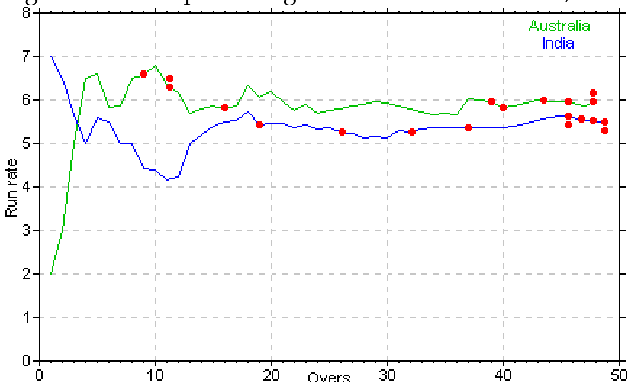
Students will be able to

- derive necessary and sufficient condition for extreme values
- apply the first derivative test to find maxima and minima
- apply the second derivative test to calculate the maximum and minimum value of a function

Introduction

In this unit, we will see one very interesting application of calculus, and it is called the maxima and minima of a function. When one says, Mount Everest or Mariana Trench, what comes to your mind? You think of a high point on the surface of Earth, and a low point on the surface of Earth. So if you can draw the Earth's topography, the highest point will refer to a place which is a mountain, and that gives you the idea of the maximum height for any object on the earth. Similarly, the minimum height or you can say the maximum depth is at the Mariana Trench, so these ideas of maximum and minimum are inherently there in our daily lives. We can see one more example to understand the topic better.

The adjacent graph is about a cricket match between Australia and India. The blue one is representing the run rate of India and the green one is representing the run rate of Australia, with respect to the overs. By mathematical modeling, we can write the run rate in terms of overs, i.e. a function can be framed or we can define some formula in such a way that run rate, say 'y', can be written in terms of 'x' where x is the overs.



Here the graph is available, on the basis of the data of the actual match. In the first over the run rate for the Australian team was two only. And in the first over the run rate for Indian team was seven. In the very first over, India's run rate was maximum and then it fell down, and then it rose up at a particular over (which one?), and then it rose then fell down then even more down then it was constant for some time, and you can see the minimum run rate was at 11th over. Just by looking at the graph it is quite clear, that for Australia, the minimum run rate was at the second over, and the maximum was around at 10th over. So this is how, if we have a function we can draw it and then from there, by just looking at the graph, we can tell about the maximum and the minimum value of the function. But there should be a mathematical technique to deal with it, without plotting the graphs! In this unit, we will look into how one can find the maximum and minimum value of a function of one variable.

A high point is called a maximum (plural *maxima*).A low point is called a minimum (plural *minima*).The general word for maximum or minimum is extremum (plural extreme).We say local maximum (or minimum) when there may be higher (or lower) points elsewhere but not nearby.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well.

11.1 Absolute and Local Maximum / Minimum

A function f has an absolute maximum (also called global maximum) at c if $f(c) \geq f(x)$ for all x in its domain, $\boldsymbol{D}$. The value $f(c)$ is called the maximum value of f. A function f has an absolute minimum (or global minimum) at c if $f(c) \leq f(x)$ for all x in its domain. Such a value $f(c)$ is called the minimum value of f. The maximum and minimum values of f are called the extreme values of f.

Whereas a function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x near c. That is, $f(c) \geq f(x)$ for all x on some open interval containing c. Similarly, f has a local minimum (or relative minimum) at c if $f(c) \leq f(x)$ when x near c.

Before proceeding, let's note two important issues regarding this definition. First, the term *absolute* here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function $f$has an absolute extremum over an interval I at $c$the absolute extremum is$f(c)$. The real number $c$is a point in the domain at which the absolute extremum occurs.

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function $f(x)$over a closed, bounded interval $[a, b]$has both an absolute maximum and an absolute minimum.

The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then there exist (at least) a point c where f attains its maximum value, $f(c)$, on the interval, and (at least) a point d where f attains its minimum value, $f(d)$, on the interval.

This means that if both of the following conditions: (1) the interval is closed, and (2)f is continuous on it, are met, then f is guaranteed to have (at least) one absolute maximum and one absolute minimum points on the interval. If either condition fails, then the existence of max / min points is not guaranteed.

11.2 A necessary Condition for Extreme Values

A necessary condition for $f(c)$ to be an extreme value of f is that $f'(c) = 0$.

Let $f(c)$ be a maximum value of f. Then there exists an open interval $(c-\delta, c+\delta)$around c, such that if $c+h$ is a number other than c in $(c-\delta, c+\delta)$, we have

$$f(c+h) < f(c)$$

Here h may be positive or negative. Thus

$$h > 0 \Rightarrow \frac{f(c+h)-f(c)}{h} < 0$$

$$h < 0 \Rightarrow \frac{f(c+h)-f(c)}{h} > 0$$

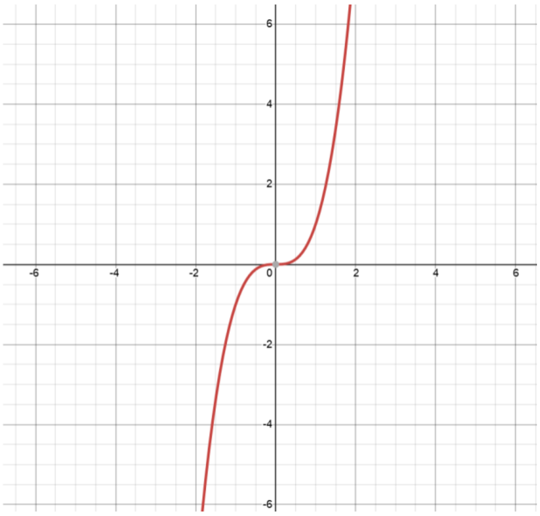
which implies that

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}\leq 0 \text{ and } \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\geq 0$$

which will be true simultaneously if and only if $f'(c)=0$.

Similarly the result holds if $f(c)$a minimum value of is f.

Note that, $f'(c)=0$ is not the sufficient condition for $f(c)$ to be an extreme value it can be explained with an example.



Consider $f(x)=x^3$ for $x=0$.

$$f'(x)=3x^2$$

$$f'(0)=0$$

Now $x>0\ \Rightarrow\ f(x)>0=f(0)$

$$x<0\ \Rightarrow\ f(x)<0=f(0)$$

Therefore it can be seen that $f(0)$ is not an extreme value even though $f'(0)=0$.

Prove that the function f defined by $f(x)=3|x|+4|x-1|\quad \forall\, x\in R$ has a minimum value 3 at $x=1$.

We can rewrite the function by using the definition of modulus function as follows:

$$f(x)=\begin{cases}4-7x, x<0\\ 4, x=0\\ 4-x, 0<x<1\\ 3, x=1\\ 7x-4, x>1\end{cases}$$

Clearly it can be seen that at $x=1$, the function has a maximum value equal to 3.

A function is said to be stationary for c and $f(c)$ a stationary value of $f$if $f'(c)=0$. The rate of change of a function is zero at the stationary point.

Find the greatest and least value of the function $f(x)=3x^4-2x^3-6x^2+6x+1$ in [0,2].

$$f'(x)=12x^3-6x^2-12x+6$$

$$=6(x-1)(x+1)(2x-1)$$

Now by the necessary condition for an extreme value, $f'(x)$ must be zero. And this gives $x=1,-1,\frac{1}{2}$. Since -1 is not in the domain of the function, this value can be ignored. The other two numbers are the candidates to be the point of maximum or minimum value of the function.

$$f(1)=2$$

$$f\left(\frac{1}{2}\right)=\frac{39}{16}=2.43$$

Moreover we must check the value of the function at the end points of the domain also.

$$f(0)=1$$

$$f(2)=21$$

Thus the function has its maximum value at $x=2$ and minimum value at $x=0$.

11.3 Sufficient Condition for Extreme Value

$f(c)$is an extreme value of f if and only if $f'(x)$ changes sign as x passes through c.

Fermat's Theorem (test for local extreme values): If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Note: Therefore, it follows that a local extreme point can only occur at places where either $f'(x) = 0$ or $f'(x)$ is undefined (i.e., either at a point where the tangent line is horizontal, or at a non-differentiable point). Examples: $f(x) = (x - 2)^2$, at $x = 2$; $g(x) = |x|$, at $x = 0$.

Note: The converse is not always true: the fact that $f'(c) = 0$, or that $f'(c)$ does not exist, does NOT guarantee that c is a local extreme point of f. Example: $f(x) = x^3$, at $x = 0$.

A **critical point** or **critical number** of a function f is a point $x = c$ in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Note: The critical points are all the *candidate* points for local maximum / minimum of f. That is, every local extreme point is a critical point, but not every critical point is a local extreme point. Naturally, the maximum/ minimum points of f have to be in the domain of f, i.e. they are points on the graph of f. Therefore, for example, if f is undefined at an infinite discontinuity then the point of discontinuity is not a critical point even though f' does not exist there.

Steps to find the absolute maximum and minimum values of a continuous function f on a closed interval:

1. Find all critical points of f in the given interval.
2. Evaluate f at the critical point(s) found in step 1, as well as at the two endpoints of the interval.
3. The point(s) of the largest value of f is the absolute maximum(s), the point(s) of the smallest value is the absolute minimum(s).

Let us understand this by an example.

Examine the polynomial $f(x) = 10x^6 - 24x^5 + 15x^4 - 40x^3 + 108$ for maximum and minimum value

Here $f'(x) = 60x^2(x^2 + 1)(x - 2)$

For maximum and minimum value $f'(x) = 0$ implies that $x = 0, 2$ are the only real values.

Now $x < 0 \Rightarrow f'(x) < 0$

$0 < x < 2 \Rightarrow f'(x) < 0$and

$$x > 2 \Rightarrow f'(x) > 0$$

Therefore $f'(x)$ does not change sign as x passes through 0, so that $f(0)$ is neither a maximum nor a minimum value and $f'(x)$ changes the sign from negative to positive as x passes through 2.

$\therefore f(2) = -100$is the minimum value.

$\therefore f(x)$has only one extreme value i.e. at 2.

Find all local maximum and minimum points for the function

$$f(x) = x^3 - x$$

The derivative is$f'(x) = 3x^2 - 1$.

This is defined everywhere and is zero at $x = \pm\frac{1}{\sqrt{3}}$.

Looking first at $x = \frac{1}{\sqrt{3}}$. we see that $f\left(\frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$.

Now we test two points on either side of $x = \frac{1}{\sqrt{3}}$, making sure that neither is farther away than the nearest critical value; since $\sqrt{3} < 3, \frac{1}{\sqrt{3}} < 1$ and we can use $x = 0$ and $x = 1$.

Since $f(0) = 0 > -\frac{2\sqrt{3}}{9}andf(1) = 0 > -\frac{2\sqrt{3}}{9}$, there must be a local minimum at$x = \frac{1}{\sqrt{3}}$.

For$x = -\frac{1}{\sqrt{3}}$., we see that$\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$.

This time we can use $x = 0$and $x = -1$.

We find that$(-1) = f(0) = 0 < \frac{2\sqrt{3}}{9}$, so there must be a local maximum at $x = -\frac{1}{\sqrt{3}}$.

This example is made very simple by our choice of points to test, for other choice of points the calculations would have been comparatively lengthy.

11.4 Second Order Derivative Test

Theorem: $f(c)$ is a minimum value of the function f if $f'(c) = 0$ and$f''(c) > 0$.

Proof:$f''(c) > 0 \Rightarrow \exists$ an open interval$(c - \delta, c + \delta)$ around c for every point x of which, the second derivative is positive.

$\Rightarrow f'(x)$is strictly increasing in $(c - \delta, c + \delta)$.

Also, $f'(c) = 0$

$\therefore f'(x) < 0 \ \forall \ x \in [c - \delta, c)$ (strictly decreasing function)

and $f'(x) > 0 \ \forall \ x \in [c, c + \delta)$ (strictly increasing function)

$\Rightarrow f \quad (c)$is a minimum value of $f(x)$.

Theorem: $f(c)$ is a maximum value of the function f if $f'(c) = 0$ and$f''(c) < 0$.

Proof: $f''(c) < 0 \Rightarrow \exists$ an open interval$(c - \delta, c + \delta)$ around c for every point x of which, the second derivative is negative.

$\Rightarrow f'(x)$is strictly decreasing in $(c - \delta, c + \delta)$.

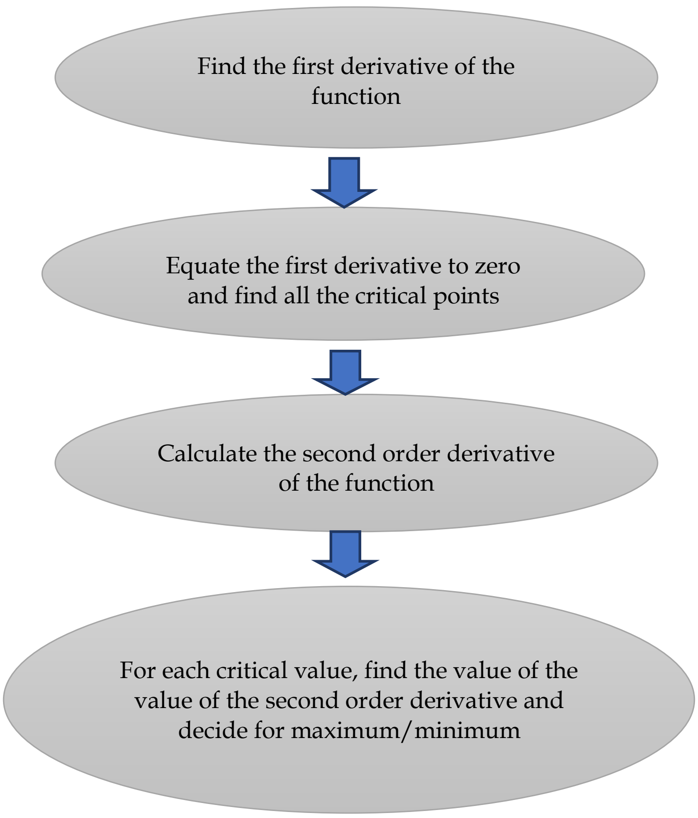
Also, $f'(c) = 0$

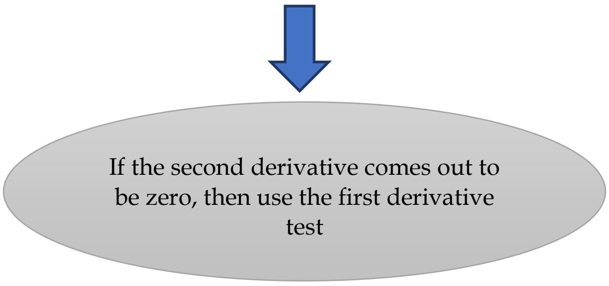
$\therefore f'(x) > 0 \ \forall \ x \in [c - \delta, c)$(strictlyincreasing function)

and $f'(x) < 0 \ \forall \ x \in [c, c + \delta)$ (strictly decreasing function)

$\Rightarrow f \quad (c)$is a maximum value of $f(x)$.

So, from now onwards we can follow the following steps to find the maximum / minimum of a function:





Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

Let $y = \left(\frac{1}{x}\right)^x$

$$\log y = -x \, logx$$

Differentiating both sides w.r.t.x.

$$\frac{dy}{dx} = -(1 + \log x)\left(\frac{1}{x}\right)^x$$

Applying the necessary condition for extreme values, we get

$$\log x = -1$$

$$x = e^{-1}$$

Now by the sufficient condition, we can check if the point $x = e^{-1}$ is the point of maximum or a point of minimum or neither of them.

At $= e^{-1}, \frac{d^2y}{dx^2} = -e.e^{\frac{1}{e}} < 0$

Therefore y has a maximum for $x = e^{-1}$ and the maximum value is $e^{\frac{1}{e}}$.

Find the maximum and minimum value of the function$f(x) = 8x^5 - 15x^4 + 10x^2$.

The given function is $f(x) = 8x^5 - 15x^4 + 10x^2$

$$f'(x) = 40x^4 - 60x^3 + 20x$$

$$= 20x(2x^3 - 3x^2 + 1)$$

$$= 20x(x-1)^2(2x+1)$$

Putting $f'(x) = 0$ for the critical points, we get

$$x = 0, 1, -\frac{1}{2}$$

Now these three points are the candidates to be the point of maximum or minimum or neither of them. Let's check with the help of the second derivative test.

We have

$$f''(x) = 160x^3 - 180x^2 + 20$$

$f''(x)(at\ x = 0) = 20 > 0 \Rightarrow x = 0$ is a point of minimum

$f''(x)(at\ x = 1) = 0 \Rightarrow x = 1$ is neither a point of minimum nor of maximumbecause$f'(x)$ does not change sign as x passes through 1.

$f''(x)\left(at\ x = -\frac{1}{2}\right) = -45 < 0 \Rightarrow x = -\frac{1}{2}$is a point of maximum.

Find the absolute maximum and minimum points of $f(x) = 4 - x^2$ on each of the intervals (i) $[-3, 1]$ and (ii) $[2, 5]$.

$f'(x) = -2x$

$f' = 0$ at $x = 0$ which is the only critical point because f is a polynomial, therefore, it has no non differentiable points.

(i) Evaluate f at the critical point 0 and the endpoints −3 and 1:

$f(-3) = -5, \quad f(0) = 4, \quad f(1) = 3$

Therefore, the absolute maximum point is (0, 4), and the absolute minimum point is (−3, −5).

(ii) The critical point $x = 0$ is not in this interval, therefore, just evaluate f at the endpoints 2 and 5:

$f(2) = 0, \quad f(5) = -21$

Therefore, the absolute maximum point is (2, 0), and the absolute minimum point is (5, −21).

Summary

In this unit we have seen how to calculate the maximum and minimum of a function if they exist.

- A function f has an absolute maximum (also called global maximum) at c if $f(c) \geq f(x)$ for all x in its domain.
- A function f has an absolute minimum (or global minimum) at c if $f(c) \leq f(x)$ for all x in its domain.
- The maximum and minimum values of f are called the extreme values of f.
- If f is continuous on a closed interval $[a, b]$, then there exist (at least) a point c where f attains its maximum value, $f(c)$, on the interval, and (at least) a point d where f attains its minimum value, $f(d)$, on the interval.
- $f(c)$ is an extreme value of f if and only if $f'(x)$ changes sign as x passes through c.
- $f(c)$is a minimum value of the function f if $f'(c) = 0$ and$f''(c) > 0$.
- $f(c)$is a maximum value of the function f if $f'(c) = 0$ and$f''(c) < 0$.

Key Words

Maxima, Minima, maximum of a function, minimum of a function, First derivative test, critical points, stationary points, second derivative test, Extreme value theorem, Fermat's theorem

Self Assessment

1. What is the saddle point?

A. Point where function has maximum value

B. Point where function has minimum value

C. Point where function has zero value

D. Point where function neither has maximum value nor minimum value

2. Which of the following is correct?

A. f(a) is an extreme value of f(x) if f '(a)=0

B. If f(a) is an extreme value of f(x), then f '(a) = 0

C. If f '(a) = 0, then f (a) is an extreme value of f(x)

D. All of these

3. The maxima and minima of the function $f(x) = 2x^3 - 15\,x^2 + 36x + 10$ occur respectively at

A. x= 3 and x = 2
B. x= 3 and x = 2
C. x = 2 and x = 3
D. x = 3 and x = 4

4. Find the maximum and minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on the interval [0, 5].

A. maximum of f is 21 and the minimum is 1
B. maximum of f is 1 and the minimum is -21
C. maximum of f is 20 and the minimum is 1
D. maximum of f is 2 and the minimum is -11

5. A necessary condition for $f(c)$ to be an extreme value of f is that

A. $f'(c) \neq 0$
B. $f(c) = 0$
C. $f'(c) = 0$
D. $f''(c) = 0$

6. The maximum value of $sinx + \cos x$ is

A. 2
B. $\sqrt{2}$
C. 1
D. $1 + \sqrt{2}$

7. The maximum value of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 1$ is

A. 3/7
B. 7/3
C. 1
D. 7

8. $f(c)$is a minimum value of the function if

A. $f'(c) = 0\; and f''(c) > 0$
B. $f'(c) = 0\; and f''(c) < 0$
C. $f'(c) = 0\; and f''(c) = 0$
D. $f''(c) > 0$

9. $f(c)$is a maximum value of the function if

A. $f'(c) = 0\; and f''(c) > 0$
B. $f'(c) = 0\; and f''(c) < 0$
C. $f'(c) = 0\; and f''(c) = 0$
D. $f''(c) < 0$

10. The maximum value of $\left(\frac{1}{x}\right)^x$ is

A. e

B. $e^{\frac{1}{e}}$

C. $\left(\frac{1}{e}\right)^{e}$

D. none of these

11. The saddle points for the function $10x^6 - 24x^5 + 15x^4 - 40x^3 + 108$ are

A. 0, 1, 2
B. -1, 0, 2
C. 1, 2
D. 0, 2

12. The saddle points of the function $8x^5 - 15\,x^4 + 10\,x^2$ are given as

A. -1/2, 0
B. 0, 1, ½
C. -1/2, 0, 1
D. none of these

13. For the function $8x^5 - 15\,x^4 + 10\,x^2, x = 0$ is

A. a point of minimum
B. a point of maximum
C. point of inflexion
D. none of these

14. For the function$8x^5 - 15\,x^4 + 10\,x^2, x = -\frac{1}{2}$ is

A. a point of minimum
B. a point of maximum
C. point of inflexion
D. none of these

15. For the function$8x^5 - 15\,x^4 + 10\,x^2, x = 1$ is

A. a point of minimum
B. a point of maximum
C. point of inflexion
D. neither a point of maximum nor a point of minimum

Answer for Self Assessment

1	D	2.	D	3.	C	4.	A	5.	C
6.	B	7.	B	8.	A	9.	B	10.	B
11.	D	12.	C	13.	C	14.	B	15.	D

Review Questions

1. Find the absolute maximum and minimum values of $f(x) = x^3 - 27x + 8$ on the interval [0, 4].
2. Find the absolute maximum and minimum values of $g(t) = t^{3/5}$ on the interval $[-32, 1]$.
3. Find the maximum and minimum values of $f(x) = x^3 - 27x + 8$.
4. Find the maximum and minimum values of $g(t) = t^{3/5}$
5. Find the greatest and least value of the function $x^4 - 4x^3 - 2x^2 + 12x + 1$ in the interval$[-2, 5]$.
6. Find the greatest and least values of the function $2x^3 - 15x^2 + 36x + 1$ in the interval $[2, 3]$ as well as in the interval$[0, 4]$.
7. Show that the function $f(x) = (x+2)(x-1)^2(2x-1)(x-3)$ changes sign from positive to negative as x passes through $\frac{1}{2}$ and from negative to positive as x passes through -2 or 3. Also show that it does not change sign as x passes through 1.
8. Show that $x^5 - 5x^4 + 5x^2 - 1$ has a maximum value when $x = 1$, a minimum value when $x = 3$ and neither when $x = 0$.
9. Show that the function$f$ defined by
$$f(x) = x^p(1-x)^q \ \ \forall\, x \in \boldsymbol{R}$$
wherep, q are positive integers, has a maximum value for
$$x = \frac{p}{p+q} \ \forall\, p, q$$
10. Find the extreme value of the expression:
$$\frac{x^3}{(x^4+1)}$$
11. Determine the value of x for which
$$\frac{x}{1 + xtanx}$$
has a maximum value.
12. Find the maximum and minimum value of$sinx\ cos2x$.
13. Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.
14. What can be said for a quadratic polynomial with respect to the critical points?

Further Reading

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Unit 12: Curvature and Asymptotes

CONTENTS

Objectives

Students will be able to

- calculate the curvature for different types of curves
- distinguish between various kinds of asymptotes
- find the asymptotes of a general algebraic curve
- find the parallel and oblique asymptotes

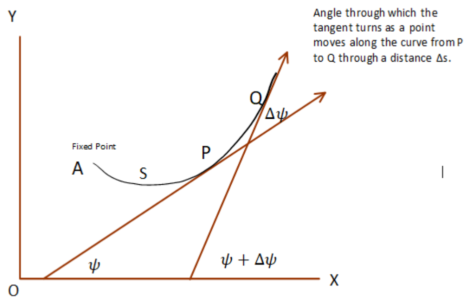
Introduction

This unit is about two important applications of derivatives namely curvature and the asymptotes. Consider that you are having a road trip in a hilly region. Imagine the roads. You have the technique to measure the distance between any two points on a straight line. But how to measure the bend happening at a particular point needs some elaboration on curvature! Similarly another crucial feature of differential calculus is the concept of asymptotes. Basically it provides a frame for any curve

(if asymptotes exist). Both these concepts are very helpful to trace the curves.

12.1 Curvature

Curvature is the numerical measure of bending of a curve. At a particular point on the curve, a tangent can be drawn. Let this line makes an angle ψ with positive x- axis. Then curvature is defined as the magnitude of rate of change of $\psi$with respect to the arc length s.



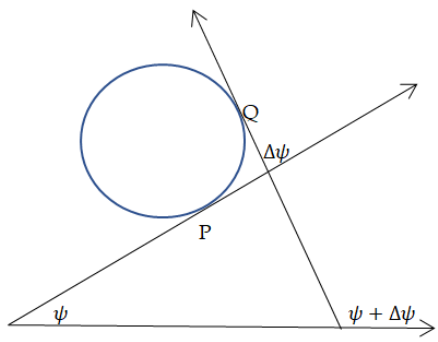
The total bending or total curvature is ArcPQ or Angle $\Delta\psi$

The average curvature is $\frac{\Delta\psi}{\Delta s}$

The curvature of the curve at P is $\lim_{Q\to P}\frac{\Delta\psi}{\Delta s}=\frac{d\psi}{ds}$

It is quite intuitive that the smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature and larger the circle, smaller will be its curvature.

Let us consider a circle with center O and radius r. Let the arc$PQ=\Delta s$



Angle $POQ=\frac{\text{Arc }PQ}{OP}$

$$\frac{\Delta\psi}{\Delta s}=\frac{1}{r}$$

$$\lim_{Q\to P}\frac{\Delta\psi}{\Delta s}=\frac{d\psi}{ds}=\frac{1}{r}$$

Therefore, curvature at any point of a circle is the reciprocalof the radius, and hence is a constant.

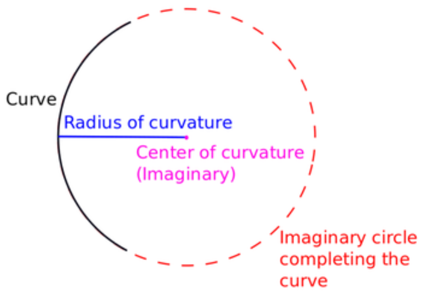
12.2 Radius of Curvature

The reciprocal of the curvature of a curve at any point in case it is non zero, is called its radius of curvature at that point. It is denoted generally by Greek alphabetρ(rho).

$$\rho=\frac{ds}{d\psi}$$

Note that the radius refers to the distance between the center of a circle and any other point on the circumference of the circle. While the radius of curvature is the radius of the circle that touches the curve at a given point. Also, it has the same tangent and curvature at that point.

The radius is of a real figure or shape whereas the radius of curvature is of an imaginary circle at a point on a given curve.



Find the radius of curvature at any point for the curve $s = c\tan\psi$

We have

$$\rho = \frac{ds}{d\psi}$$

$$\therefore \rho = c\sec^2\psi$$

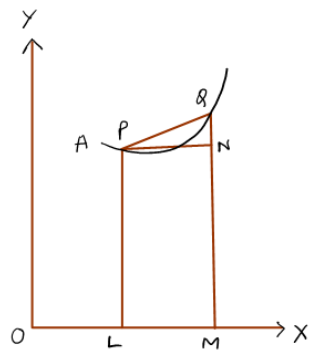
Find the radius of curvature at any point for the curve $s = c\log\sec\psi$

We have

$$\rho = \frac{ds}{d\psi}$$

$$\therefore \rho = c\,tan\,\psi$$

12.3 Length of arc as a Function



Let $y = f(x)$ be the equation of the curve. $P(x, y)$is any point on the curve such that the arc length $AP = s$. The point $Q(x + \Delta x, y + \Delta y)$ is a point near point P on the curve.

Then arc $AQ = s + \Delta s$

So arc $PQ = \Delta\, s$

In triangle PQN, $PQ^2 = (\Delta x)^2 + (\Delta y)^2$

$$\left(\frac{PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

On the left side introducing the arc PQ in numerator and the denominator, we get

$$\left(\frac{\mathrm{chord}PQ}{\mathrm{arc}PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

As $Q \to P$, the chord PQ and arc PQ become almost same making the above expression as

$$\left(\frac{\mathrm{ds}}{\mathrm{dx}}\right)^2 = 1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2$$

$$\frac{\mathrm{ds}}{\mathrm{dx}} = \sqrt{1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2}$$

Corollary: 1. $\frac{dx}{ds} = \cos\psi, \frac{dy}{ds} = sin\psi$

2. For parametric equations with parameter t, $x = x(t), y = y(t)$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$

12.4 Radius of Curvature- Cartesian Equations

Consider a curve $y = f(x)$

We have $\tan\psi = \frac{dy}{dx}$

Differentiating w.r.t. s on both sides

$$\sec^2\psi\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds}$$

$$\frac{ds}{d\psi} = \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

If $y_2 > 0, \rho > 0$ at a point, then the curve will be concave upward at that point.

If $y_2 < 0, \rho < 0$ at a point, then the curve will be concave downward at that point.

$\because$ $\rho$is independent of the choice of x-axis and y-axis $\therefore$ ρ can also be given as

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

Show that the curvature of the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the Folium $x^3 + y^3 = 3axy$is $-\frac{8\sqrt{2}}{3a}$.

Differentiation the equation of folium w.r.t. x, we get

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

$$\left(\frac{dy}{dx}\right)_{\frac{3a}{2},\frac{3a}{2}} = -1$$

And

$$\left(\frac{d^2y}{dx^2}\right)_{\frac{3a}{2},\frac{3a}{2}} = -\frac{32}{3a}$$

Therefore the curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is

$$\frac{1}{\rho} = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

$$= -\frac{8\sqrt{2}}{3a}$$

12.5 Radius of Curvature- Parametric Equations

For a curve given by $x = f(t), y = g(t)$, For $f'(t) \neq 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{f'(t)}\left(\frac{f'(t)g''(t) - g'(t)f''(t)}{f'(t)^2}\right)$$

$$\therefore \rho = \frac{(f'^2(t) + g'^2(t))^{\frac{3}{2}}}{f'(t)g''(t) - g'(t)f''(t)}$$

For the cycloid $x = a(t + sint), y = a(1 - \cos t)$, prove that $\rho = 4\,\text{a}\cos\frac{t}{2}$.

$$\frac{dx}{dt} = a(1 + \cos t)$$

$$\frac{dy}{dt} = a\sin t$$

$$\frac{dy}{dx} = \tan\frac{t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\frac{t}{2}\frac{dt}{dx}$$

$$= \frac{1}{4a}\frac{1}{\cos^4\frac{t}{2}}$$

The radius of curvature $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} = 4a\ \cos\frac{t}{2}$.

12.6 Radius of Curvature- Polar Equations

Let $r = f(\theta)$ be the given curve.

Let $x = rcos\theta, y = rsin\theta$ be the transformations.

$\frac{dx}{d\theta} = r_1 cos\theta - r\sin\theta, \frac{dy}{d\theta} = r_1\sin\theta + r\cos\theta$

$$\therefore y_1 = \frac{dy}{dx} = \frac{r_1\sin\theta + r\cos\theta}{r_1 cos\theta - r\sin\theta}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{(r_1 cos\theta - rsin\theta)^3}$$

where$r_1 = f'(\theta), r_2 = f''(\theta)$

Thus the radius of curvature can be written as

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - r\,r_2}$$

For the curve $r^m = a^m\cos m\theta$, prove that $\rho = \frac{a^m}{(m+1)r^{m-1}}$.

Taking logarithm on both sides of the given equation and then differentiating w.r.t. θ, we get

$$\frac{m}{r}\frac{dr}{d\theta} = -m\frac{sin\ m\theta}{cosm\theta}$$

$$r_1 = -r\tan m\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = -rm\ sec^2 m\theta + r\tan^2 m\theta$$

Using

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - r\,r_2}$$

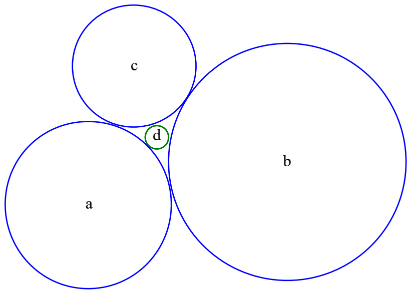
We get

$$\rho = (r^3\sec^3 m\theta)\,/(r^2\sec^2 m\theta + mr^2\sec^2 m\theta)$$

$$\rho = \frac{1}{m+1}\frac{a^m}{r^{m-1}}$$

Hence the result!

In 1643, French mathematician Rene Descartes developed a formula relating the curvatures of four circles that all touch, or are tangent, to each other.



Descartes' Circle Equation Theorem:

Given four mutually tangent circles with curvatures *a*, *b*, *c*, and *d*, then

$$(a^2 + b^2 + c^2 + d^2) = (1/2)\,(a + b + c + d)^2$$

12.7 Asymptotes of a General Algebraic Curve

The name 'asymptote' originated from Greek word asymptotes which means 'not meeting'.An asymptote of a curve is a straight line such that the distance between the curve and the line approaches to zero as one or both of the x or y coordinates tend to infinity. Simply put,an **asymptote** is a line that a graph approaches without touching.

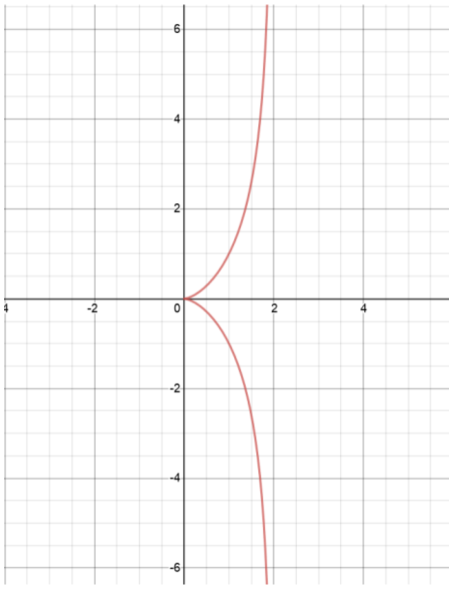
In some case a curve may have a branch or branches extending beyond the finite region. In this case let P be a point on such a branch of the curve, having its coordinates (x,y) and if P moves along the curve, so that at least one of x and y tend to + ∞ or to -∞, then P is said to tend to infinity.

A straight line is said to be an asymptote of a curve y = f (x), if the perpendicular distance of the point P (x,y) on the curve from the line tends to 0 when x or y or both tend to infinity.

An asymptote parallel to y-axis may be referred as a vertical asymptote and parallel to x-axis as a horizontal asymptote. An asymptote which is not parallel to either axis may be described as an oblique asymptote and isgiven by y = mx + c. Only open curves which have some infinite branch can have an asymptote. No closed curve can have an asymptote.The curve and its asymptote get infinitely close, but they never meet.

Their major applications involve their usage in big O notation, they are simple approximations to complex equations, and they are useful for graphing rational equations. In most cases, the asymptote(s) of a curve can be found by taking the limit of a value where the function is not defined.

For example, a cissoids given by the equation $y^2(2 - x) = x^3$, can be drawn as given below. It is clear that the infinite branches of the curve seem to meet the straight line $x = 2$.

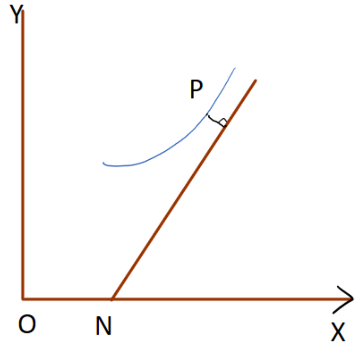


12.8 Determination of Asymptotes

The general equation of a straight line is

$$y = mx + c$$

Consider a point P on the curve and let M be the foot of the perpendicular from point P on the straight line that can be an asymptote. As $P(x, y) \to \infty, x \to \infty$



The equation $y = mx + c$ can be an asymptote of given curve if $p = PM$ and $PM \perp NM$. Then

$$p = \frac{|y - mx - c|}{\sqrt{1 + m^2}}$$

Now $p \to 0$ as $x \to \infty$

$$\therefore \lim_{x\to\infty} y - mx - c = 0$$

$$\boldsymbol{\lim_{x\to\infty} y - mx = c}$$

Also $\frac{y}{x} - m = (y - mx)\frac{1}{x}$

$$\lim_{x\to\infty}\left(\frac{y}{x} - m\right) = \lim_{x\to\infty}(y - mx).\lim_{x\to\infty}\frac{1}{x} = 0$$

This implies

$$\boldsymbol{\lim_{x\to\infty}\frac{y}{x} = m}$$

Thus knowing m and c, we can write $y = mx + c$ as the equation of asymptote .

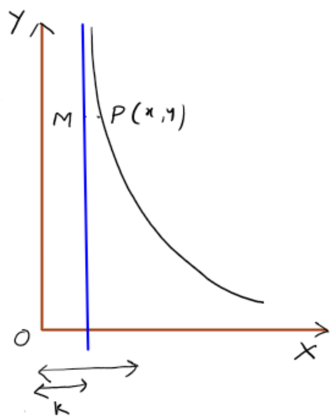
Examine the folium for asymptotes.

The folium is given by the equation $x^3 + y^3 - 3axy = 0$ ---- (1)

or$1 + \left(\frac{y}{x}\right)^3 - 3a\frac{y}{x}\frac{1}{x} = 0$

---- (2)

We will calculate the slope and the intercept in the general equation of a straight line under the definition of asymptote as derived in the section above.



Let $x \to \infty$ then $\boldsymbol{\lim_{x\to\infty}\frac{y}{x} = m}$

From equation (2) we can write $1 + m^3 = 0$

$m = -1$ is the only real root. For this value of m, we can find the associated c.

$$c = \lim_{x\to\infty}(y + x)$$

Put $y + x = p$

As $x \to \infty, p \to c$

Put $y = p - x$ in (1)

$$x^3 + (p - x)^3 - 3ax(p - x) = 0$$

$$3(p + a) - 3p(p + a)\frac{1}{x} + \frac{p^3}{x^2} = 0$$

As $x \to \infty, p \to c$

$$3(c + a) = 0$$

$$c = -a$$

$\therefore$ The equation of the asymptote is $y = -x - a$.

12.9 Asymptotes Parallel to the Coordinate Axis

For asymptote parallel to the y axis:

Let $x = k$ ----- (1)

be the asymptote of the curve. As $P(x, y) \to \infty$ along the curve $y \to \infty$ and $PM = x - k$

$$\lim_{y\to\infty} x - k = 0$$

$\lim_{y\to\infty} x = k$ which gives k.

For a general formula we arrange the given curve in descending powers of y i.e.

$y^m\phi(x) + y^{m-1}\phi_1(x) + y^{m-2}\phi_2(x) + \cdots = 0$ ---- (2)

where $\phi(x),\ \phi_1(x),\ \phi_2(x)$ ... are the polynomials in x.

Dividing (2) by y^m, we get

$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \cdots = 0$ ---- (3)

Let $y \to \infty$

then $\lim_{y\to\infty} x = k$

$$\phi(k) = 0$$

Therefore k is the root of equation $\phi(x) = 0$. Let k_1, k_2 etc. be the roots of $\phi(x) = 0$, then the asymptote parallel to y-axis are $x = k_1, x = k_2$ etc.

$\Rightarrow (x - k_1)(x - k_2)$etc. are the factors of $\phi(x)$ which is the coefficient of highest power y^m of y in the given equation.

Similarly the derivation can be done for the asymptotes parallel to the x axis and can be summarized as the following rules:

Rule 1: The asymptotes parallel to Y axis are obtained by equating to zero, the real linear factors in the coefficient of highest power of y, in the equation of the curve.

Rule 2: The asymptotes parallel to X axis are obtained by equating to zero, the real linear factors in the coefficient of highest power of x, in the equation of the curve.

Find the asymptote parallel to the coordinate axes of the curve

$$(x^2 + y^2)x - ay^2 = 0$$

The equation can be re written as

$$x^3 + y^2(x - a) = 0$$

Coefficient of highest power of x is 1 and that cannot be equated to zero. Therefore the asymptote parallel to the x-axis does not exist.

Coefficient of highest power of y is $x - a$ and equating it to zero gives $x - a = 0$.Therefore the asymptote parallel to the y-axis is $x = a$.

12.10 Oblique Asymptotes

Asymptotes of the general rational algebraic equation

Consider the equation

$U_n + U_{n-1} + U_{n-2} + \cdots + U_2 + U_1 + U_0 = 0$ ---- (1)

where U_r is a homogeneous expression of degree r in x, y and $U_r = x^r\phi_r\left(\frac{y}{x}\right)$ where $\phi_r\left(\frac{y}{x}\right)$ is a polynomial in $\frac{y}{x}$ of degree r, at the most.

Therefore (1) can be written as

$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0$ ---- (2)

Dividing by x^n

$$\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots + \frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right) + \frac{1}{x^n}\ \phi_0\left(\frac{y}{x}\right) = 0$$

and taking the limit $x \to \infty$, we get

$\phi_n(m) = 0$ ---- (3)

which determines the slope of the asymptote.

Let m_1 be one of the roots of this equation so that $\phi_n(m_1) = 0$

We can write $y - m_1 x = p_1$

i.e. $\frac{y}{x} = m_1 + \frac{p_1}{x}$

Substituting this in (2), we get

$$x^n\phi_n\left(m_1 + \frac{p_1}{x}\right) + x^{n-1}\phi_{n-1}\left(m_1 + \frac{p_1}{x}\right) + \cdots + x\phi_1\left(m_1 + \frac{p_1}{x}\right) + \phi_0\left(m_1 + \frac{p_1}{x}\right) = 0$$

Expanding each term by Taylor's theorem and re arranging the terms, we get

$$x^n\phi_n(m_1) + x^{n-1}\big(p_1\phi_n'(m_1) + \phi_{n-1}(m_1)\big) + x^{n-2}\left(\frac{p_1^2}{2}\phi_n''(m_1) + p_1\phi_{n-1}'(m_1) + \phi_{n-2}(m_1)\right) + \cdots = 0$$

Putting $\phi_n(m_1) = 0$ and dividing by x^{n-1}, we get

$$\big(p_1\phi_n'(m_1) + \phi_{n-1}(m_1)\big) + \frac{1}{x}\left(\frac{p_1^2}{2}\phi_n''(m_1) + p_1\phi_{n-1}'(m_1) + \phi_{n-2}(m_1)\right) + \cdots = 0$$

Let $x \to \infty$, we write $\lim p_1 = c_1$

$$\therefore c_1\phi_n'(m_1) + \phi_{n-1}(m_1) = 0$$

or $c_1 = -\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}$, provided $\phi_n'(m_1) \neq 0$

Therefore $y = m_1 x - \frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}$ is the asymptote corresponding to slope m_1.

Similarly, $y = m_2 x - \frac{\phi_{n-1}(m_2)}{\phi_n'(m_2)}$, $y = m_3 x - \frac{\phi_{n-1}(m_3)}{\phi_n'(m_3)}$ etc. are the asymptotes corresponding to m_2, m_3 etc. which are the roots of $\phi_n(m) = 0$, such that the denominator of the fractions is non zero.

When $\phi_n'(m_1) = 0$ and $\phi_{n-1}(m_1) \neq 0$

There does not exist any value of c_1. So there is no asymptote corresponding to the slope m_1.

Now suppose that $\phi_n'(m_1) = 0 = \phi_{n-1}(m_1)$, then we can write

$$\frac{p_1^2}{2}\phi_n''(m_1) + p_1\phi_{n-1}'(m_1) + \phi_{n-2}(m_1) + (\ldots)\frac{1}{x} + \cdots = 0$$

Taking the limit as $x \to \infty$, we get c_1 is a root of the equation,

$$\frac{c_1^2}{2}\phi_n''(m_1) + c_1\phi_{n-1}'(m_1) + \phi_{n-2}(m_1) = 0$$

which determines two values of c_1 say c_1' and c_1'', provided that $\phi_n''(m_1) \neq 0$.

Therefore $y = m_1 x + c_1'$

and $y = m_1 x + c_1''$ are the two asymptotes corresponding to the slope m_1 and this is also known as the case of parallel asymptotes.

Steps to find oblique asymptotes:

1. Put $x = 1, y = m$ in the highest degree term to get $\phi_n(m)$.
2. Similarly find $\phi_{n-1}(m), \phi_{n-2}(m)$ etc.
3. Equate $\phi_n(m) = 0$, solve for the real values of m.
4. Find the corresponding value of intercept say, c_1 for the slope m_1 using $c_1 = -\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}$
5. Then $y = mx + c$ is the required asymptote.
6. For the case of $\phi_n'(m_1) = 0 = \phi_{n-1}(m_1)$, find c_1 by the relation
$$\frac{c_1^2}{2}\phi_n''(m_1) + c_1\phi_{n-1}'(m_1) + \phi_{n-2}(m_1) = 0$$

Find the oblique asymptotes of the curve

$$2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$$

Attempting the first and second step; put $x = 1, y = m$ in the highest degree term to get $\phi_n(m)$, we get,

$$\phi_3(m) = 2 - m - 2m^2 + m^3$$

$$\phi_2(m) = -4 + 8m$$

$$\phi_1(m) = -4$$

Equate $\phi_3(m) = 0$, solve for the real values of m.

$$m = -1, 1, 2$$

When $m = 2$, $c \quad = -\frac{\phi_{n-1}(m\ \ )}{\phi_n'(m\ \ )} = -\frac{\phi_2(m)}{\phi_3'(m)} = -4$

The asymptote is $y = 2x - 4$

When $m = 1$, $c \quad = -\frac{\phi_{n-1}(m\ \ )}{\phi_n'(m\ \ )} = -\frac{\phi_2(m)}{\phi_3'(m)} = 2$

The asymptote is $y = x + 2$

Find the asymptotes of $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$

$$\phi_3(m) = 1 - m - m^2 + m^3$$

$$\phi_3'(m) = -1 - 2m + 3m^2$$

$$\phi_3''(m) = -2 + 6m$$

$$\phi_2(m) = 2 - 4m^2 + 2m$$

$$\phi_2'(m) = -8m + 2$$

$$\phi_3(m) = 1 + m$$

$$\Rightarrow m^3 - m^2 - m + 1 = 0$$

$$(m^2 - 1)(m - 1) = 0$$

$$m = 1, 1, -1$$

When $m = 1, c = -\frac{\phi_2(m)}{\phi_3'(m)}$

$$\phi_2(1) = 0,\ \phi_3'(1) = 0$$

$$\frac{c^2}{2}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$c = \frac{3 \pm \sqrt{5}}{2}$$

The asymptotes corresponding to $m = 1$ are $y = x + \frac{3+\sqrt{5}}{2}$ and $y = x + \frac{3-\sqrt{5}}{2}$

Summary

- The total bending or total curvature is ArcPQ or Angle $\Delta\psi$
- The average curvature is $\frac{\Delta\psi}{\Delta s}$
- The curvature of the curve at P is $\lim_{Q \to P} \frac{\Delta\psi}{\Delta s} = \frac{\mathrm{d}\psi}{\mathrm{d}s}$
- Radius of curvature for a Cartesian curve $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$
- Radius of curvature for a parametric curve $\rho = \frac{(f'^2(t)+g'^2(t))^{\frac{3}{2}}}{f'(t)g''(t)-g'(t)f''(t)}$
- Radius of curvature for a polar curve $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-r\,r_2}$
- The asymptotes parallel to Y axis are obtained by equating to zero, the real linear factors in the coefficient of highest power of y, in the equation of the curve.

- The asymptotes parallel to X axis are obtained by equating to zero, the real linear factors in the coefficient of highest power of x, in the equation of the curve.
- Steps to find oblique asymptotes:

1. Put $x = 1, y = m$ in the highest degree term to get $\phi_n(m)$.
2. Similarly find $\phi_{n-1}(m), \phi_{n-2}(m)$ etc.
3. Equate $\phi_n(m) = 0$, solve for the real values of m.
4. Find the corresponding value of intercept say, $c_1$for the slope m_1 using $c_1 = -\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}$
5. Then $y = mx + c$ is the required asymptote.
6. For the case of $\phi_n'(m_1) = 0 = \phi_{n-1}(m_1)$, find c_1 by the relation

$$\frac{c_1^2}{2}\phi_n''(m_1) + c_1\phi_{n-1}'(m_1) + \phi_{n-2}(m_1) = 0$$

Key Words

Curvature, radius of curvature, vertical asymptote, horizontal asymptote, oblique asymptotes

Self Assessment

1. The angle through which the tangent turns as a point moves along the curve from a point P to Q, will be large or small as compared to arc length, depends upon

A. slope of tangent
B. sharpness of bend
C. velocity
D. acceleration

2. Which of the following is true?

I. The curvature of a circle is the same at every point.

II. Larger the circle, smaller will be its curvature.

A. Only I
B. Only II
C. Both I and II
D. None is true

3. The reciprocal of the curvature of a curve at any point in case it is non- zero, is called

A. curvature
B. radius of curvature
C. bend
D. total bending

4. The radius of curvature at any point for the curve $s = c\ tan\psi$ is

A. $c\sec^2\psi$
B. $c\sec\quad\psi$
C. $c\sec^3\psi$
D. $c\cot^2\psi$

5. For a curve if the radius of curvature is negative, it means that

A. the curve is concave upwards
B. the curve is concave downwards
C. the curve has no bend
D. none of these

6. For the cycloid $x = a(t + sin\ t), y = a(1 - cos\ t)$, the radius of curvature is given as

A. $4a\cos t$
B. $4\cos\frac{t}{2}$

C. $4 \cos t$
D. $4a \cos \frac{t}{2}$

7. Which of the following is true about asymptotes?

I. An asymptote of a curve is a line to which the curve converges.
II. The curve and its asymptote get infinitely close, but they never meet.

A. Only I is true
B. Only II is true
C. Both I and II are true
D. None is true

8. For the curve $y^2(2-x) = x^3$

A. there is no asymptote
B. there exists one asymptote only
C. there are two asymptotes
D. there are three asymptotes

9. The curve $x^3 + y^3 - 3xy = 0$ has

A. one oblique asymptote
B. two oblique asymptotes
C. no asymptote
D. no oblique asymptotes

10. For the curve $y^2(2-x) = x^3$

A. there is one asymptote parallel to x-axis
B. there are two asymptotes parallel to y-axis
C. there are no asymptotes
D. there exists one asymptote parallel to y-axis

11. The curve $x^3 + y^3 - 3xy = 0$ has

A. one parallel asymptote
B. two parallel asymptotes
C. no asymptote
D. no parallel asymptotes

12. For the curve $x^2y - 3x^2 - 5xy + 6y + 2 = 0$, there are

A. one horizontal and two vertical asymptotes
B. two horizontal and two vertical asymptotes
C. two horizontal and one vertical asymptotes
D. one horizontal and one vertical asymptotes

13. A closed curve has

A. no asymptote
B. one asymptote
C. infinitely many asymptotes
D. n asymptotes where n is the degree of the curve

14. The asymptotes of the curve $xy - 2y - 3x = 0$ are given by

A. $x - 2 = 0, y + 3 = 0$
B. $x + 2 = 0, y + 3 = 0$
C. $x - 2 = 0, y - 3 = 0$
D. $x - 2 = 0, y = 0$

15. The asymptotes of the curve $x^2y^2 - a^2(x^2 + y^2) = 0$ form a

A. circle
B. square
C. pentagon

D. triangle

16. Which of the following is true about asymptotes?

I. An asymptote of a curve is a line to which the curve converges.

II. The curve and its asymptote get infinitely close, but they never meet.

A. Only I is true
B. Only II is true
C. Both I and II are true
D. None is true

17. For the curve $y^2(2-x)=x^3$

A. there is no asymptote
B. there exists one asymptote only
C. there are two asymptotes
D. there are three asymptotes

18. The curve $x^3+y^3-3xy=0$ has

A. one oblique asymptote
B. two oblique asymptotes
C. no asymptote
D. no oblique asymptotes

19. For the curve $y^2(2-x)=x^3$

A. there is one asymptote parallel to x-axis
B. there are two asymptotes parallel to y-axis
C. there are no asymptotes
D. there exists one asymptote parallel to y-axis

20. The curve $x^3+y^3-3xy=0$ has

A. one parallel asymptote
B. two parallel asymptotes
C. no asymptote
D. no parallel asymptotes

21. For the curve $x^2y-3x^2-5xy+6y+2=0$, there are

A. one horizontal and two vertical asymptotes
B. two horizontal and two vertical asymptotes
C. two horizontal and one vertical asymptotes
D. one horizontal and one vertical asymptotes

22. A closed curve has

A. no asymptote
B. one asymptote
C. infinitely many asymptotes
D. n asymptotes where n is the degree of the curve

23. The asymptotes of the curve $xy-2y-3x=0$ are given by

A. $x-2=0, y+3=0$
B. $x+2=0, y+3=0$
C. $x-2=0, y-3=0$
D. $x-2=0, y=0$

24. The asymptotes of the curve $x^2y^2-a^2(x^2+y^2)=0$ form a

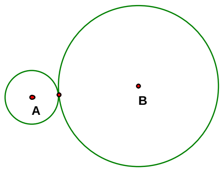
A. circle
B. square
C. pentagon
D. triangle

Answer for Self Assessment

1.	B	2.	C	3.	B	4.	A	5.	B
6.	D	7.	C	8.	B	9.	A	10.	D
11.	D	12.	A	13.	A	14.	C	15.	B
16.	C	17.	B	18.	A	19.	D	20.	D
21.	A	22.	A	23.	C	24.	B		

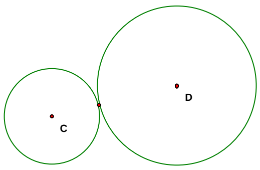
Review Questions

1. If the radius of circle A is $1/6$, then its curvature is ____________.



2. If the radius of circle B is $1/2$, then its curvature is ____________.

3. If the radius of circle C is $1/5$, then its curvature is ____________.



4. If the radius of circle D is $1/3$, then its curvature is ____________.
5. If you have a small circle and a large circle, which one will have the larger curvature?
6. Find the parallel asymptotes for the curve $2x^3 - x^2y + 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$.
7. Find the rectangular asymptotes for the curve $2x^3 - x^2y + 2xy^2 + y^3 - 4x^2 + 8xy = 0$.
8. Find the rectangular asymptotes for the curve $x^2y - 3x^2 - 5xy + 6y + 2 = 0$.
9. Find the parallel asymptotes for the curve $x^2y + xy^2 + xy + y^2 + 3x = 0$.
10. Find the oblique asymptotes for the curve $2x^3 - x^2y + 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$.
11. Find the oblique asymptotes for the curve $2x^3 - x^2y + 2xy^2 + y^3 - 4x^2 + 8xy = 0$.
12. Find the oblique asymptotes for the curve $x^2y - 3x^2 - 5xy + 6y + 2 = 0$.
13. Find the oblique asymptotes for the curve $x^2y + xy^2 + xy + y^2 + 3x = 0$.
14. Find all the asymptotes for the curve $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$.
15. Find all the asymptotes for the curve $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy = 0$.

Further Reading

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Unit 13: Concavity and Multiple Points

CONTENTS

Objectives

Students will be able to

- detect the lines of symmetry in a curve
- classify a point as a point of concavity, convexity or inflection
- find the tangents at origin
- understand various multiple points
- find the position and nature of the double points

Introduction

In this unit, we will mainly learn about various important aspects involved in the tracing of a curve. We will begin with the symmetry and its various aspects in relation to different shapes. From the differentiability we have the idea of smoothness or pointedness of the curve at a point. This idea can be extended to concavity of a function, with a special focus on the points of inflection. There will be further a discussion on the types of double points, their nature and their position.

13.1 Symmetry

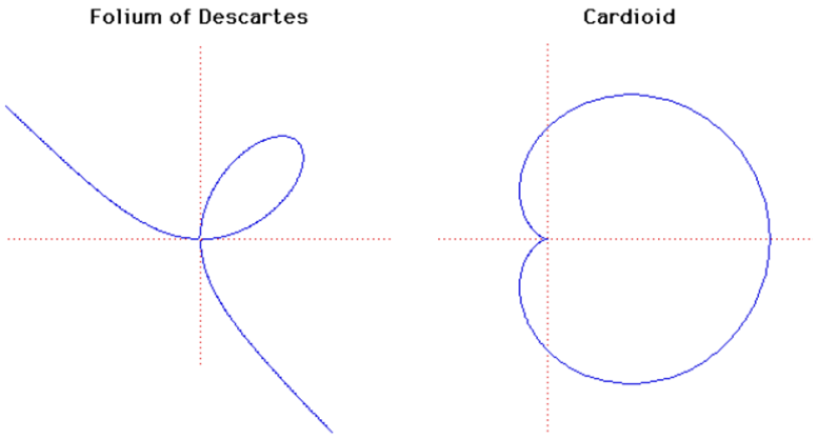
The images which can be divided into identical halves are called symmetrical. The images that cannot be divided into identical halves are asymmetrical.

Any line splitting a shape into two parts such that the two parts are the same is called a line of symmetry. These parts are also said to be symmetrical to each other.

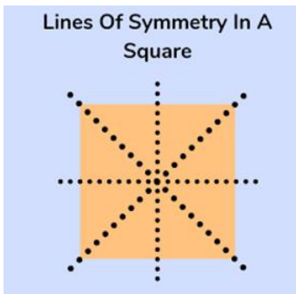
For a square there are four lines of symmetry. for a hexagon, there will be six lines of symmetry. Can you think of the lines of symmetry for a triangle and a pentagon?

You can take a piece of paper, draw the required shape on it, using the scale and pencil, cut out them and fold it in various ways to find out the lines of symmetry.

Consider the folium of Descartes and the cardioid; you can observe that there is only one line of symmetry.



13.2 Various Kinds of Symmetry



We can see that there are various lines of symmetry in various kinds of functions $f(x, y) = 0$. We can classify those lines as follows:

Symmetry about the x-axis:

In the function $f(x, y) = 0$, replace y with $-y$. If $(x, -y) = f(x, y)$, then the graph will be symmetric about the x-axis.

e.g. $y^2(2a - x) = x^3$ is symmetric about the x-axis.

Symmetry about the y-axis:

In the function $f(x, y) = 0$, replace x with $-x$. If $(-x, y) = f(x, y)$, then the graph will be symmetric about the y-axis.

e.g. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$is symmetric about the y-axis.

Symmetry about the origin:

In the function $f(x,y)=0$, replace x,y with $-x,-y$. If $(-x,-y)=f(x,y)$, then the graph will be symmetric about the origin.

e.g. $(x^2+y^2)^2=a^2(x^2-y^2)$is symmetric about the origin

Symmetry about the line,$y=x$:

In the function $f(x,y)=0$, replace x with y and y with x.If $(x,y)=f(y,x)$, then the graph will be symmetric about the line $y=x$.

e.g. $(x^2+y^2)^2=a^2(x^2-y^2)$is not symmetric about the line $y=x$.

Symmetry about the line, $y=-x$:

In the function $f(x,y)=0$, replace x with $-y$ and y with $-x$. If $(x,y)=f(-y,-x)$, then the graph will be symmetric about the line $y=-x$.

e.g. $(x^2+y^2)^2=a^2(x^2-y^2)$is not symmetric about the line $y=-x$.

Symmetry in the opposite quadrants:

In the function $f(x,y)=0$, replace x,y with $-x,-y$. If $(-x,-y)=f(x,y)$, then the graph will be symmetric in the opposite quadrants.

e.g. $xy=1$ is symmetric in first and third quadrants.

Discuss all possible lines of symmetry for the following curves:

(i) $y^2(2a-x)=x^3$

(ii) $(x^2+y^2)^2=a^2(x^2-y^2)$

(iii) $(x-b)^2(x^2+y^2)-a^2x^2=0$

(iv) $x^3+y^3=3axy$

(v) $(x^2+y^2-axy)^2=4a^2(x^2+y^2)$

(vi) $x^2+y^2=16$

(vii) $y^2=2x$

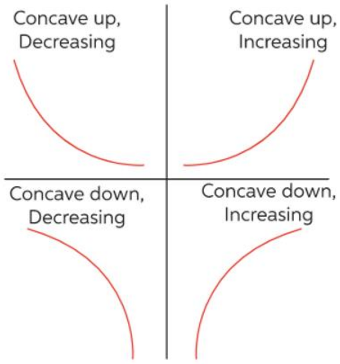
(viii) $x^2+y=2x+4$

13.3 Concavity and Convexity

Although the first derivative test determines if a function is increasing or decreasing, we would also like to know if the shape of the graph is curving upward or downward. This notion of curvature of a graph upward or downward is known as concavity.

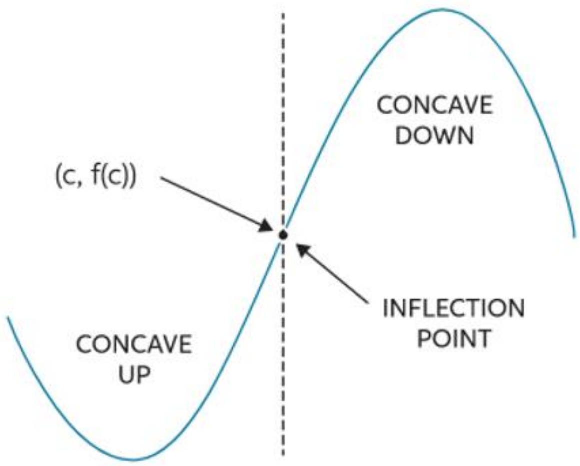
If the secant line passing through the points (x_1, f(x_1)) and (x_2, f(x_2)) is above the curve f(x)=y between these two points, then f(x) is concave up.

If the secant line passing through the points (x_1, f(x_1)) and (x_2, f(x_2)) is below the curve f(x)=y between these two points, then f(x) is concave below or concave down.



When the slope continually **increases**, the function is **concave upward**.

When the slope continually **decreases**, the function is **concave downward**.



Theorem:

Consider a function f(x) that is twice continuously differentiable on an interval I. The Function f(x) is

- concave upwards if $f''(x) > 0$ for all x in I
- concave downwards if $f''(x) < 0$ for all x in I

You need to be careful while using the following terms:

- Concave Downward is also called Concave or Convex Upward
- Concave Upward is also called Convex or Convex Downward

Discuss the curve $y = x^2$ for its concavity.

$$y'(x) = 2x$$

Clearly the first derivative is positive whenever x is positive and is negative, whenever x is negative. So by first derivative test, we can see that the slope is decreasing for negative values and is increasing for positive values of x. Therefore the curve is concave up for all x.

Alternatively, we can find the second derivative

$$y''(x) = 2$$

Clearly, it is positive for all the values of the domain, thus the curve $y = x^2$ is concave upwards $\forall\ x \in \boldsymbol{R}$.

Discuss the curve $f(x) = 5x^3 + 2x^2 - 3x$ for its concavity.

$$f'(x) = 15x^2 + 4x - 3$$

$$f''(x) = 30x + 4$$

$$f''(x) = 0 \text{ happens at } x = -\frac{2}{15}$$

Clearly $f''(x) < 0$ in $\left(-\infty, -\frac{2}{15}\right) \Rightarrow f(x)$ is concave downward in this interval and

$$f''(x) > 0 \text{ in } \left(-\frac{2}{15}, \infty\right) \Rightarrow f(x) \text{ is concave upward in this interval.}$$

Discuss the curve $f(x) = \ln(1 + x^2)$ for its concavity.

We have

$$f'(x) = \frac{2x}{1+x^2}$$

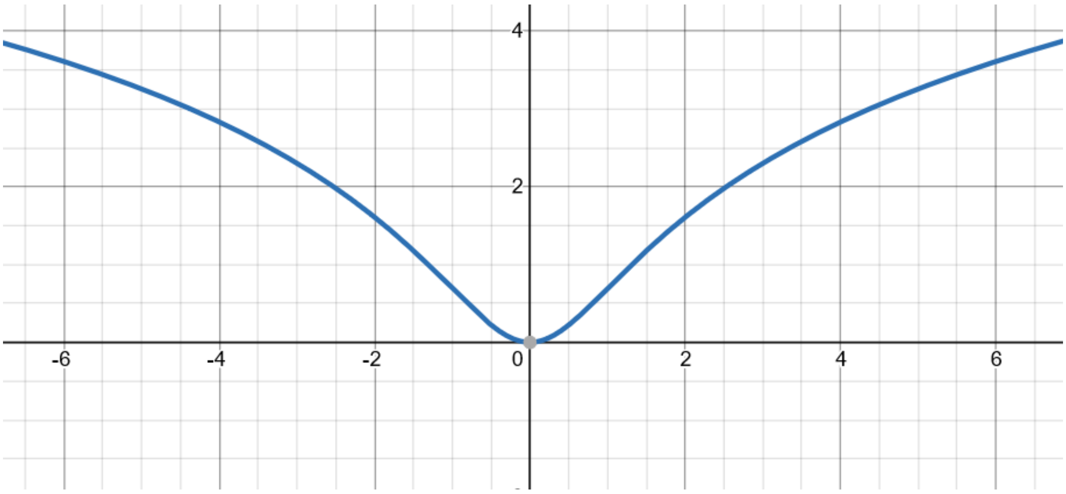
$$f''(x) = \frac{2-2x^2}{(1+x^2)^2}$$

We can look for the values of x for which the $f''(x)$ will be negative. That is

$$2 - 2x^2 < 0$$

$$2(1-x)(1+x) < 0$$

From this inequality, it is clear that, $f(x)$ is concave downward in $(-\infty,-1)U(1,\infty)$ and concave upward in $(-1,1)$. Let us also look at how the function looks like!

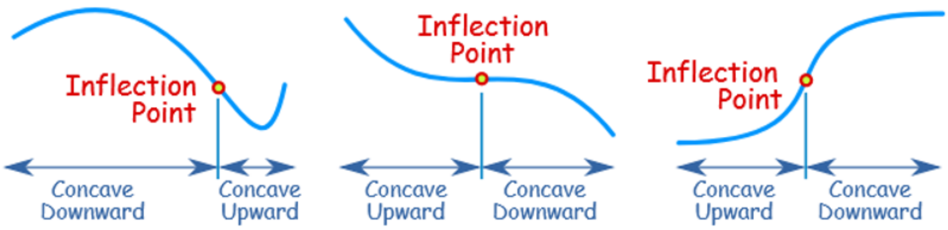


13.4 Points of Inflection

A point where a curve changes from **Concave upward** to **Concave downward** (or vice versa), is called the inflexion point.

A point (c, f(c)) is said to be an inflection point for a point c in (a, b) and for a continuous function f(x) in (a, b) if the graph of y=f(x) changes concavity at (c, f(c)).

This also implies that the first derivative changes from increasing to decreasing or decreasing to increasing at(c, f(c)).



Find the point of inflection for the curve $f(x) = x\,e^{-2x}$

Finding the derivatives, we get

$$f'(x) = e^{-2x}(-2x+1)$$

$$f''(x) = e^{-2x}(4x-4)$$

The curve changes from concave upward to concave downward when

$$f''(x) = 0$$

And so we get $x = 1$

$\therefore$ the point of inflection is $(1, e^{-2})$

Find the point of inflection for the curve $f(x) = (x-1)^3(x-5)$

Finding the derivatives, we get

$$f'(x) = 4(x-1)^2(x-4)$$

$$f''(x) = 12(x-1)(x-3)$$

The curve changes from concave upward to concave downward when

$$f''(x) = 0$$

$$\Rightarrow x = 1, 3$$

Thus, the points of inflection are (1, 0) and (3, -16)

Find the point of inflection for the curve $f(x) = x + x^{\frac{5}{3}}$

Finding the derivatives, we get

$$f'(x) = 1 + \frac{5}{3}x^{\frac{2}{3}}$$

$$f''(x) = \frac{10}{9x^{\frac{1}{3}}}$$

Here the $f''(x) = 0$ won't make sense, so using the first derivative test, we can observe that

when$x < 0,\ f''(x) < 0$ and when $x > 0, f''(x) > 0$.

So we can see the curve changes from concave downward to concave upward at (0, 0), So (0, 0) is the point of inflection of the given curve.

Find the point of inflection for the curve $f(x) = x^4 - 6x^2$

Finding the derivatives, we get

$$f'(x) = 4x^3 - 12x$$

$$f''(x) = 12x^2 - 12$$

The curve changes from concave upward to concave downward when

$$f''(x) = 0$$

$$\Rightarrow x = \pm 1$$

The points of inflection are (1, -5) and (-1, -5).

13.5 Tangents at Origin

The general equation of rational algebraic curve of the n^{th} degree which passes through the origin O, when arranged according to ascending powers of x and y is of the form

$(b_1x + b_2y) + (c_1x^2 + c_2xy + c_3y^2) + (d_1x^3 + d_2x^2y + d_3xy^2 + d_4y^3) + \cdots = 0$ ---- (1)

Let $P(x, y)$ be any point on the curve. The slope of the chord OP is $\frac{y}{x}$.

$$\lim_{P\to O} chord\ OP = \text{ Tangent at } O$$

Then when $x \to 0, y \to 0$

$\lim \frac{y}{x} = m$is the slope of the tangent.

So (1) implies

$$\left(b_1 + b_2\frac{y}{x}\right) + \left(c_1x \quad + c_2\frac{y}{x} + c_3y\frac{y}{x}\right) + \cdots = 0$$

$$b_1 + b_2m = 0$$

$$m = -\frac{b_1}{b_2}$$

$$\frac{y}{x} = -\frac{b_1}{b_2}$$

$\therefore b_1x + b_2y = 0$ is the tangent at the origin, which can be written by equating the lowest degree term to zero in equation (1).

If $b_2 = 0$ but $b_1 \neq 0$, then considering the slope of OP with reference to y-axis, it can be shown that the tangent retains the same form.

Let $b_1 = b_2 = 0$, then the equation takes the form

$$c_1x^2 + c_2xy + c_3y^2 + d_1x^3 + \cdots = 0$$

$$c_1 + c_2\frac{xy}{x^2} + c_3\frac{y^2}{x^2} + d_1x + \cdots = 0$$

As $x \to 0$

$c_1 + c_2m + c_3m^2 = 0$ ---- (2)

gives two values of m say $m_1$and m_2.

The equation of either tangent at the origin is $y = m_1x$ ---- (3)

Eliminating m_1 from (2) and (3), we get

$c_1x^2 + c_2xy + c_3y^2 = 0$ ----(4)

whichis the joint equation of two tangents at the origin, and it can also be written by equating the lowest degree term to zero in the equation of the curve.

If $c_1 = c_2 = c_3 = 0$, then (4) becomes an identity and equations of tangents can still be written by equating to zero, the terms of lowest degree, which is third, in this case.

Therefore the rule to find the tangents at origin can be summarized as:

The equation of the tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the equation of the curve.

Find the tangents at origin for the curve $x^3 + y^3 - 3axy = 0$

Clearly the origin lies on the curve. To find the tangents at origin, let us seek the lowest degree term in the curve and equate that to zero.

$$-3axy = 0$$

$$\Rightarrow x = 0, y = 0$$

are the required equations of the tangents at the origin.

Find the equation of the tangent(s) at (-1,-2) to the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$.

We can shift the origin to the point (-1, -2) by the following transformations:

$$x = X - 1$$

$$y = Y - 2$$

Using these, we get the transformed equation as:

$$X^3 - X^2 + 2XY - Y^2 = 0$$

Clearly the origin lies on the curve and we can find the tangents at origin easily by equating to zero the terms of the lowest degree in the equation of the curve.

i.e.$-X^2 + 2XY - Y^2 = 0$

i.e.$(X - Y)^2 = 0$

i.e.$X - Y = 0$

i.e.$x + 1 - (y + 2) = 0$

i.e. $x - y = 1$ is the tangent at (-1, -2) for the given curve.

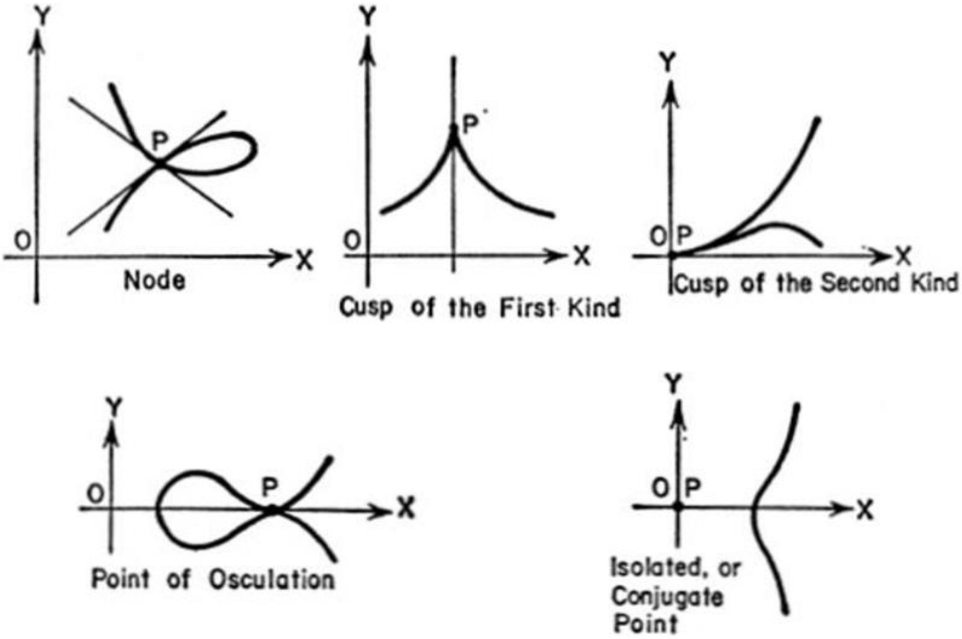
Therefore, we can find the tangent at any given point on a curve.

13.6 Multiple Points

If RS is an arc of a curve, and if at the point P on RS there exists one and only one tangent, AB, to the curve, then point P is known as an ***ordinary point*** of the curve.

If at a point P on a curve there exist two and only two distinct tangents, then that point is called a ***node***.

If the two tangents at a given point are not distinct, but coincide, we have what is called a ***cusp***.



There are several kinds of cusps:

(1) If the curve in the neighbourhood of a cusp lies partly on one side of the tangent and partly on the other side, the point is known as a *cusp of the first kind;*

(2) If the curve lies entirely on one side of the common tangent (in the region of tangency), the point is known as a *cusp of the second kind;*

(3) If there are two distinct cusps at the same point, it is known as a *point of osculation.*

All points having two and only two tangents, whether real or imaginary, distinct or coincident, are called ***double points*** of the curve.

Thus nodes and cusps of all kinds are double points.

Triple points are such points on a curve for which there are three tangents; similarly there are four for quadruple points etc.

An isolated point on a curve is also called a *conjugate point.*

All points that are not ordinary points are known as *singular points.*

13.7 Condition for any Point (x, y) to be a Multiple Point of the Curve f(x, y)=0

For a curve $f(x, y) = 0$ ---- (1)

We can write

$$\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$$

$$f_x+f_y\frac{dy}{dx}=0$$

At a multiple point of a curve, the curve has at least two tangents and $\frac{dy}{dx}$ must have at least two values at multiple points.

More than one value of $\frac{dy}{dx}$ can satisfy $f_x+f_y\frac{dy}{dx}=0$iff $f_x=0, f_y=0$.

Therefore, to find the multiple points, we have to find the values of (x,y) which simultaneously satisfy the three equations:

$$f_x(x,y)=0$$

$$f_y(x,y)=0$$

$$f(x,y)=0$$

Differentiate $f_x+f_y\frac{dy}{dx}=0$ w.r.t. x, we get

$$f_{xx}+2f_{xy}\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}+f_{yy}\left(\frac{dy}{dx}\right)^2=0$$

At the multiple points where $f_y=0,\ f_x=0$; the value of $\frac{dy}{dx}$ are the roots of the quadratic equation,

$$f_{xx}+2f_{xy}\frac{dy}{dx}+f_{yy}\left(\frac{dy}{dx}\right)^2=0$$

In case f_{xx}, f_{xy}, f_{yy} are not all zero and $f_x=0=f_y$, the point (x,y) will be a double point and will be a

node if $\boldsymbol{f_{xy}^2-f_{xx}f_{yy}}>0$

cusp if $\boldsymbol{f_{xy}^2-f_{xx}f_{yy}=0}$ **and**

conjugate point if $\boldsymbol{f_{xy}^2-f_{xx}f_{yy}}<0$

If $f_{xx}=f_{xy}=f_{yy}=0$, the point (x,y) will be a multiple point of the order higher than two, those are not in the scope of this course.

Find the multiple points on the curve

$$x^4-2ay^3-3a^2y^2-2a^2x^2+a^4=0$$

Let $f(x)=x^4-2ay^3-3a^2y^2-2a^2x^2+a^4$

$$f_x=4x^3-4a^2x$$

$$f_y=-6ay^2-6a^2y$$

$$f_x=0, f_y=0\ \Rightarrow\ x=0,a,-a\ ;y=0,-a.$$

$\therefore$ $f_x$andf_y vanish at $(0,0),(0,-a),(a,0),(a,-a),(-a,0),(-a,-a)$.

Out of these, only $(a,0),(-a,0)$& $(0,-a)$ lie on the given curve.

Now $f_{xx}=12x^2-4a^2$

$$f_{yy}=-12ay-6a^2$$

$$f_{xy}=0$$

We know,

$$f_{xx}+2f_{xy}\frac{dy}{dx}+f_{yy}\left(\frac{dy}{dx}\right)^2=0$$

At $(a,0)$

$$-6a^2\left(\frac{dy}{dx}\right)^2+4(2a^2)=0$$

$$\frac{dy}{dx} = \pm\frac{2}{\sqrt{3}}$$

Since there are two real values, the point $(a, 0)$ is a node and the tangents at $(a, 0)$ are

$$y = \pm\frac{2}{\sqrt{3}}(x - a)$$

Similarly tangents at $(-a, 0)$: $y = \pm\frac{2}{\sqrt{3}}(x + a)$

and at $(0, -a)$: $y + a = \pm\frac{\sqrt{2}}{\sqrt{3}}x$

Alternatively,

We can find tangents at $(a, 0)$, by shifting the origin to $(a, 0)$ by following transformations,

$$x = X + a, y = Y$$

$$\Rightarrow X^4 + 4X^3a - 2aY^3 + 4a^2X^2 - 3a^2Y^2 = 0$$

Tangents at origin are given by,

$$4a^2X^2 - 3a^2Y^2 = 0$$

$$X^2 = \frac{3}{4}Y^2$$

$$X = \pm\frac{3}{\sqrt{2}}Y$$

Similarly the tangents can be found at the other points too.

13.8 Position and Nature of Double Points

For the curve $f(x, y) = 0, (x, y)$ is a double point if it satisfies

$$f_x(x, y) = 0$$

$$f_y(x, y) = 0$$

$$f(x, y) = 0$$

Simultaneously.

The point (x, y) is a node, if $f_{xy}^2 - f_{xx}f_{yy} > 0$

The point (x, y) is a cusp, if $f_{xy}^2 - f_{xx}f_{yy} = 0$

The point (x, y) is isolated, if $f_{xy}^2 - f_{xx}f_{yy} < 0$

Types of cusps in terms of their position:

Single cusp of first kind: Two branches on the same side of common normal and on opposite sides of tangent.

Single cusp of second kind: Two branches lie on the same side of the normal as well as the tangent.

Double cusp of second kind: Two branches lie on the different sides of normal and on the same side of the tangent.

Point of oscu inflection: Two branches lie on different sides of normal but on one side they lie on the same and on the other on opposite sides of the common tangent.

Let us learn about the position and nature of double points by an example.

Locate the double points of the curve $y(y - 6) = x^2(x - 2)^3 - 9$ and mention their nature.

Here $f(x, y) = x^2(x - 2)^3 - 9 - y(y - 6) = 0$

$$f_x = x(x - 2)^2(5x - 4)$$

$$f_y = 6 - 2y$$

$$f_{xx} = (x-2)^2(5x-4) + 2x(x-2)(5x-4) + 5x(x-2)^2$$

$$f_{yy} = -2$$

$$f_{xy} = 0$$

Now $f_x = 0, f_y = 0$ imply $x = 0, 2, \frac{4}{5}$ and $y = 3$

Therefore possible double points are

$$(0,3), (2,3) \text{and} \left(\frac{4}{5}, 3\right)$$

The point $\left(\frac{4}{5}, 3\right)$ does not satisfy the given curve. So there are only two double points.

At $(0,3)$

$$f_{xy}^2 - f_{xx}f_{yy} < 0$$

So, $(0,3)$ is a conjugate point.

At $(2,3)$

$$f_{xy}^2 - f_{xx}f_{yy} = 0$$

So, $(2,3)$ is a cusp.

To know the nature of the cusp, let us shift the origin to the point $(2,3)$ by using

$$x = X + 2$$

$$y = Y + 3$$

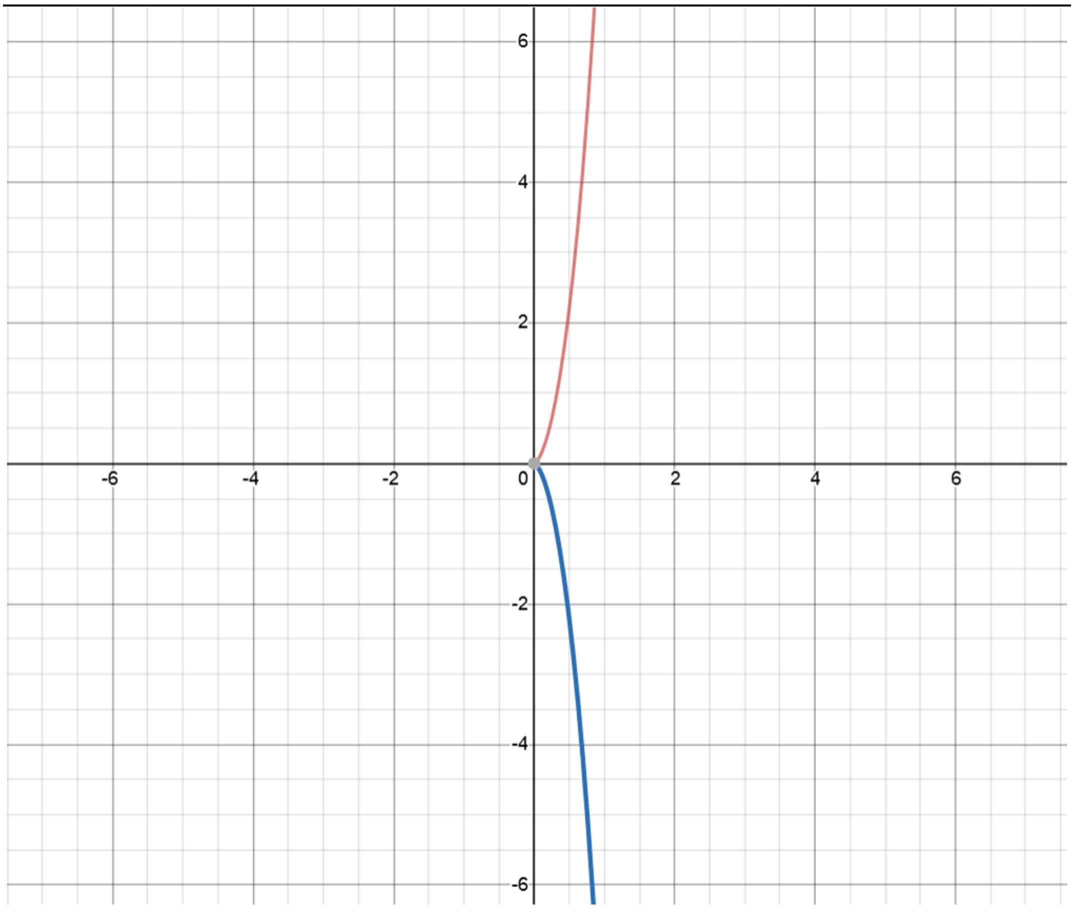
The transformed equation is $Y^2 = X^3(X+2)^2$

The tangents at origin are $Y^2 = 0$ i.e. the x-axis.

Moreover, $Y = \pm(X+2)\sqrt{X^3}$

When $X < 0, Y$ is imaginary

When $X > 0, Y$has two values, positive and negative. Therefore near the origin, the curve lies on both sides of X-axis (tangent) and only on one side of the Y-axis (normal). The graph is given below for better clarity.



Therefore, the new origin (2,3) is a single cusp of first kind.

Summary

This chapter had variety of topics, which can be summarized as follows:

- Symmetry about the x-axis: In the function $f(x,y)=0$, replace y with $-y$. If $(x,-y)=f(x,y)$, then the graph will be symmetric about the x-axis.
- Symmetry about the y-axis: In the function $f(x,y)=0$, replace x with $-x$. If $(-x,y)=f(x,y)$, then the graph will be symmetric about the y-axis.
- Symmetry about the origin: In the function $f(x,y)=0$, replace x,y with $-x,-y$. If $(-x,-y)=f(x,y)$, then the graph will be symmetric about the origin.
- Symmetry about the line, $y=x$:In the function $f(x,y)=0$, replace x with y and y with x. If $(x,y)=f(y,x)$, then the graph will be symmetric about the line $y=x$.
- Symmetry about the line, $y=-x$:In the function $f(x,y)=0$, replace x with $-y$ and y with $-x$. If $(x,y)=f(-y,-x)$, then the graph will be symmetric about the line $y=-x$.
- Symmetry in the opposite quadrants: In the function $f(x,y)=0$, replace x,y with $-x,-y$. If $(-x,-y)=f(x,y)$, then the graph will be symmetric in the opposite quadrants.
- Consider a function f(x) that is twice continuously differentiable on an interval I. The function f(x) is concave upwards if f''(x) > 0 for all x in I and concave downwards if f''(x) < 0 for all x in I
- A point where a curve changes from concave upward to concave downward (or vice versa), is called the inflexion point.

- The equation of the tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the equation of the curve.
- The point (x, y) will be a double point and will be a node if $f_{xy}^2 - f_{xx}f_{yy} > 0$, cusp if $f_{xy}^2 - f_{xx}f_{yy} = 0$ and conjugate point if $f_{xy}^2 - f_{xx}f_{yy} < 0$

Keywords

Symmetry, lines of symmetry, concave up, concave down, convex, concave, point of inflection, tangent at origin, multiple point, double point, node, cusp, isolated point

Self Assessment

1. How many lines of symmetry are there for a pentagon?

A. 2
B. 3
C. 4
D. 5

2. For a lemniscate of Bernoulii, how many lines of symmetry are there?

A. 2
B. 3
C. 4
D. 5

3. The curve $2x^2 + 2y^2 = 11$ is

A. symmetric about the x-axis
B. symmetric about the y-axis
C. symmetric about the line y=x
D. all of the above

4. The notion of curvature of a graph upward or downward is known as

A. symmetry
B. asymptotes
C. concavity
D. multiple points

5. When the slope continually increases, the function

A. is concave upwards
B. is concave downwards
C. can not be deciphered
D. is decreasing

6. The function $f(x) = 2x^2 + 3x + 4$ is

A. concave upwards on the set of real numbers
B. concave upwards on a specific interval
C. concave downwards on the set of real numbers
D. concave downwards on a specific interval

7. The point where a curve changes from concave upward to concave downward is called a

A. saddle point

B. stationary point
C. critical point
D. inflexion point

8. Which of the following is true for the point of inflexion (c, f(c))?
A. the first derivative changes from increasing to decreasing or decreasing to increasing at (c, f(c))
B. the first derivative changes from increasing to decreasing at (c, f(c))
C. the first derivative changes from decreasing to increasing at (c, f(c))
D. none of these

9. For $f(x) = xe^{-2x}$ the point of inflexion is
A. (1,1)
B. (1,e)
C. $(1, e^{-2})$
D. $(1, e^{2})$

10. The equation of the tangent or tangents at the origin is obtained by
A. equating to zero the terms of the lowest degree in the equation of the curve.
B. equating to zero the terms of the highest degree in the equation of the curve.
C. equating to one the terms of the lowest degree in the equation of the curve.
D. equating to one the terms of the highest degree in the equation of the curve.

11. For the curve $x^3 + y^3 - 3xy = 0$, there exist ____ tangent(s) at the origin.
A. one
B. two
C. three
D. four

12. For the curve $x^2(x^2 + y^2) = 5(x - y)$, the equation of the tangent at origin is
A. $x = 5y$
B. $x = -5y$
C. $x = -y$
D. $x = y$

13. If the two tangents at a given point are not distinct, but coincide, we get
A. a node
B. a cusp
C. a conjugate point
D. none of these

14. If the curve lies entirely on one side of the common tangent (in the region of tangency), the point is known as a
A. cusp of first kind
B. cusp of second kind

C. node

D. point of osculation

15. At a multiple point of a curve $f(x) = 0$, the curve has

A. one tangent

B. two tangents

C. at least one tangent

D. at least two tangents

16. If at a point on a curve there exist two and only two distinct tangents, then that point is called a

A. cusp of first kind

B. cusp of second kind

C. node

D. point of osculation

17. At a double point (x, y) of a curve $f(x, y) = 0$, if $f_{xy}^2 - f_{xx}f_{yy} = 0$, then the double point is a

A. node

B. cusp

C. isolated point

D. none of these

18. At a double point (x, y) of a curve $f(x, y) = 0$, if $f_{xy}^2 - f_{xx}f_{yy} > 0$, then the double point is a

A. node

B. cusp

C. isolated point

D. none of these

19. At a double point (x, y) of a curve $f(x, y) = 0$, if $f_{xy}^2 - f_{xx}f_{yy} < 0$, then the double point is a

A. node

B. cusp

C. isolated point

D. none of these

Answer for Self Assessment

1.	D	2.	A	3.	D	4.	C	5.	A
6.	A	7.	D	8.	A	9.	C	10.	A
11.	B	12.	D	13.	B	14.	B	15.	D
16.	C	17.	B	18.	A	19.	C		

Review Questions

1. Draw any random closed figure with 8 straight lines and discuss its lines of symmetry.
2. In the nature around, spot five things, having symmetry.
3. Discuss all the lines of symmetry for the curve $(x^2 + y^2)x - 2ay^2 = 0$
4. Discuss about the concavity of the curve $x^3 + y^3 = 3axy$
5. Discuss about the points of inflection of the curve $x^3 + y^3 = 3axy$
6. Find the tangents at origin for the curve $(x^2 + y^2)x - 2ay^2 = 0$
7. Find the tangents at origin for the curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$
8. Find the tangents at origin for the curve $2y^5 + 5x^5 - 3x(x^2 - y^2) = 0$
9. Find the tangents at origin for the curve $(x^2 + y^2)x^2 = a(x - y)$
10. Find the double points of the curve $x^3 + y^3 = 3axy$
11. Find the double points of the curve $(x^2 + y^2)x - 2ay^2 = 0$
12. Find the position and nature of the double point of the curve $x^2y^2 = (a + y)^2(b^2 - y^2)$ for $a < banda > b$.

Further Reading

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Unit 14: Tracing of Curves

CONTENTS

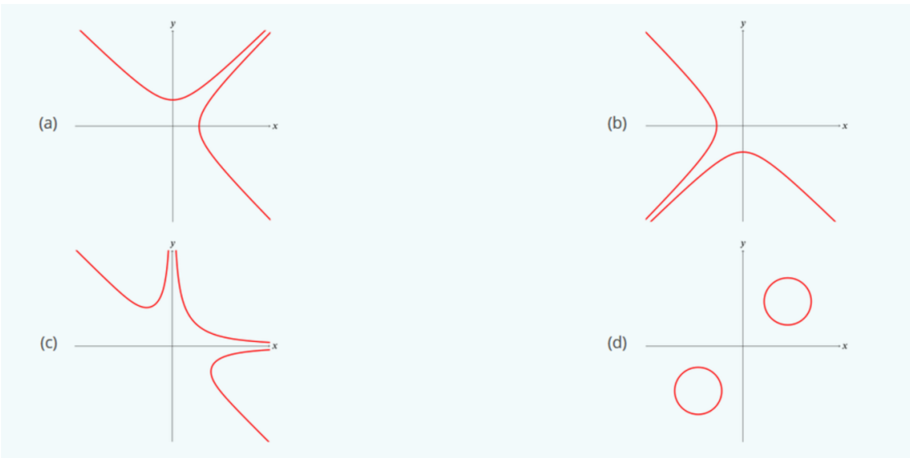
Objectives

Students will be able to

- list the properties which can be used to trace a curve
- trace the Cartesian curves
- trace the parametric curves
- trace the polar curves

Introduction

Can you recognize which of the following options depicts the curve $x^2y^2(x+y)=1$?



In order to answer the above question, we need to analyze the given Cartesian equation thoroughly. An image is worth a thousand words. A curve which is the visual synonym of a functionalrelation gives us the whole idea of information about the relation. Of course, we can also obtain this information by analysing the equation which defines the functional relation. But studying the associated curve is often easier and quicker. In addition to this, a curve which represents a relation between two quantities also helps us to easily find the value of one quantity corresponding to a specific value of the other. In this unit we shall be using many results from the previous units and will try to understand what is meant by the graph of a relation like f(x, y) = 0, and how one can draw it.

Recall that, that the set of points {(x, y): f(x, y) = 0) is known as the graph of the functional relation f(x, y) = 0. Graphing a function or a functional relation means showing the points of the corresponding set in a plane, thus, essentially curve tracing means plotting the points which satisfy a given relation. However, there are some difficulties involved in this. Let's see what these are and how to overcome them.

It is often not possible to plot all the points on a curve. The standard technique is to plot some suitable points and to get a general idea of the shape of the curve by considering tangents, asymptotes, singular points, extreme points, inflection points, concavity, monotonicity, periodicity etc. Then we draw a free hand curve as nearly satisfying the various properties as is possible.

The curves or graphs that we draw have a limitation. If the range of values of either (or both) variable is not finite, then it is not possible to draw the complete graph. In such cases the graph is not only approximate, but is also incomplete. For example, consider the simplest curve, a straight line. Suppose we want to draw the graph of $f: R \rightarrow R$such that$f(x) = c$. We know that this is in line parallel to the x-axis. But it is not possible to draw a complete graph as the line extends infinitely on both sides.

Suppose the equation of a curve is f(x, y) = 0.We shall now list some steps which, when taken, will simplify our job of tracing this curve.

14.1 Procedure for Tracing Curves Given in Cartesian Equations

Steps for curve sketching (preferably in the same order) are summarized below:

1. Domain

The first step is to determine the extentof the curve. In other words we try to find a region or regions of the plane which cannot have any point of the curve. For example, no point on the curve $x = y^2$, lies in the second or the third quadrant, as the x-coordinate of any point on the curve has to be non-negative. This means that our curve lies entirely in the first and the fourth quadrants.

Note thatit is easier to determine the extent of a curve if its equation can be written explicitly as y = f(x) or x =f(y).

2. Intercepts

The next step is to determine the points where the curve intersects the axes. If we put y = 0 in f(x, y) = 0, and solve the resulting equation for x, we get the points of intersection with the x-axis. Similarly, putting x = 0 and solving the resulting equation for y, we can find the points of intersection with the y-axis.

3. Symmetry

We find out if the curve is symmetrical about any line, or about the origin. We have already discussed symmetry of curves in the previous unit. This step reduces our workload. If the curve is symmetric about the x-axis, we can focus upon the region above x-axis only and then can replicate that for the complete curve.

4. Asymptotes

The next step is to find the asymptotes, if there are any. They indicate the trend of the branches of the curve extending to infinity. Asymptotes, if they exist, provide a frame for the curve.

5. Intervals of increase and decrease

Calculate dy/dx. This will help you in locating the portions where the curve is rising (dy/dx >0) or falling (dy/dx <0) or the points where it has a corner (dy/dx does not exist).

6. Local maximum and minimum

Calculated^2y/dx^2. This will help you in locating maxima (dy/dx = 0,$\frac{d^2y}{dx^2} < 0$) and minima (dy/dx = 0,$\frac{d^2y}{dx^2}$ >0).

7. Concavity/Convexity/Points of inflection/Multiple points

Calculated$^2y/dx^2$. This will help you in locating maxima (dy/dx = 0, $\frac{d^2y}{dx^2} < 0$) and minima (dy/dx = 0, $\frac{d^2y}{dx^2}$ >0) along with concave up or concave down nature of the curve. You will also be able to determine the points of inflection ($\frac{d^2y}{dx^2} = 0$). These will give you a good idea about the shape of the curve.

Another important step is to determine the singular points. The shape of the curve at these points is, generally, more complex, as more than one branch of the curve passes through them. Find out whether the origin lies on the curve. If it does, then find the equations of the tangents at the origin by equating to zero the lowest degree terms and we can look out for cusps and nodes.

8. Graph of the Function

All the information obtained from above steps, finally has to be put on the x-y plane. Plot as many points as you can, around the points already plotted. Also try to draw tangents to the curve at some of these plotted points. For this you will have to calculate the derivative as these points. Now join the plotted points by a smooth curve (except at points of discontinuity). The tangents will guide you in this, as they give you the direction of the curve and the graph has to be traced then.

Warm up by tracing the simple popular curves like modulus function, exponential function, parabola etc. using the above mentioned steps. Now let us trace the curve $x^2y^2(x+y) = 1$ which was asked in the introductory section.

Since the function is not an explicit function of x or y, we can skip the domain part.

When x=0, we cannot find y and when y=0, we cannot find x. It means that the curve does not make any intercepts on the axes.

The curve is symmetric about the line y=x only.

There are no horizontal and vertical asymptotes, but an oblique asymptote y=-x.

And by these many steps only, the correct answer out of four options can be inferred.

If not we would have undertaken next steps too.

Trace $y = x^3 - 12x - 16$

Let us trace the curve step by step. For your ease the steps are mentioned below:

1. Domain

   The function can take all real values of x as its domain.
2. Intercepts

   When $x = 0, y = -16$

   When $y = 0, x = -2, 4$

   Therefore the given curve meets the axes at $(0, -16), (-2,0), (4,0)$.
3. Symmetry

   No particular line of symmetry exists.
4. Asymptotes

   No asymptote exists
5. Intervals of increase and decrease

$$y' = 3x^2 - 12$$
$$y' = 0 \Rightarrow x = \pm 2$$
$$\therefore y \text{ is increasing in } (-\infty, -2)$$
$$y \text{ is decreasing in } (-2, 2)$$
$$y \text{ is increasing in } (2, \infty)$$

6. Local maximum and minimum

$$y'' = 6x$$

   At $x = -2, y'' < 0 \Rightarrow (-2, 0)$is a point of maximum.

At $x = 2, y'' > 0 \Rightarrow (2, -32)$is a point of minimum.

7. Concavity/Convexity/Points of Inflection/Multiple points

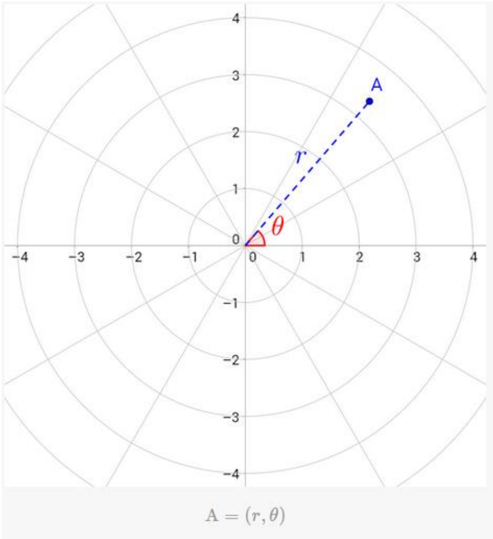
$$y'' < 0 \text{ when } x < 0 \Rightarrow \text{Concave downward}$$

$$y'' > 0 \text{ when } x > 0 \Rightarrow \text{Concave upward}$$

8. Graph of the Function

14.2 Tracing of Polar Curves

A polar equation is any equation that describes a relation between r and θ, where r represents the distance from the pole to a point on a curve, and θ represents the anti-clockwise angle made by a point on a curve, the pole, and the initial line.



One advantage of using polar equations is that certain relations that are *not* functions in Cartesian form can be expressed as functions in polar form.

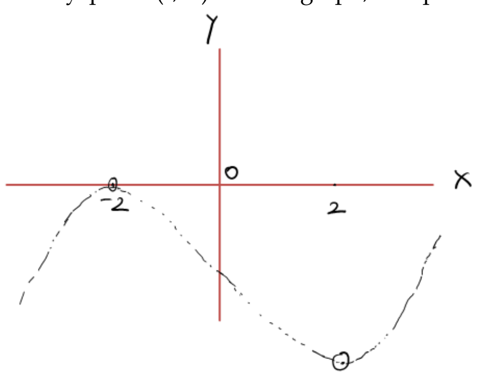
Another advantage is that certain relations are much simpler to express in polar form rather than Cartesian form.

Steps to trace a polar curve:

1. Symmetry

Consider a curve generated by the function $r=f(\theta)$ in polar coordinates.

- The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r,-\theta)$ is also on the graph. Similarly, the equation $r=f(\theta)$ is unchanged by replacing θ with −θ
- The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r,\pi+\theta)$ is also on the graph. Similarly, the equation $r=f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi+\theta$.
- The curve is symmetric about the vertical line $\theta=\pi/2$ if for every point (r, θ) on the graph, the point $(r,\pi-\theta)$ is also on the graph. Similarly, the equation $r=f(\theta)$ is unchanged when θ is replaced by $\pi-\theta$.



2. Extent

(i) Find the limits within which r must lie for the permissible values of θ. If r < a (r >a) for some a > 0, then the curve lies entirely within (outside) the circle r = a. (ii) If r^2 is negative for some values of θ, then the curve has no portion in the corresponding region.

3. Pole

The curve passes through the pole, if for r = 0, there corresponds a real value of θ

4. Asymptote

If O is the pole and $P(r, \theta)$ is any point in the polar system PY is the line which is the asymptote, and $OY \perp PY$ .The polar equation of any line is $p = r\cos(\theta - \alpha)$ where p is the length of perpendicular from the pole to the line and α is the angle which this perpendicular makes with the initial line.

$$OY = p$$

$$\angle XOY = \alpha$$

If $P(r, \theta)$ be any point on the line then $\angle POY = \theta - \alpha$

and$\frac{p}{r} = \cos(\theta - \alpha)$

$\therefore p = r\cos(\theta - \alpha)$is the required equation of the line which is the prospective asymptote.

5. Region

Find the region in which the curve does not exist, or find the greatest and least numerical value of r etc.

6. Specific points

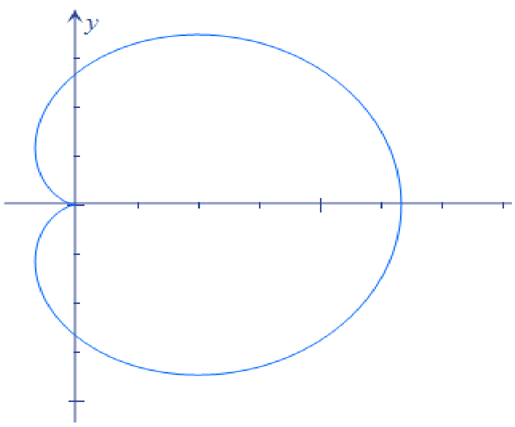
Trace the variation of r as θ varies.

Trace $r = a(1 + cos\ \theta)$

1. If we replace $\theta$to$-\theta$, the equation remains unchanged. So the given curve is symmetric about the initial line.
2. The extent can be seen from the fact that maximum value of $\cos\theta$is 1 and minimum value is -1. Thus r can take values from 0 to $2a$.
3. For $r = 0$, we can see that$\theta = \pi$. This means that the curve passes through the pole.
4. Since it's a polar curve we can find the value of r at various θ values and then plot the curve on a polar plane.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	$2a$	$\frac{(\sqrt{3}+2)}{2}a$	$\frac{(\sqrt{2}+1)}{\sqrt{2}}a$	$\frac{3}{2}a$	a	$\frac{a}{2}$	0

From all the above points, the following curve can be traced:



14.3 Tracing of Parametric Curves

When the path of a particle moving in the plane is not the graph of a function, we cannot describe it using a formula that expresses y directly in terms of x, or x directly in terms of y. Instead, we need to use a third variable t, called a parameter and write x = f(t) y = g(t).

The set of points (x, y) = (f(t), g(t)) described by these equations when t varies in an interval I form a curve, called a parametric curve, and x = f(t), y = g(t) are called the parametric equations of the curve.

Trace the curve with parametric equations:

$$x = a\cos^3\theta$$

$$y = b\sin^3\theta$$

First of all we can make a table of variation of x and $y$with θ.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	a	0	$-a$	0	a
y	0	b	0	$-b$	0

Now the (x, y)points can be plotted on the xy −plane.

For the parametric equations

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}, \qquad \frac{dx}{d\theta} \neq 0$$

Therefore $\frac{dy}{dx} = -\frac{b}{a}\tan\theta$ provided that $\frac{dx}{d\theta} \neq 0$

Moreover $\frac{dx}{d\theta} = 0$ for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

When $\theta = 0$, $\frac{dy}{dx} = 0$

At $(a, 0)$, the tangent is given as,

$$y - 0 = \left(\frac{dy}{dx}\right)_{\theta=0}(x - a)$$

$$\Rightarrow y = 0$$

When $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \infty$

At $(0, b)$, the tangent is given as,

$$y - b = \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}}(x - 0)$$

$$\Rightarrow x = 0$$

When $\theta = \pi$, $\frac{dy}{dx} = 0$

At $(-a, 0)$, the tangent is given as,

$$y - 0 = \left(\frac{dy}{dx}\right)_{\theta=\pi}(x + a)$$

$$\Rightarrow y = 0$$

When $\theta = \frac{3\pi}{2}$, $\frac{dy}{dx} = \infty$

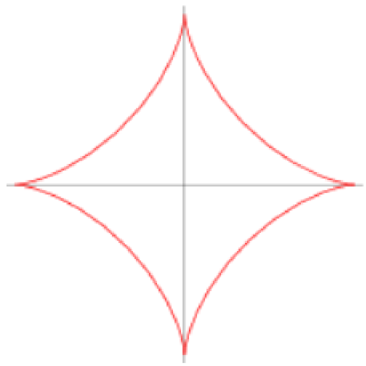
At $(0, -b)$, the tangent is given as,

$$y + b = \left(\frac{dy}{dx}\right)_{\theta=\frac{3\pi}{2}}(x - 0)$$

$$\Rightarrow\ x = 0$$

Thus, the tangent at points $(a, 0)$and $(-a, 0)$ is the x-axis and at the points $(0, b)$and $(0, -b)$ is the y-axis.

And from this information we can draw the following:



Trace the curve with parametric equations:

$$x = a(\theta + sin\theta)$$

$$y = a(1 + cos\theta)$$

First of all we can make a table of variation of x and $y$withθ.

θ	0	π	$-\pi$	3π	-3π
x	0	$a\pi$	$-\ a\pi$	$3\pi a$	$-3\pi a$
y	$2a$	0	0	0	0

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}},\quad \frac{dx}{d\theta} \neq 0$$

Here $\frac{dx}{d\theta} = a(1 + cos\theta)$

$$\frac{dy}{d\theta} = -a\ sin\theta$$

$\frac{dy}{dx} = -\tan\frac{\theta}{2}$provided$1 + cos\theta \neq 0$ or $\theta \neq \pm\pi$

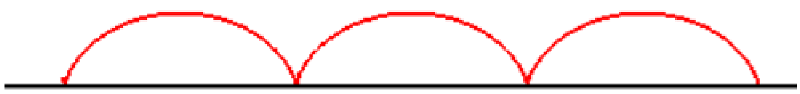
When $\theta = \pi$

The equation of the tangent is $x = a\pi$.

When $\theta = 0$

The equation of the tangent is $y = 2a$.

Similarly the tangents at the other points can be checked and with the above information we can trace the following:



Summary

- Steps for curve sketching of a Cartesian curve:
  - Domain
  - Intercepts
  - Symmetry
  - Asymptotes
  - Intervals of Increase and Decrease
  - Local Maximum and Minimum
  - Concavity/Convexity and Points of Inflection
  - Graph of the Function
  - Steps for curve sketching of a polar curve:
  - Symmetry
  - Pole
  - Asymptote
  - Region
  - Specific points

Keywords

Curve tracing, Cartesian curve, polar curve, parametric curve

Self Assessment

1. For the curve $x^2y^2(x+y)=10$, which of the following is true?

A. The curve has an intercept on x-axis
B. The curve has an intercept on y-axis
C. The curve has an intercept on x-axis and y-axis
D. The curve has no intercept on x-axis and y-axis

2. For the curve $x^2y^2(x+y)=10$, which of the following is true?

A. The curve is symmetric about x-axis only
B. The curve is symmetric about y-axis only
C. The curve is symmetric about the line $y=x$
D. The curve is symmetric about x-axis, y-axis and the line $y=x$

3. For the curve $x^2y^2(x+y)=10$

A. there is one asymptote only
B. there are two asymptotes
C. there are three asymptotes
D. there are no asymptotes

4. For the curve $y=tanx$ which of the following is false?

A. The origin is a cusp
B. $x=2a$ is an asymptote
C. Curve exists for all non-negative values of x

D. The curve is symmetrical about the $x - axis$

5. The folium of Descartes is given by

A. $x^3 + y^3 = 3ax^2y^2$

B. $x^3 + y^3 = 3axy$

C. $x^2 + y^2 = 3axy$

D. $x^4 + y^4 = 3axy$

6. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point (r, −θ) is also on the graph.

A. True

B. False

7. The curve is symmetric about the polar axis if for some point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph.

A. True

B. False

8. The curve passes through the pole, if for r = 0, there corresponds a real value of θ.

A. True

B. False

9. The equation $r = a(1 + cos\ \theta)$ represents

A. a circle

B. a lemniscate

C. a cardioid

D. a cycloid

10. The equation $p = r\ cos(\theta - \alpha)$ with usual notations, represents a

A. line

B. circle

C. cardioid

D. lemniscate

11. The equations of type x = f(t), y = g(t) are called

A. simultaneous equations

B. ordinary equations

C. parametric equations

D. none of these

12. In the equations of type x = f(t), y = g(t), t is called the parameter.

A. True

B. False

13. The curve having parametric equations $x = 5\ cos^3\theta, y = 7\ sin^3\theta$ is
A. a circle
B. an ellipse
C. a cycloid
D. an asteroid

14. The curve $r = a + bcos\ \theta$ is symmetrical about
A. initial line
B. y-axis
C. line perpendicular to the initial line
D. line $\theta = \frac{\pi}{4}$

15. Number of loops in the curve $r = a\ cos\ 2\theta$ is
A. 2
B. 3
C. 4
D. 6

Answers for Self Assessment

1. D	2. C	3. A	4. D	5. B
6. A	7. B	8. A	9. C	10. A
11. C	12. A	13. D	14. A	15. C

Review Questions

Trace the following curves

1. $y^2 = 8x$
2. $x^2 + y^2 = 9$
3. $x^2y^2(x + y) = 10$
4. $x^3 + y^3 = 3axy$
5. $y^2x^2 = x^2 - a^2$
6. $r = a(1 + cos\ \theta)$
7. $r = a(1 - cos\ \theta)$
8. $x = 5\ cos^3\theta, y = 7\ sin^3\theta$
9. $r = a \cos 2\theta, a > 0$
10. $r = a + bcos\ \theta$
11. $x = 0.5 \sec t,\ y = 1 + \cot t$
12. $x = \cos t, y = \cot t$
13. $r \log \theta = a$
14. $r = a(\theta - \sin \theta)$
15. $r = a + a \sin \theta, a > 0$

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