Discrete Structures

DEMTH136

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LOVELY PROFESSIONAL UNIVERSITY

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CONTENT

Unit 01: Set

CONTENTS

Purpose and Objectives:

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting. In this section, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set.

After this unit, you would be able to

- understand the definition of a set and describe the different ways of representing the set.
- learn different types of sets,
- describe the subset of a set, proper, improper subset, and the power set.
- draw the Venn diagram of a set and able to visualize the different counting problems using it.
- understand various set operations like union, intersection, complement, difference and symmetric difference
- prove various laws of set theory and can also use these laws to solve counting problems

Introduction

The concept of sets is used for the foundation of various topics in mathematics. To learn sets we often talk about the collection of objects, such as a set of vowels, a set of negative numbers, a group of friends, a list of fruits, a bunch of keys, etc. Sets are used to group objects together. Often, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion. We now provide a definition of a set. This definition is an intuitive definition, which is not part of a formal theory of sets.

1.1 Set

An unordered collection of well-defined distinct objects is known as a set. The word well-defined refers to a specific property which makes it easy to identify whether the given object belongs to the set or not. The word 'distinct' means that the objects of a set must be all different.

example:

1. The collection of children in class VII whose weight exceeds 35 kg represents a set.
2. The collection of all the intelligent children in class VII does not represent a set because the word intelligent is vague. What may appear intelligent to one person may not appear the same to another person.

Elements of Set:

The different objects that form a set are called the elements of a set. The elements of the set are written in any order and are not repeated. Elements are denoted by small letters.

Notation of a Set:

A set is usually denoted by capital letters and elements are denoted by small letters

If x is an element of set A, then we say $x \in A$. **[x belongs to A]**

If x is not an element of set A, then we say $x \notin A$. **[x does not belong to A]**

Example

The collection of vowels in the English alphabet.

Solution:

Let us denote the set by V, then the elements of the set are-

a, e, i, o, u or we can say, V = [a, e, i, o, u].

We say $a \in V$, $e \in V$, $i \in V$, $o \in V$, and $u \in V$.

Also, we can say $b \notin V$, $c \notin v$, $d \notin v$, etc.

1.2 How to state that whether the objects form a set or not?

A collection of lovely flowers' is not a set, because the objects (flowers) to be included are not well-defined.

Reason: The word "lovely" is a relative term. What may appear lovely to one person may not be so to the other person.

A collection of "Yellow flowers" is a set, because every red flower will be included in this set i.e., the objects of the set are well-defined.

A group of "Young singers" is not a set, as the range of the ages of young singers is not given and so it can't be decided which singer is to be considered young i.e., the objects are not well-defined.

A group of "Players with ages between 18 years and 25 years" is a set, because the range of ages of the player is given and so it can easily be decided which player is to be included and which is to be excluded. Hence, the objects are well-defined.

Now we will learn to state which of the following collections are set.

State, give a reason, whether the following objects form a set or not:

(i) All problems of this book, which are difficult to solve.

(ii) All problems of this book, which are difficult to solve for Aaron.

(iii) All the objects are heavier than 28 kg.

Solution:

(i) The given objects do not form a set.

Reason: Some problems may be difficult for one person but may not be difficult for some other persons, that is, the given objects are not well-defined. Hence, they do not form a set.

(ii) The given objects form a set.

Reason: It can easily be found that which are difficult to solve for Aaron and which are not difficult to solve for him. Hence, the objects form a set.

(iii) The given objects form a set.

Reason: Every object can be compared, in weight, with 28 kg. Then it is very easy to select objects which are heavier than 28 kg i.e., the objects are well-defined. Hence, the objects form a set.

The members (objects) of each of the following collections form a set:

(i) Students in a classroom

(ii) Books in your school-bag

(iii) Counting numbers between 5 to 15

(iv) Students of your class, which are taller than you and so on.

1.3 What are the elements of a set or members of a set?

The objects used to form a set are called its element or its members.

Generally, the elements of a set are written inside a pair of curly (idle) braces and are represented by commas. The name of the set is always written in capital letters.

Example

(i) $A = \{v, w, x, y, z\}$

Here 'A' is the name of the set whose elements (members) are v, w, x, y, z.

If a set $A = \{3, 6, 9, 10, 13, 18\}$. State whether the following statements are 'true' or 'false':

(i) $7 \in A$

(ii) $12 \notin A$

(iii) $13 \in A$

(iv) $9, 12 \in A$

(v) $12, 14, 15 \in A$

Solution:

(i) $7 \in A$. False, since element 7 does not belongs to the given set A.

(ii) $10 \notin A$. False, since element 10 belongs to the given set A.

(iii) $13 \in A$. True, since element 13 belongs to the given set A.

(iv) $9, 10 \in A$. True, since the elements 9 and 12 both belong to the given set A.

(v) $10, 13, 14 \in A$. False, since element 14 does not belong to the given set A.

1.4 Basic properties of sets

The two basic properties to represent a set are explained below using various examples.

The change in the order of writing the elements does not make any changes in the set.

In other words, the order in which the elements of a set are written is not important. Thus, the set {a, b, c} can also be written as {a, c, b} or {b, c, a} or {b, a, c} or {c, a, b} or {c, b, a}.

Example:

Set A = {4, 6, 7, 8, 9} is same as set A = {8, 4, 9, 7, 6}

i.e., {4, 6, 7, 8, 9} = {8, 4, 9, 7, 6}

Similarly, {w, x, y, z} = {x, z, w, y} = {z, w, x, y}   and so on.

If one or many elements of a set are repeated, the set remains the same.

In other words, the elements of a set should be distinct. So, if any element of a set is repeated a number of times in the set, we consider it as a single element. Thus, {1, 1, 2, 2, 3, 3, 4, 4, 4} = {1, 2, 3, 4}

Example:

The set of letters in the word 'GOOGLE' = {G, O, L, E}

The set A = {5, 6, 7, 6, 8, 5, 9} is same as set A= {5, 6, 7, 8, 9}

i.e., {5, 6, 7, 6, 8, 5, 9} = {5, 6, 7, 8, 9}

In general, the elements of a set are not repeated. Thus,

(i) if T is a set of letters of the word 'moon': then T = {m, o, n}, There are two o's in the word 'moon' but it is written in the set only once.

(ii) If U = {letters of the word 'COMMITTEE'}; then U = {C, O, M, T, E}

1.5 Representation of a Set

In the representation of a set the following three methods are commonly used:

(i) Statement form method

(ii) Roster or tabular form method

(iii) Rule or set builder form method

Statement form:

In this, a well-defined description of the elements of the set is given and the same are enclosed in curly brackets.

Example:

(i) The set of odd numbers less than 7 is written as: {odd numbers less than 7}.

(ii) A set of football players with ages between 22 years to 30 years.

(iii) A set of numbers greater than 30 and smaller than 55.

(iv) A set of students in class VII whose weights are more than your weight.

Roster form or tabular form:

In this, elements of the set are listed within the pair of brackets { } and are separated by commas.

Example:

(i) Let N denote the set of the first five natural numbers. Therefore, N = {1, 2, 3, 4, 5} → Roster Form

(ii) The set of all vowels of the English alphabet. Therefore, V = {a, e, i, o, u} → Roster Form

(iii) The set of all odd numbers less than 9. Therefore, X = {1, 3, 5, 7} → Roster Form

(iv) The set of all-natural number which divide 12. Therefore, Y = {1, 2, 3, 4, 6, 12} → Roster Form

(v) The set of all letters in the word MATHEMATICS. Therefore, Z = {M, A, T, H, E, I, C, S} → Roster Form

(vi) W is the set of the last four months of the year. Therefore, W = {September, October, November, December} → Roster Form

The order in which elements are listed is immaterial but elements must not be repeated.

Set builder form:

In this, a rule, or the formula, or the statement is written within the pair of brackets so that the set is well defined. In the set builder form, all the elements of the set must possess a single property to become a member of that set.

In this form of representation of a set, the element of the set is described by using a symbol 'x' or any other variable followed by a colon The symbol ':' or '|' is used to denote such that and then we write the property possessed by the elements of the set and enclose the whole description in braces. In this, the colon stands for 'such that' and braces stand for 'set of all.

Example:

(i) Let P is a set of counting numbers greater than 12;

The set P in set-builder form is written as :

P = {x: x is a counting number and greater than 12}

or

P = {x | x is a counting number and greater than 12}

This will be read as, 'P is the set of elements x such that x is a counting number and is greater than 12'.

The symbol ':' or '|' placed between 2 x's stands for such that.

(ii) Let A denote the set of even numbers between 6 and 14. It can be written in the set-builder form as;

A = {x | x is an even number, 6 < x < 14} or A = {x: x ∈ P, 6 < x < 14 and P is an even number}

(iii) If X = {4, 5, 6, 7}. This is expressed in roster form.

Let us express in set builder form. X = {x: x is a natural number and 3 < x < 8}

(iv) The set A of all odd natural numbers can be written as

A = {x: x is a natural number and x = 2n + 1 for n ∈ W}

Express the set of integers lying between -2 and 3. using the three methods of representation of a set:

Express the set of integers lying between -2 and 3. using the three methods of representation of a set:

Statement form: {X a set of integers lying between -2 and 3}

Roster form: X = {-1, 0, 1, 2}

Set builder form: $X = \{x : x \in I, -2 < x < 3\}$.

1.6 Types of sets:

The different types of sets are explained below with examples.

Empty Set or Null Set:

A set that does not contain any element is called an empty set, or the null set, or the void set and it is denoted by ∅ and is read as phi. In roster form, ∅ is denoted by {}. An empty set is a finite set, since the number of elements in an empty set is finite, i.e., 0.

Example:

(i) The set of whole numbers less than 0. Clearly, there is no whole number less than 0. Therefore, it is an empty set.

(ii) $N = \{x : x \in N, 3 < x < 4\}$

(iii) $A = \{x : 2 < x < 3, x \text{ is a natural number}\}$ Here A is an empty set because there is no natural number between 2 and 3.

(iv) $B = \{x : x \text{ is a composite number less than } 4\}$.

Here B is an empty set because there is no composite number less than 4.

Notes

(i) $\emptyset \neq \{0\} \therefore$ has no element.

(ii) {0} is a set that has one element 0.

(iii) The cardinal number of an empty set, i.e., $n(\emptyset) = 0$

Singleton Set:

A set that contains only one element is called a singleton set.

Example:

(i) $A = \{x : x \text{ is neither prime nor composite}\}$ It is a singleton set containing one element, i.e., 1.

(ii) $B = \{x : x \text{ is a whole number}, x < 1\}$. This set contains only one element 0 and is a singleton set.

(iii) Let $A = \{x: x \in N \text{ and } x^2 = 4\}$ Here A is a singleton set because there is only one element 2 whose square is 4.

(iv) Let $B = \{x : x \text{ is a even prime number}\}$. Here B is a singleton set because there is only one prime number which is even, i.e., 2.

Finite Set:

A set that contains a definite number of elements is called a finite set. An empty set is also called a finite set.

Example:

(i) The set of all colors in the rainbow.

(ii) $N = \{x : x \in N, x < 7\}$

(iii) $P = \{2, 3, 5, 7, 11, 13, 17, \ldots\ldots 97\}$

Infinite Set:

The set whose elements cannot be listed, i.e., a set containing never-ending elements is called an infinite set.

Example:

(i) Set of all points in a plane
(ii) $A = \{x : x \in N, x > 1\}$
(iii) Set of all prime numbers
(iv) $B = \{x : x \in W, x = 2n\}$

Cardinal Number of a Set:

The number of distinct elements in a given set A is called the cardinal number of A. It is denoted by n(A).

Example:

(i) $A \{x : x \in N, x < 5\}$ $A = \{1, 2, 3, 4\}$
Therefore, $n(A) = 4$
(ii) B = set of letters in the word ALGEBRA.
$B = \{A, L, G, E, B, R\}$ Therefore, $n(B) = 6$

Equivalent Sets:

Two sets A and B are said to be equivalent if their cardinal number is the same, i.e., $n(A) = n(B)$. The symbol for denoting an equivalent set is '$\leftrightarrow$'.

Example:

$A = \{1, 2, 3\}$ Here $n(A) = 3$

$B = \{p, q, r\}$ Here $n(B) = 3$

Therefore, $A \leftrightarrow B$

Equal sets:

Two sets A and B are said to be equal if they contain the same elements. Every element of A is an element of B and every element of B is an element of A.

Example:

$A = \{p, q, r, s\}$

$B = \{p, s, r, q\}$

Therefore, $A = B$.

1.7 Subset:

If A and B are two sets, and every element of set A is also an element of set B, then A is called a subset of B and we write it as $\mathbf{A \subseteq B}$ or $\mathbf{B \supseteq A}$

The symbol $\subset$ stands for 'is a subset of' or 'is contained in

- Every set is a subset of itself, i.e., $A \subset A$, $B \subset B$.
- Empty set is a subset of every set.
- Symbol '$\subseteq$' is used to denote 'is a subset of' or 'is contained in.

Notes

- $A \subseteq B$ means A is a subset of B or A is contained in B.
- $B \subseteq A$ means B contains A.

Example:

Let A = {2, 4, 6} , B = {6, 4, 8, 2}

Here A is a subset of B. Since, all the elements of set A are contained in set B.

But B is not the subset of A. Since, all the elements of set B are not contained in set A.

Notes

(i) If set A is a subset of set B and B is a subset of A, then A = B, i.e., they are equal sets.

(ii) Every set is a subset of itself.

(iii) Null set or $\emptyset$ is a subset of every set.

(iv) The set N of natural numbers is a subset of the set Z of integers and we write $N \subset Z$.

Example:

Let A = {2, 4, 6}

B = {x : x is an even natural number less than 8}

Here $A \subset B$ and $B \subset A$.

Hence, we can say A = B

Super Set:

Whenever a set A is a subset of set B, we say the B is a superset of A and we write, $B \supseteq A$.

The symbol $\supseteq$ is used to denote 'is a superset of'

Example:

A = {a, e, i, o, u}

B = {a, b, c, ..., z}

Here $A \subseteq B$ i.e., A is a subset of B but $B \supseteq A$ i.e., B is a superset of A

Proper Subset:

If A and B are two sets, then A is called the proper subset of B if $A \subseteq B$ but $B \supseteq A$ i.e., $A \neq B$. The symbol '$\subset$' is used to denote proper subset. Symbolically, we write $A \subset B$.

Example:

A = {1, 2, 3, 4}. Here n(A) = 4

B = {1, 2, 3, 4, 5}. Here n(B) = 5

We observe that all the elements of A are present in B but the element '5' of B is not present in A. So, we say that A is a proper subset of B. Symbolically, we write it as $A \subset B$.

Notes

(i) No set is a proper subset of itself.

(ii) Null set or $\emptyset$ is a proper subset of every set.

Example:

A = {p, q, r}

B = {p, q, r, s, t}

Here A is a proper subset of B as all the elements of set A are in set B and also $A \neq B$.

Power Set:

The collection of all subsets of set A is called the power set of A. It is denoted by P(A). In P(A), every element is a set.

Example:

If A = {p, q} then all the subsets of A will be

P(A) = {$\emptyset$, {p}, {q}, {p, q}}

Number of elements of P(A) = n[P(A)] = 4 = 2^2

Did you know?

In general, n[P(A)] =2^m where m is the number of elements in set A.

Universal Set:

A set that contains all the elements of other given sets is called a **universal set**. The symbol for denoting a universal set is **U.**

Example:

(i) If A = {1, 2, 3}

B = {2, 3, 4}

C = {3, 5, 7}

then U = {1, 2, 3, 4, 5, 7}

[Here $A \subseteq U$, $B \subseteq U$, $C \subseteq U$ and $U \supseteq A$, $U \supseteq B$, $U \supseteq C$]

(ii) If P is a set of all whole numbers and Q is a set of all negative numbers then the universal set is a set of all integers.

Notes

1.8 Venn Diagrams:

Pictorial representations of sets represented by closed figures are called set diagrams or Venn diagrams.

Venn diagrams are used to illustrate various operations like union, intersection, and difference. We can express the relationship among sets through this in a more significant way. In this,

• A rectangle is used to represent a universal set.

• Circles or ovals are used to represent other subsets of the universal set.

Venn diagrams in different situations:

• If a set A is a subset of set B, then the circle representing set A is drawn inside the circle representing set B.

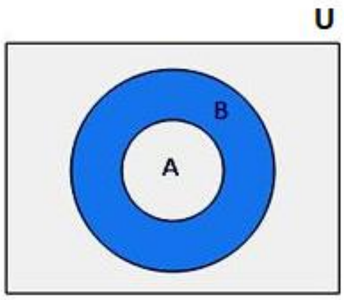


Figure 1: Venn diagram for A⊆B

• If set A and set B to have some elements in common, then to represent them, we draw two circles that are overlapping.

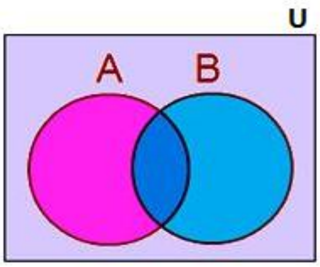


Figure 2: Venn diagram for AUB

• If set A and set B are disjoint, then they are represented by two non-intersecting circles.

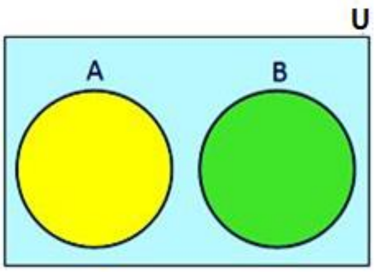


Figure 3: Venn diagram for disjoint sets

In these diagrams, the universal set is represented by a rectangular region and its subsets by circles inside the rectangle. We represented disjoint sets by disjoint circles and intersecting sets by intersecting circles.

1.9 Operations on sets:

When two or more sets combine together to form one set under the given conditions, then operations on sets are carried out.

Union of sets

Let A and B be any two sets. The union of A and B is the set that consists of all the elements of A and all the elements of B, the common elements being taken only once.

The symbol '∪' is used to denote the union. Symbolically, we write $A \cup B$ and usually read as 'A union B'.

Example:

(i) Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find $A \cup B$?.

Solution: $A \cup B = \{2, 4, 6, 8, 10, 12\}$ Note that the common elements 6 and 8 have been taken only once while writing $A \cup B$.

(ii) Let A = { a, e, i, o, u } and B = { a, i, u }. Show that $A \cup B = A$

Solution: We have, $A \cup B = \{ a, e, i, o, u \} = A$. This example illustrates that the union of sets A and its subset B is the set A itself, i.e., if $B \subset A$, then $A \cup B = A$.

The **union** of two sets A and B is the set C which consists of all those elements which are either in A or in B (including those which are in both).

In symbols, we write. $\mathbf{A \cup B = \{ x: x \in A \text{ or } x \in B \}}$ The union of two sets can be represented by a Venn diagram as shown in Figure 2. The shaded portion in Figure 2 represents $A \cup B$.

The intersection of sets:

The intersection of sets A and B is the set of all elements which are common to both A and B.

The symbol '∩' is used to denote the intersection. The intersection of two sets A and B is the set of all those elements which belong to both A and B.

Symbolically, we write $\mathbf{A \cap B = \{x: x \in A \text{ and } x \in B\}}$.

Example:

(i) Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find $A \cap B$?.

Solution: We see that 6, 8 are the only elements that are common to both A and B. Hence $A \cap B = \{ 6, 8 \}$.

(ii) Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and B = { 2, 3, 5, 7 }. Find $A \cap B$ and hence show that

$A \cap B = B$.

Solution: We have $A \cap B = \{ 2, 3, 5, 7 \} = B$. We note that $B \subset A$ and that $A \cap B = B$. Figure 4 indicates the intersection of A and B.

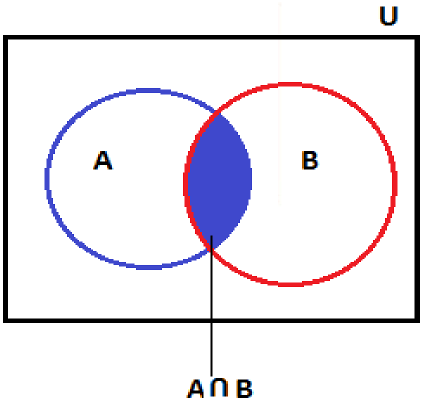


Figure 4: Venn diagram of A ∩ B

Notes

If A and B are two sets such that $A \cap B = \varphi$, then A and B are called disjoint sets.

Example: let A = { 2, 4, 6, 8 } and B = { 1, 3, 5, 7 }. A and B are disjoint sets because there are no elements that are common to A and B.

The disjoint sets can be represented by means of the Venn diagram as shown in Figure 3.

The difference of sets:

The difference between sets A and B in this order is the set of elements that belong to A but not to B.

Symbolically, we write A – B and read as " A-minus B". as $A - B = \{ x: x \in A \text{ and } x \notin B \}$.

Example:

(i) Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A?

Solution: We have, A – B = { 1, 3, 5 }.

Since the elements 1, 3, 5 belong to A but not to B. B – A = { 8 }.

Since element 8 belongs to B and not to A. We note that $A - B \neq B - A$.

(ii) Let V = { a, e, i, o, u } and B = { a, i, k, u}. Find V – B and B – V?

Solution: We have, V – B = { e, o },

Since the elements e, o belong to V but not to B. B – V = { k },

Since the element k belongs to B but not to V. We note that $V - B \neq B - V$.

The difference of two sets A and B can be represented by the Venn diagram as shown in Figure 5

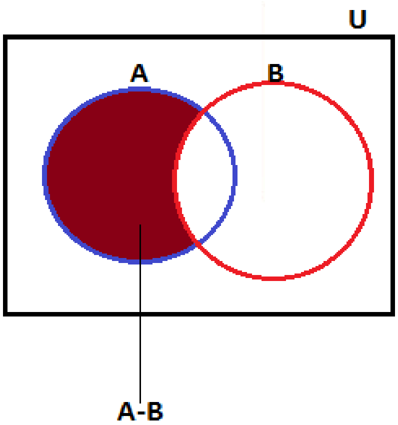


Figure 5: Venn diagram for A-B.

Symmetric difference:

The symmetric difference using the Venn diagram of two subsets A and B is a subset of U, denoted by A △ B, and is defined by

$A \triangle B = (A - B) \cup (B - A)$.

Let A and B are two sets. The symmetric difference of two sets A and B is the set

$(A - B) \cup (B - A)$ and is denoted by $A \triangle B$.

Thus, $A \triangle B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$

or, $A \triangle B = \{x : [x \in A \text{ and } x \notin B] \text{ or } [x \in B \text{ and } x \notin A]\}$

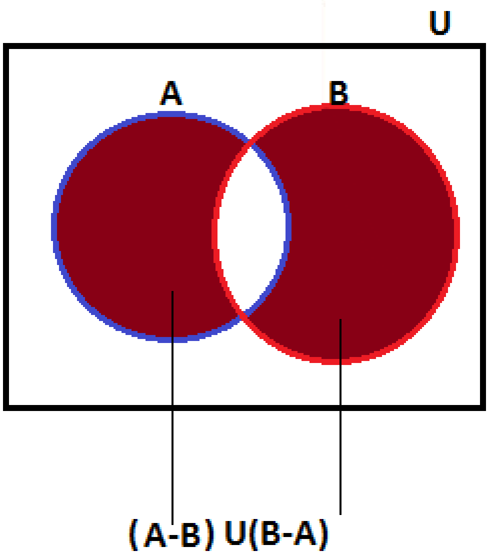


Figure 6: Venn diagram for symmetric difference

The shaded part of the given Venn diagram represents **A △ B**.

A △ B is the set of all those elements which belong either to A or to B but not to both.

A △ B is also expressed by $(A \cup B) - (B \cap A)$.

It follows that $A \triangle \emptyset = A$ for all subset A,

$A \triangle A = \emptyset$ for all subset A.

Example:

If A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 3, 5, 6, 7, 8, 9}.

then A – B = {2, 4}, B – A = {9} and A △ B = {2, 4, 9}.

Notes

The complement of set A

Let U be the universal set. The complement of the set A, denoted by A, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.

Remark: The definition of the complement of A depends on a particular universal set U.

This definition makes sense for any superset U of A. If we want to identify the universal set U, we would write "the complement of A with respect to the set U."

An element belongs to A if and only if $x \notin A$. This tells us that $\bar{A} = \{x \in U \mid x \notin A\}$.

Example:

(i) Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\bar{A}$ = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}.

(ii) Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\bar{A}$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

1.10 Laws of set theory:

Let A, B, and C are three finite subsets of universal set U. Then

- $A\cup\bar{A}=U$
- $A\cap\bar{A}=\emptyset$
- $\bar{\bar{A}}=A$
- $A\cup A=A$
- $A\cap A=A$
- $A\cup\emptyset=A$
- $A\cap\emptyset=\emptyset$
- $A\cup U=U$
- $A\cap U=A$
- $A\cup B=B\cup A$
- $A\cap B=B\cap A$
- $(A\cup B)\cup C=A\cup(B\cup C)$
- $(A\cap B)\cap C=A\cap(B\cap C)$
- $\overline{A \cap B}=\bar{A}\cup\bar{B}$
- $\overline{A \cup B}=\bar{A}\cap\bar{B}$
- $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
- $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
- $A\cap(A\cup B)=A$
- $A\cup(A\cap B)=A$

Proof of De Morgan's law: $\overline{\boldsymbol{A} \cup \boldsymbol{B}}=\bar{\boldsymbol{A}}\cap\bar{\boldsymbol{B}}$

Let P = $\overline{(A\ U\ B)}$and Q = $\bar{A}\cap\bar{B}$

Let x be an arbitrary element of P then $x \in P \Rightarrow x \in \overline{A \cup B}$

⇒ x ∉ (A U B)

⇒ x ∉ A and x ∉ B

⇒ x ∈ $\bar{A}$ and x ∈ $\bar{B}$

⇒ x ∈ $\bar{A}\cap\bar{B}$

⇒ x ∈ Q

Therefore, P ⊂ Q …………….. (i)

Again, let y be an arbitrary element of Q then y ∈ Q ⇒ y ∈ $\bar{A}\cap\bar{B}$

⇒ x ∈ $\bar{A}$ and x ∈ $\bar{B}$

⇒ y ∉ A and y ∉ B

⇒ y ∉ (A U B)

⇒ y ∈ $\overline{(A\ U\ B)}$

⇒ y ∈ P

Therefore, Q ⊂ P …………….. (ii)

Now combine (i) and (ii) we get; P = Q i.e. $\overline{A \cup B}$=$\bar{A}\cap\bar{B}$

Proof of De Morgan's law: $\overline{\boldsymbol{A} \cap \boldsymbol{B}}$=$\bar{\boldsymbol{A}}\cup\bar{\boldsymbol{B}}$

Let M = $\overline{A \cap B}$ and N = $\bar{A}\cup\bar{B}$

Let x be an arbitrary element of M then x ∈ M ⇒ x ∈ $\overline{(A\ \cap\ B)}$

⇒ x ∉ (A ∩ B)

⇒ x ∉ A or x ∉ B

⇒ x ∈ $\bar{A}$ or x ∈ $\bar{B}$

⇒ x ∈ $\bar{A}\cup\bar{B}$

⇒ x ∈ N

Therefore, M ⊂ N …………….. (i)

Again, let y be an arbitrary element of N then y ∈ N ⇒ y ∈ $\bar{A}\cup\bar{B}$

⇒ y ∈ $\bar{A}$ or y ∈ $\bar{B}$

⇒ y ∉ A or y ∉ B

⇒ y ∉ (A ∩ B)

⇒ y ∈ $\overline{(A\ \cap\ B)}$

Notes

$\Rightarrow y \in M$

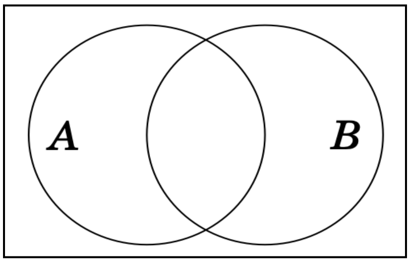
Therefore, $N \subset M$ …………….. (ii)

Now combine (i) and (ii) we get; M = N i.e. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

1.11 Principle of inclusion and exclusion:

Many counting problems involve computing the cardinality of a union A∪B of two finite sets. We examine this kind of problem now.

First, we develop a formula for $|A \cup B|$. It is tempting to say that $|A \cup B|$ must equal $|A|+|B|$, but that is not quite right. If we count the elements of A and then count the elements of B and add the two figures together, we get $|A|+|B|$. But if A and B have some elements in common, then we have counted each element in A∩B twice.



Therefore $|A|+|B|$ exceeds $|A \cup B|$ by $|A \cap B|$, and consequently

$|A \cup B| = |A| + |B| - |A \cap B|$.

Notice that the sets A, B, and A∩B are all generally smaller than A∪B. It is called the inclusion-exclusion formula because elements in A∩B are included (twice) in $|A|+|B|$, then excluded when $|A \cap B|$ is subtracted. Notice that if $A \cap B = \emptyset$ then we do in fact get

$|A \cup B| = |A| + |B|$

Conversely, if $|A \cup B| = |A| + |B|$ then it must be that $A \cap B = \emptyset$.

Example:

Among 50 patients admitted to a hospital, 25 are diagnosed with pneumonia, 30 with bronchitis, and 10 with both pneumonia and bronchitis. Determine:

(a) The number of patients diagnosed with pneumonia or bronchitis (or both).

(b) The number of patients not diagnosed with pneumonia or bronchitis.

Solution:

The first step is to formally identify the sets and indicate the number of elements in each. This can be done purely with the given information; No calculation is necessary. With this inclusion-exclusion principal question, the three sets can be defined as follows:

Let U denote the entire set of patients. Let P and B denote the set of patients diagnosed with pneumonia and bronchitis respectively. Thus:

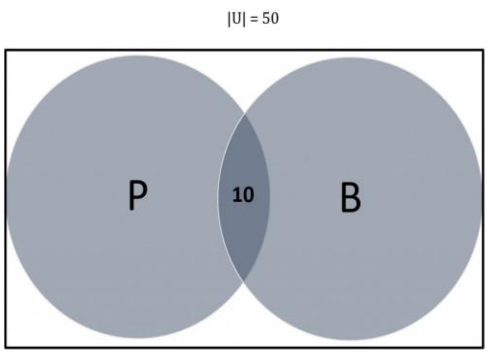
$|U| = 50$

$|P| = 25$

$|B| = 30$

$|P \cap B| = 10$

We may now create a Venn diagram. There are two sets and therefore two circles. Since we know the number of elements in the intersection of P and B ($|P \cap B|$) we can fill this in first:



Now we can calculate how many elements live only in P but not $|P \cap B|$:

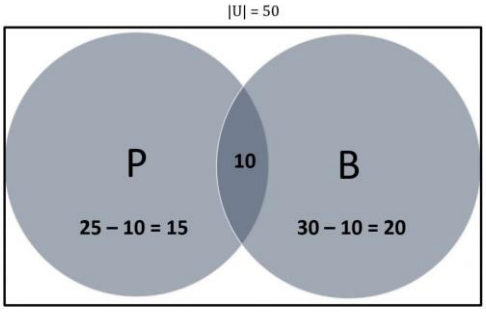
Since $|P| = 25$ and $|P \cap B| = 10$, there are 15 (25-10 = 15) elements exclusive to P.

Follow the same method to calculate the number of elements living only in B but not $|P \cap B|$:

Since $|B| = 30$ and $|P \cap B| = 10$, there are 20 (30-10 = 20) elements exclusive to B.

This new information should be added to our Venn diagram as follows:

Notes



The preliminary work is complete and we have enough information to answer the questions directly:

(a) This is the same as asking to determine |P ∪ B|. Looking at the Venn diagram, formulate the answer as follows:

|P ∪ B| = 15 + 10 + 20

= 45

Thus 45 patients are diagnosed with pneumonia or bronchitis.

The same answer can also be reached by using the inclusion-exclusion principle directly without referring to the Venn diagram:

|P ∪ B| = |P| + |B| – |P ∪ B|

= (25 + 30) – (10)

= 45

Thus 45 patients are diagnosed with pneumonia or bronchitis.

(b): This is the same as asking to determine |(P ∪ B)′|. We know that there are 50 patients altogether – of which 45 are diagnosed with pneumonia or bronchitis. Use this to solve the question:

|U| = 50.

|P ∪ B| = 45

Therefore, $|\overline{(P \cup B)}| = 50 - 45 = 5$

5 patients are not diagnosed with pneumonia or bronchitis.

Self-Assessment:

1. Let *A* be the set of students who live within one mile of a school and let *B* be the set of students who walk to classes.

Describe the students in each of these sets.

a) $A \cap B$ b) $A \cup B$ c) $A - B$ d) $B - A$

2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$.

Find a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$.

3.Write the interval (1, 4) in set-builder form.

4.Let A = {1, 2, 3, 4, 5}. Find the number of subsets and number of proper subsets of A.

5.Let A = {p, q, r, s, t}. Find the cardinality of the power set of A

6. Prove the complementation law $\overline{\overline{A}}$ =A

7. Prove the identity laws A∪∅=A and A∩U=A

8. Prove the domination laws A∩∅=∅ and A∪U=U

9. Prove the idempotent laws A∪A=A and A∩A=A

10. If X = {a, e, i, o, u} and Y = {a, b, c, d, e}, then what is Y - X?

11. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = {2, 4, 6, 8} and B = { 2, 3, 5, 7}. Verify that $\overline{A \cap B} = \bar{A} \cup \bar{B}$ and $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

12.If P = The set of whole numbers less than 5, Q = The set of even numbers greater than 3 but less than 9, R = The set of factors of 6. Then what is (P ∪ Q) ∩ (Q ∪ R)?

13. In a group of students, 100 students know Hindi, 50 know English, and 25 know both. Each of the students knows either Hindi or English. How many students are there in the group?

14. In a class of 120 students numbered 1 to 120, all even-numbered students opt for Physics, whose numbers are divisible by 5 opt for Chemistry, and those whose numbers are divisible by 7 opt for Math. How many opt for none of the three subjects?

15. The schedule of G first-year students was inspected. It was found that M was taking a Mathematics course, L was taking a Language course and B was taking both a Mathematics course and a Language course. Which of the following expression gives the percentage of the students whose schedule was inspected who were taking neither a mathematics course nor a language course?

Summary

This section deals with some basic definitions and operations involving sets. These are summarized below:

- A set is a well-defined collection of objects.
- A set that does not contain any element is called an empty set.
- A set that consists of a definite number of elements is called a finite set, otherwise, the set is called an infinite set.
- Two sets A and B are said to be equal if they have exactly the same elements.
- A set A is said to be a subset of a set B if every element of A is also an element of B.
- A power set of a set A is a collection of all subsets of A. It is denoted by P(A).
- The union of two sets A and B is the set of all those elements which are either in A or in B.
- The intersection of two sets A and B is the set of all elements which are common.
- The difference between two sets A and B in this order is the set of elements that belong to A but not to B.

Notes

- The complement of a subset A of universal set U is the set of all elements of U which are not the elements of A.
- For any two sets A and B, $\overline{A \cap B} = \bar{A} \cup \bar{B}$ and $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- If A and B are finite sets such that $A \cap B = \varphi$, then $n(A \cup B) = n(A) + n(B)$.
- If $A \cap B \neq \varphi$, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Keywords

Set: an unordered collection of distinct objects

The element of a set: an object in a set

Roster method: a method that describes a set by listing its elements

Set builder notation: the notation that describes a set by stating a property an element must have to be a member

The empty set, null set): the set with no members

Universal set: the set containing all objects under consideration

Venn diagram: a graphical representation of a set or sets

Set equality: S and *T* have the same elements

Subset: every element of *S* is also an element of *T*

Proper subset: *S* is a subset of *T* and $S \neq T$

Power set: the set of all subsets of *S*

Finite set: a set with *n* elements, where *n* is a non-negative integer

Infinite set: a set that is not finite

Cardinality: the number of elements in *S*

Further Readings

- Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education
- Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 02: Relations

CONTENTS

Purpose and Objectives:

Relations can be seen in a variety of scenarios. We work with relations regularly, such as those between a corporation and its phone number, a staff member and his or her paycheck, an individual and a relative, and so on. Relations may be found to resolve problems like deciding which combinations of cities are related by passenger flights in a network, or creating a feasible order for the various stages of a complex project, or creating a convenient way to preserve the information in databases. Functions may also be employed to describe how fast it takes a program to solve a specified problem. To measure the values of functions, several computation programs and operands are built. This section goes into the fundamentals of the relations and functions that are necessary for the field of discrete mathematics.

After this unit, you would be able to

- understand the relation on sets
- describes the different property of relations
- learn the functions as relations
- understand various types of functions and draw their graph

Introduction

Ordered pairs formed up of two related objects are the most straightforward way to describe a relationship between elements of two sets. As a result, sets of ordered pairs are referred to as binary relations. The basic language used to define binary relations is introduced in this section. The Cartesian product of sets is a concept that underpins many of the discrete constructs we'll look at in this section first and latter the relation and their properties and functions and their types.

2.1 Cross Product

Let A and B sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs $(a, b), where\ a \in A\ and\ b \in B. Hence, A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Example:

3 programming languages are offered to every student of a class for one semester

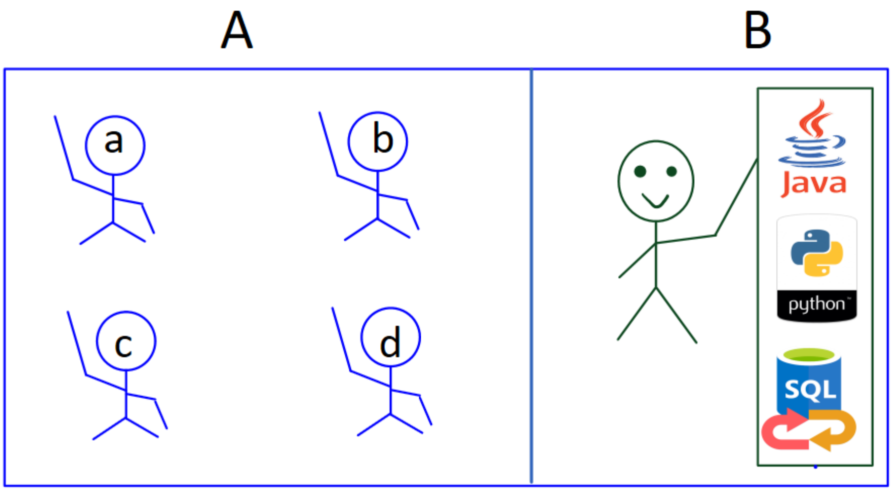


Figure 1: Represents two sets

Let A represent the set of all students in a class.

Let B represent the set of all programming languages offered

Then

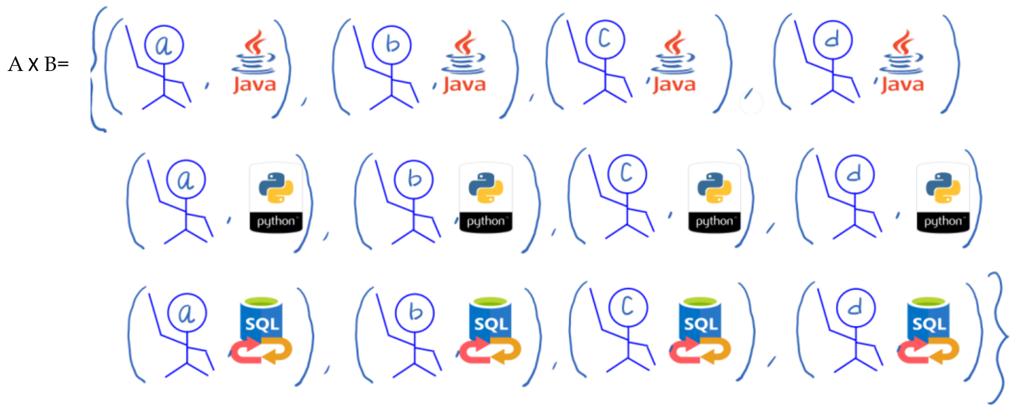


Figure 2: Cross product of set A and B

Cartesian products $A \times B \neq B \times A$ unless A = ∅ or B = ∅

Cartesian products$|A \times B| = |B \times A|$

2.2 Ordered n-tuple

The Cartesian product of the sets $A1, A2, \ldots, An$, denoted by $A1 \times A2 \times \cdots \times An$, is the set of ordered n-tuples $(a1, a2, \ldots, an)$, where ai belongs to Ai for i = 1, 2,...,n.

In other words, $A1 \times A2 \times \cdots \times An = \{(a1, a2, \ldots, an) \mid ai \in Ai \; for \; i = 1, 2, \ldots, n\}$

2.3 Relation

Let A and B set. A binary relation from A to B is a subset of $A \times B$. Hence $R \subseteq A \times B$

Example:

$Let\ A = \{1,2,3,4\}\ and\ B = \{0,5,10,15\}$ and let R be a relation from set $A\ to\ B$ is defined as

R= $\{(a,b): a^2 + b^2 \leq 20\}$ then

$$A \times B = \begin{Bmatrix} (1,0),(2,0),(3,0),(4,0), \\ (1,5),(2,5),(3,5),(4,5), \\ (1,10),(2,10),(3,10),(4,10), \\ (1,15),(2,15),(3,15),(4,15) \end{Bmatrix}$$

And

$$R = \{(1,0),(2,0),(3,0),(4,0)\}$$

2.4 Relation on a set

A relation on a set A is a relation from A to A. $R \subseteq A \times A$

Example:

Let A be the set $\{1, 2, 3, 4,5,6\}$. Which ordered pairs are in the relation $R = \{(a, b)\text{: } a\ divides\ b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 6 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1,5), (1,6), (2, 2), (2, 4), (2,6)(3, 3), (3,6), (4,4), (5,5), (6,6)\}.$$

Counting of Relation

Let A and B are two sets with $|A| = m, |B| = n$ then the total number of relations from set A to B is 2^{mn}

Let A and $|A| = n$, then the total number of relations from set A to A is 2^{n^2}

Example:

$Let\ A = \{a\}\ and\ B = \{1,2\}$ then

$$A \times B = \{(a,1),(a,2)\}$$

And $R_1 = \{(a,1)\}, R_2 = \{(a,2)\}, R_3 = \{(a,1),(a,2)\}, R_4 = \{\ \}$ are the 4 relations from set A to B.

2.5 Properties of Relations

To characterize relations on a set, many properties are used. The most significant of these will be discussed first.

Reflexive

A relation R on a set A is called reflexive if (a, a) ∈ R for every element a ∈ A.

Example:

Let $R_1, R_2, R_3, R_4, and\ R_5$ be the relations on $\{u, v, w, x\}$:

$R_1 = \{(u,u),(u,v),(v,u),(v,v),(w,x),(x,u),(x,x)\}.$

$R_2 = \{(u,u),(u,v),(v,u)\}$

$R_3 = \{(u,u),(u,v),(u,x),(v,u),(v,v),(w,w),(x,u),(x,x)\}$

$R_4 = \{(v,u),(w,u),(w,v),(x,u),(x,v),(x,w)\}$

$R_5 = \{(u,u),(u,v),(u,w),(u,x),(v,v),(v,w),(v,x),(w,w),(w,x),(x,x)\}$

The relations R_3 and R_5 are reflexive because they both contain all pairs of the form *(a, a)*, namely, *(u, u), (v, v), (w, w), and (x, x).*

The other relations are not reflexive because they do not contain all these ordered pairs. In particular, because *(w, w)* is not in any of these relations.

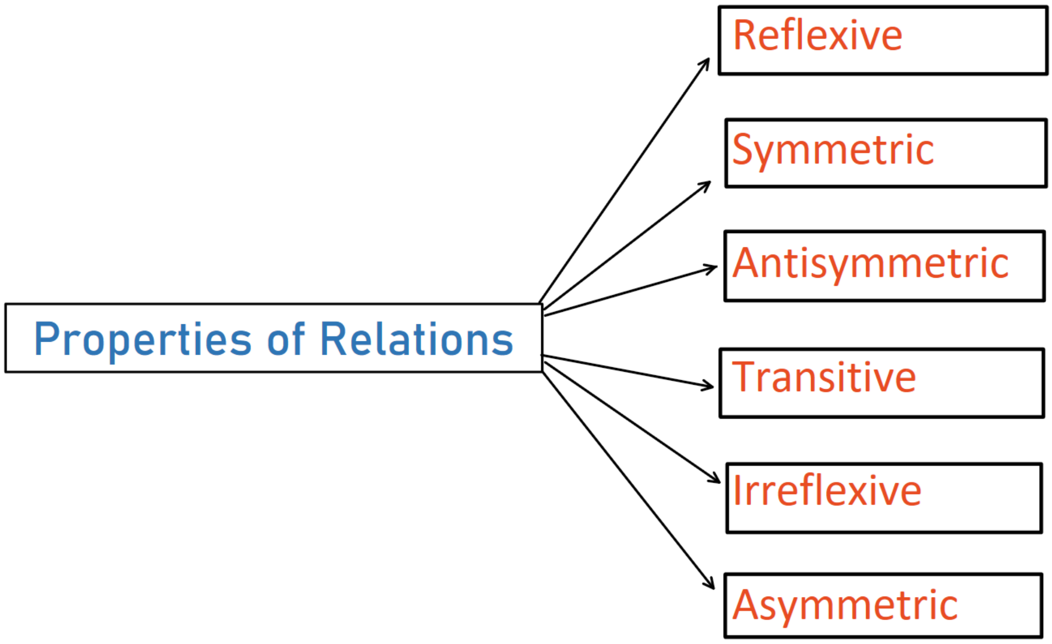


Figure 3: Different properties of the relation

Symmetric

A relation R on a set A is called symmetric $if\ (b,a)\ \in\ R\ whenever(a,b)\ \in\ R, for\ \forall\ a,b\ \in\ A.$

Example:

$R_A, and\ R_B$ be the relations on $\{u,v,w,x\}$:

$R_A = \{(u,u),(u,v),(v,u)\}$

$R_B = \{(u,u),(u,v),(u,x),(v,u),(v,v),(w,w),(x,u),(x,x)\}$

R_A is symmetric, because in each case $(a,b) and\ (b,a) \epsilon R_A, \forall\ a,b \in \{u,v,w,x\}$.

R_B is symmetric, because in each case $(a,b) and\ (b,a) \epsilon R_B, \forall\ a,b \in \{u,v,w,x\}$.

Antisymmetric

A relation R on a set A such that for all $a,b\ \in\ A, if\ (a,b)\ \in\ R\ and\ (b,a)\ \in\ R, then\ a\ =\ b$ is called antisymmetric.

Example:

$R_P, R_Q and\ R_S$ be the relations on $\{u,v,w,x\}$:

$R_P = \{(u,u),(v,v),(w,w),(x,x)\}$

$R_Q = \{(v,u),(w,u),(w,v),(x,u),(x,v),(x,w)\}$

$R_S = \{(u,u),(u,v),(u,w),(u,x),(v,v),(v,w),(v,x),(w,w),(w,x),(x,x)\}$

R_P is antisymmetric because for all $a,b \in \{u,v,w,x\}, if\ (a,b) \in R\ and\ (b,a) \in R, then\ a = b$

$R_Q and\ R_S$ are also antisymmetric. For each of these relations, there is no pair of elements such that, $if\ (a,b) \in R\ and\ (b,a) \in R, then\ a = b$.

Transitive

A relation R on a set A is called $transitive\ if\ whenever\ (a,b) \in R\ and\ (b,c) \in R, then\ (a,c) \in R, for\ all\ a,b,c \in A$.

Example:

$R_x, R_y and\ R_z$ be the relations on $\{u,v,w,x\}$:

$R_x = \{(u,v)\}$

$R_y = \{(v,u),(w,u),(w,v),(x,u),(x,v),(x,w)\}$

R_x is transitive because for all $a,b \in \{u,v,w,x\}$, $only\ (a,b) \in R$. The element v is not related to any other element of $\{u,v,w,x\}$.

R_y is transitive, because *(w, v) and (v, u), (x, v) and (v, u), (x, w) and (w, u), and (x, w) and (w, v)* are the only such sets of pairs, and *(w, u), (x, u), and (x, v)* belongs to R_y.

Irreflexive

A relation R on the set A is irreflexive if for every a ∈ A, (a, a) does not belong R. That is, R is irreflexive if no element in A is related to itself.

Example:

R_t, be the relations on $\{u,v,w,x\}$:

$R_t = \{(u,v),(u,w)\}$

Asymmetric

A relation R is called asymmetric if *(a, b) ∈ R* implies that *(b, a)* does not belong to R.

Example:

R_u, be the relations on $\{u,v,w,x\}$:

$R_u = \{(u,v),(u,w)\}$

2.6 Function

A function relates an input to an output.

Definition

A relation " f "from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

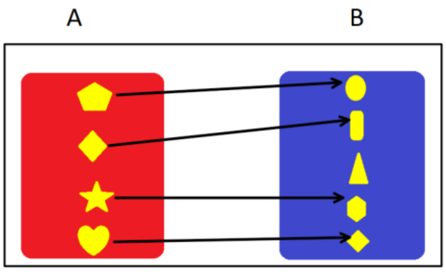


Figure 4:The Function f Maps A to B

In the function one-to-many" is not allowed, but "many-to-one" is allowed it can be checked by vertical line test.

In Figure 5 it can be seen that the vertical line crosses the left curve at more than one point it means there are two outputs (y values) for the same input (x value) which is not allowed in the function. The right curve has exactly one output (y value) for every input(x value)

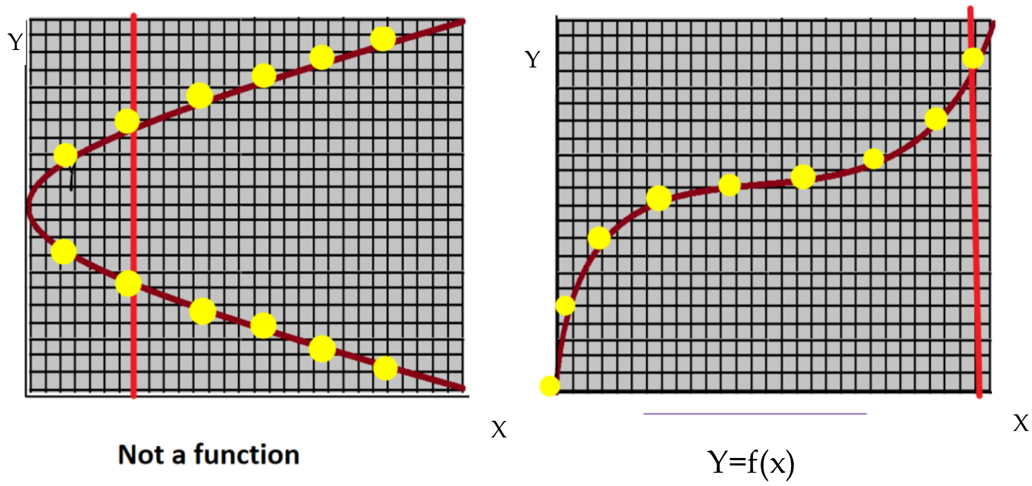


Figure 5: Vertical line test for function

Domain, Codomain, Range

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.

If f (a) = b, we say that b is the image of a and a is a preimage of b.

The range, or image, of *f* is the set of all images of elements of A.

Example:

Let A= {a, b, c} and B= {1, 2, 3, 4} $and\ f: A \to B = \{(a, 1), (b, 2), (c, 3)\}$ then the domain is {a, b, c} ,the codomain is {1, 2, 3, 4}, the range is {1, 2, 3}

Let A and B are two finite sets having m and n elements respectively then total no. of function from set A to B is n^m.

2.7 Types of the function

Constant function

The function f is called constant function if every element of A has the same image in B. Range of a constant function is a singleton set.

Example:

Let A = {x, y, u, v}, B = {2, 3, 5, 7, 11}. The function $f: A \to B$ defined by f (x) = 5 for every x belonging to A is a constant function.

Identity function

The function f: R → R defined by y = f (x) = x for each x ∈ R is called the identity function. That is, an identity function maps each element of A into itself.

Let A be the set of real numbers (R). The function $f: R \to R$ be defined by f (x) = x for all x belonging to R is the identity function on R.

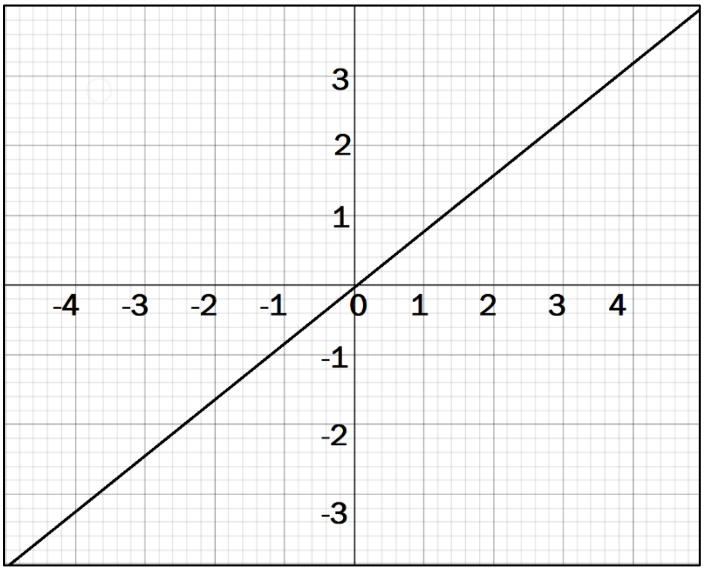


Figure 6: Graph of the identity function

The Modulus Function

The modulus of any number gives us the magnitude of that number. Using the modulus operation, we can define the modulus function as follows: $f(x) = |x|$.

$$f(x) = \begin{Bmatrix} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{Bmatrix}$$

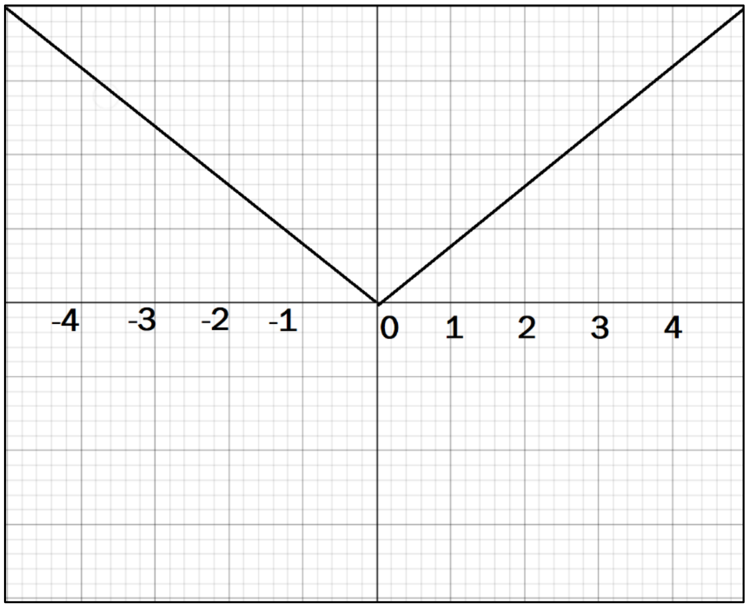


Figure 7:Graph of Modulus function

Greatest integer function

The real function *f: R → R defined by f (x) = [x], x ∈R* assumes the value of the greatest integer less than or equal to x, is called the greatest integer function.

Example:

[5.5] =5

Polynomial Function

A real valued function f: R → R defined by $y = f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots, a_nx^n$. where n ∈ N, $a_0, a_1, a_2 \cdots a_n \in R$ and for each x ∈ R, is called Polynomial functions.

Example:

$y = f(x) = 5x^0 + 10x^1 + 15x^2$, $\forall x \in [0,10]$.

One-one function

A function f: A → B is defined to be one-one (or injective) if distinct elements of A have a different image in B. $i.e., for\ every\ x_1, x_2 \in X, f(x_1) = f(x_2)\ implies\ x_1 = x_2$

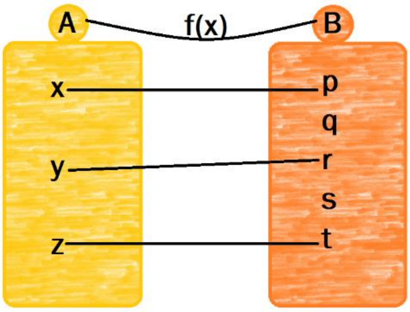


Figure 8:A One-to-One Function.

Example:

$f\ (x)\ =\ x^3$ one-one where f: R→R

Many-one

A function f: A → B is defined to be many one if there exist two or more than 2 elements in set A has the same image in set B

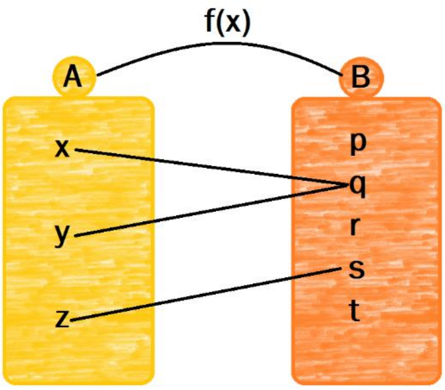


Figure 9: Many one function

Example:

$f\ (x)\ =\ Sinx$ Many-one where f: R→R .

Onto

A function f: A → B is defined to be onto (or surjective), if every element of B is the image of some element of A under f.

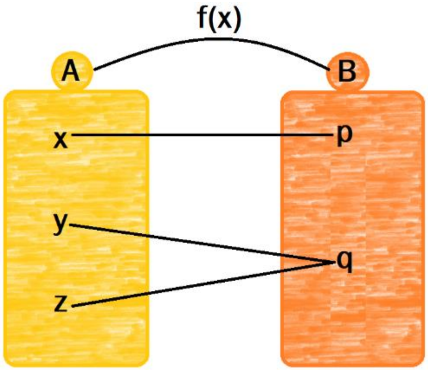


Figure 10: An onto function

Example:

Let f be the function from *{P, Q, R, S} to {10, 20, 30}* defined by

f (P) = 10, f (Q) = 20, f (R) = 10, and f (S) = 30 is an onto function.

One-one and onto

A function f: A → B is said to be one-one and onto (or bijective), if f is both one-one and onto.

Example:

$$g(x) = \begin{Bmatrix} x-1, x \text{ is odd} \\ x+1, x \text{ is even} \end{Bmatrix}$$

Self-Assessment

1. Find the relation R from $A = \{0, 1, 2, 3, 4,5 ,6,7\}$ to $B = \{0, 1, 2, 3,4\ 5, 6,7,8\}$, $where$ $(a, b) \in R$ if and only if $a^2 = b$

2. Let $A = \{x: 0 \le x \le 10, \forall x \in \mathbb{N}\}$, $B = \{x: 0 \le x + 1 \le 15, \forall x \in \mathbb{N}\}$, and R $be\ the\ relation$ $from\ A\ to\ B\ with\ (a, b) \in R$ if only if) $gcd(a, b) = 1$. Then find R.

3.Let *R={(p, p), (p, q), (q, p), (q, q), (r, r), (s, s), (t, q), (t, t), (u, u)}* be the relations on the set $\{p, q, r, s, t, u\}$, check whether it is reflexive

4.Let *R={(p, p), (p, q), (q, p), (q, q), (r, r), (s, s), (t, t), (u, u)}* be the relations on the set $\{p, q, r, s, t, u\}$, check whether it is symmetric.

5.Let *R={(p, r), (p, q), (q, r), (q, q), (r, r), (s, s), (t, t), (u, u)}* be the relations on the set $\{p, q, r, s, t, u\}$, check whether it is antisymmetric.

6.Let *R={(p, r), (p, q), (q, r), (q, q), (r, r), (s, s), (t, t), (u, u)}* be the relations on the set $\{p, q, r, s, t, u\}$, check whether it is transitive.

7. R = {(a, b) | a divides b} be the relations on the set of natural number, check whether it is irreflexive.

8. The binary relation {(a,a), (b,a), (b,b), (b,c), (b,d), (c,a) (c,b)} on the set {a, b, c} is ___________

(a) Reflective, symmetric, and transitive
(b) Irreflexive, symmetric and transitive
(c) Neither reflexive nor irreflexive but transitive
(d) Irreflexive and antisymmetric

9.If an operation is defined by $p * q = p^2 + q^2$, then $(1 * 2) * 6$ is

(a) 12
(b) 28

(c) 61
(d) None of these

10.The range of the function f(x) = $\sqrt{(x-1)(x-3)}$is
(a) [1, 3]
(b) [0, 1]
(c) [-2, 2]
(d) None of these

11.If a relation R on the set {x, y, z} be defined by R = {(x, y)}, then R is
(a) reflexive
(b) transitive
(c) symmetric
(d) None of these

12. Let $f: R \rightarrow R$ $be\ defined\ by\ f\ (x) = \frac{1}{(x-1)} \forall\, x \in R$. Then f is
(a) one-one
(b) onto
(c) bijective
(d) f is not defined

13.Let the function f (x) = $8x^2 + 2$ ∀ x ∈ R, then check whether it is onto.

14.Let the function f (x) = 8x ∀ x ∈ R, then check whether it is one-one.

15. . Let $f: R \rightarrow R$ $be\ defined\ by\ f\ (x) = \frac{1}{(x-5)} \forall\, x \in R$. Then domain of f is

Answers:

8	**c**	**9**	**c**	**10**	**d**	**11**	**b**	**12**	**d**

Summary

This section deals with some basic definitions and operations involving sets. These are summarized below:

- The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs $(a, b), where\ a \in A\ and\ b \in B$. A
- A relation on a set A is a relation from A to A. $R \subseteq A \times A$
- A relation R on a set A is called reflexive if (a, a) ∈ R for every element a ∈ A.
- A relation R on a set A is called symmetric $if\ (b, a) \in R\ whenever (a, b) \in R, for\ \forall\, a, b \in A$.
- A relation R on a set A such that for all $a, b \in A, if\ (a,b) \in R\ and\ (b,a) \in R, then\ a = b$ is called antisymmetric.
- A relation R on a set A is called $transitive\ if\ whenever\ (a,b) \in R\ and\ (b,c) \in R, then\ (a,c) \in R, for\ all\ a, b, c \in A$.
- A relation R on the set A is irreflexive if for every a ∈ A, (a, a) does not belong R. That is, R is irreflexive if no element in A is related to itself.
- A relation R is called asymmetric if *(a, b) ∈R* implies that *(b, a)* does not belong to R.
- A relation " f "from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.
- The function f is called constant function if every element of A has the same image in B. Range of a constant function is a singleton set.

- The function f: R → R defined by y = f (x) = x for each x ∈ R is called the identity function. That is, an identity function maps each element of A into itself.
- The modulus of any number gives us the magnitude of that number. Using the modulus operation, we can define the modulus function as follows*: f(x)*= | *x* | *.*
- The real function *f: R → R defined by f (x) = [x], x ∈R* assumes the value of the greatest integer less than or equal to x, is called the greatest integer function.
- A real valued function f: R → R defined by $y = f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots, a_nx^n$. where n ∈ N, $a_0, a_1, a_2 \cdots a_n \in R$ and for each x ∈ R, is called Polynomial functions.
- A function f: A → B is defined to be one-one (or injective) if distinct elements of A have a different image in B. $i.e., for\ every\ x_1, x_2 \in X, f(x_1) = f(x_2)\ implies\ x_1 = x_2$
- A function f: A → B is defined to be many ones if there exist two or more than 2 elements in set A has the same image in set B
- A function f: A → B is defined to be onto (or surjective) if every element of B is the image of some element of A under f.
- A function f: A → B is said to be one-one and onto (or bijective), if f is both one-one and onto.

Keywords

Binary relation from A to B: a subset of A × B relation on A: a binary relation from A to itself (i.e., a subset of A × A).

Reflexive: a relation R on A is reflexive if (a, a) ∈ R for all a ∈ .

Symmetric: a relation R on A is symmetric if(b, a) ∈ R whenever (a, b) ∈ R

Antisymmetric: a relation R on A is antisymmetric if a = b whenever (a, b) ∈ R and (b, a) ∈ R.

Transitive: a relation R on A is transitive if (a, b) ∈ R and (b, c) ∈ R implies that (a, c) ∈ R.

Function from A to B: an assignment of exactly one element of B to each element of A domain of f: the set A, where f is a function from A to B.

The codomain of f: the set B, where f is a function from A to B b is the image of an under f: b = f (a) a is a pre-image of b under f: f (a) = b.

Range of f: the set of images of f onto function.

A one-to-one(surjection) function: a function from A to B such that every element of B is the image of some element in A one-to-one function.

Onto(injection): a function such that the images of elements in its domain are distinct one-to-one correspondence.

Bijection: a function that is both one-to-one and onto

Further Readings

- Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education
- Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 03: Introduction to Logic

CONTENTS

Objectives:

Logic is one of humankind's oldest cognitive areas. It can be traced back to Aristotle. People included Leibniz, Boole, Russell, Turing, and others have learned it over the years. It ‘still a topic of active inquiry now a day. In "logic-enabled" computing systems, where users can display and modify logical sentences, logic is also growing more popular at the human-machine interface. Consider email readers that enable users to write rules to handle incoming mail messages depending on message attributes, such as deleting some, sending some to other mailboxes, and so on.

After this unit, you would be able to

- understand the valid and invalid argument to form a logic
- describes the different propositions, compound preposition, and negation of preposition
- learn the conditional, biconditionalstatement, and inverse, converse, and contrapositive statement

Introduction

Mathematical statements are given specific interpretations by logic laws. These laws are used to determine which mathematical statements are true and which are false. We begin our review of discrete mathematics with an introduction to logic since one of the main goals of this section to understand and develop valid mathematical arguments. Logic has many uses in computer science, in addition to its usefulness in explaining mathematical reasoning. These rules are used in a variety of areas, including the design of electronic circuits, the development of computer systems, and the checking of software correctness.

3.1 logic

Logic is thesystematic study of valid rules of inferences. More broadly logic is the analysis and appraisal of arguments.

Rules of inferences:

A relation that leads to the acceptance of some conclusion based on some premises or a hypothesis.

LetX and Y are two friends and the age of X10-year, while Y is five-year-old hence X is older than Y.So, we can frame a hypothesis *H*, logic *L*, and argument *A*that can be valid or invalid.

H: X is taller than Y

L:Footsteps of tallboys are larger than smaller ones

Now let us consider one argument

A: The X has larger footsteps than Y.

So based on hypothesis *H,* we can conclude the logic *L*,and using the logic *L* the argument *A* is valid.

The foundation of logic is the propositions so we will focus on it.

3.2 Propositions

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Example

1. Covid-19 was originated in china
2. Taj Mahal was built on the southern bank of the river Yamuna.
3. 2+10=12
4. 5+5=12
5. How are you?
6. $x^2 + y^2 = z^2$

Some of the above statements are decelerating the fact and are proposition can be seen in the following table

Table 1: Proposition and their truth value

Statement	Declaring the fact	Proposition	Promotional value
Covid-19 was originated in china	Yes	Yes	True
Taj Mahal built on the southern bank of the river Yamuna	Yes	Yes	True
2+10=12	Yes	Yes	True
5+5=12	Yes	Yes	False
How are you?	No	No	-
$x^2 + y^2 = z^2$	No	No	-

The true truth value of a proposition is denoted by T and the false truth value is denoted by F.

3.3 Compound propositions

A variety of mathematical arguments are made up of one or more propositions. Logic operators are used to creating new propositions termed compound propositions from current propositions.

Negation of the proposition

Let p be a proposition. The negation of p, denoted by ¬p (also denoted by p), is the statement "It is not the case that p."

The proposition ¬p is read "not p." The truth value of the negation of p, ¬p, is the opposite of the truth value of p.

Example

1. "Himanshu's PC runs Linux"
   p: "Himanshu's PC runs Linux"
   ¬p: "It is not the case that Himanshu's PC runs Linux."
   Or
   ¬p: "Himanshu's PC does not run Linux."

2. "Himanshu's smartphone has at least 50GB of memory"
   p: Himanshu's smartphone has at least 50GB of memory"
   ¬p: "It is not the case that Himanshu'ssmartphone has at least 50GB of memory"
   Or
   ¬p: "Himanshu'ssmartphone does not have at least 32GB of memory"
   Or
   ¬p: "Himanshu's smartphone has less than 32GB of memory."

Truth table Negation of the proposition

The truth table for the negation of a proposition p is seen in Table 2.

Table 2: Truth table of Negation of proposition

p	¬p
T	F
F	T

The conjunction of proposition p and q

Let p and q be propositions. The conjunction of p and q, denoted by p ∧ q, is the proposition "p and q." The conjunction p ∧ q is true when both p and q are true and is false otherwise.

Table 3 displays the truth table for p ∧ q.

Table 3: the truth table for p ∧ q.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Example:

Let p and q be the propositions
p:Umesh drives over 50km per hour.
q:He gets a speeding ticket.
p ∧ q: Umesh drive over 50km per hour and he gets a speeding ticket.

The disjunction of proposition p and q

Let p and q be propositions. The disjunction of p and q, denoted by p ∨ q, is the proposition "p or q." The disjunction p ∨ q is false when both p and q are false and is true otherwise.
Table 4 displays the truth table for p ∨ q.

Table 4: the truth table for p ∨ q.

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Example Let p and q be the propositions
p: Umesh drives over 50 km per hour.
q: He gets a speeding ticket.
p ∨ q: Umesh drive over 50 km per hour or he gets a speeding ticket.

The exclusive or of p and q

Let p and q be propositions. The exclusive or of p and q, denoted by p ⊕ q, is the proposition that is true when exactly one of p and q is true and is false otherwise.
Table 5 displays the truth table for p ⊕ q.

Table 5: the truth table for p ⊕ q.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example:

Students who have taken Mathematics or computer science major can take this class.

Conditional Statements

Consider the propositions p and q. The proposition "if p, then q" is the conditional declaration $p \rightarrow q$. If p is true and q is false, the conditional assertion $p \rightarrow q$ is false; otherwise, it is true.

The theory is referred to as p in the conditional declaration $p \rightarrow q$. (or antecedent or premise). The conclusion is referred to as q. (or consequence).

Table 6 displays the truth table for $p \rightarrow q$.

Example:

Let p and q be the propositions

p: Umesh drives over 50km per hour.

q: He gets a speeding ticket.

$p \rightarrow q$: If Umesh drives over 50km per hour then he gets a speeding ticket.

Table 6: the truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse

Consider the propositions p and q. The converse of conditional statement $p \rightarrow q$ is the proposition $q \rightarrow p$

Contrapositive

Consider the propositions p and q. The contrapositive of conditional statement $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

Inverse

Consider the propositions p and q. The inverse of conditional statement $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$.

Consider the propositions P and Q

P: My score is more than 80.

Q: I will get a new bike.

$P \rightarrow Q$: If my score is more than 80 then I will get a new bike

Converse ($q \rightarrow p$): If I will get a new bike then my score is more than 80.

Contrapositive ($\neg q \rightarrow \neg p$): If I will not get a new bike then my score is not more than 80.

Inverse ($\neg p \rightarrow \neg q$): If my score is not more than 80 then I will not get a new bike.

The biconditional statement

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise.

Example:

Consider the propositions P and Q

P: My score is more than 80.

Q: I will get a new bike.

$P \leftrightarrow Q$: My score is more than 80 if and only I will get a new bike

Table 7 displays the truth table for $p \leftrightarrow q$.

Table 7: the truth table for $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Summary

Logic is the systematic study of valid rules of inferences. More broadly logic is the analysis and appraisal of arguments.

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

The negation of p, denoted by $\neg p$ (also denoted by p), is the statement "It is not the case that p."

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

The proposition "if p, then q" is the conditional declaration $p \rightarrow q$. If p is true and q is false, the conditional assertion $p \rightarrow q$ is false; otherwise, it is true

The converse of conditional statement $p \rightarrow q$ is the proposition $q \rightarrow p$

The contrapositive of conditional statement $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

The inverse of conditional statement $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$.

The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise.

Keywords

Proposition: a statement that is true or false propositional.

$\neg$ p (negation of p): the proposition with truth value opposite to the truth value of p.

p $\wedge$ q (conjunction of p and q): the proposition "p and q," which is true if and only if both p and q are true.

p $\vee$ q (disjunction of p and q): the proposition "p or q," which is true if and only if at least one of p and q is true

p $\oplus$ q (exclusive or of p and q): the proposition "p XOR q," which is true when exactly one of p and q is true

p$\rightarrow$ q (p implies q): the proposition "if p, then q," which is false if and only if p is true and q is false

Self Assessment

1.What is the negation of the statement "Salman sent more than 100 text messages every day"?

(A) Salman sent more than 200 text messages every day.

(B) Salman sent less than 100 text messages but not every day.

(C) Salman did not send more than 100 text messages every day.

(D) Salman did not send any text messages every day.

2.Let Q(x, y) denote the statement "y is the capital of x." What are the truth values of i) Q(Punjab, Chandigarh), ii) Q(India, New Delhi) iii) Q(Rajasthan, Shimla), iv) Q(Nepal, Kathmandu)?

(A) T,F,T,F

(B) T,T,F,F

(C) T,T,F,T

(D) T,T,T,T

3.Let p and q be the propositions,p: Umesh drives over 50 km per hour. q: He gets a speeding ticket. Then which one of the following is correct for p and q?

(A) Umesh drives over 50 km per hour and he gets a speeding ticket.

(B) Umesh did not drive over 50 km per hour and he gets a speeding ticket.

(C) Umesh drives over 50 km per hour and he did not get a speeding ticket.

(D) Umesh drives over 50 km per hour or he gets a speeding ticket.

4.What will be the Truth values of the statement $p \leftrightarrow \neg p$ for the Truth values T, F of p?

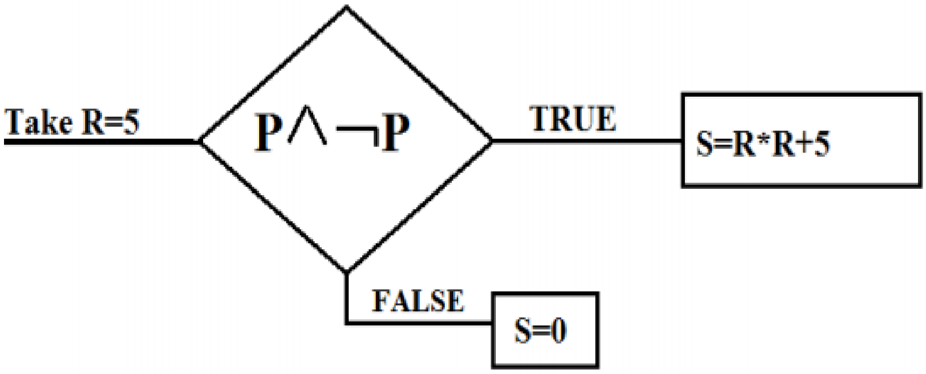
(A) T, F

(B) F, T

(C) T, T

(D) F,F

5. Let P: Dogs can fly, and consider the following flow chart of a computer program then what is the value of S



(A) 20

(B) 30

(C) 0

(D) 25

6. What will be the Truth values of the statement (p ∧ q) → (p ∨ q) for the Truth values T, T, F, F of p and T, F, T, F of q?

(A) T, F,T,F

(B) F, T,F,T

(C) T, T, T, T

(D) F,F,F,F

7. If P: "You can use the wireless network in the airport," Q: "You pay the daily fee,".Which is the right expression for the statement "To use the wireless network in the airport you must pay the daily fee ".

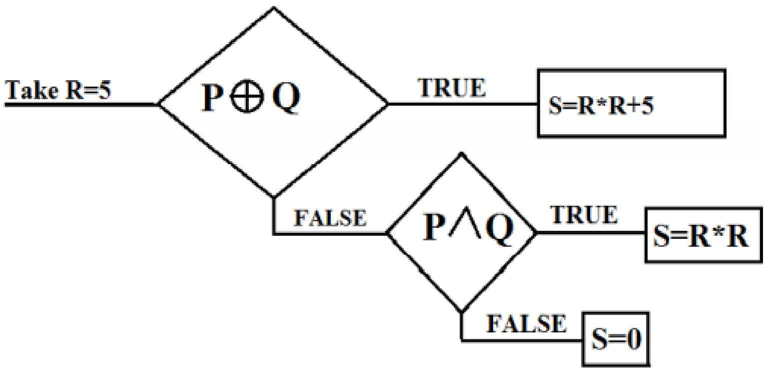
(A) Q→P

(B) P→Q

(C) Q∧P

(dD) Q∨P

8. Let P: 5+10=15, Q: 5*10=50.And consider the following flow chart of a computer program then the value of S is?



(A) 30

(B) 25

(C) 0

(D)10

9. The converse of the statement "If you are honest, then you are respected."

(A) If You are honest then he is not respected.

(B) If You are not respected then you are not honest.

(C) If you are not honest then you are not respected.

(D) If you are respected then you are honest.

10. The inverse of the statement "If you are honest, then you are respected."

(A) If You are honest then he is not respected.

(B) If You are not respected then you are not honest.

(C) If you are not honest then you are not respected.

(D) If you are respected then you are honest.

11.The contapositive of the statement "If you are honest, then you are respected."

(A) If You are honest then he is not respected.

(B) If You are not respected then you are not honest.

(C) If you are not honest then you are not respected.

(D) If you are respected then you are honest.

12. The Converse of the statement "If you are lucky, then you will get a ticket."

(A) If you are lucky then you will get a ticket.

(B) If you are not lucky then you will not get a ticket.

(C) If you will get a ticket then you are lucky.

(D) If you are respected then you are honest.

13.Consider the logic *L*: Footsteps of tallboys are larger than smaller ones then Which one of the following arguments is valid?

(A) The height of Ramesh is 6.5 feet while the height of Rakesh is 6 feet hence footsteps of Ramesh are larger than Rakesh

(B) Delhi is the capital of India hence footsteps of Ramesh are larger than Rakesh

(C) Kolkata is the capital of India hence footsteps of Ramesh are not larger than Rakesh

(D) Ramesh and Rakesh are friends hence footsteps of Ramesh are larger than Rakesh

14. Which one of the following is a proposition?

(A) How are you?

(B) What time is it?

(C) India is in Europe.

(D) 4+x=10

15.What is the truth value of the proposition “5+10=15”?

True

False

Answer for self Assessment

1.	C	2.	C	3.	A	4.	D	5.	C
6.	C	7.	A	8.	B	9.	D	10.	C
11.	B	12.	C	13.	A	14.	C	15.	True

Review Questions:

1) Which of these statements are propositions?
   a. 10+5=15
   b.2*25=40
   c. How are you?
2) What is the truth value of the following propositions?
   a. $2^5 = 10$
   b. Chandigarh is the capital of Punjab
3) What is the negation of the following propositions?
   a. His computer has more than 32 GB of memory
   b. Metro cities are more polluted than villages
4) Let P and Q are propositions
   P: Ramesh is ill.
   Q: Ramesh misses the final exam.
   Write the conjunction of P and Q as an English sentence
5) Let P and Q are propositions
   P: Ramesh is a clever boy.
   Q: Ramesh performs well in the final exam.
   Find the truth value ofthe proposition “Ramesh is not a clever boy and he performs well in the final exam”.
6) Let P and Q are propositions
   P: Julieis a computer science student.
   Q: Julie is doing the job in a software company.
   Draw the truth table for $\neg p \wedge \neg q$
7) Let P and Q are propositions
   P: Raman is a tallboy.
   Q: Raman performs well in the high jump.
   Express the proposition as an English sentence $\neg P \wedge Q$
8) Let P and Q are propositions
   P: Ramesh is ill.
   Q: Ramesh misses the final exam.
   Write the disjunction of P and Q as an English sentence
9) Let P and Q are propositions
   P: Ramesh is a clever boy.
   Q: Ramesh performs well in the final exam.
   Find the truth value of the proposition “Ramesh is not a clever boy or he performs well in the final exam”.

10) Let P and Q are propositions

    P: Julie is a computer science student.

    Q: Julie is doing the job in a software company.

    Draw the truth table for $\neg p \vee \neg q$

11) Let P and Q are propositions

    P: Raman is a tallboy.

    Q: Raman performs well in the high jump.

    Express the proposition as an English sentence $P \vee \neg Q$

12) Let P and Q are propositions

    P: Ramesh is ill.

    Q: Ramesh misses the final exam.

    Write the $P \rightarrow Q$of P and Q as an English sentence

13) Let P and Q are propositions

    P: Ramesh is a clever boy.

    Q: Ramesh performs well in the final exam.

    Find the truth value of the proposition "If Ramesh is not a clever boythen he did not perform well in the final exam".

14) Let P and Q are propositions

    P: Julie is a computer science student.

    Q: Julie is doing the job in a software company.

    Draw the truth table for contrapositive statement of $P \rightarrow Q$

15) Let P and Q are propositions

    P: Raman is a tallboy.

    Q: Raman performs well in the high jump.

    Find the converse of the conditional statement $P \rightarrow Q$

Further Readings

- Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education
- Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 04: Tautologies and Contradiction

CONTENTS

Objectives:

The substitution of a statement with another argument of the same truth meaning is an essential form of movement in a mathematical argument. As a result, in the development of mathematical statements, methods that yield propositions with the same truth value as a given compound proposition are often used. We've now covered conjunctions, disjunctions, conditional statements, and biconditional statements. Such connectives could be able to create sophisticated compound propositions for any combination of propositional variables. Our topic starts with a description of the truth table for compound propositions based on their conceivable truth values.

After this unit, you would be able to

- develop the truth table for a compound proposition.
- describes the tautologies and contradiction for a compound proposition.
- form a logical equivalence between two or more compound propositions for all possible cases

Introduction

A truth table may be used to see if a compound proposition is relatively accurate and executed. In a mathematical argument, a step is the replacement of one assertion with another claim that has the equivalent truth value. As a consequence, methods that produce propositions with the identical truth value as a given composite proposition are even sometimes used in the construction of logical statements. The final column of the table contains the truth values of the compound proposition for each combination of truth values of the propositional variables in it.

4.1 Truth Tables of Compound Propositions

We can construct compound propositions using two or more propositional variables. Every proposition has two possible truth-value i.e., true(T) or false(F). These truth values can be combined with the n truth value of other propositions. So, we can conclude there are 2^n the possible combination of compound propositional variables.

Let p and q are two propositions then the truth table of any combination of different compound propositions contains 4 rows. Using the compound propositions $(p \vee \neg q)$ and $(p \wedge q)$ we can prepare another compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$ and each compound proposition contains 4 possible combinations. Table 1 displays the truth table for $(p \vee \neg q) \rightarrow (p \wedge q)$.

Table 1: Truth table of (p ∨¬q) → (p ∧ q)

p	q	¬q	p ∨ ¬q	p ∧ q	(p ∨ ¬q) → (p ∧ q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Let the p, q, and r be the propositions then construct a truth table for the compound proposition

(p ∧ q) ∨ ¬r.

Solution

Table 2: Truth table of (p ∧ q) ∨¬r

p	q	r	¬r	p ∧ q	(p ∧ q) ∨ ¬r
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

4.2 Tautologies and contradictions

A tautology is a compound proposition that is invariably true, regardless of the truth values of the propositional variables that appear in it.

A contradiction is a collection of statements that are always false.

A contingency is a compound proposition that is neither a tautology nor a contradiction.

Let P be the proposition

P: I will help you.

Then ¬P: I will not help you.

Now if we find the compound proposition P∧¬P that is “I will help you and I will not “is a contradiction.

Table 3:Truth table of P∧¬P

P	¬P	P∧¬P
T	F	F
F	T	F

Now if we find the compound proposition P∨¬P that is "I will help you or I will not "is a tautology.

Table 4: Truth table of P∧ ¬P

P	¬P	P ∨¬P
T	F	T
F	T	T

Let p, q, and r be the propositions then prove that the compound proposition $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$ is contingency

Solution

Table 4: Truth table of $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

p	q	r	¬p	¬r	$p \leftrightarrow q$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
T	T	T	F	F	T	T	F
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	T	F	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	F

4.3 Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called logically equivalent. We can also define this notion as follows.

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example

Let p and q are two propositions then the compound proposition $\neg (p \wedge q)$ is equivalent to $\neg p \vee \neg q$, and the compound proposition $\neg (p \vee q)$ is equivalent to $\neg p \wedge \neg q$. These rules are known as the De Morgan laws

Table 5: $\neg (p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$(p \wedge q)$	$\neg (p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 6: $\neg (p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg (p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Show that the compound propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are Logically Equivalent.

Solution

Table 6:Truth table of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Summary

This section deals with some basic definitions and operations. These are summarized below:

- A tautology is a compound proposition that is invariably true, regardless of the truth values of the propositional variables that appear in it.
- A contradiction is a collection of statements that are always false.
- A contingency is a compound proposition that is neither a tautology nor a contradiction.
- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- De Morgan laws $\neg(p \wedge q) \equiv \neg p \vee \neg q$ ***and*** $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Keywords

Tautology: a compound proposition that is always true

Contradiction: a compound proposition that is always false

Contingency: a compound proposition that is sometimes true and sometimes false

Logically equivalent compound propositions: compound propositions that always have the same truth values

Self-Assessment:

1. What will be the output (Truth values) of compound proposition $(p \to q) \to (q \to p)$ for the input truth values T, T of p and T, F of q?

(A) T, F

(B) F, T

(C) T, T

(D) F, F

2. What will be the output (Truth values) of compound proposition $(p \to q) \to (q \to p)$ for the input truth values F, F of p and T, F of q?

(A) T, F

(B) F, T

(C) T, T

(D) F,F

3. What will be the output (Truth values) of compound proposition $(p \vee q) \to (p \wedge q)$ for the input truth values T, T of p and T, F of q?

(A) T, F

(B) F, T

(C) T, T

(D) F,F

4. What will be the output (Truth values) of compound proposition $(p \vee q) \to (p \wedge q)$ for the input truth values F, F of p and T, F of q?

(A) T, F

(B) F, T

(C) T, T

(D) F,F

5. What will be the output (Truth values) of compound proposition $(p \vee \neg q) \rightarrow q$ for the input truth values T, T of p and T, F of q?

(A) T, F

(B) F, T

(C) T, T

(D) F, F

6.What will be the output (Truth values) of compound proposition $(p \vee \neg q) \rightarrow q$ for the input truth values F, F of p and T, F of q?

(A) T, F

(B) F, T

(C) T, T

(D) F, F

7. If A is any statement, then which of the following is a tautology?

(A) $A \wedge F$

(B) $A \vee F$

(C) $A \vee \neg A$

(D) $A \wedge T$

8. A compound proposition that is neither a tautology nor a contradiction is called ___________

(A) Contingency

(B) Equivalence

(C) Condition

(D) Inference

9. A compound proposition that is always true called a ___________

(A) Contingency

(B) Tautology

(C) Contradiction

(D) Inference

10. A compound proposition that is always true called a ___________

(A) Contingency

(B) Tautology

(C) Contradiction

(D) Inference

11. Which one of the following compound propositions is the logical equivalent to the $\neg(p \rightarrow q)$?

(A) $p \wedge \neg q$

(B) $p \wedge q$

(C) $p \rightarrow q$

(D) $q \rightarrow p$

12. Which one of the following compound propositions is the logical equivalent to the $\neg(p \vee (\neg p \wedge q))$?

(A) $\neg p \wedge \neg q$

(B) $p \wedge \neg q$

(C) $p \rightarrow q$

(D) $q \rightarrow p$

13. Which one of the following compound propositions is the logical equivalent to the $(p \rightarrow q) \wedge (q \rightarrow p)$?

(A) $p \leftrightarrow q$

(B) $p \wedge q$

(C) $p \rightarrow q$

(D) $q \rightarrow p$

14. Which one of the following is the logical equivalent to the $p \vee (p \wedge q)$?

(A) p

(B) q

(C) $\neg p$

(D) $\neg q$

15. Which one of the following is the logical equivalent to the $p \wedge (p \vee q)$?

(A) p

(B) q

(C) $\neg p$

(D) $\neg q$

Answers for Self Assessment

1.	C	2.	B	3.	A	4.	B	5.	A
6.	A	7.	C	8.	A	9.	B	10.	C
11.	A	12.	A	13.	C	14.	A	15.	A

Review Questions:

1. How many rows appear in a truth table for the compound proposition $(p \vee \neg r) \wedge (q \vee \neg s)$?

2. How many rows appear in a truth table for the compound proposition $(p \rightarrow r) \vee (\neg s \rightarrow \neg t) \vee (\neg u \rightarrow v)$?

3. Construct a truth table for the compound proposition $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$?

4. Construct a truth table for the compound proposition $p \oplus (p \vee q)$?

5. Construct a truth table for the compound proposition $(p \oplus q) \wedge (p \oplus \neg q)$?

6. Construct a truth table for the compound proposition $p \oplus (p \to q)$?

7. Show that the conditional statement $\neg p \to (p \to q)$ is a tautology by using truth tables.

8. Show that conditional statement $(p \wedge q) \to (p \to q)$ is a tautology by using truth tables.

9. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology

10. Use De Morgan's laws to find the negation of the statements "Jan is rich and happy".

11. Show that $(p \to q) \to (r \to s)$ and $(p \to r) \to (q \to s)$ are not logically equivalent.

12. Show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.

13. Show that $(a \wedge b) \to (b \vee c)$ is a tautology

14. The compound statement $A \to (A \to B)$ is false, then what is the truth values of A, B?

15. Check whether $(a \wedge b) \to (F)$ is a tautology

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 05: Introduction to Logic Gates

CONTENTS

Objectives:

Circuits in computers and similar electrical instruments have inputs and outputs which are either 0s or 1s. Any basic feature with two specific conditions can be used to build circuits. Switches that can be turned on and off are examples of those elements. Electrical instruments that can be illuminated or unlit, and switches that can be turned on or off. The design of digital equipment is modeled using Boolean algebra. The device's inputs and outputs can be thought of as a part of the Boolean variables. The principles of Boolean algebra could be used to construct each circuit.The basic building blokes of circuits are the logic gates. We'll use the principles of Boolean algebra to configure the circuits that carry out a variety of tasks using these gates. The circuits we'll look at in this section produce a yield that is solely dependent on knowledge rather than the circuit's current state. As a result, these circuits have no memory. Combinable loops, also known as gating networks, are such circuits.

After this unit, you would be able to

- understand the functioning of various logic gates.
- understand the various operation in Boolean algebra
- construct the circuit using the logic gates
- find the output of the given circuit
- draw the Venn diagram of a set and be able to visualize the different counting problems using it.

Introduction

Claude Shannon noticed in 1938 that propositional logic could be used in the architecture of software machinery. A computerized circuit takes in input signals I_1, I_2..., In, every one of which is either 0 (off) or 1 (on), and yields the output signals O_1, O_2...,O_n. Three basic circuits (NOT, OR, and AND, referred to as logic gates, may be used to create complicated advanced circuits. The initial step in building circuitry is to describe its Boolean function with a phrase formed up of simple Boolean algebra operations. We'll show you how to make these expressions using an algorithm. The principles for dealing with the sequence 0, 1 are given by Boolean algebra. The Boolean sum, product, and complements are the three operations of Boolean algebra that we can use far more.

5.1 The Boolean operations

Complement

The complement of an element $x \in \{0,1\}$ is denoted as $\bar{x}$ and defined as $\bar{1} = 0$ and $\bar{0} = 1$.The complement of an element is corresponding to the negation of the proposition p.

Table 1: Truth table of Negation of proposition

p	¬p
T	F
F	T

Table 2: Complement of element x

x	$\bar{x}$
1	0
0	1

Boolean Sum

The Boolean sum of two elements $x, y \in \{0,1\}$ denoted by + or by OR, has the value 1 if any one of the inputs x, y are 1 otherwise 0.

Table 3 displays the Boolean sum of the inputs x and y.

Table 3: Boolean sum of the inputs x and y.

x	y	$x+y$
1	1	1
1	0	1
0	1	1
0	0	0

The Boolean sum of two elements is corresponding to the OR (∨) of the proposition p and q.

Table 4 displays the truth table for p ∨ q.

Table 4: the truth table for p ∨ q.

p	q	p ∨q
T	T	T
T	F	T
F	T	T
F	F	F

Boolean Product

The Boolean sum of two elements $x, y \in \{0,1\}$ denoted by . or by AND, has the value 1 if both the inputs x, y are 1 otherwise 0.

Table 5displays the Boolean product of the inputs x and y.

Table 5: Boolean product of the inputs x and y.

x	Y	$x.y$
1	1	1
1	0	0
0	1	0
0	0	0

The Boolean sum of two elements is corresponding to the AND (∧)of the proposition p and q.

Table 6 displays the truth table for p ∧ q.

Table 6: the truth table for p ∧ q.

p	q	p ∧q
T	T	T
T	F	F
F	T	F
F	F	F

5.2 Boolean Functions

Let B = {0, 1}. Then $B_n = \{(x_1, x_2..., x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s. The variable x is called a Boolean variable if it assumes values only from B, that is if its only possible values are 0 and 1. A function from B_n to B is called a Boolean function of degree n.

Example The function $t(x, y) = x+\bar{y}$ from the set of ordered pairs of Boolean variables to the set {0, 1} is a Boolean function of degree 2 with $t(1, 1) = 1$, $t(1, 0) = 1$, $t(0, 1) = 0$, and $t(0, 0) = 1$.

Table 7:Boolean table for $x+\bar{y}$

x	y	$\bar{y}$	$x+\bar{y}$
1	1	0	*1*
1	0	1	*1*
0	1	0	*0*
0	0	1	*1*

Table 8 displays the truth table for $x \vee \bar{y}$.

Table 8: Truth table for $x \vee \bar{y}$

x	y	$\bar{y}$	$x \vee \bar{y}$
T	T	F	*T*
T	F	T	*T*
F	T	F	*F*
F	F	T	*T*

To describe the values of Boolean function $F(x, y, z) = x + yz$, create a table.

Table 9: Boolean table for $(x + (y.z))$

x	y	z	$y.z$	$(x + (y.z))$
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

5.3 Logic Gates

Gates are the fundamental components of circuits. A Boolean operation is implemented in each type of gate.

Inverter (NOT gate)

An inverter is a tool that takes the value of one Boolean variable as inputs and emits the complement of that value. The following symbol represents the inverter (Not gate).

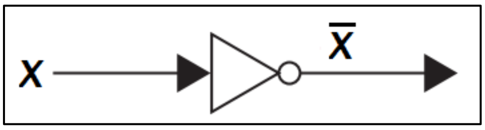


Figure 1: Inverter (NOT gate)

The following figure shows the complement of an input variable

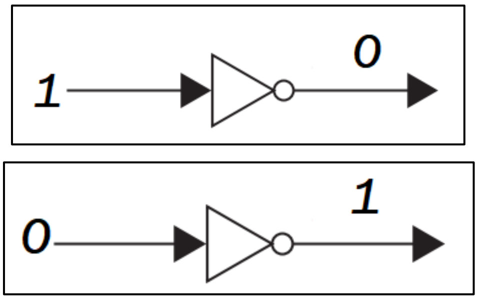


Figure 2:Complement of an input variable

OR gate

OR gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Boolean sum of these values. The following symbol represents the OR gate.

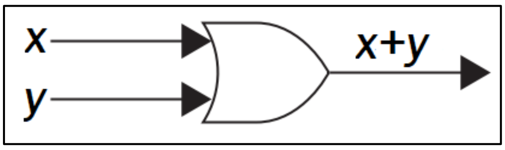


Figure 3:OR gate

The following figure shows the Boolean sum of input variables

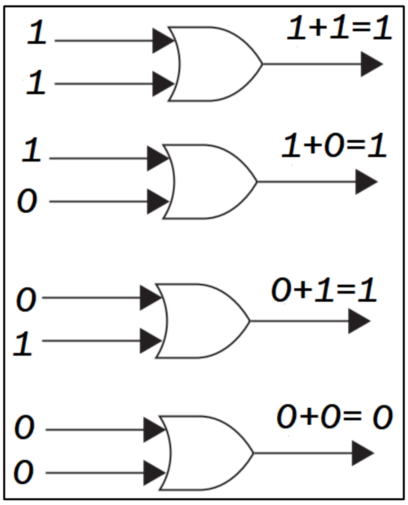


Figure 4:Boolean sum of input variable through OR gate

AND gate

AND gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Boolean product of these values. The following symbol represents the AND gate.

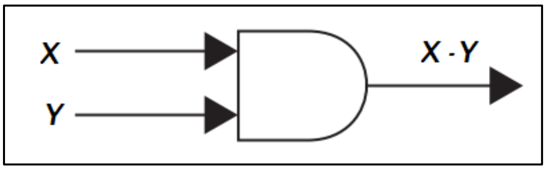


Figure 5: AND gate

The following figure shows the Boolean product of the input variables

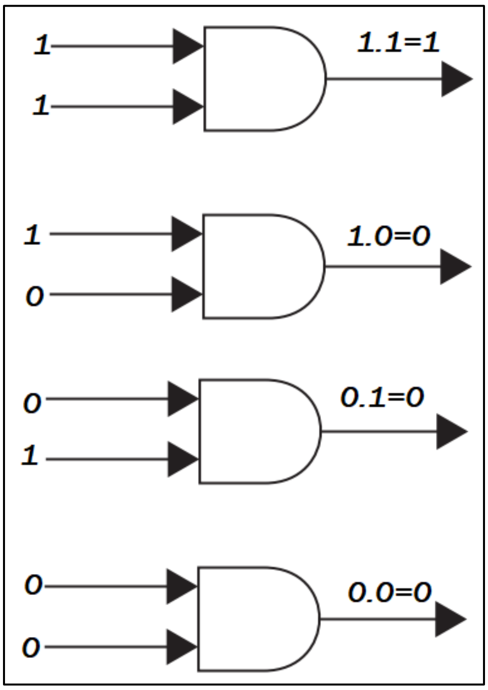


Figure 6:Boolean product of input variable through AND gate

NOR gate

NOR gate is a gate that takes the value of two or more Boolean variables as inputs and emits the complement Boolean sum of these values. The following symbol represents the OR gate.

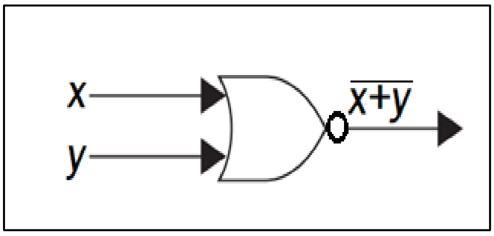


Figure 7: NOR gate

The following figure shows the complement of the Boolean sum of an input variable

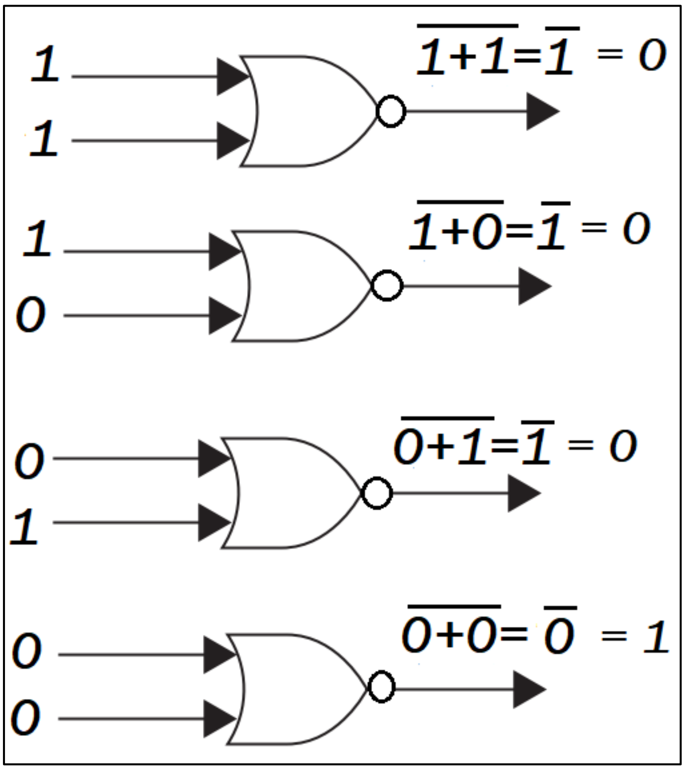


Figure 8:The complement of the Boolean sum of input variable through NOR gate

NAND gate

NAND gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Complement Boolean product of these values. The following symbol represents the NAND gate.

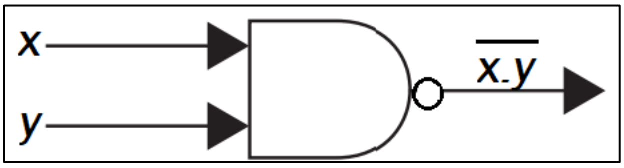


Figure 9: NAND gate

The following figure shows the complement of the Boolean product of the input variables

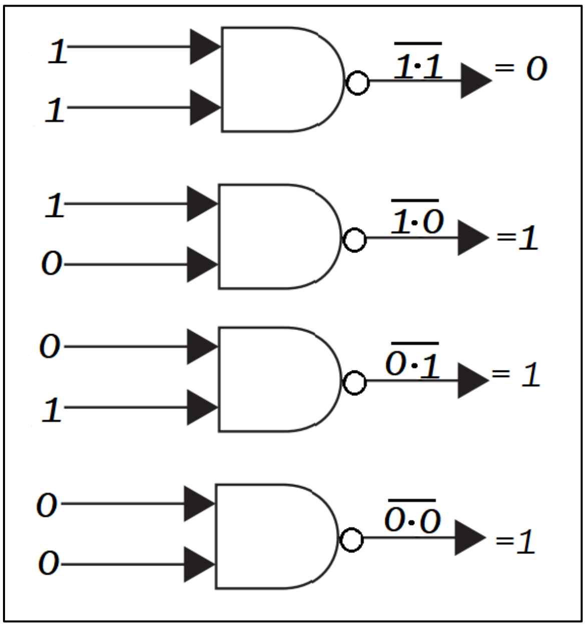


Figure 10: The complement of the Boolean product of input variable through NOR gate

5.4 Combinations of Gates

A set of inverters, OR gates, and AND gates may be used to create a new circuit. In-circuit representations, this is seen in two ways.

Specific input way: Specify the input for each gate separately

Branching: Use branching to show all of the possible outcomes.

Example Construct circuits to generate the output $x.y + \bar{x}.y$

Solution: Figure 11 shows the circuit which takes the separate input for every gate. While Figure 12 shows the circuit, which takes the input through the branching.

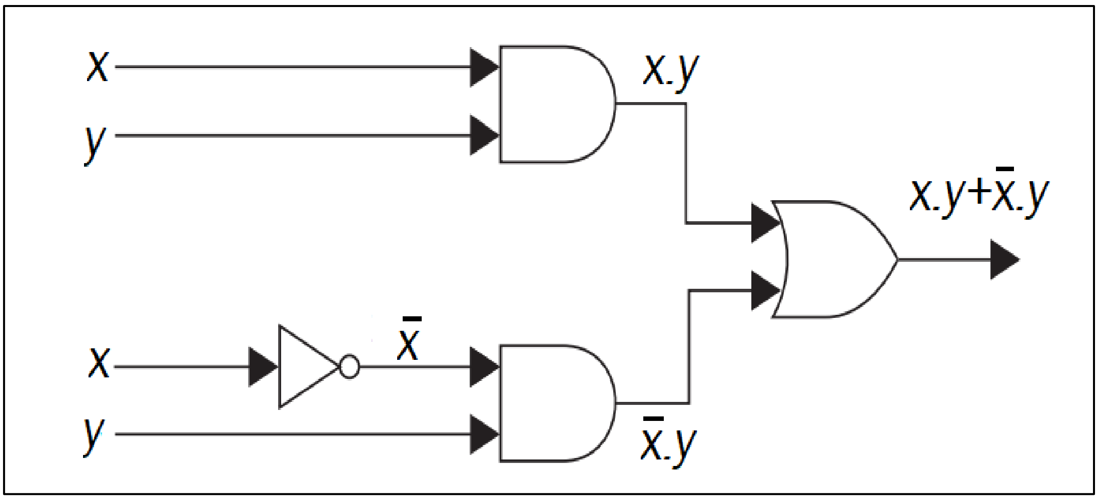


Figure 11: Circuit represents the output $x.y + \bar{x}.y$

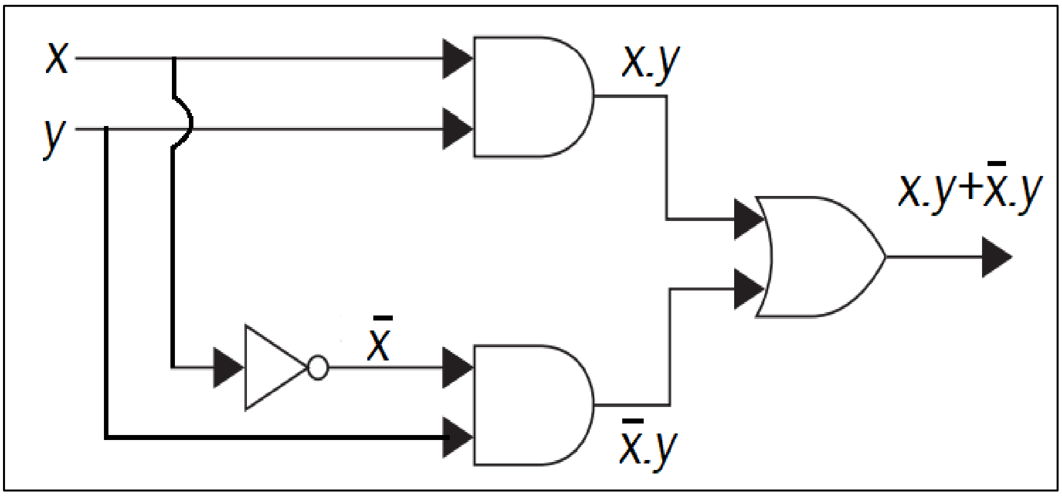


Figure 12: Circuit represents the output $x.y + \bar{x}.y$

Using the Boolean table show $x.y + \bar{x}.y = y$

Table 10: Boolean table for the $x.y + \bar{x}.y$

x	y	$x.y$	$\bar{x}$	$\bar{x}.y$	$x.y + \bar{x}.y$
1	1	1	0	0	1
1	0	0	0	0	0
0	1	0	1	1	1
0	0	0	1	0	0

It also can be proved by logical equivalence and Table 10 depict that $(x \wedge y) \vee (\bar{x} \wedge y)$ is equivalent to y.

Table 11: Truth table for the $(x \wedge y) \vee (\bar{x} \wedge y)$

x	y	$(x \wedge y)$	$\bar{x}$	$(\bar{x} \wedge y)$	$(x \wedge y) \vee (\bar{x} \wedge y)$
T	T	T	F	F	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	F	T	F	F

Construct circuits for $(x + y + z).(\bar{x}.\bar{y}.\bar{z})$

Solution

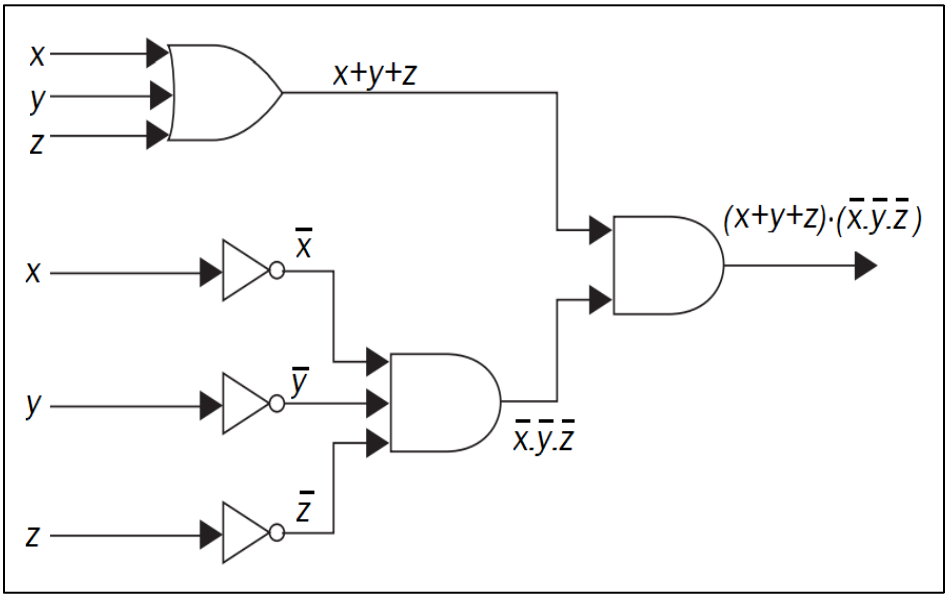


Figure 13:Circuit diagram of $(x + y + z).(\bar{x}.\bar{y}.\bar{z})$

Let a decision for the particular case is pending in the court and three judges pass that decision. Each judge votes either yes or no on the decision. The decision is passed if it receives at least two yes votes. Design a circuit that determines whether the decision passes.

Solution

Let x = 1 if the first judgepasses the decision and votes yes, and x = 0 if this judge votes no; let y = 1 if the second judge passes the decision andvotes yes, and y = 0 if this judge votes no; let z = 1 if the third judge passes the decision and votes yes, and z = 0 if this judge votes no.

The decision is passed if at least two of the votes for yes. It means (x and y) or (y and z) or (z and x) pass the decision. Figure 14 represents the circuit diagram that the decision is passed.

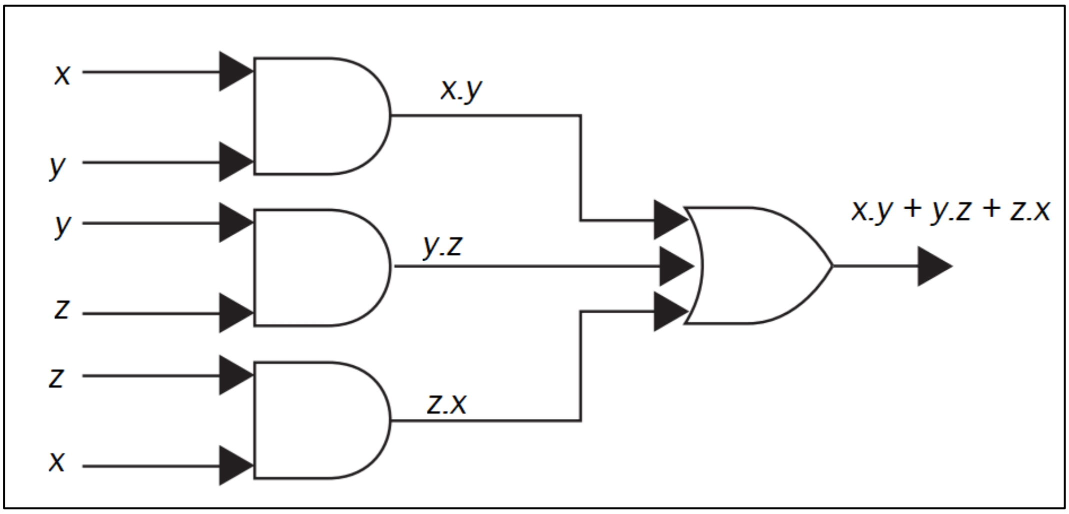


Figure 14: Circuit diagram represent that the decision is passed

It also can be visualized through the Boolean table and the last column of Table 12 shows that the decision is passed if it receives at least 2 yes votes.

Table 12: Boolean table represent that the decision is passed

x	y	z	x.y	y.z	z.x	x.y+y.z+z.x
1	1	1	1	1	1	1
1	1	0	1	0	0	1
1	0	1	0	0	1	1
1	0	0	0	0	0	0
0	1	1	0	1	0	1
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

Summary

This section deals with some basic definitions and operations involving logic gates. These are summarized below:

- The complement of an element $x \in \{0,1\}$ is denoted as $\bar{x}$ and defined as $\bar{1} = 0$ and $\bar{0} = 1$.
- The Boolean sum of two elements $x, y \in \{0,1\}$ denoted by + or by OR, provides the value 1 if any one of the inputs x, y are 1 otherwise 0.
- The Boolean sum of two elements $x, y \in \{0,1\}$ denoted by. or by AND, provides the value 1 if both the inputs x, y are 1 otherwise 0.
- A function from $B_n = \{(x_1, x_2..., x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$ to B= {0,1} is called a Boolean function of degree n.
- An inverter is a tool that takes the value of one Boolean variable as inputs and emits the complement of that value.
- OR gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Boolean sum of these values
- AND gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Boolean product of these values
- NOR gate is a gate that takes the value of two or more Boolean variables as inputs and emits the complement Boolean sum of these values
- NAND gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Complement Boolean product of these values
- A circuit can be formed using the combination of a set of inverters, OR gates, and AND gates.

Keywords

Complement: The complement of an element $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}$ is denoted as $\overline{\boldsymbol{x}}$ and defined as $\overline{\mathbf{1}} = \mathbf{0}$ and $\overline{\mathbf{0}} = \mathbf{1}$.

Boolean Sum: The Boolean sum of two elements $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$ denoted by + or by OR, provides the value 1 if any one of the inputs $\boldsymbol{x}, \boldsymbol{y}$ are 1 otherwise 0.

Boolean Product: The Boolean sum of two elements $x, y \in \{0,1\}$ denoted by . or by AND, provides the value 1 if both the inputs x, y are 1 otherwise 0.

Inverter (NOT gate): An inverter is a tool that takes the value of one Boolean variable as inputs and emits the complement of that value.

OR gate: OR gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Boolean sum of these values

AND gate: AND gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Boolean product of these values

NOR gate: NOR gate is a gate that takes the value of two or more Boolean variables as inputs and emits the complement Boolean sum of these values

NAND gate: NAND gate is a gate that takes the value of two or more Boolean variables as inputs and emits the Complement Boolean product of these values

Self Assessment:

1. The inverter is_______

A. NOT gate

B. OR gate

C. AND gate

D. NAND gate

2. The only function of NOT gate is to____

A. Stop signal

B. Invert input signal

C. Act as a universal gate

D. Do nothing

3. When an input signal 1 is applied to a NOT gate, the output is

A. 0

B. 1

C. Both0 & 1

D. Neither 0 nor 1

4. When input signals 1, 1 are applied to a AND gate, the output is

A. 0

B. 1

C. Both0 & 1

D. Neither 0 nor 1

5. When input signals 1, 0 are applied to a AND gate, the output is

A. 0

B. (b)1

C. (c)Both0 & 1

D. (d) Neither 0 nor 1

6. When input signals 0, 0 are applied to a AND gate, the output is

A. 0

B. (b)1

C. Both0 & 1

D. (d)Neither 0 nor 1

7. When input signals 1, 0 are applied to an OR gate, the output is

A. 0

B. 1

C. Both0 & 1

D. Neither 0 nor 1

8. When input signals 1, 1 are applied to a NAND gate, the output is

A. 0

B. 1

C. Either 0 & 1

D. Neither 0 nor 1

9. When input signals 1, 0 are applied to a NAND gate, the output is

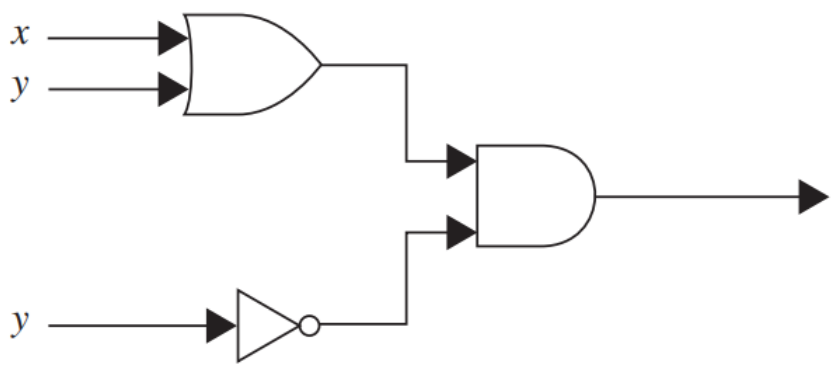
A. 0

B. 1

C. Either 0 & 1

D. Neither 0 nor 1

10. The output of the following circuit



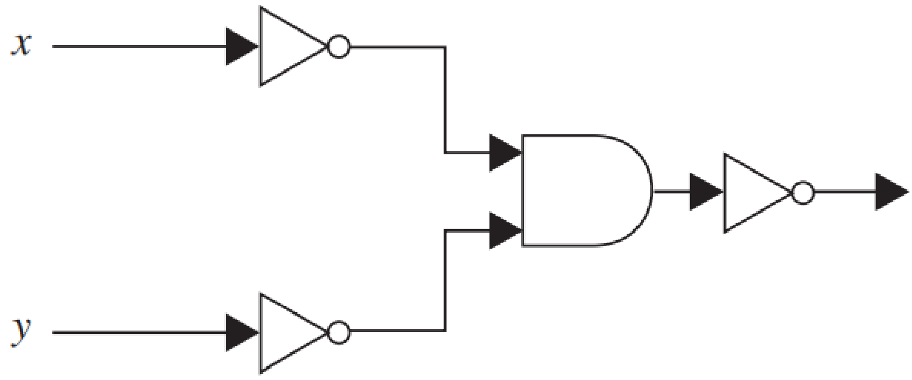
A. $(x + y).\bar{y}$

B. $(x.y).\bar{y}$

C. $(x + y) + \bar{y}$

D. $(x + y).y$

11. The output of the following circuit



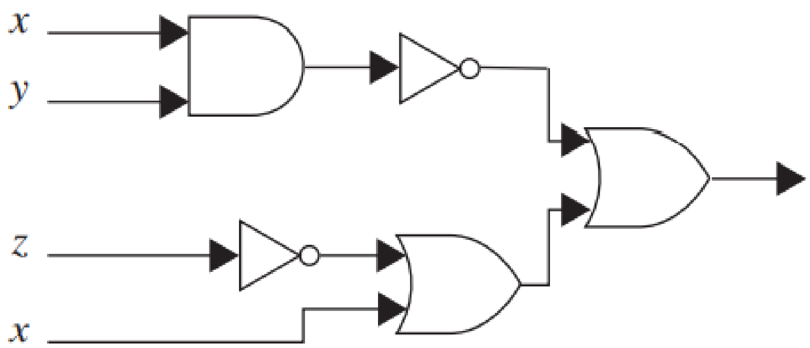
A. $\bar{x}.\bar{y}$

B. $\bar{x} + \bar{y}$

C. $x.y$

D. $x + y$

12. The output of the following circuit



A. $(x.y) + (z + x)$

B. $(x + y) + (z + x)$

C. $(\overline{x.y}) + (\bar{z} + x)$

D. $(x.y) + (z.x)$

13. Consider the input x = 1and y = 1 for $(\bar{x} . y)$ then the out is

A. 0

B. 1

C. Both 0 & 1

D. Neither 0 nor 1

14. Consider the input x = 0 and y = 0 for $(\bar{x} \,.\, y)$ then the out is

A. 0

B. 1

C. Both 0 & 1

D. Neither 0 nor 1

15. Consider the input x = 1, y = 1, and z=1 for $(\mathrm{x} + \mathrm{y}.\mathrm{z})$ then the out is

A. 0

B. 1

C. Both 0 & 1

D. Neither 0 nor 1

Answer for Self Assessment

1.	A	2.	C	3.	B	4.	B	5.	A
6.	A	7.	B	8.	A	9.	B	10.	A
11.	B	12.	C	13.	A	14.	A	15.	B

Review Questions:

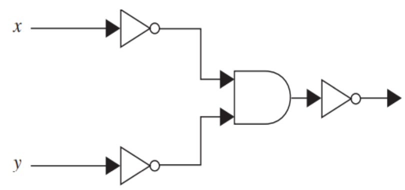
1. Show that $(\bar{1} \cdot \bar{0}) + (1 \cdot \bar{0}) = 1$.

2. Draw the Boolean table for the function F (x, y, z) = $x.y.z$

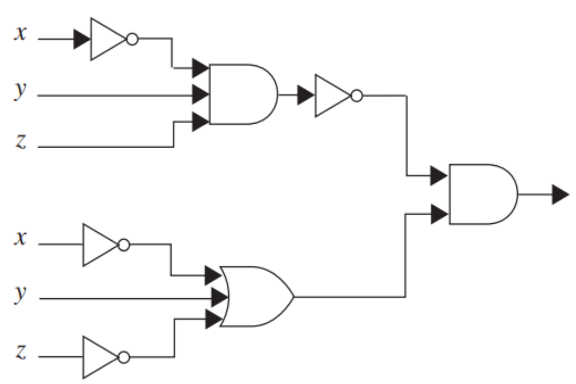
3. Draw the Boolean table for the function F (x, y, z) = $(x.y).\bar{z}$

4. Draw the Boolean table for the function F (x, y, z) = $(x + y).\bar{z}$

5. Find the output of the given circuit



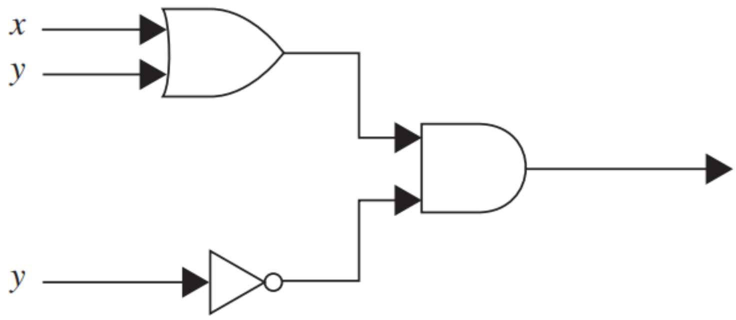
6. Find the output of the given circuit



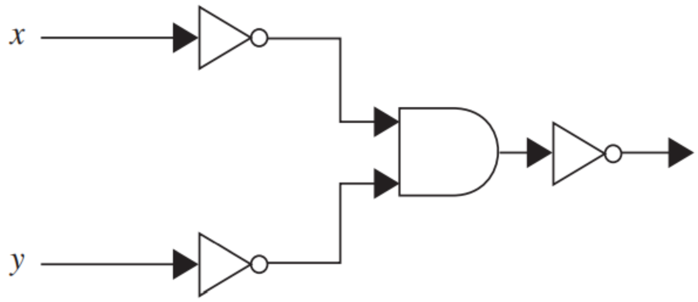
7. Construct circuits using different gates for the output $(x + y).(\overline{x + y})$

8. Construct circuits using different gates for the output $(x.y) + (\overline{x.y})$

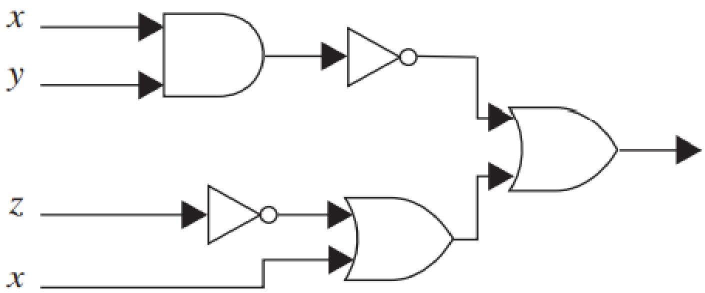
9. Find the output of the following circuit



10. Find the output of the following circuit



11. Find the output of the following circuit



12. When input signals 1, 1 are applied to a NAND gate, then what is the output?

13. When input signals 1, 0 are applied to a NAND gate, then what is the output?

14. Consider the input x = 1, y = 1, and z=1 for $(\mathbf{x} + \mathbf{y}.\mathbf{z})$then what is the output?

15. Consider the input x = 0 and y = 0 for $(\bar{\boldsymbol{x}}\,.\,\boldsymbol{y})$then what is the output?

Further Readings

- Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education
- Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 06: Introduction to Recursion

CONTENTS

Objectives:

Many computing issues are difficult to solve using the basic counting technique like how many bit strings of length n are there which do not contain two consecutive 0s. Such types of problems can be solved by recursive process. the recursion can be applied to a problem split down into a set number of non - overlapping partitions before the solution. The complexity of such algorithms can be determined with the help of a kind of recurrence relationship. Recurrence relations also can be used to model a diverse range of challenges including calculating compound interest, counting rabbits on an island, and calculating the number of steps in the Tower of Hanoi game. In this section, we'll look at several algorithms and use recurrence relations to explore their complexity.

- understand the recursive processes involved in real-life problems
- model the recurrence relation from the counting problem
- identify different types of recurrence relation
- find the solution of homogeneous and non-homogeneous recurrence relation

Introduction

A recursive description of a series defines one or more initial terms as well as a framework for defining subsequent terms, part of the terms that came before them. A recurrence relation is a combination of terms in which one sequence is related to another, and a sequence is considered a solution of a given relation if its terms satisfy the recurrence relation.

Consider the case where a virus population doubles per hour in a colony. How many viruses would be found in n hours if initially, a colony starts with 10?

To resolve, this problem, let a_1 be the total number of viruses that are present in the first hour in the colony.

Let $a_2$be the population of virus in the next hour /2nd hour so $a_2 = 2a_1$.

Similarly, let $a_3$be the population of virus in the 3rd hour so $a_3 = 2a_2$.

Here we can conclude that if a_{n+1} is the population of virus in the (n+1)th hour then it the twice of the population of virus in the nth hour.

So $a_{n+1} = 2a_n$ is the recurrence relation for the above-mentioned problem.

As it is given that $a_1 = 10$. We can solve $a_{n+1} = 2a_n$ using the a_1.

$a_1 = 10$

$a_2 = 2a_1 = 2.10$

$a_3 = 2a_2 = 2.2a_1 = 2.2.10$

$a_4 = 2a_3 = 2.2.a_2 = 2.2.2a_1$=2.2.2.10

.

.

.

,

$a_n = 2a_{n-1} = 2.2.a_{n-2} = 2.2.2a_{n-3}$ ..., $=2^{n-1}.a_1$

$a_n = 2^{n-1}.10$

6.1 Modeling With Recurrence Relations

The Tower of Hanoi

The Hanoi Tower is made up of three fixed towers on a board and disks of various sizes. These disks are first arranged on the first tower as the largest one on the bottom. Every small size disk can be laid one on the larger one see Figure 1

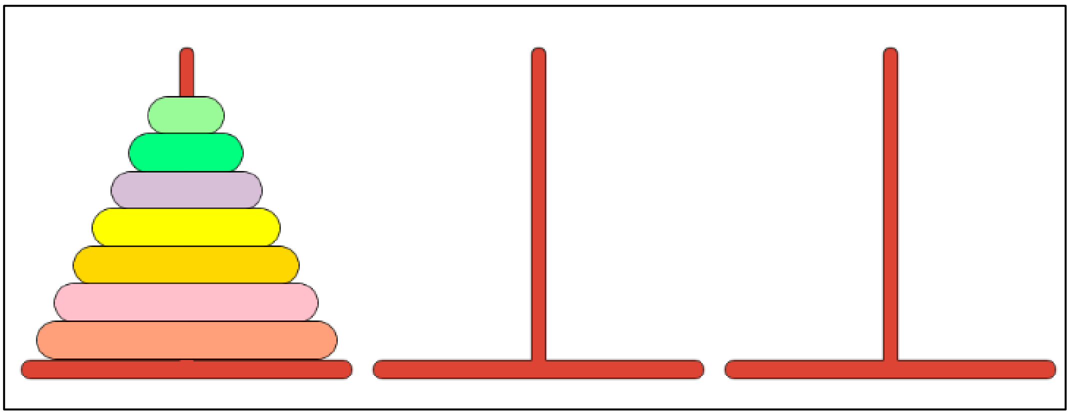


Figure 1: Tower of Hanoi game with three tower n disks

The purpose of the puzzle is to arrange all of the disks on the third tower in order of number, starting with the largest on the bottom (see figure 2).

The game's rules allow disks to be transferred from one tower to another one at a time, as long as no larger disk is put on top of a smaller disk.

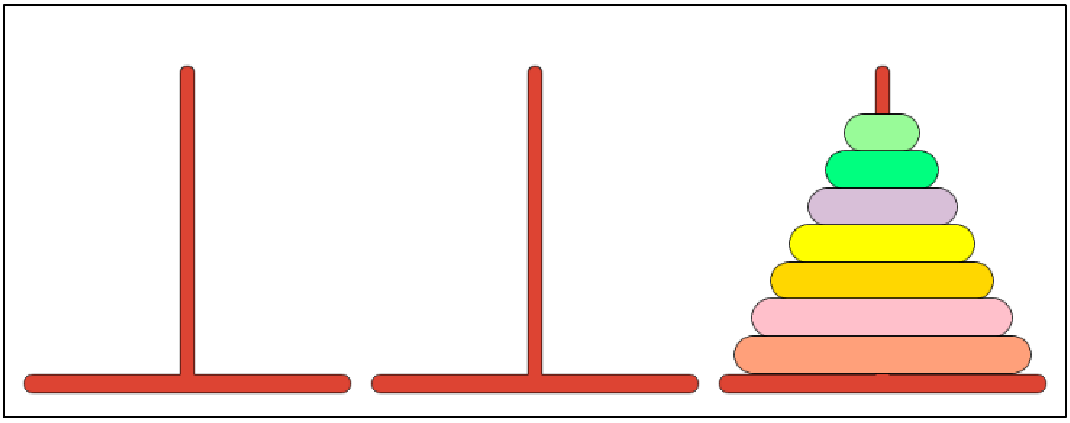


Figure 2Solution of Tower of Honai game problem with 7 disks

Now how many minimum numbers of moves are required to place n disks from the first tower to the last tower?

Or

For the n disks problem how many minimum moves are required to win the game?

We can model the recurrence relation for the n disks problem

Let M be the minimum number of moves to win the game if there is only one disk situated on the first tower. So only one move is required. Hence if $n = 1, M = 1$ (see figure 3)

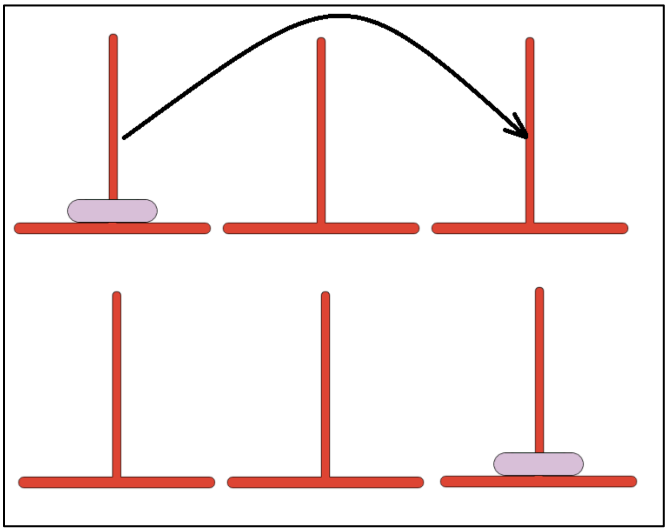


Figure 3 Shifting of a disk from tower one to three.

if $n = 2, M = 3$ (see figure 4)

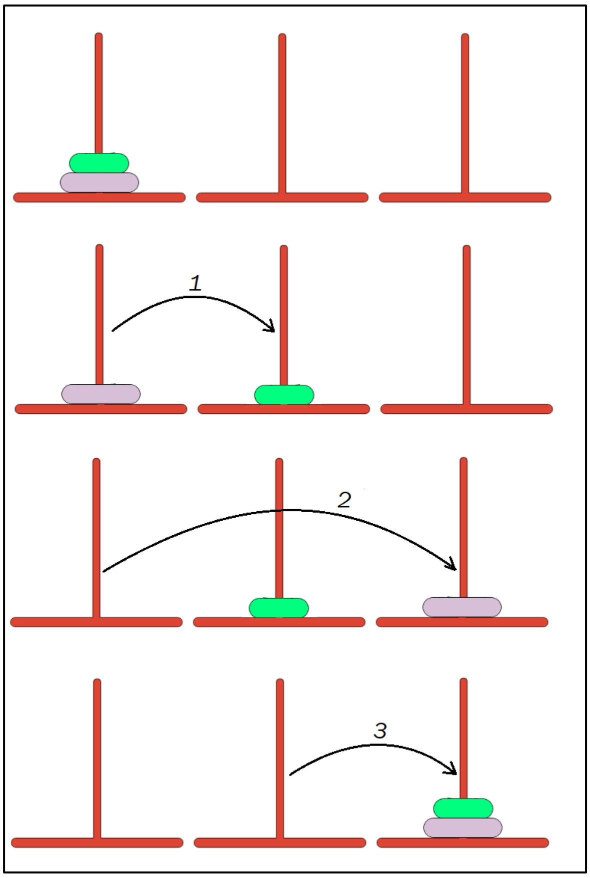


Figure 4: Shifting of 2 disks from tower one to tower 3

In the same way, if $n = 3, M = 7$. The following table represents the relation between the number of disks

Table 1:No of disk vs minimum moves in Tower of Hanoi game

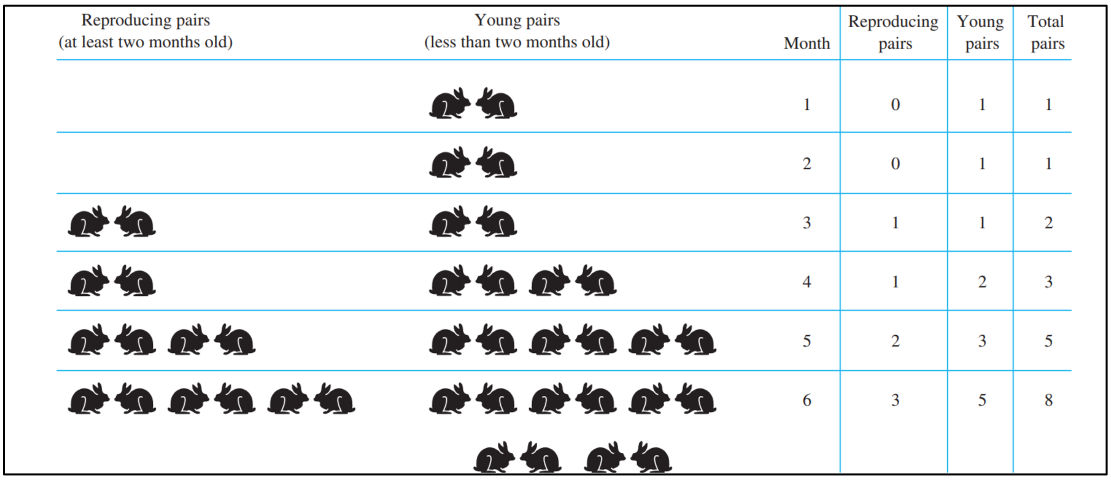
Number of disks(n)	Minimum number of moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127

Using the above table, we can conclude that "the minimum number of moves required for any disk problem is 2. (minimum number of moves required for n-1 disk problem) +1.

Let M_n be the minimum number of moves required for n disk problem and M_{n-1} be the minimum number of moves required for the n-1 disk problem then $M_n = 2M_{n-1} + 1$ is the recurrence relation for a tower of Hanoi game solution with initial condition $M_1 = 1$. The solution is $M_n = 2^n - 1$.

Rabbits and the Fibonacci Numbers

Consider the example below: A young pair of rabbits (one of each sex) is located on an island. Rabbits do not begin reproducing until they reach the age of two months. Since they reach the age of two months, each pair of rabbits grows a new pair per month, as seen in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after n months if no rabbits die.



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Figure 5: Rabbits on the island.

Let P_1 be the number of pairs after one month P_1 =1. The pair can't reproduce the child until they are two months old and let P_2 be the number of pairs after two months and P_2 =1. Now in the third month, the pair got another pair of rabbits so after the third month let P_3 be the number of pairs after the third month P_3 =2. Here we can conclude that the total number of pairs after any month is the sum of the number of pairs for the last two consecutive months. It means $P_n = P_{n-1} + P_{n-2}$ is the recurrence relation for above-mentioned rabbit problem. Moreover 1,1,2,3,5,8,13,21… are the terms of Fibonacci sequence.

Find the a_5 if $a_{n+1} = 5a_n + 10$ where $a_0 = 10$

Solution

Given that $a_{n+1} = 5a_n + 10$. Now put n=0. It becomes $a_1 = 5a_0 + 10 = 5.10 + 10 = 60$.

Now put n=1 in the given recurrence relation $a_{1+1} = 5a_1 + 10 = a_2 = 5.60 + 10 = 310$

Now put n=2 in the given recurrence relation $a_{2+1} = 5a_2 + 10 = a_3 = 5.310 + 10 = 310 = 1560$

Now put n=3 in the given recurrence relation $a_{3+1} = 5a_3 + 10 = a_4 = 5.1560 + 10 = 7760$

Now put n=2 in the given recurrence relation $a_{4+1} = 5a_4 + 10 = a_5 = 5.7760 + 10 = 310 = 38760$

6.2 Solving Linear Recurrence Relations

A linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k} = R(n)(1)$

$where\ C_0, C_2, \ldots, C_k\ are\ real\ numbers, and\ C_k \neq 0$

If any of the $C_0, C_2, \ldots, C_k$ is variable then equation (1) is known as the linear recurrence relation of degree k with variable coefficients

If any recurrence relation is expressed in form of equation (1) and then If $R(n) = 0$ then the recurrence relation is known as linear homogeneous recurrence relation of degree k with constant coefficients. If $R(n) \neq 0$ then it's known as linear nonhomogeneous recurrence relation of degree k with constant coefficients.

If every term of recurrence relation has utmost one sequence with degree ≤ 1 then it is known as linear and if any term of recurrence relation either contains more than one sequence or the degree of the term is more than 1 then it's called nonlinear recurrence relation.

The degree of recurrence relation

Let $C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k} = R(n)$ be the recurrence relation then the degree of relation is $n - (n-k) = k$. which means the largest subscripts- smallest subscripts.

The linear homogenous recurrence relation with constant coefficient and degree 2:

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

The linear homogenous recurrence relation with variable coefficient and degree 3:

$$a_{n+3} - 5a_{n+2} + 6na_{n+1} + 8a_n = 0$$

The linear nonhomogeneous recurrence relation with a constant coefficient: $a_n - 5a_{n-1} + 6a_{n-2} = 5^n$

The linear nonhomogeneous recurrence relation with a variable coefficient: $a_n - 5a_{n-1} + 6na_{n-2} = n$

The nonlinear homogenous recurrence relation with a constant coefficient: $a_n^2 - 5a_{n-1} + 6a_{n-2} = 0$

The nonlinear homogenous recurrence relation with a variable coefficient:

$$a_n - 5a_{n-1}a_{n-2} + 6na_{n-3} = 0$$

The nonlinear non homogeneous recurrence relation with a constant coefficient:

$$a_n^2 - 5a_{n-1} + 6a_{n-2} = 5^n$$

The nonlinear non homogeneous recurrence relation with a variable coefficient:

$$\boldsymbol{a_n - 5a_{n-1}a_{n-2} + 6na_{n-3} = n}$$

6.3 The shift (E) and Δ operator on sequence

Let a_n be the sequence and the shift operator applied on $a_n$as follows

$E(a_n) = a_{n+1}$

Let a_n be the sequence and the Δ operator applied on $a_n$as follows

$\Delta(a_n) = a_{n+1} - a_n$

$E(10^n) = 10^{n+1} = 10.10^n$

$E^2(10^n) = E10^{n+1} = 10.10.10^n$

$\Delta 10^n = 10^{n+1} - 10^n = 9.10^n$

$E(a_n - 5) = (a_{n+1} - 5)$

$\Delta(a_n - 5) = a_{n+1} - a_n - 5 + 5 = a_{n+1} - a_n.$

Show that, $\Delta = E - 1$

Solution

Let $f(n)$ be a sequence then $\Delta f(n) = f(n+1) - f(n)$

And $Ef(n) = f(n+1)$ so

$\Delta f(n) = Ef(n) - f(n)$ and

$\Delta = E - 1$

6.4 Solution of homogeneous linear regression with constant coefficient

Let $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$ be the homogeneous linear regression with constant coefficient. To find the solution we need to follow the following steps

Step1: Put n=0 in all the subscripts then check whether all are positive if yes then go to step 2.

If no then apply the operatorE^k both sidessuch that all subscripts become positive with n=0.Then go to step 2.

Step 2: If all subscripts are positive then $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$can be expressed as $C_0 a_{n+k} + C_1 a_{n+(k-1)} + C_2 a_{n+(k-2)} + \cdots, + C_k a_n = 0$(2)

Furthermore

$E(a_n) = a_{n+1}$

$E^2(a_n) = a_{n+1}$

$E^3(a_n) = a_{n+3}$

.

.

.

,

$E^k(a_n) = a_{n+k}.$

Hence the regression relation can be written as

$C_0 E^k(a_n) + C_1 E^{k-1}(a_n) + C_2 E^{k-2}(a_n) + \cdots + C_k E^0(a_n) = 0$(3)

Find the auxiliary/characteristic equation as $C_0 E^k + C_1 E^{k-1} + C_2 E^{k-2} + \cdots + C_k E^0 = 0$

Step 3: Find the roots of the auxiliary/characteristic equation.

Step4: Find the complementary function based on the roots as following cases-

Case1- If the roots of the auxiliary/characteristic equation are real and different (say $r_1, r_2, r_3 \ldots, r_k$) then

C.F.= $A_1 {r_1}^n + A_2 {r_2}^n + A_3 {r_3}^n + \cdots, + A_n {r_k}^n$

Case 2- If someof the roots of the auxiliary/characteristic equation are real and same (say $r_1, r_1, r_2 \ldots, r_k$) then

C.F.= $(A_1 + A_2.n){r_1}^n + A_3 {r_2}^n + A_4 {r_3}^n + \cdots, + A_n {r_k}^n$

If three roots of the auxiliary/characteristic equation are real and same (say $r_1, r_1, r_1 \ldots, r_k$) then

C.F.= $(A_1 + A_{2.}n + A_{3.}n^2)r_1{}^n + A_4 r_2{}^n + A_5 r_3{}^n + \cdots, + A_n r_k{}^n$

Case 3- If the roots of the auxiliary/characteristic equation are complex (Imaginary) say $(\alpha \pm i\beta)$

Then find $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$ and the complementary function is

C.F.= $\rho^n(Acosn\theta + Bsinn\theta)$

Step 4: the general solution of $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$ is $a_n = C.F.$

$a_n - 3a_{n-1} = 0(4)$

Here in equation 4, the degree is k=1 and for n=0, $n-1$becomes negative so we will apply E^1

Both sides of equation 4.

$E^1(a_n - 3\, a_{n-1}) = 0$

$$\Rightarrow E^1 a_n - 3\, E^1 a_{n-1}) = 0$$

$$\Rightarrow a_{n+1} - 3\, a_n) = 0$$

$$\Rightarrow (E^1 a_n - 3E^0 a_n) = 0$$

The auxiliary/characteristic equation is E-3=0 and the root is E=3.

So $a_n = C.F. = A.3^n$

$a_{n+2} - 10a_{n+1} + 25a_n = 0(5)$

Here in equation 5, the degree is k=2 and for n=0, all subscripts are positive so

$$a_{n+2} - 10\, a_{n+1} + 25a_n = E^2 a_n - 10E^1 a_n + 25E^0 a_n = 0$$

The auxiliary/characteristic equation is $E^2 - 10E^1 + 25E^0 = 0$and the roots are E=5,5.

So $a_n = C.F. = (A_1 + A_{2.}n)5^n$

$a_{n+2} - 2a_{n+1} + 2a_n = 0(6)$

Here in equation 6, the degree is k=2 and for n=0, all subscripts are positive so

$$a_{n+2} - 2a_{n+1} + 2a_n = E^2 a_n - 2E^1 a_n + 2E^0 a_n = 0$$

The auxiliary/characteristic equation is $E^2 - 2E^1 + 2E^0 = 0$ and the roots are E=1 $\pm i$

Here $(\alpha = 1$ and $\beta = 1$ so $\rho = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

So $a_n = C.F. = \sqrt{2}^n(ACos\frac{n\pi}{4} + BSin\frac{n\pi}{4})$.

Summary

This section deals with some basic definitions and operations involving the recurrence relations. These are summarized below:

- A recursive description of a series defines one or more initial terms as well as a framework for defining subsequent terms, part of the terms that came before them.

- A recurrence relation is a combination of terms in which one sequence is related to another, and a sequence is considered a solution of a given relation if its terms satisfy the recurrence relation.
- A linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form $C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k} = R(n)$
  $where\ C_0, C_2, \ldots, C_k\ are\ real\ numbers, and\ C_k \neq 0$
- Let a_n be the sequence and the shift operator applied on $a_n$as follows $E(a_n) = a_{n+1}$
- Let a_n be the sequence and the Δ operator applied on $a_n$as follows $\Delta(a_n) = a_{n+1} - a_n$
- Let $C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k} = 0$ be the homogeneous linear regression with constant coefficient. Then the auxiliaryequation is$C_0E^k + C_1E^{k-1} + \cdots + C_kE^0 = 0$
- If the roots of the auxiliary/characteristic equation are real and different (say $r_1, r_2, r_3 \ldots, r_k$) then C.F.= $A_1{r_1}^n + A_2{r_2}^n + A_3{r_3}^n + \cdots, +A_n{r_k}^n$
- If some of the roots of the auxiliary/characteristic equation are real and same (say $r_1, r_1, r_2 \ldots, r_k$) then C.F.= $(A_1 + A_{2.}n){r_1}^n + A_3{r_2}^n + A_4{r_3}^n + \cdots, +A_n{r_k}^n$
- If the roots of the auxiliary/characteristic equation are complex (Imaginary) say $(\alpha \pm i\beta)$
  Then the complementary function is C.F.= $\rho^n(Acosn\theta + Bsinn\theta)$, where $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

Keywords

Recurrence relation: a formula expressing terms of a sequence, except for some initial terms, as a function of one or more previous terms of the sequence.

Degree of recurrence relation: Highest subscripts- least subscripts

The shift operator:$E(a_n) = a_{n+1}$

The Δ operator:$\Delta(a_n) = a_{n+1} - a_n$

Linear homogeneous recurrence relation with constant coefficients: a recurrence relation that expresses the terms of a sequence, except initial terms, as a linear combination of previous terms.

Roots of auxiliary/characteristic equation: the roots of the polynomial associated with recurrence relation.

Self-assessment

Suppose F is defined recursively by F(n+1) =2*F(n)+3 with F (1) =9 then the value of F (0) is

a) 3
b) 1
c) 45
d) 21

2. Suppose F is defined recursively by F(n+1) =2*F(n)+3 with F (1) =9 then the value of F (5) is

a) 185
b) 189
c) 21
d) 45

3. Using the Euclidean algorithm, the Greatest Common Deviser (G.C.D.) of 11, 9 is similar to

a) G.C.D. (9,2)
b) G.C.D. (9,3)
c) G.C.D. (9,6)
d) G.C.D. (9,12)

4. Suppose F is defined recursively by F(n+1) =F(n)+2 with F (1) =3 then the value of F (4) is

a) 5
b) 9
c) 3
d) 4

5. Which one the following is the recurrence relation for tower of Hanoi game for three towers

a) $M_n = 2M_{n-1} + 1$
b) $M_n = 3M_{n-1} + 1$
c) $M_n = 2M_{n-1} + 2$
d) $M_n = 2M_{n-1} - 1$

6. How many minimum numbers of moves are required to finish the tower of Hanoi game for 3 towers and 5 disks?

a) 7
b) 31
c) 15
d) 63

7. How many minimum numbers of moves are required to finish the tower of Hanoi game for 3 towers and 4 disks?

a) 7
b) 31
c) 15
d) 63

8. Which one the following is the recurrence relation for Fibonacci sequence

a) $M_n = M_{n-1} + M_{n-2}$
b) $M_n = 3M_{n-1} + 1$
c) $M_n = 2M_{n-1} + 1$
d) $M_n = 2M_{n-1} - 1$

9. What is the degree of recurrence relation $M_n = 2M_{n-1} + 1$

a) 1
b) 2
c) 3
d) 4

10. What is the degree of recurrence relation $M_n = M_{n-1} + M_{n-2}$

a) 1

b) 2
c) 3
d) 4

11. Which one the following is the non-homogeneous recurrence relation with constant coefficient.

a) $M_n = M_{n-1} + M_{n-2}$
b) $M_n = 3M_{n-1}$
c) $M_n = 2M_{n-1} + 1$
d) $M_{n+1} - 2M_n$=0

12. Which one the following is the linear recurrence relation with constant coefficient.

a) $M_n = M_{n-1} + M_{n-2}$
b) $M_{n+1} = 3M_n M_{n-1}$
c) $(M_n)^2 = 2M_{n-1}$
d) $M_{n+1} - 2M_n M_{n-1}$=0

13.The solution of the recurrence relation$M_{n+1} = 5M_n - 6M_{n-1}$

a) $\boldsymbol{C_1 3^n + C_2 2^n}$
b) $\boldsymbol{C_1 3^n + C_2 5^n}$
c) $\boldsymbol{C_1 5^n + C_2 2^n}$
d) $\boldsymbol{C_1 3^n + C_2 n 3^n}$

14. The solution of the recurrence relation$\boldsymbol{M_{n+2} - 6M_{n+1} + 9M_n = 0}$

a) $\boldsymbol{C_1 3^n + C_2 2^n}$
b) $\boldsymbol{C_1 3^n + C_2 5^n}$
c) $\boldsymbol{C_1 5^n + C_2 2^n}$
d) $\boldsymbol{C_1 3^n + C_2 n 3^n}$

15. The expression $\boldsymbol{E(M_{n+2})} =$ ______

a) $\boldsymbol{M_{n+2}}$
b) $\boldsymbol{M_{n+3}}$
c) $\boldsymbol{M_{n+1}}$
d) $\boldsymbol{M_n}$

Answer for Self Assessment

1.	A	2.	B	3.	A	4.	B	5.	A
6.	B	7.	C	8.	A	9.	A	10.	B
11.	C	12.	A	13.	A	14.	D	15.	B

Review Question

1. Find $E(e^n)$
2. Find$E^2(10n - 1)$
3. Find Δx^n
4. Find $\Delta^2(e^n)$
5. Find$\Delta^2(10n + 5)$

6. Find $\Delta^5 10$

7. What is the degree of $a_n - 5a_{n-1} + 6a_{n-4} = 5^n$

8. What is the degree of $a_{n+4} - 5a_{n-1} + 6a_{n-4} = 0$

7. Find the general solution of $a_{n+2} - 15a_{n+1} + 56a_n = 0$

8. Find the general solution of $a_{n+2} - 10a_{n+1} + 21a_n = 0$

9. Find the general solution of $a_{n+2} - 6a_{n+1} + 9a_n = 0$

10. Find the general solution of $a_{n+2} - 20a_{n+1} + 100a_n = 0$

11. Find the general solution of $a_{n+2} - 4a_{n+1} + 9a_n = 0$

12. Find the general solution of $a_{n+2} - 8a_{n+1} + 17a_n = 0$

13. Find the auxiliary equation of $a_n - 16a_{n-4} = 0$

14. Find the auxiliary equation of $a_n - 25a_{n-1} + 50a_{n-2} = 0$

15. Find the auxiliary equation of $a_{n+2} - 8a_{n+1} + 17a_{n-50} = 0$

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Rajesh Kumar, Lovely Professional University

Unit 07: Introduction and Basic terminology of Graph

CONTENTS

Objectives

A storage area network can link all of the computers in a house, including embedded devices and personal computers, as well as network adapters like printers and traitors. Some of these networks use a graphical architecture in which all devices are linked to a single control node. Signals are transmitted from several different locations. A graph complete-bipartition is one way to view a local area network. Graph designs can be used to solve problems in virtually any discipline. We'll use examples to see how graphs can be used as templates in many situations. For example, we'll see how graphs are used to reflect the rivalry between various organisms in an ecological niche, who controls whom in an association, and the results of round-robin tournaments.

After this unit, you would be able to

- understand the basic terminology of graphs and their types.
- lean the handshaking theorem for the degree of vertices in the graph
- describe the digraphand special graphs
- find the isomorphism between two or more graphs

Introduction

Graphs are the structures made up of nodes and edges that link them. Graphs can be used to model interpersonal relationships, study collaboration, texts and calls between numbers, and website connections. We'll see how diagrams can be used to model roadmaps and work assignments for an organization's workers. In this segment, we go over some of the fundamental terms of graphs. We can use this vocabulary to solve a variety of problems like figuring out whether a graph can be drawn in the plane without any of its edges crossing. We'll also go through a few main graph classes that are often used as examples and in prototypes. Several relevant implementations involving these special kinds of graphs will be discussed.

7.1 Graphs

A graph G = (V, E) is made up of V, which is a non - empty set of vertices (or nodes), and E, which is a set of edges.

Each edge is connected with one or two vertices, which are referred to as its endpoints. An edge is described as a link that connects two or more endpoints.

Example

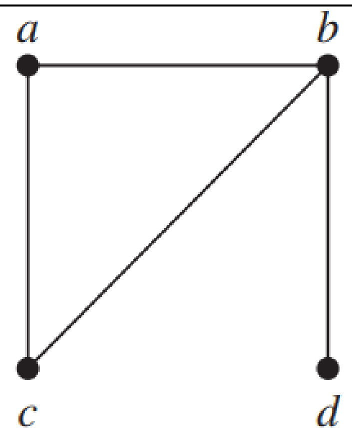


Figure 1:Graph G

Figure 1 represents the graph G which consists of non-empty set vertices V= {a, b, c d} and a set of edges

E = {a-b, a-c, b-c and b-d}.

The edges of the graph also can be described as $ab, ac, bc, and\ bd$. Infect we also can assign the particular number to each edge (Expressed in Figure 2).

$a - b = ab = e1$

$a - c = ac = e2$

$b - c = bc = e3$

$b - d = bd = e4$

G

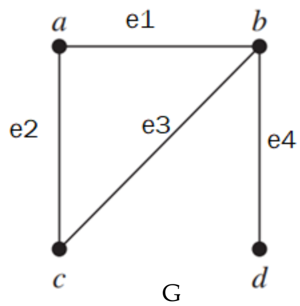


Figure 2: Edges of G are numbered

7.2 Types of graphs and basic terminology

Infinite graph

A graph is said to be infinite if it contains an infinite number of vertices or edges

Example

Consider the following pattern which never ends at any node is one of the examples of an infinite graph

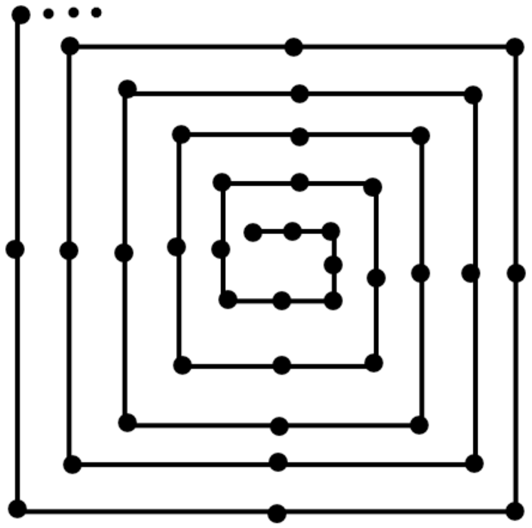


Figure 3: Example of an infinite graph

Finite graph

A graph is said to befinite if it contains a finite number of vertices and edges

Example

The graph of flight networks between some cities. In figure 4 there are finite 4 vertices as the cities and 6 edges as the air connectivity between them

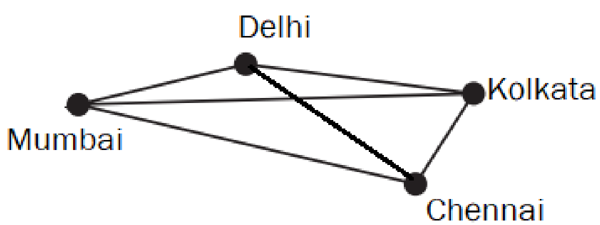


Figure 4:graph of flight network

Simple graph

A simple graph is one in which each edge connects two distinct vertices and no two edges join the same pair of nodes.

Example:

The graphs represented by Figure 2 and Figure 4 are simple graphs. Here we can see there exist a single edge between every pair of vertices

Multigraph

A Multigraph is one in which multi-edges connect two distinct vertices. This means any pair of verticescan be associated with more than one edge.

Example:

If we consider one situation in which we focus on "How two cities are linked by different kind roots like air, railway, and roads. The graph represented in Figure 5is an example of the multigraph. Here we can see there exist more than one edge between Delhi and Mumbai. And three edges are connecting the same pair of vertices Mumbai and Chennai.

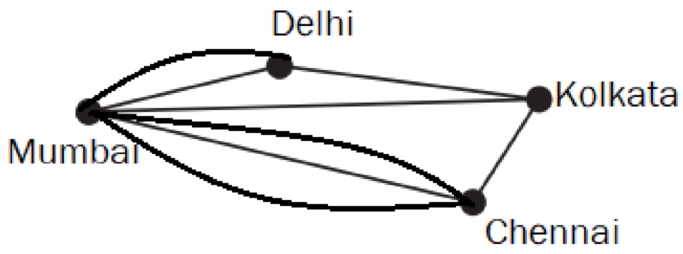


Figure 5: Example of Multigraph

Loop

A vertex is related to itself by the edge and such edge is known as the loop.

Example

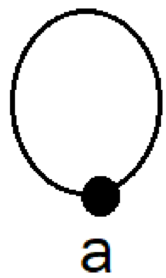


Figure 6: Example of loop

Pseudographs.

Pseudographs are graphs that can include loops and potentially several edges linking the same pair of vertices or a vertex to itself.

Example:

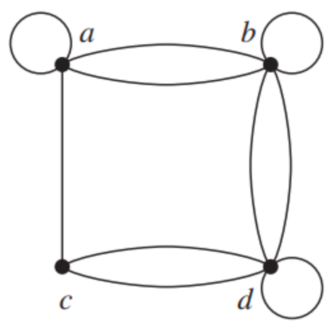


Figure 7: Example of loop graph

Undirected graph

Let G = (V, E) be a graph with a non - empty set of vertices (or nodes), and E, which is a set of edges. It is said to be undirected if every edge in G is undirected.

Example:

All the graphs which are represented in Figure 1 to Figure 7 are undirected graphs. Here we do not have any directed edge between any pair of vertices.

Directed graph

Let G = (V, E) be a graph with a non - empty set of vertices (or nodes),and E, which is a set of directed edges.

Each directed edge corresponds to an ordered pair of nodes. The ordered pair (a, b) is said to begin at a and ends at b.

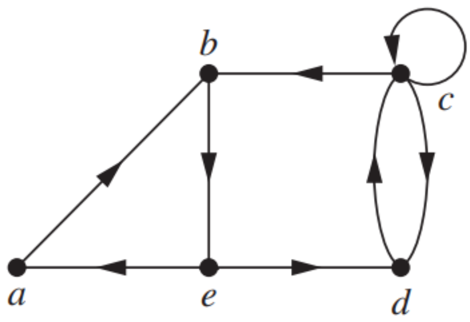


Figure 8: Digraph

All the graphs which are represented in Figure 8 are directed

The degree of a vertex in an undirected graph

In an undirected graph, the degree of a vertex is the number of edges that incident to it, so that a loop at a vertex adds twice to the degree of that vertex.

The vertex's degree v is denoted by deg (v)

Consider the graph G and H in the following Figure9.

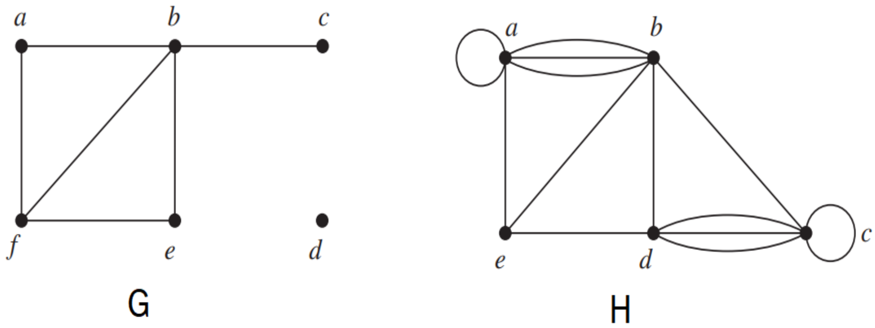


Figure 9: The Graph G and H

deg (v_i) in Graph G	deg (v_i) in Graph H
$\deg(a) = 2$	$\deg(a) = 6$
$\deg(b) = 4$	$\deg(b) = 6$
$\deg(c) = 1$	$\deg(c) = 6$
$\deg(d) = 0$	$\deg(d) = 5$

$\deg(e) = 2$	$\deg(e) = 3$
$\deg(f) = 2$	

The handshaking theorem

Let G = (V, E) be an undirected graph with $|E| = n$ edges and $|V| = m$. Then $\sum_{i=1}^{m} \deg(v_i) = 2|E|$

Let us consider the following graph G

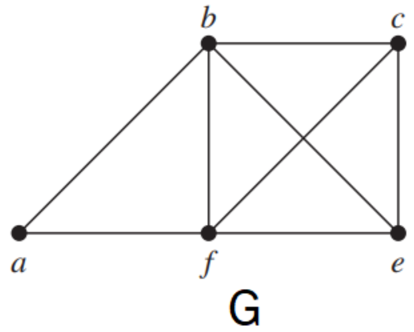


Figure 10: Graph G

Here we have 5 vertices and 8 edges

$\deg(v_i)$ in Graph G	$\sum_{i=1}^{m} \deg(v_i)$
$\deg(a) = 2$	$\sum_{i=1}^{5} \deg(v_i) = \deg(a) + \deg(b) + \deg(c) + \deg(e) + \deg(f)$
$\deg(b) = 4$	$\sum_{i=1}^{5} \deg(v_i) = 2 + 4 + 3 + 3 + 4 = 16$
$\deg(c) = 3$	$16 = 2\|E\|$
$\deg(e) = 3$	$\|E\| = 8$
$\deg(f) = 4$	

Note that this applies even if multiple edges and loops are present.

Let us consider the following graph G

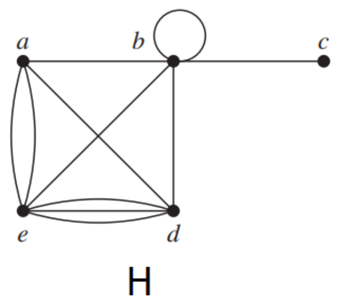


Figure 11:Graph H

Here we have 5 vertices and 11 edges

deg (v_i) in Graph G	$\sum_{i=1}^{m} \deg\ (v_i)$
$\deg(a) = 4$	$\sum_{i=1}^{5} \deg\ (v_i) = \deg(a) + \deg(b) + \deg(c) + \deg(d) + \deg(e)$
$\deg(b) = 6$	$\sum_{i=1}^{5} \deg\ (v_i) = 4 + 6 + 1 + 5 + 6 = 22$
$\deg(c) = 1$	$22 = 2\|E\|$
$\deg(d) = 5$	$\|E\| = 11$
$\deg(e) = 6$	

Adjacent vertices

If two vertices a and b in an undirected graph G are endpoints of an edge e of G, they are called adjacent in G.

An edge e of this kind is said to be incident with the vertices a and b.

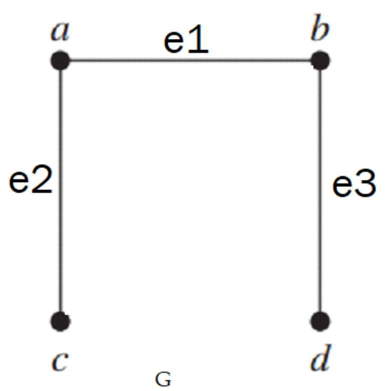


Figure 12:Graph G

In Figure 12, graph G has the vertices a, b, c, and d. The vertices a and b are the endpoints of edge e1. Hence e1 incidents to the vertices a and b.The vertices a and c are the endpoints of edge e2.

Hence e2 incidents to the vertices a and c. Similarly, the vertices d and b are the endpoints of edge e3. Hence e3 incidents to the vertices d and b.

The degree of vertices in a digraph

Let (a, b) is the directed edge of the graph G. a is said to be adjacent to b, and bis said to be adjacent from a.

The vertex a is called the initial vertex of edge (a, b), and b is calledthe terminal or end vertex of edge (a, b).

The initial vertex and terminal vertex of a loop are thesame.

The in-degree of a vertex a in a graph with directed edges, denoted by $\deg^{-}(a)$, is the number of edges with an as their terminal vertex.

The number of edges with an as their initial vertex, denoted by deg+(v), is the out-degree of v.

It's worth noting that a loop at a vertex adds 1 to both the in-degree and out-degree of the vertex.

Consider the following digraph.

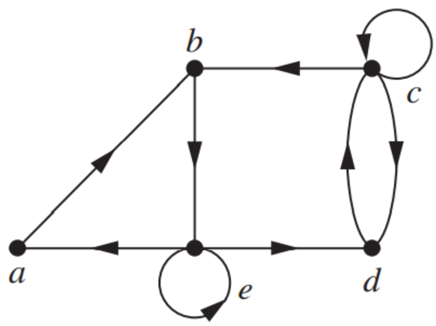


Figure 13:The digraph G

$\deg^{-}(v_i)$ in Graph G (In degree of v)	$\deg^{+}(v_i)$ in Graph G (Out degree of v)
$\deg^{-}(a) = 1$	$\deg^{+}(a) = 1$
$\deg^{-}(b) = 2$	$\deg^{+}(b) = 1$
$\deg^{-}(c) = 2$	$\deg^{+}(c) = 3$
$\deg^{-}(d) = 2$	$\deg^{+}(d) = 1$
$\deg^{-}(e) = 2$	$\deg^{+}(e) = 3$

Let G = (V, E) be a directed graph with $|E| = n$ edges and $|V| = m$. Then

$\sum_{i=1}^{m} \deg^{-}(v_i) = \sum_{i=1}^{m} \deg^{+}(v_i) = |E|$.

In the above digraph

$$\sum_{i=1}^{m} \deg^{-}(v_i) = 9 = \sum_{i=1}^{m} \deg^{+}(v_i) = |E|$$

7.3 Special simple graphs

Complete Graphs

Complete Graphs A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.

Figure 14 shows the complete graph for n=1,n=2, n=3, n=4, n=5, and n=6.

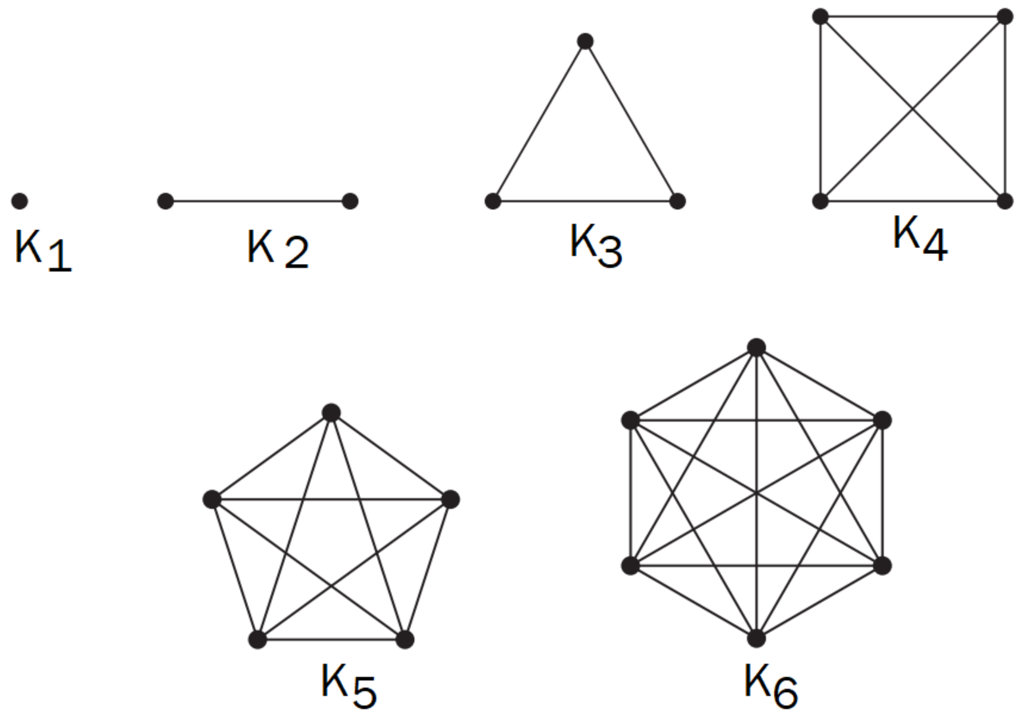


Figure 14: The complete graph for n=1,2,3...,6

Cycles

A cycle C_n, n ≥ 3, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,...,$\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$,. The cycles C_n, for n =3, n =4, and n =5 areshown in Figure 15.

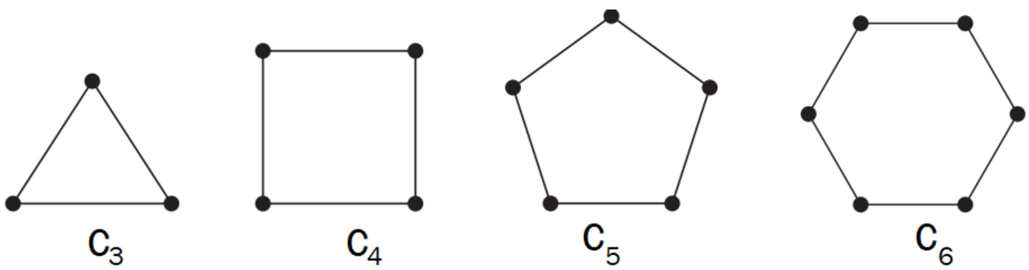


Figure 15:The cycles C_n, for n =3, n =4, and n =5

Wheels

Let C_n, n ≥ 3 be the cycle then the wheel W_n with n outer vertices is obtained from $C_n$if we add one additional(central) vertex to the cycle C_n. The added vertex is connected to every vertex of C_n

The cycles W_n, for n =3, n =4, and n =5 areshown in Figure 16

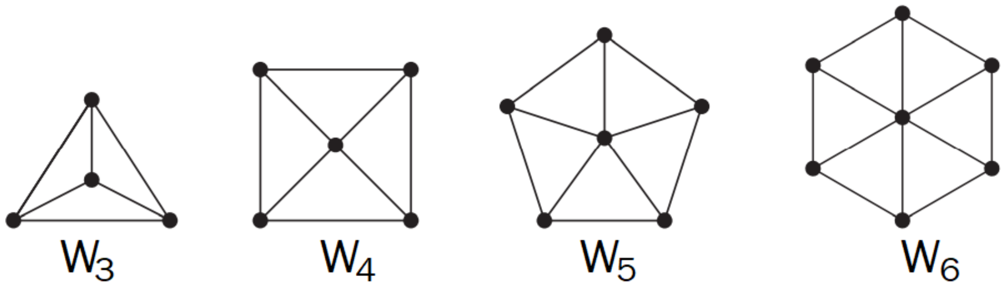


Figure 16:The WheelW_n, for n =3, n =4, and n =5

7.4 Graph Isomorphism

Adjacency Matrices

Suppose that G = (V , E) is a simple graph where |V | = n.Then the adjacency matrix is A = $[a_{ij}]$ where $a_{ij} = \begin{Bmatrix} 1 & if\ \{vi, vj\}\ is\ an\ edge\ of\ G \\ 0 & Otherwise \end{Bmatrix}$

Consider the following figure17

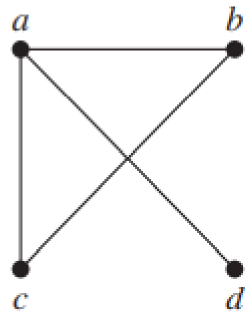


Figure 17: Graph G for adjacency matrix

$a_{ij} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is the adjacency matrix for order (a,b,c, and d)

Degree sequence

Let G be a graph with m vertices and n edges then the degree sequence of G is the sequence of the degree of the vertices in decreasing order. The graph G in Figure 17 has the degree sequence

{3,2,21}.

The Isomorphic graphs

The simple graphs$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f (a) and f (b) are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism.

It means -

The number of vertices in both graphs should be the same.

The number of edges in both graphs should be the same.

The degree sequence in both graphs should be the same.

The adjacency matrices of one-one and onto mapping should be the same.

Consider the graphs in Figure 18

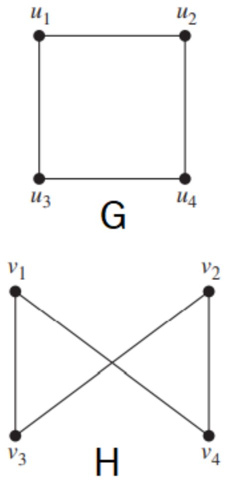


Figure 18: Graph G and H

Here

Number of vertices in G =4	Number of edges in H=4
Number of vertices in G=4	Number of edges in H=4
The degree sequence in G = {2,2,2,2}	The degree sequence in H = {2,2,2,2}
Mapping $f\ (u_1)\ =\ v_1$ $f\ (u_2)\ =\ v_4$ $f\ (u_3)\ =\ v_3$ $f\ (u_4)\ =\ v_2$	$a_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ for order$\{u_1, u_2, u_3, u_4\}$ $a_H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ for order$\{v_1, v_4, v_3, v_2\}$

Hence both Graphs G and H are isomorphic

Consider the following graphs represented in Figure 19

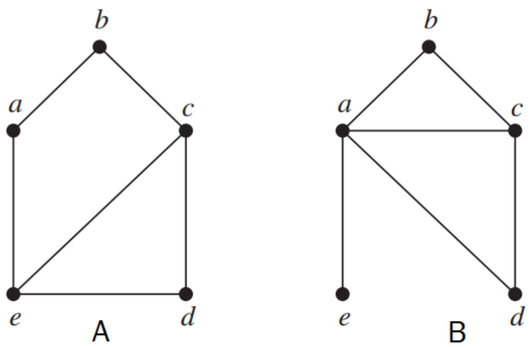


Figure 19: Graph A and B

Here

Number of vertices in A =5	Number of edges in B=5
Number of vertices in A=6	Number of edges in B=6
The degree sequence in G = {3,3,2,2,2}	The degree sequence in H = {4,3,2,2,1}

The degree sequences in both the graphs are not the same it means the one-one onto mapping is not possible. So the graphs are not isomorphic.

Summary

- A graph G = (V, E) is made up of V, which is a non - empty set of vertices (or nodes), and E, which is a set of edges.
- A graph is said to be infinite if it contains an infinite number of vertices or edges
- A graph is said to befinite if it contains a finite number of vertices and edges
- A simple graph is one in which each edge connects two distinct vertices and no two edges join the same pair of nodes.
- A Multigraph is one in which multi-edges connect two distinct vertices. This means any pair of verticescan be associated with more than one edge.
- A vertex is related to itself by the edge and such edge is known as the loop.
- Pseudographs are graphs that can include loops and potentially several edges linking the same pair of vertices or a vertex to itself.
- Let G = (V, E) be a graph with a non - empty set of vertices (or nodes), and E, which is a set of edges. It is said to be undirected if every edge in G is undirected.
- Let G = (V, E) be a graph with a non - empty set of vertices (or nodes),and E, which is a set of directed edges.Then G is a digraph
- In an undirected graph, the degree of a vertex is the number of edges that incident to it, so that a loop at a vertex adds twice to the degree of that vertex.
- Let G = (V, E) be an undirected graph with$|E|$ = n edges and $|V|$ = m. Then $\sum_{i=1}^{m} \deg(v_i) = 2|E|$
- The in-degree of a vertex a in a graph with directed edges, denoted by deg-(a), is the number of edges with an as their terminal vertex.
- The number of edges with an as their initial vertex, denoted by deg+(v), is the out-degree of v.
- Complete Graphs A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.
- A cycle C_n, n ≥ 3, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,…, $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.
- Let C_n, n ≥ 3 be the cycle then the wheel W_n with n outer vertices is obtained from $C_n$if we add one additional(central) vertex to the cycle C_n. The added vertex is connected to every vertex of C_n
- The simple graphs$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from $V_1$to V_2 with the property that a and b are adjacent in G_1 if and only if f (a) and f (b) are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism.
- The adjacency matrix of an undirected graph is A = $[a_{ij}]$

$$\text{where } a_{ij} = \begin{Bmatrix} 1 & if\ \{vi, vj\}\ is\ an\ edge\ of\ G \\ 0 & Otherwise \end{Bmatrix}$$

Keywords

Loop: an edge connecting a vertex with itself.

Undirected graph: a set of vertices and a set of undirected edges each of which is associated with a set of one or two of these vertices

Simple graph: an undirected graph with no multiple edges or loops

Multigraph: an undirected graph that may contain multiple edges but no loop

Pseudograph: an undirected graph that may contain multiple edges and loops

Directed graph: a set of vertices together with a set of directed edges each of which is associated with an ordered pair of vertices.

Degree of the vertex v in an undirected graph: the number of edges incident with v with loops counted twice

In-degree of the vertex v in a graph with directed edges: the number of edges with v as their terminal vertex

Out-degree of the vertex v in a graph with directed edges: the number of edges with v as their initial vertex

Kn (complete graph on n vertices): the undirected graph with n vertices where each pair of vertices is connected by an edge

Cn (cycle of size n), n ≥ 3: the graph with n vertices v1, v2,..., vn and edges {v1, v2}, {v2, v3},..., {vn−1, vn}, {vn, v1}

Wn (wheel of size n), n ≥ 3: the graph obtained from Cn by adding a vertex and edges from this vertex to the original vertices in Cn

Isomorphic simple graphs: the simple graphs G1 = (V1, E1) and G2 = (V2, E2) are isomorphic if there exists a one-to-one correspondence f from V1 to V2 such that {f (v1), f (v2)} ∈ E2 if and only if {v1, v2} ∈ E1 for all v1 and v2 in V1

Self-Assessment

1.Which of the following is true?

A. A graph may contain no edges but at least one vertex
B. A graph may contain many edges and no vertices
C. A graph may contain no edges and no vertices
D. A graph may contain no vertex but at least one edge

2.A graph that containsa single edge between every pair of different vertices is called___

A. a simple graph
B. a multigraph
C. a loop
D. a Pseudo graph

3.A graph that contains multiple edges between any pair of different vertices is called___

A. a simple graph
B. a multigraph
C. a loop
D. a Pseudo graph

4.How many edges are there in a graph with 10 vertices each of degree five

A. 25

B. 30

C. 20

D. 35

5.Let G be an undirected graph then the number of vertices of odd degree in G is?

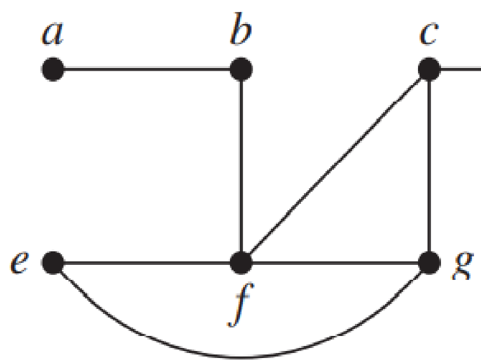
A. Always even

B. Always odd

C. Always a prime

D. Always the multiple of 5

6.The degree of vertex f in the following graph



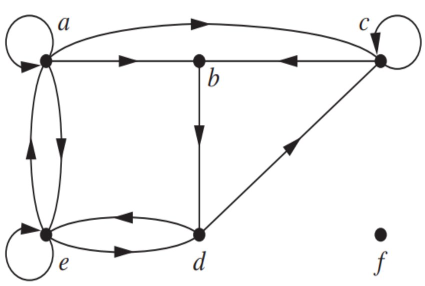
A. 2

B. 3

C. 4

D. 5

7.The in-degree of vertex b in the following graph is



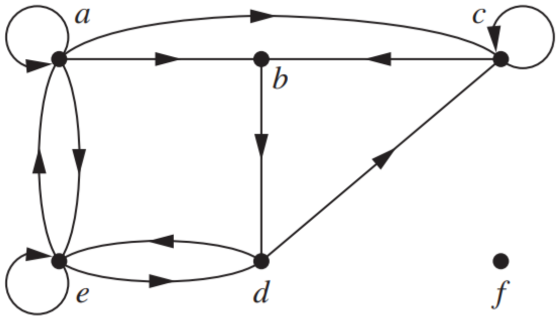
A. 1

B. 2

C. 3

D. 4

8.The out-degree of vertex d in the following graph is



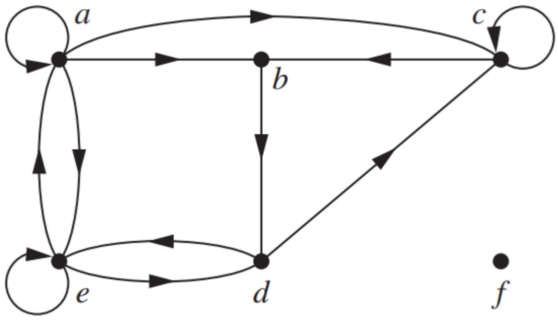
A. 1

B. 2

C. 3

D. 4

9.The sum of all out-degrees of the following graph is



A. 10

B. 12

C. 13

D. 14

10. The total number of edges in K_5

A. 20

B. 10

C. 5

D. 40

11. The total number of edges in C_5

A. 20

B. 10

C. 5

D. 40

12. The total number of vertices in W_5

A. 20

B. 6

C. 5

D. 40

13. The total number of edges in W_5

A. 20

B. 6

C. 5

D. 10

14. TheK_3 is isomorphic to

A. C_3

B. K_4

C. W_3

D. C_4

15. Two graphs are said to be isomorphic if

A. Only the number of vertices in both graphs are equal

B. Only the number of edges in both graphs are equal

C. Only the number of the degree sequences in both graphs are equal

D. There exists a one to one on to edge-preserving mapping vertices set of both the graphs

Answer for Self Assessment

1.	A	2.	A	3.	B	4.	A	5.	A
6.	C	7.	A	8.	B	9.	B	10.	C
11.	B	12.	B	13.	D	14.	A	15	D

Review Questions:

1. Find the degree sequence of $\boldsymbol{K_5}$

2. Find the degree sequence of $\boldsymbol{C_5}$

3. Find the degree sequence of $\boldsymbol{W_5}$

4. Find the $\sum_{i=1}^{m} \deg(v_i)$ for $\boldsymbol{K_{10}}$

5. Find the $\sum_{i=1}^{m} \deg(v_i)$ for $\boldsymbol{W_{10}}$

6. Find the $\sum_{i=1}^{m} \deg(v_i)$ for $\boldsymbol{C_{10}}$

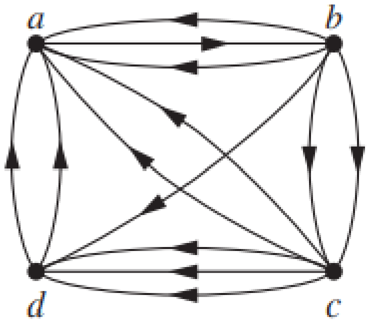
7. Find the adjacency matrix of $\boldsymbol{K_4}$

8. Find the adjacency matrix of $\boldsymbol{C_5}$

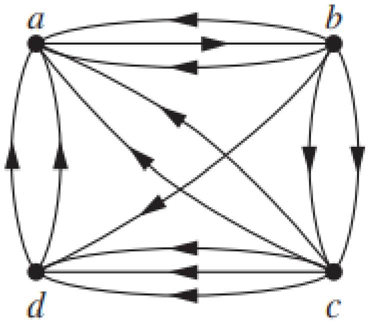
9. Find the adjacency matrix of $\boldsymbol{W_5}$

10. Check whether $\boldsymbol{C_3}$ **and** $\boldsymbol{K_3}$ **are isomorphic?**

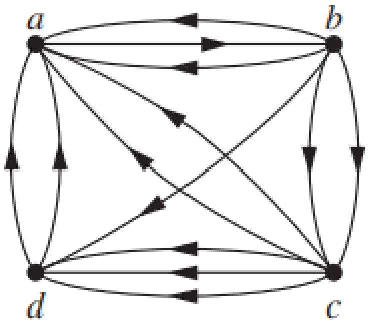
11. Find the in-degree of vertex b from the following graph



12. Find the out-degree of vertex b from the following graph



13. Verify that in the following graph sum of all the indegree is equal to the sum of all the outdegrees



14. Verify the handshaking lemma for a complete graph with 6 vertices
15. Verify the handshaking lemma for a cycle graph with 6 vertices

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 08: Paths

CONTENTS

Objectives:

If messages can be transmitted through one or more transitional devices, then what is the property of a network such that each pair of devices can exchange information? As a graph is used to describe this device network, the problem could be: When is there always a path between the pair of vertices? , What are the links/edges which are participated in the particular path?. We would, though, like to know how dependable this network is? For example,will all devices/computers be able to connect if a router or a networking connection goes down? We are now developing a new concept which is known as the path and connectivity to address these questions.

After this unit, you would be able to

- understand the path between any pair of vertices in the graph.
- describe the cut edges, cut vertices of a graph
- describe the complete bipartite graph using path length
- find the path isomorphism between two or more graphs

Introduction

Paths are created by walking along the links/edges of graphs can be used to design several problems. A graph model can be used to investigate whether a message can be transmitted between two computers over intermediate connections. Models can be used to solve problems such as reliable route preparation for mail collection, trash collection, computer network diagnostics, and so onthat include graphs and paths.

8.1 Path

Let G = (V, E) be an undirected graph with a non - empty set of vertices (V), and a set of edgesE. Let n be a non-negative integer then a path of length n between vertex (a, b) $where\ a, b \in$ V is a sequence of n edges e_i in G where i=1,2…,n. Each edge of the path is associated with the sequence of vertices such that e_1 is associated with (a,x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with $(x_{n-1}$, b).

Length of the path

The length of the path is the number of edges which are participated in the particular path.

A path is said to be simple if there is no repetition of any edge in it.

Consider the following figure for graph G and consider the vertices *a* and *b*.

Here we can see there are 3 simple paths available between vertex a and b. They can be represented based on their length as –

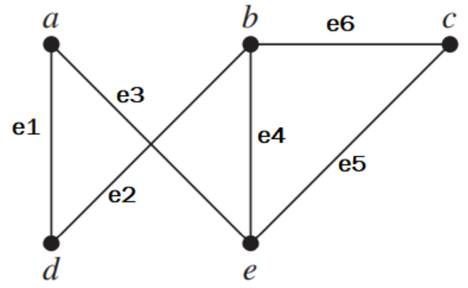


Figure 1: The graph G

Lenght	Path representation between vertex a and b						
2	a---d---b	or	*e1,e2*	or	*a,e1,d,e2,b*	or	Simply *a,d,b*
2	a---e---b	or	*e3,e4*	or	*a,e3,e,e4,b*	or	Simply *a,e,b*
3	a---e---c----b	or	*e3,e5,e6*	or	*a,e3,e,e5,c,e6,b*	or	Simply *a,e,c,b*

Circuit

A circuit is a path with a length greater than zero that starts and ends at the same vertex/node.

A circuit is said to be simple if there is no repetition of any edge in it.

Consider the following figure for graph H

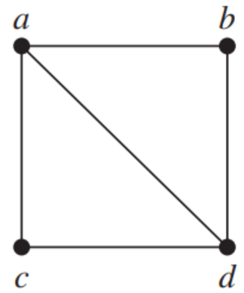


Figure 2; Graph H

Here we can see there are4 simple circuits available from vertex *a to a*. They can be represented as –

Lenght	Circuit representation from vertex *a to a*	
3	a---b---d---a	*a,d,b,a*

3	a---c---d---a	*a,c,b,a*
4	a---b---d---c---a	*a,b,d,c,a*
4	a---c---d---b---a	*a,c,d,b,a*

8.2 Connectivity in Undirected Graphs

Let G = (V, E) be an undirected graph with a non - empty set of vertices (V), and a set of edges E, then G is said to be connected if there exists a path between every pair of distinct vertices.

An undirected graph that is not connected is called disconnected. We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Consider the following figure 3.Here there are 3 graphs A,B, and C.

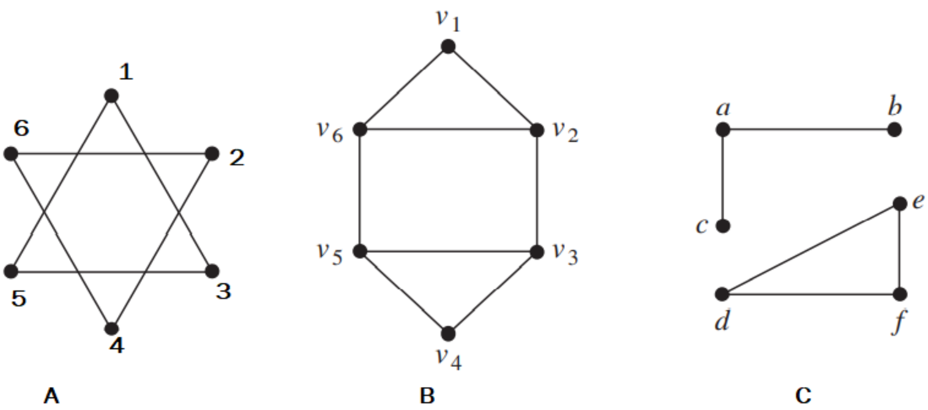


Figure 3:The connected and disconnected graph

Graph A is disconnected because we can not have any path between vertices *1 and 2*.

Graph B is said to be connected becausethere exists a path between every pair of distinct vertices.

Graph C is disconnected because we can not have any path between the vertices *a and d*.

Connected component

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

Example:

Consider the following figure 4.Here G is a disconnected graph and has 3 connected components G1,G2, and G3 of a G.These connected components are the subgraph of G

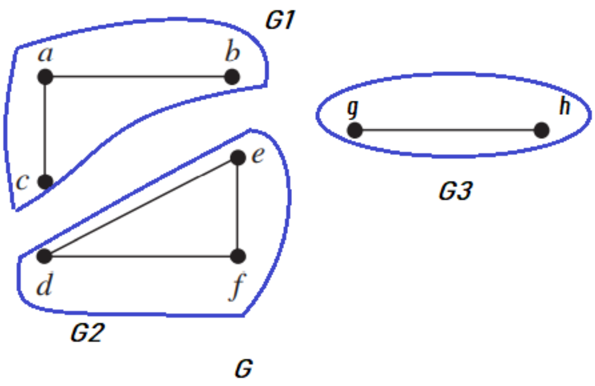


Figure 4: Example of connected components

Distance and diameter of a simple connected graph

Let G = (V, E) be a simple undirected graph with a non - empty set of vertices (V), and a set of edges E,then-

The length (number of edges) of the shortest path between two distinct vertices v1 and v2 of G is calledthe distance between v1 and v2.

The maximal distance between two distinct vertices is called the diameter of G

Example:

Consider the following figure 5.Here we will find the distance between vertices v_1 and v_2 in G and the diameter of G

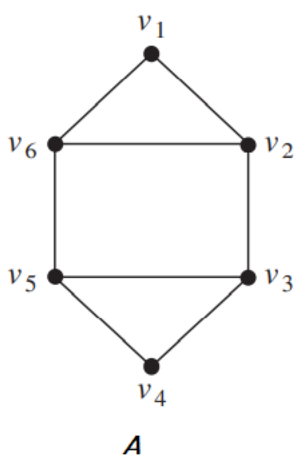


Figure 5: Graph A

There are 4 simple paths available between vertices v_1 and v_2.

The path v_1---v_2 has a length of 2

The path v_1---v_6---$v_2$has the length 3

The path v_1---v_6---v_5---v_3---v_2 has the length 4

The path v_1---v_6---v_5 ---v_4---v_3---v_2 has the length 5

Hence the distance between vertices v_1 and $v_2$is 1.

The diameter of G is 6

Cut points

Let G = (V, E) be a simple undirected graph with a non - empty set of vertices (V), and a set of edges E,then a vertex in G is said to be cut point or cut vertex if removal of that vertex converts the connected graph into disconnected one.

Cut edges or Bridges

Let G = (V, E) be a simple undirected graph with a non - empty set of vertices (V), and a set of edges E,then the cut edge or bridgeis the one edge in G whose removal converts the connected graph into a disconnected one.

While finding the cut edges and cut vertices we need to follow these rule –

- If a vertex is removed then all the incident edges towards the given vertex will also be removed.
- If anedge is removed then the endpoints of that edge will not be removed.

Consider the following figure 6 for cut vertices and edges

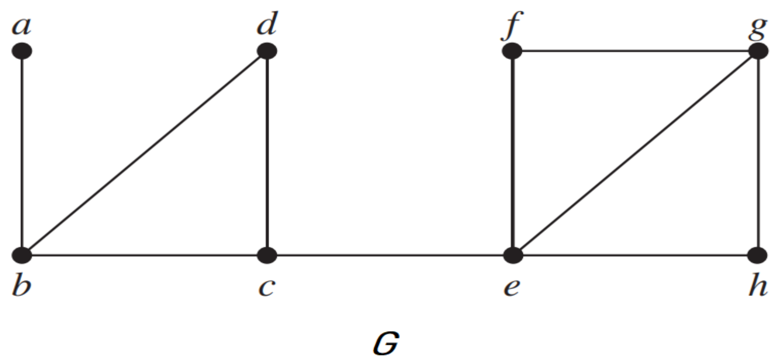


Figure 6:Graph G

Vertex	Is it a Cut vertex?	Edges	Is it a Cut Edge?
a	No	a---b	Yes
b	Yes	b---c	No
c	Yes	b---d	No
d	No	c---d	No
e	Yes	c---e	Yes
f	No	e---f	No
g	No	f---g	No
h	No	e---g	No
		g---h	No
		e---h	No

8.3 Connectedness in Directed Graphs

A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

A directed graph is **weakly connected** if there is a path between every two vertices in the underlying undirected graph

Consider the following figure 7 for strongly and weakly connected graph

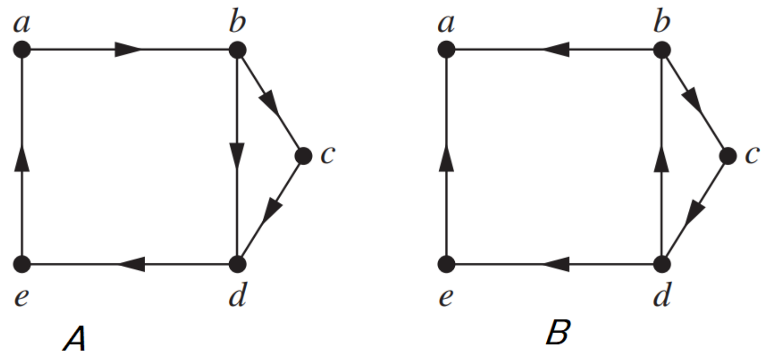


Figure 7: Strongly and weakly connected graph

A is strongly connected because there is a path from {(a to b) and (b to a)}, {(a to c) and (c to a)}, {(a to d) and (d to a)}, {(b to c) and (c to b)}, {(b to d) and (d to b)},{(b to e) and (c to d)}{(d to c), {(c to e) and (e to c)} , and {(d to e) and (e to d)}.

A is weakly connected because there is a path between every pair of distinct vertices.

There is no directed path from a to b in this graph so B is not strongly connected but it is weakly connected because There a directed path between every pair of distinct vertices.

8.4 Paths and Isomorphism

A helpful invariant that may be found to prove that two graphs are not isomorphic is the presence of a circuit of a certain length.

Paths can also be employed to build mappings for possible isomorphisms.

Two graphs are not isomorphic if they do not possess the same number of k length simple circuits.

Example 1:

Consider the following figure 8 for path isomorphism

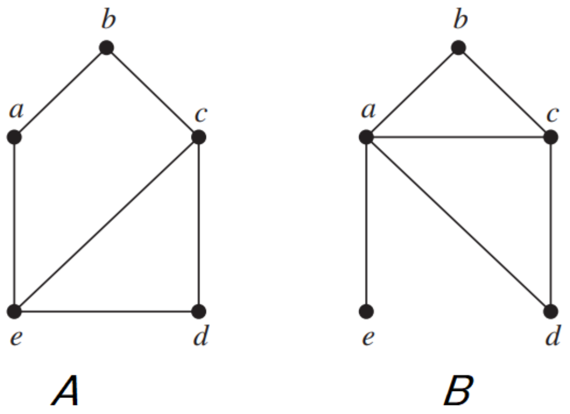


Figure 8:Non-isomorphic graph

Here a number of 3 length circuits in graph A is one while B has two such types of the circuit so both are not isomorphic.

Example2:

Consider the following figure 9 for path isomorphism.Here the $|C_4| = 2\ in\ graph\ A$ while $|C_4| = 3\ in\ graph\ B$. So A and B are not isomorphic.

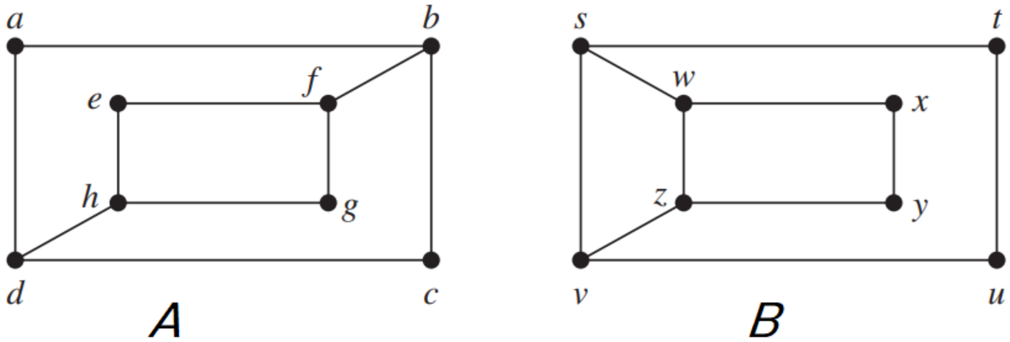


Figure 9: Non- isomorphic graph

8.5 Bipartite Graphs

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V1 and V2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Example:

The following graph is bipartite because the set vertices are divided into two groups and no two vertices of the same groups are connected to each other and every edge connects one endpoint in V1 and the other in V2.

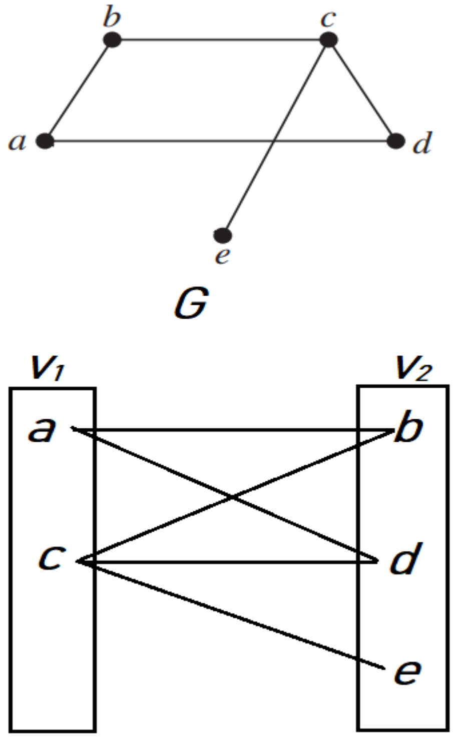
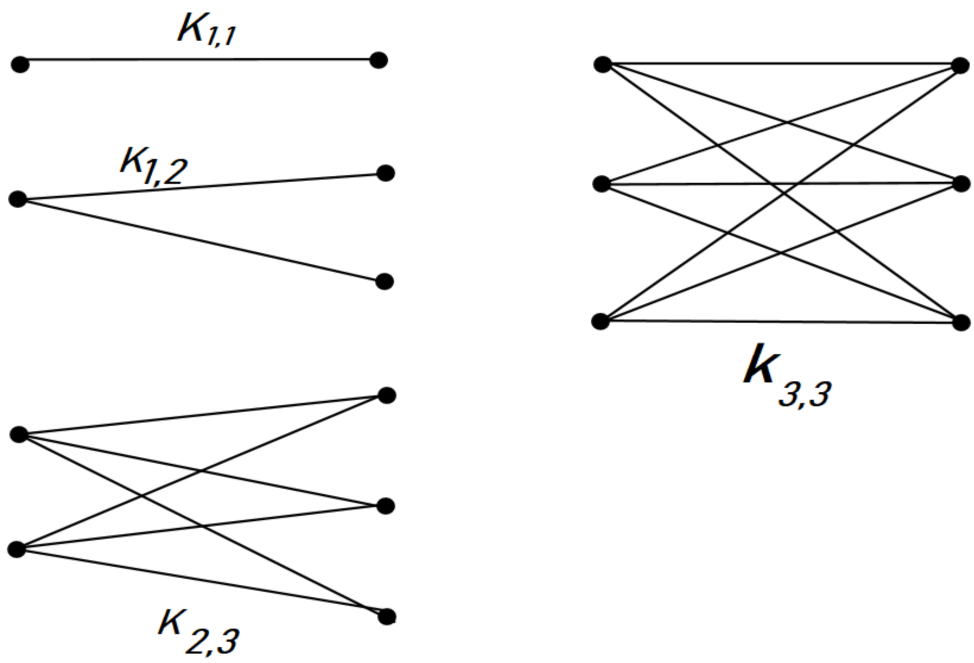


Figure 10: Example of the bipartite graph

Complete Bipartite graph

A complete bipartite graph Km,n is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

Example:



Summary

- Let G = (V, E) be an undirected graph with a non - empty set of vertices (V), and a set of edges E. Let n be a non-negative integer then a path of length n between vertex $(\boldsymbol{a}, \boldsymbol{b}) \boldsymbol{where a}, \boldsymbol{b} \in \mathbf{V}$ is a sequence of n edges $\boldsymbol{e_i}$ in G where i=1,2…,n.
- A circuit is a path with a length greater than zero that starts and ends at the same vertex/node.
- A path and circuit are said to be simple if there is no repetition of any edge in it.
- Let G = (V, E) be an undirected graph with a non - empty set of vertices (V), and a set of edges E, then G is said to be connected if there exists a path between every pair of distinct vertices.
- The length (number of edges) of the shortest path between two distinct vertices v1 and v2 of G is calledthe distance between v1 and v2.
- The maximal distance between two distinct vertices is called the diameter of G
- Let G = (V, E) be a simple undirected graph with a non - empty set of vertices (V), and a set of edges E,then a vertex in G is said to be cut point or cut vertex if removal of that vertex converts the connected graph into disconnected one.
- Let G = (V, E) be a simple undirected graph with a non - empty set of vertices (V), and a set of edges E,then the cut edge or bridgeis the one edge in G whose removal converts the connected graph into a disconnected one.
- A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- A directed graph is **weakly connected** if there is a path between every two vertices in the underlying undirected graph
- Two graphs are not isomorphic if they do not possess the same number of k length simple circuit
- A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V1 and V2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.
- A complete bipartite graph Km,n is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

Keywords

Path from u to v in an undirected graph: a sequence of edges e1, e2,...,en, where ei is associated to {xi, xi+1} for i = 0, 1,...,n, where x0 = u and xn+1 = v.

Simple path: a path that does not contain an edge more than once.

Circuit: a path of length n ≥ 1 that begins and ends at the same vertex.

Connected graph: an undirected graph with the property that there is a path between every pair of vertices.

Cut vertex of G: a vertex v such that G − v is disconnected.

The cut edge of G: an edge e such that G − e is disconnected.

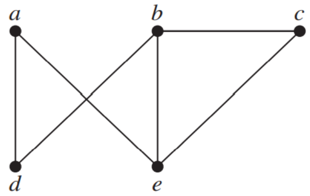
Strongly connected directed graph: a directed graph with the property that there is a directed path from every vertex to every vertex

Bipartite graph: a graph with vertex set that can be partitioned into subsets V1 and V2 so that each edge connects a vertex in V1 and a vertex in V2.

Km,n (complete bipartite graph): the graph with vertex set partitioned into a subset of m elements and a subset of n elements with two vertices connected by an edge if and only if one is in the first subset and the other is in the second subset

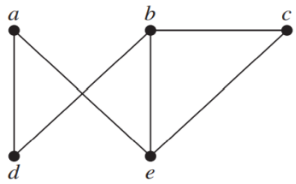
Self-Assessment

1) How many simple paths are available between vertex a to c in the following graph?



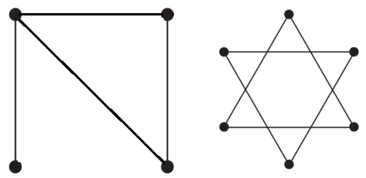
A. 4
B. 5
C. 3
D. 6

2) In the following graph, the distance between vertex a to d is?



A. 1
B. 2
C. 3
D. 5

3) Which one the following graph is disconnected

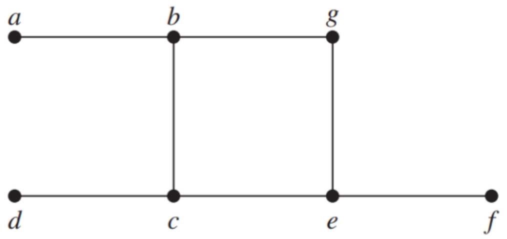


A

B

A. Only A
B. Only B
C. Both A and B
D. Neither A nor B

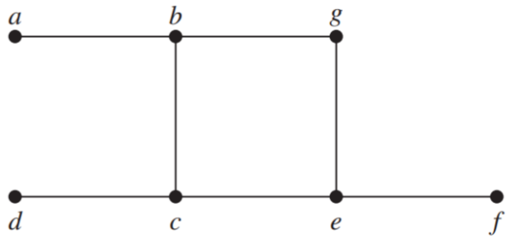
4) Which one of the following is a cut vertex of graph G?



G

A. b
B. a
C. g
D. f

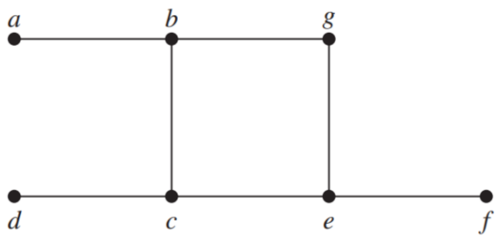
5) Which one of the following is a cut vertex of graph G?



G

A. a
B. d
C. c
D. g

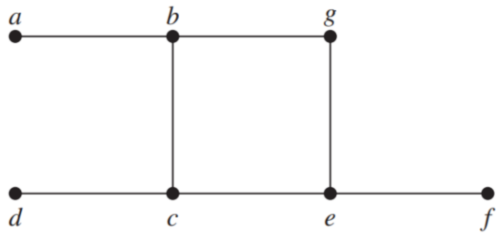
6) Which one of the following is a cut edge of graph G?



G

A. a-b
B. b-g
C. g-e
D. c-e

7) Which one of the following is a cut edge of graph G?



G

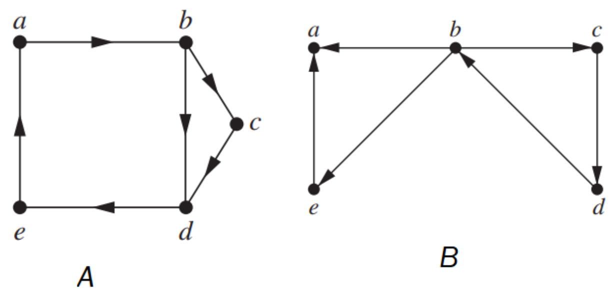
A. b-c

B. b-g

C. g-e

D. f-e

8) Which one of the following graphs is weakly connected?



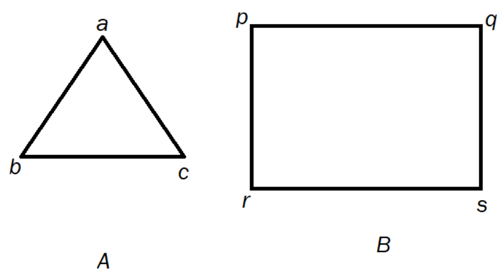
A. A

B. B

C. Both A and B

D. Neither A nor B

9) Which one of the following graphs is/are bipartite?



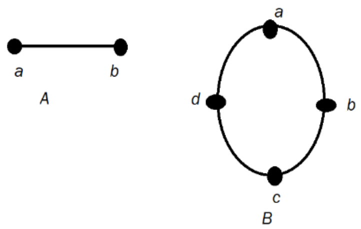
A. Only A

B. Only B

C. Both A and B

D. Neither A nor B

10) Which one of the following graphs is/are complete bipartite?



A. Only A
B. Only B
C. Both A and B
D. Neither A nor B

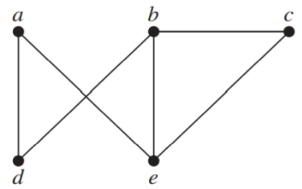
11) The number of vertices in $K_{5,10}$?
A. 5
B. 10
C. 15
D. 50

12) The number of edges in $K_{10,10}$?
A. 20
B. 100
C. 10
D. 50

13) The sum of the degree of all the vertices in $K_{5,10}$?
A. 50
B. 15
C. 150
D. 100

14) Which one of the following graphs is/are complete bipartite?
A. C_3
B. K_3
C. C_4
D. K_4

15) How many cycles of length 3 are available in the following graph?



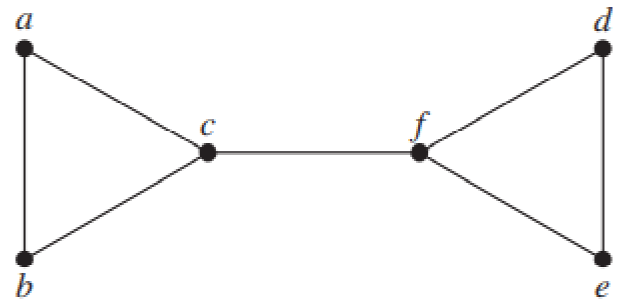
A. 1
B. 2
C. 3
D. 4

Answers for Self Assessment

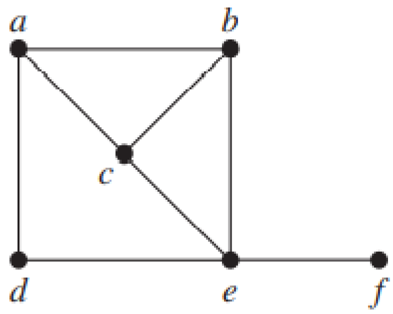
1.	A	2.	A	3.	B	4.	A	5.	C
6.	A	7.	D	8.	C	9.	B	10.	C
11.	C	12.	B	13.	D	14.	C	15.	A

Review Questions:

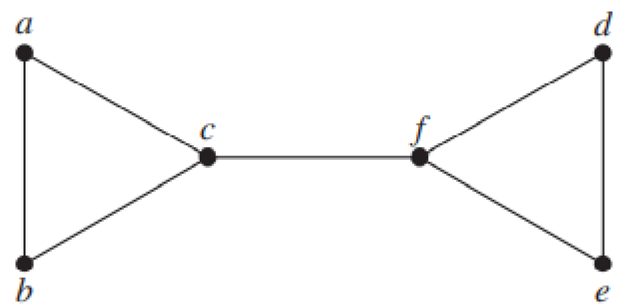
1. Find all the paths between vertices *a and d* in the following graph?



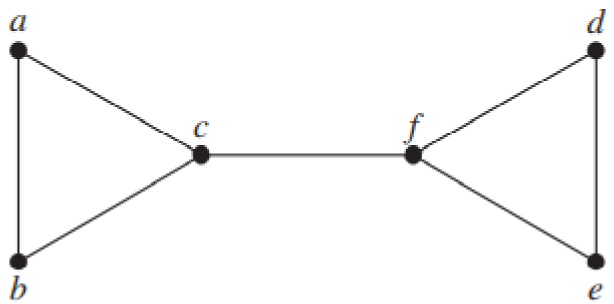
2. Find all the circuitsin the following graph?



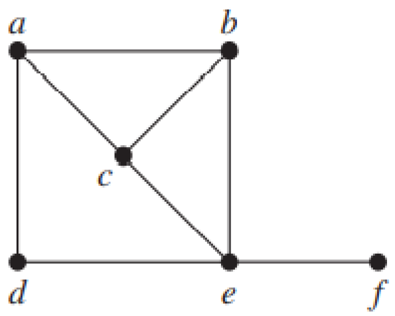
3. Find the cut edges of the following graph



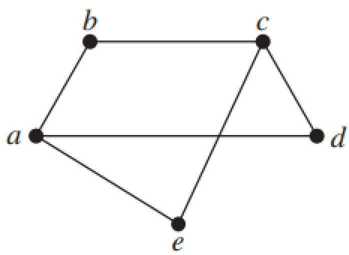
4. Find the cut vertices of the following graph?



5. Check whether the following graph is bipartite?

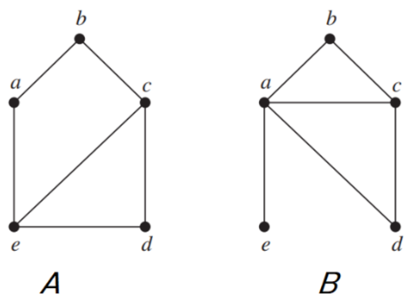


6. Check whether the following graph is completer bipartite?

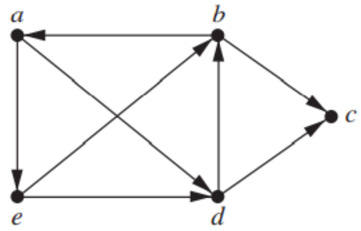


7. Check whether the graph C_4 is completer bipartite?

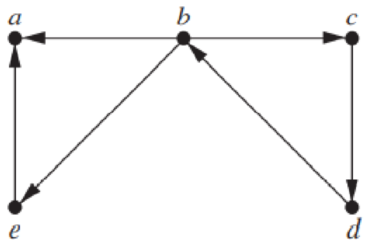
8. Check whether the following graphs are connected?



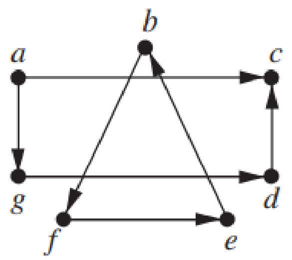
9. Check whether the following graph is strongly connected and if not, whether it is weakly connected.



10. Check whether the following graph is strongly connected and if not, whether it is weakly connected.



11. Determine whether the following graph is strongly connected and if not, whether it is weakly connected.



12. Using the path isomorphism check whether the graphs C_3 and k_3 are isomorphic?

13. Check whether the graph K_4 is completer bipartite?

14. Check whether the graph C_5 is bipartite?

15. Using the path isomorphism check whether the graphs C_4 and $k_{2,2}$ are isomorphic?

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

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Unit 09: Eulerian Graphs

CONTENTS

Objectives:

Many functional problems can be tackled using Euler paths and circuitry. Many frameworks, for example, seeking a route or circuit that precisely crosses exactly once each road in a city, each route in a transit network, each connection in a power system, or each link in a communications system. Solving such problems requires finding an Euler path or circuit throughout the required graph design If a postman will find an Euler path in the graph too that reflects the routes he wants to travel, for example, this path generates a route that crosses each street precisely once. Any streets would have to be navigated more often if no Euler route exists. In this segment, we'll look at these questions and talk about how difficult they are to answer. We'll discuss the Euler, Hamiltonian path, and circuit to see these classic puzzles as well as their new uses.

After this unit, you would be able to

understand the Euler and Hamiltonian path in the graph.

describe the Euler and Hamiltonian circuit in a graph.

lean the concept of maps and region using graphs

find the graphical representation of a map or region

Introduction

The town of Königsberg in Prussia (now Russia) was built on both ends of the Pregel River and consisted of two huge islands—Kneiphof and Loose—connected by seven crossings to each other or to the city's two mainland parts. The challenge was figuring out a route across the city that would only cross any of those bridges once.These regions and bridges are shown in Figure 1.

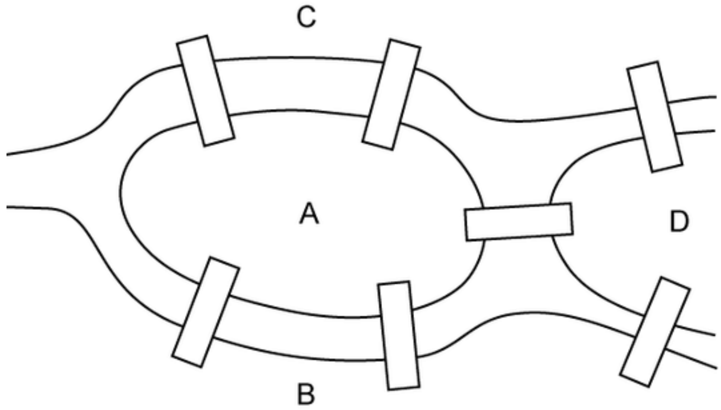


Figure 1: The Konigsberg bridge problem.

In order to describe the logical question explicitly, Euler proved that the problem is unsolvable by demonstrating that accessing a bank from a bridge other than one of the bridges is strictly forbidden.

The graphical model presented in Figure 2 can be used to solve the issue of going over every bridge without crossing it more than once.

Here the regions/ islands are converted into vertices and the bridges between two lands are being replaced by the edge. The Königsberg bridge problem can be visualized by the multi-graph.

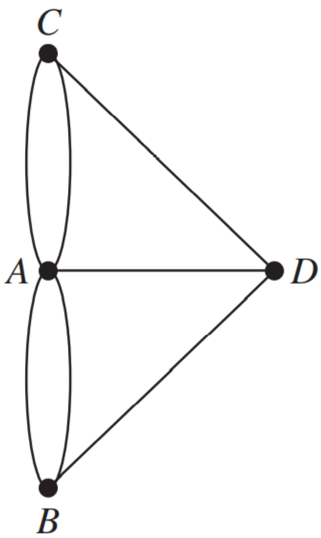


Figure 2: The graph of The Konigsberg bridge problem

9.1 Euler path and circuit

Euler path

An Euler path in G is a simple path containing every edge of G.

Euler circuit

An Euler circuit in a graph G is a simple circuit containing every edge of G.

Example1

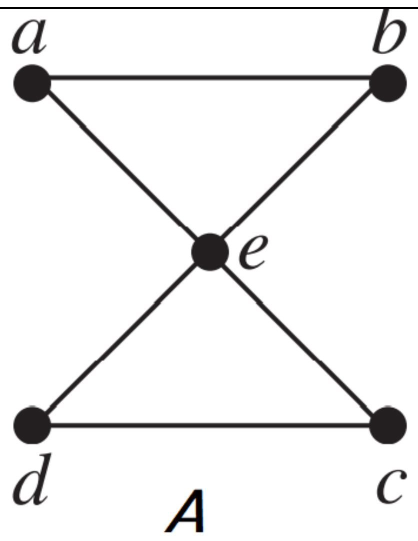


Figure 3: Graph A

Figure 3 represents graph A which has an Euler circuit, say,

$\{d - c - e - a - b - e - d\}$

Example 2:

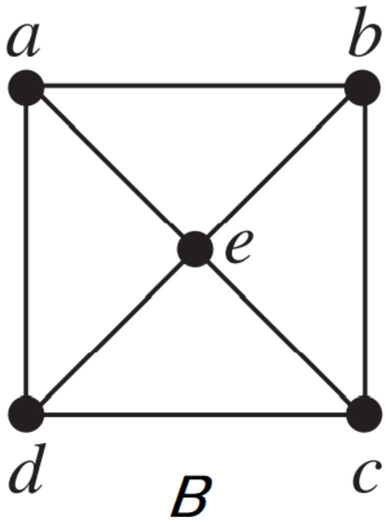


Figure 4: Graph B

Figure 4 represents graph B which does not possess any Euler circuit as well as any Euler path. In order to cover every edge of graph B, there exists a repetition of some edges.

Example 3:

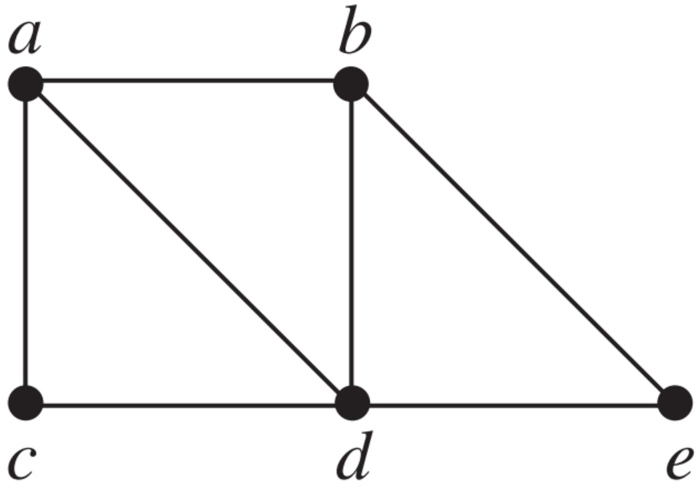


Figure 5: Graph C

Figure 5 represents graph C which does not possess any Euler circuit.In order to cover every edge for an Euler circuit of graph B, there exists a repetition of some edges. But It has an Euler path say,

$\{d - c - a - b - e - d - a\}$

Consider the following figure for digraphs D, E, and F. Check whether if any one of them is having an Euler circuit or only an Euler path.

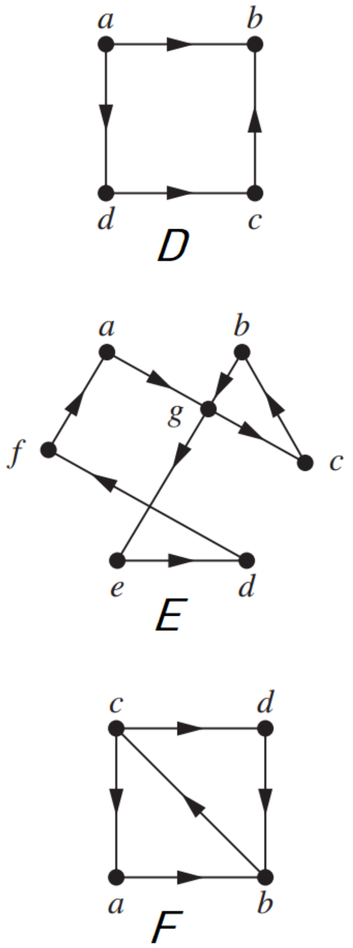


Figure 6: The digraphs D, E, and F

Solution

The digraph D is not any Euler path as well as an Euler circuit.

The digraph Ehas an Euler path as well as an Euler circuit. For example, $\{f - a - g - c - b - g - e - d - f\}$

The digraph F has only an Euler path. For example,$\{c - a - b - c - d - b\}$

Condition for multigraph:

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has an even degree

Example 1

The Königsberg bridge problem can be visualized by the multigraphrepresented in figure 2. Here we can see Every vertex does not possess an even degree. All the vertices have an odd degree .hence it does not have an Euler circuit.

Example 2

Consider the following figure for graph G.Here we can see vertex a has an odd degree so there does not exist any Euler circuit in it.

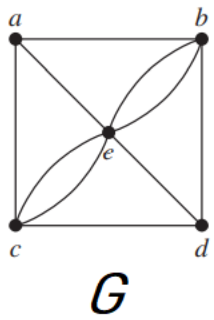


Figure 7:Graph G

Example 3

Consider the following figure for graph H.Here we can see every vertex a has an even degree so there exists an Euler circuit in it.

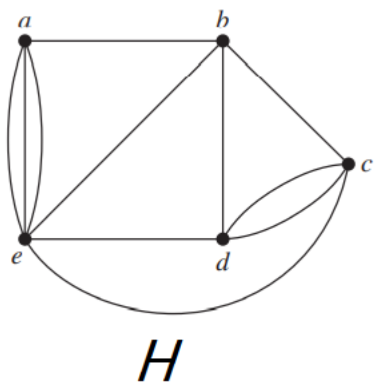


Figure 8:Graphe H

Euler circuit in graph H is

$\{a - e - a - e - b - c - e - d - c - d - b - a\}$

9.2 Hamiltonian path and circuit

Hamilton path

A path in a graph that passes through each vertex exactly once

Hamilton circuit

A circuit in a graph that passes through each vertex exactly once.

Example 1:

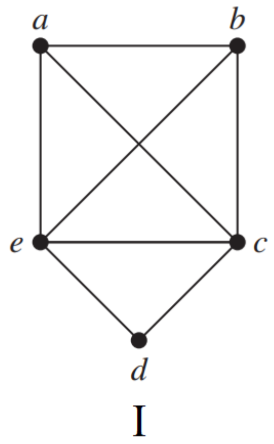


Figure 9: Graph I

Figure 9 represents graph I which has a Hamiltonian circuit, say,

$\{a - b - c - d - e - a\}$

Example 2:

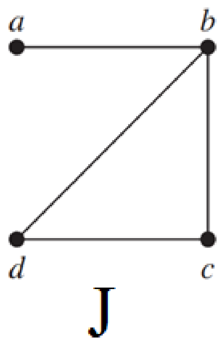


Figure 10: Graph J

Figure 10 represents graph J which does not possess any Hamiltonian circuit but has a Hamiltonianpath$\{a - b - c - d\}$

Example 3:

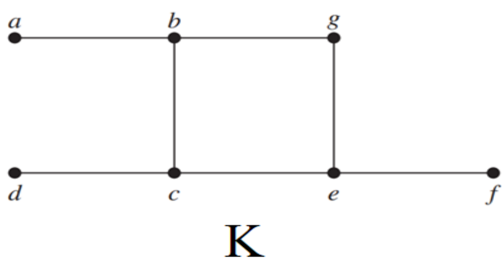


Figure 11: Graph K

Figure 11 represents graph K which does not possess any Hamiltoniancircuit as well as the Hamiltonian path.

- Every complete graph K_n with $n \geq 3$has a Hamilton circuit

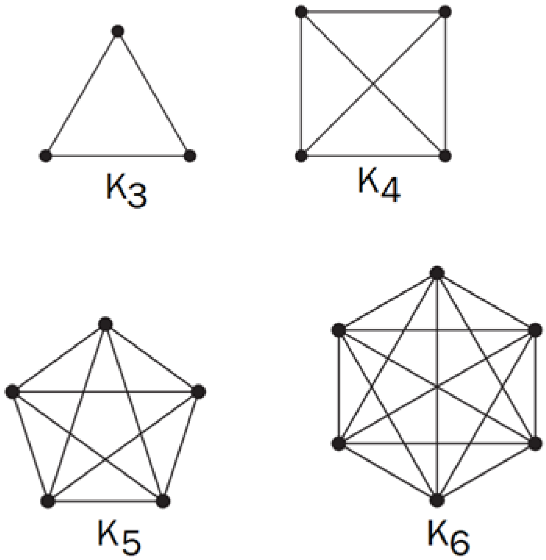


Figure 12: The complete graph for n=3,4,5, and 6.

- Every cycleC_n with has a Hamilton and an Euler circuit

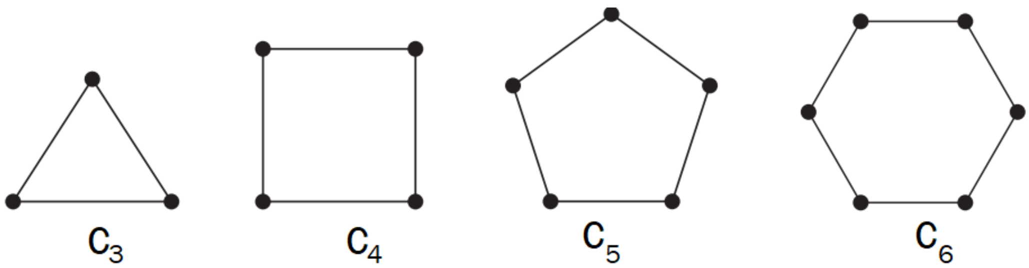


Figure 13: The cycles C_n, for n =3, n =4, and n =5

9.3 Maps and Regions

Consider the following figure 14 for the map of different regions $A, B, C, D, E, F, and\ G$

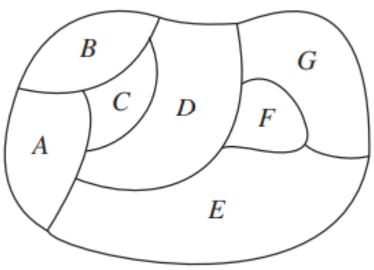


Figure 14:The map of different regions

Here Region A shares its boundary with regions B, C, D, and E. Similarly other regions also share their border with their neighbors. We can assign a vertex to every region and an edge to a boundary/border between every two adjacent regions.

Figure 15 shows the graphical representation of the map.

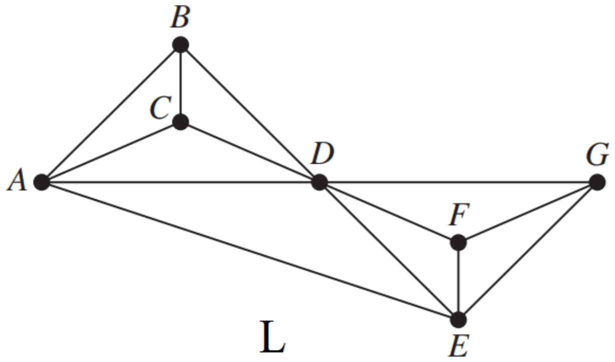


Figure 15:The graphical representation of the map.

Summary

- An Euler path in G is a simple path containing every edge of G.An Euler circuit in a graph G is a simple circuit containing every edge of G.
- A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has an even degree
- A path in a graph that passes through each vertex exactly once.A circuit in a graph that passes through each vertex exactly once.

Keywords

Euler path: a path that contains every edge of a graph exactly once.

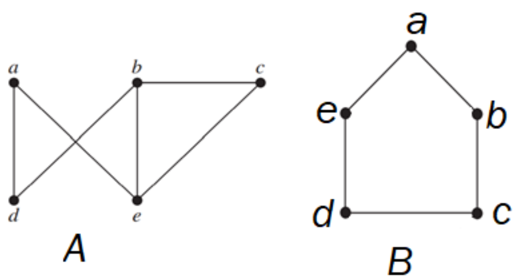
Euler circuit: a circuit that contains every edge of a graph exactly once.

Hamilton path: a path in a graph that passes through each vertex exactly once.

Hamilton circuit: a circuit in a graph that passes through each vertex exactly once.

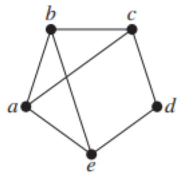
Self Assessment

1.Which one of the following graphs contains the Euler circuit?



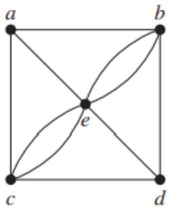
A. Only A
B. Only B
C. Both A and B
D. Neither A nor B

2.The following graph contains?



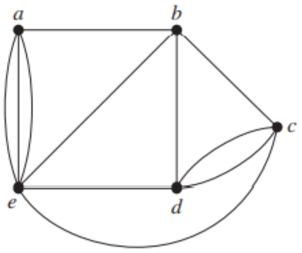
A. Euler circuit
B. Euler path
C. Both Euler circuit and path
D. Neither Euler circuit nor path

3.The following graph contains?



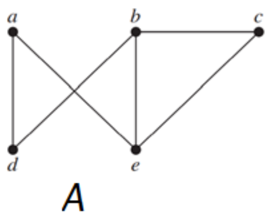
A. Euler circuit
B. Euler path
C. Both Euler circuit and path
D. Neither Euler circuit nor path

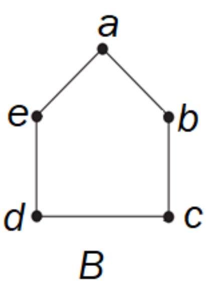
4.Which one of the following is the Euler circuit in the given graph?



A. a, b, c, d, c, e, d, b, e, a, e, a
B. a, b, c, d, c, e, d, b, d, e, a
C. a, b, c, d, a
D. a, b, c, d, e, a

5.Which one of the following graphs contains a Hamiltonian circuit?

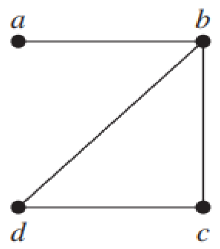




A. Only A
B. Only B
C. Both A and B

D. Neither A nor B

6.The following graph contains**?**



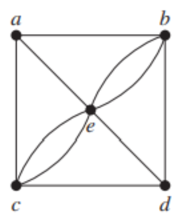
A. Hamiltonian circuit

B. Hamiltonian path

C. Both Hamiltonian circuit and path

D. Neither Hamiltonian circuit nor path

7.The following graph contains?



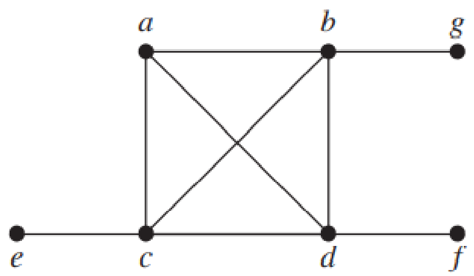
A. Hamiltonian circuit

B. Euler Circuit

C. Both Hamiltonian and Euler circuit

D. Neither Hamiltonian nor Euler circuit

8.Which one of the following is the Euler circuit in the given graph?



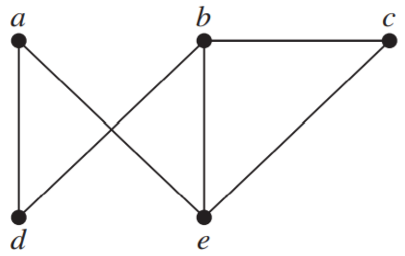
A. Hamiltonian circuit

B. Hamiltonian path

C. Both Hamiltonian circuit and path

D. Neither Hamiltonian circuit nor path

9.How many bounded regions are there in the following graph?



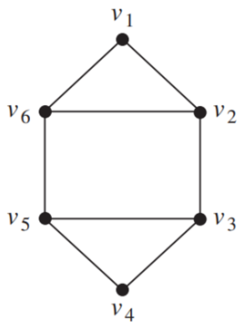
A. **1**

B. **2**

C. **3**

D. **4**

10.How many bounded regions are there in the following graph?



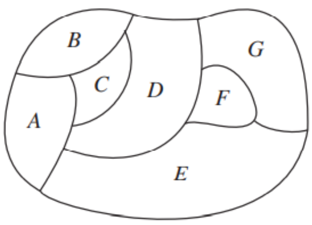
A. 1

B. 2

C. 3

D. 4

11.How many bounded regions are there in the following map?



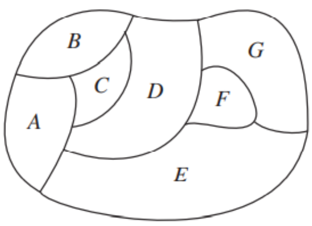
A. 6

B. 7

C. 5

D. 8

12.Regions B is adjacent to which of the following regions on the map?



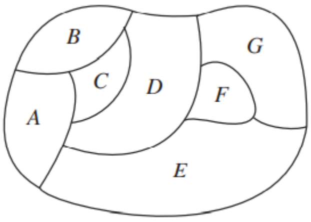
a. A, C, and D

b. A, C, D, and E

c. A, C, D, and G

d. A, C, and E

13.Regions E is adjacent to which of the following regions in the map?



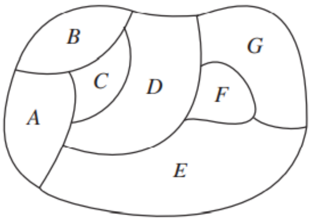
A. A, C, and D

B. A, F, D, and G

C. A, C, D, and G

D. A, C, and E

14.Regions C is adjacent to how many regions in the map?



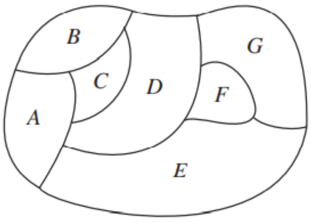
A. **2**

B. **3**

C. **5**

D. **4**

15.Regions D is adjacent to how many regions in the map?



A. 6

B. 7

C. 5

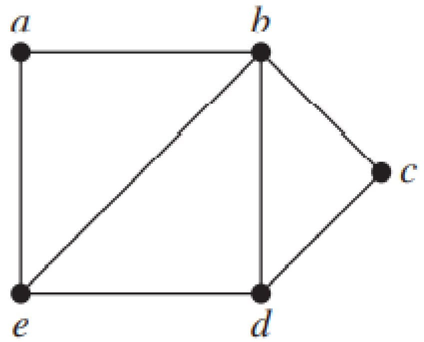
D. 8

Answers for Self Assessment

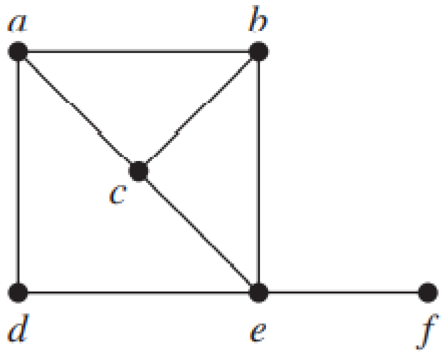
1.	C	2.	D	3.	B	4.	A	5.	B
6.	B	7.	A	8.	D	9.	D	10.	D
11.	B	12.	A	13.	B	14.	B	15.	A

Review Questions:

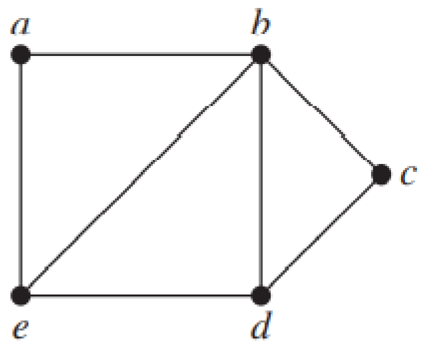
1. Check whether the following graph has an Euler circuit?



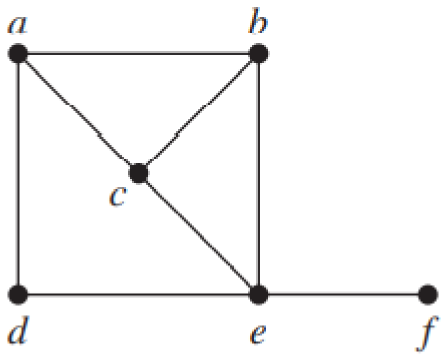
2. Check whether the following graph has an Euler path?



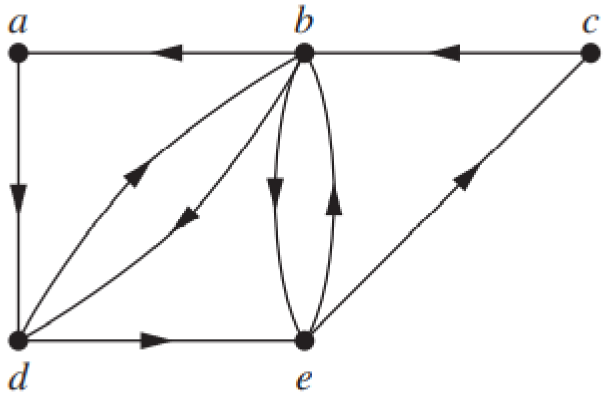
3. Check whether the following graph has a Hamiltonian circuit?



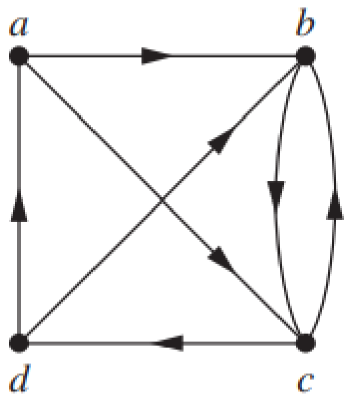
4. Check whether the following graph has a Hamiltonianpath?



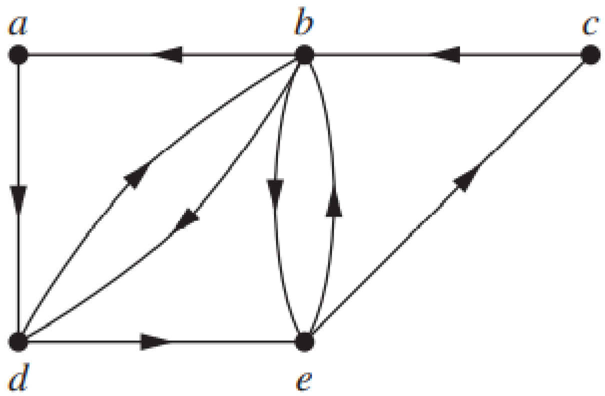
5. Check whether the following graph has an Euler circuit?



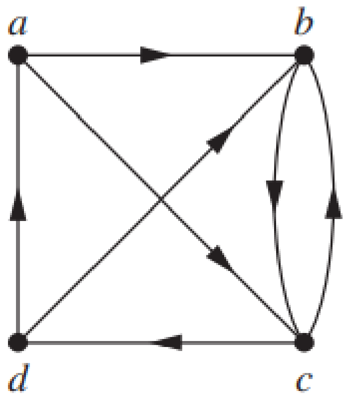
6. Check whether the following graph has an Euler path?



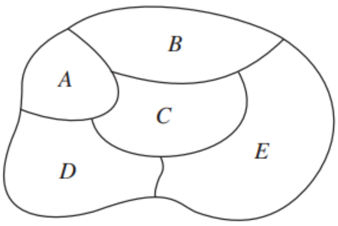
7. Check whether the following graph has a Hamiltonian circuit?



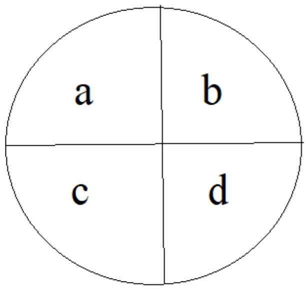
8. Check whether the following graph has a Hamiltonianpath?



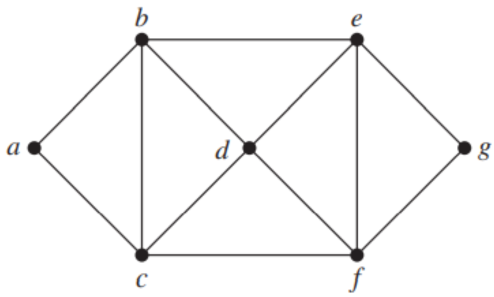
9. Let $K_{m,n}$ be the complete bipartite graph then what is the value of m and n such that it has an Euler path?
10. Let $K_{m,n}$ be the complete bipartite graph then what is the value of m and n such that it has an Euler circuit?
11. Let $K_{m,n}$ be the complete bipartite graph then what is the value of m and n such that it has a Hamiltonian path?
12. Let $K_{m,n}$ be the complete bipartite graph then what is the value of m and n such that it has a Hamiltonian circuit?
13. Draw the graphical representation of the following map?



14. Draw the graphical representation of the following map?



15. Draw the map from the following graphical representation?



Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 10: Non planar graph

CONTENTS

Purpose and Objectives:

In the development of digital circuits, the planar representation of a graph is always very significant. A graph can be used to construct a circuit, with vertices depicting circuit components and edges indicating interactions between them. A circuit can be printed on a piece of paper without any edge crossing. If the circuit is represented by a planar graph, there would be only one board with no ties crossing. We get more complicated alternatives if this graph is not planar. The planarity of graphs can also help with roadway planning. Assume we want to build a highway system to link a collection of cities. A basic graph with vertices depicting cities and edges showing highways linking them can be used to model a road network interconnecting these cities. If the eventual graph is planar, we can build this road network without tunnels or bridges.

After this unit, you would be able to

- understand the planar representation of the graph.
- describe the Eulers formula for the planar graph.
- verify whether a particular graph is planar using special results.
- find the number of regions in the planar graph.

Introduction

Consider the question of connecting three houses to three different services (see Figure 1). Is it feasible to link those houses and services in such a way that neither of the lines crosses?

The $K_{3,3}$ can be used to design this challenge. The main query can be rewritten as follows: Can $K_{3,3}$ be drawn in the plane without all of its edges intersecting?

In this segment, we'll look at whether a graph can be constructed in the plane even without any edge crossing. We'll address the problem of houses and services in detail. A graph can be represented in a variety of ways. When can there be at least one way to view this graph in a plane where no edges cross?

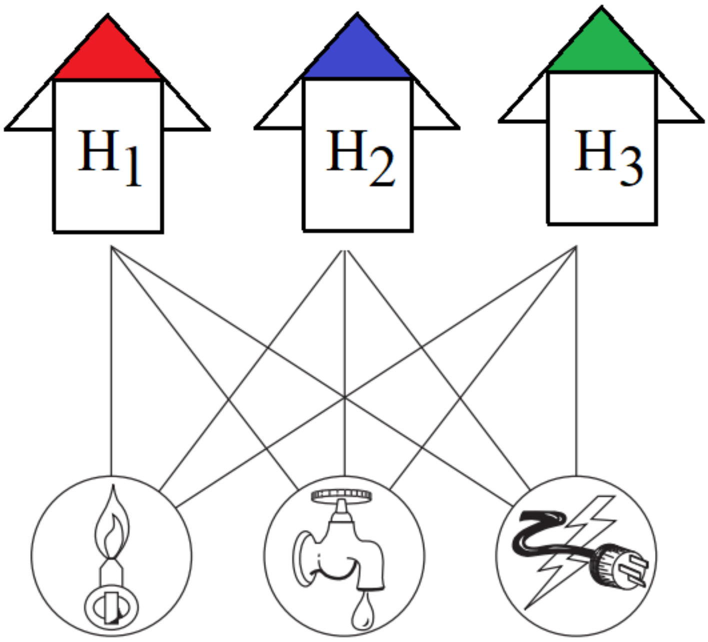


Figure 1:Three houses and 3 services problem

10.1 Planar graph

If a graph can be drawn in the plane with no edges crossing, it is said to be planar. Here crossing of edges is the intersection of the lines or link between vertices.

Consider figure2 for graph A and graph B.

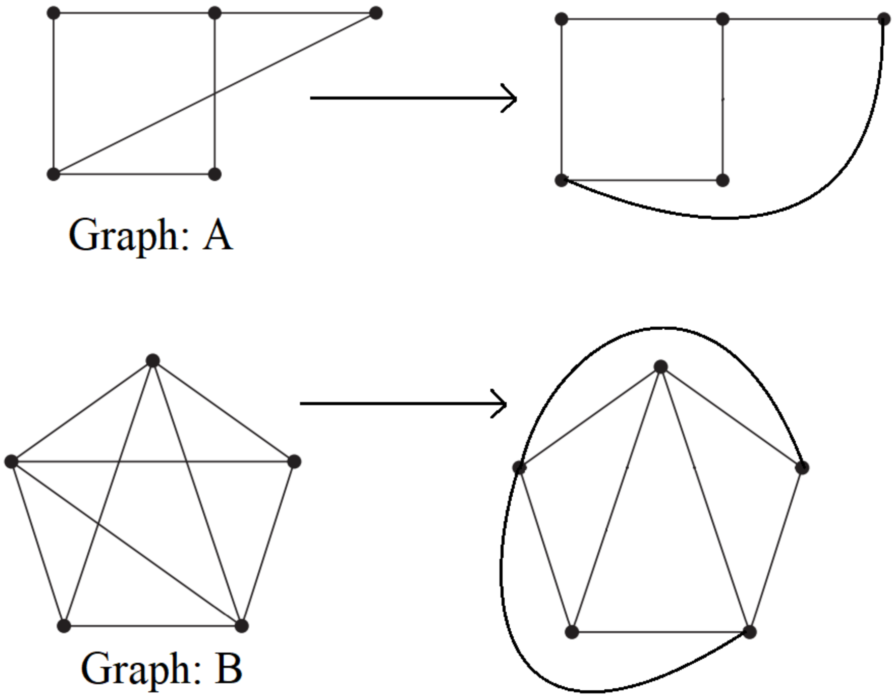


Figure 2:Planar graph A and B

Here both graphs are planar we can see: Both can be re-drawn in the plane without any edge intersection/crossing.

Check whether all cycles(C_n) are planar?

Solution

Let us draw some complete graphs shown in figure 3. Here we can see K_1, K_2, and K_3 are planers as they can be drawn in a plane without any edge crossing.

K_4 is also planar. One of the intersecting edges can be drawn from outside to the bounded region of 4 vertices. Figure 4 depict the planar representation of K_4.

While the planar representation of the K_5 can not be obtained. We are bound to cross some of the edges to make all connections.

Similarly $K_6, K_7, K_{8,...}$ are nonplanar graphs.

So we can conclude that K_n is non-planar if $n \geq 5$

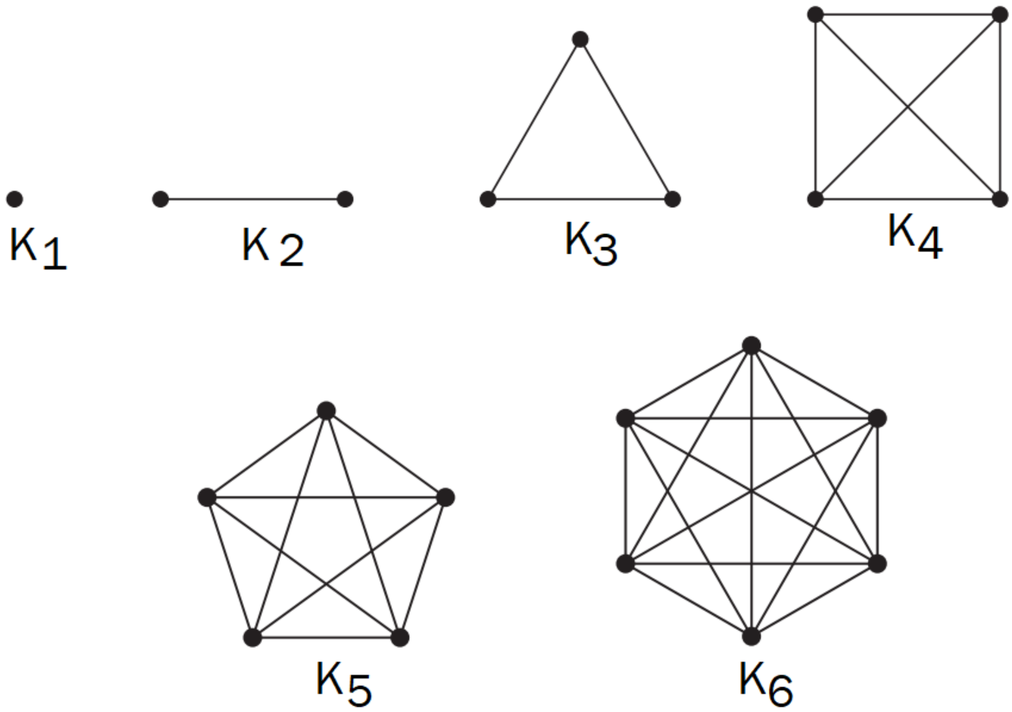


Figure 3: Complete graph with n<7

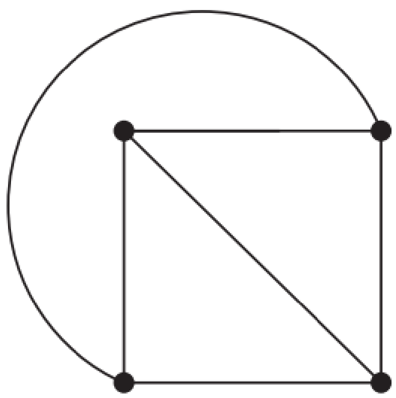


Figure 4:Planar representation of K_4.

Check whether all complete graphs(K_n) are planar?

Solution

Every cycle C_n, n ≥ 3, consists of n vertices $v_1, v_2, \ldots, v_n$ and the combination edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,..., $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$ such that there is no intersection between any two edges apart from the vertex point. Hence they are always planar. Let us draw some cycles shown in figure 4. Here we can see C_3, C_4, C_5, and C_6 are planers as they can be drawn in a plane without any edge crossing.

So we can conclude that C_n is planar

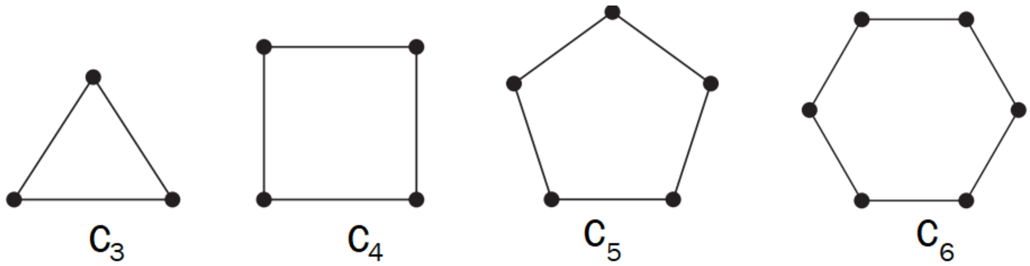


Figure 5: The cycles C_n, for n =3, 4,5, and 6.

Check whether all Wheel graphs(W_n) are planar?

Solution

Let C_n, n ≥ 3 be the cycle then the wheel W_n with n outer vertices is obtained from $C_n$if we add one additional(central) vertex to the cycle C_n. The added vertex is connected to every vertex of C_n such that there is no intersection between any two edges apart from the vertex point.

The wheel W_n, for n =3, 4,5, and 6.are shown in Figure 6

So we can conclude that W_n is planar

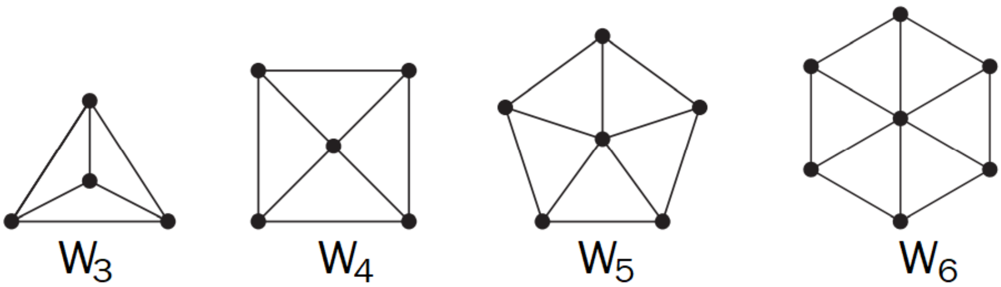


Figure 6:The Wheel W_n, for n =3, 4,5, and 6.

Is $K_{3,3}$, planar?

Solution

Consider the $K_{3,3}$ shown in figure 7. We can also relate the $K_{3,3}$ with our introductory question (see Figure 1). Here we can draw the link between house H_1 and all three utilities without any crossing. Similarly, we can draw the link between house H_2 and all three utilities without any crossing. But it is not possible to connect house H_3 with all three utilities. Somehow we face at least one intersection. So $K_{3,3}$ is a nonplanar graph. It is not feasible to link those houses and services in such a way that neither of the lines crosses.

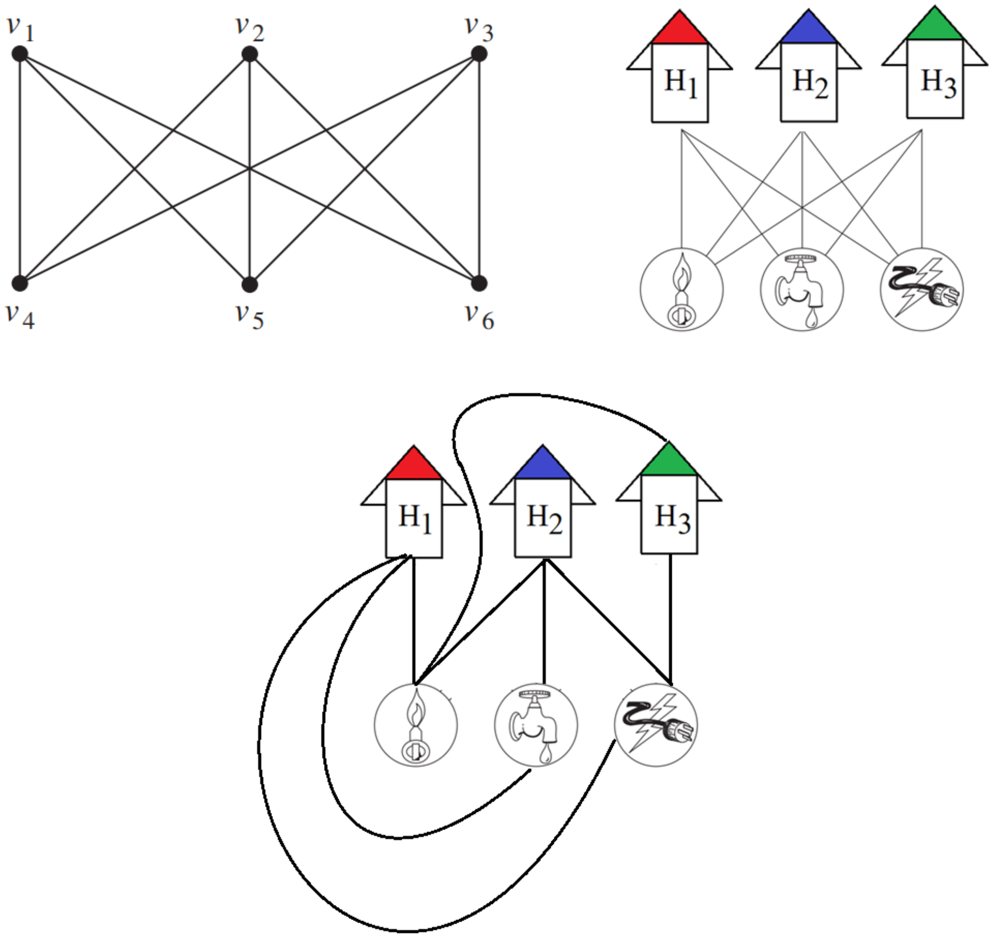


Figure 7: Representation of edge crossing in $K_{3,3}$

Regions of a planar graph: The plane is divided into bounded/feasible and open regions where the bounded region is enclosed by edges and vertices of the graph and the open region is unbounded.

Example:

Figure 8 represents a planar graph and we can see there are 3 bounded regions and 1 open region. The bounded region R_1 is enclosed by vertices a,d, and c, and the edges a-c, c-d, and d-a. The bounded region R_2 is enclosed by vertices a,b, and c, and the edges a-c, c-b, and b-a. The bounded region R_3 is enclosed by vertices a,b, and d, and the edges a-b, b-d, and d-a. While the region R_4 is the open region.

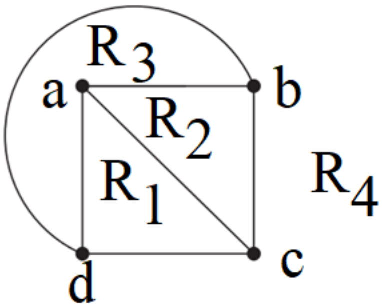


Figure 8: Regions of a planar graph

10.2 Euler's Theorem

Let G be a simple connected planar graph with $|E|$ edges and $|V|$ vertices. Let $|R|$ be the number of regions in a planar representation of G. Then $|R| = |E| - |V| + 2$.

In figure 8 we can see $|R| = 4$, $|E|=6$, and $|V|=4$. Hence $|R| = 6 - 4 + 2 = 4$.

Proof

Let $|R_k|$, $|E_k|$, and $|V_k|$, be the number of regions, edges, and vertices of the planar representation of G

There are three credentials in graph G we can start our proof with any one of them.

So let G be a simple connected and planar graph with 1 edge

Figure 9: A simple planar connected graph with 1 edge.

For k=1 (A simple planar connected graph with 1 edge) $|E_1| = 1, |V_1| = 2,$ and $|R_1| = 1.$

Here $|R_1| = |E_1| - |V_1| + 2$ is true

Let us assume that is also true for k=n (A simple planar connected graph with n edge)

$|R_n| = |E_n| - |V_n| + 2$

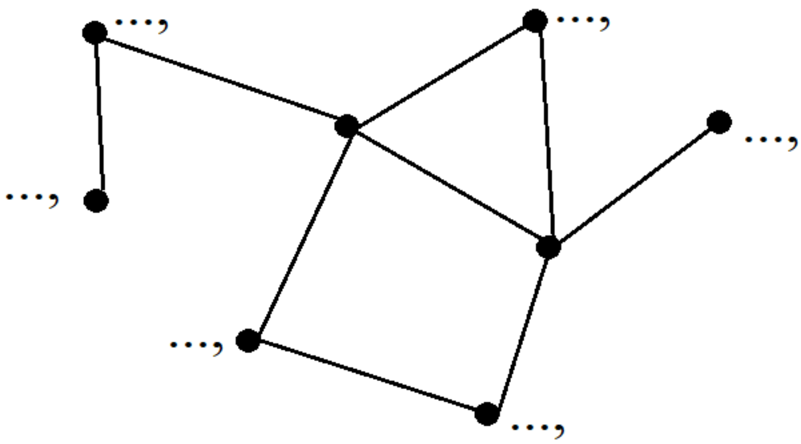


Figure 10: A simple planar connected graph with n edge.

Let us verify it for k=n+1 (A simple planar connected graph with n+1 edge). If we add one extra edge in figure 10(A simple planar connected graph with n edge) then there are 2 possibilities

Case 1: we can add one more region while adding an edge (See figure 11).Hence

$|E_k| = |E_n| + 1, |R_k| = |R_n| + 1, and\ |V_k| = |V_n|.$

$\Rightarrow |R_n| + 1 = |E_n| + 1 - |V_n| + 2$

$\Rightarrow |R_n| = |E_n| - |V_n| + 2$. So we have verified the Eulers formula for k=n+1.

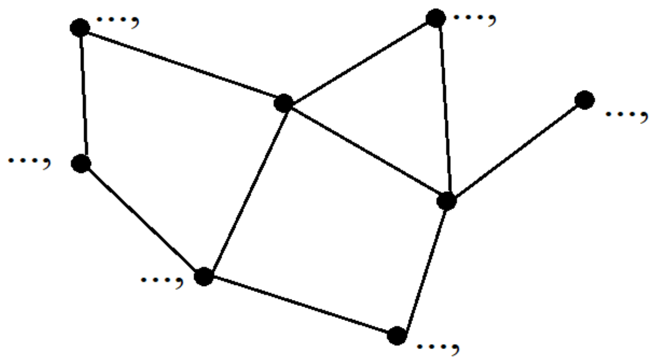


Figure 11: A simple planar connected graph with n+1 edge and n+1 regions.

Case 2: we can add one more vertex while adding an edge (See figure 12).Hence

$|E_k| = |E_n| + 1, |R_k| = |R_n|, and\ |V_k| = |V_n| + 1.$

$\Rightarrow |R_n| + 1 = |E_n| - |V_n| + 1 + 2$

$\Rightarrow |R_n| = |E_n| - |V_n| + 2$. So we have verified the Eulers formula for k=n+1.

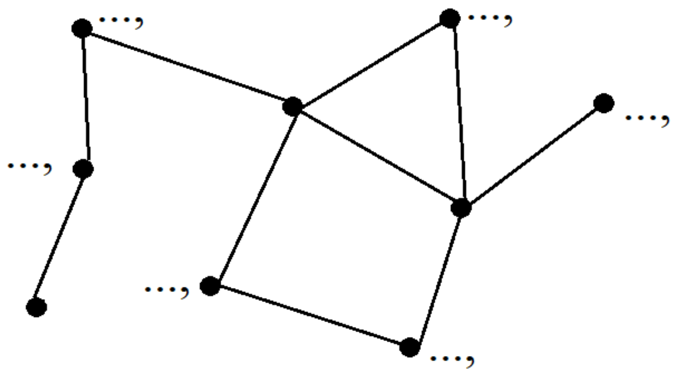


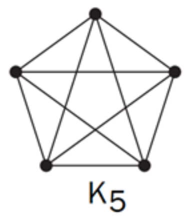
Figure 12: A simple planar connected graph with n+1 edge and n+1 vertices.

10.3 Some results for the planar graph

If G is a connected planar simple graph with e edges and v vertices, where v ≥ 3, then e ≤ 3v − 6.

Example:

K_5 is non-planar

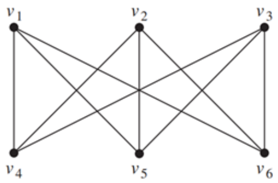


Here we can see there are 5 vertices and 10 edges. e ≤ 3v – 6 is not true.

If a connected planar simple graph has e edges and v vertices with v ≥ 3 and no circuits of length three, then e ≤ 2v – 4

Example:

$K_{3,3}$ is non-planar



Here we can see there is a circuit of length 3 and a total of 6 vertices and 9edges. e ≤ 2v – 4 is not true.

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Homeomorphic graph: The graphs G1 = (V1, E1) and G2 = (V2, E2) are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

Kuratowski's Theorem:

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5

Example:

Graph A shown in figure 13 is homeomorphic to K_5. So it's not the planar graph.

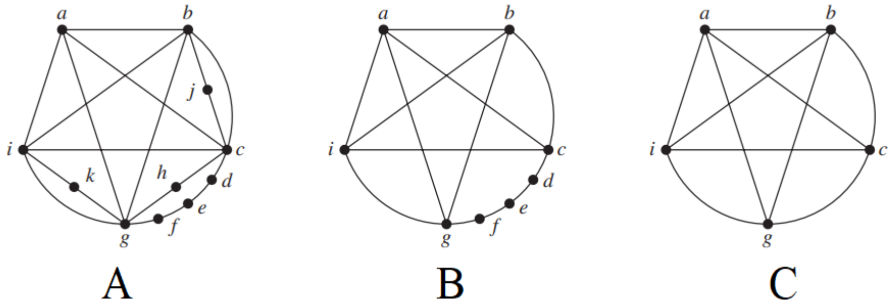
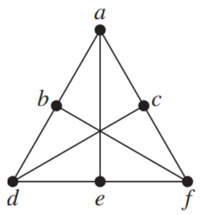


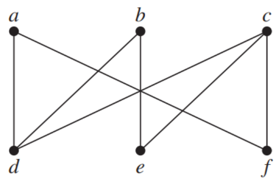
Figure 13: The nonplanar graph

<u>Review Questions:</u>

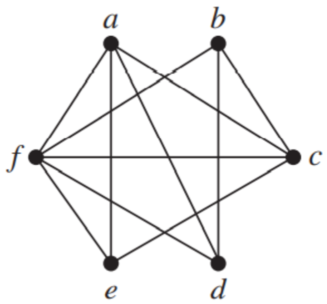
1. Check whether the following graph is planar?



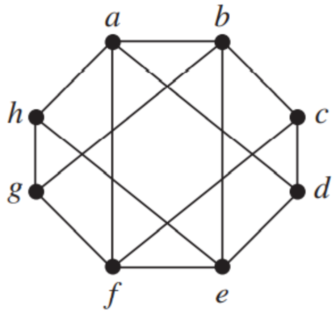
2. Check whether the following graph is planar?



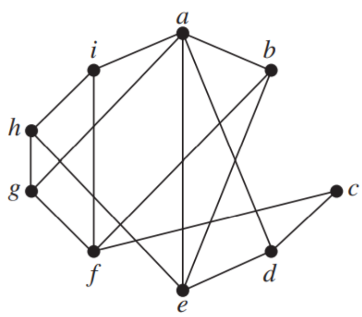
3. Check whether the following graph is planar?



4. Check whether the following graph is planar?



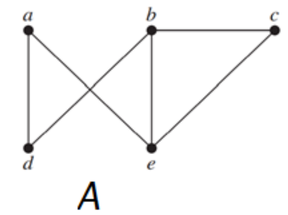
5. Check whether the following graph is planar?

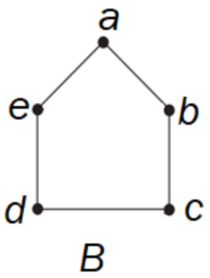


6. Prove that K_6 is nonplanar.
7. Prove that K_7 is nonplanar.
8. Prove that W_6 is planar.
9. Prove that C_8 is planar.
10. Prove that $K_{2,2}$ is planar.
11. Check whether the $K_{3,4}$ is planar.
12. Check whether the $K_{5,5}$ is planar.
13. Find the total number of bounded regions in C_{10}
14. Find the total number of bounded regions in W_{10}
15. Find the total number of bounded regions in K_3

Self-assessment

1. Which one of the following graphs is planar?





a. Only A

b. Only B

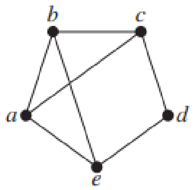
c. Both A and B

d. Neither A nor B

2. Consider the following graph and the statements-

A: The graph can be drawn in a plan without any edge crossing

B: The **graph cannot be** drawn in a plan without any edge crossing



Then which one of the following is true?

a. Only A

b. Only B

c. Both A and B.

d. Neither A nor B.

3. Which one of the following graphs is planar?

a. K_4

b. K_5

c. K_6

d. K_7

4. Which one of the following graphs is nonplanar?

a. W_4

b. C_4

c. K_4

d. $K_{4,4}$

5. Every complete graph (K_n) is nonplanar if __?

a. $n \geq 4$

b. $n > 4$

c. $n \leq 4$

d. $n < 4$

6. Consider the following graph and the statements-

A: **Every wheel graph (W_n) is planar**

B: **Every Cycle (C_n) is planar. Then which one is correct**

a. Only A

b. Only B

c. Both A and B.

d. Neither A nor B.

7. Let G be a connected planar simple graph with 10 edges and 7 vertices then the total number of regions in a planar representation of G?

a. 5

b. 6

c. 7

d. 8

8. Let G be a connected planar simple graph with a total of 3 regions and 7 vertices then the number of edges in a planar representation of G?

a. 5

b. 6

c. 7

d. 8

9. Let G be a connected planar simple graph with a total of 6 regions and 12 edges than the number of vertices in a planar representation of G?

a. 5

b. 6

c. 7

d. 8

10. Let G be a connected planar simple graph with 10 edges and 7 vertices then the total number of bounded regions in a planar representation of G?

a. 5

b. 6

c. 7

d. 8

11. Let G be a connected planar simple graph with a total of 10 regions then the number of total numbers of bounded regions in a planar representation of G?

a. 9

b. 10

c. 11

d. 8

12. Consider $\boldsymbol{K_5}$ and $\boldsymbol{K_{3,3}}$ then

a. K_5 is planar while $K_{3,3}$ is not

b. $K_{3,3}$is planar while K_5 is not

c. Both are planer

d. Both are non-planar

13. If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then

a. $e \leq 3v - 6$

b. $e \leq 3v + 6$

c. $e \leq 2v$

d. $e \leq 2v + 4$

14. If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then.

a. $e \leq 3v$

b. $e \leq 3v + 6$

c. $e \leq 2v - 4$

d. $e \leq 2v + 4$

15. To check whether K_4 is planar, which one of the following results can be used?

a. $e \leq 3v - 6$

b. $e \leq 3v + 6$

c. $e \leq 2v-4$

d. $e \leq 2v + 4$

Self-assessment Answers

1.	c	2.	a	3.	a	4.	d	5.	d
6.	c	7.	b	8.	d	9.	d	10.	a
11.	a	12.	a	13.	a	14.	c	15.	c

Summary

- Let G be a simple connected planar graph with $|E|$ edges and $|V|$ vertices.
- Let $|R|$ be the number of regions in a planar representation of G. Then $|R| = |E| - |V| + 2$.If a graph can be drawn in the plane with no edges crossing, it is said to be planar.
- **If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.**
- **If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$**
- If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

- The graphs G1 = (V1, E1) and G2 = (V2, E2) are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.
- A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5

Keywords

Planar graph: a graph that can be drawn in the plane with no crossings.

Homeomorphic: two undirected graphs are homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions

Euler's Formula: Let G be a simple connected planar graph with $|E|$ edges and $|V|$ vertices. Let $|R|$ be the number of regions in a planar representation of G. Then $|R| = |E| - |V| + 2$.

Further Readings

- Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education
- Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 11: Coloring of Complete Graph

CONTENTS

Objectives:

Many developments of graph theory have come from problems involving the coloring of maps of locations, such as maps of different areas of the world. When coloring a diagram, two regions with a similar border are usually given different colors. Using a different color for each zone is one way to guarantee that two neighboring regions cannot have the same color. However, this is impractical, because it will be difficult to discern identical colors on maps of multiple areas. Instead, if possible, use a limited number of colors. Considering the job of deciding the smallest number of colors that would be used to color a map while ensuring that neighboring areas are never the identical color.

After this unit, you would be able to

- understand the concept of graph coloring
- find the chromatic number of given graphs.
- describe the chromatic number of Complete graphs, Cycles, Wheels, and Complete-Bipartite graph

Introduction

Graph coloring can be used to solve a range of schedule and task problems. For example, there are 42 students in a Mathematics class and the teacher needs to prepare the classtest for them. Figure 1 depicts the sitting arrangement of these 42 students. The testsare being designed in a way such that no two adjacent students have the same test.This problem can be solved by graph coloring. Either teacher can prepare 42 different sets for the test or he can minimize the number of the question paper.Figure 2 shows that there are 2 colors(representing the sets) required for mentioned sitting arrangement. In this section, we will solve the various problem of coloring special graphs.

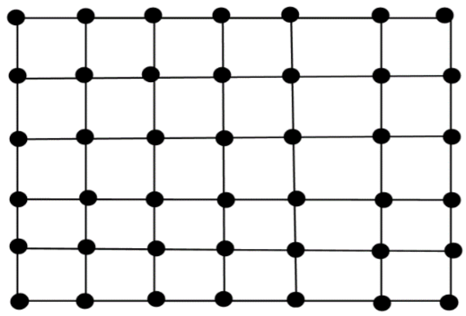


Figure 1: Sitting arrangements of 42 students

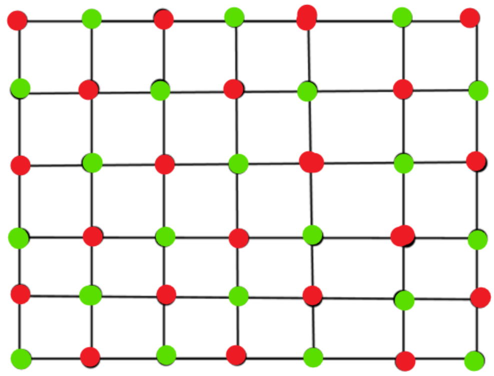


Figure 2: Assignment of papers for given sitting arrangement

11.1 Coloring

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color

The chromatic number of a graph

The chromatic number of a graph is the least number of colors needed for the coloring of this graph. The chromatic number of a graph G is denoted by χ (G). (Here χ is the Greek letter chi.)

Regular graph

A graph is said to be regular if every vertex in the graph has the same degree.

Example:

Consider the following graph depicted in figure 3

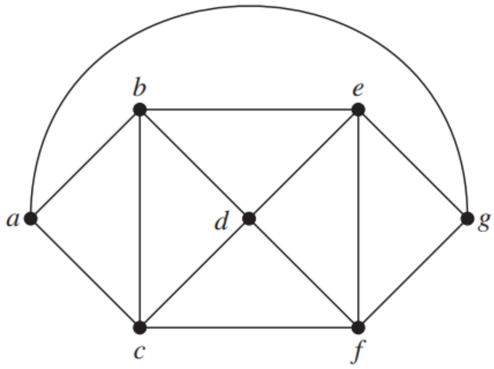


Figure 3: A simple graph with 7 vertices

Solution

There are 4 minimum number colors required for coloring. So χ (G)=4.

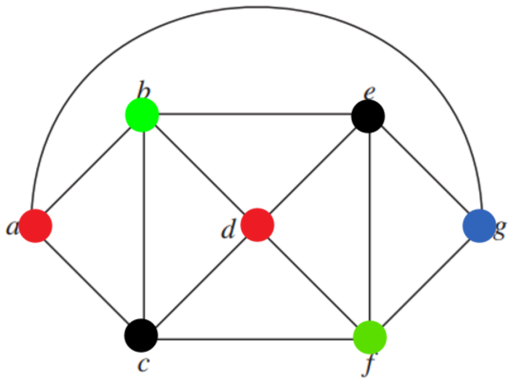


Figure 4: The coloring of the graph

Describe the chromatic number of a complete graph (K_n)?

Solution

Let us draw some complete graphs shown in figure 4. Here we can observe that every complete graph is regular. The degree of every vertex in K_n is n-1. So K_n is n-1 regular.

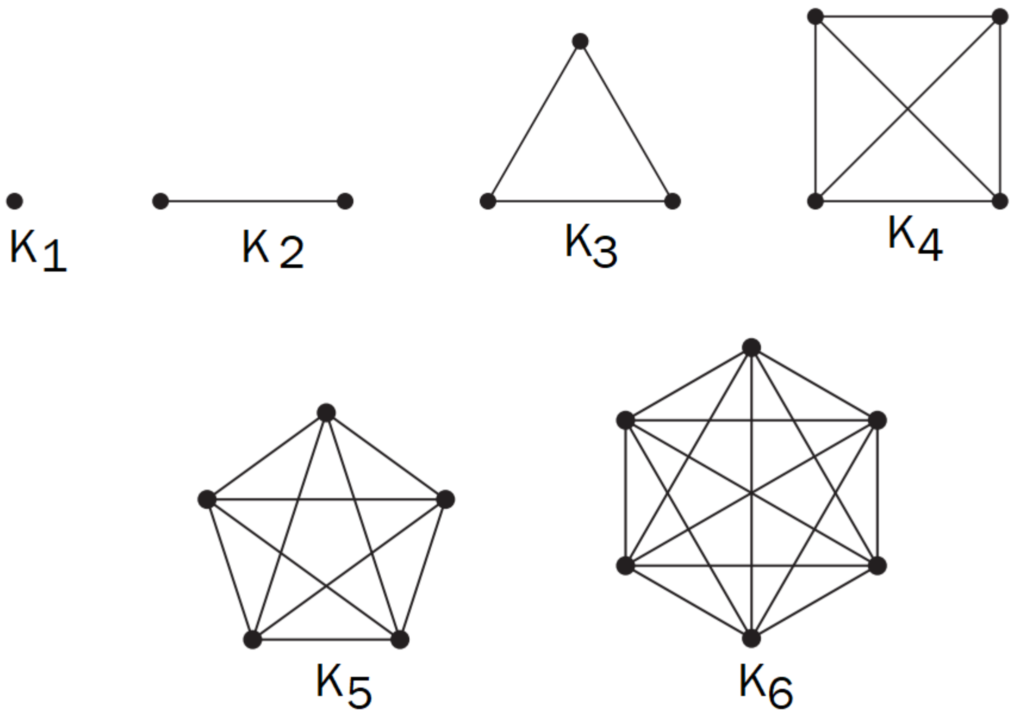


Figure 4: Complete graph with n<7

Here we can see every vertex of a complete graph is adjacent to all other vertices so we cannot assign the same color to two adjacent vertices (See figure 5). Hence there are n different colors required for coloring so

χ (K_n)=n.

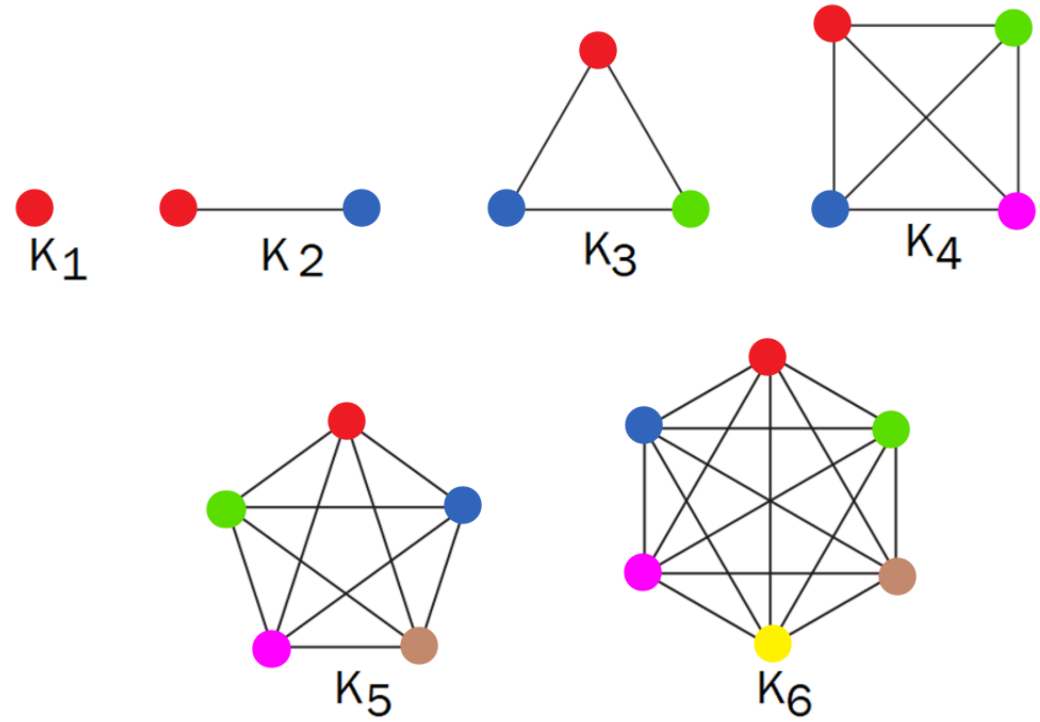


Figure 5: Chromatic number of complete graphs

Describe the chromatic number of a complete graph (C_n)?

Solution

Let us draw some cycle graphs shown in figure 6. Here we can observe that every cycle graph is regular. The degree of every vertex in C_n is 2. So C_n is 2regular.

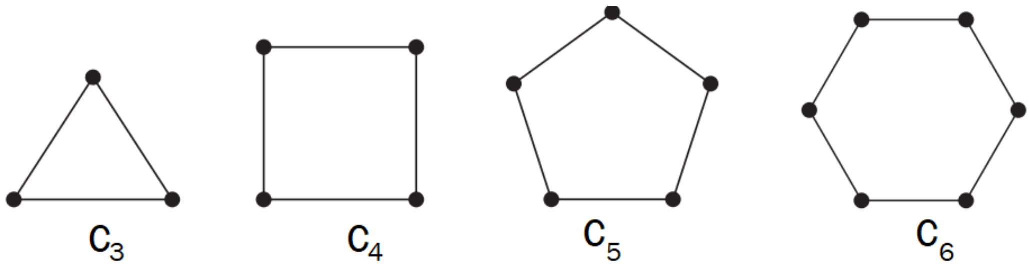


Figure 6: The cycles C_n, for n =3, 4,5, and 6.

Here we can see every vertex of the cycle is adjacent to two vertices of the graph.so we cannot assign the same color to two non-adjacent vertices (See figure 7). Hence there is 3 minimum number of colors are required for odd length cycles and 2 minimum number of colors required for even length cycle.

$$\chi\,(C_n)=\begin{Bmatrix} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{Bmatrix}$$

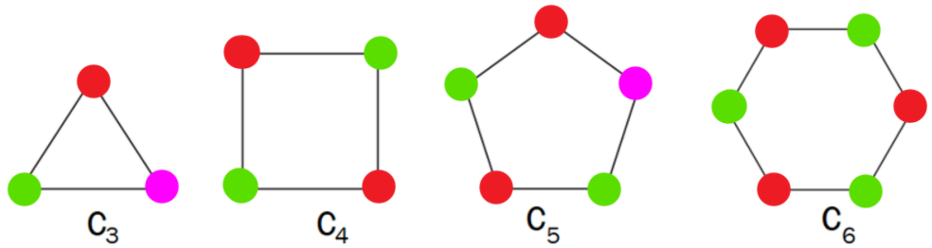


Figure 7: Chromatic number of complete graphs

Describe the chromatic number of wheel graph (W_n)?

Solution

We know that the wheel graph is obtained from the cycle. If we add one central vertex in every cycle and connect this to every vertex of the cycle then the resultant graph becomes a wheel.let us draw some wheel graphs shown in figure 8. $W_n$is the wheel with n outer vertices

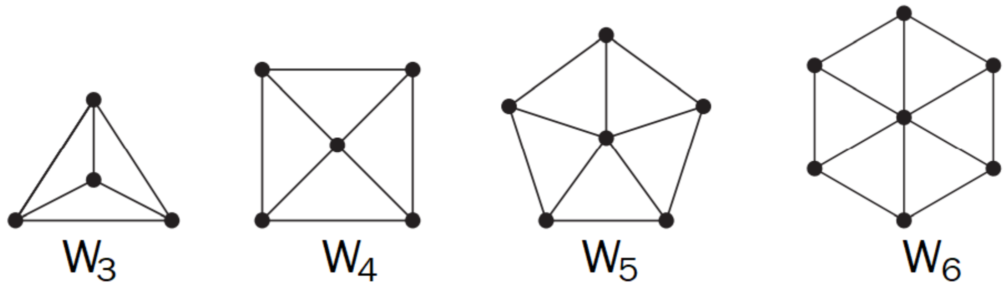


Figure 8:The WheelW_n , for n =3,4,5, and 6.

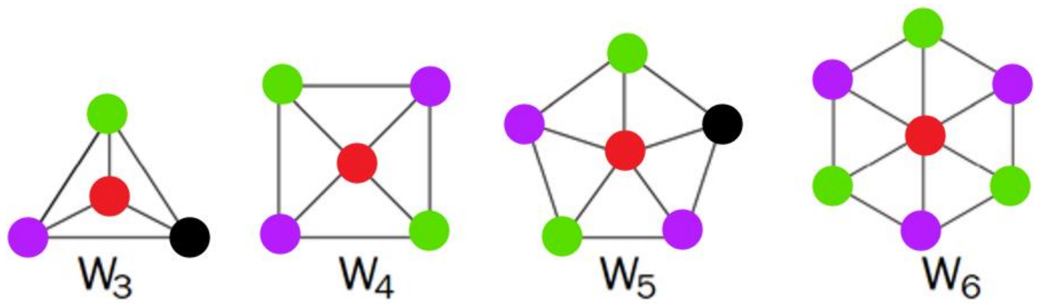


Figure 9: The chromatic number of wheels

Here we can see the central vertex of the wheel is adjacent to every outer vertex of the graph Hence there is4 minimum number of colors are required for a wheel with odd outer vertices. 3 minimum number of colors required for a wheel with even outer vertices.(See figure 9).

$$\chi\ (W_n)=\begin{Bmatrix} 3 & if\ n\ is\ even \\ 4 & if\ n\ is\ odd \end{Bmatrix}$$

Describe the chromatic number of complete bipartite ($K_{m,n}$)?

Solution

In the complete bipartite graph,we can say there are two groups of vertices and the set of vertices of one group is not adjacent to each other and all the vertices of this group are related to all the vertices of the other group.Figure 10 depicts some of the complete bipartite graphs.

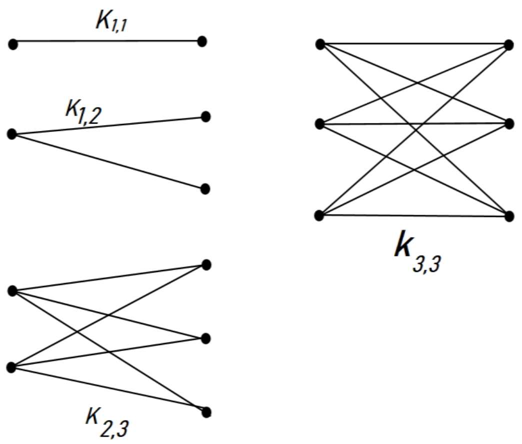


Figure 10:The$K_{1,1}$, $K_{1,2}$,$K_{2,3}$, and $K_{3,3}$

Here we can see any group vertices are not adjacent to each other so we can assign aseparate color for every group.(See figure 9). Hence the χ ($K_{m,n}$)=2

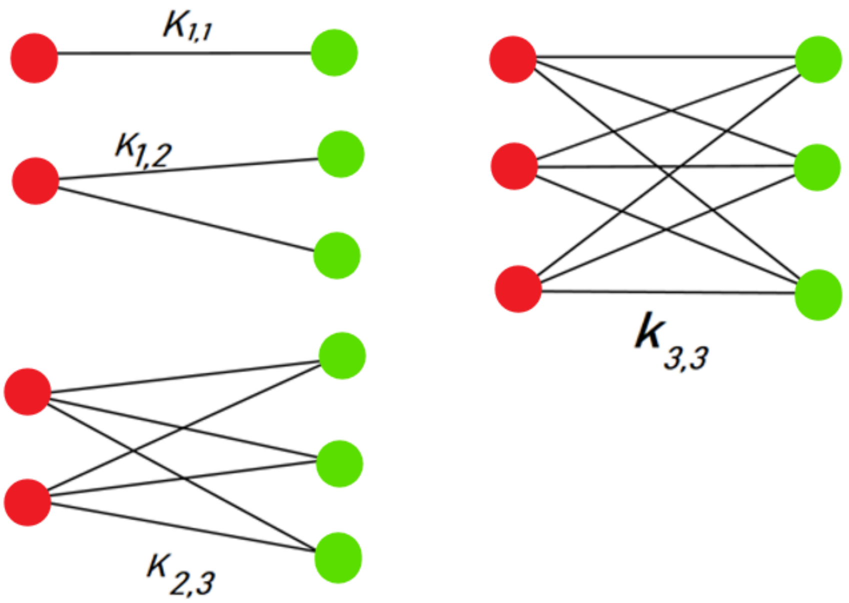


Figure 11: The chromatic number of complete bipartite graph.

The chromatic number of a planar graph is no greater than four

Summary

- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color
- The chromatic number of a graph is the least number of colors needed for the coloring of this graph. The chromatic number of a graph G is denoted by χ (G).
- A graph is said to be regular if every vertex in the graph has the same degree.
- Every complete graph is *n-1* regular.

- Every cycle is two regular.
- $\chi\ (K_n)$=n.
- $\chi\ (K_{m,n})$=2.
- $\chi\ (W_n)=\begin{cases} 3 & if\ n\ is\ even \\ 4 & if\ n\ is\ odd \end{cases}$
- $\chi\ (C_n)=\begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases}$

Keywords

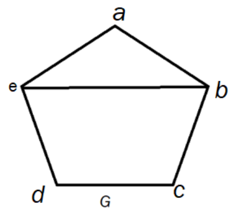
Graph coloring: an assignment of colors to the vertices of a graph so that no two adjacent vertices have the same color.

Chromatic number: the minimum number of colors needed in the coloring of a graph.

Regular graph:A graph is said to be regular if every vertex in the graph has the same degree.

Self Assessment

1. The chromatic number of G?



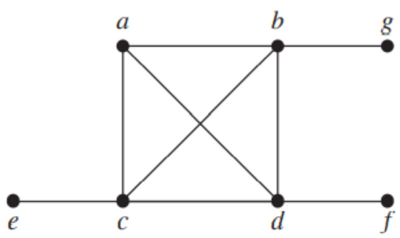
a. 2

b. 3

c. 4

d. 5

2. The chromatic number of the following graph?



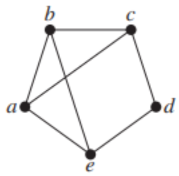
a. 2

b. 3

c. 4

d. 5

3. The chromatic number of the following graph?



A. 2

B. 3

C. 4

D. 5

4. The chromatic number of the K_5?

A. 2

B. 3

C. 4

D. 5

5. The chromatic number of the $K_{3,3}$?

A. 2

B. 3

C. 4

D. 5

6. The Chromatic number of K_{15} is

A. 10

B. 15

C. 20

D. 150

7. The K_{10} is

A. 10-regular

B. 15-regular

C. 20-regular

D. 150-regular

8. The Chromatic number of K_{105} is

A. 100

B. 105

C. 200

D. 150

9. The Chromatic number of C_{15} is

A. 2

B. 3

C. 15

D. 30

10. The $\boldsymbol{C_{10}}$ is

A. 2-regular

B. 10-regular

C. 3-regular

D. 15-regular

11. The Chromatic number of $\boldsymbol{C_{100}}$ is

A. 100

B. 2

C. 3

D. 150

12. The Chromatic number of $\boldsymbol{W_{15}}$ is

A. 3

B. 4

C. 5

D. 15

13. The Chromatic number of $\boldsymbol{W_{10}}$ is

A. 2

B. 3

C. 4

D. 5

14. The Chromatic number of $\boldsymbol{K_{15,15}}$ is

A. 3

B. 4

C. 2

D. 15

15. The $\boldsymbol{K_{3,3}}$ is

A. 2-regular

B. 6-regular

C. 3-regular

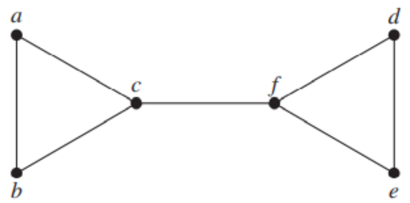
D. 9-regular

Answer for Self Assessment

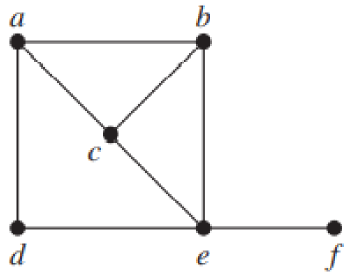
1.	B	2.	C	3	B	4.	D	5.	A
6.	B	7.	A	8.	B	9.	B	10.	A
11.	B	12.	B	13.	B	14.	C	15.	C

Review Questions:

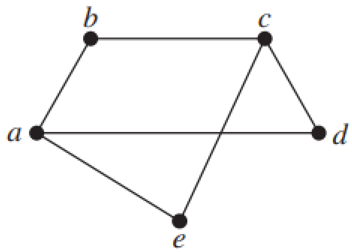
1. Find the chromatic number following the graph?



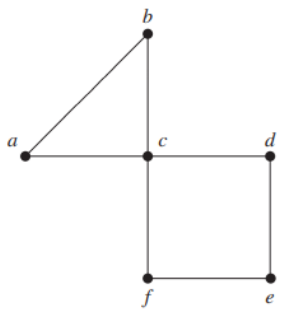
2. Find the chromatic number following the graph?



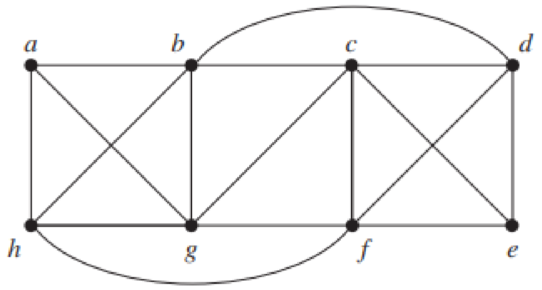
3. Find the chromatic number following the graph?



4. Find the chromatic number following the graph?



5. Find the chromatic number following the graph?



6. Check whether the graph $C_5$is regular?

7. Check whether the C_3 and K_3 are regular?

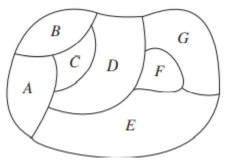
8. Find the chromatic number of C_3 and K_3?

9. Check whether the graph W_4 is regular?

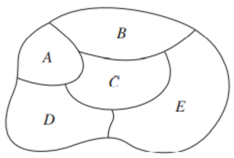
10. Check whether the C_4 and $K_{2,2}$ are regular?

11. Find the chromatic number of C_4 and $K_{2,2}$?

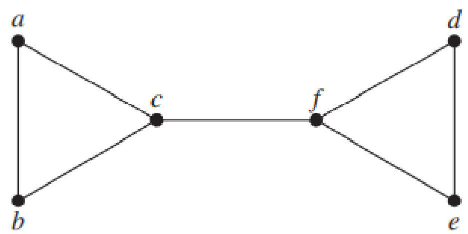
12. How many minimum colors are required for coloring of the following region.



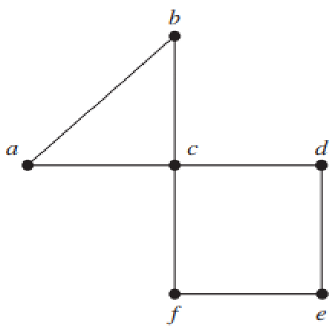
13. How many minimum colors are required for coloring of the following region.



14. Add some new edges in the following graph such that it will become a 3-regular?



15. Add some new edges in the following graph such that it will become a 5- regular?



Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 12: Labeled and Weighted Graph

CONTENTS

Objectives:

Graphs of weighted edges can be used to model a variety of problems. Think how an airline structure can be modeled as an example. By defining cities as vertices and paths as edges, we created the simple graph model. Routes among cities may be assigned to the edges to design problems concerning distances. Flying hour's problems can be modeled by applying flight times to edges. By attributing fares to each edge, problems regarding fares may be modeled.

After this unit, you would be able to

- understand the concept of a weighted graph.
- find the shortest path between any given vertices of a weighted graph.

Introduction

Web servers are modeled using weighted graphs. Weighted graphs can be used to investigate communications costs (such as the annual expense of leasing a telephone line), device reaction times over these lines, and computer reach.Weighted graphs are commonly used in a variety of problems. Choosing a route. One such problem is finding the shortest path between two vertices in a network. To be more precise, consider the path which has the least weight between the initial and ending node.

12.1 Labeled graph

A graph that has labels associated with each edge or each vertex is known as a labeled graph.

Example

Figure 1-3 shows the unlabeled and labeled graphs

Figure 1: The unlabeled graph

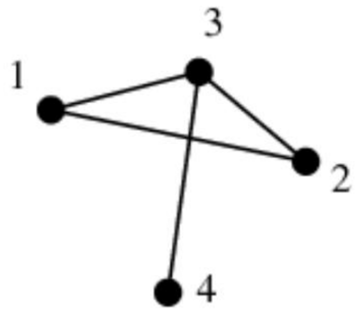


Figure 2: The vertex labeled graph

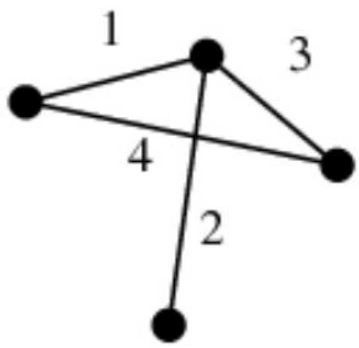


Figure 3: The edge labeled graph.

12.2 Weighted graph

Let G be a vertex labeled graph and if a number assigned to each edge of G then it called a weighted graphs

Example

Figure 4 depicts a weighted graph.

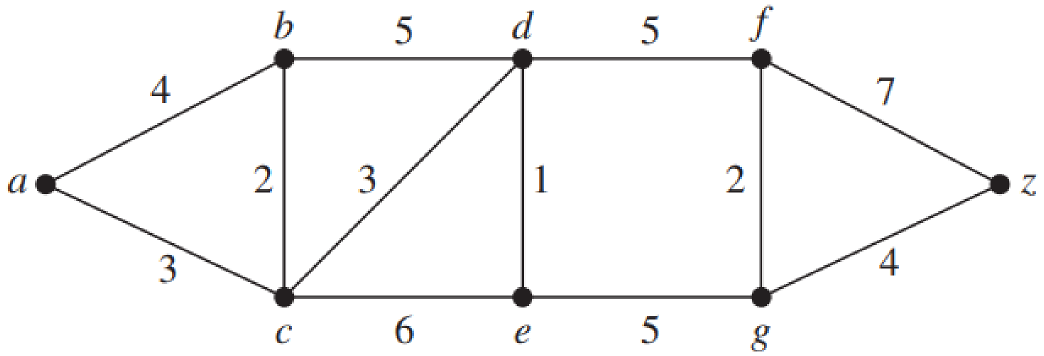


Figure 4: A weighted graph

The shortest path between two vertices

Let G be a weighted graph then the shortest path between any two vertices is the path which contains minimum weight between all possible path.

Example

Let us consider the following figure 5 for a weighted graph.

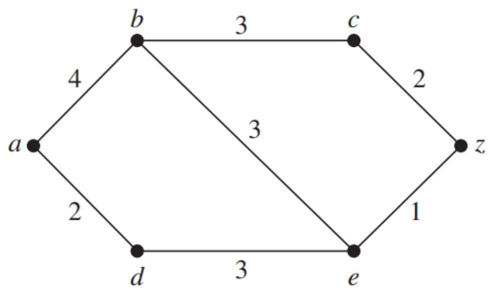


Figure 5: A weighted graph

If we want to find the shortest path between the vertex a and z then we can observe that there are 3 possible paths available between these vertices

The path $a - d - e - z$ has a total weight of 6 units.

The path $a - b - c - z$ has a total weight of 9 units.

The path $a - b - e - z$ has a total weight of 8 units.

Hence we can say the path $a - d - e - z$ is the shortest one among all three with 6 units of weight.

12.3 Dijkstra's Algorithm for shortest path

Let G be a simple connected weighted graph with the vertices $v_1, v_2 \ldots, v_n$. To find the shortest path between any two vertices of G. We need to follow the following terms.

- An open node is denoted by $\cdot$which means this node can travel towards any other open node and every open node also can travel towards this node.
- A node is closed if it is associated with the minimum sum of all the weights from the initial to this node.
- The closed node is denoted by the notation $\odot$means no open node can come/travel towards this node but this node can travel towards all other open nodes.
- A crossed node is denoted by the notation $\otimes$ means this node has already closed and all possible paths towards this node are not considered.

- The initial node is always treated as a closed $\odot$ node

Procedure

Let a and b are two vertices of weighted graph G. Now we want to find the shortest path between vertex a and b.

Step-1: let a be the initial node and we can consider this as a closed $\odot$ node.

Step-2: Find the distance (Weights) between node "a" and all other open nodes which are adjacent to "a"

Step-3: Identify a new closed node with a minimum distance from our initial one and circle it.

Step-4: Find the distance (Weights) between node "a" and all other open nodes which are adjacent to all closed nodes.

Step-5:Repeat step-3 and step-4 until you get node b as a closed node.

Consider the following figure 6 for a simple connected weighted graph where the wight represents the distance between two vertices. Find the shortest path between vertex a and f.

Solution

Step-1:Let the vertex a is the initial node then it is considered as a closed node and other nodes are open. No open node can move towards "a" but the node "a" can travel toward the open node "b" and "c". See figure 7.

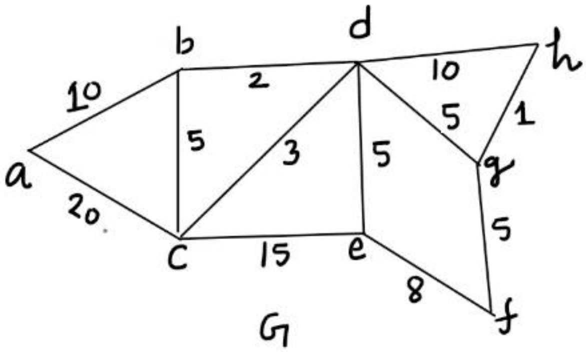


Figure 6: A weighted graph G

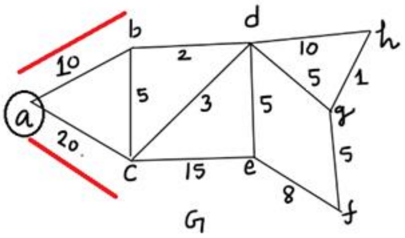


Figure 7: Node b is closed with 10 unit

Step-2: As the node "a" is closed and the node "b" and "c" are adjacent to the closed node. The distance between $a - b = 10\ unit$ and the distance between $a - c = 20\ unit$.

Step-3:Identify the new closed node with minimum distance from the initially closed node. We got the node “b” as a closed node see figure 8

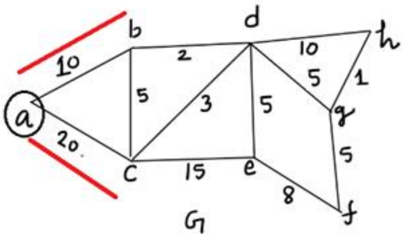


Figure 8: Node b is closed with 10 unit

Step-4: The node “b” can move towards the node “d” and the node “c” see figure 9. Hence in this step

The distance $a - c = 20\, unit.$

The distance $a - b - c = 15\, unit.$

The distance $a - b - d = 12\, unit.$

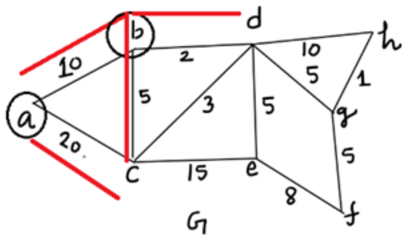


Figure 9: The node b has two possibilities c and d

Step-5:Identify the new closed node with minimum distance from the initially closed node. We got the node “d” as a closed node with a minimum weight of 12 units see figure 8

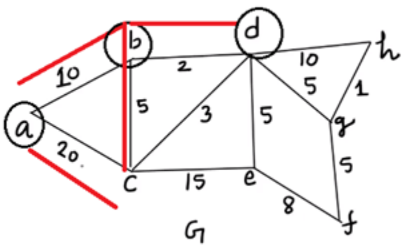


Figure 10: Node d is closed with 12 unit

Step-6: Now the node “d” is adjacent to the nodes “h”, “g”, and “c”. So node “d” can travel towards these open nodes see figure 11.

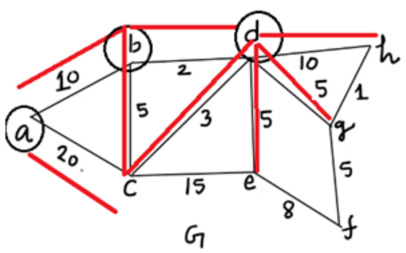


Figure 11: The open node c, e, g, and h in step 6

Hence in this step

The distance $a - c = 20\ unit$.

The distance $a - b - c = 15\ unit$.

The distance $a - b - d - c = 15\ unit$.

The distance $a - b - d - e = 17\ unit$.

The distance $a - b - d - g = 17\ unit$.

The distance $a - b - d - h = 22\ unit$.

Step-7: Identify the new closed node with minimum distance from the initially closed node. We got node "c" as a closed node with a minimum weight of 15 units. Once the node "c" is closed by any path then all other paths which close the node c are crossed. Let close node c by $a - b - c = 15\ unit$. Here the path $a - c, a - b - d - c$ would be crossed.See figure 12.

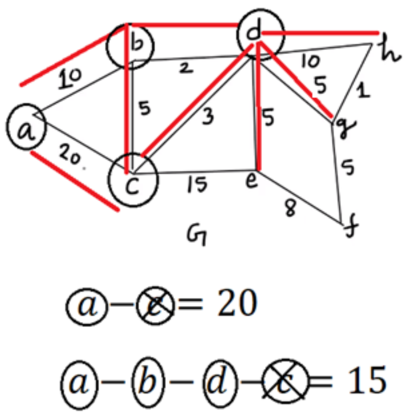


Figure 12: The node c is closed with 15 unit

Step-8: Now the node "c" is adjacent to the node "e" can travel towards this open node see figure 12.

Hence in this step

The distance $a - c - e = 35\ unit$.

The distance $a - b - d - e = 17\ unit$.

The distance $a - b - d - g = 17 unit$.

The distance$a - b - d - h = 22\ unit$.

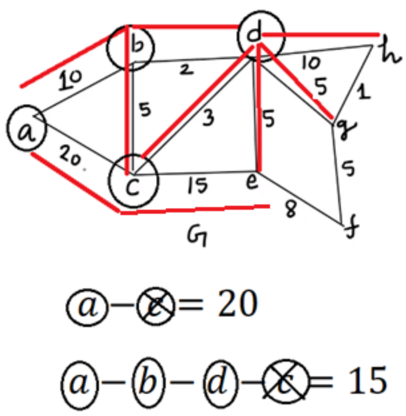


Figure 13: The open node e, g, and h in step 8

Step-9:Identify the new closed node with minimum distance from the initially closed node. We got node “e” as a closed node with a minimum weight of 17 units. Once the node “e” is closed by any path then all other paths which close the node c are crossed. The node eis closed by $a - b - d - e = 17\ unit$.Here the path $a - c - e = 35\ unit$would be crossed.See figure 14.

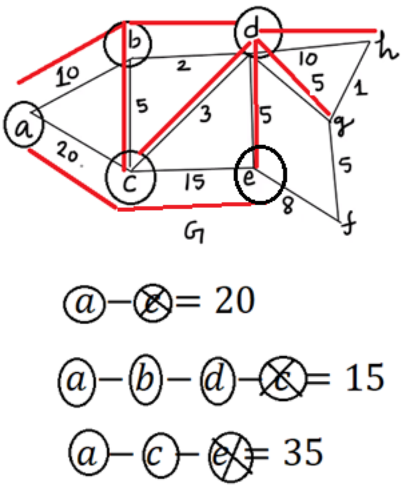


Figure 14: The node e is closed by 17 unit

Step-10: Now the node “e” is adjacent to the node “f” can travel towards this open node see figure 12.

Hence in this step

The distance $a - b - d - e - f = 25\ unit$.

The distance $a - b - d - g = 17\ unit$.

The distance $a - b - d - h = 22\ unit$.

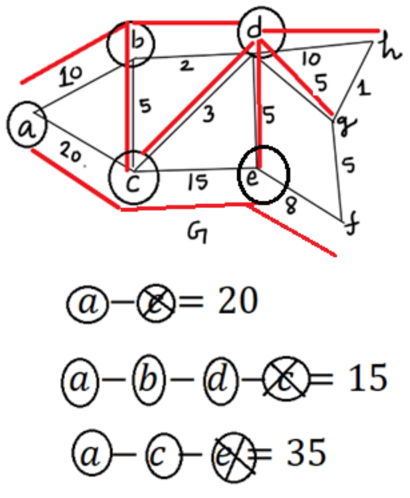


Figure 15: the open node f,g, and h

Step-10:Identify the new closed node with minimum distance from the initially closed node. We got node “g” as a closed node with a minimum weight of 17 units. The node g is closed by $a - b - d - g = 17\ unit$.See figure 16

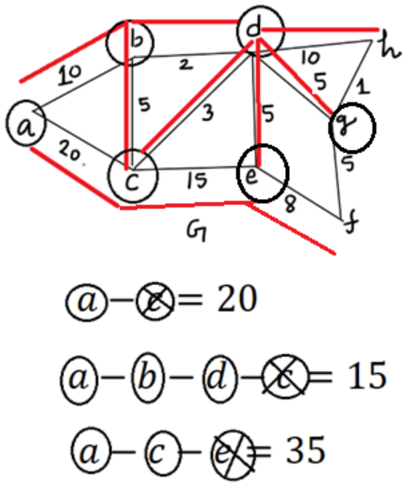


Figure 16: The node g is closed by 17 unit

Step-11: Now the node "g" is adjacent to the node "f" and node "h" and can travel towards these open nodes see figure 17.

Hence in this step

The distance $a - b - d - e - f = 25\ unit$.

The distance $a - b - d - g - f = 22\ unit$.

The distance $a - b - d - h = 22\ unit$.

The distance $a - b - d - g - h = 18 unit$.

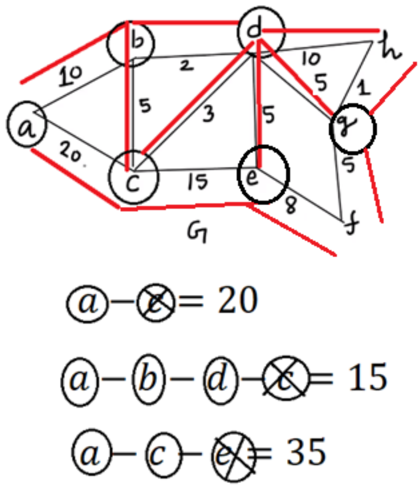


Figure 17: the open node f and h

Step-10: Identify the new closed node with minimum distance from the initially closed node. We got node "h" as a closed node with a minimum weight of 18 units but we can not travel from node h to any open node. The node "h" is traped in the algorithm. See figure 18

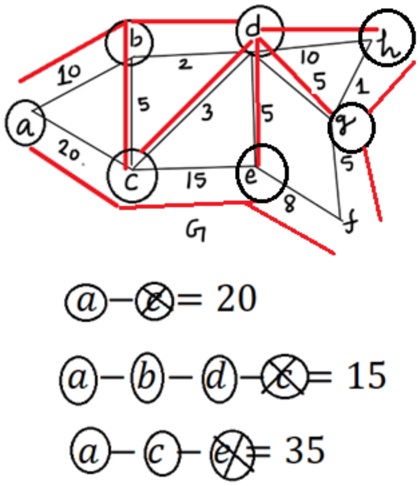


Figure 18: The node "h" is trapped

Step-11: Identify the new closed node with minimum distance from the initially closed node. We got node "f" as a closed node with a minimum weight of 22units by the path $a - b - d - g - f$. Once the node "f" is closed by any path then all other paths which close the node "f" are crossed.Here the path $a - b - d - e - f = 25\ unit$would be crossed. See figure 14

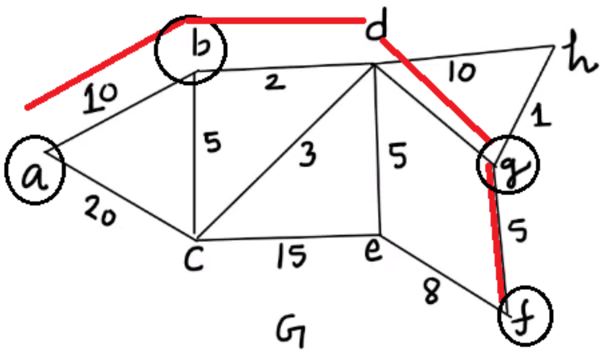


Figure 19:The destination node is closed by 22 units.

Hence the required shortest path is $a - b - d - g - f$ with a 22-unit distance.

Summary

- A graph that has labels associated with each edge or each vertex is known as a labeled graph.
- Let G be a vertex labeled graph and if a number assigned to each edge of G then it called a weighted graphs
- Let G be a weighted graph then the shortest path between any two vertices is the path which contains minimum weight between all possible path.
- Dijkstra's algorithm: a procedure for finding the shortest path between two vertices in a weighted graph

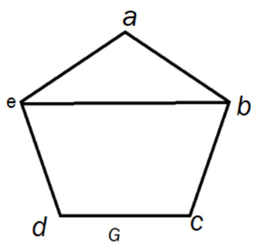
Keywords

Weighted graph: a graph with numbers assigned to its edges

Shortest-path problem: the problem of determining the path in a weighted graph such that the sum of the weights of the edges in this path is a minimum overall path between specified vertices

Self Assessment

1. Consider the following graph G, then G is__?



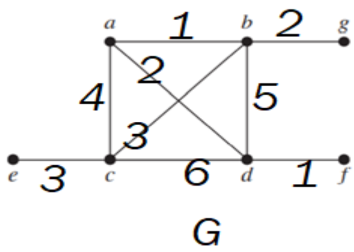
a. A vertex labeled graph

b. An edge labeled graph

c. Both vertex and edge labeled graph

d. Not a labeled graph

2. Consider the following graph G, then G is__?



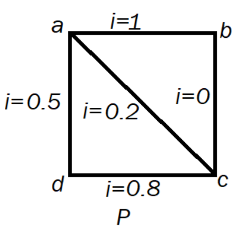
a. A vertex labeled graph

b. An edge labeled graph

c. Both vertex and edge labeled graph

d. Not a labeled graph

3. Consider the following weighted graph P. Here all the weights are associated with a connective index between two vertices, which depends on the following distance function. Then what is the distance between vertex a and b?



$$i(\mathit{Dist.}) = \begin{cases} 1 & \mathit{Dist.} = 10\ \mathit{Km.} \\ \dfrac{100 - x}{90} & 10\ \mathit{Km.} < \mathit{Dist.} < 100\ \mathit{Km.} \\ 0 & \mathit{Dist} = 100\ \mathit{Km.} \end{cases}$$

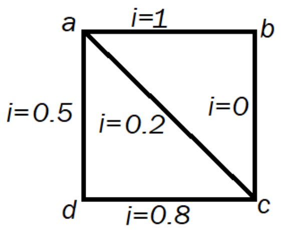
a. 10 km

b. 100 km

c. 80 km

d. 40 km

4. Consider the following weighted graph P. Here all the weights are associated with a connective index between two vertices, which depends on the following distance function. Then what is the distance between vertex b and c?



P

$$i(Dist.) = \left\{ \begin{array}{ll} 1 & Dist. = 10\ Km. \\ \dfrac{100-x}{90} & 10\ Km. < Dist. < 100\ Km. \\ 0 & Dist = 100\ Km. \end{array} \right\}$$

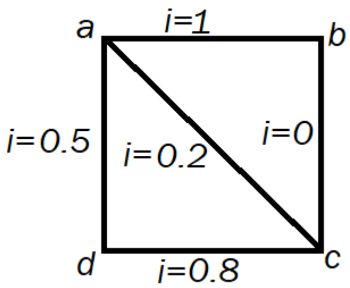
a. 10 km

b. 100 km

c. 80 km

d. 40 km

5. Consider the following weighted graph P. Here all the weights are associated with a connective index between two vertices, which depends on the following distance function. Then what is the distance between vertex a and d?



P

$$i(Dist.) = \left\{ \begin{array}{ll} 1 & Dist. = 10\ Km. \\ \dfrac{100-x}{90} & 10\ Km. < Dist. < 100\ Km. \\ 0 & Dist = 100\ Km. \end{array} \right\}$$

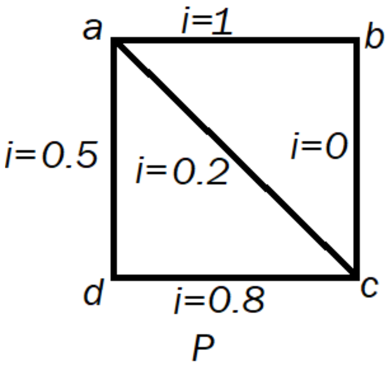
a. 10 km

b. 100 km

c. 55 km

d. 40 km

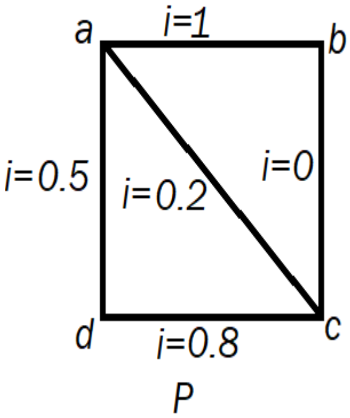
6. Consider the following weighted graph P. Here all the weights are associated with a connective index between two vertices, which depends on the following distance function. Then what is the distance between vertex a and c?



$$i(Dist.) = \begin{cases} 1 & Dist. = 10\ Km. \\ \dfrac{100 - x}{90} & 10\ Km. < Dist. < 100\ Km. \\ 0 & Dist = 100\ Km. \end{cases}$$

a. 10 km
b. 100 km
c. 28 km
d. 82 km

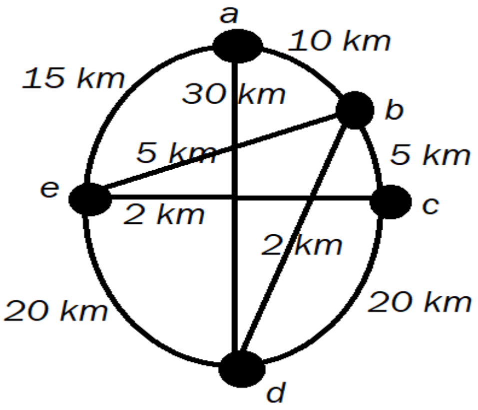
7. Consider the following weighted graph P. Here all the weights are associated with a connective index between two vertices, which depends on the following distance function. Then what is the distance between vertex a and c?



$$i(Dist.) = \begin{cases} 1 & Dist. = 10\ Km. \\ \dfrac{100 - x}{90} & 10\ Km. < Dist. < 100\ Km. \\ 0 & Dist = 100\ Km. \end{cases}$$

A. 10 km
B. 100 km
C. 28 km
D. 82 km

8. Consider the following weighted graph and then what is the shortest distance between vertex a and d



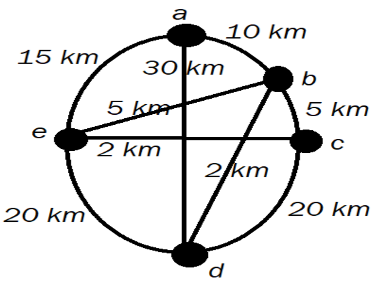
A. 12 km

B. 10 km

C. 30 km

D. 17 km

9. Consider the following weighted graph and then what is the shortest distance between vertex a and c



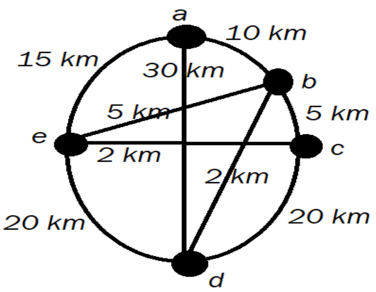
A. 12 km

B. 15 km

C. 30 km

D. 17 km

10. Consider the following weighted graph and then what is the node which is going to close/cover in the first step of Dijkstra's algorithm, to find the shortest path from vertex e to d ?



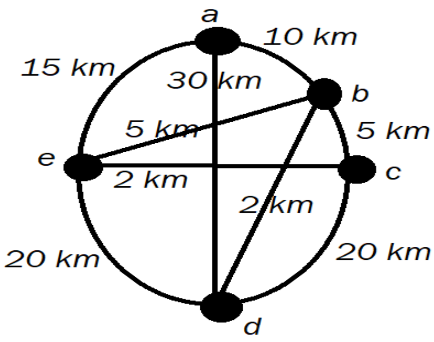
A. Node a

B. Node b

C. Node c

D. Node d

11. Consider the following weighted graph and then what is the node which is going to close/cover in the second step of Dijkstra's algorithm, to find the shortest path from vertex e to d ?



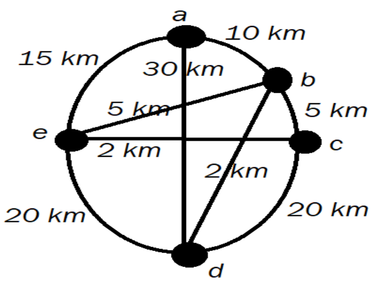
A. Node a

B. Node b

C. Node c

D. Node d

12. Consider the following weighted graph. To find the shortest path from vertex e to d by Dijkstra's algorithm, node b is closed/covered with the distance_____?



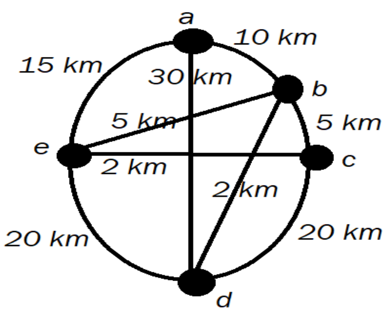
A. 5km

B. 7km

C. 2km

D. 10 km

13. Consider the following weighted graph. To find the shortest path from vertex e to d by Dijkstra's algorithm, node d is closed/covered with the distance_____?



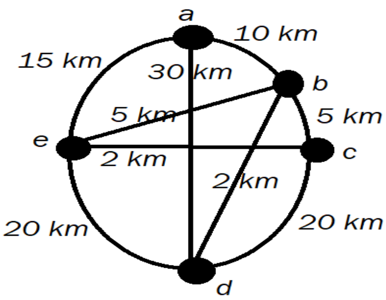
A. 5km

B. 7km

C. 2km

D. 10 km

14. Consider the following weighted graph. To find the shortest path from vertex e to d by Dijkstra's algorithm, then which node is isolated?



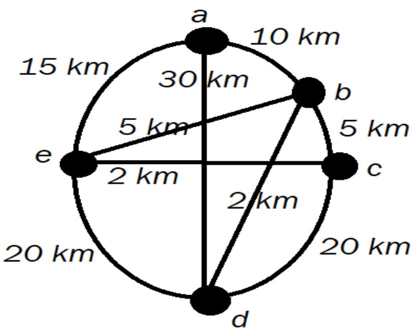
A. Node a

B. Node b

C. Node c

D. Node d

15. Consider the following weighted graph and then what is the node which is going to close/cover in the first step of Dijkstra's algorithm, to find the shortest path from vertex b to d ?



A. Node d

B. Node e

C. Node c

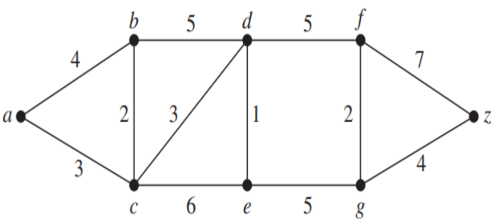
D. Node a

Answers for Self Assessment

1.	A	2.	C	3.	A	4.	B	5.	C
6.	D	7.	C	8.	B	9.	B	10.	C
11.	B	12.	A	13.	B	14.	A	15.	A

Review Questions:

Consider the following weighted graph then



1. Find the shortest path between vertex “a” and “d”?

2. Find the shortest path between vertex “a” and “e”?

3. Find the shortest path between vertex “a” and “d”?

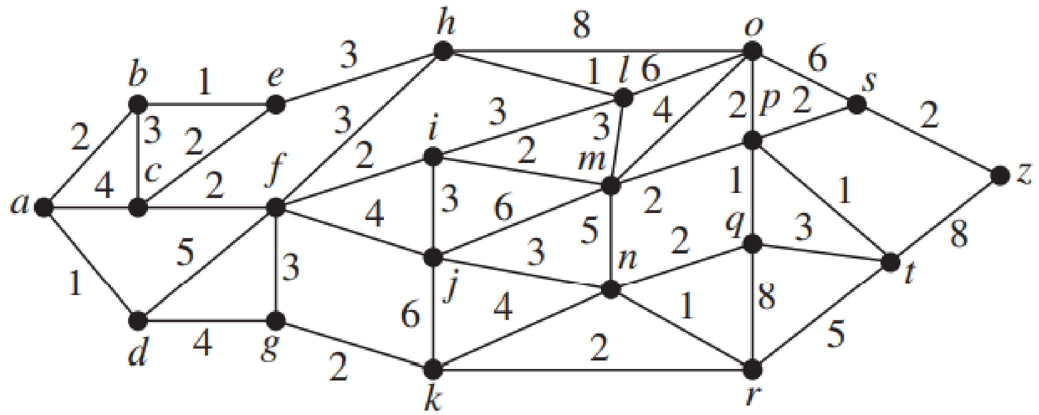
4.Find the shortest path between vertex “a” and “f”?

5.Find the shortest path between vertex “a” and “g”?

6.Find the shortest path between vertex “d” and “e”?

7. Find the shortest path between vertex “d” and “g”?

Consider the following weighted graph then



8. Find the shortest path between vertex “a” and “f”?

9. Find the shortest path between vertex “a” and “h”?

10. Find the shortest path between vertex “a” and “m”?

11.Find the shortest path between vertex “a” and “n”?

12.Find the shortest path between vertex “a” and “r”?

13.Find the shortest path between vertex “i” and “r”?

14. Find the shortest path between vertex “d” and “r”?

15. Find the shortest path between vertex “o” and “r”?

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 13: Introduction to Tree

CONTENTS

Objectives:

Trees have been reported since 1857, to measure certain kinds of chemical compounds, when the English theorist Arthur Cayley introduced them. Since then, trees have been used to solve problems in a variety of fields. As the references in this section, a broad spectrum of disciplines are involved.In computer engineering, trees are especially useful, and they are used in a variety of algorithms. Trees, for example, are used to create powerful algorithms for finding objects in a catalog. They can even be used in effective programming algorithms like Huffman coding to minimize the cost of data transfer and retrieval.

After this unit, you would be able to

- understand the definition of a tree.
- learn the rooted tree and identify different components of the rooted tree.
- describe the m-ary and full m-ary tree.
- calculate the number of total vertices, internal vertices in a full m-ary tree.

Introduction

The botanical relatives and tree data systems have a lot of similarities. A root stems and leaves are all presents. In this section, we'll look at a specific kind of structure known as a tree graph, which gets its name from the fact that it looks like a tree. Family trees, for example, are diagrams that depict genealogical tables.The members of a family are represented by vertices, and the parent-child relationships are represented by edges.

13.1 Tree

A tree is a connected undirected graph with no simple circuits.

Example:

Consider the following graphs represented in figure 1.

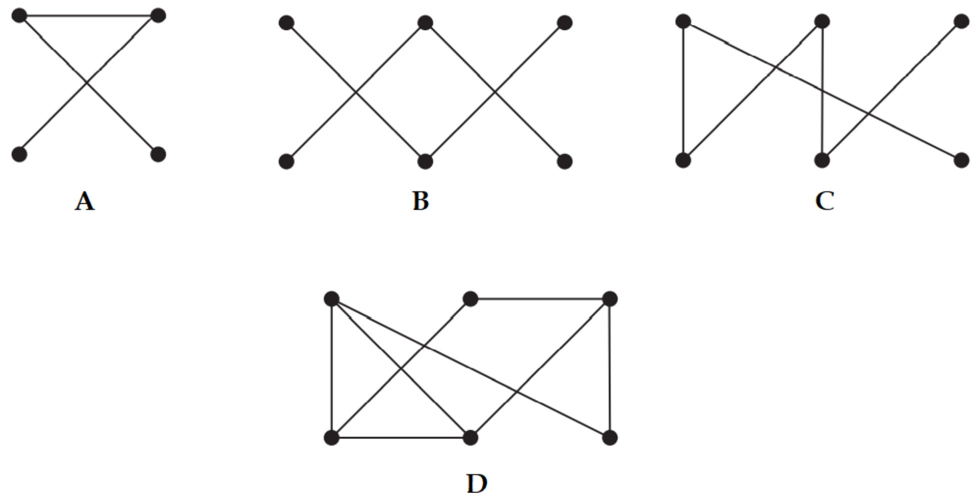


Figure 1: The graphs A, B, C, and D

Here we see the graph A is a tree as it is connected and not forming any circuit. Graph B is not a tree because it is not a connected graph. Graph C is a tree as it is connected and not forming any circuit. Finally, whatever the graph D is connected but is not a tree because it contains a circuit.

- An undirected graph is said to be a tree if there exists a unique simple path between every pair of vertices.
- A tree with n vertices has n − 1 edge

Consider figure 2 for a tree. In this graph, we can observe a unique simple path between every pair of vertices. There are 6 vertices and 5 edges.

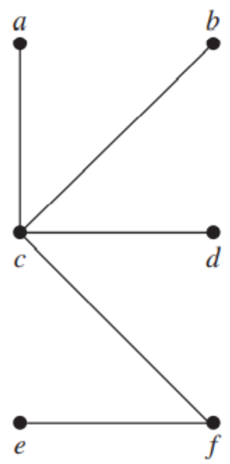


Figure 2: A tree

13.2 Rooted Trees

A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Consider the tree represented in figure 3. Figure 4 represents a tree in which the node “a” is designated as a root node. Figure 5 represents a tree in which the node “c” is designated as a root node.

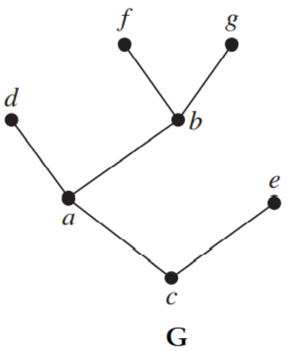


Figure 3: A tree graph

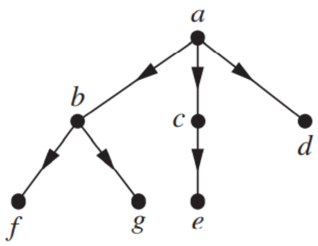


Figure 4: A rooted tree of G with "a" as a root node

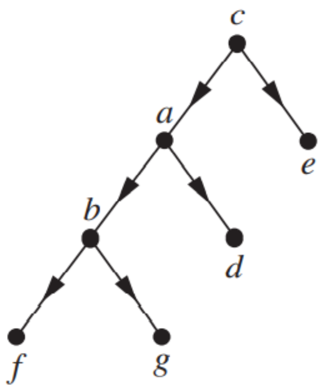


Figure 5: A rooted tree of G with "c" as a root node

Components of the rooted tree

Parent node:

Let v be a vertex other than the root node in the tree, then the unique vertex u is said to be the parent node of v if there exists a directed edge from u to v.

The child node:

Let "u" be any vertex in the tree, then a vertex "v" is said to be the child node of "u" if the node "u" is a parent node of "v".

Sibling of a vertex v:

Let the node "a" be a parent node and the nodes "b", "c"…,"z" are the child nodes of "a" then "b", "c"…,"z" are the siblings. All the child nodes that have the same parents are known as siblings.

The ancestor of a vertex v:

Let T be a rooted tree then the ancestor of any vertex "v"in T is the all the vertices on the path from the root node to "v"

The descendant of a vertex v :

Let T be a rooted tree then the descendant of any vertex "v" in T is the all the vertices that have"v" as an ancestor

Internal vertex:

Let T be a rooted tree than any vertex "v" in T is said to be an internal vertex if it has a child.

Leaf node:

Let T be a rooted tree than any vertex "v" in T is said to be a leaf node if it does not has any child.

Consider the rooted tree represented in figure 6.

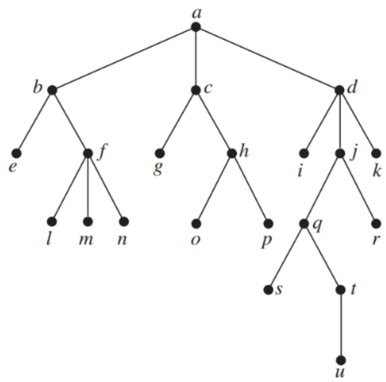


Figure 6: A rooted tree T

Here

The root node in T: "a"

Parent nodes in T:"a", "b", "c"," d"," f","h"," j"," q", and the node "t"

The child node in T: Every node except the root node "a" is the child in T

Sibling of a vertex "k": The node "i" and "j"

The ancestor of a vertex "n": The node "a", "b", and "f".

The descendant of a vertex "c": The node "g", "h", "o", and "p".

Internal vertex: The nodes "a", "b", "c","d","f","h","j","q", and "t"

Leaf node: The node "e", "l", "m", "n","g","o","p"," i","s","u", "r", and "k".

13.3 M-ary tree

A rooted tree is called an m-ary tree if every internal vertex has no more than m children.

Full m-ary tree

The tree is called a full m-ary tree if every internal vertex has exactly m children.

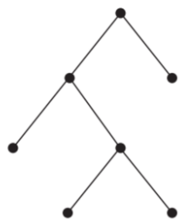
Binary tree

An m-ary tree with m = 2 is called a binary tree

Full binary tree

A three in which every parent node has exactly 2 children.

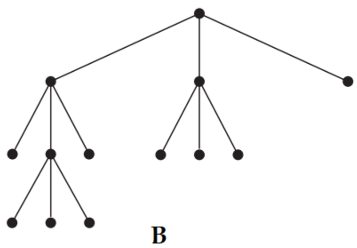
Example 1 Consider the rooted tree A represented in figure 7.It is observed that every parent node has exactly 2 children so we can say tree A is a full binary tree.



A

Figure 7: A full binary tree

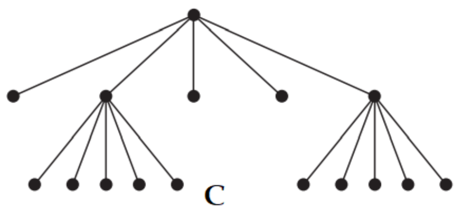
Example 2 Consider the rooted tree B represented in figure 8.It is observed that every parent node has exactly 3 children so we can say tree B is a full 3-ary tree.



B

Figure 8: A full 3-ary tree.

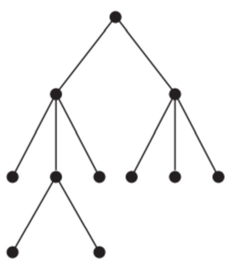
Example 3 Consider the rooted tree C represented in figure 9. It is observed that every parent node has exactly 5 children so we can say tree C is a full 5-ary tree.



C

Figure 9: A full 3-ary tree.

Example 4 Consider the rooted tree D represented in figure 10. It is observed that every parent node has a maximum of 3 children so we can say tree D is a full 3-ary tree.



D

Figure 10: A 3-ary tree.

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

The tree C represented in figure 9 is a full 5-ary tree. Here $m = 5$ and there are $i = 3$ internal vertices. So the total number of vertices $n = mi + 1 = 5 * 3 + 1 = 16$.

Summary

- A tree is a connected undirected graph with no simple circuits.
- A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.
- An undirected graph is said to be a tree if there exists a unique simple path between every pair of vertices.
- A tree with n vertices has n − 1 edge
- Let v be a vertex other than the root node in the tree, then the unique vertex u is said to be the parent node of v if there exists a directed edge from u to v.
- Let "u" be any vertex in the tree, then a vertex "v" is said to be the child node of "u" if the node "u" is a parent node of "v".
- Let the node "a" be a parent node and the nodes "b", "c" …,"z" are the child nodes of "a" then "b", "c" …,"z" are the siblings. All the child nodes that have the same parents are known as siblings.
- Let T be a rooted tree then the ancestor of any vertex "v"in T is the all the vertices on the path from the root node to "v"
- Let T be a rooted tree then the descendant of any vertex "v" in T is the all the vertices that have"v" as an ancestor
- Let T be a rooted tree than any vertex "v" in T is said to be an internal vertex if it has a child.
- A rooted tree is called an m-ary tree if every internal vertex has no more than m children.
- The tree is called a full m-ary tree if every internal vertex has exactly m children.
- An m-ary tree with m = 2 is called a binary tree
- A three in which every parent node has exactly 2 children.
- A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Keywords

Tree: a connected undirected graph with no simple circuits.

Rooted tree: a directed graph with a specified vertex, called the root, such that there is a unique path to every other vertex from this root.

The parent node of v in a rooted tree: the vertex u such that (u, v) is an edge of the rooted tree.

The child node: any vertex with v as its parent.

The sibling node: a vertex with the same parent as v.

The ancestor node of a vertex v in a rooted tree: any vertex on the path from the root to v.

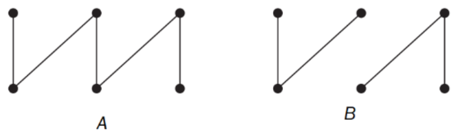
The descendant of a vertex v in a rooted tree: any vertex that has v as an ancestor.

The internal node: a vertex that has children.

The leaf node: a vertex with no children

Self Assessment

1. Which of the following graphs is/are trees?



A. Only A

B. Only B

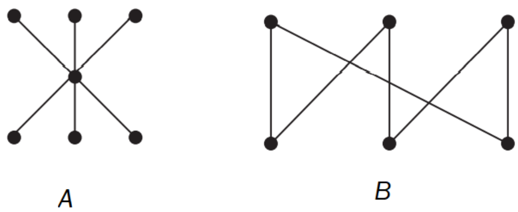
C. Both A and B

D. Neither A nor B

2. Which of the following graphs is/are trees?

A. Only A
B. Only B
C. Both A and B
D. Neither A nor B

3. Which of the following graphs is/are not a tree?



A. Only A
B. Only B
C. Both A and B
D. Neither A nor B

4. Let G be a graph with 10 vertices and G is a tree then how may edge G have?

A. 10
B. 9
C. 3
D. 0

5. Let G be a graph with 50 vertices and G is a tree then how many cycles of length 3 G have?

A. 10
B. 9
C. 3
D. 0

6. Let G be a graph with 15 vertices and G is a tree then ____?

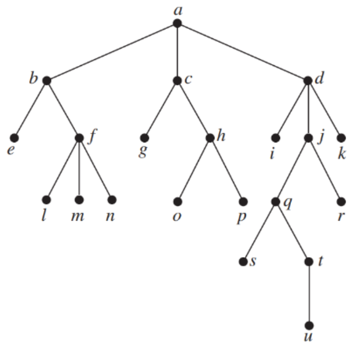
A. G is connected

B. G is not simple

C. G has a circuit of length 5

D. G is disconnected

7. Consider the following rooted tree then what is the root node?



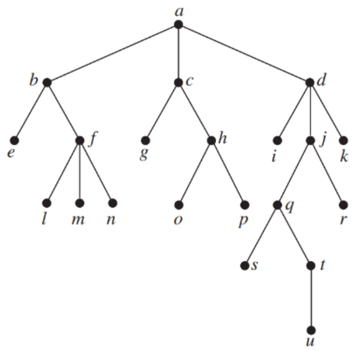
A. Node a

B. Node u

C. Node b

D. Node c

8. Consider the following rooted tree then which one of the following is an internal node?



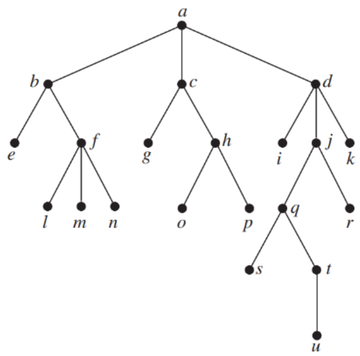
A. Node c

B. Node e

C. Node l

D. Node g

9. Consider the following rooted tree then which one of the following is the leaf node?



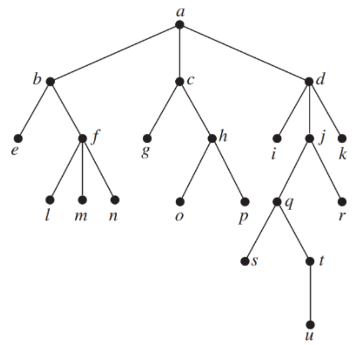
A. Node a

B. Node b

C. Node e

D. Node f

10. Consider the following rooted tree then which one of the following is the parent of node p?



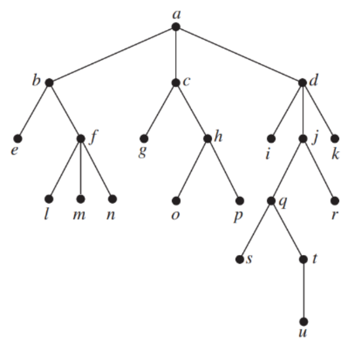
A. Node h

B. Node c

C. Node a

D. Node b

11. Consider the following rooted tree then which one of the following is the children of node t?



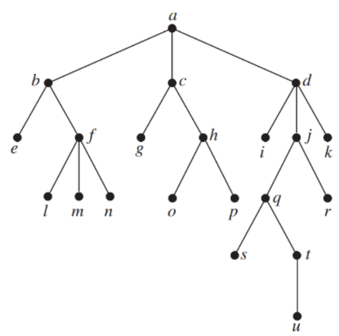
A. Node c

B. Node u

C. Node l

D. Node g

12. Consider the following rooted tree then which one of the following is the sibling of node g?



A. Node h

B. Node e

C. Node l

D. Node g

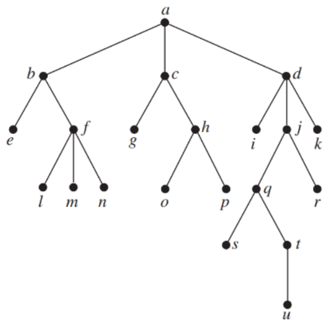
13. Let T be a full 3-ary tree with 13 vertices then the number of internal vertices in T is ____?

A. 4
B. 5
C. 6
D. 7

14. Let T be a full 3-ary tree with 3 internal vertices then the total number of vertices in T is ____?

A. 10
B. 5
C. 6
D. 7

15. Consider the following rooted tree then which one of the following is correct



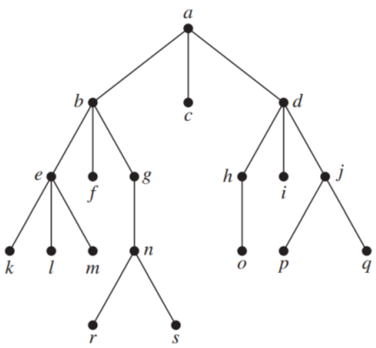
A. It's a 3-ary tree
B. It's a full 3-ary tree
C. It's a 2-ary tree
D. It's a full 2-ary tree

Answers for Self Assessment

1.	A	2.	A	3.	B	4.	B	5.	D
6.	A	7.	A	8.	A	9.	C	10.	A
11.	B	12.	A	13.	A	14.	A	15.	A

Review Questions:

Consider the following tree and answer the following questions



1. Find the root node?

2.Find the internal vertices?

3. Find leaf nodes?

4. Find the children of n?

5. Find the parent of h?

6. Find the siblings of k?

7. Find the ancestors of l?

8.Find the descendants of d?

9. Check whether it is a full m-ary tree?

10. Find the value of m if it is an m-ary tree?

11. Let T be a tree with 15 vertices then find the number of edges in T?

12.Let T be a tree with 10 edges then find the number of vertices in T?

13.Let T be a full 3-ary tree with 4 internal vertices then find the total number of vertices in T?

14.Let T be a full 3-ary tree with 4 internal vertices then find the total number of edges in T?

15.Let T be a full binary tree with 3 internal vertices then find the total number of vertices in T?

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

Unit 14: Spanning Tree

CONTENTS

Objectives:

The most common use of a spanning tree is to solve a problem like a phone network design. For example, you have a firm with many locations; you need to lease phone lines to connect them, and the phone provider charges varying rates for connecting different pairings of cities. You would like a system that links all of your branches at the lowest possible cost. It should remain a spanning tree because you can always eliminate some connections and also save cost if the structure isn't a tree. The minimal spanning tree may be applied to approximately resolving the traveling salesman challenge. Finding the optimal solution that visits each location at least once is a useful formal method of stating this problem.

After this unit, you would be able to

- understand the spanning tree in a connected graph
- learn the Prims and Kruskal algorithm to get the minimum spanning tree from a weighted connected graph
- find the minimum spanning tree from a weighted connected graph

Introduction

In the winter session, the roads between cities are blocked by snowfall. Consider the road infrastructure of cities, which is depicted in Figure 1 (a). The only way to keep the roads accessible in the winter is to plow them often. The transportation agency intends to plow the fewest highways possible so that all highways between towns are constantly clean. How can this be accomplished?

To verify that there is a passage connecting any two cities, at least five roads should be plowed. One such collection of roadways is seen in Figure 1(b). It's worth noting that the subgraph that represents these roads is a tree.

In this section first, we will embark on some important credentials of the rooted tree then will move towards the spanning tree.

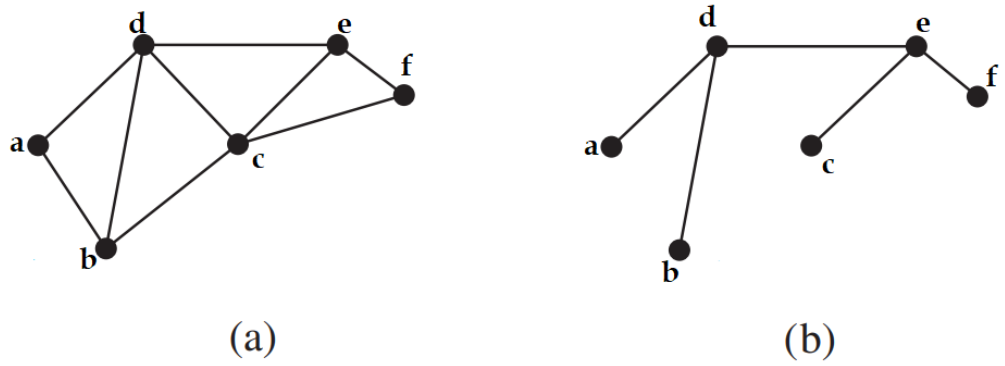


Figure 1:The graph (a) and (b)

14.1 Balanced m-ary trees

A rooted m-ary tree of height h is balanced if all leaves are at levels h or h –1.whereThe level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex and the height of a rooted tree is the maximum of the levels of vertices.

Example:

Consider figure 2 for the rooted tree. Here the level of the node "a" is 0. The level of nodes "b", "j", and "k" is 1. The level of nodes "c", "e", "f", and "l" is 2. The level of nodes "d", "g", "i", "m", and "n" is 3 and the last level of node "h" is 4.Hence the height of the tree is 4.

If the following tree with height 4 is said to be balanced if all the leaves are situated at level 4 or 3. Some of the leaves say "j" situated at level 1 so the tree is not balanced.

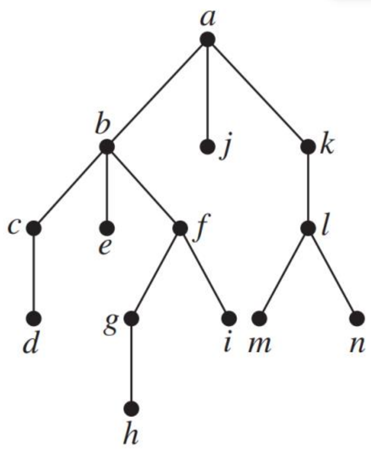


Figure 2: The rooted tree with root node "a"

While the tree represented in figure 3 is balanced because its height is 4 all the leaves are situated at level 3 or 4.

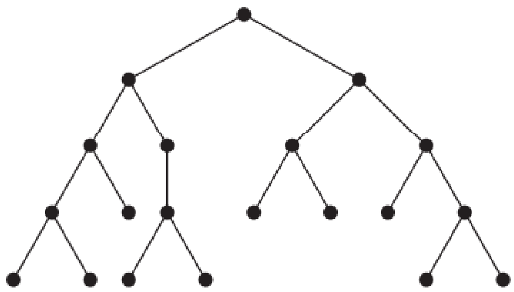


Figure 3: A 2-ary balanced tree with a height of 4.

14.2 Spanning tree

Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Example

Consider the graph represented in figure 4. There are 7 vertices and we can consider the combination of any 6 edges from the given graph such that the resulting graph is a tree. We need to avoid the edges which are forming a circuit. Figure 5 shows the 5 different spanning-tree derived from figure 4.

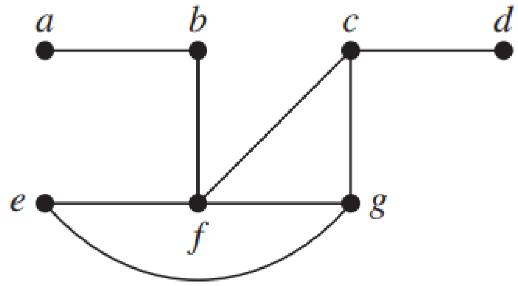


Figure 4: A simple connected graph.

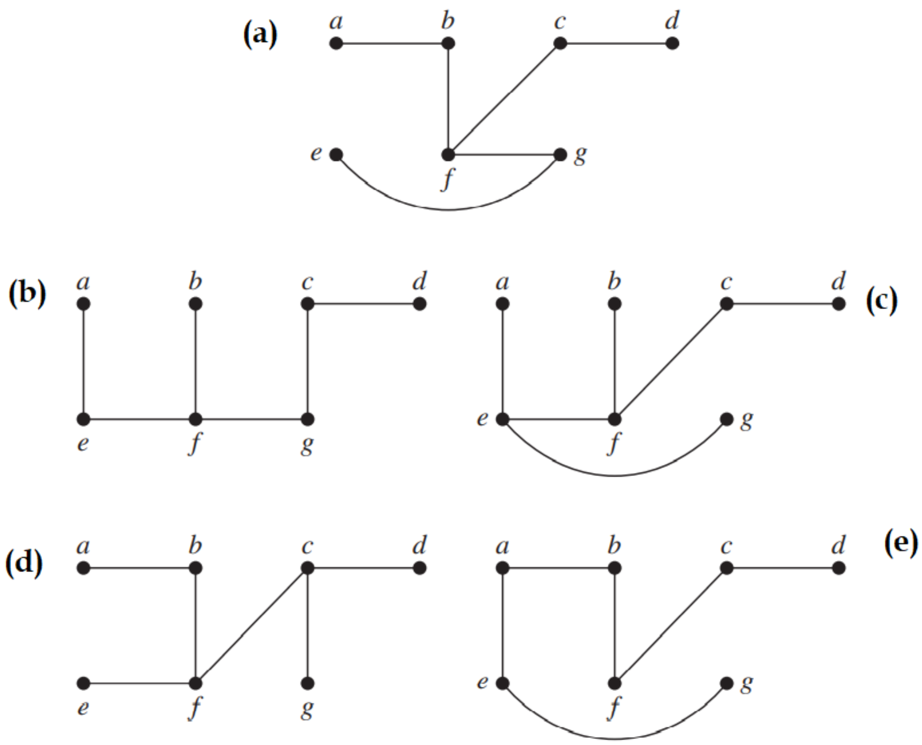


Figure 5: Different spanning trees of figure 4

14.3 Minimum Spanning Trees

A minimum spanning tree in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

Example

Consider the weighted graph 6(a). If we want to draw a minimum spanning tree from this graph then we need to avoid the maximum weight edges in every circuit of figure 6(a). Edge $a-b,$

$b-c, c-d, and\ c-f$ has heightest weight in the circuits $a-b-d-a, b-c-d-b, c-d-e-c, and\ c-f-e-c$. After removing them we get a minimum spanning tree as figure 6 (b).

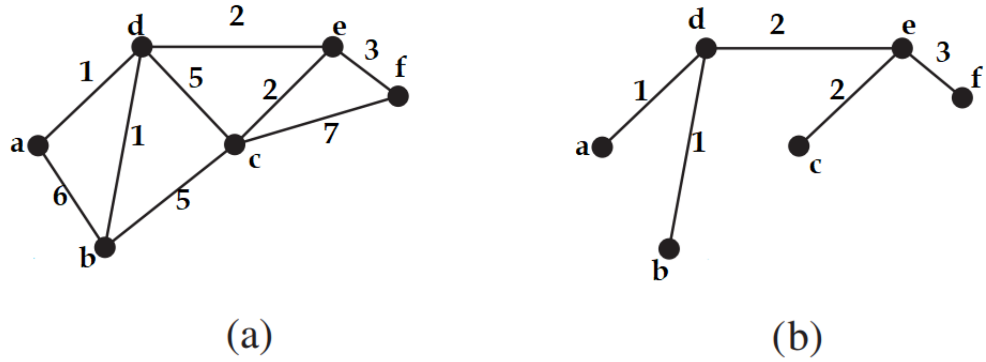


Figure 6: The weighted graph (a) and the minimum spanning tree (b)

Kruskal's Algorithm

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
for $i := 1$ **to** $n - 1$
  e := any edge in G with smallest weight that does not form a simple circuit
   when added to T
  $T := T$ with e added
return T {T is a minimum spanning tree of G}

Example

Consider a simple connected weighted graph depicted in figure 7

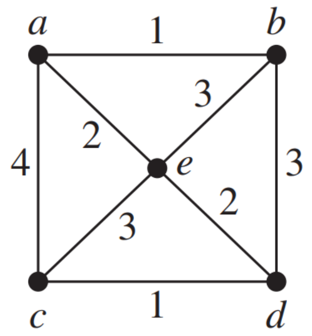


Figure 7: A simple connected weighted graph.

Let's prepare a table in which all the edges are taken according to their weight

Weight	Edge	Can this edge be considered as not forming any circuit?
1	$a - b$	Yes
	$c - d$	Yes
2	$a - e$	Yes
	$e - d$	Yes
3	$b - e$	No
	$d - b$	No
	$c - e$	No
4	$c - e$	No

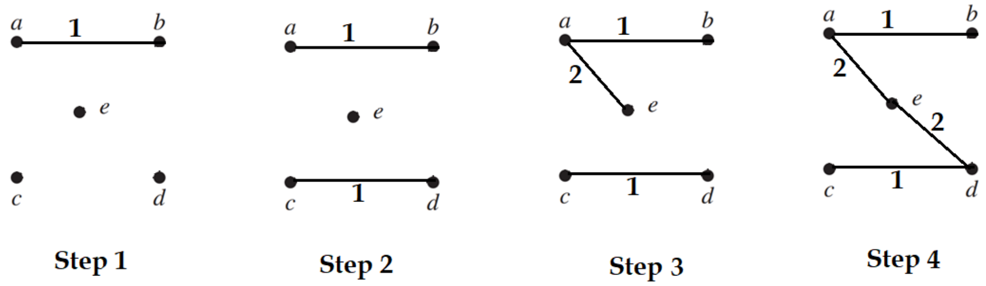


Figure 8: Steps of Krushkal algorithm.

The total weight of the minimum spanning tree is 6.

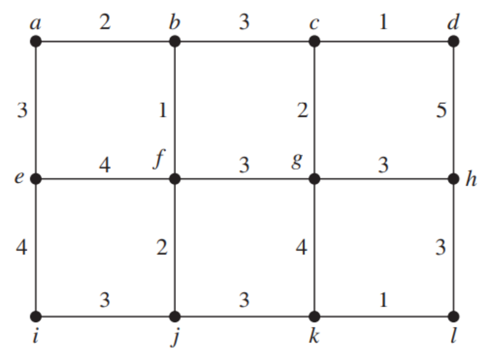
Prim's Algorithm

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for i := 1 **to** $n - 2$
    e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T
    T := T with e added
return T {T is a minimum spanning tree of G}

Example :

Consider a simple connected weighted graph depicted in figure 8.

Let's prepare a table in which all the edges are taken according to their weight



Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1

Figure 9: Asimple connected weighted graph

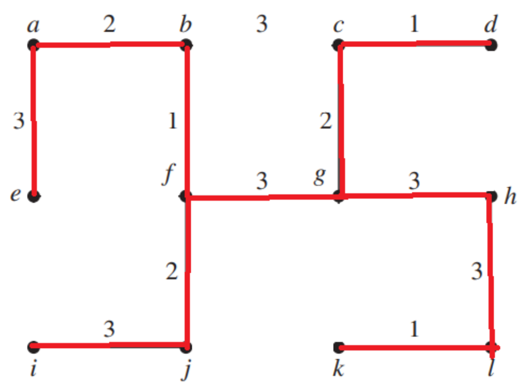


Figure 10: The minimum spanning tree of figure 9

Summary

- A rooted m-ary tree of height h is balanced if all leaves are at levels h or h –1. wherethe level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex and the height of a rooted tree is the maximum of the levels of vertices.
- Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.
- A minimum spanning tree in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

Keywords

Level of a vertex: the length of the path from the root to this vertex.

Height of a tree: the largest level of the vertices of a tree.

Balanced tree: a tree in which every leaf is at level h or h − 1, where h is the height of the tree.

Spanning tree: a tree containing all vertices of a graph.

Minimum spanning tree: a spanning tree with the smallest possible sum of weights of its edges

Review Questions:

1. Draw the one spanning tree from K_5?

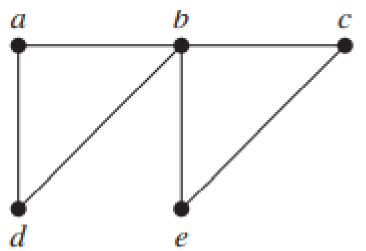
2.Draw the one spanning tree from C_6?

3. Draw the one spanning tree from $K_{3,4}$?

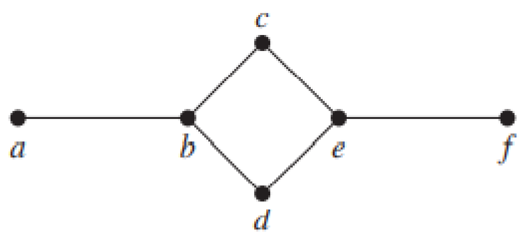
4. Draw the one spanning tree from W_4?

5.Draw the one spanning tree from $K_{1,5}$?

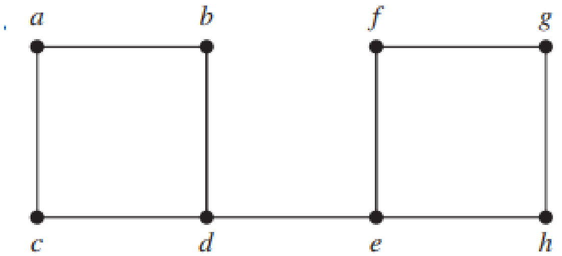
6. Draw all possible spanning trees from the following graph?



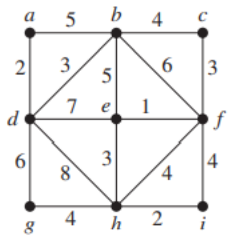
7. Draw all possible spanning trees from the following graph?



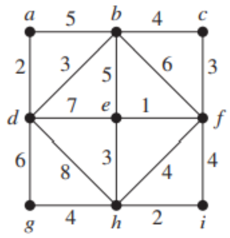
8.Draw all possible spanning trees from the following graph?



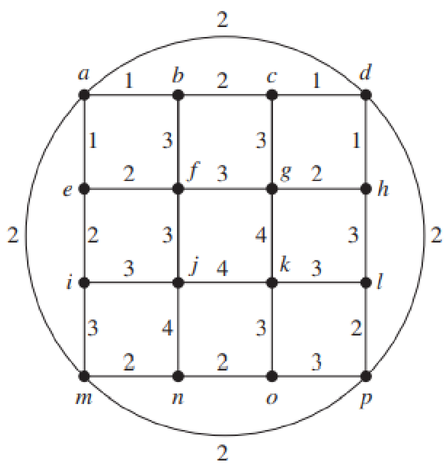
9.Find the minimum spanning trees using the Prims algorithm for the following graph?



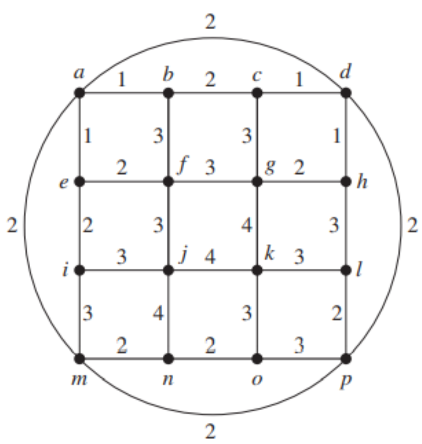
10.Find the minimum spanning trees using the Kruskal algorithm for the following graph?



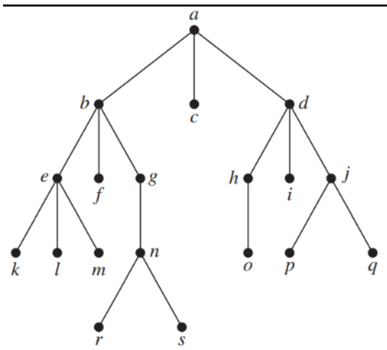
11.Find the minimum spanning trees using the Prims algorithm for the following graph?



11.Find the minimum spanning trees using the Prims algorithm for the following graph?



Consider the following tree and answer the following questions



12. Find the level of node “b”?

13. Find the level of node “f”?

12. Find the level of node “l”?

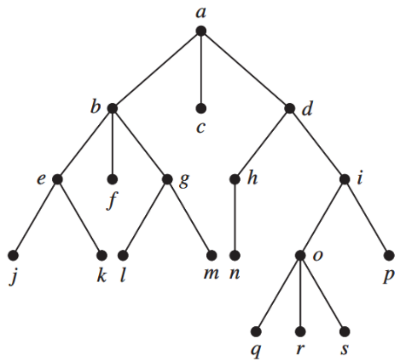
13. Find the level of node “r”?

14. Find the height of the tree?

15. Check whether the tree is balanced?

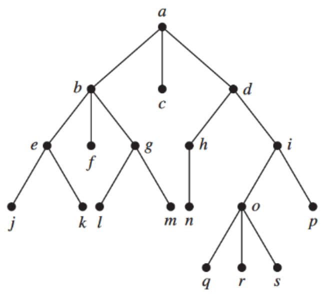
Self Assessment

1. What is the level of node m in the following tree?



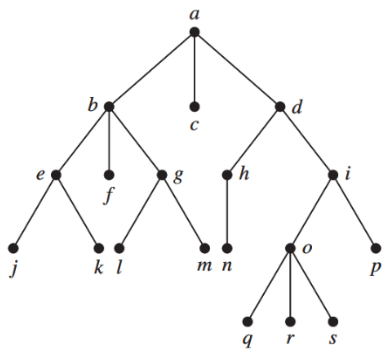
A. 2
B. 3
C. 4
D. 5

2. What is the level of node m in the following tree?



A. 2
B. 3
C. 4
D. 5

3.What is the height of the following tree?



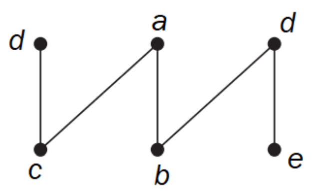
A. 2

B. 3

C. 4

D. 5

4.Consider node a as the rooted node in the following tree then what is the height of the tree?



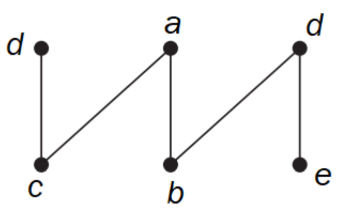
A. 2

B. 3

C. 4

D. 5

5.Consider the node a as the rooted node in the following tree then what is the level of node d is____?



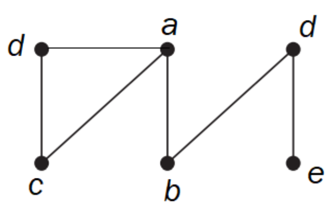
A. 2

B. 3

C. 4

D. 5

6.How many spanning trees are there in the following graph?



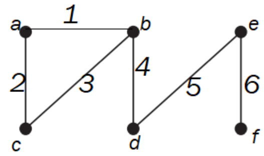
A. 2

B. 3

C. 4

D. 5

7.The following graph becomes the spanning-tree if__?



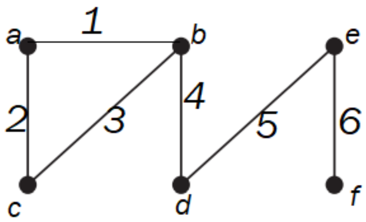
A. Edge a-b can be removed

B. Edge d-b can be removed

C. Edge d-e can be removed

D. Edge e-f can be removed

8.The following graph becomes the minimum spanning tree if__?



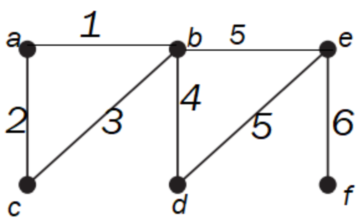
A. Edge a-b can be removed

B. Edge a-c can be removed

C. Edge b-c can be removed

D. Edge e-f can be removed

9.The weight of the minimum spanning tree in the following graph



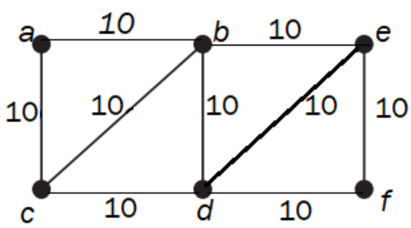
A. 18

B. 20

C. 12

D. 25

10. The weight of the minimum spanning tree in the following graph



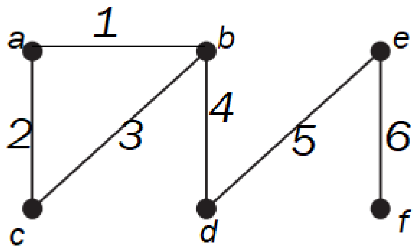
A. 40

B. 50

C. 60

D. 70

11. The following graph becomes the minimum spanning tree if__?



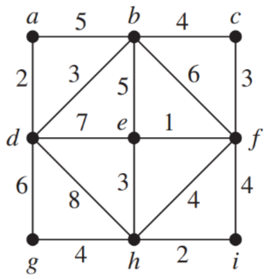
A. Edge c-b can be removed

B. Edge d-b can be removed

C. Edge d-e can be removed

D. Edge e-f can be removed

12. Consider the following weighted graph then which edge can be chosen very first to draw the minimum spanning tree by Prim's algorithm.



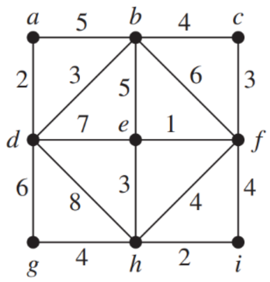
A. Edge e-f

B. Edge a-d

C. Edge h-i

D. Edge c-f

13. To draw the minimum spanning tree by Prim's algorithm from the following weighted graph. Which edge can be chosen after selecting the edge e-f



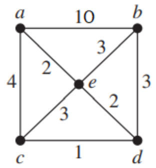
A. Edge e-f

B. Edge a-d

C. Edge h-i

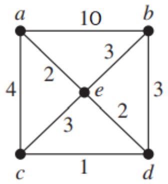
D. Edge c-f

14. Consider the following weighted graph then which edge can be chosen very first to draw the minimum spanning tree by Prim's algorithm.



A. Edge c-d
B. Edge a-b
C. Edge a-e
D. Edge b-d

15. To draw the minimum spanning tree by Prim's algorithm from the following weighted graph. Which edge can be chosen after selecting the edge c-d?



A. Edge e-d
B. Edge a-b
C. Edge a-c
D. Edge c-e

Answers for Self Assessment

1.	B	2.	C	3.	C	4.	B	5.	A
6.	B	7.	A	8.	C	9.	A	10.	B
11.	A	12.	A	13.	D	14.	A	15.	A

Further Readings

Discrete mathematics & its applications by Kenneth H Rosen, McGraw hill education

Discrete mathematics (Schaum's outlines) (sie) by Seymour Lipschutz, Marc Lipson, Varsha H. Patil, Mcgraw hill education

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