

Management Science

EOPR605

Edited by:
Dr. Anup Sharma



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Management Science

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Unit 01: Introduction to Operations Research

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Summary

Keywords

Self Assessment

Answers for Self Assessment

Review Questions

Further Reading

Objectives

After studying this unit, you will be able to:

- Understand the meaning of Operations research
- Know about the history of operations research
- Discuss the scope and application of operations research
- Explain the various types of models used in operations research

Introduction

Operations Research (OR) is a multidisciplinary field that uses mathematical modelling, statistical analysis, and optimization techniques to aid in decision-making and problem-solving related to complex systems and processes. It focuses on the use of quantitative methods to improve the effectiveness and efficiency of operations in various industries and organizations.

The main objective of Operations Research is to find the best possible solutions to real-world problems, considering the available resources, constraints, and objectives. These problems can be found in diverse fields such as manufacturing, logistics, transportation, finance, healthcare, supply chain management, telecommunications, and many others.

Key elements of Operations Research include:

Mathematical Modelling: Operations researchers use mathematical equations and models to represent real-world systems and processes. These models help in understanding the underlying structure of the problem and identifying important variables and relationships.

Optimization Techniques: One of the central aspects of OR is optimization, which involves finding the best possible solution from a set of feasible alternatives. Optimization methods aim to minimize costs, maximize profits, optimize resource allocation, or achieve other desired outcomes.

Data Analysis: Operations Research heavily relies on data analysis and statistical methods to support decision-making. Historical data and real-time information are used to calibrate models, validate results, and make informed choices.

Decision Support: OR provides decision-makers with valuable insights and recommendations to improve the quality of decisions. By considering various scenarios and potential outcomes, it helps in choosing the most suitable course of action.

Simulation: Simulation is often used to model complex systems when analytical solutions are challenging to derive. By running simulations based on probabilistic models, researchers can gain insights into system behavior and performance under different conditions.

Application Areas: Operations Research has wide-ranging applications, including inventory management, production planning, scheduling, network design, project management, resource allocation, risk analysis, and more.

Interdisciplinary Nature: Operations Research draws from various disciplines such as mathematics, statistics, computer science, economics, engineering, and management science. This multidisciplinary approach allows researchers to tackle complex problems from different angles.

Overall, Operations Research plays a crucial role in enhancing the efficiency and effectiveness of businesses and organizations, helping them make better decisions and allocate resources optimally. By combining analytical techniques with practical problem-solving skills, OR practitioners contribute to improving processes and systems across a multitude of industries.

Operations Research has gained wider acclaim in the modern complex business world. For every complex problem of an industry today, well defined Operations Research tools are the solace. Decision making in today's social and business environment has become a complex task. Well-structured problems are routinely optimized at the operational level of organizations, and increased attention is now focused on broader tactical and strategic issues. To effectively address these problems and provide leadership in the advancing global age, decision makers cannot afford to make decisions by simply applying their personal experiences, guesswork's or intuitions, because consequences of wrong decisions are serious and costly. Hence, an understanding of the applicability of the quantitative methods to decision making is of fundamental importance to decision makers. Operations research is one such tool that helps in comparison of every possible alternative (course of action) to know the potential outcomes, permits examination of the sensitivity of the solution to changes or errors. This unit introduces the basics of operation research to help students to get an overview of operations research and thereby help in better understanding of the subsequent units.

1.1 Features of Operations Research

Operations research (OR) is a field of applied mathematics, statistics, and computer science that uses quantitative methods to help make better decisions. OR has a wide range of applications, including business, healthcare, government, and the environment.

Here are some of the key features of operations research:

Systems orientation

OR takes a holistic approach to problem solving, considering the entire system rather than just individual parts. This helps to ensure that solutions are effective and sustainable. For example, let's say a company is trying to decide how to allocate its resources. If the company only considers individual parts of the system, such as the production line or the marketing department, it may make decisions that are not in the best interests of the company as a whole. However, if the company takes a systems approach, it will consider all of the different parts of the system and how they interact with each other. This will help the company to make decisions that are more effective and sustainable.

Multidisciplinary

OR draws on a variety of disciplines, including mathematics, statistics, computer science, and economics. This allows OR practitioners to develop solutions that are informed by a wide range of perspectives. For example, let's say a company is trying to design a new product. If the company only uses the expertise of engineers, it may not consider the marketing or financial implications of the product. However, if the company uses a multidisciplinary approach, it will bring together engineers, marketers, and financial experts to develop a product that is both technically feasible and commercially successful.

Quantitative

OR uses mathematical and statistical methods to analyze data and make decisions. This helps to ensure that solutions are based on evidence and are not based on intuition or guesswork. For example, let's say a company is trying to determine the optimal number of products to produce each day. If the company only uses intuition, it may make a decision that is not in the best interests of the company. However, if the company uses a quantitative approach, it will analyze data on demand, production costs, and other factors to make a decision that is based on evidence.

Modelling

OR often uses models to represent real-world systems. This allows OR practitioners to test different solutions and to understand the impact of different decisions. For example, let's say a company is trying to determine the best way to allocate its resources. The company could build a model of the company's operations and then use the model to test different allocation strategies. This would allow the company to see how different strategies would affect the company's performance.

Computational

OR often uses computers to solve complex problems. This allows OR practitioners to analyze large amounts of data and to make decisions in a timely manner. For example, let's say a company is trying to determine the optimal way to route its deliveries. The company could use a computer to solve a complex optimization problem that would take too long to solve manually. This would allow the company to make better decisions about its deliveries.

1.2 Objective of Operations Research

The primary objective of Operations Research (OR) is to apply scientific and mathematical methods to improve decision-making and problem-solving processes in various real-world situations. The basics of Operations Research revolve around achieving the following objectives:

Optimization: The central goal of OR is to find the best possible solution from a set of feasible alternatives. Optimization involves maximizing profits, minimizing costs, optimizing resource allocation, or achieving other desired outcomes. OR employs mathematical techniques to identify optimal solutions that can lead to improved efficiency and effectiveness.

Decision Support: Operations Research provides decision-makers with valuable insights and recommendations to make informed choices. By analyzing data, creating models, and running simulations, OR helps decision-makers understand the consequences of different decisions and select the most suitable course of action.

Problem-solving: OR focuses on formulating complex real-world problems in a structured and mathematical manner. By breaking down complicated issues into smaller, manageable components, OR helps in identifying the key variables, constraints, and relationships that influence the problem's outcome.

Efficiency and Effectiveness: Operations Research aims to enhance the efficiency and effectiveness of processes and systems within organizations. By optimizing workflows, resource allocation, and scheduling, OR helps achieve better utilization of resources, reduced costs, and improved overall performance.

Resource Management: OR assists in efficiently managing scarce resources, such as time, money, manpower, and materials. By considering various constraints and limitations, OR helps allocate resources optimally to achieve organizational objectives.

Forecasting and Planning: Operations Research involves the use of statistical and mathematical techniques to analyze historical data and make predictions about future events. This forecasting capability aids in strategic planning and decision-making, ensuring that organizations are better prepared for future challenges.

Risk Analysis: OR helps in assessing and managing risks associated with decision-making. By considering uncertainties and variability in data, OR models can evaluate the potential impact of different scenarios, enabling decision-makers to make risk-informed choices.

Process Improvement: Operations Research identifies inefficiencies and bottlenecks within processes and systems. By suggesting improvements and optimizations, OR contributes to streamlining operations and enhancing overall performance.

Interdisciplinary Applications: OR is a versatile field with applications in various industries, including manufacturing, logistics, healthcare, finance, transportation, supply chain management, and more. Its interdisciplinary nature allows it to address complex problems from different perspectives.

In summary, the objective of Operations Research basics is to use quantitative and analytical methods to support decision-making, optimize processes, and improve the overall performance of systems and organizations in a wide range of application areas.

1.3 Meaning of Operations Research

Operations Research (OR) is a multidisciplinary field, and as such, it has been defined in various ways by different experts and organizations over the years. Here are several definitions of Operations Research:

The **Institute for Operations Research and the Management Sciences (INFORMS)** defines Operations Research as "the application of scientific methods, techniques, and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems."

The **American Productivity and Quality Center (APQC)** defines Operations Research as "a systematic approach that uses mathematical models, statistics, and algorithms to assist in decision-making."

According to the **Operations Research Society of America (ORSA)**, OR is "the scientific study of operations for the purpose of making better decisions."

Howard Raiffa, a pioneer in decision analysis, defines Operations Research as "the professional discipline that deals with the application of scientific methods for complex decision-making."

The **European Journal of Operational Research (EJOR)** defines Operations Research as "an interdisciplinary branch of applied mathematics and formal science that uses methods such as mathematical modeling, statistics, and algorithms to arrive at optimal or near-optimal solutions to complex decision-making problems."

The **U.S. Army** defines Operations Research as "the analytical study of military problems undertaken to provide responsible commanders and staff agencies with a scientific basis for decision on action to improve military operations."

Operations Research Society of India (ORSI) defines Operations Research as "the scientific approach to managerial decision-making."

According to **Winston and Albright** in their book "Practical Management Science," Operations Research is "a scientific approach to decision-making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources."

These definitions highlight the core characteristics of Operations Research, emphasizing its use of scientific methods, mathematical modeling, and optimization techniques to support decision-making, improve processes, and find optimal solutions to complex problems across various domains.

1.4 Significance of Operations Research

Studying Operations Research (OR) offers numerous benefits and advantages for students and professionals alike. Here are some compelling reasons to consider studying Operations Research:

Problem-solving skills: OR equips individuals with analytical and quantitative problem-solving skills. By learning how to model and optimize complex systems, students can tackle real-world challenges efficiently.

Versatility: OR has applications in a wide range of industries, including manufacturing, logistics, finance, healthcare, transportation, and more. Studying OR opens up diverse career opportunities and allows individuals to make a positive impact across different sectors.

Optimization and efficiency: OR focuses on finding optimal solutions, whether it's minimizing costs, maximizing profits, or optimizing resource allocation. Understanding these principles can lead to improved efficiency and better decision-making in organizations.

Decision support: OR provides valuable insights and data-driven recommendations to support decision-makers. Being proficient in OR enables individuals to contribute to strategic planning and enhance the quality of decisions within businesses and institutions.

Interdisciplinary approach: OR draws knowledge from various disciplines such as mathematics, statistics, computer science, economics, and engineering. This interdisciplinary nature fosters a holistic understanding of complex systems and problem-solving techniques.

Data analysis and simulation: OR heavily relies on data analysis and simulation to validate models and assess different scenarios. These skills are highly sought after in today's data-driven world.

Research and innovation: OR is an evolving field that encourages research and innovation. Students studying OR have the opportunity to contribute to cutting-edge research, addressing emerging challenges, and developing new methodologies.

Competitive advantage: Organizations that employ OR methodologies often gain a competitive advantage by streamlining processes, reducing costs, and making more informed decisions.

Global relevance: The principles of OR are applicable worldwide. As businesses and governments strive to become more efficient and effective, the demand for OR expertise continues to grow on a global scale.

Personal development: Studying OR cultivates critical thinking, problem-solving, and teamwork skills. It encourages individuals to approach problems with a systematic and analytical mindset, which can be valuable in both professional and personal life.

Overall, Operations Research is a field with practical applications and the potential to make a significant impact on organizations and society. By studying OR, individuals can equip themselves with valuable skills that are in demand across various industries and contribute to solving complex problems in an ever-changing world.

1.5 Applications of Operations Research

The scope of Operations Research (OR) is broad and continually expanding due to its versatile and interdisciplinary nature. OR has applications in numerous domains, making it a valuable field for addressing complex problems and improving decision-making processes. Here are some key areas that illustrate the scope of Operations Research:

Supply Chain Management: OR plays a crucial role in optimizing supply chain networks, inventory management, transportation, and distribution. It helps organizations minimize costs, reduce lead times, and improve overall efficiency in the movement of goods and services.

Manufacturing and Production: OR is utilized to optimize production schedules, resource allocation, and capacity planning. It aims to increase productivity, minimize downtime, and enhance quality control in manufacturing processes.

Transportation and Logistics: OR is instrumental in optimizing routing and scheduling for transportation networks, such as airlines, railways, trucking, and shipping. It helps in reducing transportation costs, improving delivery times, and enhancing logistics efficiency.

Healthcare: OR is applied in healthcare operations to improve patient flow, resource allocation, hospital scheduling, and healthcare delivery. It aids in reducing waiting times, managing healthcare resources effectively, and optimizing patient care.

Finance and Investment: OR techniques are used in financial modeling, portfolio optimization, risk analysis, and credit risk assessment. It assists in making informed investment decisions and managing financial risks effectively.

Project Management: OR aids in project scheduling, resource allocation, and project planning. It helps in meeting project deadlines, optimizing resource usage, and managing project risks.

Energy and Environmental Management: OR is employed to optimize energy consumption, resource utilization, and environmental impact. It supports sustainable practices and green initiatives in various industries.

Marketing and Advertising: OR is used for market segmentation, pricing, and advertising optimization. It helps organizations reach target audiences more effectively and allocate marketing budgets efficiently.

Military and Defense: OR is applied in military operations for resource allocation, logistics planning, and strategic decision-making. It assists in optimizing military resources and improving mission success rates.

Sports Analytics: OR techniques are increasingly used in sports to optimize player selection, game strategies, and scheduling. It helps sports teams and organizations gain a competitive edge.

Public Policy and Government: OR is employed to analyze public policy issues, optimize resource allocation, and improve government operations. It supports evidence-based decision-making in the public sector.

Revenue Management: OR techniques are used in revenue management to optimize pricing and capacity allocation in industries like airlines, hotels, rental cars, and entertainment venues. It helps maximize revenue by efficiently selling perishable assets.

Risk Management: OR aids in analyzing and managing risks in various sectors, including finance, insurance, and project management. It helps assess and mitigate potential risks, improving decision-making under uncertainty.

Human Resources: OR is utilized in workforce planning, employee scheduling, and talent management. It helps organizations optimize human resource utilization and improve workforce efficiency.

Environmental Conservation: OR techniques are applied to environmental modelling and conservation efforts. It helps in designing sustainable practices, conservation planning, and natural resource management.

Agriculture: OR is used to optimize agricultural practices, including crop planning, irrigation scheduling, and logistics in the agricultural supply chain. It aids in maximizing crop yield and reducing resource waste.

Retail Operations: OR is employed in retail to optimize inventory management, store layout, and demand forecasting. It helps retailers maintain adequate stock levels and improve overall store performance.

Telecommunications: OR is used in network design, resource allocation, and traffic management for telecommunication systems. It aids in optimizing network performance and enhancing service quality.

Emergency Response and Disaster Management: OR techniques help in emergency response planning, resource allocation during disasters, and logistics in disaster relief operations. It supports efficient coordination in critical situations.

Education: OR is applied in educational institutions for resource allocation, timetabling, and student scheduling. It helps optimize classroom utilization and enhance educational efficiency.

Urban Planning: OR assists in urban transportation planning, traffic management, and infrastructure development. It supports sustainable urban growth and improves transportation systems.

Sports Scheduling: OR techniques are used to optimize scheduling in sports leagues and tournaments. It ensures fair play, minimizes travel distances, and maximizes fan engagement.

Customer Relationship Management (CRM): OR is applied in CRM to optimize customer segmentation, targeting, and customer lifetime value analysis. It helps businesses improve customer satisfaction and retention.

Energy Grid Management: OR techniques aid in optimizing power generation, transmission, and distribution in energy grids. It supports efficient energy utilization and grid stability.

The scope of Operations Research continues to evolve as new challenges arise and advancements in technology enable more sophisticated modelling and analysis. The interdisciplinary nature of OR allows it to address a wide range of problems, making it a valuable and dynamic field in today's complex world. The scope of Operations Research is not limited to these areas alone. It extends to various other fields and industries, where complex systems and decision-making processes need optimization and efficiency improvements. As technology and data analytics continue to advance, the scope of OR is likely to expand even further, providing valuable insights and solutions to increasingly complex challenges in our interconnected world.

The area of production systems presents special challenges for operations researchers. Production problems are operations research problems, hence solving them requires a solid foundation in operations research fundamentals. Additionally, the solution of production systems problems frequently draws on expertise in more than one of the primary areas of operations research, implying that the successful production researcher cannot be one-dimensional.

Production systems problems cannot be solved without an in-depth understanding of the real problem, since invoking assumptions that simplify the mathematical structure of the problem may lead to an elegant solution for the wrong problem. Common sense and practical insight are common attributes of successful production planners.

At the current time, the field of OR is extremely dynamic and ever evolving. To name a few of the contemporary (primary) research projects, current work in, primarily through the integration of the philosophies of the Theory of Constraints and Lean Manufacturing.

1.6 History of Operations Research

The history of operations research (OR) can be traced back to the early 20th century, when scientists and engineers began to apply mathematical and statistical methods to solve problems in military and industrial settings. However, the modern field of OR is generally considered to have originated during World War II, when a team of British scientists and engineers used these methods to improve the efficiency of the Royal Air Force's bombing campaign.

The term "operational research" was first coined in 1940 by two British scientists, P.M.S. Blackett and F.W.J. McCloskey. They defined OR as "the application of the scientific method, in particular the techniques of mathematical analysis, to the study of complex operational problems, with the object of improving the efficiency of the operations concerned."

During World War II, OR teams were formed in many countries to help solve military problems. These teams used a variety of methods, including linear programming, queuing theory, and game theory, to improve the effectiveness of military operations. For example, OR teams were used to optimize the placement of anti-aircraft guns, to develop strategies for convoying ships, and to improve the efficiency of bomber crews.

After the war, OR teams were disbanded, but the methods and techniques they developed continued to be used in a variety of civilian settings. OR was used to improve the efficiency of manufacturing processes, to optimize transportation networks, and to make better decisions about resource allocation.

In the 1960s, OR became a recognized academic discipline. Universities began to offer courses in OR, and professional societies were formed to promote the field. Today, OR is a thriving field with applications in a wide range of areas, including business, healthcare, government, and the environment.

Early beginnings

The origins of operations research can be traced back to the early 20th century, when scientists and engineers began to apply mathematical and statistical methods to solve problems in military and industrial settings. For example, in 1913, Harris published a paper on the application of probability theory to military problems.

World War II

The modern field of OR is generally considered to have originated during World War II, when a team of British scientists and engineers used these methods to improve the efficiency of the Royal Air Force's bombing campaign. This team, known as Blackett's Circus, developed a number of new techniques, including linear programming, queuing theory, and game theory, which were used to improve the effectiveness of bombing missions.

The post-war years

After the war, OR teams were disbanded, but the methods and techniques they developed continued to be used in a variety of civilian settings. OR was used to improve the efficiency of manufacturing processes, to optimize transportation networks, and to make better decisions about resource allocation.

The academicization of OR

In the 1960s, OR became a recognized academic discipline. Universities began to offer courses in OR, and professional societies were formed to promote the field. Today, OR is a thriving field with applications in a wide range of areas, including business, healthcare, government, and the environment.

Recent developments

In recent years, OR has benefited from the development of new technologies, such as the Internet and big data. These technologies have made it possible to collect and analyze more data, which has led to the development of new OR techniques. For example, OR is now being used to develop algorithms for self-driving cars and to optimize the use of social media data.

1.7 The Future of OR

The future of OR is bright. As the world becomes more complex, the need for OR will only grow stronger. OR can help us to make better decisions about a wide range of problems, from how to allocate resources to how to prevent wars.

The future of operations research (OR) is bright. As the world becomes more complex, the need for OR will only grow stronger. OR can help us to make better decisions about a wide range of problems, from how to allocate resources to how to prevent wars.

Here are some of the key trends that are likely to shape the future of OR:

The rise of big data

Big data is the collection of large and complex datasets that can be analyzed to reveal patterns and insights. OR is well-positioned to take advantage of big data, as it can use these data to develop new models and algorithms that can help us to make better decisions.

The growth of artificial intelligence (AI)

AI is a rapidly developing field that has the potential to revolutionize many industries. OR can play a key role in the development of AI, as it can help us to understand how AI works and how it can be used to solve problems.

The increasing importance of sustainability

Sustainability is becoming increasingly important, as we are facing a number of environmental challenges. OR can help us to develop more sustainable solutions, such as optimizing the use of resources and reducing pollution.

The need for cross-disciplinary collaboration

OR is a multidisciplinary field that draws on a variety of disciplines, including mathematics, statistics, computer science, and economics. In the future, OR will need to become even more interdisciplinary, as it will need to collaborate with other disciplines to address complex problems.

Overall, the future of OR is bright. As the world becomes more complex, the need for OR will only grow stronger. OR is well-positioned to take advantage of new technologies and trends to help us to make better decisions and solve complex problems.

Here are some specific examples of how OR is being used to solve problems in the real world:

OR is being used to develop algorithms for self-driving cars. These algorithms need to be able to make decisions in real time, and OR can help to ensure that these decisions are made in a safe and efficient manner.

OR is being used to optimize the use of social media data. This data can be used to target advertising, identify trends, and improve customer service.

OR is being used to develop more sustainable solutions. For example, OR is being used to optimize the use of energy and to reduce pollution.

These are just a few examples of how OR is being used to solve problems in the real world. As the world becomes more complex, the need for OR will only grow stronger. OR is a powerful tool that can help us to make better decisions and solve complex problems.

The rise of big data

Big data is the collection of large and complex datasets that can be analyzed to reveal patterns and insights. OR is well-positioned to take advantage of big data, as it can use these data to develop new models and algorithms that can help us to make better decisions.

For example, OR is being used to develop new algorithms for self-driving cars. These algorithms need to be able to make decisions in real time, and OR can help to ensure that these decisions are made in a safe and efficient manner. OR is also being used to optimize the use of social media data. This data can be used to target advertising, identify trends, and improve customer service.

The Growth of Artificial Intelligence (AI)

AI is a rapidly developing field that has the potential to revolutionize many industries. OR can play a key role in the development of AI, as it can help us to understand how AI works and how it can be used to solve problems.

For example, OR is being used to develop AI-powered chatbots that can provide customer service. These chatbots can be trained on large datasets of customer interactions, and they can use this data to answer customer questions and resolve issues. OR is also being used to develop AI-powered trading algorithms that can make trades in financial markets. These algorithms can be used to analyze large amounts of data and to make decisions in real time.

The increasing importance of sustainability

Sustainability is becoming increasingly important, as we are facing a number of environmental challenges. OR can help us to develop more sustainable solutions, such as optimizing the use of resources and reducing pollution.

For example, OR is being used to develop more efficient transportation systems. These systems can be designed to minimize the use of fuel and to reduce emissions. OR is also being used to develop more efficient ways to produce and use energy. These methods can help us to reduce our reliance on fossil fuels and to protect the environment.

The need for cross-disciplinary collaboration

OR is a multidisciplinary field that draws on a variety of disciplines, including mathematics, statistics, computer science, and economics. In the future, OR will need to become even more interdisciplinary, as it will need to collaborate with other disciplines to address complex problems.

For example, OR is being used to develop new methods for treating diseases. These methods can combine data from different sources, such as medical records and genetic data, to develop personalized treatments. OR is also being used to develop new methods for managing financial risks. These methods can use data from financial markets and other sources to identify and mitigate risks.

Overall, the future of OR is bright. As the world becomes more complex, the need for OR will only grow stronger. OR is well-positioned to take advantage of new technologies and trends to help us to make better decisions and solve complex problems.

1.8 Models of Operations Research

When a problem or process under investigation is simplified and represented with its typical features or characteristics, it is called as a model. The word 'model' has several meanings. All of which are relevant to Operations Research. For example, a model can act as a substitute for representing reality, such as small scale model locomotive or may act as some sort of idealization, like a model plan for recruitment procedure, etc. Constructing a model helps in bringing the complexities and possible uncertainties into a logical framework required for comprehensive analysis. In fact the model acts as a vehicle in arriving at a well-structured view of reality. An array of models can be seen in various business areas or industrial activities. The most relevant for our study purpose are portrayed below:

1. Physical Models:

- (a) Iconic Models
- (b) Analogue Models

2. Symbolic Models:

- (a) Verbal Models
- (b) Mathematical Models:
 - (i) Deterministic Models
 - (ii) Probabilistic Models

3. Combined Analogue and Mathematical Models

4. Function Models

5. Heuristic Models

1. Physical Models: To deal with specific types of problems, models like diagrams, charts, graphs and drawings are used, which are known as "Physical Models". The schematic way of representation of significant factors and interrelationship may be in a pictorial form help in useful analysis. Moreover, they help in easy observation, description and prediction. However, they over rule any manipulation. There are two types of Physical Models:

(i) Iconic Models: An image or likeness of an object or process is Icon. These models represent the system as it is by scaling it up or down. Even though use of these models in the area of management appears to be narrow, their usefulness is seen in the field of engineering and science. For example,

(a) In the field of R&D, prototype of the product is developed and tested to know the workability of the new product development.

(b) Photographs, portraits, drawings are the good example of iconic types. These models help in testifying the samples thus avoiding fullscale designing and probable loss.

(ii) Analogue Models: These models are closely associated with iconic models. However, they are not the replicas of system or process. The analogue, in constructing these models, help in analyzing the issues and forces which are in the system or process.

Because, these models use 'one set of properties' which is 'analogous to another set of properties.' For example, kids, toys, rail-road models, etc.

(2) Symbolic Models: Symbolic Models use letters, numbers, figures to represent decision variables of the system. There are two types of Symbolic Models – Verbal Models and Mathematical Models.

(a) Verbal Models: These models describe a situation in written or spoken language. Written sentences, books, etc., are examples of a verbal model.

(b) Mathematical Models: Mathematical symbols are used to represent a problem or a system under these types of models. Rules of mathematics enable the builder to make the models more abstract and precise. There are two types of Mathematical Models – Deterministic Models and Probabilistic Models.

(i) Deterministic Models: The exact statement of variables and their relationships are made under these models. The coefficients used for the mathematical formulation are known and are constant with certainty. So, to say, with a given set of data the answer will always be the same. For instance,

determination of the break-even sales volume (BEP), the volume where the total cost equals the total sales revenue earned pertaining to a product.

The sales revenue line is determined by using an equation.

$$TSR = (SV) (PU)$$

where, TSR = Total Sales revenue

SV = Sales Volume

PU = Price per Unit

The total cost line is represented as:

$$TC = TFC + SV (VC)$$

TC = Total Cost

TFC = Total Fixed Cost

SV = Sales Volume Notes

VC = Variable Cost.

The break-even point occurs when

$$TSR = TC \text{ or } (SV) (PU) = TFC + SV (VC)$$

Hence, the volume (BEP) is determined by

$$BEP = TFC / (SP - VC) \text{ VC represents the contribution to fixed cost.}$$

(ii) Probabilistic Models: The risk involved and the state of uncertainty are covered by these models. The decision variables take the form of a probability distribution and can assume more than single values. In the presence of risk and uncertainty, these models tend to yield different answers every time when attempted to. For example, uncertainty over acquisition of raw material to execute customer orders during a certain period. The purchaser has to consider both the sale and the delivery timing of the orders as they are variables. A probability distribution can be developed for the instant period for both sales delivery timings. However, the optimum selection is adhered to in accordance with the demand of the situation.

(3) Combined Analogue and Mathematical Models: When analogue models are interpreted with the use of mathematical symbols, they can be termed as physical-symbolic models. For instance, decision-makers tend to use mathematical symbols to represent their sales or profit figures.

(4) Function Models: Models which are used to represent a group of functions performed, are called function models.

(5) Heuristic Models: When intuition guides a problem-solver to find solutions, heuristic models are developed. Even though he would not be able to find an optimum solution to the problem, with his past experience he arrives at the most advantageous solution. However, it depends on how intuitive and creative the decision-maker is.

Summary

Operations Research (OR) is an analytical method of problem-solving and decision-making that is useful in the management of organizations. Thus it is also known as a mathematical or scientific analysis of a process or operation, used in making decisions. In other words, OR represents the study of optimal resource allocation. The aim of OR is to provide rational basis for decision making by seeking to understand and structure complex situations, and to utilize this understanding to predict system behavior and improve system performance.

The tools of OR are not from any one discipline, rather mathematics, statistics, economics, engineering, psychology etc, have contributed to this newer discipline of knowledge. OR is the application of the methods of science to complex problem arising in the direction and management of large system of men, machines, materials and money in industry, business, and government of defense. The mathematical model can be deterministic or probabilistic in nature. In deterministic model the parameters are considered to be defined and the outcome related to a certain course of action is known with certainty. A problem can be modeled in differing ways, and the choice of the appropriate model may be crucial to the success of the OR project.

In summary, Operations Research (OR) is a multidisciplinary field that uses mathematical modeling, statistical analysis, and optimization techniques to improve decision-making and problem-solving in various industries and domains. Its primary objectives include finding optimal solutions, enhancing efficiency, and providing decision support. OR has a broad scope and is applied in diverse areas such as supply chain management, manufacturing, healthcare, finance, transportation, project management, energy, marketing, and more. It enables professionals to develop critical problem-solving skills, work with data-driven approaches, and contribute to research and innovation. With its interdisciplinary nature and continued advancements, Operations Research continues to play a vital role in addressing complex challenges and improving systems and processes in our interconnected world.

Operations Research (OR) is an interdisciplinary field that uses mathematical, statistical, and computational techniques to analyze complex systems and optimize decision-making processes. Its main goal is to find optimal solutions to real-world problems, taking into account various constraints and objectives.

OR employs mathematical modeling to represent real-world situations as mathematical equations or algorithms. By doing so, it helps in understanding the underlying structure of a problem and identifying the key variables, relationships, and interactions involved.

Keywords

Constraint: A limitation or restriction that must be satisfied in a mathematical model or optimization problem. Constraints define the boundaries within which the solution must be found.

Feasible Solution: A solution that satisfies all the constraints of an optimization problem.

Infeasible Solution: A solution that violates one or more constraints of an optimization problem, making it impossible to achieve.

Objective Function: The mathematical expression that represents the goal to be either maximized or minimized in an optimization problem.

Pareto Optimality (Pareto Efficiency): A state in which it is not possible to improve one objective without worsening at least one other objective.

Sensitivity Report: A report that provides information about how changes in the coefficients of the objective function or the constraints affect the optimal solution of an optimization model.

Continuous Variables: Decision variables that can take on any real value within a defined range in an optimization problem.

Heuristic Algorithms: Problem-solving methods that use rules of thumb or practical strategies to find good solutions, often in situations where finding an optimal solution is computationally challenging.

Mixed-Integer Linear Programming (MILP): An optimization technique that combines linear programming with integer programming, allowing some variables to take on integer values while others are continuous.

Goal Programming: An extension of linear programming that allows for multiple objectives to be considered simultaneously, with varying degrees of importance.

Nonlinear Optimization: The process of finding the best solution for optimization problems with nonlinear objective functions or constraints.

Game Theory: A branch of OR that deals with mathematical models of strategic interactions between decision-makers, often used in economics, business, and social sciences.

Convex Optimization: An optimization technique applied to problems where the objective function and constraints satisfy specific mathematical properties, known as convexity.

Facility Location: An area of OR that deals with determining the optimal locations for facilities (e.g., factories, warehouses, service centers) to minimize costs or maximize coverage.

Combinatorial Optimization: An optimization area that focuses on finding optimal solutions within discrete sets, often involving combinatorial problems like the Traveling Salesman Problem.

Metaheuristics: High-level problem-solving strategies that guide other optimization algorithms, such as genetic algorithms, simulated annealing, and particle swarm optimization.

Linear Regression: A statistical technique used to model the relationship between a dependent variable and one or more independent variables.

Nonlinear Regression: A statistical technique used to model the relationship between variables when the relationship is not linear.

Self Assessment

1. Which of the following is NOT a key feature of operations research?
 - A. Systems orientation
 - B. Multidisciplinary approach
 - C. Quantitative methods
 - D. Intuition-based approach

2. Which of the following is NOT an application of operations research?
 - A. Inventory management
 - B. Production scheduling
 - C. Transportation routing
 - D. Patient scheduling

3. Which of the following is NOT a type of model used in operations research?
 - A. Mathematical model
 - B. Statistical model
 - C. Computer model
 - D. Analog model

4. Which of the following is NOT a type of optimization problem?
 - A. Linear programming
 - B. Integer programming
 - C. Dynamic programming
 - D. Game theory

5. Which of the following is NOT a software package used for operations research?
 - A. LINDO
 - B. Solver
 - C. CPLEX
 - D. MATLAB

6. Which of the following is NOT a type of decision-making under uncertainty?

- A. Risk analysis
 - B. Chance-constrained programming
 - C. Robust optimization
 - D. Bayesian decision theory
7. Which of the following is NOT a type of simulation?
- A. Monte Carlo simulation
 - B. System dynamics
 - C. Agent-based modeling
 - D. Petri nets
8. What is the main objective of Operations Research?
- A. To maximize profits
 - B. To minimize costs
 - C. To find optimal solutions to real-world problems
 - D. To forecast future events
9. Which of the following is NOT an application area of Operations Research?
- A. Supply Chain Management
 - B. Agriculture
 - C. Entertainment
 - D. Healthcare
10. Linear programming deals with:
- A. Optimization problems with linear objective functions and linear constraints.
 - B. Optimization problems with nonlinear objective functions and linear constraints.
 - C. Optimization problems with linear objective functions and nonlinear constraints.
 - D. Optimization problems with nonlinear objective functions and nonlinear constraints.
11. Sensitivity analysis in Operations Research involves:
- A. Evaluating the impact of changes in constraints on the objective function.
 - B. Analyzing the robustness of the optimal solution to changes in input parameters.
 - C. Solving linear programming problems using sensitivity coefficients.
 - D. Applying heuristics to find approximate solutions to complex problems.
12. What is the term used for a solution that violates one or more constraints of an optimization problem?
- A. Optimal solution
 - B. Feasible solution
 - C. Infeasible solution
 - D. Pareto optimal solution

13. Which type of decision variables can take on any real value within a defined range in an optimization problem?
- A. Integer variables
 - B. Binary variables
 - C. Continuous variables
 - D. Heuristic variables
14. What does the term "duality" refer to in linear programming?
- A. The concept of having multiple objectives in a problem.
 - B. The relationship between the primal problem and its dual problem.
 - C. The process of solving linear programming problems iteratively.
 - D. The trade-off between time and cost in optimization.
15. Combinatorial optimization deals with:
- A. Optimization problems with continuous decision variables.
 - B. Optimization problems with multiple objectives.
 - C. Optimization problems with discrete decision variables.
 - D. Optimization problems with uncertain parameters.
16. Which type of optimization problem allows for some variables to take on integer values and others to be continuous?
- A. Linear programming
 - B. Nonlinear optimization
 - C. Mixed-Integer Linear Programming (MILP)
 - D. Combinatorial optimization
17. Which area of Operations Research deals with mathematical models of strategic interactions between decision-makers?
- A. Facility Location
 - B. Game Theory
 - C. Combinatorial Optimization
 - D. Sensitivity Analysis
18. Which Operations Research technique is best suited for finding the shortest route between multiple locations?
- A. Linear Programming
 - B. Queuing Theory
 - C. Traveling Salesman Problem
 - D. Network Optimization
19. Which type of decision variables can only take on binary values (0 or 1) in an optimization problem?
- A. Integer variables
 - B. Binary variables
 - C. Continuous variables
 - D. Heuristic variables

20. What is the primary purpose of conducting sensitivity analysis in Operations Research?

- A. To evaluate the impact of changes in constraints on the objective function.
- B. To find the best solution for optimization problems with integer variables.
- C. To assess the robustness of the optimal solution to changes in input parameters.
- D. To simulate the behavior of a real-world system using computer-based models.

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. D | 2. D | 3. D | 4. D | 5. D |
| 6. D | 7. D | 8. C | 9. C | 10. A |
| 11. B | 12. C | 13. C | 14. B | 15. C |
| 16. C | 17. B | 18. C | 19. B | 20. A |

Review Questions

1. What is the main objective of Operations Research?
2. Name three application areas of Operations Research.
3. Define linear programming and explain its key components.
4. What is the difference between feasible and infeasible solutions in an optimization problem?
5. Describe the process of sensitivity analysis in Operations Research.
6. What are heuristic algorithms, and how are they used in problem-solving?
7. What is the concept of duality in linear programming?
8. Differentiate between continuous variables and integer variables in an optimization problem.
9. Explain the term "Pareto Optimality" (Pareto Efficiency) and its significance in Operations Research.
10. What are metaheuristics, and provide an example of a metaheuristic technique.
11. How is combinatorial optimization different from continuous optimization?
12. Name three areas of Operations Research that involve optimization in real-world systems.
13. Discuss the importance of simulation in Operations Research and its applications.
14. What is the purpose of facility location in Operations Research?
15. Define queuing theory and provide an example of its application.
16. What role does risk analysis play in decision-making within the field of Operations Research?
17. Describe the difference between linear regression and nonlinear regression in data analysis.
18. What are the primary steps involved in solving an optimization problem in Operations Research?
19. How does Operations Research contribute to improving supply chain management?
20. Discuss the interdisciplinary nature of Operations Research and its impact on problem-solving.



Further Reading

Operations Research: Applications and Algorithms" by Wayne L. Winston:

This book provides a comprehensive introduction to Operations Research concepts, methods, and applications. It covers linear programming, integer programming, network optimization, simulation, and more, with practical examples and case studies.

"Introduction to Operations Research" by Frederick S. Hillier and Gerald J. Lieberman:

This classic textbook offers a thorough introduction to the fundamental concepts of Operations Research. It covers linear programming, transportation and assignment problems, dynamic programming, and other optimization techniques.

"Operations Research: Principles and Practice" by A. Ravindran, D. T. Phillips, and J. J. Solberg:

This book presents a comprehensive view of Operations Research, covering the theory, methods, and applications of various OR techniques. It includes topics like decision analysis, stochastic models, and game theory.

"Operations Research: An Introduction" by Hamdy A. Taha:

This book provides a balanced introduction to Operations Research, covering linear programming, network optimization, integer programming, and more. It also includes chapters on queuing theory and simulation.

"Model Building in Mathematical Programming" by H. Paul Williams:

This book focuses on the art of formulating mathematical models for optimization problems. It provides insights into the process of creating effective models and emphasizes practical modeling techniques.

INFORMS (Institute for Operations Research and the Management Sciences):

The INFORMS website (<https://www.informs.org/>) offers a wealth of resources, including research publications, journals, conference proceedings, and webinars on various topics within Operations Research and analytics.

Unit 02: Linear Programming Applications and Model Formulation**CONTENTS**

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2.2 Types of Solutions

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2.4 Formulation of Linear Programming Problem

2.5 Two Decision Variables Problems

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Summary

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Objectives

After studying this unit, you will be able to:

- formulate general linear programming model.
- understand the application of linear programming model.

Introduction

Optimization is the process of finding the best possible solution from a set of feasible options. In linear programming, we focus on linear optimization, where both the objective function and the constraints are linear. "Linear" means that each variable appears to the first power and there are no products or powers of variables.

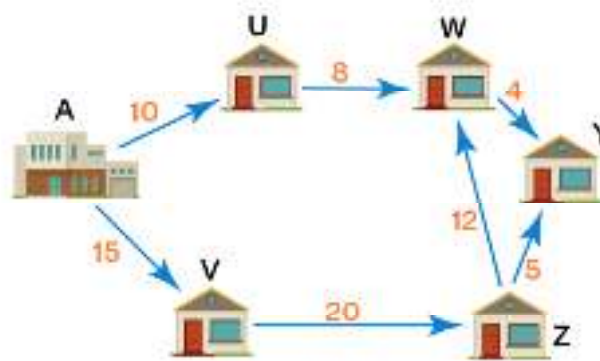
Linear programming (LP) is a mathematical optimization technique used to find the best outcome in a mathematical model with linear relationships. It is widely used in various fields, including operations research, economics, finance, engineering, and logistics, to make optimal decisions.

The fundamental idea behind linear programming is to maximize or minimize a linear objective function while satisfying a set of linear constraints. These constraints represent limitations or restrictions on the decision variables. Decision variables are the unknown quantities that the LP model seeks to determine, and they represent the quantities we want to find to optimize the objective function.

Linear programming is a simple technique where we depict complex relationships through linear functions and then find the optimum points. The important word in the previous sentence is depicted. The real relationships might be much more complex – but we can simplify them to linear relationships.

Linear Programming Examples

Suppose a postman has to deliver 6 letters in a day from the post office (located at A) to different houses (U, V, W, Y, Z). The distance between the houses is indicated on the lines as given in the image. If the postman wants to find the shortest route that will enable him to deliver the letters as well as save on fuel then it becomes a linear programming problem. Thus, LP will be used to get the optimal solution which will be the shortest route in this example.



2.1 Components of a Linear Programming Problem

A linear programming problem consists of the following components:

Decision Variables: These are the unknowns that we want to solve for. They represent quantities we can control or change to achieve the desired outcome.

Objective Function: This represents the quantity that needs to be maximized or minimized. It is a linear combination of the decision variables with associated coefficients, which indicate their importance in the optimization process.

Constraints: These are linear inequalities or equations that limit the values the decision variables can take. They represent restrictions or limitations on resources, capacities, or other factors that must be considered during optimization.

Geometric Interpretation: Geometrically, linear programming can be visualized as finding the best point (maximum or minimum) on the feasible region. The feasible region is the set of all points that satisfy all the constraints. In two dimensions, this feasible region is typically a polygon, while in three dimensions, it becomes a polyhedron. The objective function represents a family of parallel lines (in two dimensions) or planes (in three dimensions), and the optimal point is the one where these lines or planes are tangent to the feasible region.

Solving Methods: Several algorithms and techniques are available to solve linear programming problems. The most commonly used method is the Simplex algorithm, developed by George Dantzig in 1947. The Simplex algorithm efficiently moves from one vertex (corner point) of the feasible region to another until it reaches the optimal solution.

2.2 Types of Solutions

Linear programming problems can have different types of solutions:

Unique Optimal Solution: There is only one point that maximizes or minimizes the objective function.

Multiple Optimal Solutions: Several points result in the same optimal value of the objective function.

Unbounded Solution: The objective function can be increased or decreased infinitely, meaning there is no optimal solution.

Infeasible Solution: No point satisfies all the constraints, making the problem unsolvable within the given constraints.

Linear programming provides powerful tools for decision-making in a wide range of fields, allowing businesses and organizations to make efficient, data-driven choices to achieve their objectives.

2.3 Basic terminologies of LP

Linear Programming (LP) is a mathematical technique used to optimize the allocation of limited resources to achieve specific objectives. Here are some basic terminologies used in Linear Programming:

Objective Function: The mathematical expression that represents the goal to be maximized or minimized. In LP, the objective function is typically a linear equation representing the total profit, cost, or utility.

Decision Variables: The variables that need to be determined to achieve the objective. They represent the quantities or values that the LP problem aims to find.

Constraints: The limitations or restrictions on the decision variables. Constraints are represented by a system of linear inequalities or equations that define the feasible region.

Feasible Region: The region of values that satisfy all the constraints of the LP problem. It is the set of all possible solutions that meet the given limitations.

Optimal Solution: The set of values for the decision variables that maximize or minimize the objective function while satisfying all the constraints. There can be more than one optimal solution or no feasible solution at all.

Objective Coefficients: The coefficients of the decision variables in the objective function. They represent the contribution of each decision variable to the overall objective.

Constraint Coefficients: The coefficients of the decision variables in the constraints. They represent the resource requirements or limitations associated with each constraint.

Non-negativity Constraints: The requirement that decision variables must be greater than or equal to zero in LP problems. Negative values are usually not meaningful in practical contexts.

Optimization: The process of finding the best solution (optimal solution) that maximizes or minimizes the objective function while satisfying all the constraints.

Binding Constraint: A constraint that holds with equality in the optimal solution. It defines the boundary of the feasible region and determines the optimal values of some decision variables.

Non-binding Constraint: A constraint that holds with strict inequality in the optimal solution. It does not affect the optimal values of the decision variables.

Corner Point (Vertex): A point where the constraints intersect and define the boundary of the feasible region. The optimal solution is usually found at one of the corner points.

Unbounded: A situation where the objective function can be made infinitely large (for maximization) or infinitely small (for minimization) without violating the constraints. It means that the LP problem has no optimal solution.

Linear Programming provides a powerful tool for optimizing resource allocation and decision-making in various fields, such as economics, finance, operations research, and supply chain management.

2.4 Formulation of Linear Programming Problem

Formulating a linear programming (LP) problem involves defining the objective function, decision variables, and constraints to represent the real-world problem mathematically. The general steps for formulating an LP problem are as follows:

Step 1 - Identify the Decision Variables:

Decision variables represent the unknown quantities or values that need to be determined to achieve the objective. They are usually denoted by letters (e.g., x , y , z) and have specific units or limitations. Decision variables define the choices or actions that the LP problem aims to find. For example, in a production problem, decision variables might represent the quantities of different products to produce or the amount of resources to allocate to different activities.

Step 2 - Define the Objective Function:

The objective function represents the goal to be maximized or minimized in the LP problem. It is a mathematical expression that combines the decision variables with coefficients to calculate the overall value or measure of success. The objective function is usually formulated as a linear equation (for linear programming) but can be nonlinear in other types of optimization problems. The goal is to find the values of the decision variables that optimize the objective function. The objective function may be to maximize profits, minimize costs, maximize production output, minimize waste, maximize resource utilization, or any other relevant measure.

Step 3 - Set up the Constraints:

Constraints are limitations or restrictions on the decision variables that the solution must satisfy. These constraints represent the real-world limitations or requirements of the problem. Constraints are formulated as linear inequalities or equations involving the decision variables and coefficients. Each constraint defines a feasible region or boundary in which the solution must lie. The feasible region is the set of all possible values of the decision variables that satisfy all the constraints. Constraints are essential as they guide the solution towards practical and valid solutions.

Step 4 - Non-negativity Constraints:

In most LP problems, decision variables cannot take negative values since negative values do not have practical interpretations. Non-negativity constraints are included to ensure that the decision variables are greater than or equal to zero. This is represented as $x \geq 0$, $y \geq 0$, and so on, for each decision variable. Non-negativity constraints limit the feasible region to the positive or non-negative quadrant.

Step 5 - Write the Complete LP Formulation:

The complete LP formulation is the combination of the objective function and all the constraints. It is the mathematical representation of the entire LP problem. The LP formulation is usually written in standard form, where the objective function is to be maximized or minimized subject to the given constraints. The objective function and constraints are listed together in a systematic manner to represent the entire LP problem.

Once the LP problem is formulated, optimization techniques such as the Simplex method, interior-point methods, or graphical methods can be applied to find the optimal values of the decision variables that satisfy all the constraints and optimize the objective function. The optimal solution will be the set of decision variable values that maximize or minimize the objective while satisfying all the given constraints.

2.5 Two Decision Variables Problems



Example 1: Production Planning

A company produces two products, Product X and Product Y. Each unit of Product X requires 2 hours of labor and 3 units of raw material. Each unit of Product Y requires 3 hours of labor and 2 units of raw material. The company has 100 hours of labor and 120 units of raw material available. The profit per unit of Product X is \$10, and for Product Y, it is \$15. Determine the number of units of each product to produce to maximize the total profit.

Step 1 - Decision Variables:

Let x be the number of units of Product X to produce.

Let y be the number of units of Product Y to produce.

Step 2 - Objective Function:

Unit 02: Linear Programming Applications and Model Formulation

The objective is to maximize the total profit, which is given by the sum of profits from Product X ($10x$) and Product Y ($15y$).

Objective function: Maximize $Z = 10x + 15y$

Step 3 - Constraints:

Labor constraint: The total labor hours available are limited to 100 hours.

Constraint 1: $2x + 3y \leq 100$

Raw material constraint: The total raw materials available are limited to 120 units.

Constraint 2: $x + 2y \leq 120$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for both products:

$x \geq 0$

$y \geq 0$

Step 5 - Complete LP Formulation:

Maximize $Z = 10x + 15y$

Subject to:

$2x + 3y \leq 100$

$x + 2y \leq 120$

$x \geq 0$

$y \geq 0$



Example 2: Diet Planning

A nutritionist is planning a balanced diet for a patient with specific nutrient requirements. The diet includes two types of food, Food A and Food B. Each unit of Food A provides 4 units of protein, 3 units of carbohydrates, and 2 units of fats. Each unit of Food B provides 2 units of protein, 5 units of carbohydrates, and 3 units of fats. The patient requires at least 20 units of protein, 30 units of carbohydrates, and 10 units of fats daily. The cost per unit of Food A is \$2, and for Food B, it is \$3. Determine the number of units of each food to include in the diet to minimize the cost while meeting the nutrient requirements.

Step 1 - Decision Variables:

Let x be the number of units of Food A in the diet.

Let y be the number of units of Food B in the diet.

Step 2 - Objective Function:

The objective is to minimize the total cost, which is given by the sum of costs from Food A ($2x$) and Food B ($3y$).

Objective function: Minimize $Z = 2x + 3y$

Step 3 - Constraints:

Protein requirement constraint: The patient requires at least 20 units of protein.

Constraint 1: $4x + 2y \geq 20$

Carbohydrates requirement constraint: The patient requires at least 30 units of carbohydrates.

Constraint 2: $3x + 5y \geq 30$

Fats requirement constraint: The patient requires at least 10 units of fats.

Constraint 3: $2x + 3y \geq 10$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for both foods:

$$x \geq 0$$

$$y \geq 0$$

Step 5 - Complete LP Formulation:

$$\text{Minimize } Z = 2x + 3y$$

Subject to:

$$4x + 2y \geq 20$$

$$3x + 5y \geq 30$$

$$2x + 3y \geq 10$$

$$x \geq 0$$

$$y \geq 0$$



Example 3: Investment Problem

An investor has \$20,000 to invest in two investment options: Option A and Option B. Option A yields a return of 10% per year, and Option B yields 12% per year. The investor wants to maximize the total return.

Step 1 - Decision Variables:

Let x be the amount invested in Option A.

Let y be the amount invested in Option B.

Step 2 - Objective Function:

The objective is to maximize the total return, which is given by the sum of returns from Option A ($0.10x$) and Option B ($0.12y$).

$$\text{Objective function: Maximize } Z = 0.10x + 0.12y$$

Step 3 - Constraints:

Investment constraint: The total investment should not exceed \$20,000.

$$\text{Constraint 1: } x + y \leq 20000$$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for both investments:

$$x \geq 0$$

$$y \geq 0$$

Step 5 - Complete LP Formulation:

$$\text{Maximize } Z = 0.10x + 0.12y$$

Subject to:

$$x + y \leq 20000$$

$$x \geq 0$$

$$y \geq 0$$



Example 4: Production Scheduling

A factory produces two products, Product P and Product Q. Each unit of Product P requires 5 hours of production time, and each unit of Product Q requires 3 hours of production time. The factory has 50 hours of production time available. The profit per unit of Product P is \$12, and for Product Q, it is \$8. Determine the number of units of each product to produce to maximize the total profit.

Step 1 - Decision Variables:

Let x be the number of units of Product P to produce.

Let y be the number of units of Product Q to produce.

Step 2 - Objective Function:

The objective is to maximize the total profit, which is given by the sum of profits from Product P ($12x$) and Product Q ($8y$).

Objective function: Maximize $Z = 12x + 8y$

Step 3 - Constraints:

Production time constraint: The total production time available is limited to 50 hours.

Constraint 1: $5x + 3y \leq 50$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for both products:

$$x \geq 0$$

$$y \geq 0$$

Step 5 - Complete LP Formulation:

Maximize $Z = 12x + 8y$

Subject to:

$$5x + 3y \leq 50$$

$$x \geq 0$$

$$y \geq 0$$

**Example 5: Resource Allocation**

A company has two projects, Project A and Project B. Each project requires resources in terms of labor and materials. Project A requires 30 hours of labor and 10 units of materials, while Project B requires 20 hours of labor and 15 units of materials. The company has 100 hours of labor and 50 units of materials available. The benefit per unit of Project A is \$400, and for Project B, it is \$300. Determine the number of units of each project to undertake to maximize the total benefit.

Step 1 - Decision Variables:

Let x be the number of units of Project A to undertake.

Let y be the number of units of Project B to undertake.

Step 2 - Objective Function:

The objective is to maximize the total benefit, which is given by the sum of benefits from Project A ($400x$) and Project B ($300y$).

Objective function: Maximize $Z = 400x + 300y$

Step 3 - Constraints:

Labor constraint: The total labor hours available are limited to 100 hours.

Constraint 1: $30x + 20y \leq 100$

Materials constraint: The total materials available are limited to 50 units.

Constraint 2: $10x + 15y \leq 50$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for both projects:

$$x \geq 0$$

$$y \geq 0$$

Step 5 - Complete LP Formulation:

$$\text{Maximize } Z = 400x + 300y$$

Subject to:

$$30x + 20y \leq 100$$

$$10x + 15y \leq 50$$

$$x \geq 0$$

$$y \geq 0$$

2.6 Three Decision Variables Problems



Example 1: Resource Allocation

A company produces three products, Product X, Product Y, and Product Z. Each unit of Product X requires 2 hours of labor, 3 units of raw material, and 1 unit of equipment. Each unit of Product Y requires 3 hours of labor, 2 units of raw material, and 2 units of equipment. Each unit of Product Z requires 1 hour of labor, 1 unit of raw material, and 3 units of equipment. The company has 100 hours of labor, 120 units of raw material, and 50 units of equipment available. The profit per unit of Product X is \$10, for Product Y, it is \$12, and for Product Z, it is \$8. Determine the number of units of each product to produce to maximize the total profit.

Step 1 - Decision Variables:

Let x be the number of units of Product X to produce.

Let y be the number of units of Product Y to produce.

Let z be the number of units of Product Z to produce.

Step 2 - Objective Function:

The objective is to maximize the total profit, which is given by the sum of profits from Product X ($10x$), Product Y ($12y$), and Product Z ($8z$).

$$\text{Objective function: Maximize } Z = 10x + 12y + 8z$$

Step 3 - Constraints:

Labor constraint: The total labor hours available are limited to 100 hours.

$$\text{Constraint 1: } 2x + 3y + z \leq 100$$

Raw material constraint: The total raw materials available are limited to 120 units.

$$\text{Constraint 2: } x + 2y + z \leq 120$$

Equipment constraint: The total equipment units available are limited to 50 units.

$$\text{Constraint 3: } x + 2y + 3z \leq 50$$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for all products:

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

Step 5 - Complete LP Formulation:

$$\text{Maximize } Z = 10x + 12y + 8z$$

Subject to:

$$2x + 3y + z \leq 100$$

$$x + 2y + z \leq 120$$

$$x + 2y + 3z \leq 50$$

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

In this example, the LP problem is formulated with the objective function, decision variables (x , y , and z), and three constraints representing the limitations on labor, raw material, and equipment. By solving this LP problem using optimization techniques, the optimal values of x , y , and z can be obtained, representing the number of units of each product to produce to maximize the total profit, while considering the resource constraints.



Example 2: Investment Portfolio

An investor has \$50,000 to invest in three different investment options: Option X, Option Y, and Option Z. Option X yields a return of 8% per year, Option Y yields 10% per year, and Option Z yields 12% per year. The investor wants to maximize the total return while considering the following constraints:

The investment in Option X should not exceed \$20,000.

The investment in Option Y should be at least \$10,000.

The investment in Option Z should be at least \$15,000.

Step 1 - Decision Variables:

Let x be the amount invested in Option X.

Let y be the amount invested in Option Y.

Let z be the amount invested in Option Z.

Step 2 - Objective Function:

The objective is to maximize the total return, which is given by the sum of returns from Option X ($0.08x$), Option Y ($0.10y$), and Option Z ($0.12z$).

Objective function: Maximize $Z = 0.08x + 0.10y + 0.12z$

Step 3 - Constraints:

Investment constraint for Option X: The investment in Option X should not exceed \$20,000.

Constraint 1: $x \leq 20000$

Investment constraint for Option Y: The investment in Option Y should be at least \$10,000.

Constraint 2: $y \geq 10000$

Investment constraint for Option Z: The investment in Option Z should be at least \$15,000.

Constraint 3: $z \geq 15000$

Budget constraint: The total investment should not exceed \$50,000.

Constraint 4: $x + y + z \leq 50000$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for all investments:

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

Step 5 - Complete LP Formulation:

Maximize $Z = 0.08x + 0.10y + 0.12z$

Subject to:

$$x \leq 20000$$

$$y \geq 10000$$

$$z \geq 15000$$

$$x + y + z \leq 50000$$

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$



Example 3: Product Mix

A manufacturing company produces two products, Product A and Product B. Each unit of Product A requires 2 hours of labor and 3 units of raw material. Each unit of Product B requires 4 hours of labor and 1 unit of raw material. The company has 100 hours of labor and 90 units of raw material available. The profit per unit of Product A is \$10, and for Product B, it is \$15. Determine the number of units of each product to produce to maximize the total profit.

Step 1 - Decision Variables:

Let x be the number of units of Product A to produce.

Let y be the number of units of Product B to produce.

Step 2 - Objective Function:

The objective is to maximize the total profit, which is given by the sum of profits from Product A ($10x$) and Product B ($15y$).

Objective function: Maximize $Z = 10x + 15y$

Step 3 - Constraints:

Labor constraint: The total labor hours available are limited to 100 hours.

$$\text{Constraint 1: } 2x + 4y \leq 100$$

Raw material constraint: The total raw materials available are limited to 90 units.

$$\text{Constraint 2: } 3x + y \leq 90$$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for both products:

$$x \geq 0$$

$$y \geq 0$$

Step 5 - Complete LP Formulation:

Maximize $Z = 10x + 15y$

Subject to:

$$2x + 4y \leq 100$$

$$3x + y \leq 90$$

$$x \geq 0$$

$$y \geq 0$$



Example 4: Transportation Problem

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A company needs to transport goods from three warehouses to four distribution centers. The transportation costs per unit are given in the following table:

Warehouse/Distribution Center	DC1	DC2	DC3	DC4
Warehouse 1	\$10	\$7	\$12	\$5
Warehouse 2	\$8	\$6	\$9	\$10
Warehouse 3	\$11	\$5	\$13	\$8

The supply capacities of the warehouses and the demand of distribution centers are as follows:

Warehouse 1 has 50 units available.

Warehouse 2 has 40 units available.

Warehouse 3 has 30 units available.

Distribution Center 1 has a demand of 25 units.

Distribution Center 2 has a demand of 35 units.

Distribution Center 3 has a demand of 30 units.

Distribution Center 4 has a demand of 30 units.

Determine the optimal transportation plan that minimizes the total transportation cost.

Step 1 - Decision Variables:

Let x_{ij} be the number of units transported from Warehouse i to Distribution Center j .

Step 2 - Objective Function:

The objective is to minimize the total transportation cost, which is given by the sum of transportation costs from all warehouses to distribution centers.

Objective function: Minimize $Z = 10x_{11} + 7x_{12} + 12x_{13} + 5x_{14} + 8x_{21} + 6x_{22} + 9x_{23} + 10x_{24} + 11x_{31} + 5x_{32} + 13x_{33} + 8x_{34}$

Step 3 - Constraints:

Supply constraint for Warehouse 1: The total supply from Warehouse 1 should not exceed 50 units.

Constraint 1: $x_{11} + x_{12} + x_{13} + x_{14} \leq 50$

Supply constraint for Warehouse 2: The total supply from Warehouse 2 should not exceed 40 units.

Constraint 2: $x_{21} + x_{22} + x_{23} + x_{24} \leq 40$

Supply constraint for Warehouse 3: The total supply from Warehouse 3 should not exceed 30 units.

Constraint 3: $x_{31} + x_{32} + x_{33} + x_{34} \leq 30$

Demand constraint for Distribution Center 1: The total demand for Distribution Center 1 should be 25 units.

Constraint 4: $x_{11} + x_{21} + x_{31} = 25$

Demand constraint for Distribution Center 2: The total demand for Distribution Center 2 should be 35 units.

Constraint 5: $x_{12} + x_{22} + x_{32} = 35$

Demand constraint for Distribution Center 3: The total demand for Distribution Center 3 should be 30 units.

Constraint 6: $x_{13} + x_{23} + x_{33} = 30$

Demand constraint for Distribution Center 4: The total demand for Distribution Center 4 should be 30 units.

Constraint 7: $x_{14} + x_{24} + x_{34} = 30$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for all transportation units:

$x_{ij} \geq 0$ for all i and j

Step 5 - Complete LP Formulation:

Minimize $Z = 10x_{11} + 7x_{12} + 12x_{13} + 5x_{14} + 8x_{21} + 6x_{22} + 9x_{23} + 10x_{24} + 11x_{31} + 5x_{32} + 13x_{33} + 8x_{34}$

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 50$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 40$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 30$$

$$x_{11} + x_{21} + x_{31} = 25$$

$$x_{12} + x_{22} + x_{32} = 35$$

$$x_{13} + x_{23} + x_{33} = 30$$

$$x_{14} + x_{24} + x_{34} = 30$$

$x_{ij} \geq 0$ for all i and j

In each of these examples, the LP problems are formulated with the objective function, decision variables, and constraints. These LP problems can be solved using optimization techniques to find the optimal solutions that either maximize profit, minimize cost, or minimize transportation cost while satisfying all the constraints.



Example 5: Media Planning

A marketing agency is planning an advertising campaign for a client. They have three media channels available for advertising: Channel X, Channel Y, and Channel Z. The effectiveness of each channel is measured in terms of audience reach (number of potential customers reached per dollar spent). Channel X reaches 1000 potential customers per dollar, Channel Y reaches 800 potential customers per dollar, and Channel Z reaches 1200 potential customers per dollar. The total budget available for the advertising campaign is \$3000. The client wants to maximize the total audience reach while considering the following constraints:

The budget allocation for Channel X should not exceed \$1000.

The budget allocation for Channel Y should be at least \$500.

The budget allocation for Channel Z should be at least \$800.

Step 1 - Decision Variables:

Let x be the budget allocated to Channel X.

Let y be the budget allocated to Channel Y.

Let z be the budget allocated to Channel Z.

Step 2 - Objective Function:

The objective is to maximize the total audience reach, which is given by the sum of audience reach from Channel X ($1000x$), Channel Y ($800y$), and Channel Z ($1200z$).

Objective function: Maximize $Z = 1000x + 800y + 1200z$

Step 3 - Constraints:

Budget constraint for Channel X: The budget allocation for Channel X should not exceed \$1000.

Constraint 1: $x \leq 1000$

Budget constraint for Channel Y: The budget allocation for Channel Y should be at least \$500.

Constraint 2: $y \geq 500$

Budget constraint for Channel Z: The budget allocation for Channel Z should be at least \$800.

Constraint 3: $z \geq 800$

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Total budget constraint: The total budget allocation should not exceed \$3000.

Constraint 4: $x + y + z \leq 3000$

Step 4 - Non-negativity Constraints:

Non-negativity constraints for all budget allocations:

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

Step 5 - Complete LP Formulation:

Maximize $Z = 1000x + 800y + 1200z$

Subject to:

$$x \leq 1000$$

$$y \geq 500$$

$$z \geq 800$$

$$x + y + z \leq 3000$$

$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

In this example, the LP problem is formulated with the objective function, decision variables (x , y , and z), and constraints representing the budget limitations for each media channel and the total budget available. By solving this LP problem using optimization techniques, the optimal values of x , y , and z can be obtained, representing the budget allocations for each media channel that maximize the total audience reach, while considering the budget constraints.

2.7 Application of LPP

Some common real-world applications of Linear Programming (LPP) and how they are used with visuals or diagrams:

Production Planning: LPP is widely used in manufacturing industries to optimize production planning. A production manager can use LP to determine the optimal quantity of each product to produce in order to maximize profits while considering resource constraints. Visual representations like production possibility curves or production feasibility regions can help in understanding the feasible production levels and the optimal production point.

Transportation and Logistics: LPP is applied in transportation and logistics to optimize the distribution of goods from suppliers to customers while minimizing transportation costs. Visual diagrams such as transportation matrices or network flow diagrams are commonly used to represent the flow of goods and the cost associated with each route.

Dietary Planning: LPP is used in dietary planning to optimize the selection of food items in a diet to meet nutritional requirements while minimizing costs. Bar graphs or pie charts can visually represent the percentage allocation of different food items in the diet.

Investment Portfolio Management: LPP is applied in finance to optimize investment portfolios by allocating funds to different investment options to maximize returns while considering risk tolerance. Pie charts or stacked bar charts can be used to illustrate the allocation of funds to different investment instruments.

Resource Allocation in Project Management: LPP is used to optimize resource allocation in project management. Gantt charts or resource allocation bar charts can visually represent the allocation of resources like manpower and equipment to various tasks to ensure efficient project execution.

Media Planning and Advertising: LPP is used in media planning to allocate advertising budgets across different media channels to maximize the reach or impact of an advertising campaign. Pie charts or bar graphs can be used to visualize the budget allocation to different media channels.

Summary

Linear Programming (LP) finds wide application in various real-world scenarios. In production planning, LP is used by manufacturing industries to optimize their production processes. By determining the optimal quantity of each product to produce, companies can maximize their profits while adhering to resource constraints like labor, raw materials, and production capacities. Visual representations, such as production possibility curves or feasibility regions, provide a better understanding of feasible production levels and the optimal production point.

In transportation and logistics, LP helps in efficient distribution of goods from suppliers to customers while minimizing transportation costs. It assists in finding the optimal allocation of goods to different routes or vehicles, represented through transportation matrices or network flow diagrams.

Dietary planning employs LP to design optimal diets that meet nutritional requirements within budget constraints. The goal is to select the most cost-effective combination of food items to provide necessary nutrients. Visual tools like bar graphs and pie charts aid in visually depicting the proportion of different food items in the diet.

In finance, LP plays a role in investment portfolio management. It helps investors allocate funds to various investment options to maximize returns while accounting for their risk tolerance. Visual aids like pie charts or stacked bar charts illustrate the allocation of funds across different investment instruments.

Project management benefits from LP in resource allocation. By optimizing the allocation of resources like manpower, equipment, and budget to various project tasks, it ensures efficient project execution. Gantt charts and resource allocation bar charts present resource usage over time and across different tasks.

Media planning and advertising use LP to allocate advertising budgets across different media channels. By maximizing the reach and impact of advertising campaigns, LP helps to distribute budgets effectively. Pie charts or bar graphs visually illustrate the budget allocation to various media platforms.

In all these applications, LP models are formulated with specific objectives and constraints, and optimization techniques are applied to find the best possible solutions. Visual representations complement the results, aiding decision-makers in understanding and interpreting the outcomes for practical implementation.

Keywords

Linear Programming: A mathematical technique used to find the best solution for optimization problems with linear objective functions and linear constraints.

Optimization: The process of finding the best possible solution or set of values that maximize or minimize an objective function while satisfying specified constraints.

Constraints: Limitations or restrictions that define the boundaries within which a solution must lie in an optimization problem.

Objective Function: A mathematical expression representing the goal of an optimization problem, either to maximize or minimize a specific quantity.

Decision Variables: The variables in an optimization problem that represent the quantities to be determined or chosen to achieve the desired outcome.

Feasible Region: The set of all possible solutions that satisfy the constraints of an optimization problem.

Resource Allocation: The process of distributing limited resources, such as time, money, or materials, among various activities or projects to achieve the best possible outcome.

Production Planning: The process of determining the optimal quantity of products to produce, considering available resources and demand, to maximize profits or meet specific objectives.

Transportation Problem: An optimization problem that involves finding the optimal way to transport goods from suppliers to customers while minimizing transportation costs.

Portfolio Management: The process of selecting and allocating assets in an investment portfolio to optimize returns while managing risk according to the investor's goals and risk tolerance.

Self Assessment

1. Which of the following is a mathematical technique used to find the best solution for optimization problems?
 - A. Linear Regression
 - B. Linear Programming
 - C. Statistical Analysis
 - D. Data Mining

2. In Linear Programming, the objective is to:
 - A. Maximize the number of constraints
 - B. Minimize the objective function
 - C. Maximize the objective function
 - D. Ignore the constraints
3. What are decision variables in Linear Programming?
 - A. Variables representing the constraints
 - B. Variables representing the objective function
 - C. Variables representing the resources
 - D. Variables representing the quantities to be determined

4. The set of limitations that define the boundaries within which a solution must lie in an optimization problem are called:
 - A. Decision Variables
 - B. Objective Function
 - C. Constraints
 - D. Feasible Region

5. In Linear Programming, the region that contains all possible solutions that satisfy the constraints is known as the:
 - A. Objective Region
 - B. Decision Region
 - C. Feasible Region
 - D. Optimal Region

6. A linear programming problem with no feasible solution is called:
 - A. Infeasible Problem
 - B. Unbounded Problem

- C. Feasible Problem
 - D. Optimal Problem
7. Which of the following represents the mathematical expression for the goal of an optimization problem in Linear Programming?
- A. Decision Variables
 - B. Constraints
 - C. Objective Function
 - D. Feasible Region
8. Which technique is used to allocate limited resources among various activities or projects to achieve the best possible outcome?
- A. Resource Allocation
 - B. Production Planning
 - C. Transportation Problem
 - D. Portfolio Management
9. The process of determining the optimal quantity of products to produce, considering available resources and demand, is known as:
- A. Resource Allocation
 - B. Portfolio Management
 - C. Production Planning
 - D. Transportation Problem
10. Which type of Linear Programming problem involves finding the optimal way to transport goods from suppliers to customers while minimizing transportation costs?
- A. Resource Allocation
 - B. Production Planning
 - C. Transportation Problem
 - D. Portfolio Management
11. The non-negativity constraints in Linear Programming state that decision variables cannot be:
- A. Negative
 - B. Zero
 - C. Positive
 - D. Greater than one
12. Which of the following is a graphical representation used in Linear Programming to illustrate the feasible region?
- A. Production Possibility Curve
 - B. Network Flow Diagram
 - C. Gantt Chart

D. Simplex Diagram

13. What is the name of the Linear Programming technique used to solve problems with more than two variables?

- A. Simplex Method
- B. Gradient Descent
- C. Monte Carlo Simulation
- D. Greedy Algorithm

14. The Simplex Method starts at the corner point of the feasible region and moves along the edges in the direction of improving the:

- A. Objective Function
- B. Constraints
- C. Decision Variables
- D. Feasible Region

15. In an LP problem, if the objective function is to minimize and all the constraints are less than or equal to type, the feasible region is:

- A. Unbounded
- B. Empty
- C. Infeasible
- D. Bounded

16. Which of the following is not a characteristic of the Simplex Method?

- A. It always converges to the optimal solution.
- B. It moves from one corner point of the feasible region to another.
- C. It cannot handle problems with more than two variables.
- D. It improves the objective function at each iteration.

17. In LP, the dual problem is used to find:

- A. The inverse of the objective function.
- B. The optimal solution.
- C. The shadow prices of the constraints.
- D. The constraints for the primal problem.

18. When solving a maximization problem using the Simplex Method, the objective function coefficients in the final tableau are:

- A. All positive
- B. All negative
- C. Some positive and some negative
- D. All zero

19. Which type of LP problem is characterized by an unbounded feasible region?
- Infeasible Problem
 - Unbounded Problem
 - Feasible Problem
 - Optimal Problem
20. The optimal solution to an LP problem occurs at a corner point of the feasible region when the objective function is:
- Linear
 - Convex
 - Concave
 - Quadratic

Answers for self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. B | 2. C | 3. D | 4. C | 5. C |
| 6. A | 7. C | 8. A | 9. C | 10. C |
| 11. A | 12. D | 13. A | 14. A | 15. D |
| 16. C | 17. C | 18. A | 19. B | 20. A |

Review Questions

- Describe the main objective of Linear Programming and explain how it achieves optimization in real-world applications.
- Provide a step-by-step explanation of how to formulate a Linear Programming problem. Include the role of decision variables, constraints, and the objective function.
- Discuss the significance of the feasible region in Linear Programming. How is it related to the constraints and decision variables?
- Explain the concept of non-negativity constraints in Linear Programming. Why are these constraints important, and how are they incorporated into LP problems?
- Describe the Simplex Method as a popular technique to solve LP problems. How does it iteratively approach the optimal solution?
- Discuss the difference between infeasible and unbounded LP problems. Provide examples for each type and explain why they pose challenges for optimization.
- Explain how resource allocation problems can be modeled and solved using Linear Programming. Provide a real-life example of resource allocation in project management or production planning.
- Describe the transportation problem and its relevance in logistics and supply chain management. How does Linear Programming help in finding the optimal transportation plan?
- Discuss the importance of sensitivity analysis in Linear Programming. How does it help in understanding the impact of changes in constraints or objective function coefficients on the optimal solution?

10. Explain the duality concept in Linear Programming. What is the dual problem, and how is it related to the primal problem? Provide insights into the interpretation of shadow prices in the context of the dual problem.



Further Reading

Introduction to Linear Programming (MIT Open Course Ware):

Link: https://ocw.mit.edu/courses/sloan-school-of-management/15-053-optimization-methods-in-management-science-spring-2013/tutorials/MIT15_053S13_tut14.pdf

Linear Programming - Columbia University:

Link: <http://www.columbia.edu/~ks20/4404-Sigman/4404-Notes-LP.pdf>

Linear Programming - Operations Research (Stanford University):

Link: <https://web.stanford.edu/class/msande310/linearprogramming.pdf>

Introduction to Linear Programming (MIT Technology Review):

Link: <https://www.technologyreview.com/2019/11/25/138633/an-introduction-to-linear-programming/>

Linear Programming and Its Applications (Brown University):

Link: <https://www.cs.brown.edu/courses/cs053/current/lectures/CPLEXT.pdf>

A Beginner's Guide to Linear Programming (TutorialsPoint):

Link:

https://www.tutorialspoint.com/linear_programming/linear_programming_introduction.htm

An Introduction to Linear Programming (ResearchGate):

Link:

https://www.researchgate.net/publication/292548248_An_Introduction_to_Linear_Programming

Linear Programming - Introduction and Formulation (University of Kentucky):

Link: http://www.ms.uky.edu/~reinelt/programs/intro_lp/formulate.htm

Linear Programming - Concepts and Applications (University of California, Berkeley):

Link: <https://www.eecs.berkeley.edu/~satishr/cs285/notes/linear.pdf>

Unit 03: Linear Programming Problems

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Objectives

After studying this unit, you will be able to:

- formulate a general linear programming model.
- solve the LPP using graphical method.

Introduction

Linear Programming Problem (LPP) is a mathematical optimization technique that aims to find the best possible outcome in a given situation while adhering to certain constraints. The graphical method is a visual approach used to solve LPPs, particularly when dealing with two decision variables. It provides an intuitive way to understand and solve these problems by representing constraints and objectives graphically on a two-dimensional plane.

In real-world scenarios, resources are often limited, and decisions need to be made to optimize certain objectives. Linear programming serves as a powerful tool in such situations. An LPP involves maximizing or minimizing a linear objective function while satisfying a set of linear constraints. The constraints define the feasible region within which the solutions must lie, while the objective function quantifies the goal to be achieved, such as maximizing profit, minimizing costs, or optimizing resource allocation.

The graphical method takes advantage of the geometric properties of these linear relationships. It transforms the problem into a visual representation, making it easier to understand and solve. By plotting the constraints and the objective function on a graph, decision-makers can visually identify the region that satisfies all constraints, making it possible to pinpoint the optimal solution. This method is particularly beneficial when dealing with problems that have only two variables, as they can be graphed in a two-dimensional space.

The graphical method simplifies the problem-solving process by transforming complex mathematical relationships into a visual format. It provides a clear way to identify feasible solutions and the optimal outcome. However, it's important to note that this method has its

limitations. It becomes less practical as the number of variables increases, as visualizing higher-dimensional spaces becomes challenging. For problems with more than two variables, more advanced techniques like the Simplex Algorithm are employed.

In essence, the graphical method is an insightful approach to solving LPPs, especially when dealing with problems involving two variables. It transforms abstract mathematical concepts into visual representations, enabling decision-makers to make informed choices by identifying optimal solutions within the feasible region. While its application is limited to simpler cases, the graphical method remains a valuable tool for understanding the fundamental concepts of linear programming and optimization.

3.1 Solving LPP using Graphical Method

The graphical method is a visual approach used to solve Linear Programming Problems (LPPs) by graphically representing the constraints and the objective function on a two-dimensional plane. This method is particularly useful when dealing with two variables. The fundamental goal of the graphical method is to find the optimal solution - the point that maximizes or minimizes the objective function while satisfying all the given constraints.

To solve an LPP using the graphical method, several steps are involved. First, the given constraints are plotted on the graph, each creating a boundary line that represents the feasible region. This region is the intersection of all constraint lines and defines the possible values for the decision variables. The constraints can be represented as inequalities, such as $Ax + By \leq C$, where A , B , and C are constants representing the coefficients of the variables x and y .

Next, the objective function is plotted on the same graph. This function is typically a straight line representing the goal of maximizing or minimizing a certain quantity. For instance, if the objective is to maximize profit, the objective function line will have a positive slope and be shifted upwards.

The optimal solution lies at the vertex of the feasible region where the objective function line aligns to provide the maximum or minimum value. This vertex can be found by analyzing the intersection points of the objective function line with the boundary lines of the feasible region. The coordinates of these intersection points are then evaluated to determine which one yields the optimal value of the objective function.

There are three primary scenarios that can arise during this analysis:

Unique Optimal Solution: In cases where the objective function line has a clear intersection with the feasible region at a single point, this point corresponds to the optimal solution. The coordinates of this point provide the values of the decision variables that maximize or minimize the objective function.

No Feasible Solution: It's possible that the feasible region is empty, indicating that there is no valid solution that satisfies all the given constraints. This can occur if the constraint lines are parallel and do not intersect.

Infinite Optimal Solutions: Sometimes the objective function line is parallel to a boundary of the feasible region. In such cases, there are infinitely many optimal solutions along that boundary line. Each point on the boundary line will yield the same optimal value for the objective function.

In summary, the graphical method provides a geometric approach to solving Linear Programming Problems by visually analyzing the feasible region and identifying the optimal solution based on the intersection of the objective function line with the boundary lines. While this method is effective for LPPs with two variables, it becomes less practical for problems with more variables due to the complexity of visualization. For such cases, other methods, like the Simplex Algorithm, are employed to efficiently determine the optimal solution.

3.2 Review of problem formulation of Linear Programming (LP).



Example 1: Maximizing Profit

A company manufactures two types of products, Product A and Product B. Each unit of Product A requires 3 hours of labor and 2 hours of machine time, while each unit of Product B requires 2

hours of labor and 4 hours of machine time. The company has 120 hours of labor and 100 hours of machine time available each day. Product A generates a profit of \$50 per unit, and Product B generates a profit of \$40 per unit. The company wants to determine how many units of each product to produce daily to maximize its profit.

Objective: Maximize Profit = $50A + 40B$

Constraints:

Labor Constraint: $3A + 2B \leq 120$

Machine Constraint: $2A + 4B \leq 100$

Non-negativity Constraint: $A \geq 0, B \geq 0$

In this example, the objective is to maximize the profit while satisfying the labor and machine time constraints. The decision variables are the quantities of Product A (A) and Product B (B) to be produced.



Example 2: Minimizing Costs

Problem:

A farmer wants to grow two types of crops, Crop X and Crop Y, in a field. Crop X requires 4 units of fertilizer and 6 units of water per acre, while Crop Y requires 3 units of fertilizer and 5 units of water per acre. The farmer has 240 units of fertilizer and 360 units of water available. Crop X yields a profit of \$80 per acre, and Crop Y yields a profit of \$60 per acre. The farmer aims to determine the acreage of each crop to minimize the costs while maximizing profits.

Objective: Maximize Profit = $80X + 60Y$

Constraints:

Fertilizer Constraint: $4X + 3Y \leq 240$

Water Constraint: $6X + 5Y \leq 360$

Non-negativity Constraint: $X \geq 0, Y \geq 0$

Here, the objective is to maximize the profit by growing the right combination of Crop X and Crop Y, adhering to the fertilizer and water constraints.

In both examples, the decision variables are chosen to optimize a certain objective (maximize profit or minimize costs) while satisfying constraints (resource limitations or technical requirements). The graphical method involves plotting these constraints on a graph to visualize the feasible region and identify the optimal solution where the objective is optimized and the constraints are satisfied. However, the graphical method is more suitable for problems with two variables due to the simplicity of visual representation. For problems with more variables, other methods like the Simplex Algorithm are employed to efficiently find solutions.

3.3 Important Definitions

When solving Linear Programming Problems (LPPs) using the graphical method, several important definitions play a key role in understanding the process and interpreting the results:

Objective Function: The linear equation that represents the goal of the optimization problem, either to maximize or minimize a certain quantity, such as profit, cost, or efficiency. It is typically expressed as a linear combination of decision variables.

Decision Variables: The variables that need to be determined to solve the problem. These variables represent the quantities of different items or activities being optimized, such as the number of products to produce or the acres of crops to plant.

Feasible Region: The region on the graph that satisfies all the given constraints simultaneously. It is defined by the intersection of the constraint lines and often depicted as a polygonal area on the graph.

Feasible Solutions: The points within the feasible region that meet the constraints of the problem. Each feasible solution corresponds to a specific combination of decision variables.

Optimal Solution: The feasible solution within the feasible region that either maximizes or minimizes the objective function. It represents the best possible outcome based on the problem's constraints and goals.

Optimal Value: The value of the objective function at the optimal solution point. This value indicates the maximum profit or minimum cost that can be achieved while adhering to the constraints.

Binding Constraint: A constraint that is active (holds with equality) at the optimal solution. It contributes to shaping the feasible region and determines the direction of the objective function's improvement.

Non-binding Constraint: A constraint that is not active (holds with inequality) at the optimal solution. It helps define the boundary of the feasible region but does not directly affect the optimal solution.

Unbounded Solution: A situation where the feasible region extends indefinitely in one or more directions on the graph. In such cases, the optimal solution does not exist as there is no finite maximum or minimum value for the objective function.

Infeasible Solution: A situation where no point in the graph satisfies all the given constraints. This means that the problem's constraints are mutually contradictory and cannot be simultaneously satisfied.

By understanding these key definitions, one can navigate the process of solving LPPs using the graphical method, interpreting the results in terms of feasible regions, optimal solutions, and the values of the objective function.

3.4 Examples on Maximization LP Problem

Problem:

A bakery produces two types of cakes: Chocolate Cake and Vanilla Cake. The bakery has 300 cups of flour, 200 cups of sugar, and 150 cups of cocoa available. The baking process requires 2 cups of flour, 1 cup of sugar, and 3 cups of cocoa for each Chocolate Cake, and 1 cup of flour, 2 cups of sugar, and 1 cup of cocoa for each Vanilla Cake. A Chocolate Cake sells for \$10, and a Vanilla Cake sells for \$8. The bakery wants to determine the number of each type of cake to produce to maximize its profit.

Solution:

Objective: Maximize Profit = $10X + 8Y$

Constraints:

Flour Constraint: $2X + Y \leq 300$

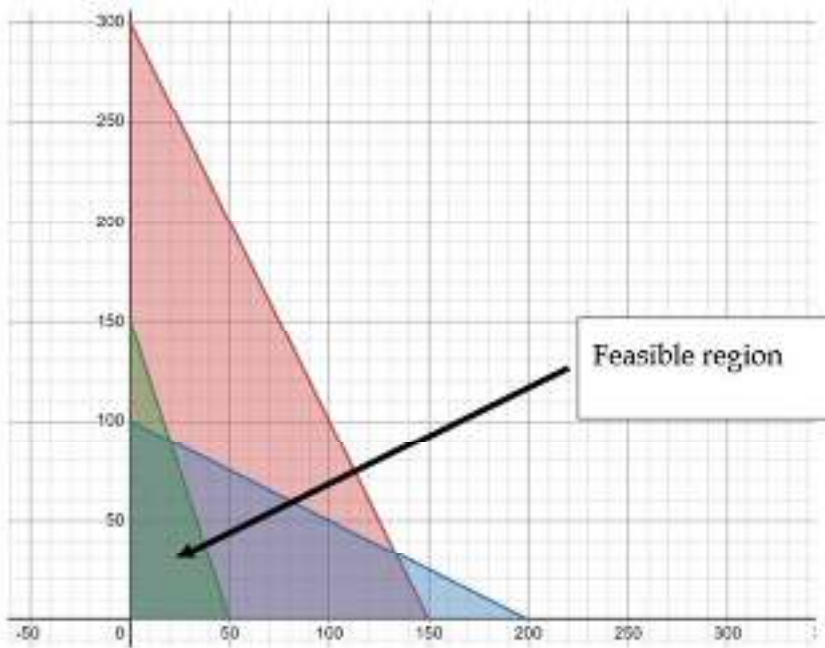
Sugar Constraint: $X + 2Y \leq 200$

Cocoa Constraint: $3X + Y \leq 150$

Non-negativity Constraint: $X \geq 0, Y \geq 0$

Graph the Constraints:

Plot each constraint on a graph using the X (Chocolate Cake) and Y (Vanilla Cake) axes.



Optimal solution is found at the intersection point ($x = 20$, $y = 90$). This means the bakery should produce 20 Chocolate Cakes and 90 Vanilla Cakes to maximize its profit. The maximum profit achievable would be $\$10 * 20 + \$8 * 90 = \$920$.

Problem 2:

A factory produces two types of products: Product X and Product Y. The factory has 200 units of labor, 150 units of material A, and 180 units of material B available. Producing one unit of Product X requires 2 units of labor, 3 units of material A, and 4 units of material B. Producing one unit of Product Y requires 3 units of labor, 2 units of material A, and 2 units of material B. Product X sells for \$10, and Product Y sells for \$12. The factory wants to determine the number of each product to produce to maximize its profit.

Solution

Objective: Maximize Profit = $10X + 12Y$

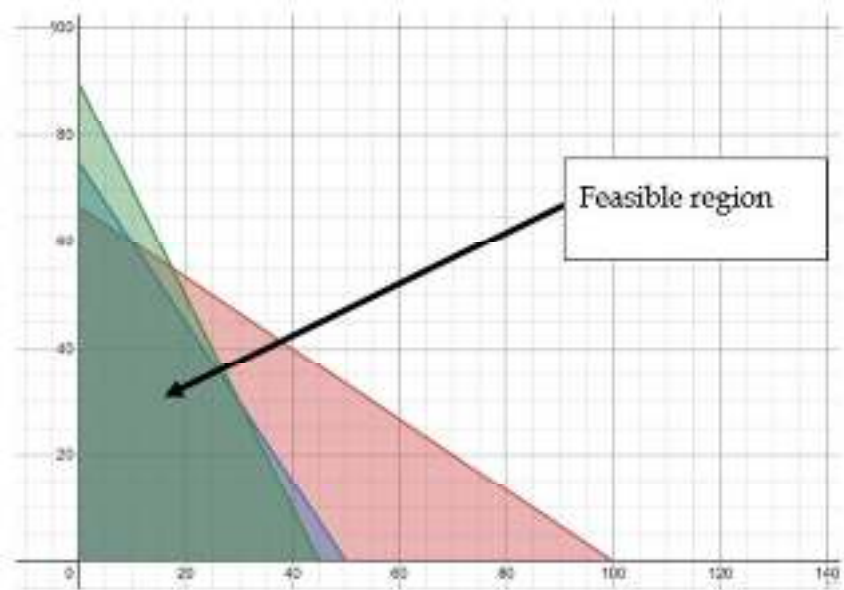
Constraints:

Labor Constraint: $2X + 3Y \leq 200$

Material A Constraint: $3X + 2Y \leq 150$

Material B Constraint: $4X + 2Y \leq 180$

Non-negativity Constraint: $X \geq 0, Y \geq 0$



Optimal solution is found at the corner point C ($X = 30$, $Y = 30$). This means the factory should produce 40 units of Product X and 30 units of Product Y to maximize its profit. The maximum profit achievable would be $\$10 * 30 + \$12 * 30 = \$660$.

Problem 3: The ABC Company has expanded into producing AM and AM-FM radios. The new plant operates 48 hours per week. Production of an AM radio requires 2 hours, and production of an AM-FM radio requires 3 hours. The profit contribution for each AM radio is Rs 40, and for each AM-FM radio, it is Rs 80. The marketing department limits the sales of AM radios to a maximum of 15 units and AM-FM radios to a maximum of 10 units per week.

(a) Formulate a linear programming model to determine the optimum production mix of AM and FM radios that will maximize profits.

(b) Solve this problem using the graphical method.

Solution:

Let's define the decision variables:

Let A represent the number of AM radios produced per week.

Let F represent the number of AM-FM radios produced per week.

The objective is to maximize profits: Profit = $40A + 80F$

Subject to the constraints:

Production time constraint: $2A + 3F \leq 48$ (hours per week)

Demand constraint for AM radios: $A \leq 15$

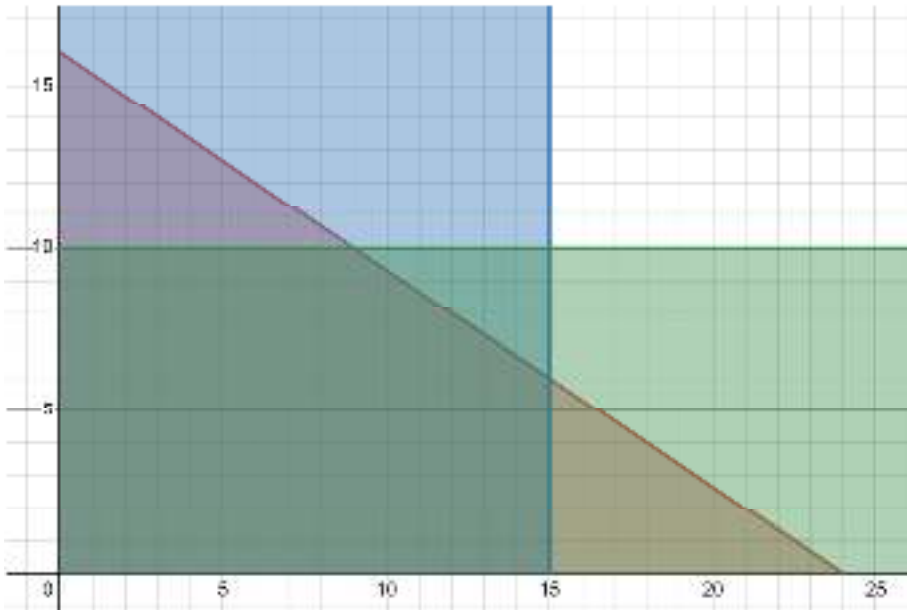
Demand constraint for AM-FM radios: $F \leq 10$

Non-negativity constraint: $A \geq 0, F \geq 0$

(Graphical Solution:

Let's graph the constraints and objective function on a graph using the A (AM radios) and F (AM-FM radios) axes.

Plot the constraints:



Production time constraint: $2A + 3F \leq 48$ (plot the line)

Demand constraint for AM radios: $A \leq 15$ (vertical line at $A = 15$)

Demand constraint for AM-FM radios: $F \leq 10$ (horizontal line at $F = 10$)

Plot the non-negativity constraint: $A \geq 0, F \geq 0$ (shaded region in the first quadrant)

Corner points of feasible region

Point A: (15, 6) - Satisfies

Point B: (15, 0) - Satisfies

Point C: (9, 10) - Satisfies

Point D: (0, 0) - Satisfies

Point D: (0, 10) - Satisfies

These five points represent the corner points of the feasible region. To determine the optimal solution, calculate the profit at each corner point:

Since the maximum value of the objective function $Z = 1,160$ occurs at the extreme point (9, 10), the optimum solution to the given LP problem is: $A = 9, B = 10$ and Max. $Z = \text{Rs } 1,160$.

Problem 4:

A company produces two types of products: Product X and Product Y. The production process for each product requires labor and materials. The company wants to minimize the cost of production while meeting certain requirements. The available resources are as follows:

Labor: 120 hours

Material A: 150 units

Material B: 180 units

The cost per unit of Product X is Rs 8, and the cost per unit of Product Y is Rs 10. The production requirements per unit are as follows:

Product X: Requires 1 hour of labor, 2 units of material A, and 3 units of material B.

Product Y: Requires 3 hours of labor, 1 unit of material A, and 2 units of material B.

The company must produce at least 10 units of Product X and 20 units of Product Y to meet market demand.

Solution:

Objective: Minimize Cost = $8X + 10Y$

Constraints:

Labor Constraint: $X + 3Y \leq 120$

Material A Constraint: $2X + Y \leq 150$

Material B Constraint: $3X + 2Y \leq 180$

Demand Constraint for X: $X \geq 10$

Demand Constraint for Y: $Y \geq 20$

(a) Graphical Solution Steps:

Plot the Constraints:

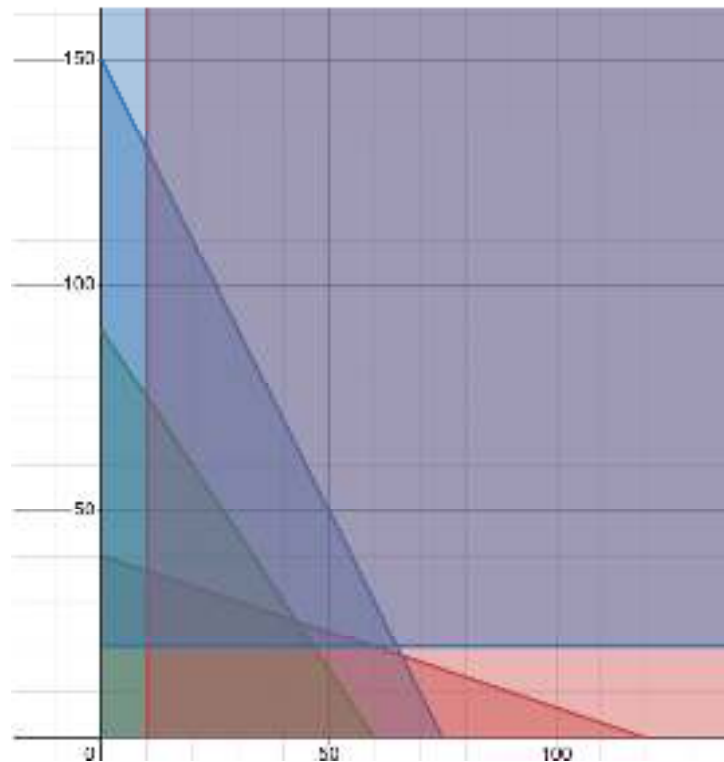
Labor Constraint: $X + 3Y \leq 120$ (plot the line)

Material A Constraint: $2X + Y \leq 150$ (plot the line)

Material B Constraint: $3X + 2Y \leq 180$ (plot the line)

Demand Constraint for X: $X \geq 10$ (vertical line at $X = 10$)

Demand Constraint for Y: $Y \geq 20$ (horizontal line at $Y = 20$)



Plot the Non-negativity Constraint:

Shade the region in the first quadrant where $X \geq 0$ and $Y \geq 0$.

Plot the Objective Function:

Create isocost lines with different values of cost (e.g., $8X + 10Y = 200, 400, 600$, etc.).

Identify the Feasible Region:

The overlapping area of the shaded region and the constraint lines.

Find Intersection Points:

Identify points where the constraint lines intersect within the feasible region.

Calculate Costs:

For each intersection point, calculate the cost using the cost function: Cost = $8X + 10Y$.

Determine Optimal Solution:

Choose the intersection point with the lowest calculated cost. This is the optimal solution.

Check Feasibility:

Verify that the optimal solution satisfies all constraints.

(b) Detailed Solution:

The feasible region's corner points are the intersections of the constraint lines:

Point A: (30, 30)

Point B: (60, 30)

Point C: (40, 60)

For each corner point, calculate the cost using the cost function:

Point A: Cost = $8(30) + 10(30) = 240 + 300 = \text{Rs } 540$

Point B: Cost = $8(60) + 10(30) = 480 + 300 = \text{Rs } 780$

Point C: Cost = $8(40) + 10(60) = 320 + 600 = \text{Rs } 920$

The optimal solution is to produce 30 units of Product X and 30 units of Product Y, resulting in a minimum cost of Rs 540.

Problem 5: Use the graphical method to solve the following LP problem.

Minimize $Z = 3x_1 + 2x_2$

subject to the constraints

(i) $5x_1 + x_2 \geq 10$,

(ii) $x_1 + x_2 \geq 6$,

(iii) $x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$.

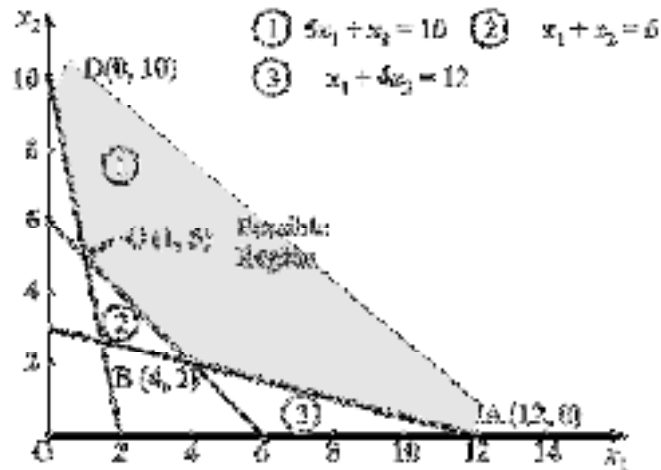
Solution:

Plot on a graph each constraint by first treating it as a linear equation. Then use inequality condition of each constraint to mark the feasible region by shaded area as shown in Fig. This region is bounded from below by extreme points A, B, C and D.

The coordinates of the extreme points of the feasible region (bounded from below) are: A = (12, 0), B = (4, 2), C = (1, 5) and D = (0, 10). The value of objective function at each of these extreme points is shown in Table.

<i>Extreme Point</i>	<i>Coordinates</i> (x_1, x_2)	<i>Objective Function Value</i> $Z = 3x_1 + 2x_2$
<i>A</i>	(12, 0)	$3(12) + 2(0) = 36$
<i>B</i>	(4, 2)	$3(4) + 2(2) = 16$
<i>C</i>	(1, 5)	$3(1) + 2(5) = \mathbf{13}$
<i>D</i>	(0, 10)	$3(0) + 2(10) = 20$

The minimum (optimal) value of the objective function $Z = 13$ occurs at the extreme point C (1, 5). Hence, the optimal solution to the given LP problem is: $x_1 = 1, x_2 = 5$, and $\text{Min } Z = 13$.



Problem 6: A firm produces two products, X and Y, and has a total daily production capacity of 9 tonnes. Both X and Y require the same production capacity. The firm has a contract to supply at least 2 tonnes of X and at least 3 tonnes of Y per day to another company. Each tonne of X requires 20 machine hours of production time, and each tonne of Y requires 50 machine hours of production time. The maximum daily available machine hours are 360. The profit is Rs 80 per tonne of X and Rs 120 per tonne of Y. Formulate and solve this problem using the graphical method to determine the production schedule that yields the maximum profit.

Solution:

Variables:

Let's define the decision variables:

Let X represent the tonnes of Product X produced per day.

Let Y represent the tonnes of Product Y produced per day.

Objective:

Maximize Profit = $80X + 120Y$

Constraints:

$X + Y \leq 9$

Contractual demand for X: $X \geq 2$

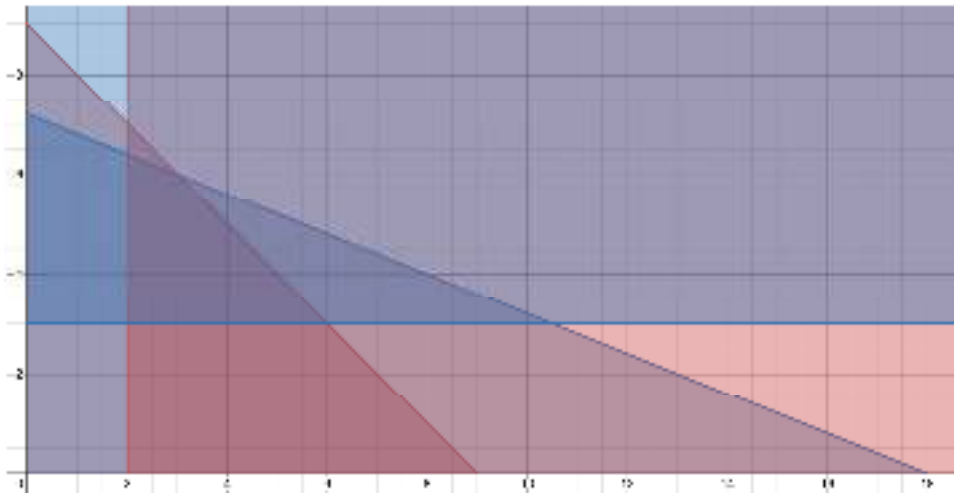
Contractual demand for Y: $Y \geq 3$

Machine hours constraint: $20X + 50Y \leq 360$

Non-negativity constraints: $X \geq 0, Y \geq 0$

Graphical Solution:

Plot the Constraints:



Determine Optimal Solution:

Choose the intersection point with the highest calculated profit. This is the optimal solution.

The coordinates of the extreme points of the feasible region are A = (2, 3), B = (6, 3), C = (3, 6), and D = (2, 6.4).

The maximum (optimal) value of the objective function, $Z = 960$ occurs at the extreme point C (3, 6). Hence the company should produce 3 tonnes of product X and 6 tonnes of product Y in order to yield a maximum profit of Rs 960.

3.5 Unbounded Solution

Sometimes an LP problem may have an infinite solution. Such a solution is referred as an unbounded solution. It happens when value of certain decision variables and the value of the objective function (maximization case) are permitted to increase infinitely, without violating the feasibility condition. It may be noted that there is a difference between unbounded feasible region and unbounded solution to a LP problem. It is possible that for a particular LP problem the feasible region may be unbounded but LP problem solution may not be unbounded, i.e. an unbounded feasible region may yield some definite value of the objective function. In general, an unbounded LP problem solution exists due to improper formulation of the real-life problem.

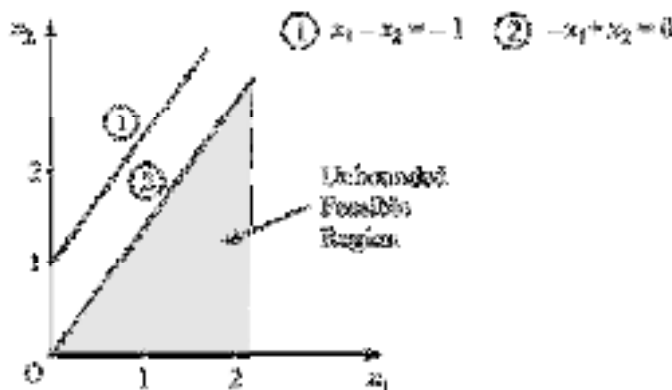
Problem 7: Use the graphical method to solve the following LP problem:

Maximize $Z = 3x_1 + 4x_2$

subject to the constraints

(i) $x_1 - x_2 = -1$ (ii) $-x_1 + x_2 \leq 0$ and $x_1, x_2 \geq 0$.

Solution: Plot on a graph each constraint by first treating it as a linear equation. Then use the inequality condition of each constraint to mark the feasible region (shaded area) as shown in Fig.



It may be noted from Fig. that there exist an infinite number of points in the convex region for which the value of the objective function increases as we move from the extreme point (origin), to the right.

That is, the value of variables x_1 and x_2 can be made arbitrarily large and accordingly the value of objective function Z will also increase. Thus, the LP problem has an unbounded solution.

3.6 Infeasible Solution

An infeasible solution to an LP problem arises when there is no solution that satisfies all the constraints simultaneously. This happens when there is no unique (single) feasible region. This situation arises when a LP model that has conflicting constraints. Any point lying outside the feasible region violates one or more of the given constraints.

Problem 8:

Use the graphical method to solve the following LP problem:

$$\text{Maximize } Z = 6x_1 - 4x_2$$

subject to the constraints

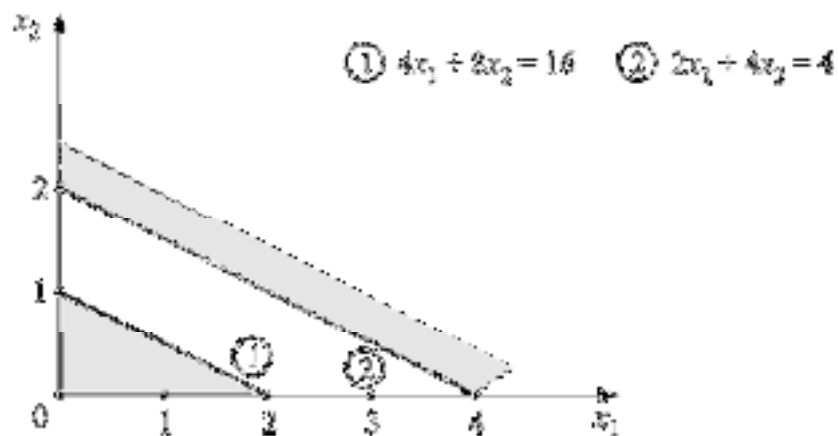
$$(i) 2x_1 + 4x_2 \leq 4$$

$$(ii) 4x_1 + 8x_2 \geq 16$$

$$\text{and } x_1, x_2 \geq 0$$

Solution:

The constraints are plotted on graph as usual as shown in Fig. Since there is no unique feasible solution space, therefore a unique set of values of variables x_1 and x_2 that satisfy all the constraints cannot be determined. Hence, there is no feasible solution to this LP problem because of the conflicting constraints.



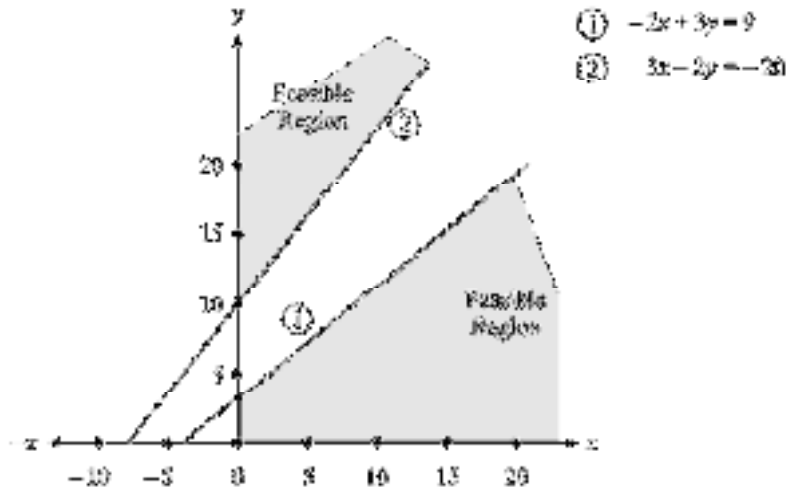
Problem 9: Use the graphical method to solve the following LP problem:

$$\text{Maximize } Z = 3x + 2y$$

subject to the constraints

$$(i) -2x + 3y \leq 9, (ii) 3x - 2y \leq -20 \text{ and } x, y \geq 0$$

Solution: There are two solution spaces (shaded areas) shown in the Fig. One of these solution spaces is satisfying the constraint $-2x + 3y \leq 9$ while the other is satisfying the constraint $3x - 2y \leq -20$. These two shaded regions in the first quadrant do not overlap and hence there is no point (x, y) common to both the shaded regions. This implies that the feasible solution to the problem does not exist. Consequently, this LP problem has one infeasible solution.



Summary

Linear Programming is a mathematical technique used for optimizing a given objective function under certain constraints. The graphical method is a graphical representation approach to solving LPPs involving two variables. The process involves formulating the objective function, constraints, and plotting them on a graph to determine the optimal solution.

In the graphical method, the first step is to formulate the objective function and constraints. The objective function represents the quantity to be maximized or minimized, while constraints represent the limitations or restrictions on the decision variables.

After formulating the problem, the constraints are plotted on a graph. Each constraint is represented as a line, and their intersection forms the feasible region, which is the set of all possible solutions satisfying the constraints. The constraints' graphical representation is based on the inequalities defined by them.

Non-negativity constraints ensure that the decision variables cannot take negative values, limiting the solutions to the first quadrant of the graph.

To determine the optimal solution, isoprofit or isocost lines are drawn parallel to the objective function line, representing different values of the objective function. The optimal solution lies at the point where the isoprofit line with the highest value intersects the feasible region. This point represents the combination of decision variables that yield the maximum (or minimum) value of the objective function while satisfying the constraints.

To ensure validity, the solution must be checked against all constraints to ensure that it falls within the feasible region. If the solution point does not violate any constraint, it is considered the optimal solution. However, if it violates any constraint, it needs to be adjusted to meet the constraints while optimizing the objective function.

In summary, the graphical method of solving LPP involves formulating the objective function and constraints, plotting them on a graph, identifying the feasible region, drawing isoprofit (or isocost) lines, and determining the intersection point that optimizes the objective function while complying with the constraints. This method provides a visual representation of the problem, aiding in understanding and solving linear programming problems with two variables.

Keywords

Linear Programming (LP): A mathematical technique for optimizing an objective function subject to linear equality and inequality constraints.

Objective Function: A mathematical expression representing the quantity to be maximized or minimized in an LP problem.

Constraints: The limitations or restrictions on decision variables that define the feasible region within which the optimal solution lies.

Feasible Region: The set of all possible solutions that satisfy the constraints of an LP problem. It is typically represented graphically.

Decision Variables: The variables in an LP problem that represent the quantities to be determined in order to optimize the objective function.

Optimal Solution: The combination of values for decision variables that results in the maximum or minimum value of the objective function within the feasible region.

Isoprofit Line: A line representing different values of the objective function in the graphical method of solving LP problems.

Non-negativity Constraint: A constraint that ensures decision variables cannot take negative values in an LP problem.

Corner Point: The intersection point of constraint lines in the graphical representation of an LP problem, often used to evaluate the objective function's value.

Binding Constraint: A constraint that is met with equality in the optimal solution, meaning it determines the shape of the feasible region.

Self Assessment

1: In linear programming, the graphical method is most suitable for solving problems with how many decision variables?

- A. 1
- B. 2
- C. 3
- D. 4

2: The feasible region in linear programming is determined by the intersection of:

- A. Objective function and constraints
- B. Constraints only
- C. Decision variables and constraints
- D. Objective function and decision variables

3: The corner points of the feasible region are the points where:

- A. The constraints intersect each other
- B. The objective function intersects the constraints
- C. The constraints intersect the axes
- D. The objective function intersects the axes

4: The graphical method finds the optimal solution by:

- A. Trying all possible combinations of decision variables
- B. Calculating the gradient of the objective function
- C. Testing points within the feasible region
- D. Applying the Simplex method

5: Which of the following is not a step in the graphical method of solving an LP problem?

- A. Plotting the objective function
- B. Plotting the constraints
- C. Calculating the cost coefficients
- D. Identifying the feasible region

6: If a corner point lies on the boundary of the feasible region, it:

- A. Is always an optimal solution
- B. Is never an optimal solution
- C. May or may not be an optimal solution
- D. Doesn't affect the optimal solution

7: The optimal solution of a linear programming problem is the point in the feasible region that has the minimum (or maximum) value of the objective function. The optimal solution of a linear programming problem with two variables can be:

- A. A vertex of the feasible region.
- B. An interior point of the feasible region.
- C. A point on an edge of the feasible region.
- D. Any point in the feasible region.

8: If the iso profit line has a positive slope, it indicates:

- A. A maximum profit
- B. A minimum profit
- C. Infeasibility
- D. Non-negativity constraints

9: The optimal solution to a linear programming problem occurs at the point where:

- A. The objective function intersects the constraint boundary
- B. The objective function is tangent to the constraint boundary
- C. The objective function is parallel to the constraint boundary
- D. None of the above

10: In a graphical LP problem, the feasible region is:

- A. Always convex
- B. Always concave
- C. Sometimes convex and sometimes concave
- D. Neither convex nor concave

11: A linear programming problem is considered infeasible when:

- A. The feasible region is empty
- B. The feasible region is non-convex
- C. There is no unique optimal solution
- D. None of the above

12: The shadow price in linear programming represents:

- A. The change in the objective function value per unit change in the right-hand side of a constraint
- B. The rate of change of the objective function
- C. The ratio of constraints' coefficients
- D. The value of the objective function at the optimal point

13: If the objective function's slope matches the slope of a constraint boundary at an interior point, then:

- A. The point is not feasible
- B. The point is on the constraint boundary
- C. The point is an optimal solution
- D. The point is infeasible

14: The feasible region in linear programming is determined by:

- A. The axes' intercepts
- B. The constraint coefficients
- C. The objective function coefficients
- D. The slope of the objective function

15: If a constraint is changed by adding a constant to both sides, the feasible region:

- A. Shifts parallel to the constraint boundary
- B. Shrinks in size
- C. Expands in size
- D. Doesn't change

16: The concept of "binding constraint" in linear programming refers to:

- A. A constraint that is not satisfied
- B. A constraint that is not plotted
- C. A constraint that doesn't affect the solution
- D. A constraint that is satisfied with equality at the optimal solution

17: In a minimization LP problem, the objective function is:

- A. To maximize profit
- B. To minimize costs
- C. To maximize utility
- D. To minimize production

18: If a constraint boundary has a negative slope, increasing the right-hand side will:

- A. Increase the feasible region
- B. Decrease the feasible region
- C. Not affect the feasible region

D. Make the feasible region empty

19: The feasible region of a linear programming problem is defined by:

- A. All the feasible solutions
- B. The intersection of the constraints
- C. The objective function coefficients
- D. The slope of the iso profit line

20: The optimal solution of a linear programming problem lies at the corner point with the:

- A. Maximum value of the objective function
- B. Minimum value of the objective function
- C. Equal values of both decision variables
- D. Largest number of constraints

21: Which of the following is not a requirement for a linear programming problem?

- A. The objective function must be linear.
- B. The constraints must be linear.
- C. The feasible region must be a polygon.
- D. The objective function must be minimized.

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. B | 2. B | 3. A | 4. C | 5. C |
| 6. A | 7. A | 8. A | 9. B | 10. A |
| 11. A | 12. A | 13. C | 14. B | 15. A |
| 16. D | 17. B | 18. A | 19. B | 20. A |
| 21. D | | | | |

Review Questions

1. What is the primary goal of Linear Programming?
2. Define the objective function in the context of a Linear Programming problem.
3. How are constraints represented in Linear Programming?
4. What is the feasible region in a Linear Programming problem?
5. Explain the concept of decision variables in Linear Programming.
6. How is the optimal solution defined in a Linear Programming problem?
7. What are is profit (is cost) lines in the graphical method of solving LP problems?
8. What purpose do non-negativity constraints serve in Linear Programming?
9. What is a corner point in the context of Linear Programming?
10. How do binding constraints impact the feasible region and optimal solution?



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Unit 4: Linear Programming Problem – Simplex Method

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4.4.2 Unbounded Solutions

4.4.3 Infeasibility

Summary

Keywords

Review Questions

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Objectives

After studying this unit, you will be able to:

- Understand the meaning of word 'simplex' and logic of using simplex method
- Know how to convert a LPP into its standard form by adding slack, surplus and artificial variables
- Learn how to solve the LPP with the help of Big M methodology
- Understand the significance of duality concepts in LPP and ways to solve duality problems

Introduction

In practice, most problems contain more than two variables and are consequently too large to be tackled by conventional means. Therefore, an algebraic technique is used to solve large problems using Simplex Method. This method is carried out through iterative process systematically step by step, and finally the maximum or minimum values of the objective function are attained.

The simplex method solves the linear programming problem in iterations to improve the value of the objective function. The simplex approach not only yields the optimal solution but also other valuable information to perform economic and 'what if' analysis.

4.1 Simplex Method of Linear Programming

Under 'Graphical solutions' to LP, the objective function obviously should have not more than two decision variables. If the decision variables are more than two, the 'Cartesian Plane' cannot accommodate them. And hence, a most popular and widely used analysis called 'SIMPLEX METHOD', is used. This method of analysis was developed by one American Mathematician by name George B. Dantzig, during 1947.

This method provides an algorithm (a procedure which is iterative) which is based on fundamental theorems of Linear Programming. It helps in moving from one basic feasible solution to another in a prescribed manner such that the value of the objective function is improved. This procedure of jumping from one vertex to another vertex is repeated.

Steps:

1. Convert the inequalities into equalities by adding slack variables, surplus variables or artificial variables, as the case may be.
2. Identify the coefficient of equalities and put them into a matrix form $AX = B$

Where "A" represents a matrix of coefficient, "X" represents a vector of unknown quantities and B represents a vector of constants, leads to $AX = B$ [This is according to system of equations].

3. Tabulate the data into the first iteration of Simplex Method.

Table 4.1: Specimen

Basic (BV) Variable	C_B	X_B	Y_1	Y_2	S_1	S_2	Minimum Ratio $X_{B_i}/Y_{ij}; Y_{ij} > 0$
S_1							
S_2							
		Z_j					
		C_j					
		$Z_j - C_j$					

- (a) C_j is the coefficient of unknown quantities in the objective function.

$$Z_j = \sum C_{B_i} Y_{ij}$$
 (Multiples and additions of coefficients in the table, i.e., $C_{B_1} \times Y_{11} + C_{B_2} \times Y_{12}$)
 - (b) Identify the Key or Pivotal column with the minimum element of $Z_j - C_j$ denoted as 'KC' throughout to the problems in the chapter.
 - (c) Find the 'Minimum Ratio' i.e., X_{B_i}/Y_{ij} .
 - (d) Identify the key row with the minimum element in a minimum ratio column. Key row is denoted as 'KP'.
 - (e) Identify the key element at the intersecting point of key column and key row, which is put into a box throughout to the problems in the chapter.
4. Reinstate the entries to the next iteration of the simplex method.
 - (a) The pivotal or key row is to be adjusted by making the key element as '1' and dividing the other elements in the row by the same number.
 - (b) The key column must be adjusted such that the other elements other than key elements should be made zero.

- (c) The same multiple should be used to other elements in the row to adjust the rest of the elements. But, the adjusted key row elements should be used for deducting out of the earlier iteration row.
- (d) The same iteration is continued until the values of $Z_j - C_j$ become either '0' or positive.

5. Find the 'Z' value given by $C_B X_B$.

4.1.1 Maximisation Cases



Example:

Maximise $Z' = 5x_1 + 3x_2$ [Subject to constraints]

$$x_1 + x_2 \leq 2$$

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12$$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of inequalities into equalities adding slack variables

$$x_1 + x_2 + x_3 = 2$$

$$5x_1 + 2x_2 + x_4 = 10$$

$$3x_1 + 8x_2 + x_5 = 12$$

Where, x_3, x_4 and x_5 are slack variables.

Step 2: Fit the data into the matrix form $AX = B$

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = B = \begin{pmatrix} 2 \\ 10 \\ 12 \end{pmatrix}$$

Step 3: Fit the data into first iteration of Simplex Method

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	2	<u>1</u>	1	1	0	0	$2/1 = 2(KR) \rightarrow$
S_2	0	10	5	2	0	1	0	$10/5 = 2$
S_3	0	12	3	8	0	0	1	$12/3 = 4$
		Z_j	0	0				
		C_j	5	3				
		$Z_j - C_j$	-5	-3				

(↑ KC)

Therefore, $Z = C_B X_B$

$$= (0 \times 2) + (0 \times 10) + (0 \times 12)$$

$$= 0$$

Notes

Step 4: Fit the data into second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	S ₃	Min. Ratio
Y ₁	5	2/1 = 2	1/1 = 1	1/1 = 1	-	-	-	-
S ₂	0	10 - 2(5) = 0	5 - 1(5) = 0	2 - 1(5) = -3	-	-	-	-
S ₃	0	12 - 2(3) = 6	3 - 1(3) = 0	8 - 1(3) = 5	-	-	-	-
		Z _j	5	5				
		C _j	5	3				
		Z _j - C _j	0	2				

Therefore, $Z = C_B X_B$
 $= (5 \times 2) + (0 \times 0) + (0 \times 6)$

Therefore, $Z = 10$

Therefore, Maximum value of 'Z' = 10



Example:

Maximise 'Z' = 2x₁ + 3x₂ [Subject to constraints]

x₁ + x₂ ≤ 1

3x₁ + x₂ ≤ 4

Where, x₁, x₂ ≥ 0

Solution:

Step 1: Conversion of inequalities into equalities by adding slack variables.

x₁ + x₂ + x₃ = 1

3x₁ + x₂ + x₄ = 4

Where x₃ and x₄ are slack variables.

Step 2: Identify the coefficients.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	Min. Ratio
S ₁	0	1	1	1	1	0	1/1 = 1 (KR)?
S ₂	0	4	3	1	0	1	4/1 = 4
		Z _j	0	0			
		C _j	2	3			
		Z _j - C _j	-2	-3			

(↑ KC)

Therefore, $Z = C_B X_B$
 $= 0 + 0 + 0$
 $= 0$

Step 4: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	Min. Ratio
y ₂	3	1/1 = 1	1/1 = 1	1/1 = 1	--	--	--
S ₂	0	4-1(1) = 3	3-1(1) = 2	1-1(1) = 0	--	--	--
		Z _j	3	3			
		C _j	2	3			
		Z _j - C _j	1	0			

Therefore, $Z = C_B X_B$
 $= (3 \times 1) + (0 \times 3)$
 $= 3$

Therefore, Maximum value of 'Z' = 3



Example:

Maximise 'Z' = 4x₁ + 3x₂ [Subject to constraints]

$$2x_1 + x_2 \leq 30$$

$$x_1 + x_2 \leq 24$$

Where, x₁, x₂ ≥ 0 [Non-negativity constraints]

Solution:

Step 1: Convert the inequalities into equalities adding slack variables.

$$2x_1 + x_2 + x_3 = 30$$

$$x_1 + x_2 + x_4 = 24$$

Where x₃ and x₄ are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B = \begin{pmatrix} 30 \\ 24 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	Min. Ratio
S ₁	0	30	2		1	0	30/2 = 15 (KR) ?
S ₂	0	24	1	1	0	1	24/1 = 24
		Z _j	0	0			
		C _j	4	3			
		Z _j - C _j	-4	-3			

(↑ KC)

Notes

Therefore, $Z = (0 \times 30) + (0 \times 24)$
 $= 0$

Step 4: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	Min. Ratio
Y ₁	4	30/2 = 15	2/2 = 1	1/2 = 0.5	-	-	15/0.5 = 30
S ₂	0	24 - 15(1) = 9	1 - 1(1) = 2	1 - 0.5(1) = 0.5	-	-	9/0.5 = 18 (KR)?
		Z _j	4	2			
		C _j	4	3			
		Z _j - C _j	0	-1			

(↑ KC)

Therefore, $Z = C_B X_B$
 $= (4 \times 15) + (0 \times 9)$
 $= 60$

Step 5: Third iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	Min. Ratio
Y ₁	4	15 - 18(0.5) = 6	1 - 0(0.5) = 1	0.5 - 1(0.5) = 0	-	-	-
S ₂	3	9/0.5 = 18	0/0.5 = 0	0.5/0.5 = 1	-	-	-
		Z _j	4	3			
		C _j	4	3			
		Z _j - C _j	0	0			

Therefore, $Z = C_B X_B$
 $= (4 \times 6) + (3 \times 18)$
 $Z = 78$



Example:

Maximise 'Z' = 5x₁ + 3x₂ [Subject to constraints]

x₁ + x₂ ≤ 12

5x₁ + 2x₂ ≤ 10

3x₁ + 8x₂ ≤ 12

Where, x₁, x₂ ≥ 0 [Non-negativity constraints]

Solution:

Step 1: Convert the inequalities into equalities adding slack variables.

x₁ + x₂ + x₃ = 12

5x₁ + 2x₂ + x₄ = 10

3x₁ + 8x₂ + x₅ = 12

Where x₃, x₄ and x₅ are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 10 \\ 12 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	12	1	1	1	0	0	12/1 = 12
S ₂	0	10	5	2	0	1	0	10/5 = 2(KR)?
S ₃	0	12	3	8	0	0	1	12/3 = 4
		Z _j	0	0				
		C _j	5	3				
		Z _j - C _j	-5	-3				

(↑ KC)

Therefore, $Z = C_B X_B$
 $= (0 \times 12) + (0 \times 10) + (0 \times 12) = 0$

Step 4: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	12 - 2(1) = 10	1 - 1(1) = 0	1 - 0.4(1) = 0.6	-	-	-	10/0.6 = 16.67
y ₁	5	10/5 = 2	5/5 = 1	2/5 = 0.4	-	-	-	-2/0.4 = 5
S ₃	0	12 - 2(3) = 6	3 - 1(3) = 0	8 - 0.4(3) = 6.8	-	-	-	6/6.8 = 0.88(KR) ?
		Z _j	5	2				
		C _j	5	3				
		Z _j - C _j	0	-1				

(↑ KC)

Therefore, $Z = C_B X_B$
 $= (0 \times 10) + (5 \times 2) + (0 \times 6) = 10$

Step 5: Third iteration of Simplex Method.

BV	C _B	X _B	y ₁	y ₂	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	10 - 0.88(0.6) = 9.47	0	0.6 - 1(0.6) = 0	-	-	-	-
y ₁	5	2 - 0.88(0.4) = 1.698	1	0.4 - 1(0.4) = 0	-	-	-	-
y ₂	3	6/6.8 = 0.88	0	6.8/6.8 = 1	-	-	-	-
		Z _j	5	3				
		C _j	5	3				
		Z _j - C _j	0	0				

Therefore, $Z = C_B X_B$
 $= (0 \times 9.47) + (5 \times 1.698) + (3 \times 0.88)$

Therefore, maximum value of Z = 10.88

Notes



Example:

Maximise $Z' = 5x_1 + 3x_2$ [Subject to constraints]
 $= 3x_1 + 5x_2 + \leq 15$
 $= 5x_1 + 2x_2 \leq 10$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities by adding the slack variables.

$$3x_1 + 5x_2 + x_3 = 15$$

$$5x_1 + 2x_2 + x_4 = 10$$

Where, x_3 and x_4 are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 3 & 5 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad B = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	Min. Ratio
S ₁	0	15	3	5	1	0	15/3 = 5
S ₂	0	10	5	2	0	1	10/5 = 2(KR) ?
		Z _j	0	0			
		C _j	5	3			
		Z _j - C _j	-5	-3			

(↑ KC)

Therefore, Maximise $Z = C_B X_B$
 $= (0 \times 15) + (0 \times 10)$
 $= 0$

Step 4: Second iteration of Simplex Method.

BV	C _B	X _B	y ₁	Y ₂	S ₁	S ₂	Min. Ratio
S ₁	0	15 - 2(3) = 9	3 - 1(3) = 0	5 - 0.4(3) = 3.8	--	--	9/3.8 = 2.37 (KR)?
S ₂	5	10/5 = 2	5/5 = 1	2/5 = 0.4	--	--	2/0.4 = 5
		Z _j	5	2			
		C _j	5	3			
		Z _j - C _j	0	-1			

(↑ KC)

Therefore, $Z = C_B X_B$
 $= (0 \times 9) + (5 \times 2)$
 $= 10$

Step 5: Third iteration of Simplex Method.

Notes

BV	C _B	X _B	y ₁	y ₂	S ₁	S ₂	Min. Ratio
y ₂	3	9/3.8 = 2.37	0	3.8/3.8 = 1	-	-	-
y ₁	5	2 - 2.37(0.4) = 1.052	1	0.4 - 1(0.4) = 0	-	-	-
		Z _j	5	3			
		C _j	5	3			
		Z _j - C _j	0	0			

Therefore, Maximise $Z = C_B X_B$
 $= (3 \times 2.37) + (5 \times 1.052)$

Maximum value of 'Z' = 12.37



Notes Real life complex applications usually involve hundreds of constraints and thousands of variables. So, virtually these problems cannot be solved manually. For solving such problems, you will have to rely on employing an electronic computer.

Self Assessment

1. Solve the following LPP problem using simplex method.

Maximize 'Z' = $7x_1 + 5x_2$ [Subject to constraints]

$x_1 + x_2 \leq 6$

$4x_1 + 3x_2 \leq 12$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

2. Solve the following LPP problem using simplex method.

Maximise 'Z' = $5x_1 + 7x_2$ [Subject to constraints]

$x_1 + x_2 \leq 4$

$3x_1 - 8x_2 \leq 24$

$10x_1 + 7x_2 \leq 35$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

4.1.2 Minimization Cases



Example:

Minimize 'Z' = $-x_1 - 2x_2$ [Subject to constraints]

$-x_1 + 3x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Notes

Solution:

Step 1: Convert the minimization problem into maximisation case by changing the signs of the decision variables in the objective function.

Therefore, 'Z' = $x_1 + 2x_2$ [Subject to constraints]

Step 2: Convert the inequalities into equalities by adding slack variables.

$$-x_1 + 3x_2 + x_3 = 10$$

$$x_1 + x_2 + x_4 = 6$$

$$x_1 - x_2 + x_5 = 2$$

Where x_3, x_4 and x_5 are slack variables.

Step 3: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ -1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = B = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

Step 4: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	10	-1	3	1	0	0	10/3 = 3.3 (KR)?
S ₂	0	6	1	1	0	1	0	6/1 = 6
S ₃	0	2	1	-1	0	0	1	-1/2 = -0.5
		Z _j	0	0				
		C _j	1	2				
		Z _j - C _j	-1	-2				

(↑ KC)

Therefore,

$$Z = C_B X_B$$

$$0 + 0 + 0 = 0$$

Step 5: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	S ₃	Min. Ratio
y ₂	2	10/3 = 3.33	-1/3 = 0.33	3/3 = 1				3.33/-0.33 = -10.09
S ₂	0	6 - 3.33 (1) 2.67	1 - 1(-0.33) (1) = 2.67	1-1(1) = 0 = 1.33				2.67/1.33 = 2.00 (KR) →
S ₃	0	2 - 3.33 (-1) = 5.33	1 - (-0.33) (-1) = 0.67	-1-1 (-1) = 0				5.33/0.67 = 8.00
		Z _j	-0.66	2				
		C _j	1	2				
		Z _j - C _j	-1.66	0				

(↑ KC)

Therefore,

$$Z = C_B X_B$$

$$= (2 \times 3.33) + (2 \times 2.67) + (0 \times 5.33)$$

$$= 6.66 + 0 + 0 = 6.66$$

Step 6: Third iteration of Simplex Method.

Notes

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	S ₃	Min. Ratio
y ₂	2	3.33 - 2(-0.33) = 3.99	-0.33 - 1(-0.33) = 0	1	-	-	-	-
y ₁	1	2.67/1.35 = 2	1.33/1.33 = 1	0	-	-	-	-
S ₃	0	5.33 - 2(0.67) = 3.99	0.67 - 1(0.67) = 0	0	-	-	-	-
		Z _j	1	2	-			
		C _j	1	2				
		Z _j - C _j	0	0	-	-		

Max. $Z = C_B X_B$
 $= (2 \times 3.99) + (1 \times 2) + (0 \times 3.99)$
 $= 7.98 + 2 + 0$
 $= .98$

Therefore, min. $Z = -9.98$



Example:

Minimise 'Z' = $-x_1 - 3x_2 + 2x_3$ [Subject to constraints]
 $3x_1 - x_2 + 3x_3 \leq 7$
 $-2x_1 + 4x_2 \leq 12$
 $-4x_1 + 3x_2 + 8x_3 \leq 10$
 Where, $x_1, x_2, x_3 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of the minimization case into maximisation case.

Therefore, Maximise $Z = -x_1 + 3x_2 - 2x_3$ [Subject to constraints]

Step 2: Convert of the inequalities into equalities by adding slack variables.

Therefore, $3x_1 - x_2 + 3x_3 + x_4 = 7$
 $-2x_1 + 4x_2 + x_5 = 12$
 $-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

Where, x_4, x_5 and x_6 are slack variables.

Step 3: Fit the data into matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 3 & -1 & 3 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$$

Notes

Step 4: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	7	3	-1	3	1	0	0	7/-1 = -7
S ₂	0	12	-2	4	0	0		0	12/4 = 3 (KR)→
S ₃	0	10	-4	3	8	0	0	1	10/3 = 3.33
		Z _j	0	0	0				
		C _j	-1	3	-2				
		Z _j - C _j	1	-3	2				

(↑ KC)

Therefore,

$$Z = C_B X_B$$

$$= 0 + 0 + 0$$

$$= 0$$

Step 5: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	7-3(-1) = 10	3 + 0.5(-1) = 2.5	-1+1(-1) = 0	3-0(-1) = 3	-	-	-	10/2.5 = 4 (KR)→
Y ₂	3	12/4 = 3	-2/4 = -0.5	4/4 = 1	0	-	-	-	3/-0.5 = -
S ₃	0	10 - 3(3) = 1	-4 + (-0.5)(3) = -5.5	3-1(3) = 0	8-0(3) = 8	-	-	-	1/2.5 = -
		Z _j	-1.5	3	0				
		C _j	-1	-3	-2				
		Z _j - C _j	-0.5	-0	2				

(↑ KC)

Therefore,

$$Z = C_B X_B$$

$$= (0 \times 0) + (3 \times 3) + (0 \times 1)$$

$$= 0 + 9 + 0$$

$$= 9$$

Step 6: Third iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	Min. Ratio
y ₁	-1	10/2.5 = 4	2.5/2.5 = 1	0	3/2.5 = 1.2	-	-	-	-
S ₂	3	3-4(-0.5) = 5	-0.5-1(-0.5) = 0	1	0	-	-	-	-
S ₃	0	1-4(-2.5) = 11	-2.5-1(-2.5) = 0	0	8-1(-2.5) = 10.5	-	-	-	-
		Z _j	-1	3	-1.2				
		C _j	-1	3	-2				
		Z _j -C _j	0	0	0.8				

Therefore, Maximise

$$Z' = C_B X_B$$

$$= -4 + 15 + 0$$

$$= 11$$

Therefore, Minimise

$$Z = -11$$



Example:

Minimise 'Z' = $-10x_1 - 12x_2 - 15x_3$ [Subject to constraints]

$$0.10x_1 + 12x_2 + 0.15x_3 \leq 36$$

$$0.06x_1 + 0.05x_2 + 0.09x_3 \leq 30$$

$$0.18x_1 + x_2 + 0.07x_3 \leq 37$$

$$0.13x_1 + 0.10x_2 + 0.08x_3 \leq 38$$

$$x_1 \leq 200$$

$$x_2 \leq 100$$

$$x_3 \leq 180$$

Where, $x_1, x_2, x_3 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of minimization case into maximisation case.

Therefore, Maximise 'Z' = $10x_1 + 12x_2 + 15x_3$ [Subject to constraints]

Step 2: Convert the inequalities into equalities adding slack variables.

$$0.10x_1 + 0.12x_2 + 0.15x_3 + x_4 = 36$$

$$0.06x_1 + 0.05x_2 + 0.09x_3 + x_5 = 30$$

$$0.18x_1 + x_2 + 0.07x_3 + x_6 = 37$$

$$0.13x_1 + 0.10x_2 + 0.08x_3 + x_7 = 38$$

$$x_1 + x_8 = 200$$

$$x_2 + x_9 = 100$$

$$x_3 + x_{10} = 180$$

Where, $x_4, x_5, x_6, x_7, x_8, x_9$ and x_{10} are slack variables.

Step 3: Fit the data into matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ 0.10 & 0.12 & 0.15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.06 & 0.05 & 0.09 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.18 & 1 & 0.07 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.13 & 0.10 & 0.08 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = B = \begin{pmatrix} 36 \\ 30 \\ 37 \\ 200 \\ 200 \\ 100 \\ 180 \end{pmatrix}$$

Notes

Step 4: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	Min. Ratio
S ₁	0	36	0.10	0.12	0.15	1	0	0	0	0	0	0	36/0.15 = 240
S ₂	0	30	0.06	0.05	0.09	0	1	0	0	0	0	0	30/0.09 = 33.3
S ₃	0	37	0.18	1	0.07	0	0	1	0	0	0	0	37/0.08 = 529
S ₄	0	38	0.13	0.10	0.08	0	0	0	1	0	0	0	38/0.08 = 475
S ₅	0	200	1	0	0	0	0	0	0	1	0	0	200/0 = -
S ₆	0	100	0	1	0	0	0	0	0	0	1	0	100/0 = -
S ₇	0	180	0	0	1	0	0	0	0	0	0	1	180/1 = 180(KR)?
		Z _j	0	0	0								
		C _j	10	12	15								
		Z _j - C _j	-10	-12	-15								

(↑ KC)

Therefore, $Z = C_B X_B$

$$= (0 \times 36) + (0 \times 30) + (0 \times 37) + (0 \times 38) + (0 \times 200) + (0 \times 100) + (0 \times 180)$$

$$= 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$$= 0$$

Step 5: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	Min. Ratio
S ₁	0	36 - 180(0.15) = 9	0.10 - 0(0.15) = 0.10	0.12 - 0(0.15) = 0.12	0.15 - 0(0.15) = 0	-	-	-	-	-	-	-	9/0.12 = 75
S ₂	0	30 - 180(0.09) = 13.8	0.06 - 0(0.09) = 0.06	0.05 - 0(0.09) = 0.05	0.09 - 1(0.09) = 0	-	-	-	-	-	-	-	13.8/0.05 = 276
S ₃	0	37 - 180(0.07) = 24.4	0.18 - 0(0.07) = 0.18	1 - 0(0.07) = 1	0.07 - 1(0.07) = 0	-	-	-	-	-	-	-	24.4/1 = 24.4(KR)?
S ₄	0	38 - 180(0.08) = 23.6	0.13 - 0(0.08) = 0.13	0.10 - 0(0.08) = 0.10	0.08 - 1(0.08) = 0	-	-	-	-	-	-	-	23.6/0.10 = 236
S ₅	0	200 - 180(0) = 200	0.10 - 0(0) = 0.10	0 - 0(0) = 0	0 - 0(0) = 0	-	-	-	-	-	-	-	(0) = 200
S ₆	0	100 - 180(0) = 100	0 - 0(0) = 0	1 - 0(0) = 1	0 - 0(0) = 0	-	-	-	-	-	-	-	100/1 = 100
Y ₃	15	180/1 = 180	0	0	1/1 = 1	-	-	-	-	-	-	-	
		Z _j	0	0	15								
		C _j	10	12	15								
		Z _j - C _j	-10	-12	0								

(↑ KC)

Therefore, $Z = C_B X_B$

$$= (0 \times 9) + (0 \times 13.8) + (0 \times 24.4) + (0 \times 23.6) + (0 \times 200) + (0 \times 100) + (15 \times 180)$$

$$= 2700$$

Step 6: Third iteration of Simplex Method.

Notes

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	Min. Ratio
S ₁	0	9 - 24.4 (0.12) = 6.07	0.10 - 0.18 (0.12) = 0.078	0.12 - 1 (0.12) = 0	0	-	-	-	-	-	-	-	6.072/0.078 = 77.85 (KR) →
S ₂	0	13.8-24.4 (0.05)=12.58	0.06 - 18 (0.05) = 0.05	0.05 - 1 (0.05) = 0	0	-	-	-	-	-	-	-	12.58/0.051 = 246.67
S ₃	0	24.4/1 = 24.4	0.18/1 = 0.18	1/1 = 1	0	-	-	-	-	-	-	-	24.4/0.18 = 135.56
S ₄	0	23.6-24.4 (0.10)=21.16	0.13 - 0.18 (0.10) = 0.112	0.10 - 1 (0.10) = 0	0	-	-	-	-	-	-	-	21.16/0.112 = 188.93
S ₅	0	200 - 24.4 (0) = 200	1-0.18 (0)=1	0	0	-	-	-	-	-	-	-	200/1 = 200
S ₆	0	100 - 24.4 (1) = 75.6	0 - 0.18 (1) = -0.18	1 - 1 (1) = 0	0	-	-	-	-	-	-	-	75.6/-0.18 = -420
Y ₃	15	180 - 0 = 180	0	0	1	-	-	-	-	-	-	-	180/0 = 0
		Z _j	2.16	12	15								
		C _j	10	12	15								
		Z _j - C _j	-7.84	0	0								

(↑ KC)

Therefore, $Z = C_B X_B$

$$= (0 \times 6.072) + (0 \times 12.58) + (12 \times 24.4) + (0 \times 21.16) + (0 \times 200) + (0 \times 75.6) + (15 \times 180)$$

$$= 2,992.8$$

Step 7: Fourth iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	Min. Ratio
y1	10	6.072/0.078 = 77.85	0.078/0.078 = 1	0	0	-	-	-	-	-	-	-	-
S2	0	12.58 - 77.85(0.051) = 8.61	0.051 - 1(0.051) = 0	0	0	-	-	-	-	-	-	-	-
y2	12	24.4 - 77.85(0.18) = 10.387	(0.18) - 1(0.18) = 0	1	0	-	-	-	-	-	-	-	-
S4	0	21.16 - 77.85(0.112) = 12.44	0.112 - 1(0.112) = 0	0	0	-	-	-	-	-	-	-	-
S5	0	200 - 77.85(1) = 122.15	1 - 1(1) = 0	0	0	-	-	-	-	-	-	-	-
S6	0	75.6 - 77.85(-0.18) = 89.613	-0.18 - 1(-0.18) = 0	0	0	-	-	-	-	-	-	-	-
y3	15	180 - 77.85(0) = 180	0	0	1	-	-	-	-	-	-	-	-
		Z _j	10	12	15								
		C _j	10	12	15								
		Z _j - C _j	0	0	0								

Notes


Maximise

$$\begin{aligned}
 Z &= C_B X_B \\
 &= (10 \times 77.85) + (12 \times 10.387) + (15 \times 180) + 0 + 0 + 0 + 0 \\
 &= 778.5 + 124.644 + 2,700 + 0 + 0 + 0 + 0 \\
 &= 3,603.144
 \end{aligned}$$

Therefore, Minimise $Z = -3,603.144$

Steps to Sum-up

1. Prepare the Table No. 1 and find the cost coefficients (Z_j) for different columns of A, i.e., for Y_j , by multiplying C_B column with entries of Y_j and adding the products, i.e., $Z_j - \sum C_{B_i} Y_{ij}$ and C_j is the most of coefficients for Z_j in objective function. Then, find the difference of Z_j and C_j ($Z_j - C_j$) for different columns of A.
 - (a) *Incoming Vector:* The j^{th} vector, i.e., y_j enters the basis if $Z_j - C_j$ is minimum where y_j is the j^{th} column of the coefficient matrix 'A'.
 - (b) *Outgoing Vector:* Find the ratio of X_{B_i}/y_{ij} (for all $Y_{ij} > 0$) for all the elements of the incoming vector. Then, the vector attached to the row having minimum ratio would be removed from the basis (if y_{ij} is greater than '0', otherwise that item should be neglected).
 - (c) *Key Element or Pivotal Element:* The element common to the incoming and outgoing vector is called key element or pivotal element.
2. The incoming vector has the coefficient of objective function in CB. Hence, make the pivotal element as "1" by dividing that row completely by the pivotal element. The other elements of the incoming vector other than pivotal element must be made "0"/"Zero". This can be done by deducting the elements of the respective rows by "K" times the adjusted pivotal row elements completely. The constant 'K' is chosen such that the pivotal columns element(s) is made "0"/"Zero". Then find Z_j and C_j in the usual manner of matrix method and if $Z_j - C_j$ is greater than or equal to zero for all columns, then the basic feasible solution is optimum otherwise the same procedure is to be continued.



Notes **Brief Steps of the simplex method:**

1. Convert the inequalities into equalities.
2. Identify the coefficients of equalities & put them into a matrix form.
3. Tabulate the data into 1st iteration of simplex method.
4. Reinstate entries in the 2nd iteration.
5. Find the 'Z' value.

Self Assessment

Fill in the blanks:

3. The element common to the incoming and outgoing vector is called
4. A variable represents unused resources and are added to original objective function with zero coefficients.
5. A Variable represents amount by which solution value exceed a resource.

4.2 Big 'M' Method

When the Linear Programming problem has greater than or equal to types of equations as constraints, it is obvious that some quantity should be deducted to convert them into equalities. The variables attached to it are known as 'surplus variables'. If the inequality is of the type greater than or equal to then add surplus variables which carry a negative sign and their cost coefficients in the objective function would be zero. As they would be considering slack or artificial variables initially as basic variables, and as the surplus variables carry negative signs, they represent vectors of identify matrix of the form,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ etc.,}$$

which cannot be taken into the basis. Hence, artificial variables along with the surplus variables would be added.

These artificial variables carry large negative values (-M) in the objective function. The artificial variables can also be added to the equation. This helps in choosing the initial variable or variables for the basis. The slack variables then would go to the basis whose cost coefficients are supposed to be zero (0) and the cost coefficients of artificial variables are supposed to be -M for maximization cases and +M for Minimization cases. The procedure for iteration follows when Simplex technique to obtain the optimum solution is used. Since the method involves artificial variables carrying -M as the cost coefficient, where M is a very large number which helps in the optimum solution finding and hence it is known as 'Big M Method'.

Steps:

1. Express the problem in the standard form by using slack, surplus and artificial variables.
2. Select slack variables and artificial variables as the initial basic variables with the cost coefficients as '0' or '-M' respectively.
3. Use simplex procedure for iterations & obtain optimum solution. During the iterations, one can notice that the artificial variables leave the basis first and then the slack variables with improved value of objective function at each iteration to obtain the optimum solution.



Example:

$$\text{Minimise 'Z' } = 4x_1 + 8x_2 + 3x_3 \quad [\text{Subject to constraints}]$$

$$x_1 + x_2 \geq 2$$

$$2x_1 + x_3 \geq 5$$

$$\text{Where, } x_1, x_2 \geq 0 \quad [\text{Non-negativity constraints}]$$

Solution:

Step 1: Conversion of minimization case into maximisation case.

$$\text{Therefore, 'Z' } = -4x_1 - 8x_2 - 3x_3 \quad [\text{Subject to constraints}]$$

Step 2: Conversion of inequalities into equalities adding slack variables and artificial variables.

$$-x_1 + x_2 - x_4 + x_6 = 2$$

$$2x_1 + x_3 - x_5 + x_7 = 5$$

Where, x_4 and x_5 are surplus variables, x_6 and x_7 are artificial variables.

Notes

Step 3: Bring the objective function into a standard form.

Therefore, Maximise 'Z' = $-4x_1 - 8x_2 - 3x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$

Step 4: Find out the matrix form of equalities.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	a ₁	a ₂	Min. Ratio
a ₁	-M	2	1	1	0	-1	0	1	0	2/1 = 2(KR) →
a ₂	-M	5	2	0	1	0	-1	0	1	5/2 = 2.5
		Z _j	-3M	-M	-M					
		C _j	-4	8	-3					
		Z _j - C _j	-3M+4	-M+8	-M+3					

(↑ KC)

Therefore,

$$\begin{aligned} Z &= C_B X_B \\ &= (-M \times 2) + (-M \times 5) \\ &= -2M - 5M \\ &= -7M \end{aligned}$$

Step 6: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	S ₁	S ₂	a ₁	a ₂	Min. Ratio
y ₁	-4	2/1 = 2	1/1 = 1	1	0	-1	0	1	0	2/0 = -
a ₂	-M	5 - 2(2) = 1	2 - 1(2) = -2	0 - 1(2) = -2	1 - 0(2) = 1	-	-	-	-	1/1 = 1(KR) →
		Z _j	-4	-4 + 2M	-M					
		C _j	-4	-8	-3					
		Z _j - C _j	0	4 + 2M	M+3					

(↑ KC)

Therefore,

$$\begin{aligned} Z &= (-4 \times 2) + (-M \times 1) \\ &= -8 - M \end{aligned}$$

Step 7: Second iteration of Simplex Method.

BV	C _B	X _B	y ₁	y ₂	y ₃	S ₁	S ₂	a ₁	a ₂	Min. Ratio
y ₁	-4	2 - 1(0) = 2	1 - 1(0) = 1	1 - (-2)(0) = 1	0	-	-	-	-	-
y ₃	-3	1/1 = 1	0	-2/1 = -2	1/1 = 1	-	-	-	-	-
		Z _j	-4	2	-3					
		C _j	-4	-8	-3					
		Z _j - C _j	0	10	0					

Therefore, Maximize

$$\begin{aligned}
 Z &= C_B X_B \\
 &= (-4 \times 2) + (-3 \times 1) \\
 &= -8 - 3 \\
 &= -11
 \end{aligned}$$

Notes

Therefore, Minimize

$$Z = 11$$



Example:

Minimise 'Z' = 3x₁ + 5x₂ [Subject to constraints]

$$2x_1 + 8x_2 \geq 40$$

$$3x_1 + 4x_2 \geq 50$$

$$x_1, x_2 \geq 0$$

[Non-negativity constraints]

Solution:

Step 1: Convert the above minimization case into maximisation case.

$$\text{Therefore, Maximise } Z = -3x_1 - 5x_2$$

Step 2: Convert the inequalities into equalities adding slack variables and artificial variables.

$$2x_1 + 8x_2 - x_3 + x_5 = 40$$

$$3x_1 + 4x_2 - x_4 + x_6 = 50$$

Where, x₃ & x₄ are surplus variables and x₅ and x₆ are artificial variables.

Step 3: Bring the objective function into a standard form.

$$\text{Maximise 'Z' } = -3x_1 - 5x_2 - 0x_3 + 0x_4 - Mx_5 - Mx_6$$

Step 4: Fit the data into a matrix form.

$$A = \begin{pmatrix} y_1 & y_2 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 2 & 8 & -1 & 0 & 1 & 0 \\ 3 & 4 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 40 \\ 50 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C _B	X _B	y ₁	y ₂	S ₁	S ₂	a ₁	a ₂	Min. Ratio
a ₁	-M	40	2	8	-1	0	1	0	40/8 = 5 (KR)→
a ₂	-M	50	3	4	0	-1	0	1	50/4 = 12.5
		Z _j	-5M	-12M					
		C _j	-3	-5					
		Z _j - C _j	-5M + 3	12M + 5					

Therefore,

$$\begin{aligned}
 Z &= C_B X_B \\
 &= (40x - M) + (50x - M)
 \end{aligned}$$

Notes

$$= -40M - 50M$$

$$= -90M$$

Step 6: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	a ₁	a ₂	Min. Ratio
y ₂	-5	40/8 = 5	2/8 = 0.25	8/8 = 1	-	-	-	-	5/0.25 = 20
a ₂	-M	50-5(4) = 30	3-0.25(4) = 2	4-1(4) = 0	-	-	-	-	30/2 = 15(KR) →
		Z _j	-2M - 1.25	-5					
		C _j	-3	-5					
		Z _j - C _j	-2M + 1.75	0					

(↑ KC)

Therefore,

$$Z = (-5 \times 5) + (-M \times 30)$$

$$= -30M - 25$$

Step 7: Third iteration of Simplex Method.

7	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	a ₁	a ₂	Min. Ratio
y ₂	-5	5-15 (0.25) = 1.25	0.25 - 1 (0.25) = 0	1-0 (0.25) = 1	-	-	-	-	-
y ₁	-3	30/2 = 15	2/2 = 1	0/2 = 0	-	-	-	-	-
		Z _j	-3	-5					
		C _j	-3	-5					
		Z _j - C _j	0	0					

$$Z = C_B X_B$$

$$= -6.25 - 45$$

Maximise

$$Z = -51.25$$

Therefore, Minimise

$$Z = 51.25$$



Example:

Minimise 'Z' = 12x₁ + 20x₂ [Subject to constraints]

$$6x_1 + 8x_2 \geq 100$$

$$7x_1 + 12x_2 \geq 120$$

Where, x₁, x₂ ≥ 0

Solution:

Step 1: Conversion of the minimization case into maximisation case.

Maximise Z = -12x₁ - 20x₂ [Subject to constraints]

Step 2: Convert the inequalities into equalities adding artificial and surplus variables.

$$6x_1 + 8x_2 - x_3 + x_5 = 100$$

$$7x_1 + 12x_2 - x_4 + x_6 = 120$$

Where, x₃ and x₄ are surplus variables and x₅ and x₆ are artificial variables.

Step 3: Bring the objective function into a standard form.

$$\text{Maximise 'Z'} = -12x_1 - 20x_2 - x_3 + x_4 - Mx_5 - Mx_6$$

Step 4: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 6 & 8 & -1 & 0 & 1 & 0 \\ 7 & 12 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 100 \\ 120 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	a ₁	a ₂	Min. Ratio
a ₁	-M	100	6	8	-1	0	1	0	100/8 = 12.5
A ₂	-M	120	7	12	0	-1	0	1	120/12 = 10(KR)→
		Z _j	-100M	-120M					
		C _j	12	-20					
		Z _j - C _j	-100M + 12	120M + 20					

(↑ KC)

Therefore,

$$\begin{aligned} Z &= C_B X_B \\ &= -100M - 120M \\ &= -220M \end{aligned}$$

Step 6: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	a ₁	a ₂	Min. Ratio
a ₁	-M	100 - 10(8) = 20	6 - 0.5(8) = 1.33	8 - 1(8) = 0	-	-	-	-	20/1.33 = 15.04(KR)→
y ₂	-20	120/12 = 10	7/12 = 0.58	12/12 = 1	-	-	-	-	10/0.58 = 17.24
		Z _j	-1.33M - 11.6 - 12	-20					
		C _j		-20					
		Z _j - C _j	-1.33M + 0.4	0					

(↑ KC)

Therefore,

$$\begin{aligned} Z &= C_B X_B \\ &= (-M \times 20) + (-20 \times 10) \\ &= -20M - 200 \end{aligned}$$

Step 7: Third iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	S ₁	S ₂	a ₁	a ₂	Min. Ratio
y ₁	-12	20/1.33 = 15.04	1.33/1.33 = 1	0	-	-	-	-	--
y ₂	-20	10 - 15.04(0.58) = 1.28	0.58 - 1(0.58) = 0	1	-	-	-	-	--
		Z _j	-12	-20					
		C _j	-12	-20					
		Z _j - C _j	0	0					

$$Z = C_B X_B$$

Notes

$$= (-12 \times 15.04) + (-20 \times 1.28)$$

$$= -180.48 - 25.6$$

Therefore, Maximise $Z = -206.08$

Therefore, Minimise $Z = 206.08$



Example: Solve the following LPP using Simplex Method.

Maximise 'Z' = $x_1 + 1.5x_2 + 2x_3 + 5x_4$ [Subject to constraints]

$$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$$

$$2x_1 + x_2 + x_3 + 5x_4 \leq 4$$

$$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$$

$$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$$

Where, $x_1, x_2, x_3, x_4 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities by adding slack variables and surplus variables.

$$3x_1 + 2x_2 + 4x_3 + x_4 + x_5 = 6$$

$$2x_1 + x_2 + x_3 + 5x_4 + x_6 = 4$$

$$2x_1 + 6x_2 - 8x_3 + 4x_4 + x_7 = 0$$

$$x_1 + 3x_2 - 4x_3 + 3x_4 + x_8 = 0$$

Where, x_5 and x_6 are slack variables and x_7 and x_8 are artificial variables.

Step 2: The standard form of objective function.

Maximise 'Z' = $x_1 + 1.5x_2 + 2x_3 + 5x_4 \pm 0x_5 \pm 0x_6 - Mx_7 - Mx_8$

Step 3: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & Y_4 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 3 & 2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 5 & 0 & 1 & 0 & 0 \\ 2 & 6 & -8 & 4 & 0 & 0 & 1 & 0 \\ 1 & 3 & -4 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = B = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Step 4: First iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	Y ₄	S ₁	S ₂	a ₁	a ₂	Min. Ratio
S ₁	0	6	3	2	4	1	1	0	0	0	6/2 = 3
S ₂	0	4	2	1	1	5	0	1	0	0	4/1 = 4
a ₁	-M	0	2	6	-8	4	0	0	1	0	0/6 = 0(KR)?
a ₂	-M	0	1	3	-4	3	0	0	0	1	0/3 = 0
		Z _j	-3M	-9M	12M	-7M					
		C _j	1	1.5	2	5					
		Z _j - C _j	-3M-1	-9M-1.5	+12M-2	-7M-5					

(↑ KC)

Therefore,

$$\begin{aligned}
 Z &= C_B X_B \\
 &= (0 \times 6) + (0 \times 4) + (-M \times 0) + (-M \times 0) \\
 &= 0
 \end{aligned}$$

Notes

Step 5: Second iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	Y ₄	S ₁	S ₂	a ₁	a ₂	Min. Ratio
S ₁	0	6-0 (2)=6 (2)	3- 0.33=2.34	2-1 (2)=0 (2)	4-1 (-1.33)=6.66 (2)	1- (0.67)=-0.34	-	-	-	-	6/-0.34 = -
S ₂	0	4-0 (1)=4 (1)	2- 0.33=1.67	1-1 (1)=0 (1)	1- (-1.33)=2.33 (1)	5- (0.67)=0.92	-	-	-	-	4/4.33
y ₂	1.5	0/6 = 0	2/6 = 0.33	6/6 = 1	-8/6 = -1.33	4/6 = 0.67	-	-	-	-	0/0.67 = 0
A ₂	-M	0-0 (3)=0	1-0.33 (3)=0.01	3-1 (3)=0	-4- (-1.33)(3)= 0.01	3 - 0.67 (3)=0.99	-	-	-	-	0/0.99 = 0 (KR)?→
		Z _j	0.5-0.01 M1	1.5	-2+0.01 M2	1.01-0.99 M5					
		C _j		1.5							
		Z _j - C _j	-0.5-0.01M	0	-4+0.01M	-3.99-0.99M					

(↑ KC)

$$\begin{aligned}
 \text{Therefore, } Z &= C_B X_B \\
 &= 0 + 0 + 0 + 0 \\
 &= 0
 \end{aligned}$$

Step 6: Third iteration of Simplex Method.

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	Y ₄	S ₁	S ₂	a ₁	a ₂	Min. Ratio
S ₁	0	6-0 (-0.34) = 6	2.34-0.01 (- 0.34) = 2.34	0	-0.34-1(-0.34) = 0	6.66- (-1) = 6.67	-	-	-	-	6/6.67 = 0.9 (KR) →
S ₂	0	4 - 0 (4.33) = 4 (4.33) = 1.63	1.67-0.01 (4.33) = 0	0	4.33-1(4.33) = 0	2.33- (-0.01)	-	-	-	-	4/2.37 = 1.69
y ₂	1.5	0-0 (0.67) = 0 (0.67) = 0.32	0.33-0.01(0.67) = 0	1	0.67-1 (0.67) = -1.32	-1.33-(-0.01) = -1.32	-	-	-	-	0/-1.32 = -
y ₄	5	0/0.99 = 0.9 = 0.01	0.01/0.9 = 1	0	0.99/0.99 = 0- 0.01	-0.01/0.09	-	-	-	-	0/-0.01 = -
		Z _j	0.53	1.5	5	2.03					
		C _j	1	1.5	5	2					
		Z _j - C _j	0.47	0	0	-4.03					

(↑ KC)

Therefore,

$$\begin{aligned}
 Z &= C_B X_B \\
 &= 0 \times 6 + 0 \times 4 + 1.5 \times 0 + 5 \times 0 \\
 &= 0
 \end{aligned}$$

Step 7: Fourth iteration of Simplex Method

BV	C _B	X _B	Y ₁	Y ₂	Y ₃	Y ₄	S ₁	S ₂	a ₁	a ₂	Min. Ratio
y ₃	2	6/6.67 = 0	2.34/6.67 = 0.35	0	6.67/6.67 = 1	0	-	-	-	-	-
S ₂	0	4- (0.9) (2.37) = 1.87	1.63 - 0.35 (2.37) = 0.80	0	2.37-1 (2.37) = 0	0	-	-	-	-	-
y ₂	1.5	0-0.9 (-1.32) = 1.19	0.32-0.35 (-1.32) = 0.78	1	-1.32 - 1 (-1.32) = 0	0	-	-	-	-	-
y ₄	5	0-0.9 (-0.01) = 0.009	0.01-0.35 (-0.01) = 0.014	0	-0.01-1 (-0.01) = 0	1	-	-	-	-	-
		Z _j	3.27	1.5	2	5					
		C _j	1	1.5	2	5					
		Z _j - C _j	2.27	0	0	0					

Notes

Therefore,

$$\begin{aligned}
 'Z' &= C_B X_B \\
 &= (2 \times 0.9) + (0 \times 1.87) + (1.5 \times 1.19) + (5 \times 0.009) \\
 &= 1.8 + 0 + 1.79 + 0.045 \\
 &= 3.64
 \end{aligned}$$



Did u know? Big 'M' Method is also known as 'Charnes' 'M' Technique.



Caution If the objective function z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The only visible difference between these two penalty is that the one will be designated by $-M$ for a maximization problem and $+M$ for a minimization problem, where $M > 0$.

Self Assessment

6. Solve the following LPP using the Big M method.

Maximise ' Z ' = $40x_1 + 60x_2$ [Subject to constraints]

$$2x_1 + x_2 \leq 70$$

$$x_1 + x_2 \geq 40$$

$$x_1 + x_2 \geq 40$$

$$x_1 + 3x_2 \leq 90$$

Where, $x_1, x_2 \geq 0$

7. Solve the following LPP using the Big M method.

Maximise ' Z ' = $5x_1 + 3x_2$ [Subject to constraints]

$$x_1 + x_2 \leq 6$$

$$2x_1 + 3x_2 \geq 3$$

$$x_1 \leq 3$$

$$x_2 \leq 3$$

Where, $x_1, x_2 \geq 0$

4.3 Unconstrained Variables

Sensitivity analysis involves 'what if?' questions. In the real world, the situation is constantly changing like change in raw material prices, decrease in machinery availability, increase in profit on one product, and so on. It is important to decision makers for find out how these changes affect the optimal solution. Sensitivity analysis can be used to provide information and to determine solution with these changes.

Sensitivity analysis deals with making individual changes in the coefficient of the objective function and the right hand sides of the constraints. It is the study of how changes in the coefficient of a linear programming problem affect the optimal solution.

We can answer questions such as,

1. How will a change in an objective function coefficient affect the optimal solution?
2. How will a change in a right-hand side value for a constraint affect the optimal solution?

For example, a company produces two products x_1 and x_2 with the use of three different materials 1, 2 and 3. The availability of materials 1, 2 and 3 are 175, 50 and 150 respectively. The profit for selling per unit of product x_1 is ₹ 40 and that of x_2 is ₹ 30. The raw material requirements for the products are shown by equations, as given below.

$$Z_{\max} = 40x_1 + 30x_2$$

Subject to constraints

$$4x_1 + 5x_2 \leq 175 \tag{a}$$

$$2x_2 \leq 50 \tag{b}$$

$$6x_1 + 3x_2 \leq 150 \tag{c}$$

where $x_1, x_2 \geq 0$

The optimal solution is

$$x_1 = ₹ 12.50$$

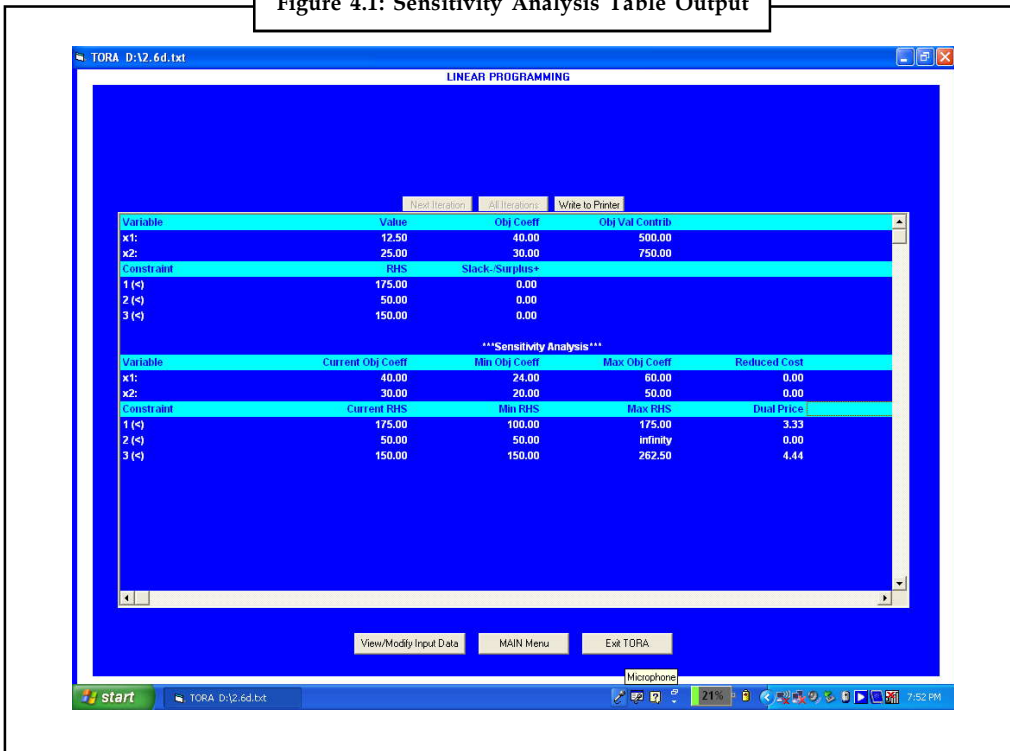
$$x_2 = ₹ 25.00$$

$$Z_{\max} = 40 \times 12.50 + 30 \times 25.00$$

$$= ₹ 1250.00$$

The problem is solved using TORA software and the output screen showing sensitivity analysis is given in Figure 4.1.


Figure 4.1: Sensitivity Analysis Table Output



Notes

4.3.1 Change in Objective Function Coefficients and Effect on Optimal Solution

Referring to the current objective coefficient of Figure 2.4, if the values of the objective function coefficient decrease by 16 (Min. obj. coefficient) and increase by 20 (Max. obj. coefficient) there will not be any change in the optimal values of $x_1 = 12.50$ and $x_2 = 25.00$. But there will be a change in the optimal solution, i.e. Z_{\max} .



Notes This applies only when there is a change in any one of the coefficients of variables i.e., x_1 or x_2 . Simultaneous changes in values of the coefficients need to apply for 100 Percent Rule for objective function coefficients.

For x_1 ,

$$\begin{aligned} \text{Allowable decrease} &= \text{Current value} - \text{Min. Obj. coefficient} \\ &= 40 - 24 \\ &= ₹ 16 \end{aligned} \tag{a}$$

$$\begin{aligned} \text{Allowable increase} &= \text{Max. Obj. coefficient} - \text{Current value} \\ &= 60 - 40 \\ &= ₹ 20.00 \end{aligned} \tag{b}$$

Similarly, For x_2 ,

$$\text{Allowable decrease} = ₹ 10.00 \tag{c}$$

$$\text{Allowable increase} = ₹ 20.00 \tag{d}$$

For example, if coefficient of x_1 is increased to 48, then the increase is $48 - 40 = ₹ 8.00$. From (b), the allowable increase is 20, thus the increase in x_1 coefficient is $8/20 = 0.40$ or 40%.

Similarly,

If coefficient of x_2 is decreased to 27, then the decrease is $30 - 27 = ₹ 3.00$.

From (c), the allowable decrease is 10, thus the decrease in x_2 coefficient is $3/10 = 0.30$ or 30%.

Therefore, the percentage of increase in x_1 and the percentage of decrease in x_2 is 40 and 30 respectively.

i.e. $40\% + 30\% = 70\%$

For all the objective function coefficients that are changed, sum the percentage of the allowable increase and allowable decrease. If the sum of the percentages is less than or equal to 100%, the optimal solution does not change, i.e., x_1 and x_2 values will not change.

But Z_{\max} will change, i.e.,

$$\begin{aligned} &= 48(12.50) + 27(25) \\ &= ₹ 1275.00 \end{aligned}$$

If the sum of the percentages of increase and decrease is greater than 100%, a different optimal solution exists. A revised problem must be solved in order to determine the new optimal values.

4.3.2 Change in the Right-hand Side Constraints Values and Effect on Optimal Solution

Notes

Suppose an additional 40 kgs of material 3 is available, the right-hand side constraint increases from 150 to 190 kgs.

Now, if the problem is solved, we get the optimal values as

$$x_1 = 23.61, x_2 = 16.11 \text{ and } Z_{\max} = 1427.78$$

From this, we can infer that an additional resources of 40 kgs increases the profit by = $1427.78 - 1250 = ₹ 177.78$

Therefore, for one kg or one unit increase, the profit will increase by

$$= 177.78/40$$

$$= ₹ 4.44$$

Dual price is the improvement in the value of the optimal solution per unit increase in the right-hand side of a constraint. Hence, the dual price of material 3 is ₹ 4.44 per kg.

Increase in material 2 will simply increase the unused material 2 rather than increase in objective function. We cannot increase the RHS constraint values or the resources. If the limit increases, there will be a change in the optimal values.

The limit values are given in Table 2.10, i.e., Min RHS and Max RHS values.

For example, for material 3, the dual price ₹ 4.44 applies only to the limit range 150 kgs to 262.50 kgs.

Where there are simultaneous changes in more than one constraint RHS values, the

100 Per cent Rule must be applied.

Reduced Cost

$$\text{Reduced cost/unit of activity} = \left(\begin{array}{c} \text{Cost of consumed} \\ \text{resources per unit} \\ \text{of activity} \end{array} \right) - \left(\begin{array}{c} \text{Profit per unit} \\ \text{of activity} \end{array} \right)$$

If the activity's reduced cost per unit is positive, then its unit cost of consumed resources is higher than its unit profit, and the activity should be discarded. This means that the value of its associated variable in the optimum solution should be zero.

Alternatively, an activity that is economically attractive will have a zero reduced cost in the optimum solution signifying equilibrium between the output (unit profit) and the input (unit cost of consumed resources).

In the problem, both x_1 and x_2 assume positive values in the optimum solution and hence have zero reduced cost.

Considering one more variable x_3 with profit ₹ 50

$$Z_{\max} = 40x_1 + 30x_2 + 50x_3$$

Subject to constraints,

$$4x_1 + 5x_2 + 6x_3 \leq 175 \quad (\text{a})$$

$$2x_2 + 1x_3 \leq 50 \quad (\text{b})$$

Notes

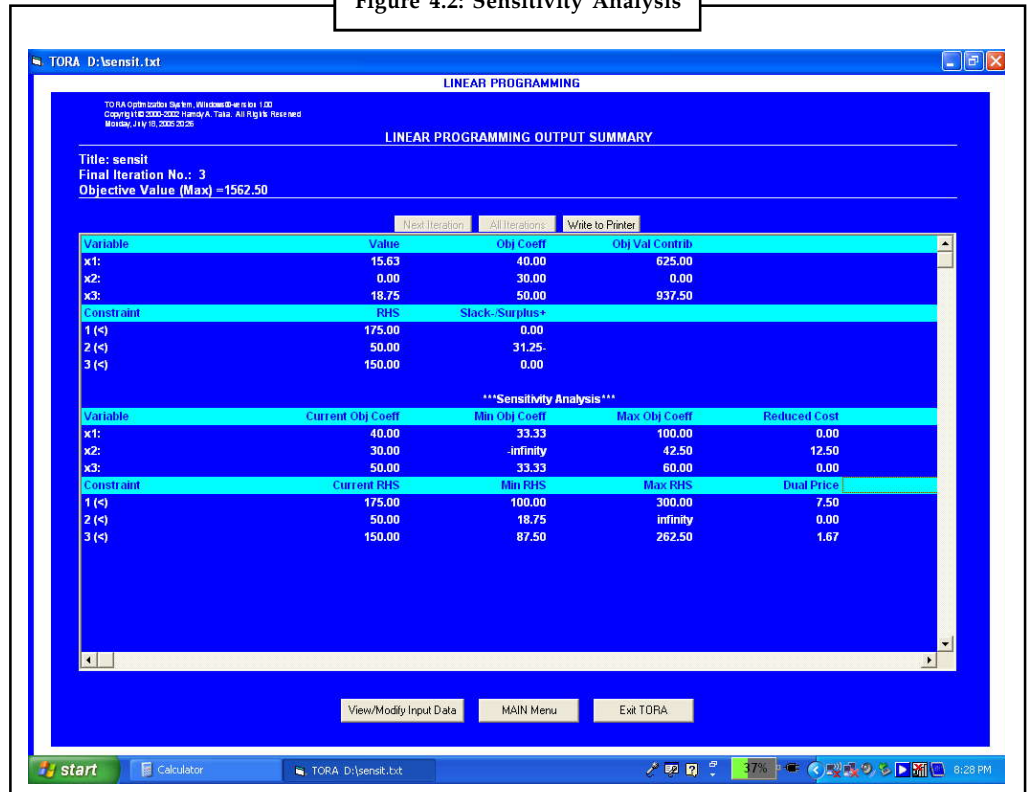
$$6x_1 + 3x_2 + 3x_3 \leq 150$$

(c)

where $x_1, x_2, x_3 \geq 0$

The sensitivity analysis of the problem is shown in the computer output below in Figure 4.2.

Figure 4.2: Sensitivity Analysis



The reduced cost indicates how much the objective function coefficient for a particular variable would have to improve before that decision function assumes a positive value in the optimal solution.

The reduced cost of ₹ 12.50 for decision variable x_2 tells us that the profit contribution would have to increase to at least $30 + 12.50 = 42.50$ before x_3 could assume a positive value in the optimal solution.

4.4 Special Cases in Linear Programming

Let us discuss special cases in linear programming such as infeasibility and unboundedness. We will examine here how these special problems can be recognised while solving linear programming problems by the simplex method.

4.4.1 Multiple or Alternative Optimal Solutions

In certain conditions, a given LPP may have more than one solution yielding the same optimal function value. Each of such optimal solutions is termed as alternative optimal Solutions.



Example:

Maximise $Z = 3x_1 + 2x_2$

Sub. to $x_1 \leq 40$
 $x_2 \leq 60$
 $3x_1 + 2x_2 \leq 180$
 $x_1, x_2 \geq 0$

Notes

Solution:

Maximise $'Z' = 3x_1 + 2x_2$
 Sub. to $x_1 + S_1 = 40$
 $x_2 + S_2 = 60$
 $3x_1 + 2x_2 + S_3 = 180$

First Iteration

BV	C _B	X _B	X ₁	X ₂	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	40	1	0	1	0	0	40/1 = 40 (KR) ?
S ₂	0	60	0	1	0	1	0	-
S ₃	0	180	3	2	0	0	1	180/3 = 60
		Z _j	0	0				
		C _j	3	2				
		Z _j - C _j	-3	-2				

Second Iteration

BV	C _B	X _B	X ₁	X ₂	Min. Ratio
X ₁	3	40	1	0	-
S ₂	0	60	0	1	60/1 = 60
S ₃	0	180-40(3) = 60	3-1(3) = 0	2-0(3) = 2	60/2 = 30 (KR) ?
		Z _j	3	0	
		C _j	3	2	
		Z _j - C _j	0	-2	

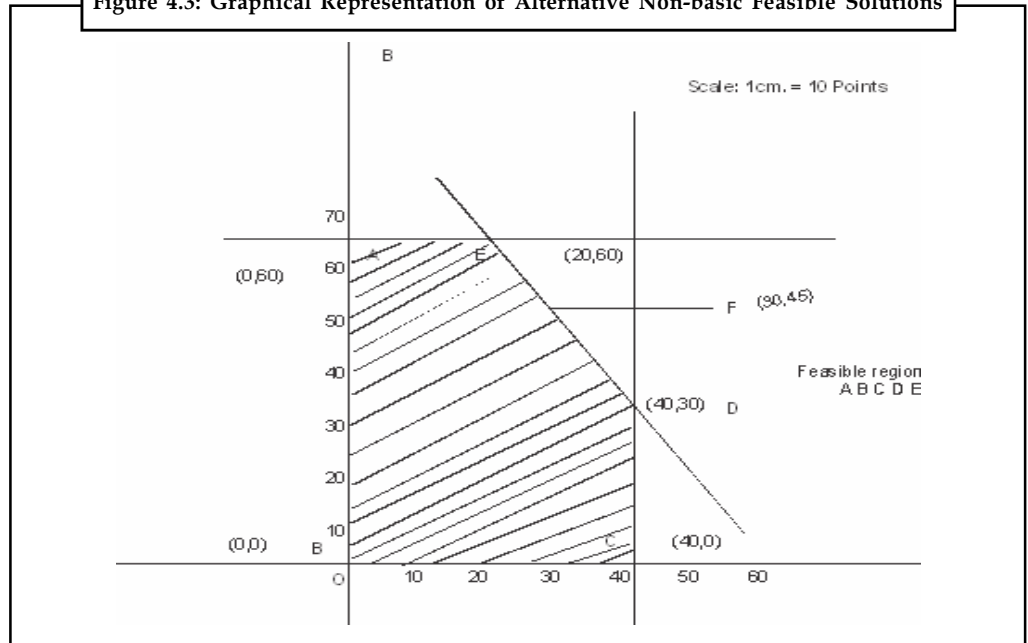
Third Iteration

BV	C _B	X _B	X ₁	X ₂	Min. Ratio
x ₁	3	40	1	0	-
S ₂	0	60-30(1) = 30	0-0(1) = 0	1-1(0) = 0	-
S ₃	0	60/2 = 30	0	2/2 = 1	-
		Z _j	3	2	
		C _j	3	2	
		Z _j - C _j	0	0	

Hence, optimum value for $Z = 180$ at $x_1 = 40$ and $x_2 = 30$. A graphical representation shows that the points D = (40, 30) and E = (20, 60) are optimal with $Z = 180$ units. Also, observe that every point on the line DE is optimal. For example, F (30, 45) gives $Z = 180$ units. Hence, the problem has infinite feasible solutions which are called alternative non-basic feasible solutions.

Notes

Figure 4.3: Graphical Representation of Alternative Non-basic Feasible Solutions



Example:

Maximise $Z' = 3x_1 + 2x_2$
 Sub. to, $x_1 \leq 4$
 $x_2 \leq 6$
 $3x_1 + 2x_2 \geq 18$
 $x_1, x_2 \geq 0$

Solution:

Maximise $Z' = 3x_1 + 2x_2$
 Sub. to $x_1 + S_1 = 4$
 $x_2 + S_2 = 6$
 $3x_1 + 2x_2 + S_3 = 18$

Where, S_1, S_2 and S_3 are slack variables.

First Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	1	0	1	0	0	$4/1 = 4$ (KR) ?
S_2	0	6	0	1	0	1	0	-
S_3	0	18	3	2	0	0	1	$18/3 = 6$
		Z_j	0	0				
		C_j	3	2				
		$Z_j - C_j$	-3	-2				

Second Iteration

Notes

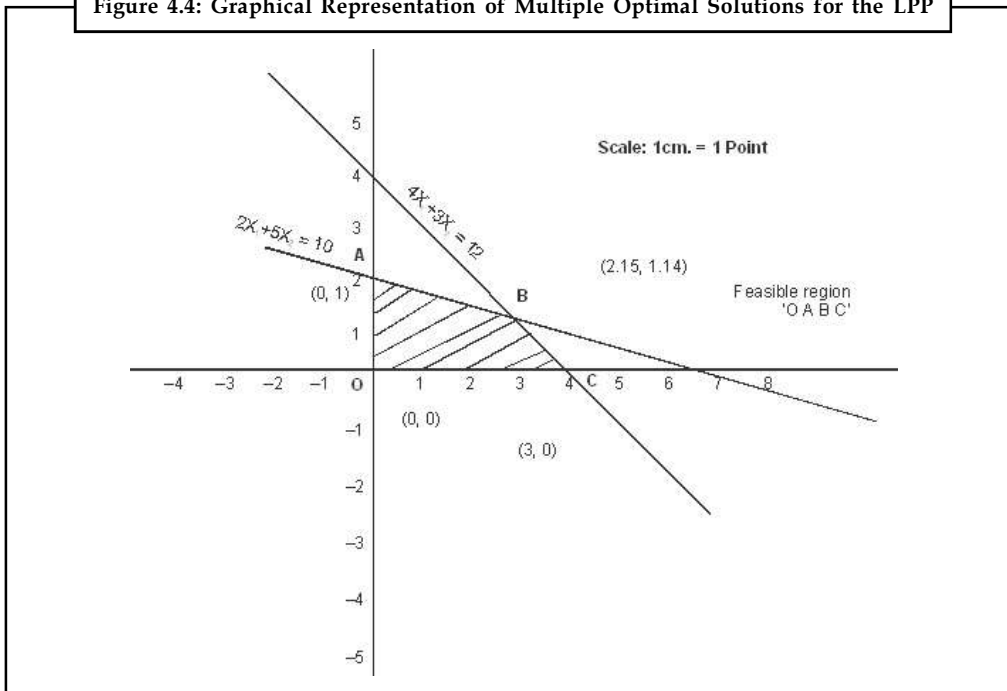
BV	C _B	X _B	X ₁	X ₂	S ₁	S ₂	S ₃	Min. Ratio
x ₁	0	4	1	0	-	-	-	-
S ₂	0	6	0	1	-	-	-	6/1 = 6
S ₃	0	18-4(3) = 6	3-1(3) = 0	2-0(3) = 2	-	-	-	6/2 = 3 (KR) →
		Z _j	3	0				
		C _j	3	2				
		Z _j - C _j	0	-2				

Third Iteration

BV	C _B	X _B	X ₁	X ₂	S ₁	S ₂	S ₃	Min. Ratio
X ₁	3	4	1	0	-	-	-	-
S ₂	0	6-3(1) = 3	0	1-1(1) = 0	-	-	-	-
x ₂	2	6/2 = 3	0	2/2 = 1	-	-	-	-
		Z _j	3	2				
		C _j	3	2				
		Z _j - C _j	0	0				

The solution is optimal with $Z = 18$ at $x_1 = 4$ and $x_2 = 3$. A graphical representation of the problem reveals the multiple optimal solutions for the LPP.

Figure 4.4: Graphical Representation of Multiple Optimal Solutions for the LPP



4.4.2 Unbounded Solutions

Sometimes an LP problem will not have a finite solution. This means when no one or more decision variable values and the value of the objective function (maximization case) are permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded.

Notes



Example:

Maximise $Z' = 3x + 2y$
 Sub. to $x - y \leq 15$
 $2x - y \leq 40$
 $x, y \geq 0$

Where, S_1 and S_2 are slack variables.

First Iteration

BV	C_B	X_B	X	y	S_1	S_2	Min. Ratio
S_1	0	15	1	-1	1	0	$15/1 = 15$ (KR) ?
S_2	0	20	2	-1	0	1	$20/2 = 10$
		Z_j	0	0			
		C_j	3	2			
		$Z_j - C_j$	-3	-2			

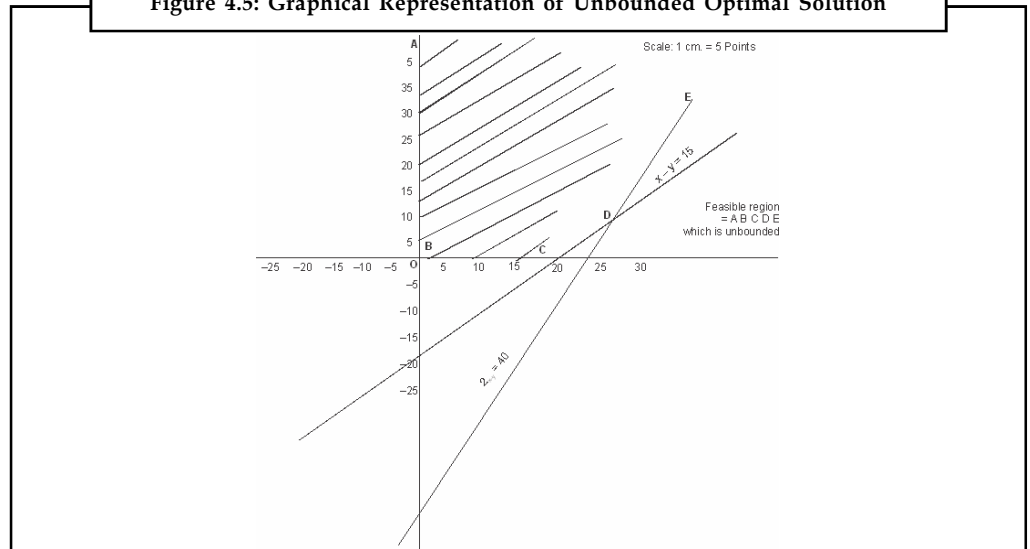
Second Iteration

BV	C_B	X_B	X	Y	S_1	S_2	Min. Ratio
S_1	0	$15 - 10(1) = 5$	$1 - 1(1) = 0$	$-1 + 1/2(1) = -0.5$	-	-	y' s are negative
X	3	10	1	-1/2	-	-	
		Z_j	3	-1.5			
		C_j	3	2			
		$Z_j - C_j$	0	-3.5			

At the end of second iteration y should enter the basis to improve the solution of Z. But there is no vector leaving the basis since y's are negative. Hence, the solution is unbounded optimum solution.

Note that the unbounded feasible region by graph is ABCDE. As the point A and E are extended, the value of Z increases. Hence, the solution is unbounded optimal solution.

Figure 4.5: Graphical Representation of Unbounded Optimal Solution





Example:

Maximise $Z' = 4x_1 + x_2 + 3x_3 + 5x_4$

Sub. to $-4x_1 + 6x_2 + 5x_3 - 4x_4 \leq 20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Solution:

Maximise $Z' = 4x_1 + x_2 + 3x_3 + 5x_4$

$-4x_1 + 6x_2 + 5x_3 - 4x_4 + S_1 = 20$

$3x_1 - 2x_2 + 4x_3 + x_4 + S_2 = 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 + S_3 = 20$

Where, S_1, S_2 and S_3 are slack variables.

First Iteration

BV	C _B	X _B	X ₁	X ₂	x ₃	x ₄	S ₁	S ₂	S ₃	Min. Ratio
S ₁	0	20	-4	6	5	-4	-	-	-	
S ₂	0	10	3	-2	4	1	-	-	-	10/1 = 10
S ₃	0	20	8	-3	3	2	-	-	-	20/2 = 10
		Z _j	0	0	0	0	0			
		C _j	4	4	1	3	5			
		Z _j - C _j	-0	4	-1	-3				

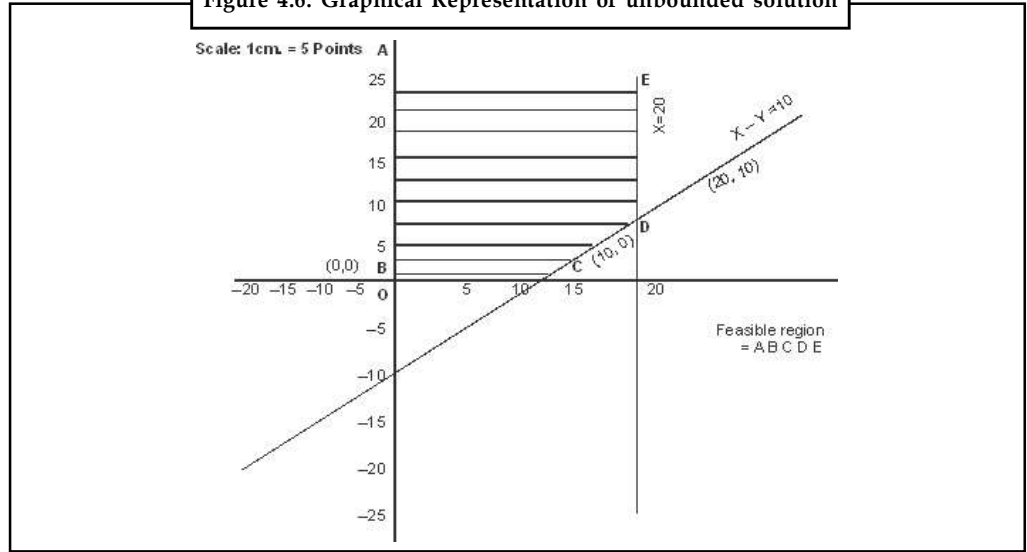
Second Iteration

BV	C _B	X _B	x ₁	x ₂	x ₃	X ₄	Min. Ratio
S ₁	0	60	12	0	11	0	
S ₂	0	10	-1	-0.5	2.5	0	10/1 = 10
S ₃	0	10	4	-1.5	1.5	1	20/2 = 10
		Z _j	20	-7.5	7.5	5	
		C _j	4	1	3	5	
		Z _j - C _j	16	-8.5	4.5	0	

At the end of second iteration, observe that x_3 should enter the basis. But there is no chance for outgoing vector from the basis since y 's are zero or negative. Hence, the solution is unbounded.

Notes

Figure 4.6: Graphical Representation of unbounded solution



4.4.3 Infeasibility

Infeasibility is a condition that arises when constraints are inconsistent (mutually exclusive) i.e. no value of the variables satisfy all the constraint simultaneously. This results in infeasible solution. If two or more constraints of a linear programming problem are mutually conflicting, it does not have a feasible solution. Let us take a problem to illustrate infeasibility.



Example: The Reddin Hardware Ltd. is producing two products, A and B. The profit contribution of product A is ₹ 5 per unit and of product B ₹ 4 per unit. Both the products go through the processing and assembly departments. Product A takes two minutes in the processing department and two minutes in the assembly department. Product B takes one minute in the processing department and one and a half minute in the assembly department. The maximum capacity of the processing department is 10,000 worker-minutes and of the assembly department 12,000 worker-hours. The marketing department has informed that a contract has been made with a hardware chain store for the supply of 6,500 units of product A and that there is no other demand for this product. There is no marketing constraint in the case of product B. What is the optimum product mix for the company?

The linear programming mode for the Reddin Hardware is as follows:

Let

$$x_1 = \text{product A}$$

$$x_2 = \text{product B}$$

Maximise

$$'Z' = 5x_1 + 4x_2$$

Objective Function

$$2x_1 + 1x_2 \leq 10,000$$

Processing Constraint

$$2x_1 + x_2 \leq 12,000$$

Assembly Constraint

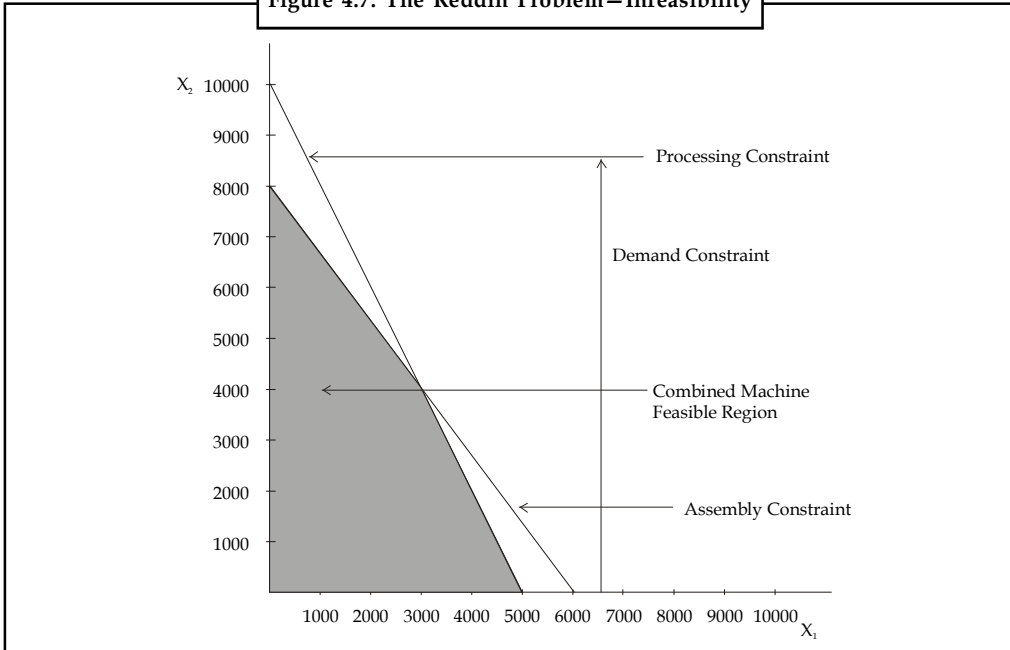
$$x_1 = 6,500$$

Demand Constraint

$$x_1 \geq 0$$

Non-negativity Constraint

Figure 4.7: The Reddin Problem – Infeasibility



The graphical presentation of the Reddin Problem in Figure shows that the combined machine feasibility region does not permit the production of 6,500 units of product A. Thus, the demand constraint and the machine capacity constraints are mutually conflicting, resulting in infeasibility.

Let us try to solve this problem by the simplex method. It is formulated below in the standard simplex form.

Row 1: P- $5x_1 - 4x_2 - 50a_1 = 0$

Row 2: $2x_1 + 1x_2 + s_1 = 10,000$

Row 3: $2x_1 + \frac{3}{2}x_2 + s_2 = 12,000$

Row 4: $x_1 + a_1 = 6,500$

Artificial variable a1 has been assigned a coefficient of 50 in the objective function.

Towards The First Basis

Row 4 × 50: $50x_1 + 50a_1 = 325,000$

Old Row 1: $Z - 5x_1 - 4x_2 - 50a_1 = 0$

New Row 1: $Z + 45x_1 - 4x_2 = 325,000$

Table 4.2: The First Basis

Row 1: P	$+ 45x_1$	$- 4x_2$		$= 325,000$
Row 2:	$2x_1$	$+ x_2$	$+ s_1$	$= 10,000$
Row 3:	$2x_1$	$+ x_2$	$+ s_2$	$= 12,000$
Row 4:	x_1		$+ a_1$	$= 6,500$

Notes

Table 4.3: The Second Basis

Row 1: P +	x_1		$-s_2$	= 357,000
Row 2:	x_1		$+s_1 - s_2$	= 2,000
Row 3:	x_1	$+x_2$	$+s_2$	= 8,000
Row 4:	x_1		$+a_1$	= 6,500

Since there is no variable with a negative coefficient in Row 1, no further improvement in profit is possible. But artificial variable a_1 continues to exist in Row 4 and it is also basic variable. This means that the optimum solution has not been reached. The problem has no solution because of infeasibility.



Case Study

Linear Programming & Technical Accounting

A company manufactures 2 products a 10 cu. Ft & 6 cu. ft. refrigerator. The demand for the former has been estimated at 15,000 and for the later at 24,000.

The production process is broken down into 3 stages:

1. Shell production
2. Motor assembly
3. Refrigerator assembly

The production manager estimates that the shell production unit can through put up to 36,000 smaller refrigerators, but only half of this quantity of the 10 cu. ft. product. Similarly, the motor assembly unit can through put 30,000 smaller motors or 80% of this quantity of the other product. The assembly department has no restrictions as labour unskilled is required. The accounting function has produced the standard cost breakdown as follows:

Standard Costs of Production

(in Dollars)

Items	10 cu. ft	6 cu. ft.
Direct materials	30	24
Direct labour	16	10
Variable overhead	14	11
Marginal cost	60	45
Selling price	90	65
Contribution	30	20

Fixed overhead charges are estimated as \$ 4,00,000.

Traditional accounting would entail the manufacture of 15,000 of the 10 cu ft. refrigerators before the shell production unit capacity is exceeded only 6,000 smaller refrigerator would be manufactured. On the basis of direct labour, \$3,20,000 of fixed overheads would be attributed to 10 cu. ft. & 80,000 to the 6 cu. ft.

Contd...

The net profit would thus be

Items	10 cu. ft	6 cu. Ft.	Total (in '000 dollars)
Contribution	450	120	570
Overhead allocation	320	80	400
Net profit	130	40	170

But this is an over simplification when a resource is limiting. The scarce resource is shell production and contribution per unit is \$ 20 for the cu. ft & \$ 15 for the 10 cu. ft. This suggests that the 6 cu. ft. is more profitable and should be produced to its demand of 24,000. For this, the net profit would stand at \$2,24,000.

But both are misleading because profitability is not directly related to contribution per unit product or per unit resource. Hence an examination from the linear programming view point was carried out to arrive at a logical solution.

Questions:

1. Formulate the given problem as a linear programming problem.
2. Solving the problem by the revised simplex technique to determine the maximum profit.

Self Assessment

Fill in the blanks:

8. The optimal solution of the primal problem appears under the variables in the row of the final simplex table associated with the dual problem.
9. The analysis involves "what if" questions.
10. The original linear programming problem is known as problem.

Summary

In this unit, you learned the mechanics of obtaining an optimal solution to a linear programming problem by the simplex method. The simplex method is an appropriate method for solving a \leq type linear programming problem with more than two decision variables. Two phase and M-method are used to solve problems of \geq or \leq type constraints. Further, the simplex method can also identify multiple, unbounded and infeasible problems.

Keywords

Artificial Variables: Temporary slack variables which re used for calculations.

Simplex Method: A method which examines the extreme points in a systematic manner, repeating the same set of steps of the algorithms until an optimal solution is reached.

Slack Variables: Amount of unused resources.

Surplus Variables: A surplus variable represents the amount by which solution exceeds a resource.

Unconstrained Variable: The variable having no non-negativity constraint.

Notes

Review Questions

1. Explain the simplex procedure to solve the Linear Programming Problem.
2. How do you recognize optimality in the simplex method?
3. Can a vector that is inserted at one iteration in simplex method be removed immediately at the next iteration? When can this occur and when is it impossible?
4. Explain briefly the term 'Artificial' variables.
5. Explain the use of artificial variables in L.P..
6. Maximize: $Z = x_1 + x_2$
 Subject to $x_1 + 5x_2 \leq 5$
 $2x_1 + x_2 \leq 4$ with x_1, x_2 non-negative.
7. Minimize: $Z = 3x_1 + 4x_2$
 Subject to $2x_1 + x_2 \leq 6$
 $2x_1 + 3x_2 \geq 9$
 With, x_1, x_2 non-negative.
8. Minimize: $Z = x_1 + 2x_2$
 Subject to $x_1 + 3x_2 \leq 11$
 $2x_1 + x_2 \geq 9$
 With, x_1, x_2 non-negative.
9. Maximize $Z = -x_1 - x_2$
 Subject to $x_1 + 2x_2 \geq 5000$
 $5x_1 + 3x_2 \geq 12000$
 With, x_1, x_2 non-negative
10. Maximize $Z = 2x_1 + 3x_2 + 4x_3$
 $x_1 + x_2 + x_3 \geq 1$
 $x_1 + x_2 + 2x_3 = 2$
 $2x_1 + 2x_2 + x_3 \geq 4$
 With all variables non-negative
11. Minimize: $Z = 14x_1 + 13x_2 + 11x_3 + 13x_4 + 13x_5 + 12x_6$
 Subject to $x_1 + x_2 + x_3 = 1200$
 $x_4 + x_5 + x_6 = 1000$
 $x_1 + x_4 = 1000$
 $x_2 + x_5 = 700$
 $x_3 + x_6 = 500$
 With all variables non-negative.

12. Discuss the role of sensitivity analysis in linear programming.
13. In sensitivity analysis, explain
 - (a) The effect of change of objective function coefficients.
 - (b) The effect of change in the right hand side of constraints.

Notes

Answers: Self Assessment

- | | |
|-----------------------------------|----------------|
| 1. $z = 21$ | 2. $z = 28$ |
| 3. key element or pivotal element | 4. slack |
| 5. surplus | 6. $z = 2,100$ |
| 7. $z = 7.45 - 3.51m$ | 8. slack, last |
| 9. sensitivity | 10. Primal |

Further Readings



Books

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Unit 05: Assignment Problem

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Objectives

After studying this unit, you will be able to:

- understand methods for solving assignment problem
- solve special cases in assignment problem

Introduction

Assignment problems in Operations Research are a critical subset of optimization that revolves around the efficient allocation of a set of resources to a corresponding set of tasks. This class of problems finds applications in various real-world scenarios, including workforce scheduling, transportation planning, project assignment, and more. The fundamental objective is to determine the optimal assignment that minimizes or maximizes a certain objective while adhering to a set of constraints. These problems are pivotal in decision-making processes across industries, as they help organizations streamline their operations, enhance resource utilization, and ultimately improve overall efficiency.

At the heart of assignment problems lies the allocation of resources to tasks in the most optimal way possible. Mathematically, these problems can be framed as a matrix, where each row represents a resource, each column signifies a task, and the entries in the matrix correspond to the cost, profit, or any other relevant measure associated with assigning a particular resource to a specific task. The challenge is to determine the assignment that leads to the optimal total cost, maximum profit, or minimum time, depending on the nature of the problem.

One of the classic forms of assignment problems is the linear assignment problem, where the objective is to find the assignment that minimizes the total cost. This problem can be solved using algorithms like the Hungarian algorithm or through network flow formulations. The Hungarian algorithm iteratively identifies a set of independent assignments that collectively cover all tasks and resources while minimizing the total cost. This algorithm's complexity is relatively low, making it suitable for practical applications involving a moderate number of resources and tasks.

In scenarios where the objective is to maximize the profit or a similar measure, the assignment problem can be adapted accordingly. This adaptation is common in cases such as job assignment, where the aim is to maximize the skills match between employees and tasks. These maximization-

based assignment problems can also be solved using algorithms similar to the Hungarian algorithm, with slight modifications to the cost matrix and assignment rules.

Furthermore, there are cases where additional constraints come into play, making the assignment problems more intricate. These constraints could involve limitations on the number of assignments for each resource or task, requirements for specific tasks to be assigned to particular resources, or even restrictions on the compatibility of certain resource-task combinations. Such constraints often lead to the formulation of variant assignment problems, including the quadratic assignment problem, the multiple-objective assignment problem, and more. These variants introduce complexities that require more advanced optimization techniques such as integer programming, heuristics, and metaheuristic algorithms.

In the realm of operations research, assignment problems have far-reaching implications. In transportation planning, these problems can help determine the most efficient way to allocate goods to distribution centers or delivery routes to vehicles, minimizing transportation costs and delivery times. In project management, assignment problems assist in allocating tasks to available team members, optimizing skill utilization and project timelines. In educational institutions, these problems can be used to match students to courses based on their preferences and capacities, ensuring efficient resource utilization and student satisfaction.

In conclusion, assignment problems constitute a crucial field within Operations Research, addressing the optimization of resource-task allocations across a wide array of applications. These problems involve finding the optimal assignment that minimizes costs, maximizes profits, or meets other specific objectives while considering various constraints. The algorithms and techniques developed for solving assignment problems have real-world significance, impacting industries ranging from logistics and manufacturing to education and healthcare. As technology advances, the scope and complexity of assignment problems continue to grow, underscoring the ongoing relevance and importance of this field in enhancing operational efficiency and decision-making processes.

5.1 Mathematical Model of Assignment Problem

The assignment problem in Operations Research can be formally defined through a mathematical model. The goal of the assignment problem is to find the optimal assignment of a set of resources to a set of tasks, such that a certain objective is minimized or maximized. Let's delve into the mathematical model of the assignment problem:

Given:

- n resources: R_1, R_2, \dots, R_n
- n tasks: T_1, T_2, \dots, T_n
- A cost (or profit) matrix C of size $n \times n$, where C_{ij} represents the cost (or profit) associated with assigning resource R_i to task T_j .

Objective: Minimize the total cost (or maximize the total profit) of the assignment.

Decision Variables: Let X_{ij} be a binary decision variable:

- $X_{ij}=1$ if resource R_i is assigned to task T_j
- $X_{ij}=0$ otherwise

Mathematical Formulation: The assignment problem can be mathematically formulated as an integer linear programming (ILP) problem. For minimization problems (e.g., minimizing cost), the objective function and constraints are as follows:

Objective Function: Minimize the total cost: $\text{Minimize } Z = \sum \sum C_{ij} \cdot X_{ij}$

Constraints: Each resource can be assigned to only one task: $\sum X_{ij} = 1$ for $i=1, 2, \dots, n$

Each task can be assigned to only one resource: $\sum X_{ij} = 1$ for $j=1, 2, \dots, n$

The decision variables are binary: $X_{ij} \in \{0, 1\}$ for $i, j=1, 2, \dots, n$

The above ILP formulation ensures that each resource is assigned to exactly one task, and each task is assigned to exactly one resource. The objective function aims to minimize the total cost by selecting appropriate assignments of resources to tasks.

For maximization problems (e.g., maximizing profit), the objective function remains the same, but the constraints need to be modified accordingly to ensure that each resource is assigned to exactly one task and each task is assigned to exactly one resource.

Solving the Assignment Problem: The assignment problem can be solved using various optimization techniques such as the Hungarian algorithm, linear programming solvers, network flow algorithms, and more. The Hungarian algorithm is particularly efficient for solving the assignment problem and has a time complexity of $O(n^3)$, making it suitable for moderate-sized problems. For larger problems or variants with additional constraints, more advanced techniques like integer programming and metaheuristic algorithms may be necessary.

In conclusion, the mathematical model of the assignment problem provides a structured framework to optimize the allocation of resources to tasks with the goal of minimizing costs or maximizing profits. This model forms the basis for solving a wide range of real-world optimization challenges across industries.

5.2 Solution Methods of Assignment Problem

The assignment problem is a fundamental optimization problem in Operations Research, focusing on the optimal allocation of a set of resources to a corresponding set of tasks, while minimizing or maximizing a certain objective. There are several solution methods for the assignment problem, each with its own advantages, limitations, and applicability. In this comprehensive overview, we'll explore various solution methods, ranging from classical techniques to modern algorithms and heuristics.

Hungarian Algorithm:

The Hungarian algorithm is one of the most widely used methods for solving the assignment problem. It provides an efficient solution for finding the optimal assignment in

$O(n^3)$ time complexity. The algorithm iteratively constructs a set of independent assignments that collectively cover all tasks and resources while minimizing the total cost.

Steps:

Subtract the smallest value in each row from every element in that row.

Subtract the smallest value in each column from every element in that column.

Draw the minimum number of horizontal and vertical lines to cover all zeros in the cost matrix. These lines indicate the potential assignments.

Adjust the matrix: Subtract the smallest uncovered value from every element that is not covered by lines and add it to elements where lines intersect.

Repeat the previous two steps until an optimal assignment is found or no more lines can be drawn.

Determine the optimal assignment based on the starred and primed zeros.

Linear Programming (LP) Relaxation:

Another approach to solving the assignment problem is by formulating it as a linear programming problem and then solving its relaxation. The relaxation allows fractional assignments, which are then rounded to obtain integer solutions. While not guaranteed to give an optimal solution for the integer problem, this approach is useful when integrated with branch-and-bound algorithms.

Augmenting Path Algorithms:

Augmenting path algorithms, such as the shortest augmenting path algorithm, are effective for solving bipartite assignment problems. They use graph theory concepts to find augmenting paths in a bipartite graph. The Ford-Fulkerson method, based on augmenting paths, can be adapted to solve the assignment problem. These algorithms generally work well for small to moderate-sized problems but may face challenges with larger instances.

Network Flow Algorithms:

The assignment problem can be framed as a network flow problem, where resources and tasks become nodes, and capacities represent the assignment limits. By transforming the problem into a network flow model, algorithms like the capacity scaling algorithm or the Edmonds-Karp algorithm can be employed. Network flow algorithms provide an efficient way to solve larger instances of the assignment problem.

Integer Linear Programming (ILP):

Formulating the assignment problem as an ILP and solving it using ILP solvers like CPLEX or Gurobi is a versatile approach. ILP formulations can handle complex constraints, including task and resource limits, compatibility constraints, and more. However, solving large ILP instances can become computationally demanding.

Branch-and-Bound Algorithm:

The branch-and-bound technique is a general-purpose approach used to solve integer programming problems, including the assignment problem. It involves recursively partitioning the solution space into smaller subproblems and pruning branches that cannot lead to an optimal solution. This method combines linear programming relaxation and enumeration to converge to the optimal solution efficiently, even for larger instances.

Heuristic Approaches:

Heuristic methods aim to find good solutions quickly without guaranteeing optimality. They are useful for large or complex instances where exact methods may be computationally infeasible. Popular heuristics for the assignment problem include the "row-reduction" method, which repeatedly reduces rows and columns, and the "column reduction" method, which reduces columns first.

Metaheuristic Algorithms:

Metaheuristic algorithms, like genetic algorithms, simulated annealing, and particle swarm optimization, provide flexible and robust approaches for solving the assignment problem. These algorithms explore the solution space by iteratively refining and searching for better solutions. While not guaranteed to find the global optimum, they often find satisfactory solutions for real-world scenarios.

Approximation Algorithms:

Approximation algorithms offer fast solutions with provable bounds on how far their solutions are from optimality. The assignment problem can be approximated using techniques like greedy algorithms or local search algorithms. These methods are particularly useful when quick solutions are required and optimality is not critical.

In conclusion, the assignment problem offers a wide range of solution methods, each catering to different problem sizes, complexities, and solution quality requirements. While exact methods like the Hungarian algorithm and ILP formulations guarantee optimality, heuristic, metaheuristic, and approximation methods provide scalable solutions for larger instances and practical scenarios. The choice of method depends on the problem characteristics, computational resources, and the desired balance between solution quality and computation time.

5.3 Hungarian Method for Solving Assignment Problem

The Hungarian method (minimization case) can be summarized in the following steps:

Step 1: Develop the cost matrix from the given problem If the number of rows are not equal to the number of columns, then add required number of dummy rows or columns. The cost element in dummy rows/columns are always zero.

Step 2: Find the opportunity cost matrix

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row, and

(b) In the reduced matrix obtained from 2(a), identify the smallest element in each column and then subtract

it from each element of that column. Each row and column now have at least one zero element.

Step 3: Make assignments in the opportunity cost matrix The procedure of making assignments is as follows:

(a) First round for making assignments

- Identify rows successively from top to bottom until a row with exactly one zero element is found.

Make an assignment to this single zero by making a square () around it. Then cross off (×) all other zeros in the corresponding column.

- Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make assignment to this single zero by making a square () around it and then cross off (×) all other zero elements in the corresponding row.

(b) Second round for making assignments

- If a row and/or column has two or more unmarked zeros and one cannot be chosen by inspection,

then choose zero element arbitrarily for assignment.

- Repeat steps (a) and (b) successively until one of the following situations arise.

Step 4: Optimality criterion

(a) If all zero elements in the cost matrix are either marked with square () or are crossed off (×) and there is

exactly one assignment in each row and column, then it is an optimal solution. The total cost associated with this solution is obtained by adding the original cost elements in the occupied cells.

(b) If a zero element in a row or column was chosen arbitrarily for assignment in Step 4(a), there exists

an alternative optimal solution.

(c) If there is no assignment in a row (or column), then this implies that the total number of assignments

are less than the number of rows/columns in the square matrix. In such a situation proceed to Step 5.

Step 5: Revise the opportunity cost matrix Draw a set of horizontal and vertical lines to cover all the zeros in the revised cost matrix obtained from Step 3, by using the following procedure: (a)

For each row in which no assignment was made, mark a tick (9)

(b) Examine the marked rows. If any zero element is present in these rows, mark a tick (9) to the respective

columns containing zeros.

(c) Examine marked columns. If any assigned zero element is present in these columns, tick (9) the respective rows containing assigned zeros.

(d) Repeat this process until no more rows or columns can be marked.

(e) Draw a straight line through each marked column and each unmarked row.

If the number of lines drawn (or total assignments) is equal to the number of rows (or columns), the current solution is the optimal solution, otherwise go to Step 6.

Step 6: Develop the new revised opportunity cost matrix

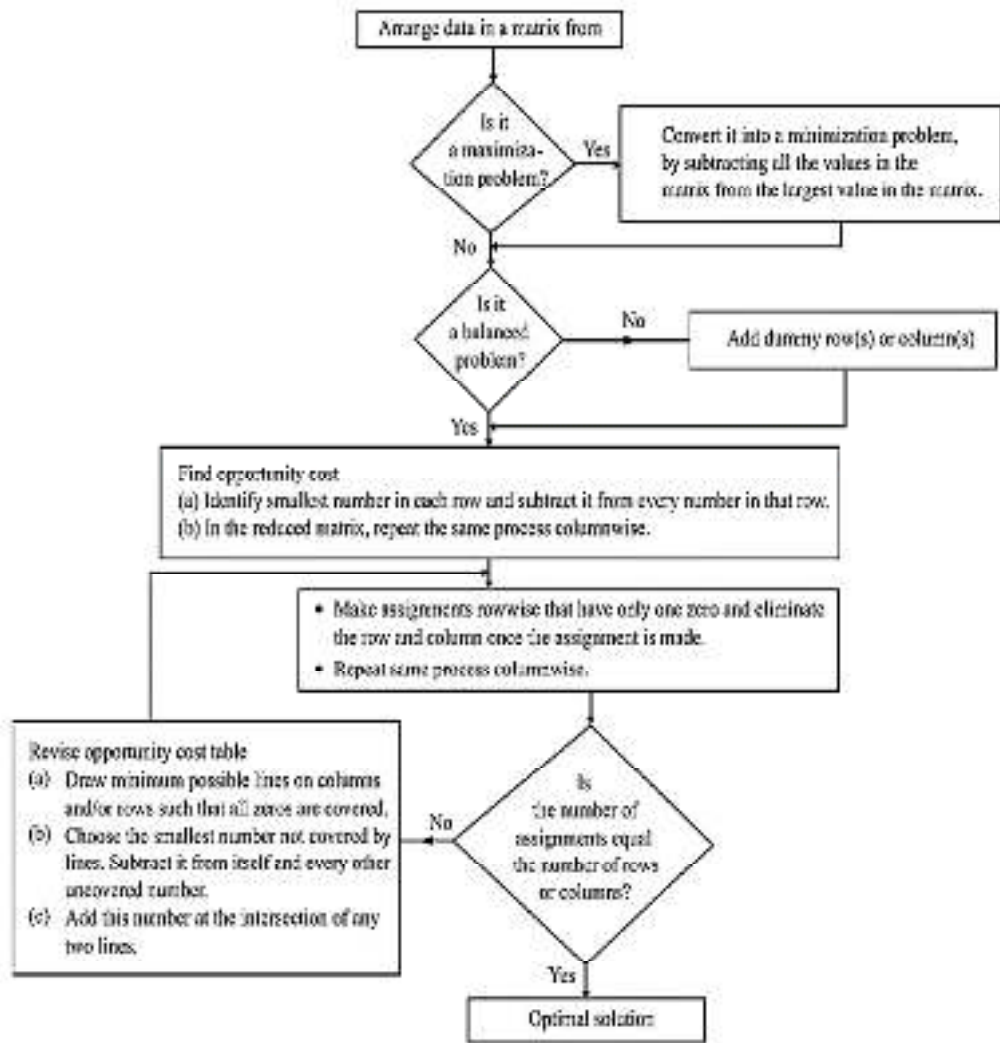
(a) Among the elements in the matrix not covered by any line, choose the smallest element. Call this value k .

(b) Subtract k from every element in the matrix that is not covered by a line.

(c) Add k to every element in the matrix covered by the two lines, i.e., intersection of two lines. (d) Elements in the matrix covered by one line remain unchanged.

Step 7: Repeat steps Repeat Steps 3 to 6 until an optimal solution is obtained.

The flow chart of steps in the Hungarian method for solving an assignment problem is shown in Fig.



Problem: A company has five jobs V, W, X, Y and Z and five machines A, B, C, D and E. The given matrix shows the return in Rs. of assigning a job to a machine. Assign the jobs to machines so as to maximize the total returns.

	Machines.				
	Returns in Rs.				
Jobs	A	B	C	D	E
V	5	11	10	12	4
W	2	4	6	3	5
X	3	12	5	14	6
Y	6	14	4	11	7
Z	7	9	8	12	5

Solution:

As the objective function is to maximize the returns, we have to convert the given problem into minimization problem.

Method 1. Here highest element in the matrix is 14, hence subtract all the element form 14 and write the relative costs. (Transformed matrix).

**Machines
Returns in Rs.**

<i>Jobs</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
V	9	3	4	2	10
W	12	10	8	11	9
X	11	2	9	0	8
Y	8	0	10	3	7
Z	7	5	6	2	9

ROCM

**Machines
Returns in Rs.**

<i>Jobs</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
V	7	1	2	0	8
W	4	2	0	3	1
X	11	2	9	0	8
Y	8	0	10	3	7
Z	5	3	4	0	7

By doing column operation on ROCM, we get the total opportunity cost matrix.
TOCM:

Machines
Returns in Rs.

<i>Jobs</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
V	3	1	2	0	7
W	0	2	0	3	0
X	7	2	9	0	7
Y	4	0	10	3	6
Z	1	3	4	0	6

Only three lines are there. So we have to go to step 7. The lowest element in uncovered cell is 1, hence subtract 1 from all uncovered cells and add this element to crossed cells and write the matrix. The resultant matrix is:

Machines
Return in Rs.

<i>Jobs</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
V	2	0	1	0	6
W	0	3	0	4	0
X	6	1	8	0	6
Y	4	0	10	4	6
Z	0	2	3	0	5

Only four lines are there, hence repeat the step 7 until we get 5 lines.

Machines
Return in Rs.

<i>Jobs</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
V	2	0	1	0	6
W	0	3	0	4	0
X	6	1	8	0	6
Y	4	0	10	4	6
Z	0	2	3	0	5

All zeros are covered by 5 lines, Hence assignment can be made. Start row wise or column wise and go on making assignment, until all assignments are over.

**Machines
Return in Rs.**

<i>Jobs</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>V</i>	2	1	0	∞	5
<i>W</i>	1	4	0	5	0
<i>X</i>	6	2	7	0	5
<i>Y</i>	3	0	8	3	4
<i>Z</i>	0	3	2	0	4

<i>Job</i>	<i>Machine</i>	<i>Return in Rs.</i>
<i>V</i>	<i>C</i>	10
<i>W</i>	<i>E</i>	5
<i>X</i>	<i>D</i>	14
<i>Y</i>	<i>B</i>	14
<i>Z</i>	<i>A</i>	7
Total in Rs.		50

Problem: A computer centre has three expert programmers. The centre wants three application programmes to be developed. The head of the computer centre, after carefully studying the programmes to be developed, estimates the computer time in minutes required by the experts for the application programmes as follows:

		Programmers		
		<i>A</i>	<i>B</i>	<i>C</i>
Programmes	<i>1</i>	120	100	80
	<i>2</i>	80	90	110
	<i>3</i>	110	140	120

Assign the programmers to the programmes in such a way that the total computer time is minimum.

Solution: Steps 1 and 2 The minimum time element in rows 1, 2 and 3 is 80, 80 and 110, respectively.

Subtract these elements from all elements in their respective row. The reduced time matrix is shown in Table.

(a)

	<i>A</i>	<i>B</i>	<i>C</i>
<i>1</i>	40	20	0
<i>2</i>	0	10	30
<i>3</i>	0	30	10

(b)

	A	B	C
1	40	10	0
2	0	0	30
3	0	20	10

In reduced Table the minimum time element in columns A, B and C is 0, 10 and 0, respectively. Subtract these elements from all elements in their respective column in order to get the reduced time matrix. This is shown in Table.

Step 3 (a) Examine all the rows starting from the first, one-by-one, until a row containing single zero element is found. In Table rows 1 and 3 have only one zero in the cells (1, C) and (3, A), respectively. Make an assignment in these cells and cross off all zero elements in the assigned column as shown in Table.

(a)

	A	B	C
1	40	20	0
2	0	0	30
3	0	30	10

(b)

	A	B	C
1	40	10	0
2	0	0	30
3	0	20	10

(b) Now examine each column starting from A in Table. There is one zero in column B in the cell (2, B). Make an assignment in this cell as shown in Table.

(c) Since the number of assignments (= 3) equals the number of rows (= 3), the optimal solution is obtained. The pattern of assignments among programmers and programmes with their respective time (in minutes) is given below:

Programmer	Programme	Time (in minutes)
1	C	80
2	B	90
3	A	110
Total		280

Problem 3: A solicitors' firm employs typists on hourly piece-rate basis for their daily work. There are five typists and their charges and speed are different. According to an earlier understanding only one job was given to one typist and the typist was paid for a full hour, even if he worked for a fraction of an hour. Find the least cost allocation for the following data:

Unit 05: Assignment Problem

Typist	Rate per hour (Rs)	No. of Pages Typed/Hour
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of Pages
P	199
Q	175
R	145
S	298
T	178

Solution:

Develop a cost matrix based on the data of the problem as shown in Table (a), where elements represent the cost to be incurred due to assignment of jobs to various typists on a one-to-one basis.

(a)	P	Q	R	S	T
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Applying Step 2 of Hungarian method, the reduced opportunity cost matrix is shown in Table (b).

(b)	P	Q	R	S	T
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

To make assignments in Table (b), examine all the rows starting from A until a row containing only one zero element is found. Rows B and D have only one zero element in the cells (B, R) and (D, R), respectively. First make assignment in cells (B, R), and cross off all zeros in the assigned columns as shown in Table (b).

Now examine each column starting from column P. There is one zero in columns P, Q, S and T in the cells (C, P), (E, Q), (E, S) and (A, T).

Assignment is made in these cells. All zeros in the matrix are either assigned or crossed off, as shown in Table.

(a)

	P	Q	R	S	T	
A	2	2	2	4	0	
B	6	4	0	10	2	✓
C	0	1	2	1	2	
D	2	4	2	4	2	✓
E	2	0	2	2	2	

✓

(b)

	P	Q	R	S	T
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

The solution shown in Table 10.8(a) is not optimal since only four assignments are made. Thus, in order to get the next best solution, apply following steps.

- (a) Mark (9) row D since it has no assignment.
- (b) Mark (9) column R since row D has zero in this column.
- (c) Mark (9) row B since column R has an assignment in row B.
- (d) Since no other rows or columns can be marked, therefore draw straight lines through the unmarked rows A, C and E and marked column R as shown in Table.

Develop the revised cost matrix by selecting the smallest element among all uncovered elements by the lines (i.e., $k = 2$) in Table. Subtract this element ($k = 2$) from all uncovered elements including itself and add it to elements in the cells (A, R), (C, R) and (E, R), respectively which lie at the intersection of two lines. Another revised cost matrix so obtained is shown in Table. Again repeat the procedure to find a new solution. The new assignments are shown in Table.

(a)

	P	Q	R	S	T	
A	2	2	2	4	0	✓
B	4	2	0	8	0	✓
C	0	1	2	1	2	✓
D	0	2	0	2	0	✓
E	2	0	2	0	2	

✓ ✓ ✓

(b)

	<i>P</i>	<i>Q</i>	<i>R</i>	<i>S</i>	<i>T</i>
<i>A</i>	2	1	2	3	0
<i>B</i>	4	1	0	7	∞
<i>C</i>	∞	∞	2	0	2
<i>D</i>	0	1	∞	1	∞
<i>E</i>	3	0	3	∞	3

The solution shown in Table 10.9(a) is also not optimal since only four assignments are made. Thus, to get the next best solution, follow Steps 6(a) to (d) of the algorithm to draw a minimum number of horizontal and vertical lines to cover all zero elements in Table 10.9(a). The new opportunity cost matrix obtained from Table by subtracting the smallest element (= 1) among all uncovered elements, including itself, by the lines and adding it to elements at the intersection of two lines is shown in Table.

The new solution obtained by repeating the procedure as explained earlier is shown in Table. Since both columns *Q* and *S* have two zeros, the arbitrary selection of a cell in any of these columns will give us an alternative solution with same total cost of assignment.

The pattern of assignments among typists and jobs, along with cost is as follows:

<i>Typist</i>	<i>Job</i>	<i>Cost (in Rs)</i>
<i>A</i>	<i>T</i>	75
<i>B</i>	<i>R</i>	66
<i>C</i>	<i>S</i>	114
<i>D</i>	<i>P</i>	80
<i>E</i>	<i>Q</i>	64
Total		399

Summary

The assignment problem, a fundamental optimization challenge within Operations Research, involves the allocation of a set of resources to a corresponding set of tasks. Its primary objective is to optimize a certain criterion, often minimizing costs or maximizing profits, while adhering to specific constraints. This problem has diverse applications across various domains, from logistics and manufacturing to education and project management, where efficient resource allocation plays a pivotal role in streamlining operations and improving overall efficiency.

In mathematical terms, the assignment problem can be represented as a matrix, where each row signifies a resource, each column denotes a task, and the entries in the matrix represent the cost, profit, or some other measure associated with assigning a particular resource to a specific task. The challenge is to find the optimal assignment that minimizes the total cost, maximizes the total profit, or achieves a similar objective.

One of the classic forms of the assignment problem is the linear assignment problem, which aims to minimize the total cost. This problem can be solved efficiently using algorithms like the Hungarian algorithm or through network flow formulations. The Hungarian algorithm, in particular, is notable for its ability to iteratively identify a set of independent assignments, minimizing the total cost. Its relatively low time complexity makes it suitable for practical applications with moderate-sized resource and task sets.

However, the assignment problem isn't limited to cost minimization; it extends to scenarios where profit maximization or other objectives are paramount. In these cases, adaptations are made to the problem's formulation to cater to different optimization goals. For instance, in job assignment

scenarios, the aim may be to maximize the skill match between employees and tasks, necessitating changes in the cost matrix and assignment rules. Solving these maximization-based assignment problems often employs similar algorithms to the Hungarian method, with adjustments to suit the specific objective.

Moreover, real-world scenarios frequently introduce additional constraints to the assignment problem, making it more intricate. Constraints can involve limitations on the number of assignments for each resource or task, requirements for specific tasks to be assigned to particular resources, or restrictions on the compatibility of certain resource-task combinations. These constraints lead to variant assignment problems, including the quadratic assignment problem, multiple-objective assignment problem, and others. Addressing these variants often requires more advanced optimization techniques, such as integer programming, heuristics, and metaheuristic algorithms.

In practice, the implications of the assignment problem are far-reaching. In transportation planning, it assists in determining the most efficient way to allocate goods to distribution centers or delivery routes, minimizing transportation costs and delivery times. In project management, it aids in task allocation to available team members, optimizing skill utilization and project timelines. In educational institutions, it can be employed to match students with courses based on their preferences and capacities, ensuring efficient resource utilization and student satisfaction.

In conclusion, the assignment problem is a fundamental topic within Operations Research, addressing the optimization of resource-task allocations in a multitude of real-world applications. It involves finding the optimal assignment that minimizes costs, maximizes profits, or meets other specific objectives, all while considering a variety of constraints. The algorithms and techniques developed for solving assignment problems have profound implications for industries and sectors, impacting decision-making processes, enhancing operational efficiency, and improving resource allocation strategies. As technology continues to advance, the significance of the assignment problem in optimizing complex and dynamic systems remains ever-present.

Keywords

Assignment Problem: A mathematical optimization problem involving the allocation of resources to tasks in a way that optimizes a specific objective, such as minimizing cost or maximizing profit.

Optimization: The process of finding the best solution from a set of feasible alternatives, typically aiming to maximize or minimize an objective function.

Hungarian Algorithm: An efficient algorithm for solving the assignment problem, known for its time complexity of $O(n^3)$, where n is the number of resources or tasks.

Linear Programming: A mathematical approach for optimizing linear objective functions subject to linear constraints, often used to formulate and solve assignment problems.

Resource Allocation: The process of assigning limited resources to various tasks or activities to achieve specific goals efficiently.

Cost Matrix: A matrix that represents the costs or benefits associated with assigning each resource to each task in the assignment problem.

Heuristic Methods: Problem-solving techniques that use rules of thumb or approximate algorithms to quickly find reasonably good solutions to complex problems.

Metaheuristic Algorithms: High-level algorithms, such as genetic algorithms or simulated annealing, used to find near-optimal solutions for assignment problems, especially in large or complex instances.

Linear Assignment Problem: A variant of the assignment problem where the objective is to minimize the total cost while ensuring that each resource is assigned to exactly one task.

Integer Linear Programming (ILP): A mathematical approach that formulates the assignment problem as a set of linear equations and inequalities with integer constraints, enabling exact solutions for small to medium-sized problems.

Quadratic Assignment Problem: An extension of the assignment problem where the cost matrix includes quadratic terms, often used in facility location and network design problems.

Network Flow Algorithms: Algorithms that solve assignment problems by modeling them as flow networks, facilitating the efficient allocation of resources.

Branch-and-Bound Algorithm: A general optimization technique used to solve integer programming problems, including the assignment problem, by partitioning the solution space and pruning suboptimal branches.

Combinatorial Optimization: A field of optimization that deals with discrete and combinatorial problems, including the assignment problem, traveling salesman problem, and knapsack problem.

Constraint Satisfaction Problem: A class of problems, including some assignment problem variants, where solutions must satisfy a set of constraints, often solved using constraint propagation or backtracking algorithms.

Self Assessment

1. The Assignment Problem deals with:
 - A. Allocating resources to activities
 - B. Optimizing linear functions
 - C. Solving nonlinear equations
 - D. Maximizing profit

2. In the context of the Assignment Problem, "assignment" refers to:
 - A. Allocating tasks to machines or individuals
 - B. Selecting the best solution method
 - C. Assigning values to variables
 - D. Designing optimal processes

3. The Hungarian Method is used to solve the Assignment Problem by:
 - A. Assigning tasks randomly
 - B. Solving linear equations
 - C. Finding the optimal assignment
 - D. Maximizing profit

4. In the Assignment Problem, the objective is to:
 - A. Minimize the number of assignments
 - B. Maximize the number of assignments
 - C. Maximize the total assignment cost
 - D. Minimize the total assignment cost

5. The matrix used to represent assignment costs is called the:
 - A. Cost matrix
 - B. Assignment matrix
 - C. Allocation matrix
 - D. Profit matrix

6. The optimal assignment in the Assignment Problem satisfies:

- A. The smallest assignment costs
 - B. The largest assignment costs
 - C. A balanced assignment matrix
 - D. The highest total profit
7. In the Hungarian Method, the initial step involves:
- A. Subtraction of row and column minima
 - B. Addition of row and column maxima
 - C. Finding the largest entry in the matrix
 - D. Rearranging the matrix entries
8. The "augmenting path" in the Hungarian Method refers to:
- A. An unbalanced assignment matrix
 - B. A path that can be used to reduce the total cost
 - C. A row or column with all zeros
 - D. The initial assignment matrix
9. The Hungarian Method terminates when:
- A. The matrix becomes a square
 - B. The optimal assignment is found
 - C. The matrix becomes unbalanced
 - D. The cost matrix becomes identity
10. When using the Hungarian Method, the optimal assignment is indicated by:
- A. The largest sum of assignment costs
 - B. The smallest sum of assignment costs
 - C. The largest entry in the matrix
 - D. The smallest entry in the matrix
11. In the context of the Assignment Problem, "dummy rows or columns" are used to:
- A. Indicate unassigned tasks or resources
 - B. Inflate the cost matrix values
 - C. Assign tasks to multiple resources
 - D. Represent non-linear assignment costs
12. The Assignment Problem can be transformed into a transportation problem by:
- A. Adding constraints to the problem
 - B. Adjusting the objective function
 - C. Introducing dummy variables
 - D. Modifying the cost matrix
13. In the Assignment Problem, if the number of tasks exceeds the number of resources:

- A. The problem becomes infeasible
 - B. Dummy tasks are added to balance the matrix
 - C. The matrix cannot be solved using the Hungarian Method
 - D. The cost matrix becomes asymmetric
14. The "optimal assignment" in the Assignment Problem refers to:
- A. The assignment with the highest costs
 - B. The assignment with the lowest costs
 - C. The assignment with the highest profit
 - D. The assignment with the fewest constraints
15. If an assignment problem has no feasible solution:
- A. It can be solved using the Hungarian Method
 - B. It has multiple feasible solutions
 - C. It has a feasible solution with infinite cost
 - D. It is not an assignment problem
16. The "opportunity cost" in the context of the Assignment Problem represents:
- A. The potential benefit of an unassigned task
 - B. The cost of an optimal assignment
 - C. The maximum profit achievable
 - D. The cost of creating dummy variables
17. The Assignment Problem is a special case of:
- A. Linear programming
 - B. Quadratic programming
 - C. Nonlinear programming
 - D. Integer programming
18. The "row reduction" step in the Hungarian Method involves:
- A. Subtracting the row minimum from each element
 - B. Adding the row minimum to each element
 - C. Dividing each element by the row minimum
 - D. Finding the maximum element in each row
19. The "column reduction" step in the Hungarian Method involves:
- A. Subtracting the column minimum from each element
 - B. Adding the column minimum to each element
 - C. Dividing each element by the column minimum
 - D. Finding the maximum element in each column

20. In the Hungarian Method, if the number of assignments is less than the number of resources:
- A. The problem cannot be solved
 - B. The matrix is unbalanced
 - C. Dummy assignments are added to balance the matrix
 - D. The matrix is infeasible
21. The Assignment Problem is commonly used in scenarios involving:
- A. Profit maximization
 - B. Network flow optimization
 - C. Task allocation
 - D. Nonlinear programming
22. The Hungarian Method is an efficient algorithm for solving the Assignment Problem with a time complexity of:
- A. $O(n^2)$
 - B. $O(n^3)$
 - C. $O(n \log n)$
 - D. $O(n!)$
23. In the Assignment Problem, if all costs are multiplied by a positive constant, the optimal assignment:
- A. Remains the same
 - B. Becomes infeasible
 - C. Requires re-solving the problem
 - D. Is affected only if the constant is prime
24. The Assignment Problem can be extended to handle cases with:
- A. Non-integer costs
 - B. Nonlinear costs
 - C. Binary constraints
 - D. Non-identical resources
25. The "shortest path" algorithm is not suitable for solving the Assignment Problem because:
- A. It is too slow
 - B. It does not handle linear costs
 - C. It cannot handle unbalanced matrices
 - D. It requires additional constraints

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. A | 2. A | 3. C | 4. D | 5. A |
| 6. D | 7. A | 8. B | 9. B | 10. B |
| 11. A | 12. C | 13. B | 14. B | 15. C |
| 16. A | 17. A | 18. A | 19. A | 20. C |
| 21. C | 22. B | 23. A | 24. D | 25. C |

Review Questions

1. What is an assignment problem? Give two applications.
2. Give the mathematical formulation of an assignment problem. How does it differ from a transportation problem?
3. Explain the conceptual justification that an assignment problem can be viewed as a linear programming problem.
4. Explain the difference between a transportation problem and an assignment problem.
5. Specify the dual of an assignment problem. What are the techniques used for solving an assignment problem?
6. State and discuss the methods of solving an assignment problem. How is the Hungarian method better than other methods for solving an assignment problem?
7.
 - (a) Give an algorithm to solve an assignment problem.
 - (b) Show that an assignment problem is a special case of a transportation problem.
8. Explain how an assignment problem can be solved by using the transportation approach?



Further Readings

"Introduction to Operations Research" by Frederick S. Hillier and Gerald J. Lieberman - This comprehensive textbook provides a detailed introduction to various topics in Operations Research, including the assignment problem and its variants.

"Combinatorial Optimization: Algorithms and Complexity" by Christos H. Papadimitriou and Kenneth Steiglitz - This book delves into the theory and algorithms behind combinatorial optimization problems, including the assignment problem.

"Network Flows: Theory, Algorithms, and Applications" by Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin - This text focuses on network flow problems, which are closely related to the assignment problem. It provides insights into optimization techniques used in solving such problems.

"Integer Programming" by Laurence A. Wolsey - This book offers an in-depth exploration of integer programming, a technique frequently used to solve assignment problems and other discrete optimization problems.

"Metaheuristics: From Design to Implementation" by El-Ghazali Talbi - For readers interested in heuristic and metaheuristic methods for solving assignment problems, this book provides valuable insights and practical implementations.

"Operations Research: Applications and Algorithms" by Wayne L. Winston - This book includes practical examples and case studies related to Operations Research problems, including the assignment problem, and demonstrates their applications in various fields.

"Transportation and Assignment Models in Operations Research" by S. C. Sharma - A focused resource that delves into transportation and assignment problems, providing detailed explanations and practical examples.

"The Assignment Problem and the Hungarian Method" by James Munkres - The original paper by Munkres that introduced the Hungarian algorithm for solving the assignment problem.

Unit 6: Transportation Problem – Initial Basic Feasible Solution

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Summary

Keywords

Review Questions

Further Readings

Objectives

After studying this unit, you will be able to:

- Understand the meaning of operations research
- Know about the history of operations research
- Discuss the scope and application of operations research
- Explain the various types of models used in operations research

Introduction

Notes

Transportation problem is a particular class of linear programming, which is associated with day-to-day activities in our real life and mainly deals with logistics. It helps in solving problems on distribution and transportation of resources from one place to another. The goods are transported from a set of sources (e.g., factory) to a set of destinations (e.g., warehouse) to meet the specific requirements. In other words, transportation problems deal with the transportation of a product manufactured at different plants (supply origins) to a number of different warehouses (demand destinations). The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. In addition, we must also know the location, to find the cost of transporting one unit of commodity from the place of origin to the destination. The model is useful for making strategic decisions involved in selecting optimum transportation routes so as to allocate the production of various plants to several warehouses or distribution centers.

The transportation model can also be used in making location decisions. The model helps in locating a new facility, a manufacturing plant or an office when two or more number of locations is under consideration. The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying the model.

6.1 Modeling of Transportation Problem

A transportation problem can be expressed in two ways.

1. Mathematical representation
2. Network representation

Obviously the method used for solving the problems is the formulation of transportation problem through mathematical methods. But for understanding of the readers, network representation is equally important.

Let us understand each of them one by one.

6.1.1 Mathematical Representation

The transportation problem applies to situations where a single commodity is to be transported from various sources of supply (**origins**) to various demands (**destinations**).

Let there be m sources of supply S_1, S_2, \dots, S_m having a_i ($i = 1, 2, \dots, m$) units of supplies respectively to be transported among n destinations D_1, D_2, \dots, D_n with b_j ($j = 1, 2, \dots, n$) units of requirements respectively. Let C_{ij} be the cost for shipping one unit of the commodity from source i , to destination j for each route. If x_{ij} represents the units shipped per route from source i , to destination j , then the problem is to determine the transportation schedule which minimizes the total transportation cost of satisfying supply and demand conditions.

The transportation problem can be stated mathematically as a linear programming problem as below:

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

Notes

Subject to constraints,

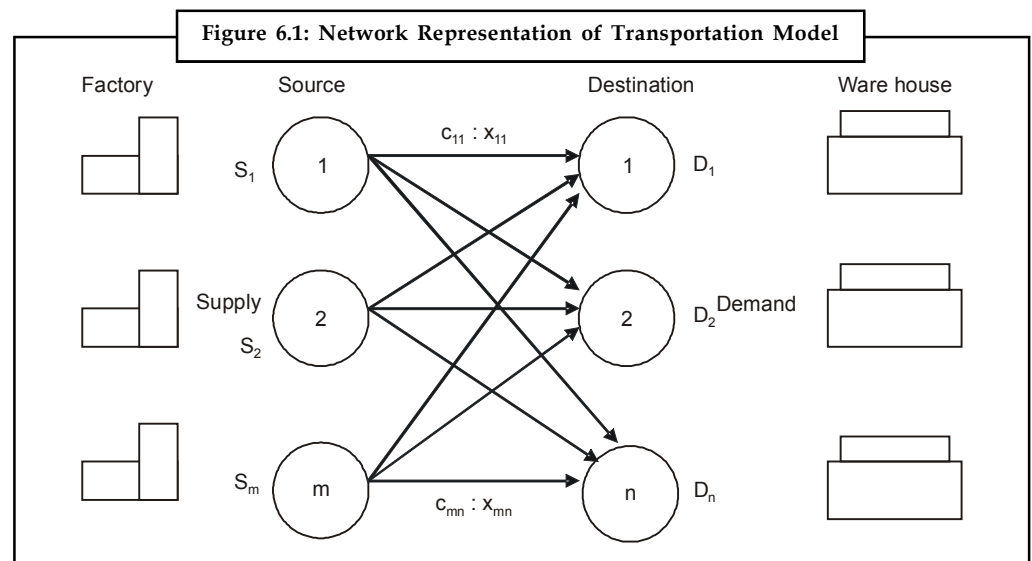
$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \text{ (supply constraints)}$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \text{ (demand constraints)}$$

and $x_{ij} \geq 0$ for all $i = 1, 2, \dots, m$ and,
 $j = 1, 2, \dots, n$

6.1.2 Network Representation of Transportation Model

The transportation model is represented by a network diagram in Figure 3.1



where,

m be the number of sources,

n be the number of destinations,

S_m be the supply at source m ,

D_n be the demand at destination n ,

c_{mn} be the cost of transportation from source m to destination n , and

x_{mn} be the number of units to be shipped from source m to destination n .

The objective is to minimize the total transportation cost by determining the unknowns x_{mn} , i.e., the number of units to be shipped from the sources and the destinations while satisfying all the supply and demand requirements.

6.1.3 General Representation of Transportation Model

Notes

The Transportation problem can also be represented in a tabular form as shown in Table 6.1

Let c_{ij} be the cost of transporting a unit of the product from i^{th} origin to j^{th} destination.

a_i be the quantity of the commodity available at source i ,

b_j be the quantity of the commodity needed at destination j , and

x_{ij} be the quantity transported from i^{th} source to j^{th} destination

Table 6.1: Tabular Representation of Transportation Model

		To				Supply
		D_1	D_2	...	D_n	
From						a_i
S_1		C_{11}	C_{12}	...	C_{1n}	a_1
	X_{11}	X_{12}	...			
S_2		C_{21}	C_{22}	...	C_{2n}	a_2
	X_{21}	X_{22}	...			
.
.
.
S_m		C_{m1}	C_{m2}	...	C_{mn}	a_m
	x_{m1}	x_{m2}	...			
Demand	b_j	b_1	b_2	...	b_n	$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

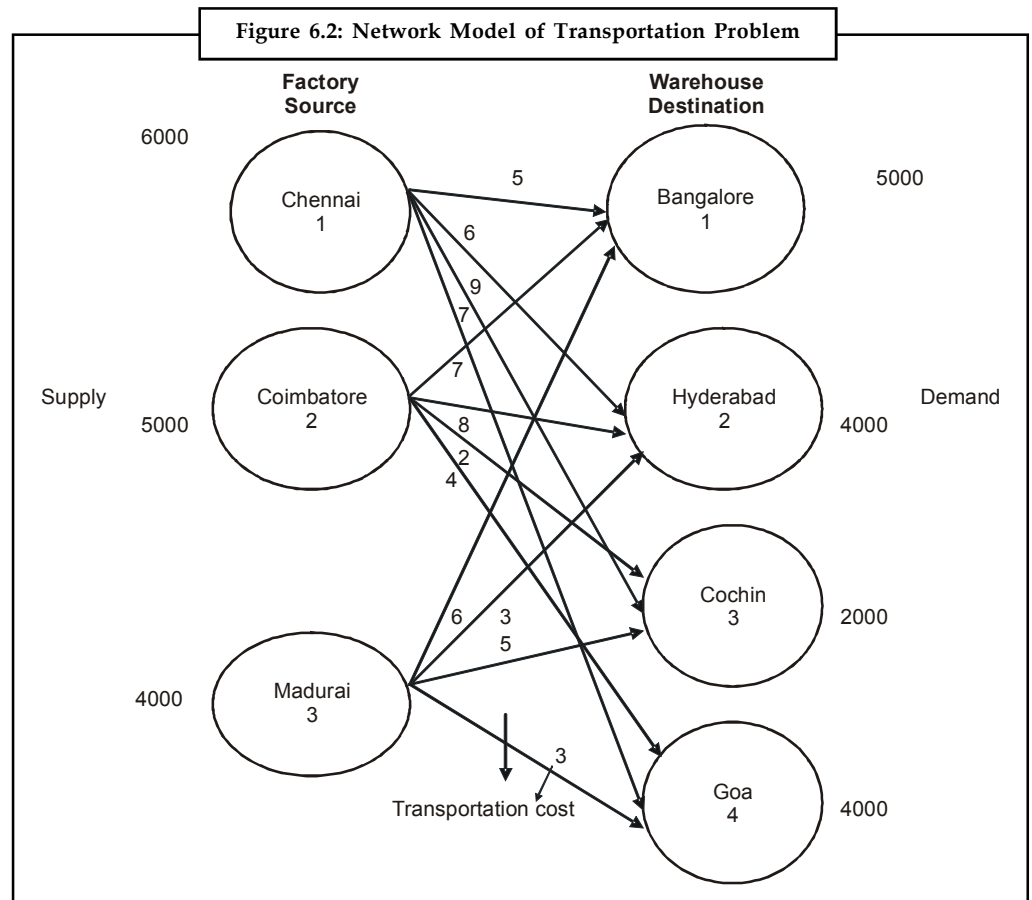
If the total supply is equal to total demand, then the given transportation problem is a balanced one.

6.2 Use of Linear Programming to Solve Transportation Problem

The network diagram shown in Figure 6.2 represents the transportation model of GM Textiles units located at Chennai, Coimbatore and Madurai. GM Textiles produces ready-made garments at these locations with capacities 6000, 5000 and 4000 units per week at Chennai, Coimbatore and Madurai respectively. The textile unit distributes its ready-made garments through four of its wholesale distributors situated at four locations Bangalore, Hyderabad, Cochin and Goa. The weekly demand of the distributors are 5000, 4000, 2000 and 4000 units for Bangalore, Hyderabad, Cochin and Goa respectively.

The cost of transportation per unit varies between different supply points and destination points. The transportation costs are given in the network diagram.

Notes



The management of GM Textiles would like to determine the number of units to be shipped from each textile unit to satisfy the demand of each wholesale distributor. The supply, demand and transportation cost are as follows:

Table 6.2 (a): Production Capacities

Supply	Textile Unit	Weekly Production (Units)
1	Chennai	6000
2	Coimbatore	5000
3	Madurai	4000

Table 6.2 (b): Demand Requirements

Destination	Wholesale Distributor	Weekly Demand (Units)
1	Bangalore	5000
2	Hyderabad	4000
3	Cochin	2000
4	Goa	4000

Table 6.2 (c): Transportation Cost per Unit

Supply	Destination			
	Bangalore	Hyderabad	Cochin	Goa
Chennai	5	6	9	7
Coimbatore	7	8	2	4
Madurai	6	3	5	3

A linear programming model can be used to solve the transportation problem.

Let,

X_{11} be number of units shipped from source1 (Chennai) to destination 1 (Bangalore).

X_{12} be number of units shipped from source1 (Chennai) to destination 2 (Hyderabad).

X_{13} number of units shipped from source 1 (Chennai) to destination 3 (Cochin).

X_{14} number of units shipped from source 1 (Chennai) to destination 4 (Goa).

:

and so on.

X_{ij} = number of units shipped from source i to destination j , where $i = 1, 2, \dots, m$ and,

$j = 1, 2, \dots, n$.

6.3 Minimizing Case

Objective Function

The objective is to minimize the total transportation cost. Using the cost data table, the following equation can be arrived at:

- Transportation cost for units shipped from Chennai = $5x_{11} + 6x_{12} + 9x_{13} + 7x_{14}$
- Transportation cost for units shipped from Coimbatore = $7x_{21} + 8x_{22} + 2x_{23} + 4x_{24}$
- Transportation cost for units shipped from Madurai = $6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$

Combining the transportation cost for all the units shipped from each supply point with the objective to minimize the transportation cost, the objective function will be,

$$\text{Minimize } Z = 5x_{11} + 6x_{12} + 9x_{13} + 7x_{14} + 7x_{21} + 8x_{22} + 2x_{23} + 4x_{24} + 6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$$

Constraints: In transportation problems, there are supply constraints for each source, and demand constraints for each destination.

Supply constraints:

$$\text{For Chennai, } x_{11} + x_{12} + x_{13} + x_{14} \leq 6000$$

$$\text{For Coimbatore, } x_{21} + x_{22} + x_{23} + x_{24} \leq 5000$$

$$\text{For Madurai, } x_{31} + x_{32} + x_{33} + x_{34} \leq 4000$$

Demand constraints:

$$\text{For B'lore, } x_{11} + x_{21} + x_{31} = 5000$$

$$\text{For Hyderabad, } x_{12} + x_{22} + x_{32} = 4000$$

Notes

For Cochin, $x_{13} + x_{23} + x_{33} = 2000$

For Goa, $x_{14} + x_{24} + x_{34} = 4000$

The linear programming model for GM Textiles will be write in the next line. Minimize $Z = 5x_{11} + 6x_{12} + 9x_{13} + 7x_{14} + 7x_{21} + 8x_{22} + 2x_{23} + 4x_{24} + 6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$

Subject to constraints,

$x_{11} + x_{12} + x_{13} + x_{14} \leq 6000$ (i)

$x_{21} + x_{22} + x_{23} + x_{24} \leq 5000$ (ii)

$x_{31} + x_{32} + x_{33} + x_{34} \leq 4000$ (iii)

$x_{11} + x_{21} + x_{31} = 5000$ (iv)

$x_{12} + x_{22} + x_{32} = 4000$ (v)

$x_{13} + x_{23} + x_{33} = 2000$ (vi)

$x_{14} + x_{24} + x_{34} = 4000$ (vii)

Where, $x_{ij} \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.



Example: Consider the following transportation problem (Table 3.3) and develop a linear programming (LP) model.

Table 6.3: Transportation Problem

Source	Destination			Supply
	1	2	3	
1	15	20	30	350
2	10	9	15	200
3	14	12	18	400
Demand	250	400	300	

Solution:

Let x_{ij} be the number of units to be transported from the source i to the destination j , where $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

The linear programming model is

Minimize $Z = 15x_{11} + 20x_{12} + 30x_{13} + 10x_{21} + 9x_{22} + 15x_{23} + 14x_{31} + 12x_{32} + 18x_{33}$

Subject to constraints,

$x_{11} + x_{12} + x_{13} \leq 350$ (i)

$x_{21} + x_{22} + x_{23} \leq 200$ (ii)

$x_{31} + x_{32} + x_{33} \leq 400$ (iii)

$x_{11} + x_{21} + x_{31} = 250$ (iv)

$x_{12} + x_{22} + x_{32} = 400$ (v)

$x_{13} + x_{23} + x_{33} = 300$ (vi)

$x_{ij} \geq 0$ for all i and j .

In the above LP problem, there are $m \times n = 3 \times 3 = 9$ decision variables and $m + n = 3 + 3 = 6$ constraints.

Notes

6.4 Maximization Transportation Problem

In this type of problem, the objective is to maximize the total profit or return. In this case, convert the maximization problem into minimization by subtracting all the unit cost from the highest unit cost given in the table and solve.



Example: A manufacturing company has four plants situated at different locations, all producing the same product. The manufacturing cost varies at each plant due to internal and external factors. The size of each plant varies, and hence the production capacities also vary. The cost and capacities at different locations are given in the following table:

Table 6.4: Cost and Capacity of Different Plants

Particulars	Plant			
	A	B	C	D
Production cost per unit (Rs.)	18	17	15	12
Capacity	150	250	100	70

The company has five warehouses. The demands at these warehouses and the transportation costs per unit are given in the Table 6.5 below. The selling price per unit is ₹ 30.

Table 6.5: Transportation Problem

Warehouse	Transportation cost (Rs) – Unit-wise				Demand
	A	B	C	D	
1	6	9	5	3	100
2	8	10	7	7	200
3	2	6	3	8	120
4	11	6	2	9	80
5	3	4	8	10	70

1. Formulate the problem to maximize profits.
2. Find the total profit.

Solution:

1. The objective is to maximize the profits. Formulation of transportation problem as profit matrix table is shown in Table 6.6. The profit values are arrived as follows.

Profit = Selling Price -Production cost -Transportation cost

Notes

Table 6.6: Profit Matrix

	Destination				Demand
	A	B	C	D	
1	6	4	10	15	100
2	4	3	8	11	200
3	10	7	12	10	120
4	1	7	13	9	80
5	9	9	7	8	70
Supply	150	250	100	70	570

Converting the profit matrix to an equivalent loss matrix by subtracting all the profit values in Table 6.6 from the highest value 13. Subtracting all the values from 13, the loss matrix obtained is shown in the Table 6.7.

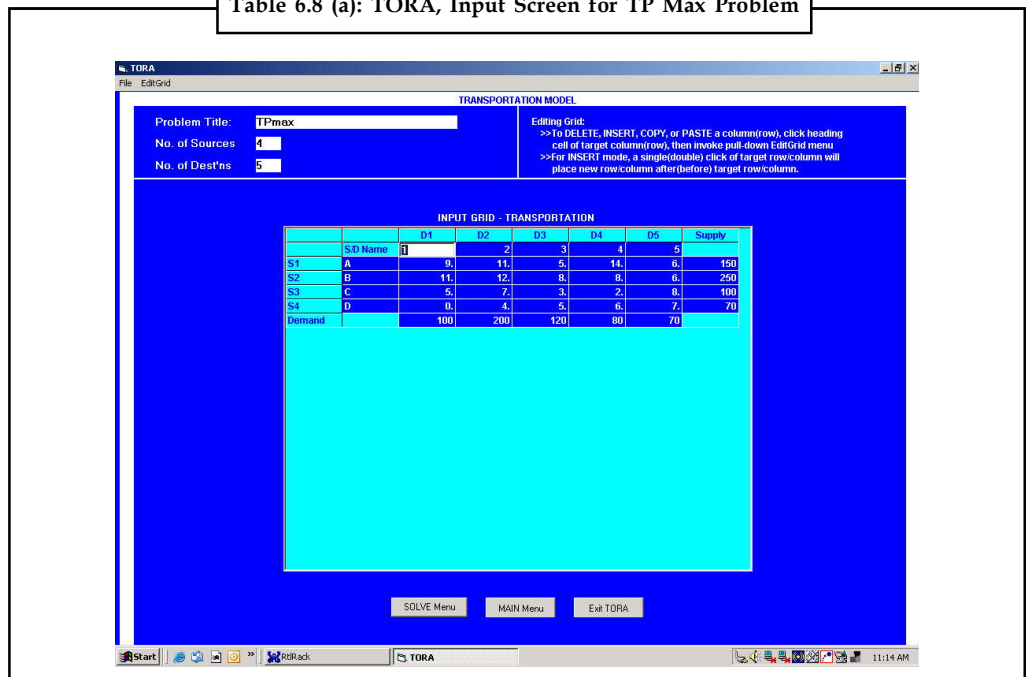
Table 6.7: Loss Matrix

	Destination				Demand
	A	B	C	D	
1	9	11	5	0	100
2	11	12	7	4	200
3	5	8	3	5	120
4	14	8	2	6	80
5	6	6	8	7	70
Supply	150	250	100	70	570

- To determine the initial solution using TORA

Input Screen:

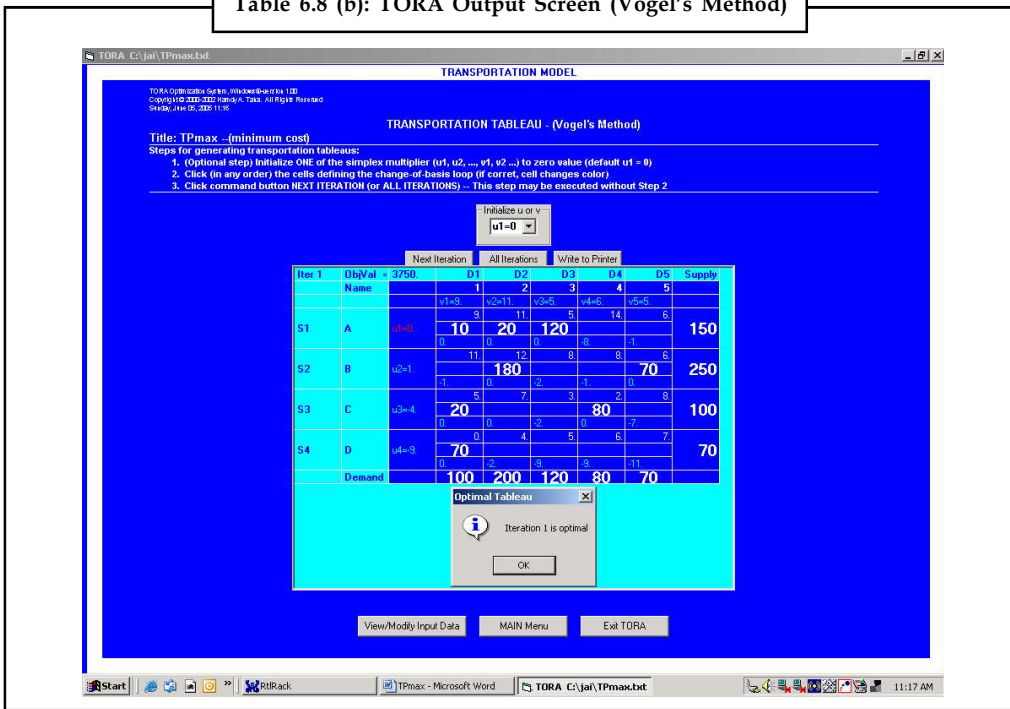
Table 6.8 (a): TORA, Input Screen for TP Max Problem



Output Screen:

Notes

Table 6.8 (b): TORA Output Screen (Vogel's Method)



The first iteration itself is optimal, hence optimality is reached.

3. To find the total cost:

The total maximization profit associated with the solution is

$$\begin{aligned}
 \text{Total Profit} &= (6 \times 10) + (4 \times 20) + (10 \times 120) + (3 \times 180) + (9 \times 70) + (10 \times 20) \\
 &\quad + (13 \times 80) + (15 \times 70) \\
 &= 60 + 80 + 1200 + 540 + 630 + 200 + 1040 + 1050 \\
 &= ₹ 4800.00
 \end{aligned}$$

6.5 Balanced Transportation Problem

When the total supplies of all the sources are equal to the total demand of all destinations, the problem is a *balanced transportation problem*.

Total supply = Total demand

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

The problem given in Example 3.1 represents a balanced transportation problem.

6.6 Unbalanced Transportation Problem

When the total supply of all the sources is not equal to the total demand of all destinations, the problem is an *unbalanced transportation problem*.

Notes

Total supply \neq Total demand

$$\sum_{i=1}^m a_i \neq \sum_{j=1}^n b_j$$

6.6.1 Demand Less than Supply

In real-life, supply and demand requirements will rarely be equal. This is because of variation in production from the supplier end, and variations in forecast from the customer end. Supply variations may be because of shortage of raw materials, labour problems, improper planning and scheduling. Demand variations may be because of change in customer preference, change in prices and introduction of new products by competitors. These unbalanced problems can be easily solved by introducing dummy sources and dummy destinations. If the total supply is greater than the total demand, a dummy destination (dummy column) with demand equal to the supply surplus is added. If the total demand is greater than the total supply, a dummy source (dummy row) with supply equal to the demand surplus is added. The unit transportation cost for the dummy column and dummy row are assigned zero values, because no shipment is actually made in case of a dummy source and dummy destination.



Example:

Check whether the given transportation problem shown in Table 6.9 is a balanced one. If not, convert the unbalanced problem into a balanced transportation problem.

Table 6.9: Transportation Model with Supply exceeding Demand

Source	Destination			Supply
	1	2	3	
1	25	45	10	200
2	30	65	15	100
3	15	40	55	400
Demand	200	100	300	

Solution:

For the given problem, the total supply is not equal to the total demand.

$$\sum_{i=1}^3 a_i \neq \sum_{j=1}^3 b_j$$

since,

$$\sum_{i=1}^3 a_i = 700 \text{ and } \sum_{j=1}^3 b_j = 600$$

The given problem is an unbalanced transportation problem. To convert the unbalanced transportation problem into a balanced problem, add a dummy destination (dummy column).

i.e., the demand of the dummy destination is equal to,

$$\sum_{i=1}^3 a_i - \sum_{j=1}^3 b_j$$

Thus, a dummy destination is added to the table, with a demand of 100 units. The modified table is shown in Table 6.10 which has been converted into a balanced transportation table. The unit costs of transportation of dummy destinations are assigned as zero.

Table 6.10: Dummy Destination Added

Source	Destination				Supply
	1	2	3	4	
1	25	45	10	0	200
2	30	65	15	0	100
3	15	40	55	0	400
Demand	200	100	300	100	700/700

Similarly,

If $\sum_{j=1}^n b_j > \sum_{i=1}^m a_i$, then include a dummy source to supply the excess demand.

6.6.2 Demand Greater than Supply



Example: Convert the transportation problem shown in Table 3.6 into a balanced problem.

Table 6.11: Demand Exceeding Supply

Source	Destination				Supply
	1	2	3	4	
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
Demand	100	200	450	250	1000/800

Solution:

The given problem is, $\sum_{j=1}^4 b_j > \sum_{i=1}^3 a_i$

$$\sum_{i=1}^3 a_i = 800 \text{ and } \sum_{j=1}^4 b_j = 1000$$

The given problem is an unbalanced one. To convert it into a balanced transportation problem, include a dummy source (dummy row) as shown in Table 6.12

Notes

Table 6.12: Balanced TP Model

Source	Destination				Supply
	1	2	3	4	
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
4	0	0	0	0	200
Demand	100	200	450	250	1000/1000

6.7 Initial Feasible Solution

Step 1: Formulate the Problem

Formulate the given problem and set up in a matrix form. Check whether the problem is a balanced or unbalanced transportation problem. If unbalanced, add dummy source (row) or dummy destination (column) as required.

Step 2: Obtain the Initial Feasible Solution

The initial feasible solution can be obtained by any of the following three methods.

1. Northwest Corner Method (NWC)
2. Row and Column Minima Method (RCMM)
3. Vogel’s Approximation Method (VAM)

The transportation cost of the initial basic feasible solution through Vogel’s approximation method, VAM will be the least when compared to the other two methods which gives the value nearer to the optimal solution or optimal solution itself. Algorithms for all the three methods to find the initial basic feasible solution are given.

6.7.1 Algorithm for North-West Corner Method (NWC)

1. Select the North-west (i.e., upper left) corner cell of the table and allocate the maximum possible units between the supply and demand requirements. During allocation, the transportation cost is completely discarded (not taken into consideration).
2. Delete that row or column which has no values (fully exhausted) for supply or demand.
3. Now, with the new reduced table, again select the North-west corner cell and allocate the available values.
4. Repeat steps (2) and (3) until all the supply and demand values are zero.
5. Obtain the initial basic feasible solution.



Example: Find out the initial feasible solution for transportation cost involved in the problem shown through Table 6.13.

Table 6.13

Retail shops					
Factories	1	2	3	4	Supply
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Solution:

Table 6.14

Retail shops					
Factories	1	2	3	4	Supply
1	3 ²⁰	5 ²⁰	7 ¹⁰	6	50
2	2	5	8 ⁴⁰	2 ³⁵	75
3	3	6	9	2 ²⁵	25
Demand	20	20	50	60	

As under the process of NWC method, we allocate $x_{11} = 20$. Now demand for the first column is satisfied, therefore, eliminate that column.

Proceeding in this way, we observe that

$$x_{12} = 20, x_{13} = 10, x_{23} = 40, x_{24} = 35, x_{34} = 25.$$

Delete the row if supply is exhausted.

Delete the column if demand is satisfied.

Here, number of retail shops (n) = 4, and

Number of factories (m) = 3

Number of basic variables = $m + n - 1 = 3 + 4 - 1 = 6$.

Initial basic feasible solution:

$$20 \times 3 + 20 \times 5 + 10 \times 7 + 40 \times 8 + 35 \times 2 + 25 \times 2 = 670$$

6.7.2 Algorithm for Row and Column Minima Method or Least Cost Method (LCM)

1. Select the smallest transportation cost cell available in the entire table and allocate the supply and demand.
2. Delete that row/column which has exhausted. The deleted row/column must not be considered for further allocation.
3. Again select the smallest cost cell in the existing table and allocate. (*Note:* In case, if there are more than one smallest costs, select the cells where maximum allocation can be made)
4. Obtain the initial basic feasible solution.

Notes



Example: Solve example 5 by least cost method.

Solution:

To make our understanding better, let us go through the Table 6.15.

Table 6.15					
Retail shops					
Factories	1	2	3	4	Supply
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Applying the least cost method-

Table 6.16					
Retail shops					
Factories	1	2	3	4	Supply
1	3	5 ²⁰	7 ³⁰	6	50
2	2 ²⁰	5	8	2 ⁵⁵	75
3	3	6	9 ²⁰	2 ⁵	25
Demand	20	20	50	60	

We observe that $c_{21} = 2$, which is the minimum transportation cost. So, $x_{21} = 20$.

Proceeding in this way, we observe that $x_{24} = 55$, $x_{34} = 5$, $x_{12} = 20$, $x_{13} = 30$, $x_{33} = 20$.

Number of basic variables = $m + n - 1 = 3 + 4 - 1 = 6$.

The initial basic feasible solution:

$$= 20 \times 2 + 55 \times 2 + 5 \times 2 + 20 \times 5 + 30 \times 7 + 20 \times 9$$

$$= 650.$$

6.7.3 Algorithm for Vogel’s Approximation Method (VAM)

1. Calculate penalties for each row and column by taking the difference between the smallest cost and next highest cost available in that row/column. If there are two smallest costs, then the penalty is zero.
2. Select the row/column, which has the largest penalty and make allocation in the cell having the least cost in the selected row/column. If two or more equal penalties exist, select one where a row/column contains minimum unit cost. If there is again a tie, select one where maximum allocation can be made.
3. Delete the row/column, which has satisfied the supply and demand.
4. Repeat steps (1) and (2) until the entire supply and demands are satisfied.
5. Obtain the initial basic feasible solution.



Caution The initial solution obtained by any of the three methods must satisfy the following conditions.

1. The solution must be feasible, i.e., the supply and demand constraints must be satisfied (also known as rim conditions).
2. The number of positive allocations, N must be equal to $m + n - 1$, where m is the number of rows and n is the number of columns.



Example: Find the initial feasible solution of the transportation problem illustrated through Table 6.17

Table 6.17

		Destination				
Origin	1	2	3	4	Supply	
1	20	22	17	4	120	
2	24	37	9	7	75	
3	34	37	20	15	25	
Demand	60	40	30	110	240	

Solution:

Solving the problem through Vogel's Approximation Method, we get the Table 6.18

Table 6.18

		Destination					
Origin	1	2	3	4	Supply	Penalty	
1	20	22 ⁴⁰	17	4	120 80	13	
2	24	37	9	7	70	2	
3	32	37	20	15	50	5	
Demand	60	40	30	110	240		
Penalty	4	15	8	3			

The highest penalty occurs in the second column. The minimum c_{ij} in this column is c_{12} (i.e., 22). Hence, $x_{12} = 40$ and the second column is eliminated.

Now again calculate the penalty.

Table 6.19

		Destination					
Origin	1	2	3	4	Supply	Penalty	
1	20	22 ⁴⁰	17	4 ⁸⁰	120	13	
2	24	37	9	7	70	2	
3	32	37	20	15	50	5	
Demand	60	40	30	110	240		
Penalty	4		8	3			

The highest penalty occurs in the first row. The minimum c_{ij} in this row is c_{14} (i.e., 4). So $x_{14} = 80$ and the first row is eliminated.

Notes

Final table:

Now assuming that you can calculate the values yourself, we reach the final table as in Table 6.20

Table 6.20											
Destination											
Origin	1	2	3	4	Supply	Penalty					
1	20	22 ⁴⁰	17	4 ⁸⁰	120	3	13	-	-	-	-
2	24 ¹⁰	37	9 ³⁰	7 ³⁰	70	2	2	2	17	24	24
3	32 ⁵⁰	37	20	15	50	5	5	5	17	32	-
Demand	60	40	30	110	240						
Penalty	4	15	8	3							
	4	-	8	3							
	8	-	11	8							
	8	-	-	8							
	8	-	-	-							
	24	-	-	-							

The initial basic feasible solution:

$$= 22 \times 40 + 4 \times 80 + 24 \times 10 + 9 \times 30 + 7 \times 30 + 32 \times 50$$

$$= 3520$$

Self Assessment

State true or false:

1. Transportation problem applies to situations where a set of commodities is to be transported from source to another.
2. The cost of transportation per unit varies between different supply points,
3. In transportation problems, there are supply constraints for each destination.
4. The most important objective of a transportation problem is to maximize the cost of shipping.
5. The transportation problem is an extension of transshipment problem.
6. There may be routes that are unavailable to transport units from one source to one or more destinations.
7. In transshipment problem, each node makes supplies to the other.
8. Degeneracy is the condition when the number of filled cells is less than the number of rows plus the number of columns minus one.

6.8 Duality

The term 'dual', in general, implies two or double. The concept of duality is very useful in mathematics, physics, statistics, engineering and managerial decision-making. For example, in a two-person game theory, one competitor's problem is the dual of the opponent's problem.

In linear programming, duality implies that each linear programming problem can be analyzed in two different ways but would have equivalent solutions. Any LP problem (either maximization and minimization) can be stated in another equivalent form based on the same data. The new LP problem is called dual linear programming problem or in short dual. In general, it is immaterial which of the two problems is called primal or dual, since the dual of the dual is primal.

For example, consider the problem of production planning. The production manager attempts to determine quantities for each product to be produced with an objective to optimize the use of available resources so that profit is maximum. But through a dual LP problem approach, he may develop a production plan that optimizes resource utilization so that marginal opportunity cost of each unit of a resource is equal to its marginal return (also called shadow price). The shadow price indicates an additional price to be paid to obtain one additional unit of the resources in order to maximize profit under the resource constraints.

If a resource is not completely used, i.e. there is slack, then its marginal return is zero. The shadow price is also defined as the rate of change in the optimal objective function value with respect to the unit change in the availability of a resource. To be more precise for any constraint, we have

Shadow price = Change in optimal objective function value / Unit change in the availability of resource

The interpretation of rate of change (increase or decrease) in the value of objective function depends on whether LP problem is of maximization or minimization type. The shadow price for a less than or equal to (\leq) type constraint will always be positive. This is because increasing the right-hand side resource value cannot decrease the value of objective function. Similarly, the shadow price for a greater than or equal to (\geq) type constraint will always be negative because increasing the right-hand side resource value cannot increase the value of the objective function.

There is no need to solve both LP problems separately. Solving one LP problem is equivalent to solving the other simultaneously. Thus, if the optimal solution to one is known, the optimal solution of the other can also be read from the $c_j - z_j$ row. In some cases, considerable computing time can be reduced by solving the dual LP problem.

6.9 Summary

- A transportation problem basically deals with that problem which aims to find the best way to fulfill the demand of various demand sources using the capacities of various supply points.
- Although the formation can be used to represent more general assignment and scheduling problems as well as transportation and distribution problems, it gets its name from its application to problems involving transporting products from several sources to several destinations.
- The two common objectives of such problems are either to minimize the cost of shipping commodity to various destinations or to maximize the profit of shipping it various destinations.
- Transportation problems are often used in, surprise, transportation planning. But it should always be remembered that the transportation problem is only a special topic of the linear programming problems.

6.10 Keywords

Balanced Transportation Problem: when the total supplies of all the sources are equal to the total demand of all destinations.

Basic Feasible Solution: A feasible solution to an 'm' origin and 'n' destination is said to be basic, if the number of positive allocations are $(m+n+1)$.

Degeneracy: When the number of filled cells is less than the number of rows plus the number of columns minus one.

Notes

11 Review Questions

- Find the initial basic feasible solution for the transportation problem given in following table.

From	To			Available
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

- Obtain an optimal solution for the transportation problem by MODI method given in this table:

		Destination				Supply
		D ₁	D ₂	D ₃	D ₄	
Source	S ₁	19	30	50	10	7
	S ₂	70	30	40	60	9
	S ₃	40	8	70	20	18
Demand		5	8	7	14	

- Solve the transportation problem

		Destination			Supply
		1	2	3	
Source	1	3	5	7	10
	2	11	8	9	8
	3	13	3	9	5
Demand		5	9	11	23 25

- Develop a network representation of the transportation problem for a company that manufactures products at three plants and ships them to three warehouses. The plant capacities and warehouse demands are shown in the following table:

The transportations cost per unit (in ₹) is given in matrix.

Plant	Warehouse			Plant Capacity (no. of units)
	W1	W2	W3	
P1	22	18	26	350
P2	12	12	10	450
P3	14	20	10	200
Warehouse demand (no. of units)	250	450	300	

- Determine whether a dummy source or a dummy destination is required to balance the model given.

- (a) Supply $a_1 = 15, a_2 = 5, a_3 = 4, a_4 = 6$
 Demand $b_1 = 4, b_2 = 15, b_3 = 6, b_4 = 10$

(b) Supply $a_1 = 27, a_2 = 13, a_3 = 10$
 Demand $b_1 = 30, b_2 = 10, b_3 = 6, b_4 = 10$

(c) Supply $a_1 = 2, a_2 = 3, a_3 = 5$
 Demand $b_1 = 3, b_2 = 2, b_3 = 2, b_4 = 2, b_5 = 1$.

6. A state has three power plants with generating capacities of 30, 40 and 25 million KWH that supply electricity to three cities located in the same state. The demand requirements (maximum) of the three cities are 35, 40 and 20 million KWH. The distribution cost (₹ in thousand) per million unit for the three cities are given in the table below:

		City		
		1	2	3
Plant	1	60	75	45
	2	35	35	40
	3	55	50	45

- (a) Formulate the problem as a transportation model.
- (b) Determine an economical distribution plan.
- (c) If the demand is estimated to increase by 15%, what is your revised plan?
- (d) If the transmission loss of 5% is considered, determine the optimal plan.

7. Find the initial transportation cost for the transportation matrix given using North-West corner method, Least cost method and Vogel's Approximation method.

		Destination				Supply
		1	2	3	4	
Source	A	5	6	7	8	25
	B	7	5	4	2	75
	C	6	1	3	2	15
Demand		50	30	20	15	

8. The Sharp Manufacturing Company produces three types of monoblock pumps for domestic use. Five machines are used for manufacturing the pumps. The production rate varies for each machine and also the unit product cost. Daily demand and machine availability are given below.

Demand Information

	Product		
	A	B	C
Demand (units)	2000	15000	700

Machine Availability Details

Machine capacity (units)					
	1	2	3	4	5
Available	700	1000	1500	1200	800

Notes

Unit Product Cost

Machine	Product		
	A	B	C
1	150	80	75
2	120	95	60
3	112	100	60
4	121	95	50
5	125	75	50

Determine the minimum production schedule for the products and machines.

9. A company has plants at locations A, B and C with the daily capacity to produce chemicals to a maximum of 3000 kg, 1000 kg and 2000 kg respectively. The cost of production (per kg) are ₹ 800, ₹ 900 and ₹ 7.50 respectively. Customer's requirement of chemicals per day is as follows:

Customer	Chemical Required	Price offered
1	2000	200
2	1000	215
3	2500	225
4	1000	200

Transportation cost (in rupees) per kg from plant locations to customer's place is given in table.

		Customer			
		1	2	3	4
Plant	A	5	7	10	12
	B	7	3	4	2
	C	4	6	3	9

Find the transportation schedule that minimizes the total transportation cost.

10. A transportation model has four supplies and five destinations. The following table shows the cost of shipping one unit from a particular supply to a particular destination.

Source	Destination					Supply
	1	2	3	4	5	
1	13	6	9	6	10	13
2	8	2	7	7	9	15
3	2	12	5	8	7	13
Demand	10	15	7	10	2	

The following feasible transportation pattern is proposed:

$$x_{11} = 10, x_{12} = 3, x_{22} = 9, x_{23} = 6, x_{33} = 9, x_{34} = 4, x_{44} = 9, x_{45} = 5.$$

Test whether these allocations involve least transportation cost. If not, determine the optimal solution.

11. A company has four factories situated in four different locations in the state and four company showrooms in four other locations outside the state. The per unit sale price, transportation cost and cost of production is given in table below, along with weekly requirement.

Factory	Showrooms				Cost of production (₹)
	1	2	3	4	
A	9	4	5	3	12
B	4	4	4	4	17
C	4	6	5	6	19
D	8	7	7	4	17

Factory	Weekly Capacity (units)	Weekly demand (units)
A	15	10
B	20	14
C	25	20
D	20	22

Determine the weekly distribution schedule to maximize the sales profits.

12. A computer manufacturer has decided to launch an advertising campaign on television, magazines and radio. It is estimated that maximum exposure for these media will be 70, 50, and 40 million respectively. According to a market survey, it was found that the minimum desired exposures within age groups 15-20, 21-25, 26-30, 31-35 and above 35 are 10, 20, 25, 35 and 55 million respectively. The table below gives the estimated cost in paise per exposure for each of the media. Determine an advertising plan to minimize the cost.

Media	Age Groups				
	15-20	21-25	26-30	31-35	above 35
TV	14	9	11	11	12
Radio	11	7	6	7	8
Magazine	9	10	7	10	8

Solve the problem and find the optimal solution, i.e., maximum coverage at minimum cost.

13. A garment manufacturer has 4 units I, II, III, and IV, the production from which are received by 4 direct customers. The weekly production of each manufacturing unit is 1200 units and all the units are of the same capacity. The company supplies the entire production from one unit to one supplier. Since the customers are situated at different locations, the transportation cost per unit varies. The unit cost of transportation is given in the table. As per the company's policy, the supply from unit B is restricted to customer 2 and 4, and from unit D to customer 1 and 3. Solve the problem to cope with the supply and demand constraints.

Manufacturing unit	1	2	3	4
A	4	6	8	3
B	4	-	5	-
C	6	5	5	9
D	-	7	-	6

Notes

14. A company dealing in home appliances has a sales force of 20 men who operate from three distribution centers. The sales manager feels that 5 salesmen are needed to distribute product line I, 6 to distribute product line II, 5 for product line III and 4 to distribute product line IV. The cost (in ₹) per day of assigning salesmen from each of the offices are as follows:

		Product Line			
		I	II	III	IV
Source	A	10	12	13	9
	B	9	11	12	13
	C	7	8	9	10

Currently, 8 salesmen are available at center A, 5 at center B and 7 at center C. How many salesmen should be assigned from each center to sell each product line, in order to minimize the cost? Is the solution unique?

15. Solve the following degenerate transportation problem:

		Destination			
		I	II	III	Supply
Source	A	7	3	4	2
	B	2	1	3	3
	C	3	4	6	5
Demand		4	1	5	

Answers: Self Assessment

- | | |
|----------------|------------|
| 1. False | 2. False |
| 3. False | 4. False |
| 5. False | 6. True |
| 7. False | 8. True |
| 9. Two | 10. Lowest |
| 11. Degeneracy | |

6.12 Further Readings



Books

G Dantzig, *Linear Programming and Extensions*, Princeton University Press.
 G Dantzig, M N Thapa, *Linear Programming 2: Theory and Extensions*, Princeton University Press.
 Robert J. Vanderbei, *Linear Programming: Foundations and Extensions*, Princeton University Press.
 S Jaisanker, *Quantitative Techniques for Management*, Excel Books.



Online links

www.adbook.net
www.mathresources.com

Unit 7: Transportation Problem – Optimality Tests

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Summary

Keywords

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Objectives

After studying this unit, you will be able to:

- Understand the significance of optimality tests in transportation problem
- Learn to drive optimal solution using Modified Distribution method and Stepping stone Method
- Construct the transshipment transportation table for transshipment problem
- Examine multiple optimal solutions, and prohibited routes in the transportation problem

Introduction

Once the initial feasible solution is reached, the next step is to check the optimality. An optimal solution is one where there is no other set of transportation routes that would reduce the total transportation cost, for which we have to evaluate each unoccupied cell (which represents unused routes) in terms of opportunity cost. In this process, if there is no negative opportunity cost, the solution is an optimal solution.

7.1 Techniques of Finding Optimal Solution

Optimality can be tested by two ways namely:

1. Stepping Stone Method
2. Modified Distribution Method

Let us understand each of them one by one.

7.2 Stepping Stone Method

It is a method for computing optimum solution of a transportation problem.

Steps Involved:

Step 1: Determine an initial basic feasible solution using any one of the following:

- (a) North West Corner Rule
- (b) Matrix Minimum Method
- (c) Vogel Approximation Method

Step 2: Make sure that the number of occupied cells is exactly equal to $m+n-1$, where m is the number of rows and n is the number of columns.

Step 3: Select an unoccupied cell.

Step 4: Beginning at this cell, trace a closed path using the most direct route through at least three occupied cells used in a solution and then back to the original occupied cell and moving with only horizontal and vertical moves. The cells at the turning points are called "Stepping Stones" on the path.

Step 5: Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, starting with the plus sign at unoccupied cell to be evaluated.

Step 6: Compute the net change in the cost along the closed path by adding together the unit cost figures found in each cell containing a plus sign and then subtracting the unit costs in each square containing the minus sign.

Step 7: Check the sign of each of the net changes. If all the net changes computed are greater than or equal to zero, an optimum solution has been reached. If not, it is possible to improve the current solution and decrease the total transportation cost.

Step 8: Select the unoccupied cell having the most negative net cost change and determine the maximum number of units that can be assigned to a cell marked with a minus sign on the closed path corresponding to this cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.

Step 9: Repeat the procedure until you get an optimum solution



Example: Consider the following transportation problem (cost in rupees). Find the optimum solution

Factory	D	E	F	G	Capacity
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Requirement	400	450	350	500	1700

Solution: First, we find out an initial basic feasible solution by Matrix Minimum Method

Notes

Factory	D	E	F	G	Capacity
A	4	6 ⁴⁵⁰	8	6 ²⁵⁰	700
B	3 ⁵⁰	5	2 ³⁵⁰	5	400
C	3 ³⁵⁰	9	6	5 ²⁵⁰	600
Requirement	400	450	350	500	1700

Here, $m + n - 1 = 6$. So the solution is not degenerate.

The cell AD (4) is empty so allocate one unit to it. Now draw a closed path from AD.

Factory	D	E	F	G	Capacity
A	4 ⁺¹	6 ⁴⁵⁰	8	6 ²⁴⁹	700
B	3 ⁵⁰	5	2 ³⁵⁰	5	400
C	3 ³⁴⁹	9	6	5 ²⁵¹	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is $+4 - 6 + 5 - 3 = 0$.

This indicates that every unit allocated to route AD will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

Choose another unoccupied cell. The cell BE is empty so allocate one unit to it.

Factory	D	E	F	G	Capacity
A	4	6 ⁴⁴⁹	8	6 ²⁵¹	700
B	3 ⁴⁹	5 ⁺¹	2 ³⁵⁰	5	400
C	3 ³⁵¹	9	6	5 ²⁴⁹	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is $+5 - 6 + 6 - 5 + 3 - 3 = 0$

This indicates that every unit allocated to route BE will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

The allocations for other unoccupied cells are:

Unoccupied cells	Increase in cost per unit of reallocation	Remarks
CE	$+9 - 6 + 6 - 5 = 4$	Cost Increases
CF	$+6 - 3 + 3 - 2 = 4$	Cost Increases
AF	$+8 - 6 + 5 - 3 + 3 - 2 = 5$	Cost Increases
BG	$+5 - 5 + 3 - 3 = 0$	Neither increase nor decrease

Since all the values of unoccupied cells are greater than or equal to zero, the solution obtained is optimum.

Minimum transportation cost is:

$$6 \times 450 + 6 \times 250 + 3 \times 250 + 2 \times 250 + 3 \times 350 + 5 \times 250 = ₹ 7350$$

Notes

Self Assessment

Multiple Choice Questions:

1. When total supply is equal to the total demand in a transportation problem, the problem is said to be
 - (a) Balanced
 - (b) Unbalanced
 - (c) Degenerate
 - (d) All of the above
2. Which of the following methods is used to verify the optimality of the current solution of the transportation problem?
 - (a) Least Cost Method
 - (b) Vogel's Approximation Method
 - (c) MODI method
 - (d) All of the above
3. MODI method and stepping stone method is used for this purpose
 - (a) To find out initial basic feasible solution
 - (b) Optimal solution
 - (c) Multiple solution
 - (d) All of the above

7.3 Modified Distribution Method (MODI)

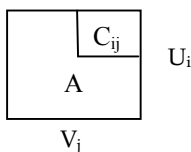
Another method to check optimality is that of MODI.

Steps Involved:

Step 1: Row 1, row 2, ..., row i of the cost matrix are assigned with variables U_1, U_2, \dots, U_i and the column 1, column 2, ..., column j are assigned with variables V_1, V_2, \dots, V_j respectively.

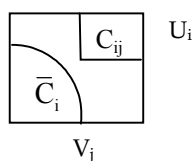
Step 2: Initially, assume any one of U_i values as zero and compute the values for U_1, U_2, \dots, U_i and V_1, V_2, \dots, V_j by applying the formula for occupied cell.

For occupied cells,

$$C_{ij} + U_i + V_j = 0$$


Step 3: Obtain all the values of C_{ij} for unoccupied cells by applying the formula for unoccupied cell. For unoccupied cells,

Opportunity Cost, $= C_{ij} + U_i + V_j$



If \bar{C}_{ij} values are > 0 then, the basic initial feasible solution is optimal.

If \bar{C}_{ij} values are $= 0$ then, the multiple basic initial feasible solution exists.

If \bar{C}_{ij} values are < 0 then, the basic initial feasible solution is not optimal.



Notes The MODI method is based on the concept of Duality.



Did u know? MODI method is also known as u-v method or method of multipliers.

7.4 Procedure for Shifting of Allocations

Select the cell which has the most negative \bar{C}_{ij} value and introduce a positive quantity called ' θ ' in that cell. To balance that row, allocate a ' $-\theta$ ' to that row in occupied cell. Again, to balance that column put a positive ' θ ' in an occupied cell and similarly a ' $-\theta$ ' to that row. Connecting all the ' θ 's and ' $-\theta$'s, a closed loop is formed.

Two cases are represented in Table 6.1 (a) and 6.1 (b). In Table 6.1 (a) if all the θ allocations are joined by horizontal and vertical lines, a closed loop is obtained.

The set of cells forming a closed loop is

$$CL = \{(A, 1), (A, 3), (C, 3), (C, 4), (E, 4), (E, 1), (A, 1)\}$$

The loop in Table 4.22(b) is not allowed because the cell (D3) appears twice.

Table 7.1(a)

	1	2	3	4
A	*		*	
B				
C			*	*
D				
E	*			*


Table 7.1(b)

	1	2	3	4
A				
B	*		*	
C				
D	*		*	*
E			*	*

Notes

Conditions for Forming a Loop

1. The start and end points of a loop must be the same.
2. The lines connecting the cells must be horizontal and vertical.
3. The turns must be taken at occupied cells only.
4. Take a shortest path possible (for easy calculations).




Notes

1. Every loop has an even number of cells and at least four cells.
2. Each row or column should have only one '+' and '-' sign.
3. Closed loop may or may not be square in shape. It can also be a rectangle or a stepped shape.
4. It doesn't matter whether the loop is traced in a clockwise or anti-clockwise direction.

Take the most negative '- q' value, and shift the allocated cells accordingly by adding the value in positive cells and subtracting it in the negative cells. This gives a new improved table. Then go to step 5 to test for optimality.

Calculate the Total Transportation Cost

Since all the \bar{C}_{ij} values are positive, optimality is reached and hence the present allocations are the optimum allocations. Calculate the total transportation cost by summing the product of allocated units and unit costs.



Task Find the initial basic solution for the transportation problem and hence solve it.

Source	Destination				Supply
	1	2	3	4	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	1200

7.5 Prohibited Routes Problem

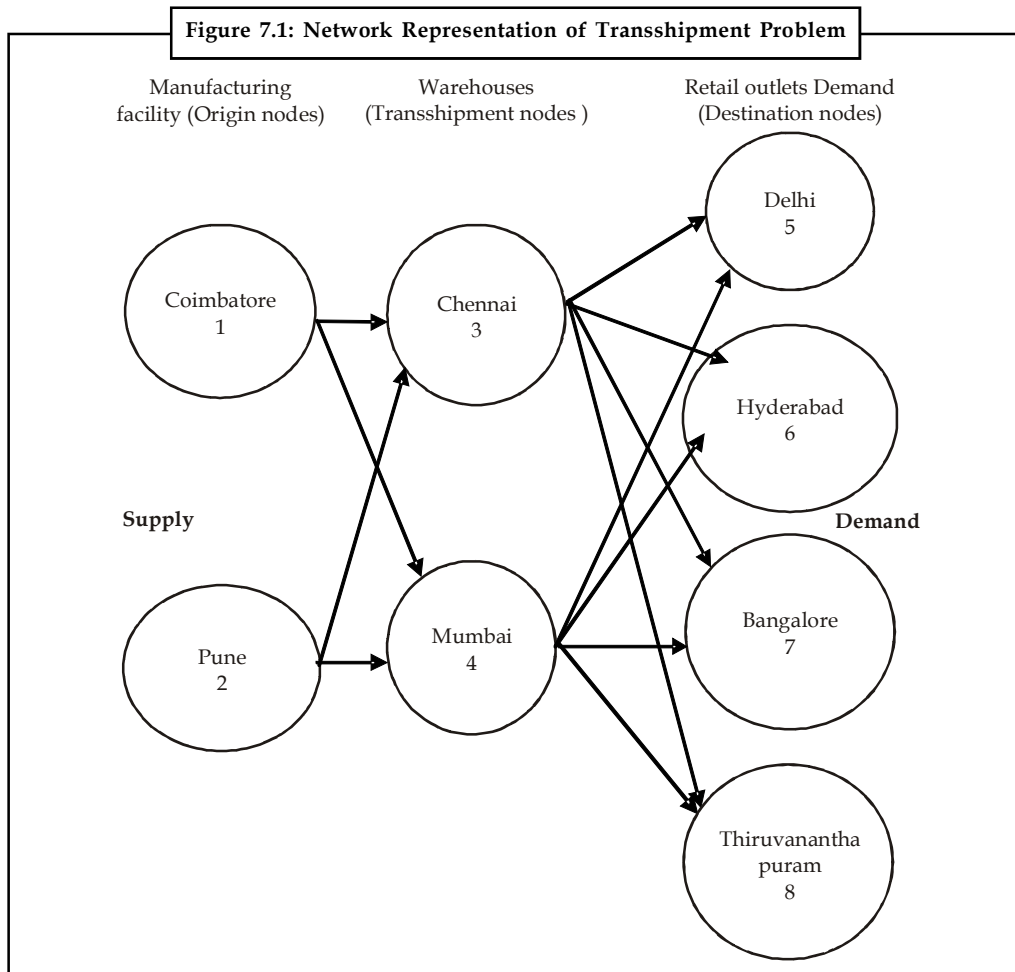
In practice, there may be routes that are unavailable to transport units from one source to one or more destinations. The problem is said to have an unacceptable or prohibited route. To overcome such kind of transportation problems, assign a very high cost to prohibited routes, thus preventing them from being used in the optimal solution regarding allocation of units.

7.6 Transshipment Problem

The transshipment problem is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.

Each of these nodes in turn supply to other destinations. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum transportation cost.

Considering a company with its manufacturing facilities situated at two places, Coimbatore and Pune. The units produced at each facility are shipped to either of the company’s warehouse hubs located at Chennai and Mumbai. The company has its own retail outlets in Delhi, Hyderabad, Bangalore and Thiruvananthapuram. The network diagram representing the nodes and transportation per unit cost is shown in Figure 7.1. The supply and demand requirements are also given.



Solving Transshipment Problem using Linear Programming

Let

X_{ij} be the number of units shipped from node i to node j ,

X_{13} be the number of units shipped from Coimbatore to Chennai,

X_{24} be the number of units shipped from Pune to Mumbai, and so on

Table 7.2 shows the unit transportation cost from sources to destination.

Notes

Table 7.2: TP of the Shipment

	Warehouse			
Facility	Chennai		Mumbai	
Coimbatore	4		7	
Pune	6		3	

Warehouses	Retail outlets			
	Delhi	Hyderabad	Bangalore	Thiruvananthapuram
Chennai	7	4	3	5
Mumbai	5	6	7	8

Objective: The objective is to minimize the total cost

$$\text{Minimize } Z = 4X_{13} + 7X_{14} + 6X_{23} + 3X_{24} + 7X_{35} + 4X_{36} + 3X_{37} + 5X_{38} + 5X_{45} + 6X_{46} + 7X_{47} + 8X_{48}$$

Constraints: The number of units shipped from Coimbatore must be less than or equal to 800. Because the supply from Coimbatore facility is 800 units. Therefore, the constraints equation is as follows:

$$X_{13} + X_{14} \leq 800 \tag{i}$$

Similarly, for Pune facility

$$X_{23} + X_{24} \leq 600 \tag{ii}$$

Now, considering the node 3,

Number of units shipped out from node 1 and 2 are,

$$X_{13} + X_{23}$$

Number of units shipped out from node 3 is,

$$X_{35} + X_{36} + X_{37} + X_{38}$$

The number of units shipped in must be equal to number of units shipped out, therefore

$$X_{13} + X_{23} = X_{35} + X_{36} + X_{37} + X_{38}$$

Bringing all the variables to one side, we get

$$-X_{13} - X_{23} + X_{35} + X_{36} + X_{37} + X_{38} = 0 \tag{iii}$$

Similarly for node 4

$$-X_{14} - X_{24} + X_{45} + X_{46} + X_{47} + X_{48} = 0 \tag{iv}$$

Now considering the retail outlet nodes, the demand requirements of each outlet must be satisfied. Therefore for retail node 5, the constraint equation is

$$X_{35} + X_{45} = 350 \tag{v}$$

Similarly for nodes 6, 7, and 8, we get,

$$X_{36} + X_{46} = 200 \tag{vi}$$

$$X_{37} + X_{47} = 400 \tag{vii}$$

$$X_{38} + X_{48} = 450 \tag{viii}$$

Linear Programming formulation,

$$\text{Minimize } Z = 4X_{13} + 7X_{14} + 6X_{22} + 3X_{24} + 7X_{35} + 4X_{36} + 3X_{37} + 5X_{38} + 5X_{45} + 6X_{46} + 7X_{47} + 8X_{48}$$

Subject to constraints,

$$\left. \begin{aligned} X_{13} + X_{14} &\leq 800 \\ X_{23} + X_{24} &\leq 600 \end{aligned} \right\} \text{origin constraints}$$

$$-X_{13} - X_{23} + X_{35} + X_{36} + X_{37} + X_{38} = 0$$

$$-X_{14} - X_{24} + X_{45} + X_{46} + X_{47} + X_{48} = 0$$

$$\left. \begin{aligned} X_{35} + X_{45} &= 350 \\ X_{36} + X_{46} &= 200 \\ X_{37} + X_{47} &= 400 \\ X_{38} + X_{48} &= 450 \end{aligned} \right\} \text{Destination constraints}$$



Caution Don't confuse between transportation and transshipment problem

A transportation problem can be converted into a transshipment problem by relaxing the restrictions on the receiving and sending the units on the origins and destinations respectively. A m-origin, n-destination, transportation problem, when expressed as transshipment problem, shall become an enlarged problem: with m + n origins and an equal number of destinations. With minor modifications, this problem can be solved using the transportation method.



Case Study

Transportation for Manufacturing Unit

A toy manufacturer wants to open a third warehouse that will supply three retail outlets. The new warehouse will supply **500 units** of backyard play sets per week. Two locations are being studied, N1 and N2. Refer to the table below for transportation costs per play set from each warehouse to each store locations.

Warehouse	Store Location		
	A	B	C
N1	\$6	\$8	\$7
N2	\$10	\$6	\$4

The existing system is shown on the following table.

Warehouse	Store Location			Capacity
	A	B	C	
1	\$8	\$3	\$7	500
2	\$5	\$10	\$9	400
Demand	400	600	400	

Question:

Which location would result in the lower transportation cost for the system?

7.7 Degeneracy in Transportation Problems

Degeneracy involves two steps:

1. **Check for degeneracy:** The solution that satisfies the above said conditions $N = m + n - 1$ is a non-degenerate basic feasible solution otherwise, it is a degenerate solution. Degeneracy may occur either at the initial stage or at subsequent iterations.

If number of allocations, $N = m + n - 1$, then degeneracy does not exist, one has to go to the next step.

If number of allocations, $N = m + n - 1$, then degeneracy does exist, and has to be resolved before going to the next step.

2. **Resolving Degeneracy:** To resolve degeneracy at the initial solution, allocate a small positive quantity ϵ to one or more unoccupied cell that have lowest transportation costs, so as to make $m + n - 1$ allocations (i.e., to satisfy the condition $N = m + n - 1$). The cell chosen for allocating ϵ must be of an independent position. In other words, the allocation of ϵ should avoid a closed loop and should not have a path.

The following Table 7.3 shows independent allocations.

Table 7.3: Independent Allocations							
				*	*		
*	*	*				*	
	*				*		*

The following Tables 7.3 (a), (b) and (c) show non-independent allocations.

Table 7.3: (a) Independent Allocations, (b) and (c)							
				*			*
	*	*					
	*	*		*			*
(a)				(b)			
*	*						
	*						
*	*						
(c)							

Notes

Self Assessment

Fill in the blanks:

- 4. is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.
- 5. A verycost is applied to prohibited routes.
- 6. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum

Summary

- An optimal solution is one where there is no other set of transportation routes that would reduce the total transportation cost, for which we have to evaluate each unoccupied cell (which represents unused routes) in terms of opportunity cost.
- Optimality can be tested by two ways namely:
 - ❖ Stepping Stone Method
 - ❖ Modified Distribution Method
- Stepping Stones is method for computing optimum solution of a transportation problem.
- The transshipment problem is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.

Keywords

MODI Method: The modified distribution method, also known as MODI method or (u - v) method provides a minimum cost solution to the transportation problem.

Optimal Solution: A feasible solution is said to be optimal if it minimizes the total transportation cost.

Stepping Stone Method: In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Transshipment: When it is possible to ship both into and out of the same node.

Review Questions

- 1. What are the conditions for forming a closed loop?
- 2. How are the maximization problems solved using transportation model?
- 3. How is optimality tested in solving transportation problems?
- 4. In what ways is a transshipment problem different from a transportation problem?

5. Solve the following transportation problems.

Notes

(a)

From	To			Available
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

(b)

From	To			Available
	A	B	C	
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Demand	6	10	15	

6. A potato chip manufacturer has three plants and four warehouses. Transportation cost for shipping from plants to warehouses, the plant availability and warehouses requirements are as follows:

Plants	Warehouses				Plant Availability (quintals)
	W ₁	W ₂	W ₃	W ₄	
F ₁	7	4	3	5	235
F ₂	6	8	7	4	280
F ₃	5	6	9	10	110
Requirements (quintals)	125	160	110	230	625

Find optimum shipping schedule.

7. Solve

	Destination				Available
	1	2	3	4	
1	21	16	25	13	11
2	17	18	14	23	13
3	32	27	18	41	19
Requirement	6	10	12	15	43

8. Find the optimal solution.

	D ₁	D ₂	D ₃	D ₄	Available
O ₁	23	27	16	18	30
O ₂	12	17	20	51	40
O ₃	22	28	12	32	53
Required	22	35	25	41	123

The cell entries are unit transportation cost.

Notes

9. Six plants with capacities of 100, 70, 80, 20, 55 and 95 respectively, supply goods to five stores requiring 50, 200, 30, 60 and 80 units respectively. The transportation costs matrix are given below. Find the optimal solution:

Plants	Stores				
	1	2	3	4	5
1	10	8	6	3	5
2	2	9	9	8	0
3	0	10	15	8	7
4	6	3	12	15	0
5	8	9	6	2	1
6	20	0	8	5	11

10. Find the minimum transportation cost.

Source	Destination					Available
	D ₁	D ₂	D ₃	D ₄	D ₅	
S ₁	4	7	3	8	2	4
S ₂	1	4	7	3	8	7
S ₃	7	2	4	7	7	9
S ₄	4	7	2	4	7	2
Requirement	8	3	7	2	2	-

11. Solve the following transportation problem for maximum profit:

Warehouse	Per unit Profit (Rs.) Market				Availability of Warehouses
	A	B	C	D	
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand in the Market	180	320	100	400	

Answers: Self Assessment

- | | |
|---------|------------------------|
| 1. (a) | 2. (c) |
| 3. (b) | 4. Transshipment |
| 5. High | 6. Transportation Cost |

Further Readings

Notes



Books

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Online links

http://myopm.net/courses/common/dt_module/tp_tutorial.pdf

www.nos.org

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Unit 08: Project Management CPM

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Objectives

After studying this unit, you will be able to:

- Understand network diagram.
- Identify critical activities.
- Schedule projects efficiently.

Introduction

Project management is the process of managing a team's efforts to meet all project objectives while staying within budget restrictions. Typically, this data is described in project documentation, which is prepared at the start of the development process. Scope, time, and budget are the three main restraints.

On a very basic level, project management includes the planning, initiation, execution, monitoring, and closing of project. Many different types of project management methodologies and techniques exist, including traditional, waterfall, agile, and lean. Project management is used across industries and is an important part of the success of construction, engineering, and IT companies.

Project management is a systematic and dynamic discipline that encompasses the planning, execution, monitoring, and control of projects to achieve specific goals and objectives within defined constraints. It is a vital process across industries and sectors, serving as the guiding framework for effectively and efficiently managing projects of varying size, complexity, and scope. The primary aim of project management is to ensure the successful completion of projects on time, within budget, and meeting the desired quality standards while maximizing the utilization of

available resources. It involves the application of knowledge, skills, tools, and techniques to navigate through challenges, risks, and uncertainties that may arise during the project's life cycle. By adopting proven methodologies, fostering collaboration, and employing effective leadership, project management empowers organizations to tackle intricate tasks, respond to dynamic environments, and deliver value to stakeholders. From large-scale infrastructure developments to innovative product launches and intricate IT implementations, project management provides the roadmap and discipline necessary to transform ideas into tangible and successful outcomes.

8.1 Project Management

Project management is a systematic and structured approach to planning, organizing, executing, controlling, and closing projects to achieve specific objectives within defined constraints such as time, budget, scope, and resources. It involves the application of knowledge, skills, tools, and techniques to meet project requirements and deliver successful outcomes. Project management is widely used across various industries and fields, ranging from construction and engineering to software development, event planning, marketing campaigns, and research initiatives.

Following definitions appropriately define project management:

- The techniques of operations research used for planning, scheduling and controlling large and complex projects are often referred to as *network analysis, network planning or network planning and scheduling techniques*.
- Project management is the discipline of using established principles, procedures and policies to successfully guide a project from conception through completion.

Project management is a comprehensive and structured approach that enables organizations to plan, execute, and deliver projects successfully. It starts with project initiation, where the project's purpose, objectives, scope, and stakeholders are identified. The planning phase follows, during which project managers create detailed schedules, define tasks, estimate resources, and allocate responsibilities. Risk management, quality assurance, communication, and stakeholder engagement strategies are also outlined. Once the plan is in place, the execution phase begins, where teams implement the project tasks and utilize resources to achieve deliverables. Concurrently, project progress is continuously monitored and controlled to ensure adherence to the project plan, budget, and scope.

Effective risk management is crucial, involving the identification, assessment, and mitigation of potential risks throughout the project's life cycle. As changes are common in projects, change management processes are essential to evaluate, approve, and implement modifications while considering their impact on project objectives. When the project is complete, the closure phase formalizes acceptance by stakeholders and finalizes all activities. Additionally, project documentation is maintained to provide transparency, accountability, and insights for future reference.

Various project management methodologies like Agile, Waterfall, Scrum, and Kanban offer different approaches based on project characteristics and requirements. Success in project management is facilitated by effective leadership, communication, teamwork, and adherence to project goals within the allocated resources and timeline. By employing project management principles and methodologies, organizations enhance efficiency, productivity, and their ability to deliver successful projects that meet stakeholders' expectations.

8.2 Application of Project Management

Project management finds widespread application across various industries and sectors. It is used to successfully plan, execute, and control a diverse range of projects. Some common applications of project management include:

Construction Projects: In the construction industry, project management ensures efficient coordination of activities, adherence to timelines, and proper allocation of resources for building infrastructure such as roads, bridges, buildings, and dams.

Information Technology (IT) Projects: IT projects, such as software development, system implementations, and website launches, rely on project management to manage requirements, development cycles, testing, and deployment.

Product Development: From conceptualization to launch, project management is utilized to guide the development of new products, ensuring that they meet customer needs and are brought to market on time.

Event Planning: Organizing events like conferences, concerts, and trade shows require meticulous planning, scheduling, and resource management, all of which fall under project management.

Marketing Campaigns: Marketing initiatives, including product launches, advertising campaigns, and social media strategies, benefit from project management to coordinate tasks, allocate budgets, and track performance.

Research Projects: In academia and scientific research, project management helps researchers plan and execute studies, manage data collection, and meet research objectives efficiently.

Healthcare Projects: Projects in healthcare, such as building new medical facilities, implementing electronic health record systems, or organizing vaccination campaigns, are managed using project management principles.

Engineering Projects: Engineers use project management techniques to develop and execute projects related to infrastructure, manufacturing, and product design.

Organizational Change Initiatives: When organizations undergo significant changes, such as mergers, restructuring, or process improvements, project management is essential to manage the transition effectively.

Non-Profit and Social Projects: Project management is applied in non-profit organizations and social initiatives to plan and execute projects that address social issues, humanitarian aid, and community development.

Film and Media Production: Filmmakers and media producers use project management to coordinate pre-production, production, and post-production activities to create movies, television shows, and other media content.

The application of project management extends to virtually any endeavor involving defined objectives, resources, and timelines. It provides a structured and disciplined approach to achieve successful outcomes in various industries, ensuring that projects are completed efficiently, within budget, and to the satisfaction of stakeholders.

8.3 Critical Path Method

The Critical Path Method (CPM) is a project management technique used to schedule activities in a project and determine the most efficient sequence of activities to complete the project as quickly as possible. It identifies the critical path, which is the sequence of activities that must be completed in the shortest time to ensure the project's overall duration is minimized. CPM is widely used in industries such as construction, engineering, software development, and any other projects that involve a series of interconnected tasks.

The Critical Path Method (CPM) is a project management technique that involves creating an activity network diagram to schedule and manage complex projects. In this method, each activity is represented as a node in the network, and the dependencies between activities are illustrated with arrows connecting the nodes. The duration of each activity is estimated, and the network diagram helps visualize the order in which activities must be completed to move forward in the project. The CPM then employs two calculation passes: the forward pass and the backward pass. The forward pass calculates the earliest start and finish times for each activity based on its predecessors, while the backward pass determines the latest start and finish times for each activity without delaying the overall project completion.

The critical path is the sequence of activities with the longest duration and has zero float, meaning any delay in these activities would delay the project's completion time. As such, the critical path represents the minimum time required to complete the entire project. Efficiently managing activities on the critical path is essential for ensuring the project is completed on time. Non-critical

activities, on the other hand, have float, meaning they can be delayed without affecting the overall project duration. By identifying the critical path, project managers can focus their efforts and resources on crucial activities to avoid delays and complete the project as quickly as possible.

8.4 Some Use Case of CPM Application

CPM provides project managers with a comprehensive tool for scheduling, resource allocation, and time management. It helps them identify potential bottlenecks, allocate resources efficiently, and make informed decisions to keep the project on track. In case the project's scope changes or unexpected delays occur, project managers can recalculate the critical path and adapt their strategies accordingly. With the aid of project management software and various CPM tools, project managers can effectively plan, monitor, and control complex projects, ultimately improving project success rates and enhancing overall project efficiency.

Highway Construction:

In a highway construction project, CPM is applied to plan and schedule various activities such as surveying, earthwork, asphalt laying, and bridge construction. By identifying the critical path, the project manager can focus on activities that directly impact the project's duration. For instance, delays in earthwork or bridge construction could significantly impact the overall project timeline, leading to increased costs and potential penalties for late completion.

New Product Development:

In a product development project, CPM is used to schedule tasks like product design, prototyping, testing, manufacturing, and marketing. By identifying the critical path, the product development team can prioritize tasks and allocate resources effectively. For example, if the testing phase is on the critical path, the team needs to ensure that tests are conducted promptly to meet the planned launch date.

Event Planning for a Conference:

When planning a conference, CPM helps in organizing tasks such as venue selection, speaker invitations, event marketing, and logistics. By identifying the critical path, the event planner can ensure that key activities, like confirming speakers and securing the venue, are completed on time. Delays in these critical tasks could impact the entire event schedule and lead to last-minute changes.

Movie Production:

In a film production project, CPM is utilized to schedule tasks like scriptwriting, casting, shooting, post-production, and distribution. Identifying the critical path helps the film production team focus on crucial activities like shooting and editing to meet the movie's release date. Delays in the post-production phase, for example, could lead to a delayed movie release and affect box office revenue.

Research Study:

In a scientific research project, CPM assists in planning tasks such as literature review, data collection, analysis, and report writing. By identifying the critical path, researchers can prioritize activities with strict deadlines, such as conference paper submissions. Delays in data collection or analysis could affect the research project's timely completion and dissemination of findings.

IT System Implementation:

During an IT system implementation, CPM helps in organizing tasks like requirements gathering, system configuration, testing, and user training. By identifying the critical path, the IT team can allocate resources effectively and ensure that tasks like user training are completed before the system goes live. Delays in user training could lead to challenges during system adoption.

Event Management for a Wedding:

In organizing a wedding, CPM is applied to schedule tasks such as venue booking, catering arrangements, invitation distribution, and floral decorations. By identifying the critical path, the wedding planner can ensure that tasks like venue booking and catering confirmation are completed early to secure preferred choices and avoid last-minute hassles.

In all these examples, the Critical Path Method proves instrumental in optimizing project scheduling, resource allocation, and decision-making. By identifying critical activities and potential

bottlenecks, project managers and teams can proactively address challenges, adhere to timelines, and achieve successful project outcomes.

8.5 Key features of the Critical Path Method

Activity Network Diagram:

The first step in CPM is to create an activity network diagram, also known as a network diagram or PERT (Program Evaluation and Review Technique) chart. The diagram represents the project's activities as nodes (or circles) and the connections between activities as arrows (or lines). Each activity has a duration and is usually labeled with a unique identifier.

Activity Durations:

For each activity in the network, the project manager or team estimates the time it will take to complete the activity. These estimated durations are crucial for calculating the project's overall duration and identifying the critical path.

Dependencies and Precedence Relationships:

Activities in the network have dependencies, meaning some activities must be completed before others can start. The network diagram visually represents these precedence relationships between activities.

Forward Pass:

The forward pass is a calculation process that starts from the project's beginning and moves forward through the network diagram. It determines the early start (ES) and early finish (EF) times for each activity. ES represents the earliest time an activity can start, and EF represents the earliest time it can finish, assuming all preceding activities are completed as early as possible.

Backward Pass:

The backward pass is another calculation process that starts from the project's end and moves backward through the network diagram. It determines the late start (LS) and late finish (LF) times for each activity. LS represents the latest time an activity can start without delaying the project's overall completion time, and LF represents the latest time it can finish.

Float or Slack:

The difference between an activity's late start (LS) and early start (ES) times (or late finish and early finish times) is known as float or slack. Float indicates the amount of time an activity can be delayed without affecting the project's overall duration. Activities on the critical path have zero float, meaning any delay in these activities will directly delay the entire project.

Critical Path:

The critical path is the longest sequence of activities that have zero float. It is the path through the network diagram that determines the shortest possible duration for the project. If any activity on the critical path is delayed, the entire project's completion time will be delayed.

Project Duration and Time Management:

The sum of durations of activities on the critical path represents the project's overall duration. By focusing on the critical path and managing activities along it, project managers can ensure the project is completed on time.

CPM provides a structured approach to project scheduling, enabling project managers to identify potential delays, allocate resources efficiently, and make informed decisions to keep projects on track. The method is highly effective in complex projects with many interconnected activities and dependencies. Project management software often includes CPM tools to help manage and optimize project schedules.

8.6 Network Components

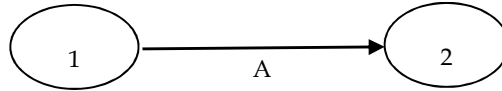
Network diagram is represented using following components:

Nodes: Nodes are breakpoints representing end of an activity and starting of another one.

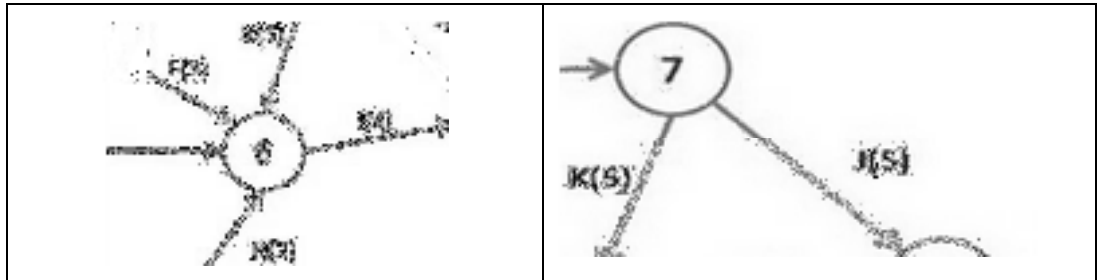
8.7 AOA and AON Representation

These are two different ways of representing activities, involved in a project.

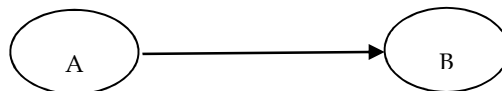
Activity on Arrow (AOA): Nodes are connected by arrows, each representing an activity. Nodes have different incremental numbers representing flow of the project.



If an activity is preceded by multiple activities, multiple activities may be converging on a single node. If multiple activities have a common preceding activity, multiple arrows burst out of a single node.



Activity on Node (AON): Arrows connect nodes, each representing an activity. Precedence condition can be easily represented without using dummy activities.



Both AOA and AON can be used for representation. We will be using AOA further in this text.

8.8 Errors and Dummies in PERT/CPM Network

Certain representations are not used to represent project flow. These errors are:

<p>A diagram showing a sequence of activities A, B, C, and E connected by arrows. Activity D is shown as a loop between nodes B and C, with arrows pointing from B to D and from D to C.</p>	<p>Looping: This is considered as an error because every project is expected to start and end within a time period. Looping results in never ending project representation</p>
<p>A diagram showing a sequence of activities A, B, C, and E connected by arrows. Activity D is shown as a vertical arrow pointing down from node C, but it does not connect to any other node.</p>	<p>Dangling: If one activity is dangling, its completion or non-completion doesn't affect project duration. This simply means that this activity is not a part of the project. Thus, such activities should either be removed or should be connected to end node.</p>
<p>A diagram showing two parallel activities, B and C, between the same two nodes. Activity B is represented by a top arrow and activity C by a bottom arrow.</p>	<p>Parallel Activities: Parallel activities should not be represented with same starting and ending node. Alternate representations can be picked based on the precedence conditions:</p> <p>Two diagrams showing alternate representations for parallel activities B and C between nodes A and D. The first diagram uses a dummy activity (represented by a dashed arrow) to connect the start of B to the start of C. The second diagram uses a dummy activity to connect the end of C to the end of B.</p>

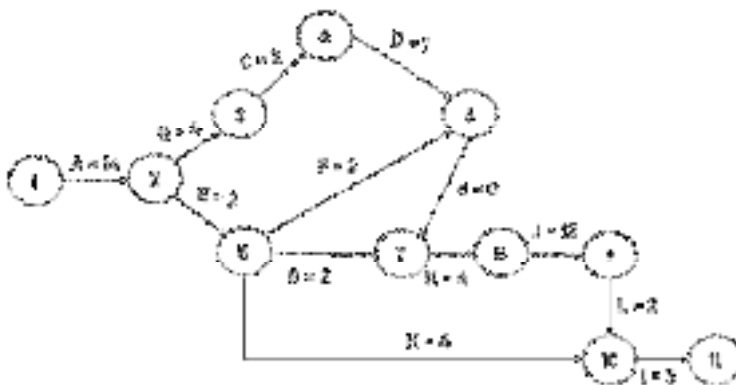
8.9 CPM Process

Critical path method involves following steps:

- Precedence diagram
- Forward Pass (Early Start & Early Finish)
- Backward Pass (Late Start & Late Finish)
- Critical Path (Connected activities with Zero Slack)

Activity	Precedence Activity	Duration
A	-	14
B	A	4
C	B	2
D	C	1
E	A	2
F	E	3
G	E	2
H	E	4
I	H,L	3
J	K	12
K	D,F,G	4
L	J	2

Step 1: Precedence diagram:



Here, $d=0$ represents dummy activity with duration = 0. Activities are represented on arrows.

Step - 2: Forward Pass:

Early start is a time by which any activity can start. Early finish is completion time of the activity and is calculated as:

$$\text{Early finish} = \text{Early start} + \text{Activity duration.}$$

Early start of following activity is calculated as:

$$\text{Early start (i+1)} = \text{Early finish (i)}$$

In case of multiple precedence activities, early start of following activity is calculated as:

$$\text{Early start (i+1)} = \text{Max. (Early finish of precedent activities)}$$

Early start and early finish calculations for forward pass is as followed:

$$E1=0$$

$$E2=E1+t_{1,2} [t_{1,2}=A=14]=0+14=14$$

$$E3=E2+t_{2,3} [t_{2,3}=B=4]=14+4=18$$

$$E4=E3+t_{3,4} [t_{3,4}=C=2]=18+2=20$$

$$E5 = E2 + t_{2,5} [t_{2,5} = E = 2] = 14 + 2 = 16$$

$$E6 = \text{Max}\{E_i + t_{i,6}\} [i=4,5] = \text{Max}\{E4 + t_{4,6}; E5 + t_{5,6}\} = \text{Max}\{20 + 1; 16 + 3\} = \text{Max}\{21; 19\} = 21$$

$$E7 = \text{Max}\{E_i + t_{i,7}\} [i=5,6] = \text{Max}\{E5 + t_{5,7}; E6 + t_{6,7}\} = \text{Max}\{16 + 2; 21 + 0\} = \text{Max}\{18; 21\} = 21$$

$$E8 = E7 + t_{7,8} [t_{7,8} = K = 4] = 21 + 4 = 25$$

$$E9 = E8 + t_{8,9} [t_{8,9} = J = 12] = 25 + 12 = 37$$

$$E10 = \text{Max}\{E_i + t_{i,10}\} [i=5,9] = \text{Max}\{E5 + t_{5,10}; E9 + t_{9,10}\} = \text{Max}\{16 + 4; 37 + 2\} = \text{Max}\{20; 39\} = 39$$

$$E11 = E10 + t_{10,11} [t_{10,11} = I = 3] = 39 + 3 = 42$$

Step - 3: Backward Pass

Late finish is maximum possible completion time of the activity. Late start is a time by which any activity should start and is calculated as:

$$\text{Late finish} - \text{Activity duration} = \text{Late start}$$

Late finish of following activity is calculated as:

$$\text{Late finish (i)} = \text{Late start (i+1)}$$

In case of multiple precedence activities, early start of following activity is calculated as:

$$\text{Late finish (i)} = \text{Min. (Late start of following activities)}$$

Late start and late finish calculations for forward pass is as followed:

$$L11 = E11 = 42$$

$$L10 = L11 - t_{10,11} [t_{10,11} = I = 3] = 42 - 3 = 39$$

$$L9 = L10 - t_{9,10} [t_{9,10} = L = 2] = 39 - 2 = 37$$

$$L8 = L9 - t_{8,9} [t_{8,9} = J = 12] = 37 - 12 = 25$$

$$L7 = L8 - t_{7,8} [t_{7,8} = K = 4] = 25 - 4 = 21$$

$$L6 = L7 - t_{6,7} [t_{6,7} = d = 0] = 21 - 0 = 21$$

$$L5 = \text{Min}\{L_j - t_{5,j}\} [j=10,7,6] = \text{Min}\{L10 - t_{5,10}; L7 - t_{5,7}; L6 - t_{5,6}\} = \text{Min}\{39 - 4; 21 - 2; 21 - 3\} = \text{Min}\{35; 19; 18\} = 18$$

$$L4 = L6 - t_{4,6} [t_{4,6} = D = 1] = 21 - 1 = 20$$

$$L3 = L4 - t_{3,4} [t_{3,4} = C = 2] = 20 - 2 = 18$$

$$L2 = \text{Min}\{L_j - t_{2,j}\} [j=5,3] = \text{Min}\{L5 - t_{2,5}; L3 - t_{2,3}\} = \text{Min}\{18 - 2; 18 - 4\} = \text{Min}\{16; 14\} = 14$$

$$L1 = L2 - t_{1,2} [t_{1,2} = A = 14] = 14 - 14 = 0$$

Step-4: Critical Path

Critical path is identified by connecting starting and end node with set of activities where early start = late start (or early finish = late finish).

The critical path of the project is: 1-2-3-4-6-7-8-9-10-11 and critical activities are A, B, C, D, d, K, J, L, I. The total project time is 42.

8.10 Floats in CPM

It is the amount of activity time that can increase or delayed without delaying project completion time. This float is calculated as the difference between the latest finish time and the earliest finish time for the activity. Three types of floats are:

Total float (TF_{ij}) = (L _j - E _i) - t _{ij} = LS _{ij} - ES _{ij} = LF _{ij} - EF _{ij}	This is the length of time by which an activity can be delayed until all preceding activities are completed at their earliest possible time and all successor activities can be delayed until their latest permissible time.
Free float (FF_{ij}) = (E _j - E _i) - t _{ij} = Min {ES _{ij} , for all immediate successors of activity (i, j)} - EF _{ij}	This is the length of time by which the completion time of any non-critical activity can be delayed without causing any delay in its immediate successor activities.
Independent float (IF_{ij}) = (E _j - L _i) - t _{ij} = {ES _{ij} - LS _{ij} } - t _{ij}	This is the length of time by which completion time of any non-critical activity (i, j) can be delayed without causing any delay in its predecessor or the successor activities.

Floats in the above example can be calculated as:

Unit 08: Project Management

Activity (i,j)	Duration (t_{ij})	Earliest time start (E_i)	(E_j)	(L_i)	Latest time Finish (L_j)	Earliest time Finish ($E_i + t_{ij}$)	Latest time Start ($L_j - t_{ij}$)	Total Float ($L_j - E_i$) - t_{ij}	Free Float ($E_j - E_i$) - t_{ij}	Independent Float ($E_j - L_i$) - t_{ij}
2-5	3	14	16	14	16	17	16	2	0	0
5-6	1	16	21	18	21	19	18	2	3	0
6-7	2	16	21	18	21	18	19	3	3	1
5-10	4	16	30	18	30	20	30	10	10	17

Summary

Critical Path Method (CPM) is a valuable project management technique used in various industries and scenarios to effectively plan, schedule, and manage projects. It helps identify the critical activities that directly impact the project's duration, ensuring that they are completed efficiently and on time. In highway construction, CPM assists in organizing tasks like surveying and bridge construction to avoid delays and increased costs. For new product development, CPM prioritizes tasks like testing and manufacturing to meet launch dates. In event planning, CPM ensures key activities like venue selection and speaker invitations are completed promptly for a successful conference. In movie production, CPM focuses on shooting and post-production to meet release dates. In research studies, CPM aids in data collection and analysis for timely completion. For IT system implementation, CPM allocates resources efficiently for smooth user training and system adoption. In event management for weddings, CPM schedules crucial tasks like venue booking and catering arrangements for a seamless event. Overall, CPM optimizes project scheduling and resource allocation, helping achieve successful project outcomes.

Keywords

Critical Path: The longest sequence of dependent activities in a project, which determines the minimum time required to complete the project.

Activity Network Diagram: A visual representation of project activities and their dependencies, often using nodes and arrows to illustrate the project's flow.

Forward Pass: The process of calculating early start (ES) and early finish (EF) times for activities in the project network based on their dependencies and durations.

Backward Pass: The process of calculating late start (LS) and late finish (LF) times for activities, working backward from the project's end, to identify the total float or slack available.

Total Float/Slack: The amount of time an activity can be delayed without affecting the project's overall duration or critical path.

Critical Activities: Activities on the critical path with zero total float, meaning any delay in these activities will directly impact the project's completion time.

Resource Allocation: The process of assigning resources (human, material, and equipment) to project activities efficiently, considering their availability and skillset.

Project Schedule: A timeline that outlines the planned start and finish dates for all project activities, helping monitor progress and ensure timely completion.

Dependency: The relationship between project activities that determines their sequencing and determines which activities must be completed before others can start.

Project Management Software: Tools and applications used to assist project managers in planning, scheduling, and managing projects, often incorporating CPM functionality.

Optimization: The process of finding the best possible sequence of activities to achieve project goals while minimizing resource usage and project duration.

Milestones: Specific points in a project's timeline that represent significant achievements or deliverables, helping track progress and monitor project milestones.

Risk Management: The process of identifying, assessing, and mitigating potential risks that could impact project outcomes, schedule, or budget.

Project Execution: The phase of the project where planned activities are performed, and deliverables are produced, based on the project schedule.

Project Monitoring and Control: The continuous process of comparing actual project progress against the planned schedule and making adjustments as needed to keep the project on track.

Project Closure: The final phase of the project, involving formal acceptance of deliverables, documentation of lessons learned, and the conclusion of project activities.

Fast-Tracking: A project management technique that involves overlapping activities to accelerate the project schedule and reduce the overall duration.

Resource Leveling: A technique used to adjust project activities and resources to resolve resource conflicts and avoid overallocation or underutilization of resources.

Crashing: A project management technique that involves adding more resources or extending work hours to reduce the time needed for certain critical activities.

Float Analysis: The examination of total float values to identify non-critical activities with substantial float, which can be delayed without affecting the project's overall duration.

Self Assessment

1. What is the Critical Path Method (CPM) used for in project management?
 - A. Resource allocation
 - B. Risk assessment
 - C. Project scheduling
 - D. Quality control

2. Which of the following represents the critical path in a project network diagram?
 - A. The shortest path to project completion
 - B. The longest path to project completion
 - C. The path with the most activities
 - D. The path with the fewest dependencies

3. Total Float (Slack) in the Critical Path Method refers to:
 - A. The amount of time an activity can be delayed without delaying the project's completion.
 - B. The total time allocated for project execution.
 - C. The buffer time added to critical activities to ensure on-time delivery.
 - D. The difference between the actual project duration and the planned project duration.

4. Which project management technique involves overlapping activities to accelerate the project schedule?
 - A. Resource leveling
 - B. Fast-tracking
 - C. Crashing
 - D. Risk management

5. What is the purpose of a Gantt chart in project management?
 - A. To show the project's critical path.
 - B. To identify risks and uncertainties in the project.
 - C. To graphically represent the project schedule and progress.

- D. To allocate resources to project activities.
6. The process of identifying, assessing, and mitigating potential risks that could impact project outcomes is known as:
- A. Risk avoidance
 - B. Risk analysis
 - C. Risk management
 - D. Risk acceptance
7. Which project management phase involves formally accepting project deliverables and concluding project activities?
- A. Project initiation
 - B. Project planning
 - C. Project execution
 - D. Project closure
8. What is the purpose of resource leveling in project management?
- A. To optimize the use of resources and resolve conflicts.
 - B. To identify the critical path and its dependencies.
 - C. To accelerate project activities to meet tight deadlines.
 - D. To analyze the risks and uncertainties in the project.
9. What does the term "Milestone" represent in project management?
- A. The project's estimated completion date.
 - B. The longest path in the project network diagram.
 - C. Specific points in the project timeline that represent significant achievements.
 - D. The critical path activities that must be completed first.
10. Which project management tool is used to estimate the minimum and maximum time required for an activity?
- A. Gantt chart
 - B. PERT (Program Evaluation and Review Technique) chart
 - C. Network diagram
 - D. Resource histogram
11. What is the purpose of the Critical Path Method (CPM) in project management?
- A. To identify the least important activities in a project.
 - B. To determine the shortest path to project completion.
 - C. To allocate resources to project activities.
 - D. To prioritize high-risk activities in the project.
12. Which project management technique involves delaying non-critical activities to optimize resource utilization?

- A. Resource levelling
 - B. Fast-tracking
 - C. Crashing
 - D. Risk management
13. In the context of CPM, what does "ES" stand for?
- A. Early Start
 - B. Estimated Schedule
 - C. Efficient Scheduling
 - D. Event Sequence
14. In a project network diagram, what do arrows (lines) between nodes (circles) represent?
- A. The critical path
 - B. Project milestones
 - C. Project dependencies
 - D. Project resources
15. What is the purpose of a Work Breakdown Structure (WBS) in project management?
- A. To allocate resources to project tasks.
 - B. To identify critical activities in the project.
 - C. To organize and break down the project into manageable components.
 - D. To track project progress using graphical representation.
16. Which project management document outlines the project's purpose, objectives, scope, and stakeholders?
- A. Gantt chart
 - B. Project schedule
 - C. Project charter
 - D. Risk register
17. What does the term "Float" refer to in the context of the Critical Path Method?
- A. The amount of time an activity can be delayed without delaying the project.
 - B. The difference between the planned budget and the actual budget.
 - C. The total duration of the project.
 - D. The number of critical activities in the project.
18. In project management, what is the "Triple Constraint"?
- A. The critical path, total float, and resource allocation in a project.
 - B. The balance between project scope, schedule, and budget.
 - C. The project's critical activities, milestones, and dependencies.
 - D. The project manager's responsibility to manage time, cost, and quality.
19. Which project management tool is used to allocate resources to project tasks over time?

- A. Gantt chart
- B. Network diagram
- C. PERT chart
- D. Resource histogram

20. What is the purpose of a Project Risk Register in project management?

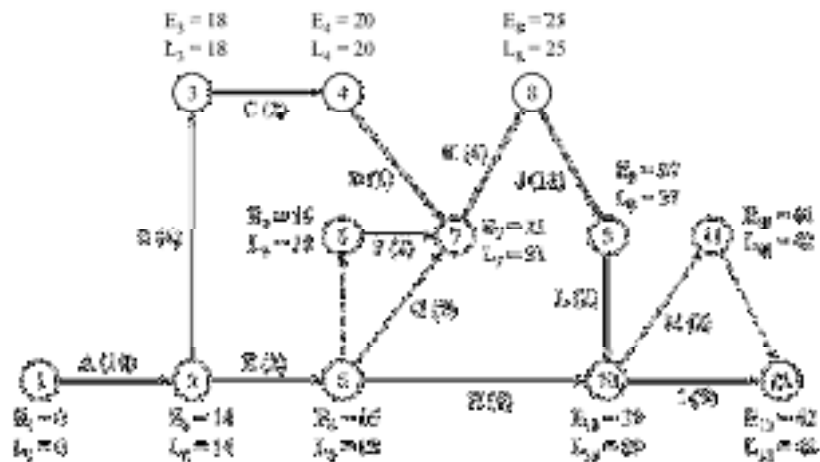
- A. To identify the critical path in the project.
- B. To allocate resources to high-risk activities.
- C. To track and manage project risks and their associated responses.
- D. To monitor the progress of project milestones.

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. C | 2. B | 3. A | 4. B | 5. C |
| 6. C | 7. D | 8. A | 9. C | 10. B |
| 11. B | 12. A | 13. A | 14. C | 15. C |
| 16. C | 17. A | 18. B | 19. D | 20. C |

Review Questions

1. Explain a real scenario where CPM technique can be applied.
2. Identify a real-life project where project duration and critical path can be calculated with PERT technique.
3. Observe any queue in public area. Identify number of servers and based on the concepts of queuing theory, suggest how number of servers can affect queue length.
4. What is traffic intensity? If traffic intensity is 0.40, what is the percentage of time a system remains idle?
5. Give two examples to illustrate the applications of queuing theory in business and industry.
6. A cold store has only one loading dock with 3 loaders. Trucks arrive at dock at an average rate of 4 trucks/hour and the arrival rate is Poisson distributed. The loading takes 10 minutes on an average and can be assumed to be exponentially distributed. The operating cost is Rs 20 /hour/truck and the loaders are paid Rs 6 each per hour. Would you advise the truck owner to add another crew of three loaders?
7. Refer the following diagram and calculate all floats:



8. In above question, if activity F is removed, what will be effect on:
 - a. Total project duration
 - b. Critical path
 - c. Floats
9. What are different floats in CPM. Explain significance of all three floats.
10. What is significance of expected activity duration and variance of activity in PERT?



Further Readings

- Operations Research, By Sivarethina mohan, McGraw-Hill Education (India) Pvt Limited
- Quantitative Techniques, Theory and Problems, By P. C. Tulsian, Pearson Education



Web Links

- <https://www.geeksforgeeks.org/project-evaluation-and-review-technique-pert/>
- <http://people.brunel.ac.uk/~mastjib/jeb/or/queue.html>

Unit 09: Project Management PERT

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Objective

After studying this chapter student will be able to

- understand the difference between PERT and CPM
- learn the about various time estimates of PERT
- appreciate the concept of floats

Introduction

In the dynamic landscape of project management, where timelines, resources, and uncertainties converge, the need for sophisticated tools to ensure efficient execution has never been more pronounced. The Project Evaluation and Review Technique (PERT) stands as a stalwart in this realm, offering project managers a systematic approach to tackle intricate projects with precision and strategic planning. Rooted in a rich history of development and refinement, PERT has emerged as a cornerstone methodology that empowers organizations to manage complexity, optimize resource allocation, and effectively mitigate risks. With its foundational principles deeply entrenched in rigorous analysis and a probabilistic perspective, PERT provides an invaluable framework for orchestrating projects of varying scales and complexities.

The origins of PERT can be traced back to the late 1950s, a pivotal period characterized by rapid technological advancements and the growing demand for managing large-scale projects with utmost efficiency. Born out of collaboration between the United States Navy, Lockheed Corporation, and consulting firm Booz Allen Hamilton, PERT was initially conceived to address the unique challenges posed by complex projects, notably exemplified in the development of the Polaris missile program. As the enormity of these projects necessitated a departure from traditional project management approaches, PERT emerged as a groundbreaking methodology, incorporating innovative principles that would redefine project management strategies.

The initial application of PERT in the development of the Polaris submarine-launched ballistic missile system served as a catalyst for its subsequent widespread adoption across various industries. The methodology's innovative approach to managing complexity and uncertainty resonated deeply, particularly in endeavors characterized by intricate interdependencies and dynamic environments. Over time, PERT underwent refinement and adaptation, solidifying its position as a versatile and potent tool capable of addressing a diverse spectrum of project complexities.

At the heart of the PERT framework lies a structured and analytical approach to project management. This framework provides project managers with a comprehensive toolkit for project planning, scheduling, and execution, encompassing three fundamental components: task analysis, time estimation, and network visualization.

Task analysis, the foundational step of PERT, involves the meticulous breakdown of a project into discrete tasks or activities. Each task's sequence, dependencies, and interrelationships are carefully delineated, creating a blueprint that facilitates a thorough comprehension of the project's structure. This granular deconstruction is instrumental in establishing logical task sequences, enabling seamless execution, and identifying critical junctures where deviations can have far-reaching implications.

PERT introduces a novel dimension to time estimation by embracing a probabilistic outlook. Unlike conventional methods that rely on singular time estimates, PERT incorporates three distinct time estimates for each task: optimistic, pessimistic, and most likely. These estimates capture the spectrum of potential outcomes, acknowledging best-case, worst-case, and most probable scenarios. Through a weighted average calculation known as the Expected Time (TE), project managers arrive at a nuanced and comprehensive estimate for each task, integrating the inherent variability and uncertainty associated with project timelines.

Central to PERT's effectiveness is its visual representation, commonly conveyed through network diagrams known as PERT charts or PERT networks. These visual tools provide an intuitive depiction of the project's architecture, presenting task sequences, dependencies, and critical paths in a clear and accessible manner. PERT networks serve as a panoramic lens through which project managers can discern bottlenecks, resource constraints, and opportunities for optimization. This visual language enhances communication, fosters collaboration among stakeholders, and facilitates collective decision-making grounded in a shared understanding of the project's intricacies.

One of PERT's defining attributes is its proactive stance toward uncertainty. PERT's integration of probabilistic thinking equips project teams to navigate the inherent variability of project tasks with agility and insight. By accounting for potential deviations and anticipating risks, project managers are better equipped to allocate resources judiciously, devise contingency plans, and maintain project resilience in the face of unforeseen challenges.

Integral to PERT's methodology is the concept of the critical path, a central artery within the project that dictates the minimum completion time. Tasks situated along the critical path are characterized by zero slack, signifying that any delay in their execution directly extends the project's overall timeline. The identification of the critical path empowers project managers with unparalleled visibility into time-sensitive elements, enabling focused resource allocation, meticulous scheduling, and strategic decision-making to ensure the timely accomplishment of project milestones.

As projects continue to evolve in complexity and scope, PERT remains an enduring testament to the power of structured methodology in project management. Its systematic approach, grounded in rigorous analysis and a probabilistic perspective, offers project managers a robust framework to navigate the intricate web of project intricacies. By providing a clear roadmap for task execution, integrating a probabilistic outlook, and fostering transparent communication through visual representation, PERT empowers project teams to tackle complex projects with precision and confidence. In a world where effective project management is integral to success, PERT stands as a beacon of innovation and a steadfast guide for steering projects toward their desired outcomes.

9.1 Contemporary Issues in Project Management: Navigating Complexities with PERT

In the dynamic landscape of project management, the pursuit of successful project outcomes has become increasingly intricate and challenging. Rapid technological advancements, global connectivity, and evolving stakeholder expectations have ushered in a new era of project

complexities. As organizations strive to remain competitive and innovative, they encounter a range of contemporary issues that demand adaptive strategies and robust methodologies. One such invaluable tool that addresses these challenges is the Project Evaluation and Review Technique (PERT). In this discourse, we delve into some of the pressing contemporary issues in project management and explore how PERT serves as a guiding light, enabling project managers to navigate complexities, optimize resources, and enhance decision-making for successful project delivery.

1. Technological Advancements and Digital Transformation: The pervasive influence of technology has revolutionized the business landscape, leading to a surge in digital transformation initiatives. While these endeavors promise increased efficiency and agility, they also introduce intricate challenges. Project managers must grapple with intricate interdependencies among various technological components, evolving software platforms, and the need for seamless integration. PERT's systematic approach to task analysis and network visualization proves invaluable in understanding the digital ecosystem, identifying critical pathways, and ensuring that technological advancements are harnessed to their fullest potential. By applying PERT, project managers can orchestrate the deployment of complex technological solutions with a strategic understanding of their impact on project timelines and outcomes.

2. Globalization and Virtual Teams: The rise of globalization has enabled organizations to tap into diverse talent pools and collaborate across borders. However, managing virtual teams dispersed across different time zones and cultures poses distinct challenges. Effective communication, coordination, and resource allocation become paramount. PERT's emphasis on visual representation and critical path analysis serves as a unifying language for virtual teams, enabling clear communication of project structures and interdependencies. By incorporating probabilistic time estimates, project managers can account for potential delays arising from differences in work cultures, time zones, and communication challenges, ensuring that virtual teams remain aligned and focused on key project milestones.

3. Resource Scarcity and Optimization: The efficient allocation of resources is a perennial concern in project management. With the increasing demand for specialized skills, materials, and budgets, project managers must navigate the complexities of resource scarcity and optimize their utilization. PERT's approach to time estimation, particularly the integration of optimistic, pessimistic, and most likely estimates, enables project managers to factor in the potential variability in resource availability. By identifying critical paths and resource constraints, PERT facilitates informed decision-making, allowing project managers to allocate resources judiciously and optimize task sequencing to minimize resource bottlenecks.

4. Sustainability and Environmental Considerations: In an era of heightened environmental awareness, sustainable project management practices have gained prominence. Projects are now evaluated not only based on financial success but also their ecological impact. Project managers must navigate intricate regulatory frameworks, ethical considerations, and sustainable procurement practices. PERT's probabilistic approach accommodates uncertainties associated with environmental compliance and sustainability requirements. By incorporating various time estimates, project managers can devise strategies that consider potential delays arising from unforeseen sustainability-related challenges, ensuring that projects adhere to ethical and ecological standards.

5. Agile and Iterative Project Management: The Agile methodology has gained traction as organizations seek more adaptive and iterative approaches to project management. Agile methodologies prioritize flexibility, collaboration, and frequent iterations. Integrating Agile principles with PERT can prove beneficial, allowing project managers to visualize and plan for iterative cycles while maintaining a clear understanding of critical paths. PERT's network visualization aids in identifying dependencies between Agile sprints, facilitating seamless transitions and optimized resource allocation for continuous delivery.

6. Risk Management and Uncertainty: Uncertainty is an inherent aspect of project management, and contemporary projects are often subject to a myriad of unforeseen risks. Traditional project management methods may struggle to adequately address these uncertainties. PERT's probabilistic approach is well-suited to tackle this challenge. By incorporating optimistic, pessimistic, and most likely time estimates, project managers can quantify and manage risks more effectively. PERT's emphasis on critical path analysis aids in identifying high-impact risks that could potentially disrupt project timelines, enabling project managers to devise proactive mitigation strategies.

In the dynamic landscape of contemporary project management, navigating complexities is a constant endeavor. From technological advancements to global collaboration, from resource optimization to sustainability considerations, project managers grapple with multifaceted challenges that demand innovative approaches. The Project Evaluation and Review Technique (PERT) stands as a potent ally, offering a structured and adaptable framework to address these contemporary issues. By providing a systematic approach to task analysis, incorporating probabilistic time estimates, visualizing critical paths, and promoting transparent communication, PERT equips project managers to navigate the intricacies of modern projects with confidence and precision. As organizations continue to embrace change and innovation, PERT remains a steadfast guide, enabling project managers to achieve successful project outcomes in the face of evolving complexities.

9.2 Unveiling the PERT Framework: Methodology and Components

At its core, PERT is characterized by its systematic and analytical approach to project management. It offers a comprehensive framework that enables project managers to plan, schedule, and execute projects by breaking them down into discrete tasks, estimating task durations, and identifying critical paths. The methodology revolves around three key components: task analysis, time estimation, and network visualization.

1. Task Analysis: Central to the PERT methodology is the meticulous analysis of project tasks. Each project is dissected into a series of individual tasks or activities, with their interdependencies and sequences clearly delineated. This granular deconstruction facilitates a holistic understanding of the project's structure and the relationships between its constituent parts. Task analysis involves identifying both the immediate predecessors and successors of each task, enabling project managers to establish a logical flow and ensure seamless execution.

2. Time Estimation: PERT introduces a probabilistic dimension to time estimation, acknowledging the inherent variability in project task durations. Unlike traditional project management methods that rely on single-point estimates, PERT integrates three distinct time estimates for each task: optimistic, pessimistic, and most likely. These estimates capture a range of potential outcomes, accounting for best-case, worst-case, and most probable scenarios. Through a weighted average calculation, known as the Expected Time (TE), project managers can derive a more accurate and nuanced estimate of each task's duration.

3. Network Visualization: PERT's emphasis on visual representation manifests through its network diagrams, often referred to as PERT charts or PERT networks. These diagrams provide an intuitive depiction of the project's architecture, showcasing task sequences, dependencies, and critical paths. PERT networks offer a panoramic view of the project's progression, facilitating the identification of bottlenecks, resource constraints, and opportunities for optimization. This visual tool enhances communication, comprehension, and collaboration among project stakeholders, fostering a shared understanding of the project's intricacies.

Probabilistic Outlook: Mitigating Uncertainty and Enhancing Decision-Making

What sets PERT apart from conventional project management approaches is its embrace of uncertainty and its integration of probabilistic thinking. PERT's acknowledgment of the variability inherent in project tasks and its incorporation of optimistic, pessimistic, and most likely time estimates enable project managers to make informed decisions that account for potential deviations. This probabilistic outlook equips project teams to anticipate risks, allocate resources judiciously, and devise contingency plans that bolster the project's resilience.

Unveiling the Critical Path: Engine of Project Timelines

A fundamental tenet of PERT is the concept of the critical path, an essential pathway through the project's tasks that dictates the project's minimum completion time. Tasks situated along the critical path possess zero slack, meaning any delay in their execution directly extends the project's overall timeline. By identifying the critical path, project managers gain unparalleled insights into the project's time-sensitive elements and can allocate resources with surgical precision. This focused approach minimizes the risk of delays, ensures efficient resource utilization, and empowers project managers to optimize time-critical tasks.

Enhanced Communication and Collaboration: A Visual Language

PERT's visualization capabilities transcend mere aesthetics, serving as a potent medium for effective communication and collaboration. The visual representation offered by PERT networks simplifies the conveyance of complex project information, enabling project managers to articulate project structures, dependencies, and timelines to diverse stakeholders. This visual language promotes transparency, aligns expectations, and fosters a collaborative environment where collective decision-making is based on a shared understanding of the project's nuances.

Caveats and Considerations: Precision Amidst Complexity

While PERT offers a robust framework for managing complex projects, its efficacy hinges on accurate data collection, thorough expert input, and meticulous time estimation. Inaccurate estimations can ripple through the project, potentially leading to misjudgments in critical path determination and overall project scheduling. As such, project managers must exercise diligence and sound judgment during the data collection and estimation phases, considering a wide range of factors that may influence task durations.

In an era marked by unprecedented technological advancements, global connectivity, and intricate project undertakings, the Project Evaluation and Review Technique (PERT) stands as an indispensable beacon for project managers. PERT's systematic framework, rooted in rigorous analysis, probabilistic thinking, and visual representation, equips project managers to navigate complexity with precision and finesse. By embracing uncertainty, identifying critical paths, and fostering transparent communication, PERT empowers project teams to orchestrate projects of diverse scales and intricacies while minimizing risks and optimizing outcomes. As the world of project management continues to evolve, PERT remains an enduring testament to the power of innovation and structured methodology in the pursuit of project success.

9.3 Three Time Estimates in PERT

In the realm of project management, uncertainty is a constant companion. The ability to predict how long each task will take is often elusive due to the complexities, dependencies, and potential obstacles inherent in any project. The Project Evaluation and Review Technique (PERT) offers a dynamic solution to this challenge by introducing a probabilistic approach to estimating task durations. At the heart of PERT's effectiveness are its three time estimates: optimistic, pessimistic, and most likely. These estimates provide project managers with a comprehensive view of potential outcomes, enabling them to plan, allocate resources, and make informed decisions with heightened precision. In this exploration, we delve into the significance and nuances of these three time estimates in PERT, unraveling how they empower project managers to navigate project uncertainty while optimizing outcomes.

Optimistic Time Estimate: The Best-Case Scenario

The optimistic time estimate represents the rosiest scenario, where everything goes according to plan, and there are no hurdles or delays. It reflects the shortest possible time needed to complete a task, assuming favorable conditions and ideal circumstances. This estimate is derived by considering factors that might accelerate task completion, such as the availability of ample resources, minimal disruptions, and optimal efficiency. For instance, in a software development project, the optimistic estimate for a coding task might assume that the developer encounters no technical challenges and works at maximum speed.

Pessimistic Time Estimate: Accounting for the Worst-Case Scenario

Conversely, the pessimistic time estimate reflects the worst-case scenario. It accounts for all conceivable challenges, delays, and unforeseen obstacles that could impede task completion. This estimate is often influenced by external factors like unexpected resource shortages, technical difficulties, or regulatory constraints. The pessimistic estimate provides an upper bound on task duration, serving as a contingency for extreme scenarios. In the context of manufacturing a physical product, the pessimistic estimate might encompass delays caused by supply chain disruptions, equipment failures, or unexpected quality control issues.

Most Likely Time Estimate: Balancing Realism and Pragmatism

Between the optimistic and pessimistic estimates lies the most likely time estimate, which embodies a balanced and realistic view of task duration. It takes into account the typical conditions and challenges that a task might encounter, reflecting a middle-ground perspective. This estimate is derived from experience, historical data, and expert judgment. It encapsulates the time frame that is

most likely to occur under normal circumstances, considering factors like past project performance, team capabilities, and known challenges. For instance, in a construction project, the most likely estimate for pouring concrete might consider factors such as weather conditions and the proficiency of the construction crew.

Expected Time: Merging the Estimates for Precision

The culmination of these three estimates results in the calculation of the expected time (TE) for each task. The expected time provides a weighted average that synthesizes the optimistic, pessimistic, and most likely estimates into a singular projection. Mathematically, it is calculated using the formula:

$$\text{Expected Time (TE)} = (\text{Optimistic} + 4 \times \text{Most Likely} + \text{Pessimistic}) / 6$$

By incorporating the three estimates, PERT introduces a layer of complexity to the estimation process that transcends traditional deterministic approaches. This probabilistic perspective acknowledges the inherent uncertainty in project environments and equips project managers to anticipate a range of potential outcomes.

Utilizing Probabilistic Analysis for Decision-Making:

The amalgamation of the three time estimates and the calculation of the expected time serve as a foundation for probabilistic analysis in PERT. This approach enables project managers to go beyond mere point estimates and embrace a more comprehensive understanding of the project's temporal landscape. The inherent variability introduced by the optimistic, pessimistic, and most likely estimates facilitates a nuanced evaluation of potential risks, enabling project managers to allocate resources more effectively, plan contingencies, and make informed decisions based on a probabilistic outlook.

The Power of Three Estimates in Action: Critical Path and Risk Mitigation:

Perhaps one of the most powerful applications of the three time estimates in PERT is the identification of the critical path. The critical path is the sequence of tasks that collectively determine the minimum time required to complete a project. It traverses through the project network, encompassing tasks that possess zero slack - meaning any delay in these tasks would directly impact the project's overall timeline.

The three time estimates play a pivotal role in the determination of the critical path. By considering the expected time for each task and evaluating the interdependencies among tasks, project managers can discern the critical path's trajectory. The critical path becomes a focal point for resource allocation, scheduling, and risk mitigation efforts. Tasks along the critical path are particularly sensitive to variations in duration, underscoring the importance of accurately estimating the optimistic, pessimistic, and most likely times.

Moreover, the three time estimates empower project managers to mitigate risks proactively. By identifying tasks with significant variations between their optimistic and pessimistic estimates, project managers can pinpoint areas of potential uncertainty and allocate resources or implement contingency plans accordingly. For instance, if a task has a wide range between its optimistic and pessimistic estimates, it signals that the task is particularly sensitive to external factors and requires vigilant monitoring or additional resources to mitigate potential delays.

In the dynamic landscape of project management, the ability to make informed decisions and allocate resources judiciously is pivotal to success. The Project Evaluation and Review Technique (PERT) offers a transformative approach by introducing three time estimates - optimistic, pessimistic, and most likely - that capture the multifaceted nature of project uncertainty. Through these estimates, project managers gain a panoramic view of potential outcomes, enabling them to plan, strategize, and optimize project execution. The convergence of these estimates not only fuels the calculation of the expected time but also underpins critical path analysis and risk mitigation efforts.

PERT's embrace of the probabilistic perspective empowers project managers to navigate the complexities of project environments with finesse, providing a robust framework for decision-making that accounts for uncertainty. As projects continue to evolve in intricacy and scale, the three

time estimates in PERT stand as a beacon of precision, guiding project managers towards optimal resource allocation, timely delivery, and successful project outcomes.

9.4 Comparison between PERT (Project Evaluation and Review Technique) and CPM (Critical Path Method)

In the realm of project management, both PERT (Project Evaluation and Review Technique) and CPM (Critical Path Method) stand as cornerstone methodologies that have revolutionized how projects are planned, executed, and controlled. While they share common objectives of optimizing project timelines and resource allocation, PERT and CPM exhibit distinct characteristics and approaches. In this comparison, we delve into the similarities and differences between PERT and CPM, unraveling their unique contributions and applications.

Aspect	PERT (Project Evaluation and Review Technique)	CPM (Critical Path Method)
Methodological Foundation	Developed for complex and uncertain projects.	Developed for routine, well-defined projects.
Time Estimation	Utilizes three time estimates: Optimistic, Pessimistic, and Most Likely.	Relies on a single-point estimate, typically the Most Likely duration.
Expected Time Formula	$(\text{Optimistic} + 4 \times \text{Most Likely} + \text{Pessimistic}) / 6$	N/A (Not used in CPM)
Network Representation	Uses events (milestones) and activities in network representation.	Uses only activities in network representation.
Critical Path	Probabilistic approach to identify the critical path, considering uncertainty.	Deterministic approach to identify the critical path, based on fixed durations.
Slack	Activities on the critical path have zero slack, indicating no flexibility in timing.	Activities on the critical path have zero slack, same as in PERT.
Dependency Types	Dependent tasks are linked by predecessor-successor relationships.	Dependent tasks are linked by predecessor-successor relationships.
Application	Suitable for projects with uncertainty, such as research, development, and innovation.	Suited for routine projects, like construction, manufacturing, and routine operations.
Risk Management	Focuses on addressing project uncertainty and managing risks.	Less emphasis on addressing uncertainty, more on efficient resource allocation.
Resource Allocation	Accounts for uncertainties in resource allocation due to project complexity.	Focuses on efficient resource allocation within well-defined project scope.
Visual Representation	Utilizes PERT networks for visualization, emphasizing events and activities.	Uses activity-on-node (AON) diagrams, focusing solely on activities.

Resource Management	Offers flexibility in resource allocation to accommodate project variability.	Emphasizes managing resources within predefined constraints and schedules.
Adaptability	More adaptable to changing project circumstances and uncertainties.	Less adaptable due to deterministic nature, better suited for stable environments.

This detailed table provides an extensive comparison between PERT and CPM, considering various aspects such as their methodologies, time estimation approaches, critical path determination, suitability for different project types, risk management emphasis, resource allocation strategies, visual representation methods, resource management, and adaptability to changing project conditions. Understanding these nuances can guide project managers in selecting the most appropriate methodology based on the unique characteristics of their projects.

Problem-1

Given the tasks their dependencies and three-time estimates of task calculate the critical path and total expected duration of critical tasks.

Task	Dependencies	Optimistic	Most Likely	Pessimistic
Market Research (MR)	None	5 days	9 days	8 days
Requirement Analysis (RA)	MR	7 days	9 days	12 days
UI/UX Design (UX)	RA	10 days	13 days	18 days
Database Development (DB)	None	14 days	18 days	22 days
Front-End Development (FE)	UI/UX Design (UX)	12 days	16 days	20 days
Back-End Development (BE)	Database Development (DB)	10 days	12 days	16 days
Integration and Testing (IT)	Front-End Development (FE)	9 days	11 days	14 days
Quality Assurance (QA)	Integration and Testing (IT), Back-End Development (BE)	6 days	8 days	10 days
User Acceptance Testing (UAT)	Quality Assurance (QA)	5 days	7 days	9 days
Deployment (D)	User Acceptance Testing (UAT)	4 days	5 days	6 days

Solution**Calculation of Expected Duration**

Market Research (MR): Expected Duration = (Optimistic + 4 x Most Likely + Pessimistic) / 6 = (5 + 4 x 5 + 8) / 6 = 6.17 days

Requirement Analysis (RA): Expected Duration = (7 + 4 x 9 + 12) / 6 = 9 days

UI/UX Design (UX): Expected Duration = (10 + 4 x 13 + 18) / 6 = 13.17 days

Database Development (DB): Expected Duration = 14 days (No probabilistic calculation as there's only one estimate)

Front-End Development (FE): Expected Duration = (12 + 4 x 16 + 20) / 6 = 16 days

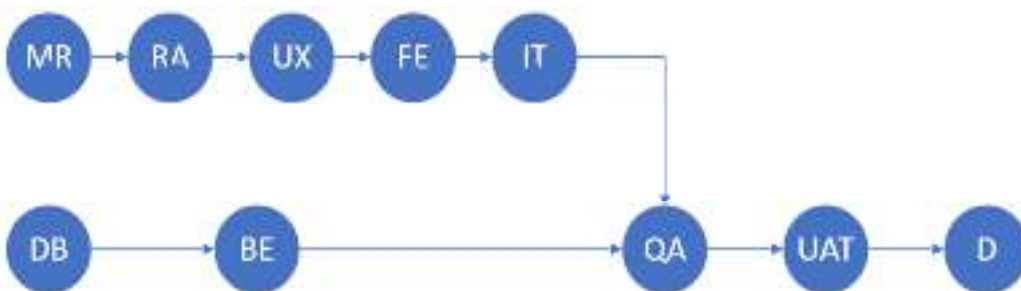
Back-End Development (BE): Expected Duration = (10 + 4 x 12 + 16) / 6 = 12.67 days

Integration and Testing (IT): Expected Duration = (9 + 4 x 11 + 14) / 6 = 11 days

Quality Assurance (QA): Expected Duration = (6 + 4 x 8 + 10) / 6 = 8 days

User Acceptance Testing (UAT): Expected Duration = (5 + 4 x 7 + 9) / 6 = 7 days

Deployment (D): Expected Duration = (4 + 4 x 5 + 6) / 6 = 5 days

Network Diagram

Critical Path: MR - RA - UX - FE - IT - QA - UAT - D

Total Expected Duration of Critical Path: 6.17 + 9 + 13.17 + 16 + 11 + 8 + 7 + 5 = 75.34 days

The critical path is the sequence of tasks that determines the minimum time needed for the project's completion. It involves Market Research (MR), Requirement Analysis (RA), UI/UX Design (UX), Front-End Development (FE), Integration and Testing (IT), Quality Assurance (QA), User Acceptance Testing (UAT), and Deployment (D). The total expected duration of completing all these critical tasks is approximately 75.34 days. This critical path is crucial for resource allocation and scheduling to ensure the timely completion of the software development project.

9.5 Float in Project Management: Navigating Flexibility and Efficiency

In the intricate realm of project management, the concept of "float" emerges as a crucial element that orchestrates the delicate dance between flexibility and efficiency. Float, often referred to as slack, embodies the breathing room within a project schedule – a margin of time that tasks can be delayed without jeopardizing the overall project timeline. This concept serves as a dynamic tool that empowers project managers to optimize resource allocation, identify critical tasks, and strategically manage project uncertainties. In this exploration, we delve into the intricacies of float, unraveling its significance, classifications, calculations, and practical applications in ensuring successful project delivery.

Understanding Float:

Float is an inherent feature of project schedules that acknowledges the inherent variability and unpredictability of project execution. It reflects the amount of time a non-critical task can be delayed without affecting the project's final deadline. In other words, float quantifies the degree of flexibility available within the project schedule, allowing project managers to navigate uncertainties, allocate resources judiciously, and prioritize tasks strategically.

Float comes in two primary forms:

Total Float (Total Slack): This type of float represents the amount of time a task can be delayed without affecting the project's completion date. It pertains to tasks that are not on the critical path, which means their delay won't impact the overall project timeline. Total float highlights the extent to which non-critical tasks can be postponed, providing a buffer for unforeseen delays or resource constraints.

Free Float: Free float is a more focused variant of total float. It refers to the amount of time a task can be delayed without affecting the start of any subsequent dependent task. Unlike total float, which considers the project's completion date, free float considers only the relationships between dependent tasks. It enables project managers to identify opportunities for optimizing task sequences and resource utilization within a specific pathway.

Calculating Float: A Mathematical Insight

Float calculation involves intricate mathematical formulations that leverage the principles of the Critical Path Method (CPM). To compute float, project managers need to ascertain the critical path and durations of tasks. The critical path comprises tasks with zero total float, indicating that any delay in these tasks would directly affect the project's timeline. Float is then determined by the difference between the late start and early start times for non-critical tasks:

$$\text{Total Float (Slack)} = \text{Late Start} - \text{Early Start}$$

This formula yields the total float for each non-critical task, enabling project managers to pinpoint the tasks with the most flexibility. Tasks with positive float represent those that can be delayed without causing delays in the project's completion.

Free Float: Free float is a more focused variant of total float. It refers to the amount of time a task can be delayed without affecting the start of any subsequent dependent task. Unlike total float, which considers the project's completion date, free float considers only the relationships between dependent tasks. It enables project managers to identify opportunities for optimizing task sequences and resource utilization within a specific pathway.

$$\text{Free Float (FF)} = \text{Early Start (ES) of Next Task} - \text{Finish (F)}$$

9.6 Practical Applications of Float: Unveiling Opportunities

Float plays a pivotal role in several crucial aspects of project management, each contributing to the project's success and efficient resource allocation:

Task Prioritization: Float assists project managers in identifying critical tasks that require vigilant monitoring and immediate attention. Tasks with limited or zero float are potential bottlenecks, necessitating proactive resource allocation and careful execution to prevent project delays.

Resource Allocation: Float provides insights into tasks with flexibility, enabling project managers to allocate resources strategically. Tasks with substantial float can accommodate minor delays without endangering the project timeline, allowing for more efficient resource allocation across the project.

Risk Management: By highlighting tasks with float, project managers can anticipate and manage risks more effectively. Tasks with significant float provide a buffer to absorb unexpected delays, minimizing the impact of uncertainties on the project's overall schedule.

Optimal Pathway Selection: Free float aids project managers in optimizing task sequences. Identifying tasks with free float allows for the rearrangement of dependent tasks, potentially shortening critical paths and enhancing overall project efficiency.

Contractual Agreements: Float can play a role in contractual agreements with stakeholders or contractors. Clear communication regarding the availability and utilization of float can help manage expectations and negotiations regarding project completion.

Resource Overallocation: Float assists in identifying periods of resource underutilization, where resources could potentially be allocated to tasks with float, optimizing resource utilization and reducing project costs.

Problem-2

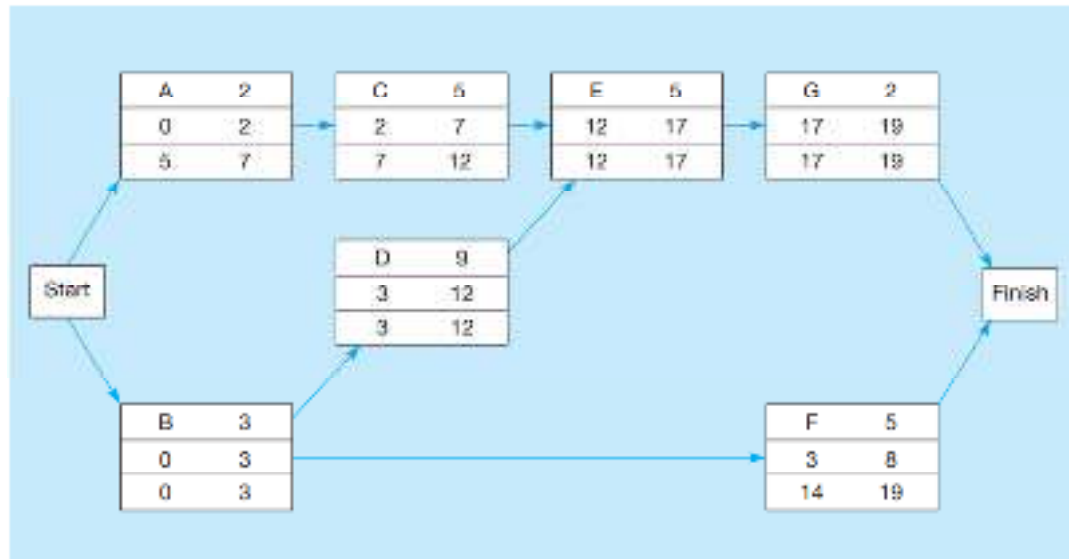
To complete the wing assembly for an experimental aircraft, Scott DeWitte has laid out the major steps and seven activities involved. These activities have been labelled A through G in the following table, which also shows their estimated completion times (in weeks) and immediate predecessors. Determine the expected time and variance for each activity.

ACTIVITY	<i>a</i>	<i>m</i>	<i>b</i>	IMMEDIATE PREDECESSORS
A	1	2	3	—
B	2	3	4	—
C	4	5	6	A
D	8	9	10	B
E	2	5	8	C, D
F	4	5	6	B
G	1	2	3	E

Solution

The critical path, earliest start times, earliest finish times, latest start times, latest finish times, and slack times can be determined using the procedures outlined in the chapter.

ACTIVITY	ACTIVITY TIME				SLACK
	ES	EF	LS	LF	
A	0	2	5	7	5
B	0	3	0	3	0
C	2	7	7	12	5
D	3	12	3	12	0
E	12	17	12	17	0
F	3	8	14	19	11
G	17	19	17	19	0



The activities along the critical path are B, D, E, and G. These activities have zero slack, as shown in the table. The expected project completion time is 19 weeks. The earliest and latest start and finish times are shown in the table.

Summary

In the dynamic world of project management, the concept of "float" emerges as a pivotal element that orchestrates the intricate interplay between flexibility and efficiency. Float, often referred to as slack, introduces a crucial buffer of time within project schedules, allowing tasks to be delayed without impacting the overall project timeline. This concept serves as a strategic asset that empowers project managers to optimize resource allocation, prioritize tasks, and navigate uncertainties effectively.

At its core, float embodies the inherent flexibility within project schedules. It represents the breathing room, acknowledging that projects often encounter unforeseen delays or challenges. Float is categorized into two primary forms: total float and free float. Total float provides a cushion for non-critical tasks, allowing them to be delayed without affecting the project's completion date. On the other hand, free float zooms in on task interdependencies, measuring the time a task can be delayed without affecting the start of subsequent dependent tasks.

Float calculations, rooted in the Critical Path Method (CPM), involve determining early start, late start, and early finish times for tasks. Total float is calculated by subtracting the early start time from the late finish time for non-critical tasks. Free float, a more nuanced variant, is calculated by measuring the time between task finish and the early start of its dependent task. These calculations provide quantitative insights into task scheduling flexibility, enabling strategic decision-making.

The practical applications of float are manifold. It guides task prioritization, allowing project managers to identify critical tasks that demand immediate attention. Float also informs resource allocation, enabling optimal utilization based on available task flexibility. Moreover, it plays a pivotal role in risk management, acting as a buffer against uncertainties. Free float aids in optimizing task sequences, potentially shortening critical paths for enhanced project efficiency. Additionally, float contributes to effective contractual agreements and aids in mitigating resource overallocation.

In conclusion, float serves as a guiding principle that empowers project managers to navigate the complexities of project execution with finesse. Beyond its numerical representation, float embodies the spirit of adaptability, ensuring projects remain resilient in the face of uncertainty. By harnessing the power of float, project managers unlock the potential to deliver successful outcomes, leveraging the delicate equilibrium between adaptability and efficiency to navigate the ebb and flow of projects with precision. In the dynamic landscape of project management, the concept of "float" emerges as a pivotal element that orchestrates the intricate interplay between flexibility and efficiency. Float, often referred to as slack, introduces a crucial buffer of time within project

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Keywords

Float: The buffer of time that tasks can be delayed without affecting the project's overall timeline.

Total Float: The amount of time a non-critical task can be delayed without impacting the project's completion date.

Free Float: The time a task can be delayed without affecting the start of subsequent dependent tasks.

Critical Path: The sequence of tasks with zero float, determining the shortest project duration.

Resource Allocation: Efficient distribution of resources to tasks to optimize project execution.

Risk Management: Identifying and mitigating potential uncertainties that could impact the project.

Task Prioritization: Ranking tasks based on their significance and potential impact on project outcomes.

Early Start: The earliest time a task can begin based on its dependencies and constraints.

Late Start: The latest time a task can start without delaying the project's completion.

Early Finish: The earliest time a task can be completed based on its start time and duration.

Late Finish: The latest time a task can be completed without delaying the project's end date.

Optimization: Maximizing efficiency and effectiveness in project planning and execution.

Dependency: Relationship between tasks where one task relies on the completion of another.

Flexibility: The capacity to adapt to changes and challenges during project execution.

Efficiency: Achieving goals with minimal waste of resources, time, and effort.

Uncertainty: The presence of unknown factors that could impact project outcomes.

Task Sequence: The order in which tasks are performed based on their dependencies.

Bottleneck: A task or resource that limits the flow of work in a project.

Project Timeline: The schedule that outlines the start and end dates of project tasks.

Adaptability: The ability to adjust and modify project plans in response to changing circumstances.

What does "float" refer to in project management?

Self Assessment

1. A type of boat used in construction projects
 - A. A buffer of time tasks can be delayed without affecting the project timeline
 - B. A type of task dependency
 - C. The project manager's decision-making process
 - D. None of these

2. Total Float is the:
 - A. Maximum time a task can be delayed without delaying the project.
 - B. Maximum time a task can be delayed without delaying the early start of the next task.
 - C. Time by which a task can be accelerated.
 - D. Time a task takes longer than initially estimated.

3. Free Float is the:
 - A. Maximum time a task can be delayed without delaying the project.
 - B. Maximum time a task can be delayed without delaying the early start of the next task.
 - C. Time by which a task can be accelerated.
 - D. Time a task takes longer than initially estimated.

4. What is the Critical Path?
 - A. The most challenging tasks in the project.
 - B. The sequence of tasks that are considered critical.
 - C. The path that takes the least amount of time.
 - D. The longest sequence of tasks that determines the project's minimum duration.

5. Which float helps in optimizing task sequences?
 - A. Total Float
 - B. Free Float
 - C. Negative Float
 - D. Total Slack

6. Which term describes the time a task can be delayed without delaying its dependent tasks?
 - A. Total Float
 - B. Critical Path
 - C. Free Float

- D. Resource Allocation
7. The concept of "float" is most closely associated with:
- A. Resource allocation
 - B. Risk management
 - C. Task prioritization
 - D. Task dependency
8. Which formula is used to calculate Total Float?
- A. Total Float = Late Finish - Early Start
 - B. Total Float = Early Finish - Late Start
 - C. Total Float = Late Start - Early Finish
 - D. Total Float = Early Start + Late Finish
9. Float provides a buffer for:
- A. Accelerating tasks
 - B. Delaying critical tasks
 - C. Managing resource overallocation
 - D. Delaying non-critical tasks
10. What does "Critical Path Method (CPM)" primarily focus on?
- A. Identifying the tasks with the most float
 - B. Allocating resources efficiently
 - C. Determining the sequence of tasks
 - D. Managing project risks
11. What does the term "early start" refer to in float calculations?
- A. The earliest time a task can begin
 - B. The latest time a task can start without delaying the project
 - C. The earliest time a task can be completed
 - D. The time by which a task can be accelerated
12. Which type of float considers the project's completion date?
- A. Total Float
 - B. Free Float
 - C. Total Slack
 - D. Critical Float
13. What does "Resource Allocation" in project management involve?
- A. Calculating float for tasks
 - B. Identifying critical tasks
 - C. Distributing resources to tasks efficiently

- D. Delaying tasks on the critical path
14. What does "Risk Management" in project management aim to do?
- A. Allocate resources optimally
 - B. Prioritize tasks based on their impact
 - C. Identify and mitigate potential uncertainties
 - D. Calculate float for all tasks
15. The Critical Path represents:
- A. The shortest sequence of tasks in the project
 - B. The sequence of tasks with the most float
 - C. The tasks with the least importance
 - D. The tasks with the highest risk
16. What is the primary purpose of calculating float in project management?
- A. To determine the earliest start time for tasks
 - B. To identify tasks that can be completed early
 - C. To allocate resources optimally
 - D. To assess task flexibility and prioritize activities
17. What does "Optimization" in project management involve?
- A. Accelerating all tasks to complete the project earlier
 - B. Maximizing efficiency and effectiveness in planning and execution
 - C. Allocating all available resources to critical tasks
 - D. Delaying tasks with the most float
18. What is a "Dependency" between tasks?
- A. A relationship between tasks where one task relies on the completion of another
 - B. A relationship between tasks with the least float
 - C. A type of float calculation
 - D. A type of task classification
19. What does "Efficiency" refer to in project management?
- A. Completing tasks quickly without considering quality
 - B. Achieving goals with minimal waste of resources, time, and effort
 - C. Allocating more resources than necessary to tasks
 - D. Accelerating tasks without considering dependencies
20. What does "Uncertainty" mean in the context of project management?
- A. A situation where all tasks have the same duration
 - B. The presence of unknown factors that could impact project outcomes
 - C. The lack of float in critical tasks

D. The total duration of the project

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. B | 2. A | 3. B | 4. D | 5. B |
| 6. C | 7. D | 8. A | 9. D | 10. C |
| 11. A | 12. A | 13. C | 14. C | 15. A |
| 16. D | 17. B | 18. A | 19. B | 20. B |

Review Questions

1. Explain the concept of "float" in project management. How does it contribute to balancing flexibility and efficiency within project schedules? Provide an example to illustrate its significance.
2. Differentiate between "total float" and "free float." How are these types of float calculated, and how do they influence task sequencing and resource allocation decisions? Provide a real-world scenario where each type of float would be particularly useful.
3. Discuss the role of the "critical path" in project management. How is it determined, and why is it crucial for project planning and execution? Elaborate on how changes to tasks along the critical path can impact the project timeline.
4. In the context of project management, how does "risk management" intersect with the concept of float? Provide examples of how identifying tasks with float can contribute to effective risk mitigation strategies.
5. Examine the practical applications of "float" in optimizing project outcomes. How does float impact task prioritization, resource allocation, and efficient utilization of project resources? Provide insights into how project managers can leverage float for better decision-making.
6. Describe the steps involved in calculating "total float" and "free float" for tasks in a project. How do these calculations provide project managers with valuable insights into task scheduling and sequencing? Illustrate with a hypothetical project scenario.



Further Readings

A Guide to the Project Management Body of Knowledge (PMBOK® Guide)" by Project Management Institute

"Project Management: A Systems Approach to Planning, Scheduling, and Controlling" by Harold Kerzner

"Critical Chain" by Eliyahu M. Goldratt

"Project Management for Dummies" by Stanley E. Portny

Project Management Institute (PMI) website (www.pmi.org)

"Understanding Float in Project Management" by Bright Hub PM (www.brighthubpm.com)

"A Comprehensive Guide to Float in Project Management" by Smartsheet (www.smartsheet.com)

Unit 10: Game Theory

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Objectives

After studying this unit, you will be able to:

- Understand how best strategies are formulated in conflicting scenario
- Understand principles of game theory
- Make decisions between pure and mixed strategies

Introduction

John von Neuman and Oskar Morgenstern developed the first general mathematical formulation of game theory, as it is known to economists, social scientists, and biologists (1944). Due to limitations in their formal structure, the theory was initially only applicable under specific and constrained circumstances. When an analyst encounters situations in which what counts as one agent's best action (for her) is dependent on expectations about what one or more other agents will do, and what counts as their best actions (for them) is similarly dependent on expectations about her, game theory is the most important and useful tool in her toolbox.

The game, which serves as a model of an interaction scenario among rational participants, is at the centre of game theory. The key to understanding game theory is that one player's payoff is dependent on the other player's strategy. The game determines the players' identities, preferences, and accessible options, as well as the impact of these strategies on the outcome. Various more needs or assumptions may be required depending on the model.

10.1 Terminologies used in Game theory

Number of Players:

If a game involves only two players (competitors), then it is called a **two-person game**.

If the number of players are more, the game is referred to as **n-person game**

Sum of gains and losses:

If, in a game, the sum of the gains to one player is exactly equal to the sum of losses to another player, so that, the sum of the gains and losses equals zero, then the game is said to be a **zero-sum game**

If the sum of the gains and losses equals zero, then the game is said to be a **zero-sum game**.

Strategy:

For each reward, a player's strategy is a list of all conceivable actions (moves, decision alternatives, or courses of action) that he is likely to do (outcome). The players are supposed to be aware of the game rules that control their decision options (or strategies). The outcome of a given strategy is also known in advance by the participants and is expressed in numerical values (e.g. money, per cent of market share or utility).

A particular strategy that a player chooses to play again and again regardless of other player's strategy, is referred as **pure strategy**

A set of strategies that a player chooses on a particular move of the game with some fixed probabilities are called **mixed strategies**.

10.2 Problem Solving Approach

A problem can be solved using game theory through following approach:

1. Prepare pay-off matrix
2. Apply Maxi-min and Mini-max principle (Pure strategy)
3. If saddle point exist, identify value of game and write optimal strategy.
4. If no saddle point exist in above step, it is mixed strategy game.
5. If it is 2x2 matrix, use analytical or matrix method to solve.
6. If it is mxn or mx2 or 2xn matrix, reduce it to 2x2 matrix using dominance method.
7. Step 5 can be followed if matrix is reduced to 2x2 matrix format.
8. If step 6 can't be executed to reduce mx2 or 2xn matrix, use graphical method to solve the problem.
9. If mxn matrix can't be solved by any of the above methods, use sub-game method or solve problem as LPP.

10.3 Pure Strategy (Minimax and Maximin scenario)

It's difficult for each player to choose the best strategy without knowing what the other player is doing. Because any player's payout table has all of the necessary information, only one player's payoff table is required to evaluate the decisions. Steps involve in pure strategy are:

- For player A the minimum value in each row represents the least gain (payoff) to him, if he chooses his particular strategy. These are written in the matrix by row minima.
- Select the strategy that gives the largest gain among the row minimum values.
- This choice of player A is called the **maximin principle**, and the corresponding gain is called the maximin value of the game.
- For player B, who is assumed to be the loser, the maximum value in each column represents the maximum loss to him, if he chooses his particular strategy. These are written in the payoff matrix by column maxima.
- Select the strategy that gives the minimum loss among the column maximum values. This choice of player B is called the minimax principle, and the corresponding loss is the **minimax value** of the game.
- If the maximin value equals the minimax value, then the game is said to have a **saddle (equilibrium) point** and the corresponding strategies are called **optimal strategies**.

- **Value of the game** is the expected payoff at the end of the game, when each player uses his optimal strategy, i.e. the amount of payoff, V , at an equilibrium point. A game may have more than one saddle points. A game with no saddle point is solved by choosing strategies with fixed probabilities.



Example: For the following matrix, find value of the game:

		Player B		
		B_1	B_2	B_3
Player A	A_1	20	15	12
	A_2	25	14	8
	A_3	40	2	10
	A_4	-5	4	11

We apply the maximin (minimax) principle to analyze the game.

		Player B			Row Minimum
		B_1	B_2	B_3	
Player A	A_1	20	15	12	12
	A_2	25	14	8	8
	A_3	40	2	10	2
	A_4	-5	4	11	-5
Column Maximum		40	15	12	

Select minimum from the maximum of columns

$$\text{Column MiniMax} = (12)$$

Select maximum from the minimum of rows

$$\text{Row MaxiMin} = [12]$$

Here, Column MiniMax = Row MaxiMin = 12

This game has a saddle point and value of the game is 12

The optimal strategies for both players are

The player A will always adopt strategy 1

The player B will always adopt strategy 3

10.4 Mixed Strategy

In certain cases, no saddle point exists, i.e. maximin value \neq minimax value. In all such cases, players must choose the mixture of strategies to find the value of game and an optimal strategy. The value of game obtained by the use of mixed strategies represents the least payoff, which player A can expect to win and the least which player B can expect to lose.

1. Dominance Method:

Dominance method is used to reduce payoff matrix to $m \times 2$ or $2 \times n$ or preferably 2×2 matrix. Following steps are involved:

- For player B, who is assumed to be the loser, if each element in a column, say C_r is greater than or equal to the corresponding element in another column, say C_s in the payoff matrix, then the column C_r is said to be dominated by column C_s and therefore, column C_r can be

deleted from the payoff matrix. In other words, player B will never use the strategy that corresponds to column Cr because he will loose more by choosing such strategy.

- For player A, who is assumed to be the gainer, if each element in a row, say R_r , is less than or equal to the corresponding element in another row, say R_s , in the payoff matrix, then the row R_r is said to be dominated by row R_s and therefore, row R_r can be deleted from the payoff matrix. In other words, player A will never use the strategy corresponding to row R_r , because he will gain less by choosing such a strategy.
- A strategy say, k can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy k dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination would be decided as per rules 1 and 2 above



Example: Following matrix can be reduced using dominance theory

		Player B		
		B_1	B_2	B_3
Player A	A_1	3	5	4
	A_2	5	6	2
	A_3	2	1	4
	A_4	3	3	5

row-3 \leq row-4, so remove row-3, ($A_1 \leq A_4: 2 \leq 3, 1 \leq 3, 4 \leq 5$)

		Player B		
		B_1	B_2	B_3
Player A	A_1	3	5	4
	A_2	5	6	2
	A_4	3	3	5

column-2 \geq column-1, so remove column-2. ($B_2 \geq B_1: 5 \geq 3, 6 \geq 5, 3 \geq 3$)

		Player B	
		B_1	B_3
Player A	A_1	3	4
	A_2	5	2
	A_4	3	5

row-1 \leq row-3, so remove row-1, ($A_1 \leq A_4: 3 \leq 3, 4 \leq 5$)

		Player B	
		B_1	B_3
Player A	A_2	5	2
	A_4	3	5

This reduced matrix can be used as a base for further calculations.

2. Algebraic Method:

This method is used to determine the probability of using different strategies by players A and B. This method becomes quite lengthy when a number of strategies for both the players are more than two.



Example: Using algebraic method, following matrix can be evaluated:

		Player B	
		B_1	B_2
Player A	A_1	10	70
	A_2	60	20

We apply the maximin (minimax) principle to analyze the game.

		Player B		
		B_1	B_2	Row Minimum
Player A	A_1	10	70	10
	A_2	60	20	20
	Column Maximum	60	70	

Select minimum from the maximum of columns. Column MinMax = (60)

Select maximum from the minimum of rows. Row MaxiMin = [20] Here, Column MinMax \neq Row MaxiMin \therefore This game has no saddle point.

P_i is probability that player A selects i th strategy. Similarly, Q_j is probability that player B selects j th strategy. Once probabilities are calculated, expected payoffs can be calculated for each player.

Here $a = 10, b = 70, c = 60, d = 20$

$$P_1 = \frac{d - c}{(a + d) - (b + c)} = \frac{20 - 60}{(10 + 20) - (70 + 60)} = \frac{-40}{30 - 130} = 0.4$$

$$P_2 = 1 - P_1 = 1 - 0.4 = 0.6$$

$$Q_1 = \frac{d - b}{(a + d) - (b + c)} = \frac{20 - 70}{(10 + 20) - (70 + 60)} = \frac{-50}{30 - 130} = 0.5$$

$$Q_2 = 1 - Q_1 = 1 - 0.5 = 0.5$$

$$V = \frac{a \cdot d - b \cdot c}{(a + d) - (b + c)} = \frac{(10 \times 20) - (70 \times 60)}{(10 + 20) - (70 + 60)} = \frac{200 - 4200}{30 - 130} = 40$$

3. Arithmetic Method:

The arithmetic method (also known as short-cut method) provides an easy method for finding optimal strategies for each player in a payoff matrix of size 2×2 , without saddle point.

Steps involved in this method is:

Find the differences between the two values in the first row and put it against the second row of the matrix, neglecting the negative sign (if any). Find the difference between the two values in the second row and put it against first row of the matrix, neglecting the negative sign (if any).

		Firm B	
		B ₂	B ₃
Firm A	A ₂	12	15
	A ₃	14	10

$14 - 10 = 4$
 $15 - 12 = 3$
 $15 - 10 = 5$
 $14 - 12 = 2$

Probabilities may be calculated from the above weightages.



Example: For the following matrix, value of the game need to be calculated.

		Player B	
		B ₁	B ₂
Player A	A ₁	1	7
	A ₂	6	2

Probabilities can be calculated as:

		Player B	
		B ₁	B ₂
Player A	A ₁	1	7
	A ₂	6	2

$$|6 - 2| = 4 \therefore p_1 = \frac{4}{6 + 4} = 0.4$$

$$|1 - 7| = 6 \therefore p_2 = \frac{6}{6 + 4} = 0.6$$

$$|7 - 2| = 5 \quad |1 - 6| = 5$$

$$\therefore q_1 = \frac{5}{5 + 5} = 0.5 \quad \therefore q_2 = \frac{5}{5 + 5} = 0.5$$

Value of the game can be calculated as:

$$\therefore p_1 = \frac{4}{6 + 4} = 0.4 \quad \therefore q_1 = \frac{5}{5 + 5} = 0.5$$

$$\therefore p_2 = \frac{6}{6 + 4} = 0.6 \quad \therefore q_2 = \frac{5}{5 + 5} = 0.5$$

Expected gain of Firm A

(1) $1 \times 0.4 + 6 \times 0.6 = 4$, Firm B adopt B1

(2) $7 \times 0.4 + 2 \times 0.6 = 4$, Firm B adopt B2

Expected loss of Firm B

(1) $1 \times 0.5 + 7 \times 0.5 = 4$, Firm A adopt A1

(2) $6 \times 0.5 + 2 \times 0.5 = 4$, Firm A adopt A2

4. Graphical Method:

This method is applicable for $2 \times m$ or $n \times 2$ matrix format for two person zero sum game.

Consider the following $2 \times n$ payoff matrix of a game, without saddle point.

Player A	Player B				Probability
	B_1	B_2	...	B_n	
A_1	a_{11}	a_{12}	...	a_{1n}	p_1
A_2	a_{21}	a_{22}	...	a_{2n}	p_2
Probability	q_1	q_2	...	q_n	

The expected payoff for player A will be:

B's Pure Strategies	A's Expected Payoff
B_1	$a_{11}p_1 + a_{21}p_2$
B_2	$a_{12}p_1 + a_{22}p_2$
...	...
B_n	$a_{1n}p_1 + a_{2n}p_2$



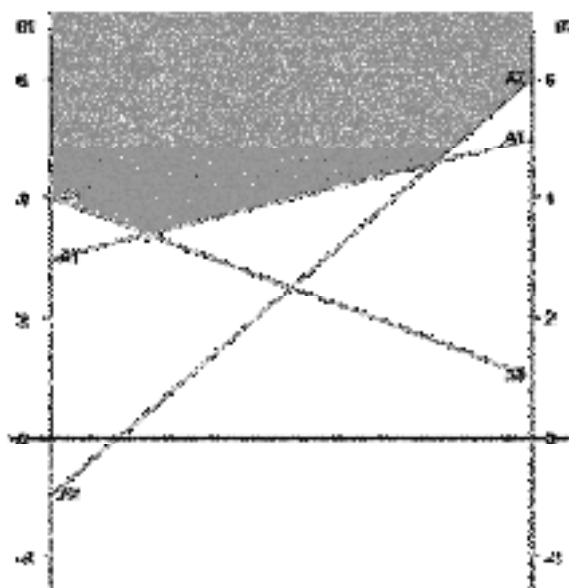
Example:

Following matrix may be solved using graphical method:

Player A	Player B	
	B_1	B_2
A_1	3	5
A_2	-1	6
A_3	4	1
A_4		

First, we draw two parallel lines 1 unit distance apart and mark a scale on each. The two parallel lines represent strategies of player B. If player A selects strategy A1, player B can win 3 or 5 units depending on B's selection of strategies. The value 3 is plotted along the vertical axis under strategy B1 and the value 5 is plotted along the vertical axis under strategy B2. A straight line joining the two points is then drawn. Similarly, we can plot strategies A2, A3 also.

The problem is graphed in the following figure.



The lowest point V in the shaded region indicates the value of game. From the above figure, the value of the game is 3.4 units.

The point of optimal solution occurs at the intersection of two lines

$$E1=3p1+5p2$$

$$E3=4p1+p2$$

Comparing the above two equations, we have

$$3p1+5p2=4p1+p2$$

Substituting $p2=1-p1$

$$3p1+5(1-p1)=4p1+(1-p1)$$

Solving $p1=0.8$

$$p2=1-p1=1-0.8=0.2$$

Substituting the values of $p1$ and $p2$ in equation E1

$$V=3(0.8)+5(0.2)=3.4$$

5. Linear Programming Method:

The two-person zero-sum games can also be solved by linear programming. The major advantage of using linear programming technique is that it helps to solve the mixed-strategy games of larger dimension payoff matrix. Following matrix may be converted to LPP format.

		Player B		
		B_1	B_2	Probability
Player A	A_1	8	1	p_1
	A_2	3	9	p_2
Probability		q_1	q_2	

Player A's objective is to maximize the expected gains, which can be achieved by maximizing V, i.e., it might gain more than V if company B adopts a poor strategy.

The expected gain for player A will be as follows

$$8p1+3p2 \geq V$$

$$p1+9p2 \geq V$$

Dividing the above constraints by V, we get

$$8(p1/V)+3(p2/V) \geq 1$$

$$(p1/V)+9(p2/V) \geq 1$$

To simplify the problem, we put

$$p1/V=x1, p2/V=x2$$

In order to maximize V, player A can

$$\text{Minimize } Zp=1/V=x1+x2$$

subject to

$$8x1+3x2 \geq 1$$

$$x1+9x2 \geq 1$$

$$\text{and } x1, x2 \geq 0$$

player B's objective is to minimize its expected losses, which can be reduced by minimizing V, i.e., player A adopts a poor strategy.

The expected loss for player B will be as follows

$$8q1+q2 \leq V$$

$$3q_1 + 9q_2 \leq V$$

Dividing the above constraints by V , we get

$$8(q_1/V) + (q_2/V) \leq 1$$

$$3(q_1/V) + 9(q_2/V) \leq 1$$

To simplify the problem, we put

$$q_1/V = y_1, q_2/V = y_2$$

In order to minimize V , player B can

Maximize $Z = 1/V = y_1 + y_2$

subject to

$$8y_1 + y_2 \leq 1$$

$$3y_1 + 9y_2 \leq 1$$

and $y_1, y_2 \geq 0$

This LPP can be solved using simplex or graphical method (if dealing with two variables).

Summary

- A game is a contest involving two or more competitors each of whom wants to win. A theory of games provides a series of mathematical models that may be useful in explaining interactive decision-making concepts, where two or more competitors are involved under conditions of conflict and competition.
- Models provide an opportunity to a competitor to evaluate not only his personal decision alternatives (courses of action), but also the evaluation of the competitor's possible choices in order to win the game.
- The strategy for a player is the list of all possible actions (moves, decision alternatives or courses of action) that are likely to be adopted by him for every payoff (outcome). It is assumed that the players are aware of the rules of the game governing their decision alternatives (or strategies).
- A game with only two players, say A and B, is called a two-person zero-sum game, only if one player's gain is equal to the loss of other player, so that total sum is zero.

Keywords

Mixed strategies: A set of strategies that a player chooses on a particular move of the game with some fixed probability are called mixed strategies.

Payoff matrix: The payoffs (a quantitative measure of satisfaction that a player gets at the end of the play) in terms of gains or losses, when players select their particular strategies (courses of action), can be represented in the form of a matrix, called the payoff matrix.

Pure strategy: A particular strategy that a player chooses to play again and again regardless of other player's strategy, is referred as pure strategy.

Zero-sum game: If, in a game, the sum of the gains to one player is exactly equal to the sum of losses to another player, so that, the sum of the gains and losses equals zero, then the game is said to be a zero-sum game.

Saddle point: Saddle point is the payoff value that represents both minimax and maximin value of the game.

Dominance rules: It is the procedure to reduce the size of the payoff matrix according to the tendency of players.

Self Assessment

- 1) A common assumption about the players in a game is that
 - A. Neither player knows the payoff matrix
 - B. The players have different information about payoff matrix
 - C. Only one of the players pursues a rational strategy
 - D. The specific identity of the players is irrelevant to the play of the game

- 2) In a zero sum game,
 - A. What one player wins, the other loses
 - B. The sum of each player's winnings if the game is played many times must be zero
 - C. The game is fair-each person has an equal chance of winning
 - D. Long run profits must be zero

- 3) The prisoner's dilemma is not a constant sum game because
 - A. Some outcomes are better than others for both players
 - B. The prisoner's sentences are necessarily non-zero
 - C. The game does not have a nash equilibrium
 - D. The sum of the prisoner's sentences is non zero

- 4) The saddle point in a payoff matrix is always the
 - A. Largest number in the matrix
 - B. Smallest in its column and largest in row
 - C. Largest in column and smallest in row
 - D. Smaller number in the matrix

- 5) Mixed strategies are only used when a matrix game has np
 - A. Best point
 - B. Maximum point
 - C. Minimum point
 - D. Saddle point

- 6) The principle of dominance are applicable when
 - A. The payoff matrix is a profit matrix for the player A and a loss matrix for the player B
 - B. The payoff matrix is a profit matrix for the player B and a loss matrix for the player A
 - C. The payoff matrix is a profit matrix
 - D. The payoff matrix is a square matrix

- 7) Which of the following statements are true?
 - A. In a two person zero sum game, gains of one player are equal to the losses of the other player
 - B. Every matrix game has a saddle point
 - C. Use of the concept of dominance in reducing the size of a matrix game may lead to the loss of the saddle point
 - D. A matrix game is not the same as a rectangular game

- 8) Which strategy can be eliminated by using dominance theory in the following matrix:

	Y_1	Y_2
X_1	9	13
X_2	12	8
X_3	6	14

- A. X1
- B. X2
- C. Y1
- D. None of the above

	Y_1	Y_2
X_1	6	3
X_2	2	8

9) What is value of the following game?

- A. 11
- B. 19
- C. 4.67
- D. 4.75

10) A two-person zero-sum game means that the

- A. the sum of losses to one player is equal to the sum of gains to other
- B. the sum of losses to one player is not equal to the sum of gains to other
- C. no any player gains or losses
- D. none of these

11) A game is said to be fair if

- A. both upper and lower values of the game are the same and zero
- B. upper and lower values of the game are not equal
- C. upper value is more than the lower value of the game
- D. None of These

12) What happens when maximin and minimax values of the game are same?

- A. no solution exists
- B. solution is mixed
- C. saddle point exists
- D. none of these

13) A mixed strategy game can be solved by

- A. algebraic method
- B. matrix method
- C. graphical method
- D. all of these

14) When no saddle point is found in a payoff matrix of a game, the value of the game is then found by

- A. reducing the size of the game to apply the algebraic method
- B. solving any one 2x2 subgame
- C. finding the average of all the values of the payoff matrix
- D. none of these

15) One of the assumptions in the game theory is

- A. All players act rationally and intelligently
- B. The winner alone acts rationally
- C. Loser acts intelligently
- D. Both the players believe luck

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. D | 2. A | 3. A | 4. C | 5. D |
| 6. A | 7. A | 8. D | 9. C | 10. A |
| 11. A | 12. C | 13. D | 14. A | 15. D |

Review Questions

Q1. Explain the following terms:

- (i) Two-person zero-sum game,
- (ii) Principles of dominance,
- (iii) Pure strategy in game theory

Q2. Explain the theory of dominance in the solution of rectangular games.

Q3. How is the concept of dominance used in simplifying the solution of a rectangular game?

Q4. Explain: Minimax and Maximin principle used in the theory of games.

Q5. Define: (i) Competitive game; (ii) Pure strategies; (iii) Mixed strategies (iv) Two-person zero-sum (or rectangular) game, (v) Payoff matrix.

Q6. What is a game in game theory? What are the properties of a game? Explain the 'best strategy' on the basis of minimax criterion of optimality.

Q7. Define: (i) competitive game, (ii) payoff matrix, (iii) pure and mixed strategies, (iv) saddle point, (v) optimal strategies, and (vi) rectangular (or two-person zero-sum) game.

Q8. State the major limitations of the game theory.

Q9. Explain the difference between pure strategy and mixed strategy.

Q10. Game theory provides a systematic quantitative approach for analysing competitive situations in which the competitors make use of logical processes and techniques in order to determine an optimal strategy for winning

**Further Readings**

Operations Research, By Sivarethinamohan, McGraw-Hill Education (India) Pvt Limited

Quantitative Techniques, Theory and Problems, By P. C. Tulsian, Pearson Education

**Web Links**

Game theory- <https://www.britannica.com/science/game-theory>

Game Theory- <https://www.investopedia.com/terms/g/gametheory.asp>

Unit 11: Queuing Theory

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Objectives

After studying this unit, you will be able to:

- Identify and examine situations that generate queuing problems
- Describe the trade-off between cost of service and cost of waiting line
- Analyze a variety of performance measures of a queuing system
- Understand the concept of single server Queuing model

Introduction

Queuing theory deals with problems that involve waiting (or queuing). It is quite common that instances of queue occurs everyday in our daily life. Examples of queues or long waiting lines might be

- Waiting for service in banks and at reservation counters.
- Waiting for a train or a bus.
- Waiting for checking out at the Supermarket.
- Waiting at the telephone booth or a barber's saloon.

Whenever a customer arrives at a service facility, some of them usually have to wait before they receive the desired service. This forms a queue or waiting line and customers feel discomfort either mentally or physically because of long waiting queue.



Notes: We infer that queues form because the service facilities are inadequate. If service facilities are increased, then the question arises how much to increase? For example, how many buses would be needed to avoid queues? How many reservation counters would be needed to reduce the queue? Increase in number of buses and reservation counters requires additional resource. At the same time, costs due to customer dissatisfaction must also be considered.

In designing a queuing system, the system should balance service to customers (short queue) and also the economic considerations (not too many servers). Queuing theory explores and measures the performance in a queuing situation such as average number of customers waiting in the queue, average waiting time of a customer and average server utilization.

11.1 Queuing System

The customers arrive at service counter (single or in groups) and are attended by one or more servers. A customer served leaves the system after getting the service. In general, a queuing system comprises with two components, the queue and the service facility. The queue is where the customers are waiting to be served. The service facility is customers being served and the individual service stations. A general queuing system with parallel server is shown in Figure 11.1 below:

Characteristics of Queuing System

In designing a good queuing system, it is necessary to have a good information about the model. The characteristics listed below would provide sufficient information.

1. The arrival pattern.
2. The service mechanism.
3. The queue discipline.
4. The number of customers allowed in the system.
5. The number of service channels.

1. The Arrival Pattern: The arrival pattern describes how a customer may become a part of the queuing system. The arrival time for any customer is unpredictable. Therefore, the arrival time and the number of customers arriving at any specified time intervals are usually random variables. A Poisson distribution of arrivals correspond to arrivals at random. In Poisson distribution, successive customers arrive after intervals which independently are and exponentially distributed. The Poisson distribution is important, as it is a suitable mathematical model of many practical queuing systems as described by the parameter “the average arrival rate”.

2. The Service Mechanism: The service mechanism is a description of resources required for service. If there are infinite number of servers, then there will be no queue. If the number of servers is finite, then the customers are served according to a specific order. The time taken to serve a particular customer is called the service time. The service time is a statistical variable and can be studied either as the number of services completed in a given period of time or the completion period of a service.

3. The Queue Discipline: The most common queue discipline is the “First Come First Served” (FCFS) or “First-in, First-out” (FIFO). Situations like waiting for a haircut, ticket-booking counters follow FCFS discipline. Other disciplines include “Last In First Out” (LIFO) where last customer is serviced first, “Service In Random Order” (SIRO) in which the customers are serviced randomly irrespective of their arrivals. “Priority service” is when the customers are grouped in priority classes based on urgency. “Preemptive Priority” is the highest priority given to the customer who enters into the service, immediately, even if a customer with lower priority is in service. “Non-preemptive priority” is where the customer goes ahead in the queue, but will be served only after the completion of the current service.

4. The Number of Customers Allowed in the System: Some of the queuing processes allow the limitation to the capacity or size of the waiting room, so that the waiting line reaches a certain length, no additional customers is allowed to enter until space becomes available by a service completion. This type of situation means that there is a finite limit to the maximum queue size.

5. The Number of Service Channels: The more the number of service channels in the service facility, the greater the overall service rate of the facility. The combination of arrival rate and service rate is critical for determining the number of service channels. When there are a number of service channels available for service, then the arrangement of service depends upon the design of the system's service mechanism.

Parallel channels means, a number of channels providing identical service facilities so that several customers may be served simultaneously. Series channel means a customer go through successive ordered channels before service is completed. The arrangements of service facilities are illustrated in Figure 11.2. A queuing system is called a one-server model, i.e., when the system has only one server, and a multi-server model i.e., when the system has a number of parallel channels, each with one server.

Attitude of Customers

Patient Customer: Customer arrives at the service system, stays in the queue until served, no matter how much he has to wait for service.

Impatient Customer: Customer arrives at the service system, waits for a certain time in the queue and leaves the system without getting service due to some reasons like long queue before him.

Balking: Customer decides not to join the queue by seeing the number of customers already in service system.

Reneging: Customer after joining the queue, waits for some time and leaves the service system due to delay in service.

Jockeying: Customer moves from one queue to another thinking that he will get served faster by doing so.

11.2 Waiting Line Process

Waiting in lines is a part of our everyday life. Waiting in lines may be due to overcrowded, overfilling or due to congestion. Any time there is more customer demand for a service than can be provided, a waiting line forms. We wait in lines at the movie theater, at the bank for a teller, at a grocery store. Wait time depends on the number of people waiting before you, the number of servers serving line, and the amount of service time for each individual customer. Customers can be either humans or an object such as customer orders to be process, a machine waiting for repair. Mathematical analytical method of analyzing the relationship between congestion and delay caused by it can be modeled using Queuing analysis.

A waiting line process or queuing process is defined by two important elements:

- the population source of its customers and
- the process or service system.

The customer population can be considered as finite or infinite. The customer population is finite when the number of customers affects potential new customers for the service system already in the system. When the number of customers waiting in line does not significantly affect the rate at which the population generates new customers, the customer population is considered infinite. Customer behavior can change and depends on waiting line characteristics. In addition to waiting, a customer can choose other alternative.

When customer enters the waiting line but leaves before being serviced, process is called Reneging.

When customer changes one line to another to reduce wait time, process is called Jockeying. Balking occurs when customer do not enter waiting line but decides to come back latter.



Notes: Another element of queuing system is service system. The number of waiting lines, the number of servers, the arrangements of the servers, the arrival and service patterns, and the service priority rules characterize the service system. Queue system can have channels or multiple waiting lines.



Example: Single waiting line : bank counter, airline counters, restaurants, amusement parks.

In these examples multiple servers might serve customers. In the single line multiple servers has better performance in terms of waiting times and eliminates jockeying behavior than the system with a single line for each server. System serving capacity is a function of the number of service facilities and server proficiency. In waiting line system, the terms server and channel are used interchangeably. Waiting line systems are either single server or multiple servers.



Example:

1. Single server: gas station, food mart with single checkout counter, a theater with a single person selling tickets and controlling admission into the show.
2. Multiple server: gas stations with multiple gas pumps, grocery stores with multiple cashiers, multiple tellers in a bank.

Services require a single activity or services of activities called phases. In a single-phase system, the service is completed all at once, such as a bank transaction or grocery store checkout counter. In a multiphase system, the service is completed in a series of phases, such as at fast-food restaurant with ordering, pay, and pick-up windows.

The process of waiting line is characterized by rate at which customers arrive and are served by service system. Arrival rate specifies the average number of customers per time period. The service rate specifies the average number customers that can be serviced during a time period. The service rate governs capacity of the service system. It is the fluctuation in arrival and service patterns that causes wait in queuing system. A general waiting line system with parallel server is shown in Figure 8.1.

11.3 Poisson and Exponential Distributions

Both the Poisson and Exponential distributions play a prominent role in queuing theory.

Considering a problem of determining the probability of n arrivals being observed during a time interval of length t , where the following assumptions are made.

1. Probability that an arrival is observed during a small time interval (say of length v) is proportional to the length of interval. Let the proportionality constant be λ , so that the probability is λv .
2. Probability of two or more arrivals in such a small interval is zero.
3. Number of arrivals in any time interval is independent of the number in non-overlapping time interval.

These assumptions may be combined to yield what probability distributions are likely to be, under Poisson distribution with exactly n customers in the system.

Suppose function P is defined as follows:

$P(n \text{ customers during period } t) =$ the probability that n arrivals will be observed in a time interval of length t

$$\text{then, } P(n, t) = (n = 0, 1, 2, \dots) \quad (1)$$

This is the Poisson probability distribution for the discrete random variable n , the number of arrivals, where the length of time interval, t is assumed to be given. This situation in queuing theory is called Poisson arrivals. Since the arrivals alone are considered (not departures), it is called a pure birth process.

The time between successive arrivals is called inter-arrival time. In the case where the number of arrivals in a given time interval has Poisson distribution, inter-arrival times can be shown to have the exponential distribution. If the inter-arrival times are independent random variables, they must follow an exponential distribution with density $f(t)$ where,

$$f(t) = \lambda e^{-\lambda t} \quad (t > 0) \quad (2)$$

Thus for Poisson arrivals at the constant rate λ per unit, the time between successive arrivals (inter-arrival time) has the exponential distribution. The average Inter - arrival time is denoted by $E(t)$.

By integration, it can be shown that $E(t) = \frac{1}{\lambda}$ (3)

If the arrival rate $\lambda = 30/\text{hour}$, the average time between two successive arrivals are $1/30$ hour or 2 minutes.

For example, in the following arrival situations, the average arrival rate per hour, λ and the average inter arrival time in hour, are determined.

1. One arrival comes every 15 minutes.

Average arrival rate, $\lambda = 4$ arrivals per hour.

Average inter arrival time $= 15$ minutes $= \frac{1}{4}$ or 0.25 hour.

2. Three arrivals occur every 6 minutes. Average arrival rate, $\lambda = 30$ arrivals per hour.

Average Inter-arrival time, $= 2$ minutes $= \frac{1}{3}$ or 0.33 hr.

3. Average interval between successive intervals is 0.2 hour.

Average arrival rate, $\lambda = 5$ arrivals per hour. Average Inter-arrival time, $= 0.2$ hour.

Similarly, in the following service situations, the average service rate per hour, μ and average service time in hours are determined.

1. One service is completed in 10 minutes.

Average service rate, $\mu = 6$ services per hour.

Average service time, $= 10$ minutes or 0.166 hour.

2. Number of customers served in 15 minutes is 4.

Average service rate, $\mu = 4 \times 60 = 240$ services per hour.

Average services time, $= 15$ mins or 0.25 hour.

3. Average service time is 0.25 hour.

Average service rate, $\mu = 4$ services per hour. Average service time $= 0.25$ hour.



Example: In a factory, the machines break down and require service according to a Poisson distribution at the average of four per day. What is the probability that exactly six machines break down in two days?

Solution:

Given $\lambda = 4$, $n = 6$, $t = 2$

$P(n, t) = P(6, 4)$ when $\lambda = 4$

□□

we know, $P(n, t) =$

□

$P(6,2) =$

=

= 0.1221



Example: On an average, 6 customers arrive in a coffee shop per hour. Determine the probability that exactly 3 customers will reach in a 30 minute period, assuming that the arrivals follow Poisson distribution.

Solution:

Given, $\lambda = 6$ customers / hour

$t = 30$ Minutes = 0.5 hour $n = 2$

□□

we know, $P(n, t) =$

$P(6,2) = = 0.22404$

Similarly, when the time taken to serve different customers are independent, the probability that no more than t periods would be required to serve a customer is given by exponential distribution as follows:

$p(\text{not more than } t \text{ time period}) = 1 - e^{-\lambda t}$ where λ = average service rate

11.4 Symbols and Notations

The symbols and notations used in queuing system are as follows:

n = Number of customers in the system (both waiting and in service).

λ = Average number of customers arriving per unit of time.

μ = Average number of customers being served per unit of time.

ρ / λ = ρ , traffic intensity.

C = Number of parallel service channels (i.e., servers).

L_s = Average or expected number of customers in the system (both waiting and in service).

L_q = Average or expected number of customers in the queue.

W_s = Average waiting time in the system (both waiting and in service). W_q = Average waiting time of a customer in the queue.

P_n = Time independent probability that there are n customers in the system (both waiting and in service).

$P_n(t)$ = Probability that there are n customers in the system at any time t (both waiting and in service).



Did you know?

Different models in queuing theory are classified by using special notations described initially in 1953 in the form $(a/b/b)$. Later A.M. Lee in 1966 added the symbols d and c to the Kendall notation. Now in the literature of queuing theory the standard format used to describe the main characteristics of parallel queues is as follows:

$\{(a/b/c) = (d/c)\}$

Where a = arrivals distribution b = service time

c = number of service channels (servers)

d = maximum number of customers allowed in the system e = queue discipline

11.5 Model 1: (MM1) : (∞ / FIFO)

Assumptions

This model is based on the following assumptions:

1. The arrivals follow Poisson distribution, with a mean arrival rate λ .
2. The service time has exponential distribution, average service rate μ .
3. Arrivals are infinite population ∞ .
4. Customers are served on a First-in, First-out basis (FIFO).
5. There is only a single server.

System of Steady-state Equations

In this method, the question arises whether the service can meet the customer demand. This depends on the values of λ and μ .

If $\lambda \geq \mu$, i.e., if arrival rate is greater than or equal to the service rate, the waiting line would increase without limit. Therefore for a system to work, it is necessary that $\lambda < \mu$.

As indicated earlier, traffic intensity $\rho = \lambda / \mu$. This refers to the probability of time. The service station is busy. We can say that, the probability that the system is idle or there are no customers in the system, $P_0 = 1 - \rho$.

From this, the probability of having exactly one customer in the system is $P_1 = \rho P_0$.

Likewise, the probability of having exactly 2 customers in the system would be

$$P_2 = \rho P_1 = \rho^2 P_0$$

The probability of having exactly n customers in the system is

$$P_n = \rho^n P_0 = \rho^{n(1-\rho)} = (\lambda / \mu)^n P_0$$

The expected number of customers in the system is given by,

The expected number of customers in the system is given by,

$$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n (\lambda / \mu)^n (1 - \rho)$$

$$= \frac{\lambda}{\mu - \lambda} \quad (1)$$

The expected number of customers in the queue is given by,

$$L_q = \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \sum_{n=1}^{\infty} n P_n - \sum_{n=1}^{\infty} P_n$$

$$= \frac{\lambda}{\mu(\mu - \lambda)} - \rho$$

With an average arrival rate λ , the average time between the arrivals is $1/\lambda$. Therefore, the mean waiting time in queue, w_q , is the product of the average time between the arrivals and the average queue length.

$$W_q = \frac{1}{\lambda} \left[\frac{\lambda}{\mu - \lambda} \right] \quad (4)$$

$$= \frac{1}{\mu - \lambda}$$

substituting

$$\left[\frac{\lambda}{\mu - \lambda} \right] = \frac{\rho}{\mu - \lambda}$$

Similarly the average waiting time in the system,

$$W = \left[\frac{\lambda}{\mu - \lambda} \right] \left[\frac{\rho}{\mu - \lambda} \right] \quad (5)$$

putting

$$\lambda = \mu \rho$$

we get

$$W = \frac{1}{\mu - \lambda}$$

Queuing Equations

The evaluation of Model I is listed below

1. **Expected number of customers in the system:**

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$$

2. **Average number of customers in the queue:**

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho}$$

3. **Average waiting time in the system:**

$$W_s = \frac{1}{\mu - \lambda}$$

4. **Average waiting time in the queue:**

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

5. **Average waiting time for a customer:**

$$W(m/m > 0) = \frac{1}{\mu(1 - \rho)} \text{ or } \frac{1}{\mu - \lambda}$$

6. **Expected length of non-empty queue:**

$$L(m/m > 0) = \frac{\lambda}{\mu - \lambda}$$

7. **Probability that there are n customers in the system:**

$$P_n = \frac{[\lambda]^n}{n!} P_0 = \frac{[\lambda]^n}{n!} \left[1 - \frac{\lambda}{\mu} \right]$$

8. **Probability that there is nobody in the system:**

$$P_0 = \frac{\mu}{\mu + \lambda}$$

9. **Average number of customers in the queue when the server is busy:**

$$L_q = \rho L_s$$

10. **Traffic intensity or utilization factor:**

$$\rho = \frac{\lambda}{\mu}$$



Example: Consider a situation where the mean arrival rate (λ) is one customer every 4 minutes and the mean service time (m) is $2\frac{1}{2}$ minutes. Calculate the average number of customers in the system, the average queue length and the time taken by a customer in the system and the average time a customer waits before being served.

Solution:

Given, Average Arrival Rate $\lambda = 1$ customer every 4 minutes or 15 customers per hour

Service Rate $\mu = 1$ customer every $2\frac{1}{2}$ minutes or 24 customers per hour

1. The average number of customers in the system:

$$L_s = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{15}{24 - 15} = 1.67 \text{ customers}$$

2. The average queue length:

$$L_q = \frac{\lambda}{\mu} \left[\frac{\lambda}{\mu - \lambda} \right]$$

$$= \frac{15}{24} \times \frac{15}{24 - 15}$$

$$= 1.44 \text{ customers}$$

3. The average time a customer spends in the system:

$$W_s = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{24 - 15}$$

$$= 0.011 \times 60 = 0.67 \text{ minutes}$$

4. The average time a customer waits before being served:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{15}{24(24 - 15)}$$

$$= 0.069 \times 60$$

$$= 4.14 \text{ minutes}$$



Example: Trucks at a single platform weigh-bridge arrive according to Poisson probability distribution. The time required to weigh the truck follows an exponential probability distribution. The mean arrival rate is 12 trucks per day, and the mean service rate is 18 trucks per day. Determine the following:

1. What is the probability that no trucks are in the system?
2. What is the average number of trucks waiting for service?
3. What is the average time a truck waits for weighing service to begin?
4. What is the probability that an arriving truck will have to wait for service?

Solution:

Given $\lambda = 12$ trucks per days, $\mu = 18$ trucks per day.

1. Probability that no trucks are waiting for service.

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{12}{18}$$

$$= 0.3333 \text{ or } 33.33\%$$

2. Average number of trucks waiting for service.

$$W_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{12^2}{18(18 - 12)}$$

$$= 1.33 \text{ trucks}$$

3. Average time a truck waits for waiting service in hours.

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{12}{18(18 - 12)}$$

$$= 0.1111 \text{ days or } 33.3 \text{ minutes}$$

4. Probability that an arriving truck will have to wait for service. $P_w = 1 - P_0$

$$= 1 - 0.333$$

$$= 0.667 \text{ or } 66.7\%$$

Summary

- Queuing Theory is a collection of mathematical models of various queuing systems.
- It is used extensively to analyze production and service processes exhibiting random variability in market demand (arrival times) and service time.
- Queues or waiting lines arise when the demand for a service facility exceeds the capacity of that facility, that is, the customers do not get service immediately upon request but must wait, or the service facilities stand idle and wait for customers.
- The type of queuing system a business uses is an important factor in determining how efficient the business is run.
- As the size of the population increases the world over, the number of queues and their queue length also increase.
- In the business world, more customers mean more business transactions.
- Out of the many ways to attract customers, an efficient queuing system plays a significant role as it reduces a customer's waiting time. The shorter waiting time makes customers happy, and in all probabilities, a happy customer will come back for business again.
- In a queuing system, the calling population is assumed to be infinite.

- This means that if a unit leaves the calling population and joins the waiting line or enters service, there will be no change in the arrival rate.
- The arrivals occur one at a time in a random order and once the customer joins the queuing system he will eventually receive the service.
- The arrival rate and services are modeled as variables that follow statistical distributions. If the arrival rate is greater than the service rate, the waiting line will grow without bound.
- Waiting line models that assume that customers arrive according to a Poisson probability distribution, and service times are described by an exponential distribution.
- The Poisson distribution specifies the probability that a certain number of customers will arrive in a given time period.
- The exponential distribution describes the service times as the probability that a particular service time will be less than or equal to a given amount of time.
- A waiting line priority rule determines which customer is served next. A frequently used priority rule is first-come, first-served.
- Other rules include best customers first, high-test profit customer first, emergencies first, and so on.
- Although each priority rule has merit, it is important to use the priority rule that best supports the overall organization strategy.
- The priority rule used affects the performance of the waiting line system.
 - Basic single server model assumes customers are arriving at Poisson arrival rate with exponential service times, and first come, first serviced queue discipline, and infinite queue length, and infinite calling population.
- By adding additional resources to single server system either service rate can be increased or arrival rate at each server can be decreased with additional cost overhead.
- In Single server single-phase system, customer is served once completed.
- In single server queuing system wait time or performance of system depends on efficiency of serving person or service machine.
- Single server single-phase queuing system is most commonly automated system found in our regular life.

Keywords

Arrival Rate: The number of customers arriving at an average.

Calling Population: Source of customers or messages, the nature of customer numbers.

Queue: A waiting line.

Service Time: The time taken to serve a customer.

Waiting Line System: A system that consists of arrivals in queues, servers and waiting line structures.

Self Assessment

1. Which of the following characteristics apply to queueing system
 - A. Customer perception
 - B. Arrival process
 - C. Both a and b
 - D. none of the above
2. The cost of providing service in a queueing system decreases with

-
- A. Decreased average waiting time in a queue
 B. Decreased arrival time
 C. Increased arrival rate
 D. None of the above
3. Priority queue discipline may be classified as
- A. Finite or infinite
 B. Limited or unlimited
 C. Preemptive or non preemptive
 D. All of the above
4. A bank transaction or grocery store checkout counter is an example of _____ system.
5. _____ specifies the average number of customers per time period.
6. The..... governs capacity of the service system.
7. It is the fluctuation in and patterns that causes wait in queuing system.
8. When customer changes one line to another to reduce wait time, process is called
9. An important assumption in Single server Queuing model 1 is that the customers are served on a First-in, First-out basis (FIFO).
- A. True
 B. False
10. If arrival rate is lesser than or equal to the service rate, the waiting line would increase without limit.
- A. True
 B. False
11. Traffic intensity in a system of steady state is given by $\rho = \lambda / \mu$.
- A. True
 B. False
12. For Poisson arrivals at the constant rate λ per unit, the time between successive arrivals (inter-arrival time) has the exponential distribution.
- A. True
 B. False

Answers for Self Assessment

1. C
2. D
3. C
4. Single Phase
5. Arrival
6. Service
7. Arrival, Service
8. Jockeying
9. A
10. B
11. A
12. A

Review Questions

1. What is the queuing theory?
2. Define arrival rate and service rate.
3. Explain the characteristics of MM1 queuing model.
4. Briefly explain Service Mechanism and Queue Discipline.
5. What is system of steady state?
6. What is multi-server system?
7. Give any four applications of MM1 queuing model.
8. How the time spent in the system is calculated?
9. What is queue discipline?
10. What is traffic intensity?



Further Readings

Books D. Jerry Banks, John S. Carson, Discrete-Event System Simulation, Prentice Hall
K. Watkins, Discrete Event Simulation in C, McGraw-Hill.



Web Links

www.waitinglinetheory.com
orms.czu.cz www.usfca.edu

Unit 12: Decision Theory

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Objectives

After studying this unit, you will be able to:

- Understand how decisions can be made when risk is involved.
- Understand how decisions can be made when uncertainty is involved.

Introduction

Whether it's a simple decision like whether to take the bus or a taxi, or a more complex decision like whether to pursue a difficult political career, decision theory is concerned with the reasons behind an agent's choices. (Note that "agent" refers to an entity capable of thought and action, which is normally a human person.) Standard thinking holds that an agent's actions are entirely determined by her beliefs, desires, or values on any particular occasion, but this is not without controversy, as will be seen below. In any event, decision theory is a theory of beliefs, desires, and other relevant attitudes as much as it is a theory of choice; what counts is how these diverse attitudes (dubbed "preference attitudes") cohere together.

Decision theory is a comprehensive and powerful framework that guides rational decision-making in situations of uncertainty. At its core, it revolves around three fundamental components: the decision maker, the choices or actions available to them, and the potential outcomes that follow. The decision maker can be an individual, a group, or even an automated system. They are assumed to have clear preferences and strive to make choices that maximize their expected utility, which represents their satisfaction or well-being associated with each outcome.

One of the classical approaches in decision theory is Expected Utility Theory (EUT), which posits that decision makers assign probabilities to each possible outcome and compute the expected utility for each alternative based on these probabilities. By choosing the option with the highest expected utility, they aim to optimize their decision. However, EUT has been challenged by Prospect Theory, proposed by Daniel Kahneman and Amos Tversky, which recognizes that individuals perceive and evaluate potential gains and losses differently. It suggests that people tend to be risk-averse when dealing with gains but risk-seeking when dealing with losses, leading to behaviors that deviate from traditional rationality.

Another essential branch of decision theory is Game Theory, which focuses on strategic interactions and decision-making in situations where the outcome of one person's choice depends on the decisions of others. Game theory finds applications in various fields, including economics, politics, and biology, to analyze competitive scenarios and devise optimal strategies.

Bayesian Decision Theory is another valuable approach that incorporates Bayesian probability theory to update beliefs and make decisions in light of new information. It is particularly useful in situations where decision-making involves sequential choices and learning from observed data.

Decision theory has far-reaching implications and practical applications across diverse domains. In business, it helps with investment decisions, pricing strategies, and risk management. In medicine, it aids in treatment choices and resource allocation. Environmental policy-making benefits from decision theory's tools in assessing the impact of different policies. Moreover, automated systems, including self-driving cars and AI algorithms, employ decision theory principles to make informed choices and optimize performance.

By providing a structured and analytical framework for decision-making, decision theory enables individuals and organizations to navigate complex and uncertain situations with more confidence and achieve better outcomes. It continues to be a crucial field of study with evolving methodologies and insights that impact our understanding of human behavior and the design of intelligent systems.

Preferences and prospects are the two most important ideas in decision theory (or equivalently, options). When we say (in this article) that an agent "prefers" the "option" A over B, we imply that the agent believes A is more desirable or worthy of consideration than B. Preference is a comparison attitude, as this crude definition demonstrates. Beyond that, there is room for debate regarding what preferences over options actually entail, or, to put it another way, what it is about an agent (possibly oneself) that causes us anxiety when we discuss his or her preferences over options.

12.1 Steps Involved in Decision Making

Decision making involves following steps:

Identify and define the problem.

List all possible future events (not under the control of decision-maker) that are likely to occur.

Identify all the courses of action available to the decision-maker.

Express the payoffs (p_{ij}) resulting from each combination of course of action and state of nature.

Apply an appropriate decision theory model to select the best course of action from the given list on the basis of a criterion (measure of effectiveness) to get optimal (desired) payoff.

12.2 Decision-Making Environments

To arrive at an optimal decision it is essential to have an exhaustive list of decision-alternatives, knowledge of decision environment, and use of appropriate quantitative approach for decision-making. The three types of decision-making environments are:



The knowledge of these environments helps in choosing the quantitative approach for decision-making.

12.3 Decision Making under Un-Certainty

When probability of any outcome can not be quantified, the decision-maker must arrive at a decision only on the actual conditional payoff values, keeping in view the criterion of effectiveness (policy). Different criteria are:

Criterion	Objective	Method
Optimism (Maximax or Minimin)	The opportunity to achieve the largest possible profit (maximax) or the lowest possible cost (minimin).	<ol style="list-style-type: none"> 1. Locate the maximum (or minimum) payoff values corresponding to each decision alternative. 2. Select a decision alternative with best payoff value (maximum for profit and minimum for cost)
Pessimism (Maximin or Minimax)	Earn no less (or pay no more) than some specified amount. decision alternative that represents the maximum of the minima (or minimum of the minima in case of loss) payoff in case of profits.	<ol style="list-style-type: none"> 1. Locate the minimum (or maximum in case of profit) payoff value in case of loss (or cost) data corresponding to each decision alternative. 2. Select a decision alternative with the best payoff value (maximum for profit and minimum for loss or cost).
Equal Probabilities (Laplace)	Assumed that all states of nature will occur with equal probability	<ol style="list-style-type: none"> 1. Equal probability value to each state of nature 2. Compute the expected (or average) payoff for each alternative (course of action) 3. Select the best expected payoff value (maximum for profit and minimum for cost).
Coefficient of Optimism (Hurwicz)	A decision-maker should be neither completely optimistic nor pessimistic and, therefore, must display a mixture of both	<ol style="list-style-type: none"> 1. Decide the coefficient of optimism α (alpha) and then coefficient of pessimism $(1 - \alpha)$. 2. For each decision alternative select the largest and lowest payoff value and multiply these with α and $(1 - \alpha)$ values, respectively. 3. Then calculate the weighted average, H by using above formula. 4. Select an alternative with best weighted average payoff value.
Regret (Savage)	Decision-maker regrets for choosing wrong decision alternative resulting in an opportunity loss of payoff. He always intends to minimize this	<ol style="list-style-type: none"> 1. Find the best payoff corresponding to each state of nature. 2. Subtract all other payoff values in that

	regret.	row from this value.
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Example 1: Selecting the best strategy in the following condition:

State of Nature	Strategies		
	A	B	C
X	70	50	30
Y	30	45	30
Z	15	0	30

Maximin Criterion

State of Nature	Strategies		
	A	B	C
X	70	50	30
Y	30	45	30
Z	15	0	30
Column Min.	15	0	30

Max. (Min. column payoff) = 30. C strategy may be adopted.

Maximax Criterion

State of Nature	Strategies		
	A	B	C
X	70	50	30
Y	30	45	30
Z	15	0	30
Column Max.	70	50	30

Max. (Max. column payoff) = 70. A strategy may be adopted.

Minimax Regret Criterion

Opportunity loss table

State of Nature	Strategies		
	A	B	C
X	$70 - 70 = 0$	$70 - 50 = 20$	$70 - 30 = 40$

Y	$45 - 30 = 15$	$45 - 45 = 0$	$45 - 30 = 15$
Z	$30 - 15 = 15$	$30 - 0 = 30$	$30 - 30 = 0$
Column Max. Regret	15	0	30

Min. (Max. column payoff) = 15. A strategy may be adopted.

Laplace Criterion

State of Nature	Strategies		
	A	B	C
X	70	50	30
Y	30	45	30
Z	15	0	30
Column Average	38.3	31.7	30

Max. (Avg. payoff) = 38.3. A strategy may be adopted

Hurwicz Criterion (Alpha = 0.6)

State of Nature	Strategies		
	A	B	C
X	70	50	30
Y	30	45	30
Z	15	0	30
Column Max.	70	50	30
$\alpha \cdot \text{Max}$	42	30	18
Column Min.	15	0	30
$(1 - \alpha) \cdot \text{Min}$	6	0	12
$[\alpha \cdot \text{Max}] + [(1 - \alpha) \cdot \text{Min}]$	48	30	30

Max. (Wt. Avg. payoff) = 48. A strategy may be adopted.



Example 2: A company is planning to develop a new product, but it faces uncertainty about the potential market acceptance. The company has three options: proceed with a high investment in Research and Development (R&D), proceed with a moderate investment in R&D, or cancel the project altogether. The potential outcomes (profits in millions of dollars) under different market scenarios (favorable, moderate, and unfavorable) are as follows:

Decision Option (R&D Investment)	Favorable Market	Moderate Market	Unfavorable Market

High Investment	\$30	\$15	-\$10
Moderate Investment	\$20	\$10	-\$5
Cancel Project	\$0	\$0	\$0

Maximax:

In Maximax, we choose the option that maximizes the best possible outcome for each decision alternative. For each R&D investment option, we select the highest payoff under the best scenario.

Maximax for High Investment: \$30 (favorable market)

Maximax for Moderate Investment: \$20 (favorable market)

Maximax for Cancel Project: \$0 (favorable market)

Based on the Maximax criterion, the company should proceed with the High Investment option.

Maximin:

In Maximin, we choose the option that maximizes the minimum payoff under each decision alternative. For each R&D investment option, we select the highest minimum payoff across all scenarios.

Maximin for High Investment: -\$10 (unfavorable market)

Maximin for Moderate Investment: -\$5 (unfavorable market)

Maximin for Cancel Project: \$0 (neutral in all scenarios)

Based on the Maximin criterion, the company should cancel the project as it guarantees no loss.

Minimax Regret:

In Minimax Regret, we calculate the regret for each decision alternative by finding the difference between the maximum payoff and the payoff under each scenario. Then, we select the option with the lowest maximum regret.

Regret Table:

Decision Option (R&D Investment)	Favorable Market	Moderate Market	Unfavorable Market
High Investment	0	5	20
Moderate Investment	10	5	15
Cancel Project	30	15	10

Minimax Regret for High Investment: Maximum Regret = 20

Minimax Regret for Moderate Investment: Maximum Regret = 15

Minimax Regret for Cancel Project: Maximum Regret = 30

Based on the Minimax Regret criterion, the company should proceed with the High Investment option.

Hurwicz Criterion:

In the Hurwicz criterion, a coefficient of optimism (α) is used to balance between optimism and pessimism. The decision is made by taking the weighted average of the best and worst payoffs for each option.

Suppose the company chooses a coefficient of optimism (α) of 0.7, indicating that it is moderately optimistic about favorable outcomes.

Hurwicz for High Investment: $(0.7 * \$30) + (0.3 * -\$10) = \$21.50$

Hurwicz for Moderate Investment: $(0.7 * \$20) + (0.3 * -\$5) = \$14.50$

Hurwicz for Cancel Project: $(0.7 * \$0) + (0.3 * \$0) = \$0$

Based on the Hurwicz criterion with $\alpha = 0.7$, the company should proceed with the High Investment option.

12.4 Decision Making under Risk

In this decision-making environment, the decision-maker has enough knowledge to assign a probability to each possible outcome (state of nature). Knowing the probability distribution of events (natural states), the decision-maker must choose a course of action that yields the highest expected (average) payoff value. The expected payout is the total of all possible weighted payoffs as a result of selecting a choice option.

Expected monetary value

The expected monetary value (EMV) for a given course of action is obtained by adding payoff values multiplied by the probabilities associated with each state of nature. Steps involved are:

- Create a payoff matrix that lists all possible actions and natural states.
- Calculate the conditional payoff values for each possible combination of course of action and state of nature, as well as the odds of each state of nature occurring.
- Multiply the conditional payoffs by the related probabilities and sum these weighted values for each path of action to get the EMV for each course of action.
- Decide on the line of action that will result in the best EMV.

Expected opportunity loss

The expected opportunity loss (EOL), often known as the expected value of regret, is a different decision criterion for risky decisions. The difference between the highest profit (or payout) and the actual profit as a result of taking a given course of action in a particular condition of nature is known as the EOL. As a result, EOL is the amount of payment lost as a result of failing to choose a course of action that yields the minimal payout in a given state of nature. It is preferable to take a course of action that results in the shortest possible EOL.

Steps involved are:

- Create a conditional payoff values matrix with the accompanying probability for each course of action and state of nature combination.
- Subtract each reward from the maximum payoff to get the conditional opportunity loss (COL) values for each state of nature.
- Multiply the likelihood of each state of nature with the COL value, then add the values to get the EOL for each course of action.
- Choose a course of action with the shortest possible EOL.

Expected value of perfect information

The utmost amount of money required to obtain extra information about the occurrence of distinct states of nature before making a decision is known as the expected value of perfect information (EVPI). It is simple to choose a course of action that delivers the intended payoff in the presence of any state of nature if decision-makers have perfect (full and precise) information about the occurrence of various states of nature.

$EVPI = (\text{Expected profit with perfect information}) - (\text{Expected profit without perfect information})$



Example: Let's consider a decision-making problem involving a manufacturing company that needs to decide whether to introduce a new product into the market. The company has conducted market research and identified three possible scenarios with corresponding probabilities, expected profits, and expected opportunity losses:

Scenario	Probability	Expected Profit (in	Expected
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		millions)	Opportunity Loss (in millions)
Favorable	0.4	\$10	\$0
Moderate	0.5	\$5	\$3
Unfavorable	0.1	-\$5	-\$2

The company can either launch the new product (Decision A) or not launch the product (Decision B).

"What is the optimal decision for a manufacturing company considering launching a new product, based on Expected Monetary Value (EMV), Expected Opportunity Loss (EOL), and Expected Value of Perfect Information (EVPI) criteria, given three possible market scenarios with corresponding probabilities, expected profits, and expected opportunity losses?"

Expected Monetary Value (EMV) Criterion:

To calculate the EMV for each decision alternative, we multiply the probabilities of the scenarios by their corresponding expected profits and sum them up.

EMV for Decision A (Launch the new product):

$$(0.4 * \$10 \text{ million}) + (0.5 * \$5 \text{ million}) + (0.1 * -\$5 \text{ million}) = \$4 \text{ million}$$

EMV for Decision B (Do not launch the new product):

$$(0.4 * \$0 \text{ million}) + (0.5 * \$3 \text{ million}) + (0.1 * -\$2 \text{ million}) = \$1.1 \text{ million}$$

According to the EMV criterion, the company should decide to launch the new product (Decision A), as it has a higher expected monetary value of \$4 million compared to \$1.1 million for not launching the product (Decision B).

Expected Opportunity Loss (EOL) Criterion:

Expected Opportunity Loss represents the expected difference between the best possible outcome and the expected outcome under each decision alternative.

EOL for Decision A (Launch the new product):

$$(0.4 * \$10 \text{ million}) + (0.5 * \$3 \text{ million}) + (0.1 * \$2 \text{ million}) = \$5.3 \text{ million}$$

EOL for Decision B (Do not launch the new product):

$$(0.4 * \$0 \text{ million}) + (0.5 * \$0 \text{ million}) + (0.1 * \$0 \text{ million}) = \$0 \text{ million}$$

According to the EOL criterion, the company should decide not to launch the new product (Decision B), as it has a lower expected opportunity loss of \$0 million compared to \$5.3 million for launching the product (Decision A).

Expected Value of Perfect Information (EVPI) Criterion:

EVPI represents the maximum amount the company should be willing to pay for perfect information about the outcome of the scenarios.

$$\text{EVPI} = \text{EMV with Perfect Information} - \text{EMV without Perfect Information}$$

EMV with Perfect Information:

$$(0.4 * \$10 \text{ million}) + (0.5 * \$5 \text{ million}) + (0.1 * -\$5 \text{ million}) = \$4 \text{ million (same as EMV for Decision A)}$$

EMV without Perfect Information:

$$(0.4 * \$0 \text{ million}) + (0.5 * \$3 \text{ million}) + (0.1 * -\$2 \text{ million}) = \$1.1 \text{ million (same as EMV for Decision B)}$$

$$\text{EVPI} = \$4 \text{ million} - \$1.1 \text{ million} = \$2.9 \text{ million}$$

According to the EVPI criterion, the company should be willing to pay up to \$2.9 million for perfect information about the outcomes of the scenarios to make an optimal decision.

In conclusion, each criterion provides valuable insights into the decision-making process:

EMV suggests that the company should launch the new product due to its higher expected monetary value.

EOL suggests that the company should not launch the new product to minimize expected opportunity loss.

EVPI indicates the maximum value the company should be willing to pay for perfect information about the scenarios to make the best decision.



Example: Let's consider another decision-making problem for a restaurant owner who wants to introduce a new menu item. The owner is unsure about the potential success of the new item due to uncertain customer demand. The restaurant owner has identified three possible scenarios, each with corresponding probabilities, expected profits, and expected opportunity losses:

Scenario	Probability	Expected Profit (in dollars)	Expected Opportunity Loss (in dollars)
High Demand	0.3	\$5,000	\$0
Moderate Demand	0.5	\$2,000	\$1,500
Low Demand	0.2	-\$1,000	-\$500

The owner can either introduce the new menu item (Decision A) or stick with the existing menu (Decision B).

What is the optimal decision for the restaurant owner regarding introducing a new menu item, considering Expected Monetary Value (EMV), Expected Opportunity Loss (EOL), and Expected Value of Perfect Information (EVPI) criteria, given three possible demand scenarios with corresponding probabilities, expected profits, and expected opportunity losses?

Expected Monetary Value (EMV) Criterion:

To calculate the EMV for each decision alternative, we multiply the probabilities of the scenarios by their corresponding expected profits and sum them up.

EMV for Decision A (Introduce the new menu item):

$$(0.3 * \$5,000) + (0.5 * \$2,000) + (0.2 * -\$1,000) = \$1,500 + \$1,000 - \$200 = \$2,300$$

EMV for Decision B (Stick with the existing menu):

$$(0.3 * \$0) + (0.5 * \$0) + (0.2 * \$0) = \$0$$

According to the EMV criterion, the optimal decision for the restaurant owner is to introduce the new menu item (Decision A), as it has a higher expected monetary value of \$2,300 compared to \$0 for sticking with the existing menu (Decision B).

Expected Opportunity Loss (EOL) Criterion:

Expected Opportunity Loss represents the expected difference between the best possible outcome and the expected outcome under each decision alternative.

EOL for Decision A (Introduce the new menu item):

$$(0.3 * \$5,000) + (0.5 * \$1,500) + (0.2 * \$500) = \$1,500 + \$750 + \$100 = \$2,350$$

EOL for Decision B (Stick with the existing menu):

$$(0.3 * \$0) + (0.5 * \$1,500) + (0.2 * \$500) = \$750 + \$100 = \$850$$

According to the EOL criterion, the optimal decision for the restaurant owner is to stick with the existing menu (Decision B), as it has a lower expected opportunity loss of \$850 compared to \$2,350 for introducing the new menu item (Decision A).

Expected Value of Perfect Information (EVPI) Criterion:

EVPI represents the maximum amount the restaurant owner should be willing to pay for perfect information about the demand scenarios to make an optimal decision.

EVPI = EMV with Perfect Information - EMV without Perfect Information

EMV with Perfect Information:

$$(0.3 * \$5,000) + (0.5 * \$2,000) + (0.2 * -\$1,000) = \$2,300 \text{ (same as EMV for Decision A)}$$

EMV without Perfect Information:

$$(0.3 * \$0) + (0.5 * \$0) + (0.2 * \$0) = \$0 \text{ (same as EMV for Decision B)}$$

$$\text{EVPI} = \$2,300 - \$0 = \$2,300$$

According to the EVPI criterion, the maximum value the restaurant owner should be willing to pay for perfect information about the demand scenarios is \$2,300.

In conclusion, each criterion provides different perspectives on the optimal decision:

EMV suggests introducing the new menu item due to its higher expected monetary value (\$2,300).

EOL suggests sticking with the existing menu to minimize the expected opportunity loss (\$850).

EVPI indicates the maximum value the restaurant owner should be willing to pay for perfect information about the demand scenarios to make the best decision (\$2,300).

Ultimately, the restaurant owner needs to consider their risk tolerance, objectives, and preferences while making the final decision.



Example: In the following scenario, value of perfect information may be calculated as:

Nature	Probability	Decision		
		D1	D2	D3
N1	0.3	-20	-50	200
N2	0.4	200	-100	-50
N3	0.3	400	600	300

EMV Calculation:

Nature	Probability	Decision			Expected Payoff		
		D1	D2	D3	D1	D2	D3
N1	0.3	-20	-50	200	-6	-15	60
N2	0.4	200	-100	-50	80	-40	-20
N3	0.3	400	600	300	120	180	90
				EMV (Sum)	194	125	130

D1 decision may be taken.

In case of perfect information, highest payoff may be targeted.

Nature	Probability	Expected Payoff			Highest Expected Payoff
		D1	D2	D3	
N1	0.3	-6	-15	60	60
N2	0.4	80	-40	-20	80
N3	0.3	120	180	90	180
	Total	194	125	130	320 = EPPI

Expected Payoff with perfect information (EPPI) = 320

Highest Expected monetary value (EMV) = 194

Expected value of perfect information (EVPI)

$$= \text{EPPI} - \text{EMV} = 320 - 194 = 126$$

Summary

- Decision analysis is a method of analysing decision alternatives in terms of projected outcomes.
- Decision theory refers to a set of approaches for making decisions in the face of ambiguity and risk.
- In circumstances where an issue entails a series of decisions, a decision tree graphically depicts the path of decision and random events (including a decision on whether to obtain additional information).
- Decision theory is a descriptive and prescriptive business modelling approach for categorizing knowledge levels and comparing predicted outcomes from various courses of action.

Keywords

Decision making under certainty: It is an environment in which future outcomes or states of nature are known.

Decision making under risk: It is an environment in which the probability of outcomes or states of nature can be quantified

Decision making under uncertainty: It is an environment in which the probability of outcomes or states of nature can not be quantified.

Decision tree: Decision tree is the graphical display of the progression of decision and random events

EMV: Expected monetary value is obtained by adding payoffs for each course of action, multiplied by the probabilities associated with each state of nature.

EVPI: Expected value of perfect information is an average (or expected) value of an additional information if it were of any worth

Posterior probabilities: Posterior probabilities are the revised probabilities of the states of nature obtained after conducting a test to improve the prior probabilities of respective nature

Self Assessment

1) A type of decision-making environment is

- A. certainty
- B. uncertainty
- C. risk
- D. all of the above

2) Decision theory is concerned with

- A. methods of arriving at an optimal decision
- B. selecting optimal decision in sequential manner
- C. analysis of information that is available
- D. all of the above

3) Which of the following criterion is not used for decision-making under uncertainty?

- A. maximin
- B. maximax
- C. minimax

- D. minimize expected loss
- 4) Which of the following criterion is not applicable to decision making under risk?
- A. maximize expected return
 - B. maximize return
 - C. minimize expect regret
 - D. knowledge of likelihood occurrence of each state of nature
- 5) The minimum expected opportunity loss (EOL) is
- A. equal to EVPI
 - B. minimum regret
 - C. equal to EMV
 - D. both (a) and (b)
- 6) The decision-making criterion that should be used to achieve maximum long-term payoff is
- A. EOL
 - B. EMV
 - C. Hurwicz
 - D. Maximax
- 7) Essential characteristics of a decision model are
- A. states of nature
 - B. decision alternatives
 - C. payoff
 - D. all of the above
- 8) The difference between the expected profit under conditions of risk and the expected profit with perfect information is called
- A. expected value of perfect information
 - B. expected marginal loss
 - C. expected opportunity loss
 - D. none of the above
- 9) The value of the coefficient of optimism (a) is needed while using the criterion of
- A. equally likely
 - B. maximin
 - C. realism
 - D. minimax
- 10) A situation in which a decision maker knows all of the possible outcomes of a decision and also knows the probability associated with each outcome is referred to as
- A. certainty.
 - B. risk.

- C. uncertainty.
D. strategy.
- 11) Which of the following methods of selecting a strategy is consistent with risk averting behavior?
- A. If two strategies have the same expected profit, select the one with the smaller standard deviation.
B. If two strategies have the same standard deviation, select the one with the smaller expected profit.
C. Select the strategy with the larger coefficient of variation.
D. All of the above are correct.
- 12) Strategy A has an expected value of 10 and a standard deviation of 3. Strategy B has an expected value of 10 and a standard deviation of 5. Strategy C has an expected value of 15 and a standard deviation of 10. Which one of the following statements is true?
- A. A risk averse decision maker will always prefer A to B, but may prefer C to A.
B. A risk neutral decision maker will always prefer C to A or B.
C. A risk seeking decision maker will always prefer C to A or B.
D. All of the above are correct.
- 13) A situation in which a decision maker must choose between strategies that have more than one possible outcome when the probability of each outcome is unknown is referred to as
- A. diversification.
B. certainty.
C. risk.
D. uncertainty.
- 14) If a decision maker is risk averse, then the best strategy to select is the one that yields the
- A. highest expected payoff.
B. lowest coefficient of variation.
C. highest expected utility.
D. lowest standard deviation
- 15) A risk-return tradeoff function
- A. shows the minimum expected return required to compensate an investor for accepting various levels of risk.
B. slopes upward for a risk averse decision maker.
C. is horizontal for a risk neutral decision maker.
D. All of the above are correct.

Answers for Self Assessment

1. D 2. D 3. D 4. B 5. D

6. D 7. D 8. A 9. C 10. B
11. A 12. B 13. D 14. C 15. D

Review Questions

- Q1. Given the complete set of outcomes in a certain situation, how is the EMV determined for a specific course of action? Explain in your own words.
- Q2. Explain the difference between expected opportunity loss and expected value of perfect information.
- Q3. Indicate the difference between decision-making under risk, and uncertainty, in statistical decision theory.
- Q4. Briefly explain 'expected value of perfect information' with examples.
- Q5. Describe a business situation where a decision-maker faces a decision under uncertainty and where a decision based on maximizing the expected monetary value cannot be made. How do you think the decision-maker should make the required decision?
- Q6. Discuss the difference between decision-making under certainty, under uncertainty and under risk.
- Q7. What techniques are used to solve decision-making problems under uncertainty? Which technique results in an optimistic decision? Which technique results in a pessimistic decision?
- Q8. Explain the various quantitative methods that are useful for decision-making under uncertainty.
- Q9. What is a scientific decision-making process? Discuss the role of the statistical method in such a process.
- Q10. Give an example of a good decision that you made, which resulted in a bad outcome. Also give an example of a good decision that you made and that had a good outcome. Why was each decision good or bad?



Further Readings

- Operations Research, By Sivarethina mohan, McGraw-Hill Education (India) Pvt Limited
Quantitative Techniques, Theory and Problems, By P. C. Tulsian, Pearson Education



Web Links

1. Decision Theory Approach in Management-<https://commercemates.com/decision-theory-approach-in-management-2/>
2. Decision Theory- <https://www.merospark.com/content/388/decision-theory/>

Unit 13: Decision Tree

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Objective

After studying this student will be able to

- understand decision trees
- solve problem using decision tree analysis
- understand decision making with utilities

Introduction

Decision tree analysis is a powerful and widely used tool in the field of Operations Research (OR) for making decisions under uncertainty. It is a graphical representation of a decision-making process that involves multiple sequential decisions and their potential outcomes. This analytical technique is employed in a variety of fields, including business, finance, engineering, healthcare, and more, to optimize decision-making and achieve the best possible outcomes.

At its core, a decision tree is structured as a tree-like diagram with nodes representing decision points, branches representing potential choices or outcomes, and leaves representing the final results or consequences of those decisions. Each decision and outcome is associated with a probability and a corresponding payoff or cost, allowing analysts to quantify the uncertainty and potential impact of various choices.

The key components of a decision tree are as follows:

Decision Nodes: These are represented as squares or rectangles and indicate points in the process where decisions must be made. Each decision node corresponds to a specific choice that a decision-maker can select.

Chance Nodes: Chance nodes are represented as circles or ellipses and represent uncertain events or random variables. Each chance node is associated with a probability distribution that describes the likelihood of each possible outcome.

Terminal Nodes: Terminal nodes, also known as end nodes or leaves, represent the final outcomes or consequences of the decision process. They are represented as triangles and contain the final payoffs or costs associated with each possible path.

13.1 Steps in Decision Tree Analysis

Decision tree analysis involves a systematic process to construct and analyze decision trees. Below are the steps typically followed in decision tree analysis:

Problem Formulation and Objective Setting:

Clearly define the decision problem and the objective you want to achieve.

Identify the decisions to be made and the potential outcomes or alternatives.

Determine the relevant uncertainties or events that may affect the outcomes.

Data Collection:

Gather data related to the decisions, outcomes, and uncertainties.

Ensure that the data is accurate, reliable, and sufficient for analysis.

Construct the Decision Tree:

Start by drawing a square or rectangle at the top of the page to represent the initial decision point.

For each decision, draw a line or arrow emanating from the decision node to the possible choices or alternatives. Label each branch with the corresponding decision.

For each chance event (uncertain event), draw a circle or ellipse, and label it with the event's name. Connect the chance node to the branches representing possible outcomes.

Continue this process for subsequent decision points and chance events until you reach the terminal nodes.

Assign Probabilities:

Estimate or obtain the probabilities of the potential outcomes at each chance node.

These probabilities represent the likelihood of each outcome occurring given the associated event.

Assign Payoffs (or Costs):

Assign numerical values (payoffs) to each terminal node, representing the resulting payoff or cost associated with the chosen path.

Analyze the Tree:

Use mathematical techniques, typically backward induction, to calculate expected values for each decision node and identify the optimal decisions at each stage.

Calculate the expected payoff or cost for each decision based on the probabilities and payoffs associated with the possible outcomes.

Decision Making:

Choose the optimal decision strategy by selecting the path that leads to the highest expected value or the lowest expected cost.

Sensitivity Analysis:

Perform sensitivity analysis to assess the impact of varying probabilities or payoffs on the optimal decision.

Identify critical parameters that significantly influence the decision and explore different scenarios.

Implement and Monitor:

Implement the recommended decision strategy in practice.

Continuously monitor the outcomes and update the decision tree as new information becomes available or circumstances change.

Interpret and Communicate Results:

Interpret the findings and communicate the results to stakeholders or decision-makers.

Explain the rationale behind the recommended decision strategy and any insights gained from the analysis.

By following these steps, decision tree analysis provides a structured and systematic approach to decision-making, allowing decision-makers to make informed choices and optimize outcomes under uncertainty.



Example

We use the Rayon cabins case as an example to illustrate these decision tree steps.

Zain is the founder and president of Rayon cabins, a profitable firm located in Mumbai, Maharashtra.

The problem that Zain identifies is whether to expand his product line by manufacturing and marketing a new product i.e., backyard storage sheds. He explored the alternatives that are available to him. Zain decides that his alternatives are to construct (1) a large new plant to manufacture the storage sheds, (2) a small plant, or (3) no plant at all (i.e., he has the option of not developing the new product line).

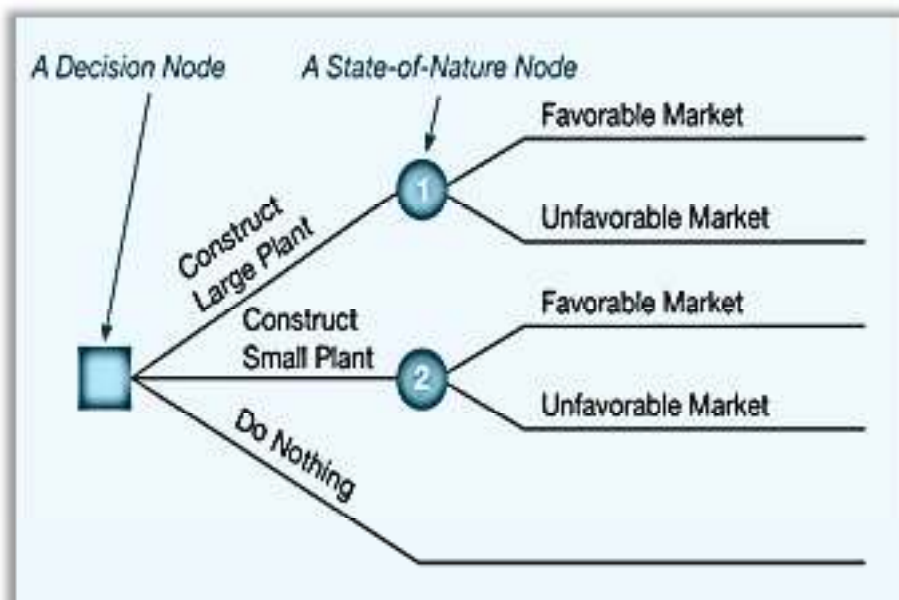
Zain determines that there are only two possible outcomes: the market for the storage sheds could be favorable, meaning that there is a high demand for the product, or it could be unfavorable, meaning that there is a low demand for the sheds.

Because Zain wants to maximize his profits, he can use profit to evaluate each consequence.

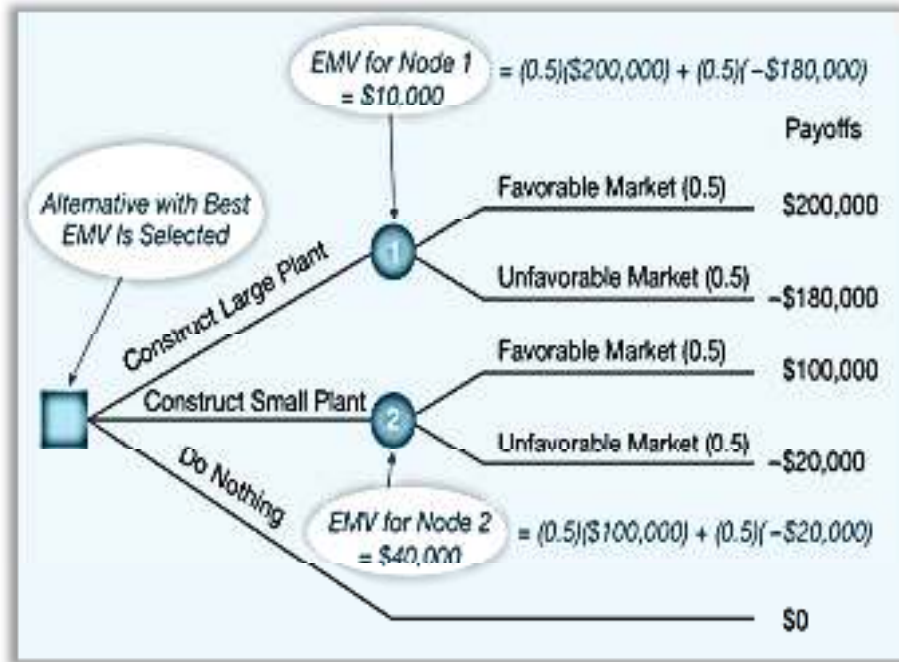
ALTERNATIVE	STATE OF NATURE	
	FAVORABLE MARKET (\$)	UNFAVORABLE MARKET (\$)
Construct a large plant	200,000	-180,000
Construct a small plant	100,000	-20,000
Do nothing	0	0

The next step is to put the payoffs and probabilities on the tree and begin the analysis.

Zain's Decision Tree



Completed and solved Decision Tree



Note that the payoffs are placed at the right side of each of the tree's branches.

The probabilities are shown in parentheses next to each state of nature. Beginning with the payoffs on the right of the figure, the EMVs for each state-of-nature node are then calculated and placed by their respective nodes.

The EMV of the first node is \$10,000.

This represents the branch from the decision node to construct a large plant. The EMV for node 2, to construct a small plant, is \$40,000.

Building no plant or doing nothing has, of course, a payoff of \$0.

The branch leaving the decision node leading to the state-of-nature node with the highest EMV should be chosen. **In Zain's case, a small plant should be built.**

A MORE COMPLEX DECISION FOR Rayon's Cabins – SAMPLE INFORMATION

When **sequential decisions** need to be made, decision trees are much more powerful tools than decision tables.

Let's say that Zain has two decisions to make, with the second decision dependent on the outcome of the first.

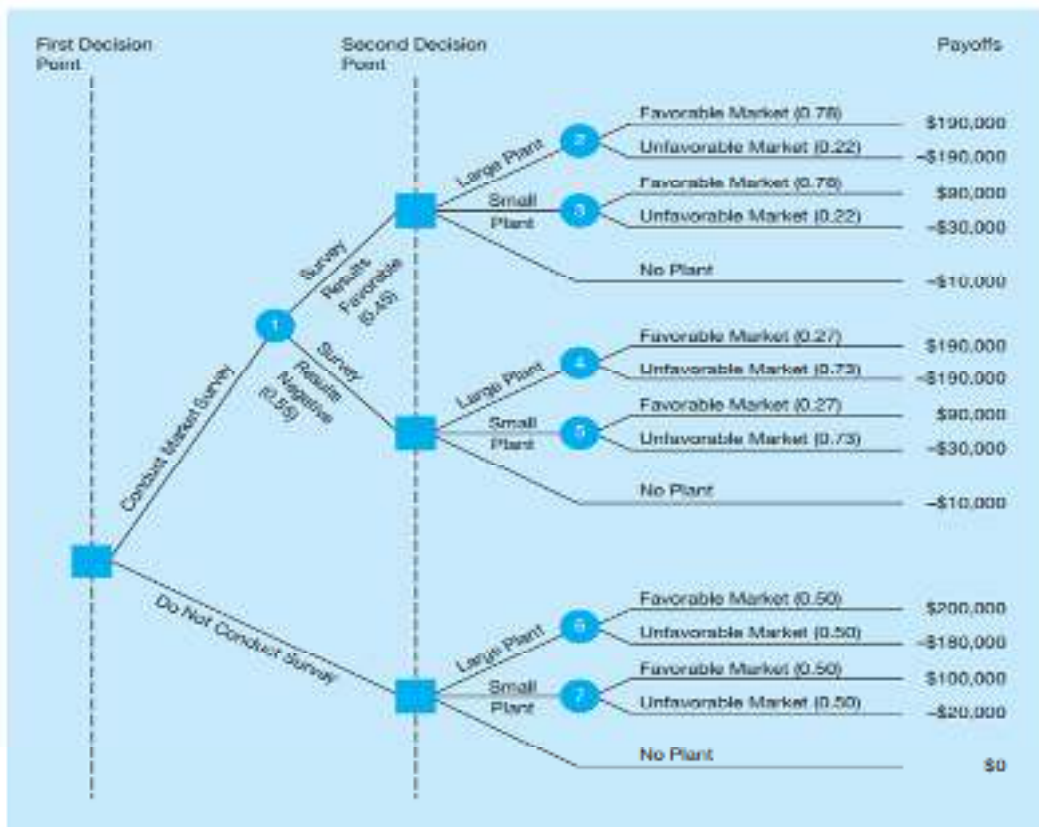
Before deciding about building a new plant, Zain has the option of conducting his own marketing research survey, at a cost of \$10,000.

When sequential decisions need to be made, decision trees are much more powerful tools than decision tables.

Let's say that Zain has two decisions to make, with the second decision dependent on the outcome of the first.

Before deciding about building a new plant, Zain has the option of conducting his own marketing research survey, at a cost of \$10,000.

Zain's new decision tree is represented in Figure.



Let’s take a careful look at this more complex tree.

Note that all possible outcomes and alternatives are included in their logical sequence.

This is one of the strengths of using decision trees in making decisions.

The user is forced to examine all possible outcomes, including unfavorable ones.

He or she is also forced to make decisions in a logical, sequential manner.

Examining the tree, we see that Zain’s first decision point is whether to conduct the \$10,000 market survey.

If he chooses not to do the study (the lower part of the tree), he can construct a large plant, a small plant, or no plant.

This is Zain’s second decision point.

The market will be either favorable (0.50 probability) or unfavorable (also 0.50 probability) if he builds.

The payoffs for each of the possible consequences are listed along the right side.

As a matter of fact, the lower portion of Zain’s tree is identical to the simpler decision tree shown in earlier Figure.

Why is this so?

The upper part of Figure reflects the decision to conduct the market survey.

State-of nature node 1 has two branches.

There is a 45% chance that the survey results will indicate a favorable market for storage sheds.

We also note that the probability is 0.55 that the survey results will be negative.

The rest of the probabilities shown in parentheses in Figure are all conditional probabilities or posterior probabilities

For example, 0.78 is the probability of a favorable market for the sheds given a favorable result from the market survey.

Of course, you would expect to find a high probability of a favorable market given that the research indicated that the market was good.

Don't forget, though, there is a chance that Zain's \$10,000 market survey didn't result in perfect or even reliable information.

Any market research study is subject to error. In this case, there is a 22% chance that the market for sheds will be unfavourable given that the survey results are positive.

We note that there is a 27% chance that the market for sheds will be favorable given that.

Zain's survey results are negative.

The probability is much higher, 0.73, that the market will actually be unfavourable given that the survey was negative.

Finally, when we look to the payoff column in Figure, we see that \$10,000, the cost of the marketing study, had to be subtracted from each of the top 10 tree branches.

Thus, a large plant with a favorable market would normally net a \$200,000 profit.

But because the market study was conducted, this figure is reduced by \$10,000 to \$190,000. In the unfavourable case, the loss of \$180,000 would increase to a greater loss of \$190,000.

Similarly, conducting the survey and building no plant now results in a -\$10,000 payoff.

With all probabilities and payoffs specified, we can start calculating the EMV at each state-of-nature node.

We begin at the end, or right side of the decision tree, and work back toward the origin.

When we finish, the best decision will be known.

Given favorable survey results,

$$\begin{aligned} \text{EMV}(\text{node 2}) &= \text{EMV}(\text{large plant} \mid \text{positive survey}) \\ &= (0.78)(\$190,000) + (0.22)(-\$190,000) = \$106,400 \\ \text{EMV}(\text{node 3}) &= \text{EMV}(\text{small plant} \mid \text{positive survey}) \\ &= (0.78)(\$90,000) + (0.22)(-\$30,000) = \$63,600 \end{aligned}$$

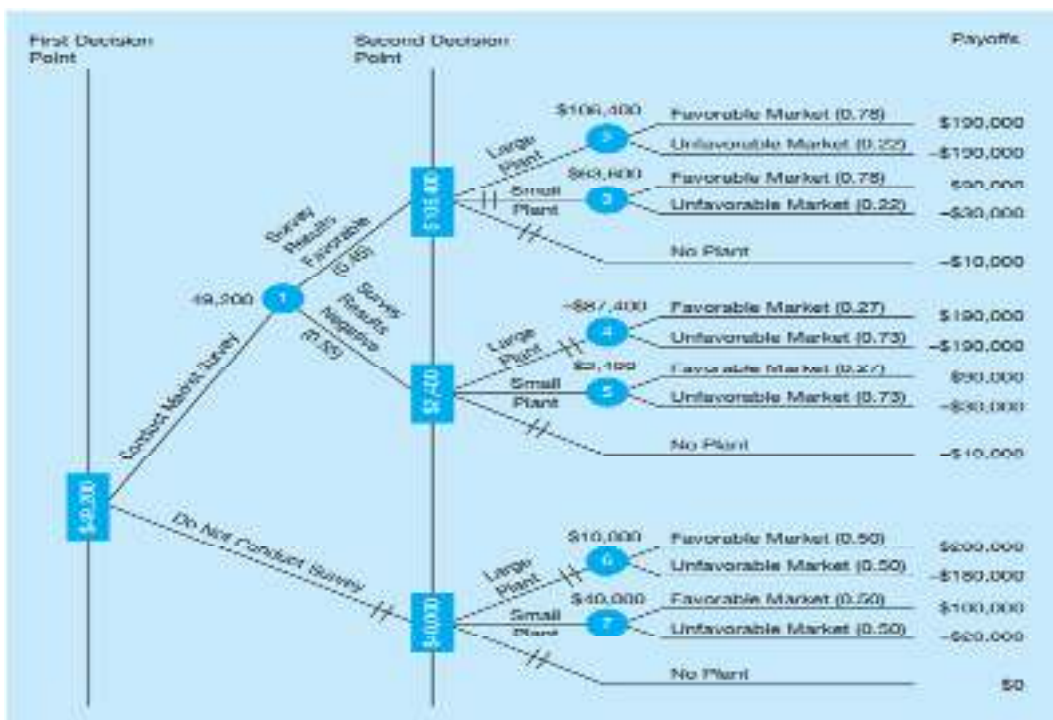
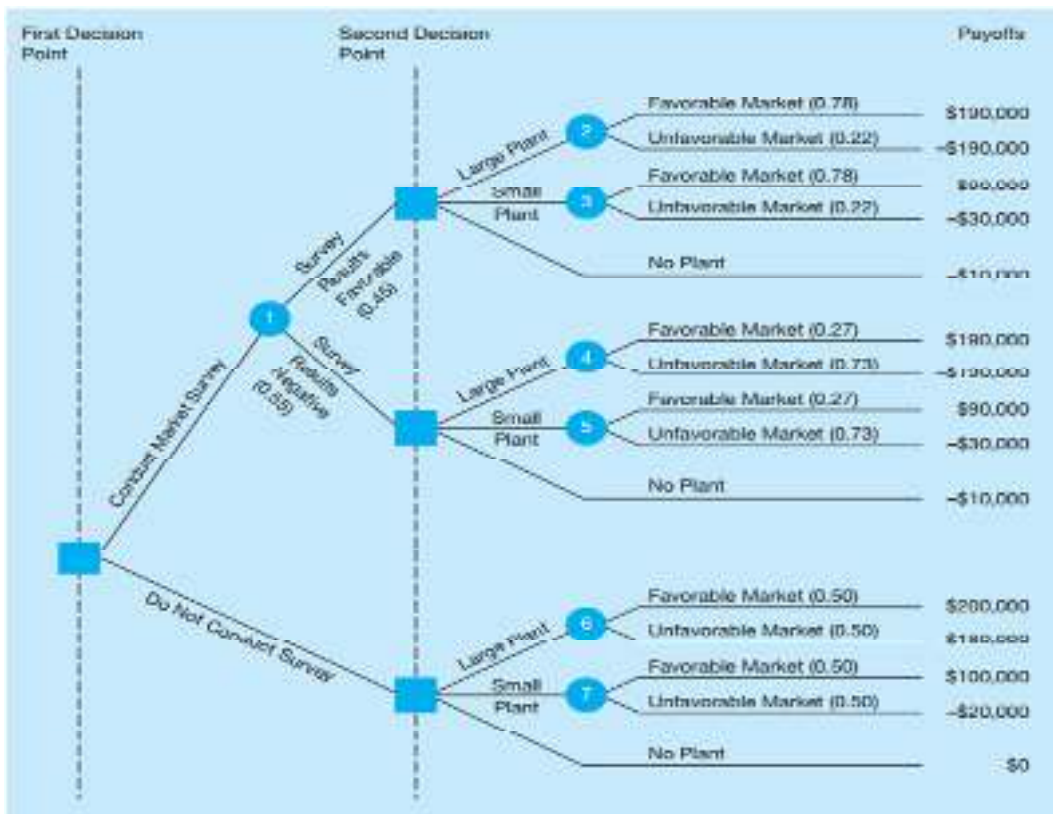
The EMV of no plant in this case is -\$10,000. Thus, if the survey results are favorable, a large plant should be built.

Note that we bring the expected value of this decision (\$106,400) to the decision node to indicate that if the survey results are positive, our expected value will be \$106,400. This is shown in Figure.

Given negative survey results,

$$\begin{aligned} \text{EMV}(\text{node 4}) &= \text{EMV}(\text{large plant} \mid \text{negative survey}) \\ &= (0.27)(\$190,000) + (0.73)(-\$190,000) = -\$87,400 \\ \text{EMV}(\text{node 5}) &= \text{EMV}(\text{small plant} \mid \text{negative survey}) \\ &= (0.27)(\$90,000) + (0.73)(-\$30,000) = \$2,400 \end{aligned}$$

Unit 13: Decision Tree



The EMV of no plant is again -\$10,000 for this branch.

Thus, given a negative survey result, Zain should build a small plant with an expected value of \$2,400, and this figure is indicated at the decision node.

Continuing on the upper part of the tree and moving backward, we compute the expected value of conducting the market survey:

$$\begin{aligned} \text{EMV}(\text{node 1}) &= \text{EMV}(\text{conduct survey}) \\ &= (0.45)(\$106,400) + (0.55)(\$2,400) \\ &= \$47,880 + \$1,320 = \$49,200 \end{aligned}$$

If the market survey is *not* conducted,

$$\begin{aligned} \text{EMV}(\text{node 6}) &= \text{EMV}(\text{large plant}) \\ &= (0.50)(\$200,000) + (0.50)(-\$180,000) \\ &= \$10,000 \end{aligned}$$

$$\begin{aligned} \text{EMV}(\text{node 7}) &= \text{EMV}(\text{small plant}) \\ &= (0.50)(\$100,000) + (0.50)(-\$20,000) \\ &= \$40,000 \end{aligned}$$

The EMV of no plant is \$0.

We move back to the first decision node and choose the best alternative.

The EMV of conducting the survey is \$49,200, versus an EMV of \$40,000 for not conducting the study, so the best choice is to seek marketing information.

If the survey results are favorable, Zain should construct a large plant, but if the research is negative, Zain should construct a small plant.



Example: Monica Britt has enjoyed sailing small boats since she was 7 years old, when her mother started sailing with her.

Today, Monica is considering the possibility of starting a company to produce small sailboats for the recreational market.

Unlike other mass-produced sailboats, however, these boats will be made specifically for children between the ages of 10 and 15. The boats will be of the highest quality and extremely stable, and the sail size will be reduced to prevent problems of capsizing.

Her basic decision is whether to build a large manufacturing facility, a small manufacturing facility, or no facility at all.

With a favorable market, Monica can expect to make \$90,000 from the large facility or \$60,000 from the smaller facility.

If the market is unfavourable, however, Monica estimates that she would lose \$30,000 with a large facility and she would lose only \$20,000 with the small facility.

Because of the expense involved in developing the initial Molds and acquiring the necessary equipment to produce fiberglass sailboats for young children, Monica has decided to conduct

a pilot study to make sure that the market for the sailboats will be adequate.

She estimates that the pilot study will cost her \$10,000.

Because of the expense involved in developing the initial Molds and acquiring the necessary equipment to produce fiberglass sailboats for young children, Monica has decided to conduct

a pilot study to make sure that the market for the sailboats will be adequate.

She estimates that the pilot study will cost her \$10,000.

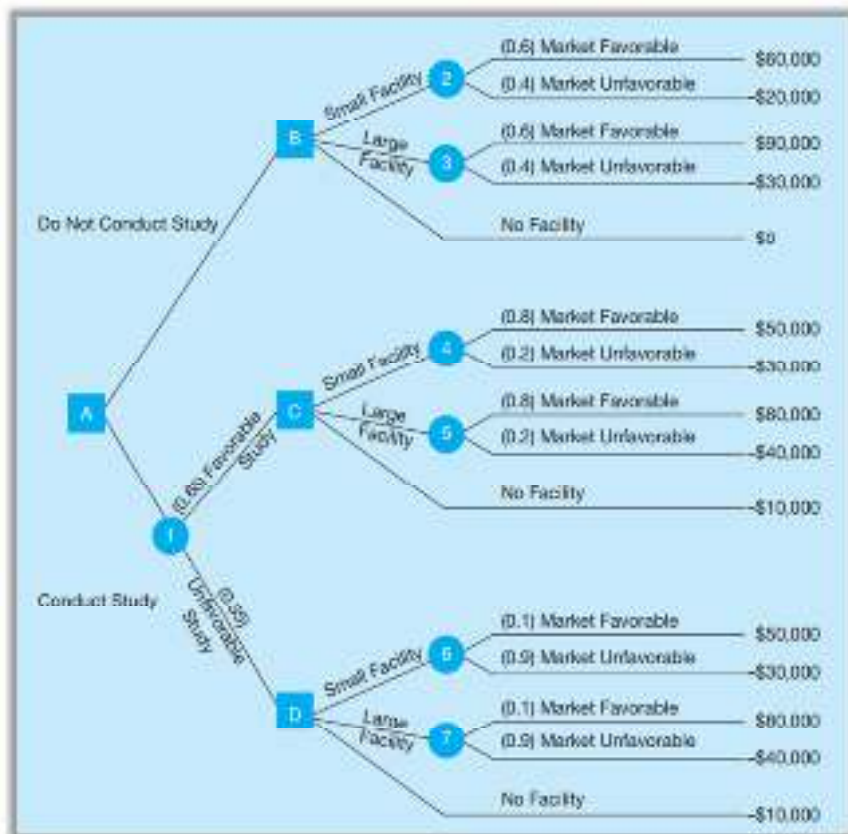
Furthermore, the pilot study can be either favorable or unfavourable.

Monica estimates that the probability of a favorable market, given a favorable pilot study, is 0.8. The probability of an unfavourable market, given an unfavourable pilot study, is estimated to be 0.9.

Monica feels that there is a 0.65 chance that the pilot study will be favorable.

Of course, Monica could bypass the pilot study and simply make the decision as to whether to build a large plant, small plant, or no facility at all.

Without doing any testing in a pilot study, she estimates that the probability of a favorable market is 0.6. What do you recommend?



The EMV at each of the numbered nodes is calculated as follows:

$$\text{EMV}(\text{node } 2) = 60,000(0.6) + (-20,000)0.4 = 28,000$$

$$\text{EMV}(\text{node } 3) = 90,000(0.6) + (-30,000)0.4 = 42,000$$

$$\text{EMV}(\text{node } 4) = 50,000(0.8) + (-30,000)0.2 = 34,000$$

$$\text{EMV}(\text{node } 5) = 80,000(0.8) + (-40,000)0.2 = 56,000$$

$$\text{EMV}(\text{node } 6) = 50,000(0.1) + (-30,000)0.9 = -22,000$$

$$\text{EMV}(\text{node } 7) = 80,000(0.1) + (-40,000)0.9 = -28,000$$

$$\text{EMV}(\text{node } 1) = 56,000(0.65) + (-10,000)0.35 = 32,900$$

At each of the square nodes with letters, the decisions would be:

Node B: Choose Large Facility, since the EMV = \$42,000.

Node C: Choose Large Facility, since the EMV = \$56,000.

Node D: Choose No Facility, since the EMV = -\$10,000.

Node A: Choose Do Not Conduct Study, since the EMV (\$42,000) for this is higher than the EMV (\$32,000) for node 1.

- Based on the EMV criterion, Monica would select Do Not Conduct Study and then select Large Facility.
- The EMV of this decision is \$42,000. Choosing to conduct the study would result in an EMV of only \$32,900.

13.2 Utility Theory

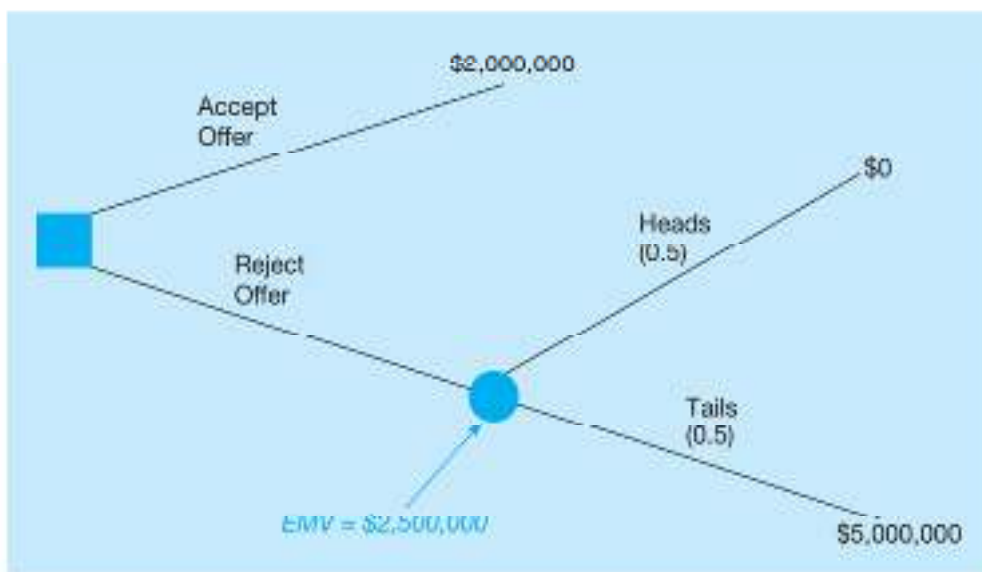
We have focused on the EMV criterion for making decisions under risk. However, there are many occasions in which people make decisions that would appear to be inconsistent with the EMV criterion. When people buy insurance, the amount of the premium is greater than the expected payout to them from the insurance company because the premium includes the expected payout, the overhead cost, and the profit for the insurance company. A person involved in a lawsuit may choose to settle out of court rather than go to trial even if the expected value of going to trial is greater than the proposed settlement. A person buys a lottery ticket even though the expected return is negative. Casino games of all types have negative expected returns for the player, and yet millions of people play these games. A businessperson may rule out one potential decision because it could bankrupt the firm if things go bad, even though the expected return for this decision is better than that of all other alternatives.

Why do people make decisions that don't maximize their EMV? They do this because the monetary value is not always a true indicator of the overall value of the result of the decision. The overall worth of a particular outcome is called utility, and rational people make decisions that maximize the expected utility. Although at times the monetary value is a good indicator of utility, there are other times when it is not. This is particularly true when some of the values involve an extremely large payoff or an extremely large loss. Suppose that you are the lucky holder of a lottery ticket. Five minutes from now a fair coin could be flipped, and if it comes up tails, you would win \$5 million. If it comes up heads, you would win nothing. Just a moment ago a wealthy person offered you \$2 million for your ticket. Let's assume that you have no doubts about the validity of the offer. The person will give you a certified check for the full amount, and you are absolutely sure the check would be good.

A decision tree for this situation is shown in Figure. The EMV for rejecting the offer indicates that you should hold on to your ticket, but what would you do? Just think, \$2 million for *sure* instead of a 50% chance at nothing. Suppose you were greedy enough to hold on to the ticket, and then lost. How would you explain that to your friends? Wouldn't \$2 million be enough to be comfortable for a while?

Most people would choose to sell the ticket for \$2 million. Most of us, in fact, would probably be willing to settle for a lot less. Just how low we would go is, of course, a matter of personal preference. People have different feelings about seeking or avoiding risk. Using the EMV alone is not always a good way to make these types of decisions.

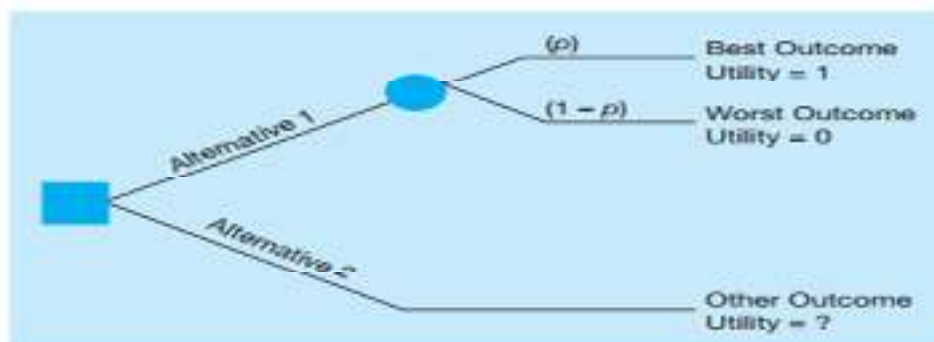
One way to incorporate your own attitudes toward risk is through **utility theory**. In the next section, we explore first how to measure utility and then how to use utility measures in decision making.



13.3 Measuring Utility and Constructing a Utility Curve

The first step in using utility theory is to assign utility values to each monetary value in a given situation. It is convenient to begin utility assessment by assigning the worst outcome a utility of 0 and the best outcome a utility of 1. Although any values may be used as long as the utility of the best outcome is greater than the utility for the worst outcome, using 0 and 1 has some benefits. Because we have chosen to use 0 and 1, all other outcomes will have a utility value between 0 and 1. In determining the utilities of all outcomes, other than the best or worst outcome, a standard gamble is considered. This gamble is shown in Figure.

In Figure, p is the probability of obtaining the best outcome, and $1 - p$ is the probability of obtaining the worst outcome. Assessing the utility of any other outcome involves determining the probability (p) that makes you indifferent between alternative 1, which is the gamble between the best and worst outcomes, and alternative 2, which is obtaining the other outcome for sure. When you are indifferent between alternatives 1 and 2, the expected utilities for these two alternatives must be equal. This relationship is shown as



Expected utility of alternative 2 = Expected utility of alternative 1

$$\text{Utility of other outcome} = (p)(\text{Utility of best outcome, which is 1}) \\ + (1 - p)(\text{Utility of the worst outcome, which is 0})$$

$$\text{Utility of other outcome} = (p)(1) + (1 - p)(0) = p \quad (3-7)$$

Now all you have to do is to determine the value of the probability (p) that makes you indifferent between alternatives 1 and 2. In setting the probability, you should be aware that utility assessment

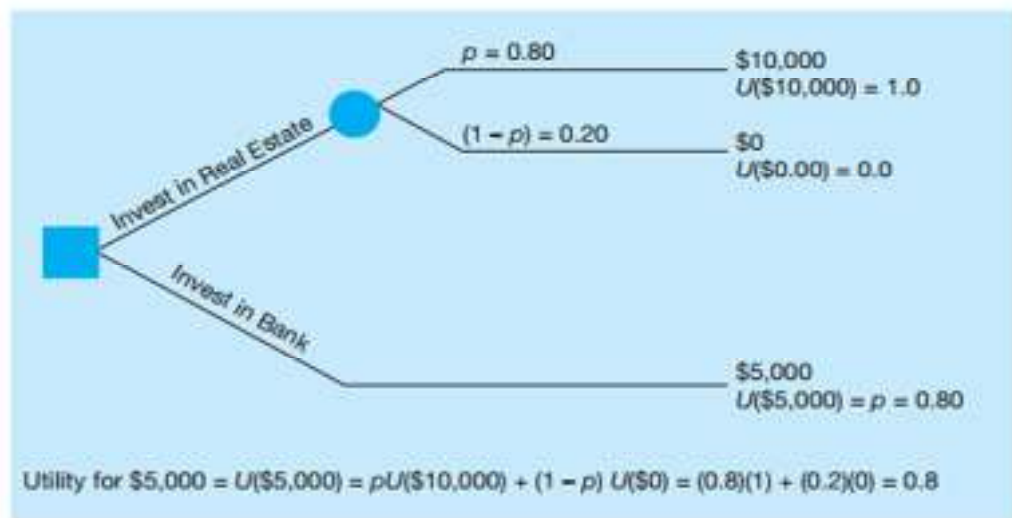
is completely subjective. It's a value set by the decision maker that can't be measured on an objective scale. Let's take a look at an example.

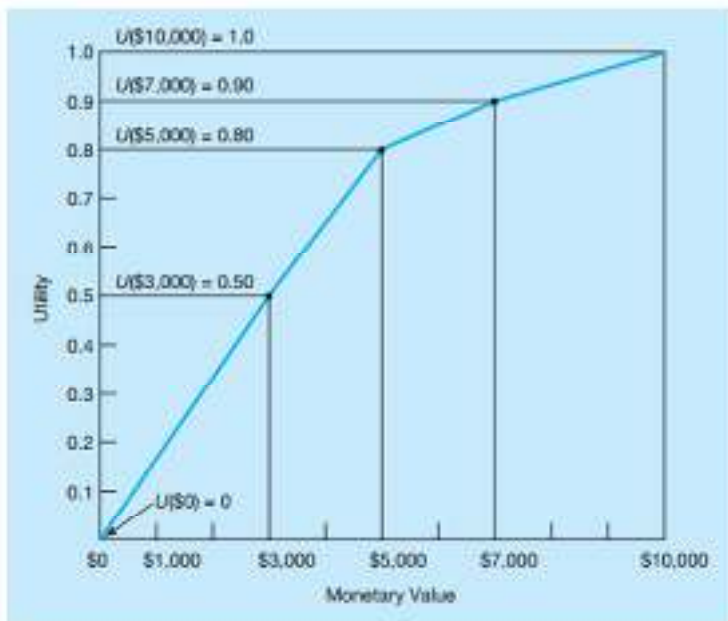
Jane Dickson would like to construct a utility curve revealing her preference for money between \$0 and \$10,000. A utility curve is a graph that plots utility value versus monetary value. She can invest her money either in a bank savings account or in a real estate deal.

If the money is invested in the bank, in 3 years Jane would have \$5,000. If she invested in the real estate, after 3 years she could either have nothing or have \$10,000. Jane, however, is very conservative. Unless there is an 80% chance of getting \$10,000 from the real estate deal, Jane would prefer to have her money in the bank, where it is safe. What Jane has done here is to assess her utility for \$5,000. When there is an 80% chance (this means that p is 0.8) of getting \$10,000, Jane is indifferent between putting her money in real estate and putting it in the bank.

Jane's utility for \$5,000 is thus equal to 0.8, which is the same as the value for p . This utility assessment is shown in Figure.

Other utility values can be assessed in the same way. For example, what is Jane's utility for \$7,000? What value of p would make Jane indifferent between \$7,000 and the gamble that would result in either \$10,000 or \$0? For Jane, there must be a 90% chance of getting the \$10,000. Otherwise, she would prefer the \$7,000 for sure. Thus, her utility for \$7,000 is 0.90. Jane's utility for \$3,000 can be determined in the same way. If there were a 50% chance of obtaining the \$10,000, Jane would be indifferent between having \$3,000 for sure and taking the gamble of either winning the \$10,000 or getting nothing. Thus, the utility of \$3,000 for Jane is 0.5. Of course, this process can be continued until Jane has assessed her utility for as many monetary values as she wants. These assessments, however, are enough to get an idea of Jane's feelings toward risk. In fact, we can plot these points in a utility curve, as is done in Figure 3.9. In the figure, the assessed utility points of \$3,000, \$5,000, and \$7,000 are shown by dots, and the rest of the curve is inferred from these.





Jane's utility curve is typical of a risk avoider. A risk avoider is a decision maker who gets less utility or pleasure from a greater risk and tends to avoid situations in which high losses might occur. As monetary value increases on her utility curve, the utility increases at a slower rate.

Figure 3.10 illustrates that a person who is a risk seeker has an opposite-shaped utility curve. This decision maker gets more utility from a greater risk and higher potential payoff. As monetary value increases on his or her utility curve, the utility increases at an increasing rate. A person who is indifferent to risk has a utility curve that is a straight line. The shape of a person's utility curve depends on the specific decision being considered, the monetary values involved in the situation, the person's psychological frame of mind, and how the person feels about the future. It may well be that you have one utility curve for some situations you face and completely different curves for other.

Summary

Decision trees are powerful and versatile tools used in various fields to facilitate decision-making under uncertainty. They provide a graphical and intuitive representation of complex decision processes, making them easy to interpret and communicate. The decision tree structure consists of decision nodes, chance nodes, and terminal nodes. Decision nodes represent points in the process where choices must be made, while chance nodes represent uncertain events with associated probabilities. Terminal nodes signify the final outcomes or consequences of following specific paths through the tree. By assigning probabilities and payoffs to each chance node and terminal node, decision tree analysis enables the calculation of expected values at decision nodes, aiding in the identification of optimal decision strategies.

One of the primary advantages of decision trees is their ability to handle both categorical and continuous data, making them suitable for classification and regression tasks. Additionally, decision trees are capable of handling large datasets and can be applied to both structured and unstructured data. Despite their simplicity and ease of interpretation, decision trees may suffer from overfitting, where the model becomes too tailored to the training data and performs poorly on unseen data. To address this issue, techniques such as pruning and sensitivity analysis are employed to ensure the model's robustness and generalization capability.

Decision tree algorithms, like ID3, C4.5, and CART, are used to construct decision trees from the given data. These algorithms utilize measures like information gain and Gini index to identify the most informative attributes for splitting the data at each node. Moreover, decision tree analysis goes beyond making single-point predictions by incorporating probabilities and considering multiple potential outcomes. It is especially valuable in domains with significant uncertainty, such as finance, healthcare, and project management, where making informed and optimal decisions is crucial for success.

In conclusion, decision trees provide a systematic and structured approach to decision-making, facilitating optimal choices in complex and uncertain scenarios. Their simplicity, interpretability, and flexibility have made them a popular tool in the fields of data science, machine learning, and operations research. By enabling visualization and analysis of various decision paths, decision trees empower decision-makers to make well-informed choices that maximize rewards and minimize risks, leading to better outcomes and improved efficiency in a wide range of applications.

Keywords

Decision Tree: A graphical representation of a decision-making process that uses branches to depict choices and outcomes, aiding in making optimal decisions under uncertainty.

Decision Node: A square or rectangle in a decision tree representing a point where decisions must be made based on available choices.

Chance Node: A circle or ellipse in a decision tree representing an uncertain event or random variable with associated probabilities.

Terminal Node: A triangle-shaped node in a decision tree representing the final outcome or consequence of a specific path through the tree.

Probability: A measure indicating the likelihood of an event or outcome occurring, often represented as a value between 0 and 1.

Payoff: A numerical value associated with a terminal node in a decision tree, representing the result or cost of a specific decision path.

Uncertainty: The lack of complete knowledge or predictability in decision-making, addressed through probabilities in decision tree analysis.

Backward Induction: A mathematical technique used to calculate expected values and determine optimal decisions by starting from terminal nodes and working backward through the tree.

Pruning: The process of removing branches or nodes from a decision tree to reduce complexity and prevent overfitting to the training data.

Sensitivity Analysis: The examination of how changes in input parameters, such as probabilities or payoffs, impact the decision tree's optimal strategy.

Overfitting: A situation where a decision tree becomes too specific to the training data, leading to reduced accuracy when applied to new or unseen data.

Information Gain: A measure used in certain decision tree algorithms to evaluate the usefulness of a particular attribute for splitting the data.

Entropy: A measure of the impurity or uncertainty in a set of data, commonly used to assess the effectiveness of attribute splits in decision tree construction.

Classification Tree: A type of decision tree used for categorizing data into discrete classes or categories.

Regression Tree: A type of decision tree used to predict continuous numerical values or outcomes.

Self Assessment

1. Which of the following represents a decision point in a decision tree?
 - A. Square
 - B. Circle
 - C. Triangle
 - D. Ellipse

2. What do chance nodes in a decision tree represent?
 - A. Decisions
 - B. Outcomes
 - C. Uncertain events
 - D. Payoffs

3. In decision tree analysis, the process of calculating expected values starts from:
 - A. Terminal nodes
 - B. Chance nodes
 - C. Decision nodes
 - D. None of the above

4. What is the objective of constructing a decision tree?
 - A. Visual representation of data
 - B. Identify probabilities
 - C. Optimize decision-making under uncertainty
 - D. Represent random variables

5. Which mathematical technique is commonly used for decision tree analysis?
 - A. Forward induction
 - B. Sensitivity analysis
 - C. Backward induction
 - D. Regression analysis

6. What does a branch represent in a decision tree?
 - A. A decision point
 - B. A potential outcome
 - C. A probability value
 - D. A cost or payoff

7. The process of assigning numerical values to the final outcomes in a decision tree is known as:
 - A. Probability estimation
 - B. Sensitivity analysis
 - C. Payoff assignment
 - D. Decision making

8. In a decision tree, what do terminal nodes represent?
 - A. Probabilities
 - B. Payoffs
 - C. Decision points
 - D. Uncertain events

9. Decision trees are commonly used in which of the following fields?
- A. Medicine
 - B. Finance
 - C. Engineering
 - D. All of the above
10. The process of analyzing the impact of varying probabilities or payoffs on the optimal decision is called:
- A. Decision analysis
 - B. Sensitivity analysis
 - C. Risk assessment
 - D. Probabilistic modelling
11. Which type of decision tree allows for outcomes with continuous variables?
- A. Classification trees
 - B. Regression trees
 - C. Categorical trees
 - D. Dichotomous trees
12. The primary advantage of decision trees is:
- A. Simplicity and ease of interpretation
 - B. Ability to handle large datasets
 - C. Accuracy in all scenarios
 - D. Low computational complexity
13. What is the main drawback of decision trees?
- A. They are time-consuming to construct.
 - B. They cannot handle uncertain events.
 - C. They tend to overfit the data.
 - D. They can only handle binary outcomes.
14. Which technique is used to handle overfitting in decision trees?
- A. Sensitivity analysis
 - B. Pruning
 - C. Forward induction
 - D. Regression analysis
15. The process of identifying critical parameters that significantly influence the decision is part of:
- A. Sensitivity analysis
 - B. Decision analysis
 - C. Probability assessment
 - D. Payoff assignment

16. In decision tree analysis, what does the expected value represent?
- A. The most probable outcome
 - B. The highest payoff
 - C. The average outcome weighted by probabilities
 - D. The lowest cost
17. Which of the following is NOT a type of decision tree algorithm?
- A. ID3
 - B. CART
 - C. C4.5
 - D. Naive Bayes
18. The technique of dividing the data into subsets based on the values of specific attributes is called:
- A. Pruning
 - B. Splitting
 - C. Overfitting
 - D. Entropy
19. Which attribute selection measure is used in the ID3 algorithm for decision tree construction?
- A. Gini index
 - B. Information gain
 - C. Variance reduction
 - D. Mean squared error
20. Decision tree analysis falls under which category of methods in Operations Research?
- A. Optimization
 - B. Simulation
 - C. Heuristics
 - D. Probabilistic modelling

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. A | 2. C | 3. A | 4. C | 5. C |
| 6. B | 7. C | 8. B | 9. D | 10. B |
| 11. B | 12. A | 13. C | 14. B | 15. A |
| 16. C | 17. D | 18. B | 19. B | 20. D |

Review Questions

1. You are the manager of a retail store, and you have two possible locations to open a new branch. One location is in a busy shopping mall with high foot traffic, while the other is in a quieter neighbourhood with less competition. How would you use a decision tree to determine the best location for the new branch?
2. A pharmaceutical company is developing a new drug, and they need to decide whether to proceed with clinical trials based on the drug's success in preclinical studies. How could a decision tree assist in making this critical decision?
3. A manufacturing company needs to choose between two suppliers for a crucial component of their products. Supplier A offers lower prices, but their delivery times are inconsistent, while Supplier B has higher prices but guarantees reliable and timely deliveries. How would a decision tree help the company choose the best supplier?
4. An investment firm is considering investing in two different projects. Project A promises high returns, but it is also associated with higher risks, while Project B offers moderate returns with lower risks. How can a decision tree aid the firm in evaluating and selecting the most suitable investment opportunity?
5. A healthcare provider wants to implement a new electronic medical records system. They have two options: to develop an in-house system tailored to their specific needs or to purchase a ready-made system from a reputable vendor. How would a decision tree analysis help them decide which option to pursue?
6. A marketing team is planning a new advertising campaign and needs to choose between traditional media channels, such as TV and radio, or digital marketing platforms, such as social media and email marketing. How could a decision tree assist in allocating the marketing budget effectively?
7. A transportation company is considering expanding its fleet by either purchasing new vehicles or leasing them. Purchasing offers long-term ownership benefits, while leasing provides more flexibility in fleet management. How would a decision tree help the company make the best choice for their business?
8. A university is deciding whether to invest in upgrading its campus facilities or to allocate funds for hiring top-notch professors. Upgrading the facilities might attract more students, while hiring renowned faculty can enhance the university's reputation. How can a decision tree analysis aid in making this complex decision?
9. An insurance company needs to assess whether to approve an insurance policy for an applicant based on their risk profile. They have to weigh factors like age, health condition, and driving history. How would a decision tree assist the underwriting process and determine the policy's feasibility?
10. A food chain is planning to expand its menu with new dishes. They need to decide which items to introduce, considering factors like customer preferences, cost of ingredients, and potential profit margins. How could a decision tree help in determining the most profitable and appealing additions to the menu?



Further Reading

Decision Trees in Machine Learning: A comprehensive guide on decision trees, their algorithms, and their application in machine learning. <https://towardsdatascience.com/decision-trees-in-machine-learning-641b9c4e8052>

Introduction to Decision Tree Analysis: A detailed tutorial on decision tree analysis,

including construction, calculation of expected values, and sensitivity analysis. https://www.tutorialspoint.com/operations_research/operations_research_decision_tree_analysis.htm

Decision Trees: A Practical Guide for Machine Learning and Data Science: An in-depth article that covers decision tree concepts, splitting criteria, pruning, and handling missing data. <https://www.analyticsvidhya.com/blog/2020/04/decision-tree-algorithm-explained/>

Decision Trees for Business Decision Making: A business-oriented guide on how to use decision trees for strategic decision-making and resource allocation. https://www.researchgate.net/publication/308539266_Decision_Trees_for_Business_Decision_Making

A Comparison of Decision Tree Pruning Techniques: A research paper that discusses various pruning methods to prevent overfitting in decision trees. <https://www.sciencedirect.com/science/article/abs/pii/S0164121217303451>

Unit 14: Contemporary Issues in Operations Research

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Objective

After studying this student will be able to

- understand contemporary issues in operations research
- learn the about various software packages of optimization
- appreciate the concept of goal programming and integer programming

Introduction

In the intricate tapestry of modern decision-making, where multiple objectives compete for attention and resources, the discipline of operations research emerges as a guiding light. Seamlessly blending mathematical analysis, computational power, and real-world problem-solving, operations research equips decision-makers with tools to untangle complexity, optimize processes, and craft strategies that shape industries and societies.

At the heart of operations research lies the concept of Goal Programming, a technique meticulously designed to address the intricate web of conflicting objectives. In a world where choices rarely lead to a singular outcome, Goal Programming offers a compass to navigate the trade-offs, enabling decision-makers to reach a nuanced compromise. By quantifying goals, setting priorities, and minimizing deviations, this methodology transforms complex problems in fields like project management, manufacturing, finance, and more into solvable equations. The result: balanced decisions that respect the intricacies of a multi-objective reality.

Parallely, Integer Programming stands as a robust pillar within the operations research landscape. With its focus on discrete solutions and integer values, this technique emerges as a guiding force where outcomes must align with real-world constraints. From optimizing routes for delivery fleets to making investment choices in a portfolio, Integer Programming orchestrates decisions that reflect the practicalities of whole units and discrete choices. This mathematical discipline bridges the gap between theory and reality, ensuring that solutions are not just optimal but also feasible in a world defined by discrete quantities.

In this exploration, we embark on a journey to uncover the intricacies of Goal Programming and Integer Programming. We traverse through their methodologies, observing how mathematical

constructs translate into actionable insights. Moreover, we delve into a diverse array of applications that bear testimony to the relevance and transformative power of these techniques. From healthcare to manufacturing, supply chain logistics to finance, the footprints of Goal Programming and Integer Programming are indelibly imprinted on the path of progress.

The following chapters will unravel the mechanics of Goal Programming, decoding how priorities are assigned, deviations are minimized, and solutions harmonize competing goals. Simultaneously, we'll dive deep into the realm of Integer Programming, uncovering how the interplay of whole numbers and discrete choices shapes optimal solutions in contexts as varied as telecommunications, agriculture, and sports scheduling.

As we journey through the realms of optimization, we'll witness how these techniques illuminate the crossroads of decision-making. By seamlessly blending mathematical precision with pragmatic considerations, Goal Programming and Integer Programming stand as vanguards of informed choices, propelling industries, organizations, and societies toward a future where complexity is embraced, and decisions are guided by the power of optimization.

14.1 Contemporary Issues in Operations Research

Operations research can play a significant role in addressing merging issues, particularly in situations where efficient resource allocation, decision-making under uncertainty, and optimization are crucial. Merging issues can arise in various contexts, such as traffic management, supply chain integration, company mergers, and more. Here's how operations research techniques can be applied to solve merging issues:

Supply Chain Disruptions and Resilience:

Risk Management: Operations researchers are developing models to identify potential supply chain risks and vulnerabilities. These models consider factors like supplier dependencies, geopolitical events, and natural disasters.

Resilience Optimization: Researchers work on strategies to enhance supply chain resilience by designing flexible networks, multi-sourcing strategies, and adaptive logistics plans that can respond effectively to disruptions.

Sustainability and Green Operations:

Carbon Footprint Reduction: Operations research is used to optimize transportation and distribution networks to minimize carbon emissions and transportation costs simultaneously.

Closed-Loop Supply Chains: Researchers design closed-loop supply chain systems that focus on recycling, remanufacturing, and reducing waste, while considering economic feasibility.

Data Analytics and Big Data:

Predictive Maintenance: Operations research techniques are applied to analyze sensor data from machinery and equipment to predict maintenance needs and reduce downtime.

Demand Forecasting: Advanced analytics and machine learning are used to forecast demand more accurately, leading to better inventory management and resource allocation.

Healthcare Operations:

Patient Flow Optimization: Operations researchers develop models to streamline patient flow through hospitals, reducing waiting times and improving resource utilization.

Surgical Scheduling: OR techniques are employed to optimize surgical schedules, taking into account surgeon availability, operating room utilization, and patient priorities.

Smart Manufacturing and Industry 4.0:

Production Optimization: Operations research is used to optimize production processes, considering factors like machine allocation, production sequencing, and maintenance scheduling.

Real-time Monitoring: IoT-enabled sensors provide real-time data, which is then used for dynamic optimization of manufacturing processes and quality control.

Urban Operations and Smart Cities:

Traffic Management: Operations research helps optimize traffic signal timing, route planning, and congestion management to improve urban mobility.

Waste Management: Researchers design efficient waste collection and disposal systems to minimize environmental impact and operational costs.

Decision-making under Uncertainty:

Scenario Analysis: Operations research involves developing models that consider multiple possible scenarios and their associated outcomes, enabling decision-makers to make informed choices in uncertain environments.

Risk-based Optimization: Techniques such as robust optimization are employed to find solutions that perform well across a range of uncertain conditions.

Ethical Considerations in Decision-making:

Fairness and Bias: Operations researchers work on algorithms that are fair and unbiased, especially in applications like hiring, lending, and resource allocation.

Transparency: Developing transparent optimization models allows stakeholders to understand how decisions are made and ensures accountability.

Humanitarian Logistics:

Pre-positioning of Supplies: Operations research helps determine optimal locations for storing emergency supplies to enable swift response during crises.

Resource Allocation: Researchers develop models to allocate resources effectively in humanitarian operations, considering factors like urgency and available capacity.

Dynamic Pricing and Revenue Management:

Dynamic Pricing Strategies: Operations research is used to design pricing algorithms that adjust in real-time based on factors like demand, competitor pricing, and market conditions.

Yield Management: Techniques optimize the allocation of limited resources (such as hotel rooms or airline seats) to maximize revenue while meeting customer demand.

Traffic Merging and Flow:

Traffic Signal Timing: Operations research can optimize traffic signal timings at merging points to ensure smooth traffic flow and minimize congestion.

Lane Management: By analyzing traffic patterns and vehicle counts, operations researchers can determine optimal lane configurations to handle merging and reduce bottlenecks.

Supply Chain Integration:

Inventory Optimization: When companies merge or collaborate, they often face challenges related to inventory management. Operations research helps in optimizing inventory levels and order quantities to ensure seamless supply chain integration.

Distribution Network Design: Operations researchers can design efficient distribution networks that account for merged supply chains, optimizing routes, warehouse locations, and transportation modes.

Company Mergers and Acquisitions:

Resource Allocation: Operations research models can aid in allocating resources such as employees, equipment, and budget across merged departments or divisions to achieve optimal productivity and synergy.

Facility Location: If two companies merge and have duplicate facilities, operations research can assist in determining the best locations for consolidated facilities, minimizing costs and maximizing efficiency.

Data Integration and Decision-Making:

Data Analytics: Operations research techniques can help merge and analyze data from disparate sources, leading to better insights and informed decision-making during the merging process.

Decision Support Systems: Operations research models can provide decision-makers with scenarios and recommendations for merging strategies, considering factors like costs, benefits, and risks.

Project and Portfolio Management:

Project Integration: Operations research can optimize project scheduling, resource allocation, and task assignment when multiple projects or teams need to merge.

Portfolio Optimization: In financial or investment contexts, operations research can help optimize investment portfolios when two financial entities merge, considering risk, return, and diversification.

Merging IT Systems:

Data Migration and Integration: Operations research can assist in planning and optimizing the migration of data from different IT systems during a merger, ensuring data consistency and accuracy.

System Interoperability: Operations research techniques can be applied to design interoperable IT systems that merge seamlessly, reducing downtime and compatibility issues.

Crisis and Emergency Response:

Resource Coordination: During emergencies or crises, operations research can assist in coordinating the merging of resources from different agencies or organizations to provide an effective response.

Scenario Planning: Operations research models can be used to simulate various merging scenarios in emergency situations, helping decision-makers anticipate challenges and plan accordingly.

These issues represent a subset of the diverse challenges that operations research professionals are currently addressing. The field's interdisciplinary nature allows it to contribute solutions to a wide range of industries and domains. To stay updated on the latest developments, consider exploring academic journals, attending conferences, and engaging with experts in the field.

In all these contexts, operations research methods involve mathematical modeling, optimization, simulation, and data analysis to find the most efficient and effective solutions for merging issues. The specific techniques and approaches used will depend on the nature of the merging problem, the available data, and the goals of the merging entities.

14.2 Software Packages for Optimization

There are several software tools and packages commonly used in operations research to model, analyze, and solve complex optimization problems. These tools offer a range of functionalities, including linear and nonlinear programming, integer programming, simulation, and more. Here are some widely used software options along with brief details about each:

IBM ILOG CPLEX:

Description: CPLEX is a powerful optimization software suite that offers a wide range of optimization techniques, including linear programming, mixed-integer programming, quadratic programming, and more.

Features: It provides solvers for various optimization problems, supports modeling languages like AMPL, OPL, and others, and is commonly used in academia and industry for complex optimization tasks.

Gurobi:

Description: Gurobi is another popular optimization software package known for its efficiency in solving large-scale linear, integer, and quadratic programming problems.

Features: It offers state-of-the-art optimization algorithms, a user-friendly interface, and integration with various programming languages and modeling tools.

AMPL:

Description: AMPL (A Mathematical Programming Language) is a modeling language that facilitates the formulation and solving of optimization problems.

Features: AMPL allows users to define optimization models in a natural and concise way. It supports a variety of solvers, including CPLEX and Gurobi, and provides a flexible environment for testing different problem formulations.

MATLAB Optimization Toolbox:

Description: MATLAB's Optimization Toolbox provides a wide range of optimization algorithms and tools for solving linear and nonlinear programming problems, as well as constrained and unconstrained optimization problems.

Features: MATLAB offers a user-friendly environment for prototyping and solving optimization problems, making it suitable for both beginners and experts.

SAS/OR:

Description: SAS/OR is part of the SAS software suite and focuses on operations research and optimization.

Features: SAS/OR offers solvers for linear, nonlinear, and integer programming, as well as simulation and network optimization. It's commonly used in industries like manufacturing, logistics, and finance.

AnyLogic:

Description: AnyLogic is a simulation software that can be used for various operations research applications, including discrete-event simulation, agent-based modeling, and system dynamics.

Features: AnyLogic is known for its flexibility in modeling complex systems, making it suitable for analyzing dynamic and stochastic processes.

LINDO/LINGO:

Description: LINDO and LINGO are optimization software packages that provide solvers for linear programming, integer programming, quadratic programming, and nonlinear programming problems.

Features: They offer user-friendly interfaces, modeling languages, and are widely used for optimization tasks in business and engineering.

Excel Solver:

Description: Microsoft Excel includes a built-in optimization tool called Solver that is capable of solving linear and nonlinear programming problems.

Features: While not as advanced as specialized OR software, Excel Solver is widely accessible and suitable for basic optimization tasks.

These software options offer various features and capabilities to cater to different types of operations research problems. The choice of software depends on factors like the complexity of the problems you're dealing with, your familiarity with the software, and the specific requirements of your project.

Python Libraries:

PuLP: A library for defining and solving linear and integer programming problems.

Pyomo: A flexible optimization modeling language for linear, integer, nonlinear, and mixed-integer programming.

SciPy.optimize: Part of the SciPy library, it provides optimization algorithms for various problem types.

CVXPY: A domain-specific language for convex optimization problems.

R Libraries:

lpSolve: Provides functions to solve linear and integer programming problems.

ROI: Offers an interface to various optimization solvers for linear, nonlinear, and mixed-integer programming.

Julia Packages:

JuMP: A modeling language for mathematical optimization problems, compatible with various solvers.

Convex.jl: A library for convex optimization problems.

Goal Programming is a decision-making and optimization technique used to solve problems with multiple, conflicting objectives or goals. Unlike traditional optimization methods that aim to find a single optimal solution, Goal Programming allows decision-makers to balance the trade-offs between different objectives and reach a compromise solution that meets multiple goals to the best possible extent.

14.3 Goal Programming Models

In today's business environment, maximizing profit (or minimizing cost) is not always the only objective that a firm may specify. In many cases, maximizing profits is just one of several objectives that may include maximizing machine utilization, maintaining full employment, providing quality ecological management, minimizing noise level in the neighborhood, and meeting numerous other noneconomic targets. Often, some of these objectives are conflicting (i.e., it may not be possible to simultaneously achieve these objectives).

Mathematical programming techniques such as LP and IP have the shortcoming that their objective function can deal with only a single criterion, such as profit, cost, or some such measure. To

overcome this shortcoming, an important technique that has been developed to handle decision models involving multiple objectives is called goal programming (GP).

How do LP/IP and GP models differ? In LP/IP models, we try to find the best possible value for a single objective. That is, the aim is to *optimize* a single measure. In GP models, on the other hand, we first set a goal (or desired target) for each objective. In most decision modeling situations, some of these goals may be achievable only at the expense of other goals. Therefore, we establish a hierarchy or rank of importance among these goals so that lower-ranked goals are given less prominence than high-ranked goals. Based on this hierarchy, GP then attempts to reach a “satisfactory” level for each goal. That is, GP tries to satisfy the multiple objectives (i.e., come as close as possible to their respective goals) rather than optimize them.

How does GP satisfy the goals? Instead of trying to maximize or minimize the objective functions directly, as in LP/IP, with GP we try to minimize *deviations* between the specified goals and what we can actually achieve for the multiple objective functions within the given constraints. Deviations can be either positive or negative, depending on whether we overachieve or underachieve a specific goal. These deviations are not only real decision variables in the GP model, but they are also the only terms in the objective function. The objective is to minimize some function of these deviation variables.

Application of Goal Programming

Goal Programming has a wide range of applications across various domains where decision-makers need to balance and prioritize multiple conflicting objectives. Here are some common application areas:

Project Management:

In project management, goals often conflict, such as completing a project within a certain time frame while minimizing costs. Goal Programming helps project managers find a compromise solution that meets time and budget constraints to the best possible extent.

Manufacturing and Production Planning:

Manufacturers may need to optimize production schedules to meet demands for different products while minimizing costs, inventory levels, and production delays. Goal Programming assists in finding a production plan that balances these objectives.

Supply Chain Management:

Companies face challenges in optimizing supply chain operations to minimize transportation costs, inventory holding costs, and delivery times. Goal Programming helps in finding solutions that address these conflicting goals.

Financial Portfolio Management:

Investors aim to achieve high returns while managing risks. Goal Programming helps in constructing investment portfolios that strike a balance between maximizing returns and minimizing risk exposure.

Energy Management and Sustainability:

Organizations aim to reduce energy consumption to meet sustainability goals while minimizing operational costs. Goal Programming assists in finding energy usage patterns that satisfy environmental targets and financial constraints.

Healthcare Resource Allocation:

In healthcare, goals include maximizing patient care quality while minimizing costs and resource utilization. Goal Programming helps allocate resources efficiently to achieve these objectives.

Transportation and Route Planning:

In transportation, goals may involve minimizing travel time, costs, and emissions while delivering goods. Goal Programming helps in designing optimal routes that consider these conflicting objectives.

Education and Course Scheduling:

Educational institutions need to schedule classes to meet student demand while ensuring efficient use of resources. Goal Programming helps in creating schedules that balance class availability, faculty preferences, and room allocation.

Environmental Management:

Environmental management involves balancing objectives like pollution reduction, resource conservation, and regulatory compliance. Goal Programming assists in finding strategies that satisfy these goals.

Facility Location and Expansion:

When deciding where to open new facilities or expand existing ones, businesses must consider factors such as market demand, transportation costs, and land prices. Goal Programming helps in making location decisions that address these objectives.

Personnel Scheduling:

In industries with shift work, employee scheduling aims to optimize workforce utilization while considering fairness and employee preferences. Goal Programming helps in creating schedules that meet staffing requirements and employee satisfaction.

Agricultural Planning:

Farmers need to make planting decisions to maximize crop yield while minimizing resource use and risks. Goal Programming assists in finding planting strategies that balance these objectives.

These applications demonstrate the versatility of Goal Programming in solving real-world problems that involve multiple, often conflicting, objectives. It provides decision-makers with a systematic approach to making informed decisions while considering the priorities and trade-offs inherent in complex decision environments.

14.4 Integer Programming

If linear programs with the additional requirement that one or more variables must be integer. Such problems are called **integer linear programs**. If all variables must be integer, we have an all-integer linear program. If some, but not all, variables must be integer, we have a mixed-integer linear program. In many applications of integer linear programming, one or more integer variables are required to equal either 0 or 1. Such variables are called 0-1 or *binary variables*. If all variables are 0-1 variables, we have a 0-1 integer linear program.

Integer variables—especially 0-1 variables—provide substantial modeling flexibility. As a result, the number of applications that can be addressed with linear programming methodology is expanded. For instance, the Management Science in Action, *Optimizing the Transport of Oil Rig Crews*, describes how Petrobras uses a model with 0-1 variables for assigning helicopters to flights for transporting crews to and from its oil rigs. Later Management Science in Actions describe how the Virginia Court of Appeals uses a 0-1 integer program for scheduling panels of judges to preside over appeal hearings and how the modeling flexibility provided by 0-1 variables helped Ketron build a customer order allocation model for a sporting goods company.

For example, when an airline decides how many flights to operate on a given sector, it can't decide to operate 5.38 flights; it must operate 5, 6, or some other integer number.

An **integer programming** model is a model that has constraints and an objective function identical to that formulated by LP. The only difference is that one or more of the decision variables has to take on an integer value in the final solution. There are three types of integer programming problems:

1. Pure integer programming problems are cases in which all variables are required to have integer values.
2. Mixed-integer programming problems are cases in which some, but not all, of the decision variables are required to have integer values.
3. Zero-one integer programming problems are special cases in which all the decision variables must have integer solution values of 0 or 1.

Solving an integer programming problem is much more difficult than solving an LP problem. The solution time required to solve some of these may be excessive even on the fastest computer.

Application of Integer Programming

Integer Programming is a subset of mathematical optimization that deals with problems where decision variables are required to take on integer values. This characteristic makes it particularly useful for solving problems where solutions must be discrete or represent whole units. Here are some common application areas for Integer Programming:

Production and Manufacturing:

Production Scheduling: IP is used to optimize the allocation of resources and machines to different production tasks while considering constraints and minimizing costs.

Cutting Stock Problem: In industries like paper manufacturing, IP helps determine how to cut raw materials (e.g., rolls of paper) into smaller pieces to fulfill demand while minimizing waste.

Transportation and Logistics:

Vehicle Routing: IP is applied to optimize routes and schedules for delivery vehicles, taking into account customer locations, vehicle capacity, and distance.

Facility Location: IP assists in determining the best locations for warehouses or distribution centers to minimize transportation costs and fulfill demand efficiently.

Network Design:

Telecommunications Network Design: IP helps in designing optimal routes and locations for network infrastructure like cell towers or data centers to minimize costs and maximize coverage.

Project Management:

Project Scheduling: IP can be used to optimize the scheduling of project tasks with different precedence constraints and limited resources.

Personnel Scheduling:

Employee Shift Scheduling: IP is employed to create optimal schedules for employees while adhering to staffing requirements and employee preferences.

Portfolio Optimization:

IP assists in constructing investment portfolios by selecting a subset of assets to invest in while considering constraints on budget, risk, and expected returns.

Capital Budgeting:

Organizations use IP to allocate limited funds among various projects or investments to maximize returns or achieve other objectives.

Agricultural Planning:

IP helps farmers decide which crops to plant on available farmland to maximize yield, considering factors like crop rotation, resource availability, and market demand.

Combinatorial Optimization:

IP is often used in various combinatorial optimization problems, such as the Traveling Salesman Problem, where the goal is to find the shortest route visiting a set of cities exactly once.

Binary Network Design:

In telecommunication and computer network design, IP helps decide which links to include or exclude in a network to optimize connectivity and reliability.

Sports Scheduling:

IP can create schedules for sports leagues that optimize factors like travel distances, rest days, and venue availability.

Quadratic Assignment Problem:

IP addresses problems like assigning facilities to locations in a way that minimizes distances and associated costs.

These are just a few examples of how Integer Programming is applied across various domains to solve discrete optimization problems. IP is particularly valuable when the decision variables inherently take on discrete values, such as yes/no, on/off, or quantities that represent whole units.

Summary

We began by exploring the realm of operations research and optimization, which involves employing mathematical techniques to solve complex decision-making challenges. Within this field, contemporary issues such as sustainability, data analytics, machine learning integration, and dynamic optimization were highlighted as areas of growing importance. We then delved into how operations research can address merging issues, applying its methodologies to scenarios like traffic management and supply chain coordination, demonstrating its real-world applicability.

Transitioning to the software landscape, we introduced optimization software tools like Gurobi, IBM ILOG CPLEX, and AMPL, which serve as powerful aids in solving intricate optimization problems across various domains. Further, programming-based tools such as PuLP and Pyomo, along with heuristic algorithms and metaheuristics, emerged as versatile assets for tackling complex optimization challenges efficiently.

Our discussion turned to two key optimization techniques: Goal Programming and Integer Programming. We unveiled Goal Programming's role in balancing conflicting objectives through compromise solutions, finding relevance in project management, manufacturing, and beyond. On the other hand, Integer Programming, with its focus on discrete values, was shown to excel in solving problems requiring whole-number solutions, spanning areas like production, logistics, and network design.

To solidify understanding, we provided a set of review questions, probing comprehension of Goal Programming, Integer Programming, heuristic algorithms, and optimization concepts. Lastly, we offered guidance on seeking further knowledge through references and reading materials, steering you toward academic databases, official software websites, and reputable resources to deepen your exploration of operations research, optimization techniques, and associated tools.

Keywords

Linear Programming: Linear Programming is a mathematical method used to find the best possible outcome in a model represented by linear relationships. It involves optimizing a linear objective function subject to linear constraints.

Integer Programming: Integer Programming is an optimization technique where decision variables are restricted to integer values. It is used when solutions must be composed of whole units, making it suitable for discrete optimization problems.

Goal Programming: Goal Programming is an optimization method that addresses problems with multiple, conflicting objectives. It seeks to minimize deviations from predefined goals for each objective while considering their priority weights.

Optimization Software: Description: Optimization Software refers to computer programs and tools designed to solve complex optimization problems. These software packages provide algorithms and solvers to find optimal solutions for various applications.

Decision Variables: Decision Variables are the variables in an optimization problem that represent choices or decisions to be made. They are adjusted to find the best solution to the problem.

Sensitivity Analysis: Sensitivity Analysis assesses how changes in input parameters affect the optimal solution of an optimization problem. It provides insights into the stability and robustness of the solution.

Constraints: Constraints are the limitations or restrictions placed on decision variables in an optimization problem. They define the feasible region within which the optimal solution must lie.

Objective Function: The Objective Function quantifies the goal to be optimized in an optimization problem. It is a mathematical expression that needs to be either minimized or maximized based on the problem's context.

Heuristic Algorithms: Heuristic Algorithms are optimization techniques that provide near-optimal solutions in a reasonable amount of time, even for complex problems. They prioritize speed over guaranteed optimality.

Metaheuristic Methods: Metaheuristic Methods are high-level strategies used to solve complex optimization problems. They guide the search process by exploring multiple solution spaces and are particularly effective for large-scale problems.

Self Assessment

1. What is the primary goal of Goal Programming?
 - A. To find a single optimal solution
 - B. To balance conflicting objectives
 - C. To maximize only one objective
 - D. To minimize deviations from a single goal

2. Which type of programming is used when decision variables must take on integer values?
 - A. Linear Programming
 - B. Nonlinear Programming
 - C. Integer Programming
 - D. Goal Programming

3. What does a priority weight in Goal Programming represent?
 - A. The ideal solution
 - B. The importance of the constraint
 - C. The importance of the objective
 - D. The trade-offs in the solution

4. Which of the following is NOT an application of Goal Programming?
 - A. Project scheduling
 - B. Network design
 - C. Employee shift scheduling
 - D. Portfolio optimization

5. In Integer Programming, what type of values do decision variables take on?
 - A. Fractional values
 - B. Real numbers
 - C. Whole numbers
 - D. Complex numbers

6. Which optimization technique is used when goals conflict, and a compromise solution is sought?
 - A. Linear Programming
 - B. Quadratic Programming
 - C. Goal Programming

- D. Nonlinear Programming
7. What is the main benefit of Integer Programming in solving real-world problems?
- A. It works for continuous variables only.
 - B. It provides exact solutions for complex problems.
 - C. It guarantees a global optimal solution in all cases.
 - D. It handles constraints involving fractions efficiently.
8. Which of the following is NOT a limitation of Goal Programming?
- A. Difficulty in assigning priority weights
 - B. Complexity in modeling nonlinear problems
 - C. Inability to handle continuous variables
 - D. Sensitivity to changes in goal values
9. Which type of optimization is suitable when there are multiple, discrete decision points?
- A. Linear Programming
 - B. Integer Programming
 - C. Nonlinear Programming
 - D. Continuous Programming
10. Which of the following is NOT a characteristic of Goal Programming?
- A. Minimizing deviations from goals
 - B. Handling continuous decision variables
 - C. Incorporating priority weights
 - D. Balancing multiple objectives
11. Which optimization technique is commonly used in vehicle routing problems?
- A. Goal Programming
 - B. Integer Programming
 - C. Linear Programming
 - D. Quadratic Programming
12. What role do priority weights play in Goal Programming?
- A. They represent the constraints in the problem.
 - B. They indicate the optimal solution.
 - C. They influence the importance of each objective.
 - D. They help model nonlinear functions.
13. In which application might Goal Programming be used to allocate resources while minimizing costs and meeting demand constraints?
- A. Energy Management
 - B. Project Management

- C. Portfolio Optimization
 - D. Telecommunications Network Design
14. Which of the following is NOT a suitable domain for Integer Programming?
- A. Project Scheduling
 - B. Employee Shift Scheduling
 - C. Continuous Process Optimization
 - D. Network Design
15. What is the primary advantage of Goal Programming over Linear Programming?
- A. It provides faster solutions.
 - B. It guarantees global optimality.
 - C. It handles continuous decision variables.
 - D. It accommodates multiple objectives.
16. Which of the following problems involves optimizing routes for delivery vehicles with constraints on capacity?
- A. Cutting Stock Problem
 - B. Portfolio Optimization
 - C. Quadratic Assignment Problem
 - D. Vehicle Routing Problem
17. What is the key challenge addressed by Goal Programming?
- A. Maximizing a single objective
 - B. Handling continuous decision variables
 - C. Balancing conflicting objectives
 - D. Solving linear programming problems
18. Which optimization technique aims to minimize deviations from desired goals?
- A. Integer Programming
 - B. Linear Programming
 - C. Goal Programming
 - D. Network Design
19. In Integer Programming, when are decision variables allowed to take fractional values?
- A. Always
 - B. Never
 - C. Only in special cases
 - D. Only when priority weights are assigned
20. Which optimization technique can be applied to allocating resources among projects while considering project precedence constraints?

- A. Goal Programming
 - B. Integer Programming
 - C. Linear Programming
 - D. Network Design
21. Which software is known for its optimization capabilities and offers solvers for linear, integer, and quadratic programming problems?
- A. MATLAB Optimization Toolbox
 - B. AnyLogic
 - C. GAMS
 - D. Gurobi
22. Which programming language offers libraries like PuLP and Pyomo for modeling and solving optimization problems?
- A. R
 - B. Julia
 - C. Python
 - D. MATLAB
23. AMPL is a modeling language that integrates with various solvers for optimization problems. What does AMPL stand for?
- A. Advanced Modeling Programming Language
 - B. A Mathematical Programming Language
 - C. Applied Mathematical Programming Language
 - D. Algorithmic Modeling Programming Language
24. SAS/OR is a part of which software suite that focuses on operations research and optimization?
- A. SPSS
 - B. SAS
 - C. MATLAB
 - D. R
25. Which software provides a modeling language for mathematical optimization problems and is compatible with various solvers?
- A. AMPL
 - B. GAMS
 - C. MATLAB
 - D. PuLP
26. Which optimization software is known for its speed and ease of use in scientific and numerical computing?
- A. MATLAB Optimization Toolbox

- B. Gurobi
- C. AnyLogic
- D. Julia

27. Which optimization software provides a collection of open-source libraries for various optimization-related tasks?

- A. IBM ILOG CPLEX
- B. GAMS
- C. COIN-OR
- D. LINDO

28. Which software package includes a built-in optimization tool called Solver that can be used in Microsoft Excel?

- A. Gurobi
- B. SAS/OR
- C. IBM ILOG CPLEX
- D. Excel

29. In which programming language is the Gurobi Python API used to interact with Gurobi's optimization solvers?

- A. R
- B. Python
- C. Julia
- D. C++

30. Which software offers the Optimization Toolbox for linear, nonlinear, and integer programming problems?

- A. MATLAB
- B. GAMS
- C. R
- D. Any Logic

Answers for Self Assessment

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. B | 2. C | 3. C | 4. B | 5. C |
| 6. C | 7. B | 8. C | 9. B | 10. B |
| 11. B | 12. C | 13. A | 14. C | 15. D |
| 16. D | 17. C | 18. C | 19. B | 20. B |
| 21. D | 22. C | 23. B | 24. B | 25. A |
| 26. D | 27. C | 28. D | 29. B | 30. A |

Review Questions

1. What is the main purpose of Goal Programming, and how does it address the challenge of conflicting objectives? Provide an example scenario where Goal Programming could be applied.
2. Explain the key difference between Linear Programming and Integer Programming. When is Integer Programming particularly useful, and what types of problems is it well-suited for solving?
3. Describe the components of a Goal Programming model, including decision variables, objective functions, goals, deviations, and priority weights. How are these components integrated to formulate and solve a Goal Programming problem?
4. Compare and contrast sensitivity analysis and heuristic algorithms in the context of optimization problems. How does sensitivity analysis help decision-makers, and what role do heuristic algorithms play in finding solutions?
5. Briefly explain the concept of an "objective function" in optimization. How does the objective function differ in Linear Programming and Goal Programming, and how does each type of optimization approach contribute to decision-making?



Further Readings

<https://www.sciencedirect.com/topics/computer-science/goal-programming>

<https://www.informs.org/>

<https://www.mathworks.com/help/optim/>

<https://www.gurobi.com/resources/>

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