# Functional Analysis

DEMTH642

Edited by:
Dr. Kulwinder Singh

LOVELY PROFESSIONAL UNIVERSITY

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# Content

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# Unit 01: Normed Linear Space and Banach Spaces I

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Normed linear spaces.
- Banach spaces.
- Properties of normed space.

## Introduction

The notion of a norm is an abstract generalization of the length of a vector. It is axiomatically stated that the norm is any real valued function that satisfies specific requirements. The linear space together with the norm is called a normed linear space. Moreover, the Banach spaces is a type of normed linear spaces that possess the additional property of completeness.

In what follows, K will denote the field of R (real numbers) or C (complex numbers). We shall always assume that R and C have their usual metrics and that all the linear spaces that we consider will be defined over K (R or C).

## 1.1 Normed Linear Space

In this section, we first introduce the formal definition of norm, which serves as the building block of the subsequent sections.

**Definition**. A norm $||\cdot||$ is defined as the function $||\cdot||: X \to R$ on a linear space $X$ satisfying the following properties:

(i) $||x|| \geq 0,\ \forall\, x \in X;$

(ii) $||x\,|| = 0$ if and only if $x = 0;$

(iii) $||\,x + y\,|| \leq ||x|| + \ ||y||, \forall x, y \in X;$

(iv) $||\,\alpha x|| \leq |\alpha|\,||x||, \forall x \in X, \alpha \in K.$

A linear space $X$ over $K$ with norm $||.||$ defined on $X$ is called a normed linear space or simply a normed space over , written as $(X, ||.||)$ or $X$. The normed linear space is real or complex accordingly as the field $K$ is $R$ or $C$.

Next, we provide some examples for the lucid illustration of the normed spaces.

**Example.** Consider a linear space $X$ together with the norm defined by $||x|| = |x|$. Then, it is easy to verify that the properties (i)-(iv) of the Definition holds. In particular, triangle inequality follows by the fact $||x + y|| = |x + y| \leq |x| + |y|, \forall\, x, y \in X$.

**Example.** The_space $R^n$ (n-dimensional Euclidean space**)** and $C^n$ (n-dimensional unitary space) of all n-tuples of real and complex numbers are normal linear spaces with the norm defined by

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} = (|x_1|^p + |x_2|^p + \cdots |x_n|^p)^{\frac{1}{p}};\ 1 \leq \mathrm{p} < \infty$$

**Example.** Let $\ell_p$ be the space of all sequences $x = \{x_n\}$ satisfying $\sum_{i=1}^{\infty} |x_i|^p < \infty,\ p \geq 1$. Then, this space is a normed linear space with the norm $||x||_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}, \forall\, x \in \ell_p$.

**Example.** Consider $\ell_\infty$ space, that is, the space of all bounded sequences $x = \{x_n\}$. Then, this space is a normed linear space with the norm $||x||_\infty = \sup |x_i|,\ \ 1 \leq i \leq \infty$.

**Example.** Find $||x||_1, ||x||_2$ and $||x||_\infty$ for the vector $x = (2, 3, 1, -4) \in \mathrm{R}^4$.

**Solution:** We have $||x||p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$.

Then, $||x||_1 = |x_1| + |x_2| + |x_3| + |x_4| = |2| + |3| + |1| + |-4| = 10$

$$||x||_2 = \left(\sum_{i=1}^{4} |x_i|^2\right)^{\frac{1}{2}} = (|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2)^{\frac{1}{2}}$$

$$= (|2|^2 + |3|^2 + |1|^2 + |-4|^2)^{\frac{1}{2}} = \sqrt{30}.$$

Also, $||x||_\infty = sup\{|x_1|, |x_2|, |x_3|, |x_4|\} = sup\{|2|, |3|, |1|, |-4|\} = 4$.

**Example.** Let $C[a, b]$denotes the space of continuous real valued functions defined on $[a, b]$. Then, $C[a, b]$ defines a normed linear space with the norms:

1. $||f|| = \sup |f(x)|, \forall f \in C[a, b], x \in [a, b]$.
2. $||f|| = \int_a^b |f(x)| dx, \forall\, f \in C[a, b]$.

## 1.2 Properties of Normed Linear Space

We now recall some basic definitions, which shall be frequently used in the remaining part of this section.

**Definition .** Let $(X, ||.||)$ be the normed linear space on $X$. Then, we have the following definitions.

- An open sphere (or open ball) with center $x_0$ and radius $r > 0$ is the set $B(x_0; r) = \{x \in X: ||x - x_0|| < r\}$. By the surface (or boundary) of this ball, we mean the set $S(x_0; r) = \{x \in X: ||x - x_0|| = r\}$.
- The set $B[x_0; r] = \{x \in X: ||x - x_0|| \leq r\}$ denoted by $S[x_0, r\ ]$ or $S_r[x_0]$ is called the closed sphere or closed ball with radius $r$ and center $x_o$.
- A set $D$ in $X$ is said to be open if for every $x \in D$, there exists a ball with center $x$ which is contained in D.
- A set $D$ in $X$ is said to be closed if for any sequence $\{x_n\}$ in $D$ with $x_n \to x$ implies that $x \in D$.
- A set $D$ in $X$ is said to be bounded in $X$ if there exists a constant $M$ such that $||x|| < M, \forall x \in D$.
- A set $D$ in $X$ is said to be compact if whenever $\{x_n\} \in D$, there exist a convergent subsequence of $\{x_n\}$ whose limit is in $D$.
- A sequence $\{x_n\}$ is said to be bounded, if there exists a real constant $K > 0$ such that $||x_n|| < K, \forall\ n$.
- A sequence $\{x_n\} \subset X$ is said to be convergent if there exists an $x \in X$ such that

$$\lim_{n\to\infty} ||x_n - x||=0$$

- A sequence $\{x_n\} \subset X$ is said to be Cauchy sequence if for given $\epsilon > 0, \exists$ a positive integer N such that

$$\lim_{n\to\infty} ||x_m - x_n|| < \epsilon\ \forall\ \text{m, n} \geq \text{N}$$

  That is, $x_n$ is said to be Cauchy sequence in X iff

$$\lim_{n\to\infty} ||x_m - x_n|| \to 0\ as\ m, n\ \to \infty$$

- The space X is said to be complete if every Cauchy sequence in $X$ converges to an element in $X$.

**Theorem .** Show that Every normed linear space is a metric space w.r.to the metric $(x, y) = ||x - y||\ ; \forall\ x, y\ \epsilon\ X$. But the converse may not be true.

**Proof.** Let $X$ be a normed linear space. Define a mapping $d: X \times X \to R$ by:

$$d(x, y) = ||x - y||; \forall\ x, y \epsilon X.$$

We show that $d$ is a metric on $X$.

Since (i) $(x, y) = ||x - y|| \geq 0$ .

That is, $d(x, y) \geq 0$.

(ii) $d(x,y) = ||x-y|| = 0$, iff $x - y = 0$ iff $x = y$.

That is, $d(x,y) = 0$, iff $x = y$.

(iii) $d(x,y) = ||x-y| = ||y-x|| = d(y,x)$.

That is, $d(x,y) = d(y,x)$.

(iv) $d(x,z) = ||x-z| = ||x-y+y-z| \leq ||x-y|| + ||y-z||$,

$=d(x,y) + d(y,z)$.

Thus, $d(x,z) \leq d(x,y) + d(y,z)$.

Hence, $d$ is a metric on normed linear space $X$, known as metric induced by norm and hence $X$ with $d$ is a metric space.

Now, we show that the converse of above theorem need not be true. For this consider a linear space $X$ with metric $d$ defined as $d(x,y) = \frac{|x-y|}{1+|x-y|}$.

We can clearly verify the above metric satisfies all the conditions of metric space.

If we take $(x,y) = \left||x-y|\right| = \frac{|x-y|}{1+|x-y|}$ .

Or we can write $\left||z|\right| = \frac{|z|}{1+|z|}$ $\forall\ z = x - y\ \in X$

Thus for any α scalar $\left||\alpha z|\right| = \frac{|\alpha z|}{1+|\alpha z|} = |\alpha| \frac{|z|}{1+|\alpha||z|} \neq |\alpha|\left||z|\right|$.

This shows that $X$ is not normed linear space.

**Theorem.** Show that For any normed space ,

$$|||x|| - ||y||| \leq ||x-y||; \forall\ x, y \epsilon X.$$

**Proof .** We now prove that $|||x|| - ||y||| \leq ||x-y||; \forall\ x, y \epsilon X$.

We can write $x = x - y + y$.

So, $\left||x|\right| = \left||x-y+y|\right| \leq \left||x-y|\right| + \left||y|\right|$.

Implies, $||x|| - ||y|| \leq ||x-y||$.

Similarly, we can write $y = y - x + x$.

So, $||y|| = ||y-x+x|| \leq ||y-x|| + ||x||$.s

This implies, -$||y-x|| \leq ||x|| - ||y||$.

Or $-||x-y|| \leq ||x|| - ||y||$.

So, from the above relations, we get

$-||x-y|| \leq ||x|| - ||y|| \leq ||x-y||$.

Implies, $|||x|| - ||y||| \leq ||x-y||$.

**Definition .** Let $X$ and $Y$ are normed linear spaces, then $f: X \rightarrow Y$ is said to be continuous at $x_0 \in X$ , If for given ϵ>0, thereexists $\delta > 0$ such tha $\left||f(x) - f(x_0)|\right| < \epsilon$ whenever $\left||x - x_0|\right| < \delta$.

That is, $||x - x_0|| < \delta$, implies $||f(x) - f(x_0)|| < \epsilon$ .

**Theorem.** Show that a norm function is continuous.

**Proof.** Let $< x_n > \to$ x in normed linear space X.

That is, $x_n \to x$ as $n \to \infty$.

Or, $x_n - \text{x} \to 0$ as $n \to \infty$.

This gives $||x_n - \text{x}|| \to 0$ as $n \to \infty$.

Now, $|\ ||x_n|| - ||x||\ | \leq ||x_n - \text{x}|| \to 0$ as $n \to \infty$.

This implies, $|\ ||x_n|| - ||x||\ | \to 0$ as $n \to \infty$.

Or, $||x_n|| - ||x|| \to 0$ as $n \to \infty$.

Or, $||x_n|| \to ||x||$ as $n \to \infty$.

This Shows that $||.||$ is a continuous function.

**Theorem.** Show that every convergent sequence is a cauchys sequence.

**Proof.** Let $< x_n >$ be a sequence convergent to $x$, then for given $\epsilon > 0$, there exists $n_0 \in$ N such that

$$||x_n - x|| < \frac{\epsilon}{2} \quad \forall \text{ n} \geq \text{n}_0.$$

In particular, $||x_n - x|| < \frac{\epsilon}{2}$ for fixed m $> \text{n}_0$.

Now, $||x_n - x_m|| = ||x_n - x + x - x_m||$

$$= ||(x_n - x) + (x - x_m)||$$

$$\leq ||x_n - x|| + ||x - x_m||$$

$$= ||x_n - x|| + |-1|||x_m - x||$$

$$= ||x_n - x|| + ||x_m - x|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ \forall\, m, n \geq n_o$$

That is, $||x_n - x_m|| < \epsilon\ \forall m, n \geq n_o$.

Therefore, $< \text{x}_\text{n} >$ is a Cauchy sequence in Normed linear space $X$.

**Remark:** Converse of above result need not be true in general. As every normed linear space is a metric space, we will show this for metric space

For this , Consider X = (0, 1], with $d(x, y) = |x - y|$ and consider sequence $< x_n > = < 1/n >$ , where $x_n \in X$. then, $< 1/n >$ is a Cauchy sequence as

$d(x_n, y_m) = |1/n - 1/m| \to 0$ as $m, n \to \infty$.

However, $d(1/n, 0) = |1/n - 0| \to 0$ as $n \to \infty$.

But 0 $\notin X$. Therefore, $< x_n >$ is a Cauchy sequence in X but not convergent in X.

**Theorem .** Let $X$ be a normed linear space over the field $K$. Let $< x_n >$ and $< y_n >$ be sequences in $X$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, respectively , and $\{\alpha_n\}$ be a sequence in $K$ with $\lim_{n\to\infty} \alpha_n = \alpha$.

Then

(i) $\lim_{n\to\infty} (x_n + y_n) = x + y$.

(ii) $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

**Proof.** We know by definition of norm

$$||(x_n + y_n) - (x + y)|| = ||(x_n - x) + (y_n - y)||$$
$$\leq ||(x_n - x)|| + ||(y_n - y)||$$
$$\to 0 \text{ as } n \to \infty.$$

Also, $||\alpha_n x_n - \alpha x|| = ||\alpha_n(x_n - x) + (\alpha_n - \alpha)x||$

$$\leq |\alpha_n|||x_n - x|| + |\alpha_n - \alpha|||x||$$

$$\to 0 \text{ as } n \to \infty.$$

## 1.3 Banach Space (Complete normed Space)

**Definition.** A normed linear space $X$ is called complete if every Cauchy sequence in $X$ converges to a limit point in $X$.

A complete normed linear space is called a Banach space.

(OR )

A normed linear space which is complete as a metric space is called a Banach space.

**Example.** The Spaces $R$ and $C$ of reals and complex numbers are Banach spaces. These are the consequences from the real analysis result that every Cauchy sequence is convergent.

**Example.** The spaces $\boldsymbol{R^n}$ and $C^n$ are Banach spaces . Here we prove that $R^n$ is complete.

Let $\{x^{(p)}\}$ be a Cauchy sequence in $\boldsymbol{R^n}$**,**

$$x_i^{(p)} = \left(x_1^{(p)}, x_2^{(p)}, \dots, x_n^{(p)}\right), p = 1,2, \dots$$

Then, given any $\epsilon > 0$, there is a natural number $n_0$ such that

$\forall\, p, q; p, q \geq \mathrm{n}_0$ , implies $||x^{(p)} - x^{(q)}|| = \sqrt{\sum_{i=1}^{n} \left|x_i^{(p)} - x_i^{(q)}\right|^2} < \epsilon$

Hence, $\forall\, p, q; p, q \geq n_0$ , implies $\left|x_i^{(p)} - x_i^{(q)}\right| \leq ||x^{(p)} - x^{(q)}|| < \epsilon$

So, for each i, $x_i^{(p)}$ is a cauchy sequence of real numbers. Since R is complete, $x_i^{(p)}$ converge to a real number $x_i$ for all $i = 1, 2, 3, \dots,$ n. But this implies that for already choosen $\epsilon$, there exist a natural number $p_i$ such that

$$\forall\ p; p \geq p_{\mathrm{i}}\ \text{, implies } \left|x_i^{(p)} - x_i\right| < \epsilon/\sqrt{n}$$

Take $x = (x_1, x_2, \dots, x_n)$, where $x_i = \lim_{n\to\infty} x_i^{(p)}$. Then $x \in R^n$.

We show that $x = \lim_{n\to\infty} x_i^{(p)}$. Let $p_o = max(p_1, p_2, \dots, p_{\mathrm{n}})$

$\forall\ p; p \geq p_0 \geq n_0$ , implies $||x^{(p)} - x^{(q)}|| = \sqrt{\sum_{i=1}^{n}\left|x_i^{(p)} - x_i^{(q)}\right|^2} < \epsilon$ By (1)

Hence $\{x^{(p)}\}$ converges to $x \in R^n$, as required. Thus, $R^n$ is complete and hence is a Banach Space.

**Example.** The space $l^\infty$ is a Banach space . This space consists of all bounded sequences $x = \{x_i\}$ of real or cpmplex numbers with addition and scalar multiplication defined by:

$$x + y = \{x_i + y_i\},$$

$$ax = \{ax_i\}.$$

The norm in $l^\infty$ is defined by:

$$||x|| = sup_{i=1}^{\infty}|x_i|$$

We show that $l^\infty$ is a Banach space.

Let $\{x^{(p)}\}$ be any Cauchy sequence in $l^\infty$, $x^{(p)} = x_i{}^{(p)}$. Then,given any $\epsilon > 0$, there is a natural number $n_0$ such that:

$\forall\ p, q; p, q \geq \mathrm{n}_0 \Longrightarrow ||x^{(p)} - x^{(q)}|| = sup_{i=1}^{\infty}\left|x_i^{(p)} - x_i^{(q)}\right| < \epsilon.$

So, for each $i = 1, 2, \dots,$

$\forall\ p, q; p, q \geq \mathrm{n}_0 \Longrightarrow \left|x_i^{(p)} - x_i^{(q)}\right| \leq ||x^{(p)} - x^{(q)}|| < \epsilon.$

Hence, $x_i^{(p)}$ is a Cauchy sequence of real (or complex ) numbers, since $R(orC)$ is complete , $x_i^{(p)}$ converges to $x_i$ for all $i = 1, 2, 3, \dots$

Take $x = x_i$. We show that $x \in l^\infty$ and$\lim_{p\to\infty} x^{(p)} = x$.

Since $x_i^{(p)} \to x_i$, there is a natural number $n_1$ such that

$\forall p, ; p, \geq \mathrm{n}_1 \Longrightarrow \left|x_i^{(p)} - x_i\right| < \frac{\epsilon}{2}, 1 = 1, 2, \dots$ (1)

That is,

$\forall p, ; p, \geq \mathrm{n}_1 \Longrightarrow ||x^{(p)} - x|| = sup_{i=1}^{\infty}\left|x_i^{(p)} - x_i\right| \leq \frac{\epsilon}{2} < \epsilon.$

Hence $x^{(p)} \to x$. Also, from (1)

$$|x_i| = |x_i - x_i^{(p)} + x_i^{(p)}|$$

$$\leq |x_i - x_i^{(p)}| + |x_i^{(p)}|$$

$$< \frac{\epsilon}{2} + k_p$$

Now $\frac{\epsilon}{2} + k_p$ is a finite number, independent of $i$. This proves completeness of $l^\infty$.

**Example. The space $c$ that is** the Space of all convergent real (or complex ) sequences is a Banach space . It is a subspace of $l^\infty$.

**Example. The space $c_o$** that is the space of all sequences which converges to 0 is a Banach space

**Example. The space $l_p, (p \geq 1)$ :** This is the space of all sequences $x = \{x_i\}$ such that $||x|| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$ is a Banach space.

**Example. The space $C[a, b]$.** This is the space of all continuous functions from $[a, b]$ to $R$ $(or\ C)$ is a Banach space. The norm in $C[a, b]$ is $||f|| = \sup_{x \in [a,b]} |f(x)|, f \in C[a, b]$.

## Summary

- A linear space $X$ over $K$ with norm $||.||$ defined on $X$ is called a normed linear space or simply a normed space over , written as $(X, ||.||)$ or $X$. The normed linear space is real or complex accordingly as the field $K$ is $R$ or $C$.
- An open sphere (or open ball) with center $x_0$ and radius $r > 0$ is the set $B(x_0; r) = \{x \in X: ||x - x_0|| < r\}$. By the surface (or boundary) of this ball, we mean the set $S(x_0; r) = \{x \in X: ||x - x_0|| = r\}$.
- The set $B[x_0; r] = \{x \in X: ||x - x_0|| \leq r\}$ denoted by $S[x_0, r\ ]$ or $S_r[x_0]$ is called the closed sphere or closed ball with radius $r$ and center $x_o$.
- A set $D$ in $X$ is said to be open if for every $x \in D$, there exists a ball with center $x$ which is contained in D.
- A set $D$ in $X$ is said to be closed if for any sequence $\{x_n\}$ in $D$ with $x_n \rightarrow x$ implies that $x \in D$.
- A set $D$ in $X$ is said to be bounded in $X$ if there exists a constant $M$ such that $||x|| < M, \forall x \in D$.
- A set $D$ in $X$ is said to be compact if whenever $\{x_n\} \in D$, there exist a convergent subsequence of $\{x_n\}$ whose limit is in $D$.
- A sequence $\{x_n\}$ is said to be bounded, if there exists a real constant $K > 0$ such that $||x_n|| < K, \forall\, n$.
- A sequence $\{x_n\} \subset X$ is said to be convergent if there exists an $x \in X$ such that

  $\lim_{n\rightarrow\infty} ||x_n - x|| = 0$

- A sequence $\{x_n\} \subset X$ is said to be Cauchy sequence if for given $\epsilon > 0, \exists$ a positive integer N such that

$$\lim_{n\to\infty} \|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$$

That is, $x_n$ is said to be Cauchy sequence in X iff

$$\lim_{n\to\infty} \|x_m - x_n\| \to 0 \; as \; m, n \; \to \infty$$

- The space X is said to be complete if every Cauchy sequence in $X$ converges to an element in $X$.

## Keywords

- Normed linear space
- Open sphere
- Closed sphere
- Bounded set
- Compact set
- Convergent sequence
- Cauchy Sequence
- Banach space

## Self Assessment

1. Which of the following is not a requirement for a normed linear space.

A. Associativity
B. Linearity
C. Triangle inequality
D. Homogeneity

2. If two vectors in a normed linear space have norms equal to zero , then

A. The vectors must be orthogonal
B. The vectors must be the zero vector
C. The vectors must be linearly dependent.
D. The vectors must be equal

3. In a normed linear space , the zero vector is unique because:

A. It satisfies the homogeneity property
B. It is defined as the multiplicative identity
C. It is defined as the additive identity
D. None of the above is correct

4. Which of the following is not a norm on a normed linear space.

A. Euclidean norm
B. Taxicab norm
C. Supremum norm
D. Inner product norm

5. Which of the following is not a property of norm in general.

A. $||x|| \geq 0$
B. $||x + y|| \leq ||x|| + ||y||$
C. $||kx|| = k||X||$
D. $||x|| = 0$, iff $x = 0$

6. Which of the following is true about a Banach space

A. It is a finite dimensional vector space
B. It is a normed linear space that is complete
C. It is a vector space with finite no of elements
D. None of the above

7. Which of the following statement is true about a complete normed linear space.

A. Every Cauchy sequence converges within the space.
B. Every bounded sequence converges within the space.
C. Every convergent sequence is bounded within the space.
D. None of the above

8. In a normed linear space , if the norm of a vector is zero, then the vector must be:

A. Zero vector
B. Unit Vector
C. A non zero vector
D. An infinite vector

9. A complete normed space is known as a:

A. Hilbert space
B. Banach space
C. Compact space
D. Euclidean Space

10. Which of the following is a Banach space.

A. Space of all polynomial functions on $[a, b]$ with supremum norm
B. Space of all continuous functions on $[a , b]$ with supremum norm.
C. Space of all polynomial functions on $[a, b]$ with the p- norm.
D. Space of all continuous functions on $[a, b]$ with the p- norm.

11. Consider the statements.

(i) Every finite dimensional normed linear space is a Banach space.

(ii) Every Banach space is finite dimensional linear space.

A. Only (i) is true
B. Only (ii) is true
C. Both (i) and (ii) are true
D. Neither (i) nor (ii) is true.

12. Which of the following is true in normed space.

A. Union of any family of open sets is open.
B. Intersection of any family of open sets is open.
C. Union of any family of closed sets is closed.
D. Intersection of any family of closed sets is open.

13. If $p \geq q \geq 1$, which of the following is true.

A. $l_p \subset l_q$
B. $l_p \supset l_q$
C. $l_p = l_q$
D. None of the above

14. Consider the statements:

(i) Every normed space is complete.

(ii) Every normed space can be identified as a dense subspace of a Complete normed space

A. Only (i) is true.
B. Only (ii) is true.
C. Both (i) and (ii) are true.
D. Neither (i) nor (ii) is true.

15. The linear span of empty set equals:

A. Zero subspace
B. Empty set

C. The whole space

D. None of the above

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | B | 3. | C | 4. | D | 5. | C |
| 6. | B | 7. | A | 8. | A | 9. | B | 10. | B |
| 11. | A | 12. | A | 13. | B | 14. | B | 15. | A |

## Review Questions

1. Define a normed linear space.
2. What is definition of norm in normed linear space.
3. State triangle inequality property for a normed linear space.
4. Explain the concept of convergence in a normed linear space.
5. Define a Cauchy sequence in a normed linear space.
6. What is the difference between a normed linear space and a metric space.
7. Define a Banach Spaces.
8. What are the key properties that a space must satisfy to be considered a Banach space.
9. What is the difference between a normed space and a Banach space.

## Further Readings

- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
- C. Goffman G Pedrick, A First Course In Functional Analysis.
- B.V. Limaya, Functional Analysis.

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# Unit 02: Normed Linear Spaces and Banach spaces II

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Finite dimensional Normed Space and Subspaces.
- Quotient space and its completeness
- Dual space and completeness
- Equivalent Norms.

## Introduction

In this chapter, We introduce the idea of finite dimensional normed spaces and subspaces . These spaces have some pleasant and useful properties . Such spaces are all Banach spaces. Further, we also discuss quotient space, dual space and their completeness. Finally, we shall see that any two norms on finite dimensional normed spaces are equivalent.

## 2.1 Finite Dimensional Normed Space and Subspaces

In this section, we first introduce the formal definition of Schauder basis of a normed space which serves as the building block of the subsequent section.

**Definition**. A collection $B = \{e_1, e_2, \dots, e_n, \dots\}$ of elements of a normed space $X$ is said to be a basis for $X$ if :

(i) $B$ is linearly independent set and

(ii) For each $x \in X$, there are uniquely determined scalers

$$\alpha_1, \alpha_2, \dots, \alpha_n \dots$$

such that

$$\lim_{n\to\infty} \|x - \sum_{i=1}^{n} \alpha_i e_i\| = 0.$$

If $B$ is a basis for $X$, then each $x \in X$ is uniquely expressed as:

$$x = \sum_{i=1}^{\infty} \alpha_i \, e_i$$

A normed space $X$ is said to be finite dimensional if it has finite basis, otherwise $X$ is said to be infinite dimensional.

**Example.** The space $R^n$ has
$e_1 = (1,0,0, \dots ,0), e_2 = (0,1,0, \dots ), \dots , e_n = (0,0, \dots ,1)$ as a basis.

**Definition.** A non empty subset $Y$ of a normed space $X$ is said to be a subspace of $X$ if

(i) $Y$ is a (linear) subspace of $X$ considered as a linear space, and
(ii) $Y$ is equipped with the norm $||.||_Y$ induced by the norm $||.||$ on $X$.
i,e $\|x\|_Y = \|x\|, \forall\, x \in Y$. We may denote the subspace $(Y, ||.||_Y)$ simply by $Y$.

**Theorem.** Let $Y$ be a subspace of a normed space $X$. Then $Y$ is complete $\Rightarrow Y$ *is closed*.

**Proof.** Suppose $Y$ is complete and let $x$ be a limit point of $Y$. Then every open sphere centered at contains points of $Y$ (other than x). In particular, the open sphere $S_{\frac{1}{n}}(x)$, where n is a positive integer contains a point $x_n$ of $Y$ other than $x$. Thus $\{x_n\}$ is a sequence in $Y$ such that

$$\|x_n - x\| < \frac{1}{n}, \forall \text{n}.$$

$\Rightarrow \lim_{n\to\infty} x_n = x$ in X.
$\Rightarrow \{x_n\}$ is a Cauchy sequence in $X$ and hence in $Y$. But $Y$ being complete , it follows that $x \in Y$. Thus $Y$ is closed.

**Theorem.** Let $Y$ be a subspace of Banach space $X$. Then $Y$ is closed $\Rightarrow$ $Y$ is complete.

Proof. Let $Y$ be closed and let $\{x_n\}$ be Cauchy sequence in $Y$. Then it is Cauchy in $X$. But $X$ being complete, $\Rightarrow \exists\, x \in X$ such that $x_n \to x$. Either $x \in Y$, then we are done, or each neighbourhood of $x$ contains points $x_n (\neq x)$. As such, $x$ is a limit point of $Y$. But $Y$ being closed , implies $x \in Y$. Thus $Y$ is complete.

**Corrolory.** Let $Y$ be a subspace of a Banach space $X$. Then $Y$ is complete iff $Y$ is closed.

## 2.2 Quotient Space of a Normed Space and its Completeness

In this Section, we shall consider one of the most useful methods of constructing new Banach spaces from the given Banach spaces.
Let $X$ be a normed space and $Y$ a subspace of . For any $x \in X$, the set

$$x + Y = \{x + y : y \in Y\}$$

is called a coset of $Y$ determined by $x$ or a translate of $S$ by $x$. The set

$$\{x + Y : x \in \text{X}\}$$

Of all cosets of $Y$ in $X$ is a linear space under addition and scalar multiplication defined by

$$x + Y + y + Y = x + y + Y, x, y \in X$$

And $\alpha(x + Y) = \alpha x + Y, x \in X, \alpha \in F,$

This set of cosets of $Y$ in $X$ is called a quotient space of $X$ by $Y$ and is denoted by $X/Y$.
For any subspace $Y$ of a linear space , the dimension of $X/Y$ is called the deficiency of $Y$.
We can make $X/Y$. a normed linear space as follows:
Let $||.||$ be the norm in $X$. For an $x + Y \in X/Y$, put

$$\|x + Y\|_1 = \inf_{y\in Y} ||x + y|| = d(x, Y)$$

Where $d$ is the metric induced by the norm $||.||$ on X.
**Theorem.** If $Y$ is a closed subspace of a normed space $(X, ||.||)$,then $X/Y$ is also a normed space under the norm defined by

$$\|x + Y\|_1 = \inf_{y\in Y} ||x + y|| = d(x, Y).$$

Proof. It is obvious from the definition that

$$\|x+Y\|_1 \geq 0$$

Also $\|x+Y\|_1 = 0$ if and only if $\inf_{y\in Y}||x+y||$, so that by the property of infimum , there is a sequence $\{y_n\}$ in $Y$ such that

$||x+y_n||_1 \to 0$ as $n \to \infty$

But then $x+y_n \to 0$ that is $y_n \to -x$ as $n \to \infty$. Since $Y$ is closed subspace , $x \in Y$

Hence

$x+Y=Y$, the zeroth element of $X/Y$.

Now, let $x+Y, y+Y \in X/Y, x, y \in$ X .Then

$$x+Y+y+Y = x+y+Y \in X/Y$$

By definition of $||.||_1$ in $X/Y$, there are sequences $\{x_n\}$ and $\{y_n\}$ in $Y$ such that

$\lim_{n\to\infty}||x+x_n|| = \|x+Y\|_1$, $\lim_{n\to\infty}||y+y_n|| = ||y+Y||_1$,

Hence , for any $x, y \in X$ and the definition of infimum,

$$||x+Y+y+Y||_1 = ||x+y+Y||_1 \leq ||x+y+x_n+y_n||$$

$$\leq ||x+x_n|| + ||y+y_n||$$

Taking limits as $n \to \infty$, we have :

$$||x+Y+y+Y||_1 = ||x+y+Y||_1$$
$$\leq \lim_{n\to\infty}||x+x_n|| + \lim_{n\to\infty}||y+y_n||$$
$$\leq ||x+Y||_1 + ||y+Y||_1$$

So that condition (ii) is satisfied.

Now to prove (iii), For any scalar $\alpha$ and $x+Y \in$ X/Y, consider an element

$$\alpha(x+Y) = \alpha\mathrm{x} + \mathrm{Y}$$

If $\alpha = 0$, then

$||\alpha(x+Y)||_1 = ||0.x+Y||_1 = ||Y||_1 = 0 = |\alpha|||x+Y||_1$.

Let $\alpha \neq 0$. Then

$$||\alpha x+Y||_1 = \inf_{y\in Y}||\alpha x+y||$$

$$= \inf_{y'\in Y}||\alpha x+\alpha y'||$$
$$= |\alpha|\inf_{y'\in Y}||x+y'||$$
$$== |\alpha|||x+Y||_1.$$

Hence $(X/Y, ||.||_1)$ is a normed space.

Now, we discuss the question of completeness of $X/Y$ if $X$ is complete. In a support of this we prove the following theorem.

**Theorem.** Let $Y$ be a closed subspace of a Banach space $X$. Then $X/Y$ with the norm defined by

$$||x+Y||_1 = \inf_{y\in Y}||x+y|| = d(x,Y)$$

is also a Banach space.

**Proof.** To prove that $\boldsymbol{X/Y}$ is a Banach space **,** we have to prove that every Cauchy sequence in $\boldsymbol{X/Y}$ Converges to a point $\boldsymbol{X/Y}$**.** Since a Cauchy sequence convergent if and only if it has a convergent subsequence , we shall show that every Cauchy sequence in $\boldsymbol{X}/Y$ contains a convergent subsequence .

Let $x_n+Y, x_n \in X$ be a Cauchy sequence in $X/Y$. Then , given any $\epsilon > 0$, there is a natural number $n_1$ such that:

$$\forall m,n; m,n \geq n_1 \Rightarrow ||x_m+Y-(x_n+Y)||_1 = ||x_m-x_n+Y||_1 < \epsilon$$

Take $\epsilon = \frac{1}{2}$ and $m = n_1, n = n_{1+}1$. Then

$$||x_{n_1}+Y-(x_{n_1}+Y)|| = ||x_{n_1}-x_{n_1+1}+Y||_1 < \frac{1}{2}$$

If we choose $\epsilon = \frac{1}{4}$, then there is a natural number $n_2$ such that

$$\left|\left|x_{n_2} + Y - (x_{n_2+1} + Y)\right|\right|_1 = \left|\left|x_{n_2} - x_{n_{2+1}} + Y\right|\right|_1 < \frac{1}{4}$$

Continuing in this way, we see that, in general there is a natural number $n_k$ such that

$$\left|\left|x_{n_k} + Y - (X_{n_k+1} + Y)\right|\right|_1 = \left|\left|x_{n_k} - x_{n_k+1} + Y\right|\right|_1 < \frac{1}{2^k}$$

In each $x_{n_k} + Y$ and $x_{n_k+1} + Y$, select vectors $y_k, y_{k+1}$ respectively such that

$$||y_k - y_{k+1}|| < \frac{1}{2^k}$$

Then, for any $k' > k$,

$$\begin{aligned} ||y_k - y_{k'}|| &= ||y_k - y_{k+1} + y_{k+1} - y_{k+2} + \cdots y_{k'-1} - y_{k'}|| \\ &\le ||y_k - y_{k+1}|| + ||y_{k+1} - y_{k+2}|| + \cdots ||y_{k'-1} - y_{k'}|| \\ &< \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k'}} \\ &< \frac{\frac{1}{2^k}}{1-\frac{1}{2}} = \frac{1}{2^{k-1}} \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus $\{y_k\}$ is a Cauchy sequence in $X$. Since $X$ is complete, $\{y_k\}$ converges to a point of $X$. Hence $\left|\left|x_{n_k} + Y - (y + Y)\right|\right|_1 \le \left|\left|y_k - y\right|\right| \to 0$ as $k \to \infty$

So that the subsequece

$$x_{n_k} + Y \to y + Y \in X/Y$$

But then $x_n + Y \to$ y + Y. Hence $X/Y$ is complete.

## 2.3 Dual Space and Completeness

Let $X$ be a normed linear space and let $K$ be a scaler field associated with $X$. This field is also a normed linear space with norm defined as

$$\|x\| = |x|; x \in K,$$

then

1. A linear operator $x': X \to K$ is called a functional.
2. A functional $x': X \to K$ is said to be continuous at a point $x_0$ of $X$, if for each $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $|x'(x) - x'(x_0)| < \epsilon$. We say that $x'$ is continuous on $X$ if and only if it is continuous on each point of $X$.
3. A functional $x': X \to K$ is said to be linear if

$$x'(\alpha x_1 + \beta x_2) = \alpha x'(x_1) + \beta x'(x_2), \forall x_1, x_2 \in X; \alpha, \beta \in K.$$

4. A linear functional $x'$ is said to be bonded if there exists $M > 0$ such that

$$|x'(x)| \le M\|x\|, \forall x \in X.$$

5. The set of all linear functionals defined on $X$ is itsel a linear space , if addition and scaler multiplication are defined by:

$$(x_1' + x_2')(x) = x_1'(x) + x_2'(x)$$

$$(\alpha x')(x) = \alpha x'(x)$$

And is denoted by $x^f$, called the algebraic dual ( conjugate) space of $X$.

6. A norm of a linear functional $x' \in x^f$ is defined as:

$$\begin{aligned} \|x'\| &= \sup_{\|x\|=1} |x'x| \\ &= \sup_{\|x\|\le 1} |x'x| \\ &= sup_{x\neq 0} \frac{|x'x|}{\|x\|} \end{aligned}$$

Note that $|x'x| \le \|x'\| \, \|x\|; \ \forall x \in X$.

7. The set of all bounded (continuous) linear functionals defined on $X$ is a linear subspace of $x^f$ and is denoted by $X'$.

A norm on $X'$ is given by (6). The linear space $X'$ normed in this way is called normed conjugate of $X$. Sometimes it is denoted by $X^*$.

**Remark:-** Since $X^f$ is a linear space, we may also consider its algebraic dual (or conjugate) space which we denote by $\left(X^f\right)^f$ or $X^{ff}$, that is the class of all linear functionals on $X^f$.We shall denote elements of $X^{ff}$ by $x''$ (i, e $x'': x^f \to K$, the scaler field associated with $x^f$) and we shall use the notation $x''(x')$ for the value of $x''$ at $x'$.

**Theorem.** Let $X$ be a norm linear space, then the norm conjugate space $X'$ of $X$ is complete.

**Proof.** Let $\{x_n'\}$ be a Cauchy sequence in $X'$, then by definition of Cauchy sequence , for every $\epsilon > 0$, there exists positive integer $N$ such that

$$\|x_m' - x_n'\| < \epsilon \text{whenever } m, n \ \geq N.$$

Consequently for each $x \in X$,

$|x_m'(x) - x_n'(x)| = |(x_m' - x_n')x| \leq \|x_m' - x_n'\|\|x\| < \epsilon\|x\|$ , $\forall m, n \geq N$**…………(1)**

Which shows that $\{x_n'(x)\}$ ia a Cauchy sequence in the space $R \text{ or } C$ for each $x \in X$. Since the scaler field $R \text{ or } C$ is complete, so $\{x_n'(x)\}$ converges to a limit depending on $X$ which we denote by $x'(x)$.

That is $\lim_{n\to\infty} x_n'(x) = x'(x)$.

Thus defining a functional $x'$ on $X$. We show that $x' \in X'$ and for this it is enough to show that $x'$ is linear and bounded.

First we show that $x'$ is linear, since for scalers $\lambda_1, \lambda_2$ and vectors $x_1, x_2$ in $X$, we have

$$x'(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n\to\infty} x_n'(\lambda_1 x_1 + \lambda_2 x_2)$$

$$= \lim_{n\to\infty} x_n'(\lambda_1 x_1) + \lim_{n\to\infty} x_n'(\lambda_2 x_2)$$

$$= \lambda_1 \lim_{n\to\infty} x_n'(x_1) + \lambda_2 \lim_{n\to\infty} x_n'(x_2)$$

$$= \lambda_1 x'(x_1) + \lambda_2 x'(x_2).$$

Which shows that $x'$ is linear.

Now we show that $x'$ is bounded and hence continuous. Since $\{x_n'\}$ is a Cauchy sequence, so it is bounded . Therefore by definition, there exists a constant $K > 0$ such that $\|x_n'\| \leq K$; $\forall n$.

For $x \in X$, we have

$$|x_n'(x)| \leq \|x_n'\|\|x\|$$

$$\leq K\|x\|;\ \forall\ n.$$

Taking limit as $n \to \infty$, we get

$$|x'(x)|\ K\|x\|; \forall\ n.$$

Which shows that $x'$ is bounded and hence continuous . Hence $x' \in X'$.

To complete the proof , it remains to show that $x_n' \to x'$.

By (1), we have

$|x_m'(x) - x_n'(x)| = \epsilon\|x\|$ , $\forall m, n \geq N$.

Since the results holds for every $m \ \geq N$.

$$x_m'(x) \to x'(x)$$

We may let $m \to \infty$. Thus letting $\underset{n\to\infty}{lim}$, we get

$$|x'(x) - x_n'(x)| \leq \epsilon\|x\|;\ \forall n \geq N$$

Implies $|(x' - x_n')(x)| \leq \epsilon\|x\|$; $\forall n \geq N$

By taking Sup over all $x$ of norm, we have

$\|x' - x_n'\| \leq \epsilon; \forall\, n \geq N.$

Which shows that $\{x_n'\}$ converges to $x'$. Consequentely $X'$ is complete.

## 2.4 Equivalent Norms

A norm $||.||_1$ on a vector space $X$ is said to be equivalent to a norm $||.||_2$ on $X$ if there are positive numbers $\alpha$ and $\beta$ such that $\forall\, x \in X$ we have

$$\alpha||x||_2 \leq ||x||_1 \leq \beta||x||_2.$$

This concept is motivated by the fact that Equivalent norms on $X$ define the same topology for $X$.

**Example.** Let $X = R^2$ with norm $\|\boldsymbol{x}\|_1 = |x_1| + |x_2|$ ; $\boldsymbol{x} = (x_1, x_2) \in R^2$ and $\|\boldsymbol{x}\|_2 = (\sum_{i=1}^{2}|x_i|^2)^{\frac{1}{2}}$, then show that $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_2$ are equivalent norms.

Solution:- we have

$$\|\boldsymbol{x}\|_1 = |x_1| + |x_2| = \sum_{i=1}^{2}|x_i| = \sum_{i=1}^{2}(1)|x_i|$$

$$\leq (\sum_{i=1}^{2}(1)^2)^{\frac{1}{2}} (\sum_{i=1}^{2}|x_i|^2)^{\frac{1}{2}}$$

$$= \sqrt{2}\|\boldsymbol{x}\|_2$$

This implies $\|\boldsymbol{x}\|_1 \leq \sqrt{2}\ \|\boldsymbol{x}\|_2$

Or $\frac{1}{\sqrt{2}}\ \|\boldsymbol{x}\|_1 \leq \|\boldsymbol{x}\|_2$……………(1)

Now,

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{2}|x_i|^2\right)^{\frac{1}{2}} = \sqrt{|x_1|^2 + |x_2|^2}$$

$$\leq |x_1| + |x|_{2=} \|\boldsymbol{x}\|_1$$

Or

$$\|\boldsymbol{x}\|_2 \leq (1)\ \|\boldsymbol{x}\|_1……………(2)$$

From (1) and (2)

$$\frac{1}{\sqrt{2}}\ \|\boldsymbol{x}\|_1 \leq \|\boldsymbol{x}\|_2 \leq (1)\ \|\boldsymbol{x}\|_1$$

This shows that $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_2$ are equivalent norms.

**Theorem.** The relation of ′ being equivalent to ′ among the norms that can be defined on a linear space $X$ is an equivalence relation.

**Proof.** In order to show that relation of ′ being equivalent to ′ among the norms is an equivalence relation, we have to show that it is reflexive, symmetric and transitive,

**Reflexive.** We have for any norm $\|.\|$ on $X$ and for any $x \in X$

$$\alpha\|x\| \leq \|x\| \leq \beta\|x\|$$

Is satisfied for $\alpha = \beta = 1$. Hence $\|.\| \sim \|.\|$.

**Symmetric.** If $\|.\|_1 \sim \|.\|_2$ then there are positive numbers $\alpha$ and $\beta$ such that $\forall\, x \in$ X, we have

$$\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2$$

$$\Rightarrow \frac{1}{\beta}\|x\|_1 \le \|x\|_2 \le \frac{1}{\alpha}\|x\|_1$$

Hence $\|.\|_2 \sim \|.\|_1$.

**Transitive.** If $\|.\|_1 \sim \|.\|_2$ and $\|.\|_2 \sim \|.\|_3$ then there are positive numbers $\alpha, \beta, \alpha_1$ and $\beta_1$ such that $\forall\, x \in X$, we have

$\alpha\|x\|_2 \le \|x\|_1 \le \beta\|x\|_2$ and $\alpha_1\|x\|_3 \le \|x\|_2 \le \beta_1\|x\|_3$

$\Rightarrow \alpha_1\|x\|_3 \le \|x\|_2 \le \frac{1}{\alpha}\|x\|_1 \le \frac{\beta}{\alpha}\|x\|_2 \le \frac{\beta}{\alpha}.\beta_1\|x\|_3$

$\Rightarrow \alpha_1\|x\|_3 \le \frac{1}{\alpha}\|x\|_1 \le \frac{\beta}{\alpha}.\beta_1\|x\|_3$

$\Rightarrow \alpha\alpha_1\|x\|_3 \le \|x\|_1 \le \beta\beta_1\|x\|_3$

Since $\alpha, \beta, \alpha_1, \beta_1 > 0$.

Hence $\|.\|_1 \sim \|.\|_3$ .

Consequently the relation of ' being equivalent to ' among the norms that can be defined on a linear space $X$ is an equivalence relation.

**Theorem.** Any two equivalent norms on a linear space $X$ define the same topology on $X$.

**Proof.** Let $\|.\|_1 \sim \|.\|_2$ then there are positive numbers $\alpha$ and $\beta$ such that $\forall\, x \in X$, we have

$$\alpha\|x\|_2 \le \|x\|_1 \le \beta\|x\|_2$$

We show that every basic open ball in $(X, \|.\|_1)$ is open in $(X, \|.\|_2)$ and conversely.

For an $x \in X$, let $B(x; r)$ be an open ball in $(X, \|.\|_1)$, then we show that it is open ball in $(X, \|.\|_2)$.

For this let $y \in B(x; r)$ then $\|x - y\|_1 = r_1 < r$

Consider $B_1(y; r')$ in $(X, \|.\|_2)$ where $r' = \frac{r - r_1}{\beta}$

Then for any $z \in B_1(y; r')$ we have $\|z - y\|_2 < r'$ then

$\|z - x\|_1 = \|z - y + y - x\|_1 \le \|z - y\|_1 + \|y - x\|_1$

$\|z - x\|_1 \le \beta\|z - y\|_2 + r_1$ Since $\|.\|_1 \sim \|.\|_2$ and $\|x - y\|_1 = r_1 < r$

$$\|z - x\|_1 < \beta r' + r_1 = \beta\left(\frac{r - r_1}{\beta}\right) + r_1 = r \Rightarrow \|z - x\|_1 < r$$

Hence $z \in B(x; r)$ implies $\in B_1(y; r') \subseteq B(x; r)$ . Hence $B(x; r)$ is open ball in $(X, \|.\|_2)$. Similarly we can show that every basic open ball in $(X, \|.\|_2)$ is open in $(X, \|.\|_1)$. Hence any two equivalent norms on a linear space $X$ define the same topology on $X$.

The next theorem shows that equivalent norms preserve Cauchy property of sequence.

**Theorem.** Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms on a linear space $X$, then every Cauchy sequence in $(X, \|.\|_1)$ is also Cauchy sequence in in $(X, \|.\|_2)$ and conversely.

**Proof.** Let $\{x_n\}$ be a Cauchy sequence in in $(X, \|.\|_1)$, then for given $\epsilon > 0$ there exists $n_0 \in N$

Such that

$$\|x_m - x_n\|_1 < \epsilon; \forall \mathrm{m}, \mathrm{n} > \mathrm{n}_0$$

$$\|x_m - x_n\|_2 \le \frac{1}{\alpha}\|x_m - x_n\|_1 < \frac{\epsilon}{\alpha}; \forall \mathrm{m}, \mathrm{n} > \mathrm{n}_0$$

$$\|x_m - x_n\|_2 < \epsilon' \ \forall \mathrm{m}, \mathrm{n} > \mathrm{n}_0$$

Hence $\{x_n\}$ is a Cauchy sequence in in $(X, \|.\|_2)$. Similarly, we can prove converse.

## Summary


- A normed space $X$ is said to be finite dimensional if it has finite basis, otherwise $X$ is said to be infinite dimensional.
- Let $X$ be a normed space and $Y$ a subspace of . For any $x \in X$, the set

$$x + Y = \{x + y: y \in Y\}$$

  is called a coset of $Y$ determined by $x$ or a translate of $S$ by $x$. The set

$$\{x + Y : x \in X\}$$

  Of all cosets of $Y$ in $X$ is a linear space under addition and scalar multiplication defined by

$$x + Y + y + Y = x + y + Y, x, y \in X$$

  And $$\alpha(x + Y) = \alpha x + Y, x \in X, \alpha \in F,$$

  This set of cosets of $Y$ in $X$ is called a quotient space of $X$ by $Y$ and is denoted by $X/Y$.
- For any subspace $Y$ of a linear space , the dimension of $X/Y$ is called the deficiency of $Y$.


## Keywords

- Subspace
- Basis
- Dimmension
- Finite dimension normed space
- Quotient space
- Dual space
- Completeness
- Norm

## Self Assessment

1: What is normed space?

A. A vector space equipped with a norm.
B. A vector space equipped with an inner product.
C. A vector space equipped with a metric.
D. None of the above.

2: Which of the following statements is true about a normed space?

A. Every normed space is finite-dimensional.
B. Every normed space is infinite-dimensional.
C. A normed space can be either finite-dimensional or infinite-dimensional.
D. A normed space cannot have a dimension.

3: Which of the following is true about subspaces of a normed space?

A. Every subspace is finite-dimensional.
B. Every subspace is infinite-dimensional.
C. A subspace can be either finite-dimensional or infinite-dimensional.
D. A subspace cannot have a dimension.

4: Which of the following statements is true about finite-dimensional normed spaces?

A. Every finite-dimensional normed space is complete.
B. Every finite-dimensional normed space is incomplete.
C. A finite-dimensional normed space can be either complete or incomplete.
D. None of the above.

5: What is the definition of a quotient space?

A. A space obtained by dividing a normed space by a subspace.
B. A space obtained by dividing a normed space by a linear transformation.
C. A space obtained by dividing a normed space by a scalar.
D. A space obtained by dividing a normed space by a scalar multiple.

6: Which of the following conditions ensures the completeness of the quotient space?

A. The subspace is open.
B. The subspace is dense.
C. The subspace is closed.
D. The subspace is connected.

7: Two norms $\|.\|_1$ and $\|.\|_2$ on a normed linear space $X$ are equivalent, then there exists positive constants α and β such that:

A. $\alpha\|.\|_1 \leq \|.\|_2 \leq \beta\|.\|_1$ for all $x$ in $X$.
B. $\|.\|_1 = \alpha\|.\|_2$ for all $x$ in $X$.
C. $\|.\|_1 = \beta\|.\|_2$ for all $x$ in $X$.
D. None of the above.

8: Two norms $\|.\|_1$ and $\|.\|_2$ on a normed linear space $X$ are said to be equivalent if :

A. They induce the same topology on X.
B. They have the same dimension.
C. They have the same norm constant.
D. None of the above.

9: Which of the following statements about the quotient space is true?

A. The quotient space is always finite-dimensional.
B. The quotient space is always a normed linear space.
C. The quotient space is isomorphic to the original normed linear space .
D. The quotient space is always a complete space.

10: Which of the following is a necessary condition for the quotient space to be finite-dimensional?

A. The original normed linear space must be finite-dimensional.
B. The subspace must be finite-dimensional .
C. The original normed linear space and the subspace must have the same dimension.
D. (D)None of the above.

11: Which of the following statements is true about a subspace of a normed linear space?

A. It must contain all the vectors of the normed linear space.
B. It must contain the zero vector.
C. It must be a finite-dimensional space.
D. It must be a closed set.

12: Let V be a normed linear space and W be a subspace of V. Which of the following statements is true about the dimension of W?

A. The dimension of W is always greater than or equal to the dimension of V.
B. The dimension of W is always less than or equal to the dimension of V.
C. The dimension of W is always equal to the dimension of V.
D. None of the above.

13: What is the dual space of a normed linear space?

A. The space of all linear transformations from the given space to its scalar field .
B. The space of all linear functionals from the given space to its scalar field.
C. The space of all continuous linear transformations from the given space to its scalar field
D. The space of all continuous linear functionals from the given space to its scalar field.

14: Which of the following statements is true about the dual space of a normed linear space?

A. The dual space is always finite-dimensional.
B. The dual space is always infinite-dimensional.
C. The dual space is always a Banach space.
D. The dual space can be finite-dimensional or infinite-dimensional.

15: Which of the following statements is true regarding the completeness of the dual space of a normed linear space?

A. The dual space is always complete.
B. The dual space is never complete.
C. The dual space is complete if and only if the original normed linear space is finite-dimensional.
D. The completeness of the dual space depends on the specific norm chosen for the original space.

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | C | 3. | C | 4. | A | 5. | A |
| 6. | C | 7. | A | 8. | A | 9. | B | 10. | C |
| 11. | B | 12. | B | 13. | D | 14. | D | 15. | C |

## Review Questions

1. Define a normed space and give an example of a finite dimensional normed space.
2. Define subspace of a normed space.
3. Is the Zero subspace always a proper subspace.
4. Define Quotient space of a normed linear space.
5. State the definition of equivalent norms.
6. Give an example of two norms that are equivalent.
7. Define dual space of a normed linear space.

## Further Readings

- Introductory Functional Analysis With Applications By Erwin Kreyszig.

- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
- C. Goffman G Pedrick, A First Course In Functional Analysis.
- B.V. Limaya, Functional Analysis

Dr. Arshad Ahmad Khan, Lovely Professional University

# Unit 03: Bounded Linear Operator and its Properties

CONTENTS



## Objectives

After studying this unit, you will be able to understand:

- Bounded and continuous linear operator
- The null space of a linear operator
- Norm of a bounded linear operator
- The space of bounded linear operators
- Linear functional
- Compactness and finite dimensional space

## Introduction

In this chapter, We introduce the idea of bounded and continuous linear operators. The study of bounded and continuous linear operators serves as a powerful tool to analyze and understand the behavior of functions between normed vector spaces. Further, we also discuss kernel or null space of a linear operator , norm of a linear operator and the space of bounded linear operators. Finally we discuss linear functional and the Compactness and finite dimensional space.

## 3.1 Bounded and Continuous Linear Operators

In calculus we consider the real line $R$ and real-valued functions on $R$ (or on a subset of $R$). Obviously, any such function is a mapping of its domain into $R$ . In functional analysis we consider more general spaces, such as metric spaces and normed spaces, and mappings of these spaces.In functional analysis, an operator is defined as a mapping between two vector spaces. Specifically, let $X$ and $Y$ be two vector spaces (typically normed vector spaces or Banach spaces) over the same field (usually $R \ or \ C$). An operator $T$ is a function that maps elements from $X$ to elements in $Y$.

Different types of operators are commonly encountered in functional analysis, such as linear operators, bounded operators, compact operators, self-adjoint operators, unitary operators, and many others. Each type of operator has its own set of properties and characteristics, which are studied to understand the behavior and structure of the operator. Operators play the fundamental role in functional analysis, as they provide a way to study the relationship between vector spaces, mathematical objects in a functional-analytic setting.

Before defining a bounded linear operator , we recall some definitions and results.

**Definition.** Let $X$ and $Y$ be normed spaces over a field $F$. We say that $T: X \to Y$ is a linear operator if $T$ is linear (that is $T(x+y) = T(x) + T(y)\ \forall\ x, y \in X$ *and* $T(\lambda x) = \lambda T(x))\ \forall x \in X$ and $\lambda \in F$.

**Example.** Let $X$ be any normed space, then the identity function $I: X \to X$ defined by :

$$I(x) = x, x \in X$$

Is a linear operator .

Here for $\lambda_1, \lambda_2 \in F$ and $x_1, x_2 \in X$,

$$\begin{aligned} I(\lambda_1 x_1 + \lambda_2 x_2) &= \lambda_1 x_1 + \lambda_2 x_2 \\ &= \lambda_1 I(x_1) + \lambda_2 I(x_2). \end{aligned}$$

**Example.** For any linear spaces $X, Y$, the function $0: X \to Y$ defined by:

$$0(x) = 0, x \in X$$

Is a linear operator.

Note that zero operator is also called null operator or trivial operator.

**Example.** In the space $C[a,b]$, define a function $I: C[a,b] \to C[a,b]$ by:

$$I(f) = \int_a^x f(t)dt\, , f \in C[a,b].$$

Then $I$ is a linear operator.

**Example.** Let $K$ be the space of all analytic functions over $C$ and $D: K \to K$ be defined by:

$D(f) = f'$, $f \in K$ and $f'$ is the derivative of $f$. Then $D$ is a linear operator.

**Example.** Consider the linear space $P$ of all polynomials $p(x)$ with real coefficients defined on [0,1], then the mapping $D$ defined by $D(p) = \frac{dp}{dx}$, is a linear operator from $P$ into itself.

### The kernel or null space of a linear operator

Let $T: X \to Y$ be a linear operator. Then the set of those elements of $X$ which are mapped onto the zero element of $Y$ is a subspace of $X$ called the kernel or null space of $T$ and is denoted by $Ker\ T$. To see the $Ker\ T$ is a subspace of $X$ , let $x_1, x_2 \in Ker\ T$.

Then $Tx_1 = 0, Tx_2 = 0$ and for any $\lambda_1, \lambda_2 \in F$,

$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 Tx_1 + \lambda_2 Tx_2$, by linearity of $T$,

So that $\lambda_1 x_1 + \lambda_2 x_2 \in Ker\ T$.

Now we define the Continuous linear operator. Of special interest among the class of all linear operators are those which are continuous . Since every normed space is also a metric space , continuity of an operator is always with respect to the metric defined by the norm. Let $X$ and $Y$ be normed spaces . A linear operator $T: X \to Y$ is said to be continuous at a point $x_0 \in X$ if given $\epsilon > 0$, there is a real no $\delta = \delta(\epsilon) > 0$ such that

$\forall x \in X, \|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \epsilon$.

$T$ is said to be continuous on $X$ if it is continuous at every point of $X$. $T: X \rightarrow Y$ is continuous if and only if, for all sequences $\{x_n\}$ which converges to $x$, $Tx_n$ converges to $Tx$.

Another concept associated with linear operator defined on a normed space is that of boundness which is equivalent to continuity of the operator.

A linear operator $T: X \rightarrow Y$ is said to be bounded if there is a constant $k > 0$ such that $\|Tx\| < k\|x\| \; \forall x \in X$.

The concepts of continuity and boundness of a linear operator are equivalent is shown in the following theorem.

**Theorem.** Let $T: X \rightarrow Y$ be a linear operator . Then

(a) $T$ is continuous on $X$ if and only if $T$ is bounded .

(b) $T$ is continuous if and only if it is continuous at $0 \in X$.

(c) If $T$ is continuous on $X$ then $Ker\ T$ is closed in $X$.

**Proof.** Suppose that $T$ is continuous on $X$. Then it is continuous at each $x_0 \in X$. So given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\forall x \in X, \|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \epsilon.$$

Let $y \in X$ and put

$x = x_0 + \frac{\delta}{2\|y\|} y$, i,e $x - x_0 = \frac{\delta}{2\|y\|} y$

Then, using the linearity of $T$ and $\|x - x_0\| = \left\| \frac{\delta}{2\|y\|} y \right\| = \frac{\delta}{2} < \delta$,

we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T\left(\frac{\delta}{2\|y\|} y\right) \right\|$$

$$= \frac{\delta}{2\|y\|} \|Ty\| < \epsilon$$

So that, $\|Ty\| < \frac{2\epsilon}{\delta} \|y\|$

$< k\|y\|, k = \frac{2\epsilon}{\delta}$.

Hence $T$ is bounded.

Alternatively, suppose that $T$ is continuous but not bounded. Then for each natural number n , there is an $x_n$ in $X$, such that

$$\|Tx_n\| > n\|x_n\|.$$

Let $y_n = \frac{1}{n\|x_n\|} x_n$

Then $\|Ty_n\| = \frac{1}{n\|x_n\|} \|Tx_n\|$

$> \frac{n\|x_n\|}{n\|x_n\|} = 1$

Now $\|y_n\| = \frac{1}{n\|x_n\|} \|x_n\| = \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$.

By continuity of $T$,

$$\|y_n\| \rightarrow 0 \Rightarrow \|Ty_n\| \rightarrow 0$$

But $\|Ty_n\| \geq 1 \; \forall n$, a contradiction . Hence $T$ is bounded.

Conversely suppose that $T$ is bounded . Then there is a real number $k > 0$ such that

$$\|Tx\| \leq k\|x\| \; \forall x \in X.$$

So, for any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{k}$. Then

$$\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| = \|T(x - x_0)\|$$

$$\leq k\ \|x - x_0\|$$
$$< \epsilon.$$

Hence $T$ is continuous.

**(b)** Suppose that $T$ is continuous on $X$, then it is continuous on $0 \in X$.

Coversely suppose that $T$ is continuous at $0 \in X$. Then, with $x_0 = 0$, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$\forall x \in X, \|x - x_0\| = \|x\| < \delta \Rightarrow \|Tx - Tx_0\| = \|Tx\| < \epsilon.$

Hence, for any $x_0 \in X$,

$$\|x - x_0\| \leq \delta \Rightarrow \|Tx - Tx_0\| = \|T(x - x_0)\| < \epsilon$$

So $T$ is continuous at $x_0$ and therefore also on $X$.

**(c)** Suppose that $T$ is continuous and let $\boldsymbol{x}$ be a limit point of $Ker\ T$. Then there is a sequence $\{x_n\}$ in $Ker\ T$ such that

$$\lim_{n\to\infty} x_n = x$$

By the continuity of $T$,

$$0 = \lim_{n\to\infty} Tx_n = Tx.$$

Hence $x \in ker\ T$ and $ker\ T$ is closed.

**Norm of a Bounded Linear Operator**

Let $T: X \to Y$ be a bounded linear operator . Then there is a real number $k > 0$ such

$$\|Tx\| \leq k\ \|x\|\ \forall x \in X$$

Suppose that $x \neq 0$.Then $\frac{\|Tx\|}{\|x\|} \leq k\ \forall x \in X, x \neq 0.$

So $k$ is an upper bound for $\frac{\|Tx\|}{\|x\|}$. The leaset upper bound $\sup\limits_{x\neq 0\in x} \frac{\|Tx\|}{\|x\|}$ is called the norm of $T$ and is denoted by $\|T\|$. Thus

$$\|T\| = \sup_{x\neq 0\in x} \frac{\|Tx\|}{\|x\|}.$$

**Note:-** If $X = \{0\}$, then $\|T\| = 0.$

It is clear from definition of $\|T\|$, $\|T\| \geq 0$ and $\|T\| = 0$ if and only if $\|Tx\| = 0\ \forall x \in X$, that is $T = 0.$

**Note:-** We have another relation for a bounded linear operator namely

$$\|Tx\| \leq \|T\|\|x\|\ \forall x \in X.$$

As we have $\|T\| = \sup\limits_{x\neq 0\in x} \frac{\|Tx\|}{\|x\|} \leq k$

Also by definition of supremum, $\frac{\|Tx\|}{\|x\|} \leq \|T\|\ \forall x \neq 0 \in X$

This gives $\|Tx\| \leq \|T\|\|x\| \forall x \neq 0 \in X$

But for $x = 0, T\,0 = 0$ so that $\|T0\| = \|0\| \leq \|T\|\|0\|.$

Thus $\|Tx\| \leq \|T\|\|x\|\ \forall x \in X.$

**Note:-** We can also write $\|T\| = \sup_{\|x\|=1}\|Tx\|$=$\sup_{\|x\|\leq 1}\|Tx\|$

As $\|T\| = \sup\limits_{x\neq 0\in X} \frac{\|Tx\|}{\|x\|} = \sup\limits_{y\neq 0\in X} \left\|T\left(\frac{y}{\|y\|}\right)\right\| = \sup_{\|x\|=1}\|Tx\|$ , when $x = \frac{y}{\|y\|}$

Because of various equivalent forms of norms of a linear operator yields the following definition of boundedness.

A linear operator $T: X \to Y$ is said to be bounded if and only if $\|T\|$ is finite.

**Example.** The identity operator $I: X \to X$ defined by :

$$I(x) = x \;\forall\; x \in X$$

is bounded as $\|I\| = \sup_{x \neq 0 \in X} \frac{\|I(x)\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1.$

**Example.** The null operator (zero operator ) $0: X \to Y$ defined by:

$$0(x) = 0 \;\forall\; x \in X$$

is bounded as as $\|0\| = \sup_{x \neq 0 \in X} \frac{\|0(x)\|}{\|x\|} = \frac{\|0\|}{\|x\|} = 0.$

**Example.** The operator $\boldsymbol{A}: \boldsymbol{R^m} \to \boldsymbol{R^n}$ defined by

$$x = Ay = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij}\, x_i \right) e^{j'}$$

Where $\boldsymbol{x} = (x_1, x_2, \dots, x_m)$, $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$

And $(e_1, e_2, \dots e_m)$ and $(e_1', e_2', \dots e_n')$ are basis of $\boldsymbol{R^m}$ and $\boldsymbol{R^n}$ respectively is bounded.

As $y_j = \sum_{i=1}^{m} a_{ij}\, x_i$, so that

$$\|A\boldsymbol{x}\|^2 = \|\boldsymbol{y}\|^2 = \sum_{j=1}^{n} |y_j|^2 = \sum_{j=1}^{n} \left| \sum_{i=1}^{m} a_{ij}\, x_i \right|^2$$

$$\leq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} |a_i j|^2 . \sum_{i=1}^{m} |x_1|^2 \right)$$

( By Minkowski's Inequality)

$$\leq \left( \sum_{j=1}^{n} \sum_{i=1}^{m} |a_i j|^2 \right) . \sum_{i=1}^{m} |x_i^2|$$

$$\leq k^2 \|x\|^2$$

where $k^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} |a_i j|^2$ and $\|x\|^2 = \sum_{i=1}^{m} |x_i^2|$

Hence $\|A\boldsymbol{x}\| \leq k\|\boldsymbol{x}\|$.

Therefore $A$ is a bounded and hence a continuous linear operator .

**Note:-** $\|A\| = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} |a_i j|^2}$ is called the norm of matrix operator $A$.

**Note:-** if **(** $X, \|.\|$**)** be a normed space and $\{x_1, x_2, x_3, \dots, x_n\}$ be a linearly independent set of vectors in $X$. Then, there is a real number $c > 0$ such that for all scalers $a_1, a_2, \dots, a_n$

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \geq c \sum_{i=1}^{n} |a_i| \;\dots\dots\dots\dots\mathbf{(1)}$$

**Theorem.** Show that every linear operator on a finite dimensional normed space is bounded .

**Proof.** Let $X$ be a finite dimensional normed space and let $B = (e_1, e_2, \dots e_n)$ be a basis of $X$. Let $T: X \to Y$ be a linear operator . For any $x \in X$,

$$x = \sum_{i=1}^{n} x_i e_i$$

So that, by linearity of $T$

$$Tx = \sum_{i=1}^{n} x_i \, Te_i$$

Hence

$$\|Tx\| = \left\| \sum_{i=1}^{n} x_i \, Te_i \right\|$$

$$\leq \sum_{i=1}^{n} |x_i| \|Te_i\|$$

$$\leq b \sum_{i=1}^{n} |x_i| \, , b = sup_{i=1}^{n} \|Te_i\| \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(2)}$$

Also by **(1)** , there is a positive real number $c$ such that

$$\|x\| = \|\sum_{i=1}^{n} x_i \, e_i\| \geq c \sum_{i=1}^{n} |x_i| \ldots\ldots\ldots\ldots \mathbf{(3)}$$

From (2) and (3),we have

$$\|x\| \geq c \sum_{i=1}^{n} |x_i| \geq c . \frac{1}{b} \|Tx\|$$

I,e

$$\|Tx\| \leq \frac{b}{c} \|x\|$$

Or,

$$\|Tx\| \leq k \, \|x\|, \text{ where } k = \frac{b}{c} > 0.$$

Hence $T$ is bounded linear operator.

**Theorem.** if $T_1: X \to Y$ and $T_2: Y \to Z$ be bounded linear operators .Then $T_2 T_1$ is bounded and $\|T_2 T_1\| \leq \|T_2\| \|T_1\| = \|T_1\| \|T_2\|$.

In particular , if $T: X \to X$ is a linear operator , then

$$\|T^n\| \leq \|T\|^n.$$

**Proof.** Since $T_1, T_2$ are bounded , then $\|T_1\|, \|T_2\|$ exists and are finite. Moreover, for any $x \in X$

$$\|(T_2 T_1)(x)\| = \|T_2(T_1 x)\|$$

$$\leq \|T_2\| \|T_1 x\|$$

$$\leq \|T_2\| \|T_1\| \|x\|$$

Hence $\|T_2 T_1\| \leq \|T_2\| \|T_1\| = \|T_1\| \|T_2\|$.

In particular, if $T: X \to X$ is a linear operator , then by induction on $n$, we have

$$\|T^n\| \leq \|T\|^n.$$

**The space of bounded Linear operators.**

Let $X, Y$ be normed linear spaces and $B(X, Y)$ denote the space of all bounded linear operators fron $X$ to $Y$.

**Theorem.** The space $B(X, Y)$ of all bounded (hence continuous) linear operators from $X$ to $Y$ is a normed space under the norm defined by

$$\|T\| = \sup_{\|x\|=1} \|T\,x\|, x \in X.$$

**Proof.** First we show that $B(X, Y)$ is a linear space .

For this , let $S, T \in B(X, Y)$. Define $S + T : X \to Y$ by

$$(S + T)(x) = Sx + Tx \;\forall x \in X.$$

For any α, β scalers and $x, y \in X$,then

$$\begin{aligned}(S + T)(\alpha x + \beta y) &= S(\alpha x + \beta y) + T(\alpha x + \beta y)\\ &= \alpha Sx + \beta Sy + \alpha Tx + \beta Ty\\ &= \alpha(Sx + Tx) + \beta(Sy + Ty)\\ &= \alpha(S + T)x + \beta(S + T)y\end{aligned}$$

Therefore $S + T$ is a linear operator.

Morover, for any $x \in X$,

$$\begin{aligned}\|S + T\| &= \sup_{\|x\|=1} \|(S + T)x\|\\ &= \sup_{\|x\|=1} \|S\,x + T\,x\|\\ &\le \sup_{\|x\|=1} \|S\,x\| + \sup_{\|x\|=1} \|T\,x\|\\ &\le \|S\| + \|T\| \text{ ………….}\textbf{(1)}\end{aligned}$$

Hence $S + T$ is a bounded linear operator and so is in $B(X, Y)$.

It is easy to see that the commutative and associative laws of addition are satisfied in $B(X, Y)$. The function $0: X \to Y$ defined by:

$$0(x) = 0$$

Is linear and bounded. Also for any $T \in B(X, Y)$,

$$0 + T = T + 0 = T$$

Next , for each $T \in B(X, Y)$, the function $(-T)x = -Tx, x \in X$

Is linear and satisfies

$$T + (-T) = 0$$

$$\text{Also } \|-T\| = \|T\|$$

So $-T \in B(X, Y)$. Hence $B(X, Y)$is an additive abelian group .

Define the scaler multiplication in $B(X, Y)$ as

$(\alpha T)(x) = \alpha Tx$ ; $\forall x \in X, T \in B(X, Y)$, and α is scaler.

For $x, y \in X$and $a, b \in F$

$$\begin{aligned}(\alpha T)(ax + by) &= \alpha.T(ax + by)\\ &= \alpha(aTx + bTy)\\ &= a\alpha Tx + b\alpha Ty)\\ &= a(\alpha T)x + b(\alpha Ty)\end{aligned}$$

Also for any $x \in X$, $\|x\| = 1$

$$\|\alpha\, T\| = \ sup\|(\alpha T)x\|$$
$$= sup\|\alpha.\,\mathrm{T}x\|$$
$$= |\alpha| sup\|\mathrm{T}x\|$$

So that

$$\|\alpha\, T\| = |\alpha|\|\mathrm{T}\| \ldots\ldots\ldots\ldots\mathbf{(2)}$$

Therefore, $\alpha T \in B(X,Y)$. Thus $B(X,Y)$ is a linear space.

Since, for any $T \in B(X,Y)$, $\|\mathrm{T}\| \geq 0$

And $\|\mathrm{T}\| = 0$ if and only if $T = 0$ **…………(3)**

Hence (1), (2) and (3) show that $B(X\,,Y)$ is a normed space.

In the next theorem we discuss properties of $B(X\,,Y)$ in relation to the properties of $Y$.

**Theorem.** Show that, if $Y$ is a Banach space , then so is $B(X,Y)$ under the norm defined by

$$\|\mathrm{T}\| = \sup_{\|x\|=1} \|\mathrm{T}x\|, x \in X, T \in B(X,Y).$$

**Proof.** Suppose $Y$ is a Banach space and let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$, then for given $\epsilon > 0 \ni \mathrm{n}_0 \in \mathrm{N}$ such that $\|\mathrm{T_n} - T_m\| = sup\{\|\mathrm{T_n}x - T_m x\|: x \in X, \|\mathrm{x}\| = 1\} < \epsilon\ \forall n,m \geq n_0$**………..(1)**

That is ,

$$\|\mathrm{T_n}x - T_m x\| < \epsilon\ \forall \mathrm{x}\ \in \mathrm{X}, n,m \geq n_0$$

So for any $x \in X$,$\{T_n x\}$ is a cauchy sequence in $Y$. Since $Y$ is complete, ,$\{T_n x\}$ converges in $Y$.

Let $\lim_{n\to\infty} T_n x = y = Tx$

Where $T: X \to Y$, which takes $x \to Tx$. We will show that $T$ is bounded linear operator .

Since $T_n$ is linear , for any $\alpha, \beta \in \mathrm{F}, \mathrm{x} \in \mathrm{X}$

$$T_n(\alpha x + \beta y) = \alpha T_n x + \beta T_n y$$

Thus, $T_n(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$

$$= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y$$
$$= \alpha T x + \beta \mathrm{T_y}$$

Also letting $m \to \infty$ in (1) and using the continuity of norm function, we have

$$\left\|T_n - \lim_{m\to\infty} T_m\right\| = \ sup\{\left\|T_n x - \lim_{m\to\infty} T_m x\right\| \leq \epsilon\ \forall x \neq 0 \in X\}$$

That is ,

$$\|T_n - T\| = sup\|T_n x - Tx\|: x \neq 0 \in X \leq \epsilon \forall \mathrm{n} \geq \mathrm{n}_0$$

Hence $T_n - T \in B(X,Y)$. But then

$T = T_n - (T_n - T)$, as difference of two elements of $B(X,Y)$

That is $\mathrm{T_n} \to \mathrm{T}$ as $n \to \infty$ as $\mathrm{T_n}$ is a Cauchy sequence in $B(X,Y)$ is in $B(X,Y)$.

Hence $B(X,Y)$ is a Banach space.

**Note.** Converse of above theorem is also true . That is , if $B(X,Y)$ ia a Banach space , so is $Y$.

To prove this we make use of Hahn-Banach Theorem on normed spaces , which we will discuss later in this chapter.

## 3.2 Linear Functional

In the previous section we considered functions called linear operators from one normed space into another normed space defined over the same field . In this section we shall deal with a special type of linear operators called **linear functionals.** These are linear operators from a normed space $X$

Over $F$ to $F$, where $F$ is $R$ or $C$ and is itself a normed space under the usual norm defined by on $R$ or $C$.

Thus a function $f: X \to F$ is said to be linear functional if, for any $x, y \in X$ and $\alpha, \beta \in F$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

A linear functional $f: X \to F$ is said to be continuous at a point $x_0 \in X$, if fgiven $\epsilon > 0$, there is a real number $\delta > 0$ such that

$\|x - x_0\| < \delta$ implies $|f(x) - f(x_0)| < \epsilon \; \forall x \in X$.

$f$ is said to be continuous on $X$, if $f$ is continuous at every point of $X$.

$f$ is said to be bounded if there is a real number $k \geq 0$ such that

$$|f(x)| \leq k\|x\|, \forall x \in X.$$

As in the case of linear operators we define the norm of a linear $f: X \to F$ by :

$$\|f\| = sup\left\{\frac{|f(x)|}{\|x\|} : x \neq 0 \in X\right\}$$

Then , if $f$ is a bounded linear functional , so that $\|f\| \leq k$, then

$$|f(x)| \leq \|f\|\|x\|, \forall x \in X.$$

It is now easy to establish the following equivalent forms of the norm of a linear functional $f$:

$$\|f\| = sup\left\{\frac{|f(x)|}{\|x\|} : x \neq 0 \in X\right\}$$

$$= sup\left\{\frac{|f(x)|}{\|x\|} : ||x|| \leq 1 \in X\right\}$$

$$= sup\{|f(x)| : ||x|| = 1 \in X\}$$

**Theorem**. Let $f: X \to F$ be a linear functional . Then:

(i) $f$ is continuous if and only if $f$ is bounded.

(ii) $f$ is continuous on $X$ if and only if it is continuous at $0 \in X$.

For a linear functional $f: X \to F$, the kernel or null space denoted by $Ker\ f$ is defined by:

$$Kerf = \{x \in X : f(x) = 0\}$$

and is a subspace of $X$.

**Proof.** The proof of this theorem is same as in the case of linear operators.

**Examples of Bounded linear functional**

**Example.** Let $R^n$ be the n-dimmensional real normed space with with the norm defined by :

$$||\boldsymbol{x}|| = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \boldsymbol{x} = (x_1, x_2, \dots, x_n) \in R^n$$

For any $a = (a_1, a_2, \dots, a_n) \in R^n$, define a function $f_{\boldsymbol{a}}: R^n \to R$ by:

$$f_{\boldsymbol{a}}(x) = \sum_{i=1}^{n} a_i x_i, x \in R^n$$

It is easy to verify that $f_a$is linear functional . By schwartz inequality

$$|f_a(\boldsymbol{x})| \leq \sqrt{\sum_{i=1}^{n} |a_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\leq \|a\|\|x\|, x \in R^n$$

Hence $f_a$ is a bounded and hence a continuous linear functional on $R^n$. Also

$$\|f_a\| \leq \|a\| \text{..........}\mathbf{(1)}$$

However if we take $x = a$, we obtain

$$f_a(a) = \sum_{i=1}^{n} |a_i|^2 = \|a\|^2$$

So that $$\|f_a\| = \sup_{x \neq 0} \frac{|f_a(x)|}{\|x\|} \geq \frac{f_a(a)}{\|a\|} = \|a\| \text{..........}\mathbf{(2)}$$

From (1) and (2), we get

$$\|f_a\| = \|a\|.$$

**Example.** For the space $C[a,b]$ of all real continuous functions from $[a,b] \to R$ with the sup norm ,define a function $I: C[a,b] \to R$ by :

$$I(f) = \int_a^b f(t)dt$$

Then $I$ is a linear functional . Also

$$|I(f)| \leq \int_a^b |f(t)|dt$$

$$\leq \sup_{t \in [a,b]} |f(t)| \int_a^b dt$$

$$\leq (b-a)\|f\|$$

Hence,

$$\|I\| \leq (b-a) \text{..........}\mathbf{(3)}$$

Also, taking $f_0(t) = 1 \forall\, t \in [a,b]$, we get

$$\|I\| = \sup_{f \neq 0} \frac{|I(f)|}{\|f\|} \geq \frac{I(f_0)}{\|f_0\|} = b - a \text{..........}\mathbf{(4)}$$

From (3) and (4) we get

$$\|I\| = (b-a).$$

**Example.** Let $c$ be the space of all convergent real sequences $x = \{x_n\}$. Let $f: c \to R$ be defined by $f(x) = \lim_{n \to \infty} x_n$

Then $f$ is a bounded linear functional with

$$\|f\| = 1$$

## 3.3 Compactness and Finite Dimensional Space

Compactness is one of the most important concepts in analysis. We now define compact linear operator.

**Definition.** Let $X$ and $Y$ be normed spaces . An operator $T: X \to Y$ is called a compact linear operator (or completely continuous linear operator) if $T$ is linear and if for every bounded subset $M$ of $X$, the image $T(M)$ is relatively compact, that is the closure $\overline{T(M)}$ is compact .

The term compact is suggested by the definition . The older term "completely continuous " can be motivated by the following lemma, which shows that a compact linear operator is continuous , where as the converse is not generally true.

Before proving the lemma, we first note the following .

I. A compact subset $M$ of a metric space is closed and bounded .

II. If a normed space $X$ has the property that the closed unit ball $M = \{x \in X: \|x\| \leq 1\}$

is compact , then $X$ is finite dimensional.

**Lemma.** Let $X$ and $Y$ be normed spaces. Then:

**(a)** Every compact linear operator $T: X \to Y$ is bounded, hence continuous.

**(b)** If $dim\, X = \infty$, the identity operator $I: X \to X($ which is continuous ) is not compact.

**Proof of (a).** The unit sphere $U = \{x \in X: \|x\| = 1\}$ is bounded. Since $T$ is compact , $\overline{T(U)}$

Is compact and is bounded by (I), so that

$$\sup_{\|x\|=1} \|Tx\| < \infty$$

Hence $T$ is bounded and shows that it is continuous.

**Proof of (b).** The closed unit ball $M = x \in X: \|x\| \leq 1$ is bounded. If $dim\, X = \infty$, then by (II) $M$ cannot be compact, thus $I(M) = M = \overline{M}$ is not relatively compact.

We now prove tha compactness criteria for operators in the following theorem.

**Theorem.** Let $X$ and $Y$ be normed spaces and $T: X \to Y$ a linear operator .Then $T$ is compact if and only if it maps every bounded sequence $\{x_n\}$ in $X$ onto a sequence $\{Tx_n\}$ in $Y$ which has a convergent subsequence.

**Proof.** If T is compact and $\{x_n\}$ is bounded, then the closure of $\{Tx_n\}$ in Y is compact and shows that $\{Tx_n\}$ contains a convergent subsequence.

Conversely, assume that every bounded sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$

such that $\{Tx_{n_k}\}$ converges in Y. Consider any bounded subset $B \subset X$, and let $\{y_n\}$ be any sequence in T(B). Then $y_n = Tx_n$ for some $x_n \in B$ , and $\{x_n\}$ is bounded since $B$ is bounded. By assumption, $\{Tx_n\}$ contains a convergent subsequence. Hence $\overline{T(B)}$ is compact because $\{y_n\}$ in T(B) was arbitrary, by definition, this shows that $T$ is compact.

Next, we study the compactness of finite dimensional linear operator. Prior to that, we recall the following results.

**Result 1. ( Compactness)** In a finite dimensional normed space $X$, any subset $M \subset X$ is compact if and only if $M$ is closed and bounded.

**Result 2.** If a normed space $X$ is finite dimensional, then every linear operator on $X$ is bounded.

**Result 3.** Let $T$ be a linear operator . Then, if $dim\, D(T) = n < \infty$, then $dim\, R(T) \leq n$.

**Theorem. (** Finite dimensional domain or range**)** Let $X$ and $Y$ be normed spaces and $T: X \to Y$

be a linear operator. Then:

(a) If $T$ is bounded and $dim\, T(X) < \infty$, the operator $T$ is compact.

**(b)** $dim\, X < \infty$, the operator $T$ is compact.

**Proof.** Let $\{x_n\}$ be any bounded sequence in $X$. Then

$\|Tx_n\| \leq \|T\|\|x_n\|$ shows that $\{Tx_n\}$ is bounded . Hence $\{Tx_n\}$ is relatively compact by result 1. Since $dim\, T(X) < \infty$. It follows that $\{Tx_n\}$ has a convergent subsequence. Since $\{x_n\}$ was arbitrary bounded sequence in $X$, the operator $T$ is compact.

**Proof of b.** It follows from (a) by noting that $dim\,X < \infty$ implies boundedness of $T$ by result 2 above $dimT(X) \leq dim\,X$ by result 3.

## Summary

- Let $X$ and $Y$ be normed spaces over a field $F$. We say that $T: X \to Y$ is a linear operator if $T$ is linear (that is $T(x + y) = T(x) + T(y)\ \forall\, x, y\ \in X\ and\ T(\lambda x) = \lambda T(x))\ \forall x \in X$ and $\lambda \in F$.
- Let $X$ be any normed space, then the identity function $I: X \to X$ defined by :

$$I(x) = x, x\ \in X$$

is a linear operator .

- For any linear spaces $X, Y$, the function $0: X\ \to Y$ defined by:

$$0(x) = 0, x \in X$$

is a linear operator.

- Zero operator is also called null operator or trivial operator.
- In the space $C[a, b]$, define a function $I: C[a, b] \to C[a, b]$ by:

$$I(f) = \int_a^x f(t)dt\,, f\ \in C[a, b].$$

Then $I$ is a linear operator.

- Let $T: X \to Y$ be a linear operator. Then the set of those elements of $X$ which are mapped onto the zero element of $Y$ is a subspace of $X$ called the kernel or null space of $T$ and is denoted by $Ker\ T$.
- Let $X$ and $Y$ be normed spaces . A linear operator $T: X\ \to Y$ is said to be continuous at a point $x_0 \in X$ if given $\epsilon > 0$, there is a real no $\delta = \delta(\epsilon) > 0$ such that $\forall x\ \in X, \|x - x_0\| <\ \delta \Rightarrow \|Tx - Tx_0\| <\ \epsilon$.
- $T$ is said to be continuous on $X$ if it is continuous at every point of $X$.
- A linear operator $T: X\ \to Y$ is said to be bounded if there is a constant $k > 0$ such that $\|Tx\| < k\|x\| \forall x\ \in X$.
- Let $T: X\ \to Y$ be a bounded linear operator . Then there is a real number $k > 0$ such

$$\|Tx\| \leq k\ \|x\|\ \forall x\ \in X$$

Suppose that $x \neq 0$.Then $\frac{\|Tx\|}{\|x\|} \leq k\ \forall x\ \in X, x \neq 0$.

So $k$ is an upper bound for $\frac{\|Tx\|}{\|x\|}$. The leaset upper bound $\sup\limits_{x\neq 0 \in x} \frac{\|Tx\|}{\|x\|}$ is called the norm of $T$ and is denoted by $\|T\|$. Thus

$$\|T\| = \sup_{x\neq 0\in x} \frac{\|Tx\|}{\|x\|}.$$

- $\|A\| = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} |a_i j|^2}$ is called the norm of matrix operator $A$.
- Every linear operator on a finite dimensional normed space is bounded .
- if $T_1: X \to Y$ and $T_2: Y \to Z$ be bounded linear operators .Then $T_2 T_1$ is bounded and $\|T_2 T_1\| \leq \|T_2\|\|T_1\| = \|T_1\|\|T_2\|$.
- The space $B(X, Y)$ of all bounded (hence continuous) linear operators from $X$ to $Y$ is a normed space under the norm defined by

$$\|T\| = \sup_{\|\,x\|=1} \|T\,x\|, x \in X.$$

- If $Y$ is a Banach space , then so is $B(X,Y)$ under the norm defined by

$$\|T\| = \sup_{\|x\|=1} \|Tx\|, x \in X, T \in B(X,Y).$$

- A linear functional $f: X \to F$ is said to be continuous at a point $x_0 \in X$, if given $\epsilon > 0$, there is a real number $\delta > 0$ such that

$$\|x - x_0\| < \delta \text{ implies } |f(x) - f(x_0)| < \epsilon \, \forall x \in X.$$

- Let $X$ and $Y$ be normed spaces . An operator $T: X \to Y$ is called a compact linear operator (or completely continuous linear operator) if $T$ is linear and if for every bounded subset $M$ of $X$, the image $T(M)$ is relatively compact, that is the closure $\overline{T(M)}$ is compact

## Keywords

- Bounded linear operator
- Continuous linear operator
- Null space of a linear operator
- Norm
- Closed
- Compact
- Finite dimmensional

## Self Assessment

1: If T is a bounded linear operator, then:

A. $\|Tx\| \leq \|T\|. \|x\|$

B. $\|Tx\| \geq \|T\|. \|x\|$

C. $\|Tx\| = \|T\|. \|x\|$

D. None of the above

2: Which of the following statements is true about a bounded linear operator?

A. Every bounded linear operator is continuous.

B. Every continuous operator is bounded.

C. Every bounded linear operator is compact.

D. None of the above

3: Which of the following is NOT a property of a bounded linear operator?

A. Preserving the zero vector: $T(0) = 0$

B. Homogeneity

C. Additivity

D. Surjective

4: What is the null space of a linear operator?

A. The set of all inputs for which the linear operator is not defined.

B. The set of all inputs that map to the zero vector under the linear operator.

C. The set of all outputs for which the linear operator is not defined.

D. The set of all outputs that map to the zero vector under the linear operator.

5: Which property holds true for the norm of a linear operator with respect to scalar multiplication.

A. $\|kT\| = k\|T\|$

B. $\|kT\| = \frac{1}{k}\|T\|$

C. $\|kT\| = k^2\|T\|$

D. $\|kT\| = |k|\|T\|$

6: Which of the following statements is true regarding compactness in a normed linear space?

A. Every closed and bounded subset is compact.

B. Every open and bounded subset is compact.

C. Every closed and unbounded subset is compact.

D. Every open and unbounded subset is compact.

7: Which of the following statements about the norm of a linear operator is true?

A. The norm of a linear operator is always zero.

B. The norm of a linear operator is always one.

C. The norm of a linear operator can be negative.

D. The norm of a linear operator is always positive.

8: If the norm of a linear operator T is zero, what can we conclude?

A. T is the zero operator (T(x) = 0 for all x).

B. T is not a linear operator.

C. T is an invertible operator.

D. None of the above.

9: If $X$ and $Y$ are normed spaces, then the space of bounded linear operators $B(X,Y)$ is a Banach space if and only if:

A. $X$ is a Banach space.

B. $Y$ is a Banach space.

C. Both $X$ and $Y$ are Banach spaces.

D. Both $X$ and $Y$ are finite dimensional spaces.

10: If E is a normed space and if $d$ is the metric induced by the norm, then for any scalar $k, d(kx, ky)$ equals

A. $d(x,y)$

B. $|k|\ d(x,y)$

C. $k\ d(x,y)$

D. $k^2 d(x,y)$

11: Let $X$ be a normed space and $f$ be a bounded, non-zero linear functional on $X$. Then, which of the following is not true?

A. $f$ is onto.

B. $f$ is continuous

C. $Ker\ f$ is a close subspace of $f$.

D. $f$ is an open map.

12: If f is a linear functional on a normed space X, then $Ker\ f$ is:

A. Closed in $X$

B. Dense in $X$

C. Either dense or closed in $X$

D. None of the above.

13: Every complete subspace of a normed space is:

A. finite

B. open

C. closed

D. None of the above

14: Every bounded operator of finite rank is :

A. Compact

B. Open

C. Has a zero adjoint

D. None of the above

15: Rank of a linear operator A equals:

A. $dim(Im\ A)$

B. $dim(KerA)$

C. $dim(ImA^*)$

D. $dim(KerA^*)$

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | B | 3. | D | 4. | B | 5. | A |
| 6. | A | 7. | D | 8. | A | 9. | B | 10. | B |
| 11. | D | 12. | C | 13. | C | 14. | A | 15. | A |

## Review Questions

1. What is a linear operator between two normed spaces?
2. Define a bounded linear operator between normed spaces.
3. What is kernel or null space of a linear operator.

4. Define norm of a linear operator.
5. Define what a linear functional on a normed linear operator is.
6. Define kernel or null space of a linear operator.
7. Define norm of a linear operator.
8. Show that every linear operator on a finite dimensional normed space is bounded.
9. Define compact linear operator.
10. Let $X$ and $Y$ be normed spaces and $T: X \to Y$ be a linear operator. Prove that $Ker\, T$ is a subspace of $X$.

## Further Readings


- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
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# Unit 04: Hahn-Banach Theorem and its Consequences

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Conjugate of an operator
- Convex functional
- The Hahn- Banach theorem for Real spaces
- The Hahn- Banach theorem for complex spaces
- The Hahn- Banach theorem for normed spaces

## Introduction

In this chapter, we introduce the idea of conjugate of an operator. Further , we also discuss convex functional . Finally we discuss different forms of Hahn- Banach theorem and its consequences.

## 4.1 Conjugate of an Operator

In the context of normed spaces and linear operators, the conjugate of an operator is a concept related to the duality between a normed space and its dual space. To understand the conjugate of an operator, we first need Knowledge of normed spaces, dual spaces and linear operators, which we have already discussed in previous chapters. We now define the conjugate of an operator.

**Definition.** Let $X$ and $Y$ be normed spaces. Let $B(X,Y)$ be the space of all bounded linear operators defined from $X$ to $Y$. Let $X^*$ and $Y^*$ be the conjugate spaces of $X$ and $Y$respectively. Let $T \in B(X,Y)$, then we define an operator $T' : Y^* \to X^*$ as follows:

For each $f \in Y^*, f.T$ is a mapping from $X$ to $F$. It is bounded because both $T$ and $f$ *are* bounded .

So $f.T \in X^*$. Then we put $T'(f) = f.T$ **…………..(1)**

So, for each $x \in X, Tx \in Y$ and $(f.T)x \in F$ while $T(f) \in X^*$ so that, for each $x \in X, T'(f)x \in$ F.

Hence we can write (1) as $T'(f)x = (f.T)x = f(Tx)$ **…………..(2)**

The operator $T'$, defined by (1) or (2), is called the conjugate (or sometimes, the adjoint ) of the linear operator $T$.

We now discuss some properties of $T'$.

I. $T'$ is linear:

For this, let $f_1, f_2 \in Y^*$ and $\alpha_1, \alpha_2 \in F$.Then

$$\begin{aligned} T'(\alpha_1 f_1 + \alpha_2 f_2)x &= (\alpha_1 f_1 + \alpha_2 f_2)(Tx), Tx \in Y \\ &= (\alpha_1 f_1)(Tx) + \alpha_2 f_2(Tx) \\ &= \alpha_1(f_1(Tx)) + \alpha_2(f_2(Tx)) \\ &= \alpha_1(f_1.T)x + \alpha_2(f_2.T)x \\ &= \alpha_1 T'(f_1)x + \alpha_2 T'(f_2)x \\ &= [\alpha_1 T'(f_1) + \alpha_2 T'(f_2)]x, \forall x \in X. \end{aligned}$$

Hence,

$$T'(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 T'(f_1) + \alpha_2 T'(f_2) \ldots\ldots\ldots\ldots\ldots\textbf{(3)}$$

II. $T'$ is bounded . Here, for any $f \in Y^*$ and $x \in X$,

$$\begin{aligned} \|T'\| = \sup_{\|f\| \le 1} \|T'(f)\| &= \sup_{\|f\| \le 1 \, \|x\| \le 1} \|T'f(x)\| \\ &= \sup_{\|f\| \le 1 \, \|x\| \le 1} \|f(Tx)\| \\ &\le \|f\| \sup_{\|x\| \le 1} \|Tx\| \\ &\le \sup_{\|x\| \le 1} \|Tx\| , \|f\| \le 1 \\ &\le \|T\| \ldots\ldots\ldots\ldots\ldots\textbf{(4)} \end{aligned}$$

Hence $T'$ is bounded.

III. The mapping $\phi: B(X, Y) \to B(Y^*, X^*)$ defined by

$$\phi(T) = T'$$

Is an isometry.

IV. If $X = Y$ then $\phi$ preserves identity and reverses products.

That is :

$\phi(I) = I'$ and $\phi(T_1 T_2) = T_2' T_1'$

Here $I$ is the identity mapping defined on $X$.

## 4.2 Hahn-Banach Theorem (Real and Complex Form) and its Consequences

The Hahn-Banach theorem is an extension theorem for linear functionals. It guarantees that a normed space is richly supplied with bounded linear functionals and makes possible an adequate theory of dual spaces, which is an essential part of the general theory of normed spaces. In this way the Hahn-Banach theorem becomes one of the most important theorems in connection with bounded linear operators. Furthermore, our discussion will show that the theorem also characterizes the extent to which values of a linear functional can be preassigned. The theorem was discovered by H. Hahn (1927), rediscovered in its present more general form by S. Banach (1929) and generalized to complex vector spaces by H. F. Bohnenblust and A. Sobczyk (1938).

Now we shall prove this theorem and also discuss some of its important implications.

Before proving the theorem we recall some important definitions .

- ✓ Let $V$ be a linear space and $F$ is the field of real or complex numbers . A functional $p: V \to F$ is said to be finite , if $p(x)$ is finite for all $x \in V$.
- ✓ A functional $p: V \to F$ is said to be convex functional (or seminorm) if:

  (i) $p(x) \geq 0 \ \forall \ x \in V,$

  (ii) $p(ax) = ap(x) \ \forall \ x \in V$ and real $a \geq 0$ (positive homogeneous property),

  (iii) $p(x + y) \leq p(x) + p(y) \ \forall x, y \ \in V$ (sub- additive property).

**Example.** The norm function $\|.\|: V \to F$, where $V$ is the normed space, is a convex functional.

A linear functional $f$ defined on $V$ is called an extension of a linear functional $f_0$ defined on a subspace $U$ of $V$ if

$$f(x) = f_0(x) \ \forall x \in U.$$

## Theorem. (The Hahn- Banach Theorem for Real spaces).

Let $p$ be a finite convex functional defined on a real vector space $V$ and let $U$ be a subspace of $V$. Let $f_0: U \to R$ be a linear functional such that

$$f_0(x) \leq \mathrm{p(x)} \ \forall x \in U \qquad \textbf{..........(1)}$$

Then $f_0$ can be extended to a linear functional $f$ defined on $V$ such that

$$f(x) \leq p(x) \ \forall x \in V$$

**Proof.** For $U = \mathrm{V}$ the result is trivial, so we suppose that $U \neq \mathrm{V}$ .

Step I:

We first prove that $f_0$ can be extended onto a large subspace without violating condition (1).

Let $z \in V \backslash U$ and put

$$V_1 = \{x + \alpha z: x \in U, \alpha \in R\}.$$

Then $V_1$ is a subspace of $V$ and contains $U$ properly.

That is, $U \subset V_1 \subseteq V$.

Define a function $f': V_1 \to R$ by ;

$$f'(x + \alpha z) = f_0(x) + \alpha f'(z)$$
$$= f_0(x) + \alpha \mathrm{c}, \ \ \mathrm{c} = f'(z) \qquad \textbf{..........(2)}$$

Then $f'$ is a linear functional on $V_1$.

We show that it is possible to choose a real number $c$ such that the majorization condition

$$f'(x + \alpha z) \leq p(x + \alpha z)$$

Is satisfied . That is there exists a real number $c$ such that

$$f_0(x) + \alpha c \leq p(x + \alpha z)$$

i,e $$f_0\left(\frac{x}{\alpha}\right) + c \leq p\left(\frac{x}{\alpha} + z\right)$$

i,e $$c \leq p\left(\frac{x}{\alpha} + z\right) - f_0\left(\frac{x}{\alpha}\right) \qquad \textbf{..........(3)}$$

if $\alpha > 0$, and

$$f_0\left(\frac{x}{\alpha}\right) + c \geq -\left(\frac{1}{-\alpha}\right)p(x + \alpha z) = -p\left(\frac{-x}{\alpha - z}\right)$$

$$c \geq -p\left(\frac{-x}{\alpha - z}\right) - f_0\left(\frac{x}{\alpha}\right) \quad \ldots\ldots\ldots\textbf{(4)}$$

if $\alpha < 0$.

Now for any two arbitrary points $y', y''$ of $U$, we have

$f_0(y'') - f_0(y') = f_0(y'' - y') \leq p(y'' - y')$

$\leq p(y'' + z - (y' + z))$

$\leq p(y'' + z + (-y' - z))$

$\leq p(y'' + z) + p(-y' - z)$

Hence

$$-f_0(y') - p(-y' - z) \leq p(y'' + z) - f_0(y'') \quad \ldots\ldots\ldots\textbf{(5)}$$

Put $\quad c' = \sup_{y \in U}\{-f_0(y') - p(-y' - z)\}$

$c'' = \inf_{y \in U}\{p(y'' + z) - f_0(y'')\}$

Then $c' \leq c''$

By (5) and the fact that $y', y''$ are arbitrary.

Now choose a $c$ such that

$$c' \leq c \leq c''$$

Then, for this value of $c$, the linear functional $f'$ defined on $V_1$ by (2) satisfies the condition that

$$f'(x) \leq p(x) \forall x \in V_1 \quad \ldots\ldots\ldots\textbf{(6)}$$

as condition (3) and (4) are satisfied . Hence $f'$ is an extension of $f_0$ to a subspace $V_1$ containing $U$ properly and satisfying condition (1).

Step II.

Now suppose that $V$, as a linear space is generated by a countable set of elements $x_1, x_2, x_3, \ldots x_n, \ldots$, in $V$. Then we construct a linear functional on $V$ by induction on $n$. That is, we construct a sequence of subspaces

$$V_1 = \{x_1, U\}, \qquad V_2 = \{x_2, V_1\}, \ldots V_n = \{x_n, V_n - 1\}, \ldots$$

each contained in the next . This prosess extends the functional $f_0$ onto the whole space $V$, since every $x$ in $V$ is in some subspace $V_n$.

Step III. For the general case , that is, when no countable set generates $V$, THE theorem is proved by applying Zorn's lemma as follows:

Let $F$ be the class of all possible extensions $f^*$ of $f_0$ satisfying the condition

$$f^*(x) \leq p(x)\ \forall x \in D(f^*)$$

And

$$f^*(x) = f_0(x)\ \forall x \in D(f_0)$$

Here $D(f_0)$ is the domain of $f_0$. Then $f$ is non empty because $f'$ constructed above is in $F$. We partially order $F$ as follows :

For $f, g \in F$, we say that

$$f \leq g$$

If and only if $g$ is an extension of $f$, that is

$$D(g) \supseteq D(f)$$

and $\qquad g(x) = f(x)\ \forall x \in D(f)$.

Now let $C$ be a chain in $F$. Define a linear functional $\bar{f}$ as follows:

(i) Domain of $\bar{f} = \cup_{g \in c} D(g)$,

(ii) For $x \in D(\bar{f})$,

$$\bar{f}(x) = g(x)\ \forall x \in D(g), g \in C.$$

It is clear that $\bar{f}$ is a linear extension of $f_0$ and

$$\bar{f}(x) \leq p(x)\ \forall x \in D(\bar{f})\ .$$

So $\bar{f} \in F$ and is an upper bound for $C$. By Zorn's lemma, $F$ has a maximal element $f$ which is an extension of $f_0$ and

$$f(x) \leq p(x) \forall x \in D(f).$$

We claim that $D(f) = V$, otherwise let $z \in V \backslash D(f)$.

Then as in step I , there is an extension $f'$of $f$ to $D(f), z$, contracdicting the maximality of $f$. Hence $f$ is the required extension of $f_0$. This proves the theorem completely.

Before discussing the complex version of the Hahn-Banach theorem we need the following concept.

✓ A functional $p$ defined on a complex linear space $V$ is said to be convex if:

(i) $p(x) \geq 0\ \forall x \in V$,

(ii) $p(\alpha x) = |\alpha| p(x) \forall\ \alpha \in \mathrm{C}$ and $x \in V$,

(iii) $p(x+y) \leq p(x) + p(y) \forall x, y\ \in V$.

**Theorem. (Hahn-Banach theorem for complex space)**

Let $p$ be a finite convex functional defined on a complex linear space $V$ and let $U$ be a subspace of $V$. Let $f_0$ be alinear functional defined on $U$ satisfying the condition:

$$|f_0(x)| \leq p(x)\ \forall x\ \in U \quad \textbf{……..(1)}$$

Then $f_0$ can be extended to a linear functional $f$ on $V$ such that

$$|f(x)| \leq p(x)\ \forall x \in V$$

**Proof.** Since $V$ is a complex linear space, for each $v \in V$ and $\alpha = \alpha_1 + i\alpha_2\ \in C, \alpha v \in V$. If we restrict the scalers to real numbers only then $V$ is a real vector space . Denote this space by $V_R (= V)$ and the correspondind subspace by $U_R (= U)$. Clearly $p$ is a finite convex functional defined on $V_R$ while $f_0'$ given by:

$$f_0'(x) = \text{real part of } f_0(x), x \in U_R$$

Is a real linear functional on $U_R$. Hence, by the Hahn-Banach theorem for real spaces, there is a linear extension $f_1$ defined on all $V_R$ satisfying the condition:

$$f_1(x) \leq p(x)\ \forall x\ \in V_R (= V) \quad \textbf{……..(2)}$$

and $f_1(x) = f_0'(x) \forall x \in U_R (= U)$

Also, $-f_1(x) = f_1(-x) \leq p(-x) = |-1| p(x) = p(x)$

Thus $f_1(x) \geq -p(x) \forall x \in V_R \quad$ **……..(3)**

From (2) and (3), we obtain

$$|f_1(x)| \leq p(x) \forall x \in V_R \quad \textbf{……..(4)}$$

Now we consider $f_0$ as a linear functional on the complex space $U$. So

$$f_0(x) = f_0'(x) + i f_0''(x) \quad \textbf{……..(5)}$$

Since $f_0$ is linear on $U$,

$$i f_0(x) = f_0(ix) = f_0'(ix) + i f_0''(ix) \quad \textbf{……..(6)}$$

Multiplying (5 ) by $i$, we obtain

$$if_0(x) = -f_0''(x) + if_0'(x) \qquad \text{……..(7)}$$

Comparing (6) and (7), we have

$$f_0''(x) = -f_0'(ix)$$

Hence

$$f_0(x) = f_0'(x) - if_0'(ix) \qquad \text{……..(8)}$$

If $\boldsymbol{f_1}$ denotes the linear extension of $f_0'$ to the whole of $V$, as a real linear space , then put

$$f(x) = f_1(x) - if_1(ix) \qquad \text{……..(9)}$$

We show that the function $f$ defined by (9) is the required linear extension of $f_0$

To $V$ and satisfies the given condition.

Obviously $f$ is an extension of $f_0$ to the whole of $V$. Also

$$\begin{aligned} f(x+y) &= f_1(x+y) - if_1\big(i(x+y)\big) \\ &= f_1(x) + f_1(y) - if_1(ix+iy) \\ &= f_1(x) + f_1(y) - if_1(ix) - if_1(iy) \\ &= f(x) + f(y) \qquad \text{……..(10)} \end{aligned}$$

And $f(\alpha x) = f\big((\alpha_1 + i\alpha_2)x\big)$

$$\begin{aligned} &= f(\alpha_1 x + \alpha_2 ix) \\ &= f(\alpha_1 x) + f(\alpha_2 ix) \\ &= f_1(\alpha_1 x) - if_1(\alpha_1 ix) + f_1(\alpha_2 ix) - if_1(-\alpha_2 x) \\ &= \alpha_1 f_1(x) - i\alpha_1 f_1(ix) + \alpha_2 f_1(ix) + i\alpha_2 f_1(x) \\ &= \alpha_1(f_1(x) - if_1(ix)) + i\alpha_2(f_1(x) - if_1(ix)) \\ &= (\alpha_1 + i\alpha_2)(f_1(x) - if_1(ix)) \\ &= \alpha f(x) \; \forall x, y \in V \text{ and } \alpha \in C. \end{aligned}$$

Hence $f$ is a linear extension of $f_0$.

Finally, we show that

$$|f(x)| \le p(x) \forall x \in V.$$

Suppose, on the contrary, that $|f(x_0)| > p(x_0)$ for some $x_0 \in V$.

Then,

$$f(x_0) = \rho e^{i\phi}, \rho > 0$$

If we put

$$y_0 = e^{-i\phi} x_0$$

Then $y_0 \in V$ and using $|f(x_0)| = \rho$, we have

$$\begin{aligned} f_1(y_0) = Re f(y_0) &= Re\left(e^{-i\phi} f(x_0)\right) \\ &= \rho > p(x_0) = p(y_0) \end{aligned}$$

Which contradicts (4). Hence

$$|f(x)| \le p(x) \; \forall x \in V.$$

This completes the proof of the theorem.

**Theorem. (The Hahn- Banach Theorem for normed spaces).**

Let $V$ be a normed space and $U$ be a subspace of $V$. Let $f_0$ be a bounded linear functional on $U$ with norm $\|f_0\|$. Then $f_0$ has a continuous linear extension $f$ defined on $V$ such that

$$\|f\| = \|f_0\|.$$

**Proof.** Since $f_0$ is a bounded linear functional , $\|f_0\|$ is finite . Put

$$p(x) = \|f_0\|\|x\| \ \forall x \in V$$

We show that $p$ is a convex functional defined on $V$.

Clearly $p(x) \geq 0$. Also for any $\alpha \in F$,

$$p(\alpha x) = \|f_0\|\|\alpha x\| = |\alpha|\|f_0\|\|x\| == |\alpha| \mathrm{p(x)}, x \in V.$$

Moreover , for $x, y \in V$,

$$\begin{aligned} p(x+y) &= \|f_0\|\|x+y\| \\ &\leq \|f_0\|(\|x\| + \|y\|) \\ &\leq \|f_0\|(\|x\| + \|f_0\|\|y\|) \\ &\leq p(x) + p(y). \end{aligned}$$

Also $|f_0(x)| \leq \|f_0\|\|x\|$

$\leq p(x)$.

Thus, by the complex version of Hahn-Banach Theorem, there is a linear functional $f$ defined on $V$ such that

$|f(x)| \leq \mathrm{p(x)} = \|f_0\|\|x\| \ \forall x \in V$

And $f(x) = f_0(x) \ \forall x \in U$ **……….(1)**

From (1), we have

$\|f\| \leq \|f_0\|$ **……….(2)**

Also $\|f\| = \sup\limits_{x \neq 0_{x \in V}} \frac{|f(x)|}{\|x\|} \geq \sup\limits_{x \neq 0_{x \in U}} \frac{|f_{0(x)}|}{\|x\|} = \|f_0\|$ **……….(3)**

Hence, from (2) and (3)

$\|f\| = \|f_0\|$

This proves the proof of the theorem for normed spaces.

Next, we prove an important deduction of the Hahn-Banach theorem for normed spaces and show that a non-trivial normed space $X$ always have enough bounded linear functionals to distinguish between the points of $X$.

**Corrolory.** Let $X$ be a non-trivial normed space and $x_0 \neq 0$ be any point of $X$ . Then there is a continuous ( and so bounded ) linear functional $f$ defined on $X$ such that

$$\|f\| = 1 \text{ and } \|f(x_0)\| = \|x_0\|$$

**Proof.** Let $0 \neq x_0 \in X$. Consider the subspace $Y$ generayted by $x_0$. An arbitrary element of $Y$ is of the form $ax_0, a \in F$. Define a functional $f_0: Y \to F$ by :

$$f_0(y) = f_0(ax_0) = a\|x_0\|, y = ax_0 \in Y, a \in F \quad \text{………..(1)}$$

Then $f_0$ is linear because for $y = ax_0$ and $y' = a'x_0$ in $Y$ and $\alpha, \alpha' \in F$, we have

$$\begin{aligned} f_0(\alpha y + \alpha' y') &= f_0((\alpha a + \alpha' a')x_0) \\ &= (\alpha a + \alpha' a')\|x_0\| \quad \text{by (1)} \\ &= \alpha a\|x_0\| + \alpha' a'\|x_0\| \\ &= \alpha f_0(y) + \alpha' f_0(y') \end{aligned}$$

Also $\|f_0\| = \sup\limits_{y \neq 0_{y \in Y}} \frac{f_0(y)}{\|y\|} = \sup\limits_{\alpha \in F} \frac{|a|\|x_0\|}{|a|\|x_0\|} = 1$, as $y \neq 0$ so that $a \neq 0$

So $f_o$ is a bounded linear functional defined on $Y$. By the Hahn- Banach theorem for normed spaces , there is a linear extension $f$ of $f_0$ to $X$ such that

$$\|f\| = \|f_0\| = 1, f(y) = f_0(y) = a\|x_0\|, y = ax_0 \in Y$$

Thus $\|f\| = 1$ and $f(x_0) = \|x_0\|$ as required.

**Corrolory.** Every non trivial normed space has a non zero linear functionals defined on it.

**Corrolory.** Let $X$ be a normed space. Then, for any $x, y \in X, x \neq y$ , there is a bounded linear functional $f$ such that

$$f(x) = f(y).$$

## Summary

- Let $V$ be a linear space and $F$ is the field of real or complex numbers . A functional $p: V \to F$ is said to be finite , if $p(x)$ is finite for all $x \in V$.
- A functional $p: V \to F$ is said to be convex functional if:

(i) $p(x) \geq 0 \; \forall \; x \in V,$

(ii) $p(ax) = ap(x) \; \forall \; x \in V$ and real $a \geq 0$ .

(iii)$p(x+y) \leq p(x) + p(y) \; \forall x, y \in V.$

- A linear functional $f$ defined on $V$ is called an extension of a linear functional $f_0$ defined on a subspace $U$ of $V$ if

$$f(x) = f_0(x) \; \forall x \in U.$$

- (Hahn- Banach theorem for Real spaces)

Let $p$ be a finite convex functional defined on a real vector space $V$ and let $U$ be a subspace of $V$. Let $f_0: U \to R$ be a linear functional such that

$$f_0(x) \leq \text{p(x)} \; \forall x \in U$$

Then $f_0$ can be extended to a linear functional $f$ defined on $V$ such that

$$f(x) \leq p(x) \; \forall x \in V.$$

- (Hahn- Banach theorem for Complex spaces)

Let $p$ be a finite convex functional defined on a complex linear space $V$ and let $U$ be a subspace of $V$. Let $f_0$ be alinear functional defined on $U$ satisfying the condition:

$$|f_0(x)| \leq p(x) \; \forall x \; \in U$$

Then $f_0$ can be extended to a linear functional $f$ on $V$ such that

$$|f(x)| \leq p(x) \; \forall x \in V$$

- (Hahn- Banach theorem for normed spaces)

Let $V$ be a normed space and $U$ be a subspace of $V$. Let $f_0$ be a bounded linear functional on $U$ with norm $\|f_0\|$. Then $f_0$ has a continuous linear extension $f$ defined on $V$ such that

$$\|f\| = \|f_0\|.$$

## Keywords

- Conjugate
- Bounded linear operator
- Subspace
- Convex functional
- Linear functional
- Seminorm
- Maximality

## Self Assessment

1: In the context of normed spaces, what is the conjugate of a bounded linear operator?

A. The adjoint operator.
B. The inverse operator.
C. The transpose operator.
D. None of the above.

2: For a bounded linear operator $T$ on a normed space $X$, the operator's conjugate, denoted by $T'$, satisfies which property?

A. $T'T = I$.
B. $T'T = T$.
C. $T'T = -T$.
D. None of the above.

3: Consider two normed spaces $X$ and $Y$, and let $T: X \to Y$ be a bounded linear operator. Which of the following statements is false?

A. If $T$ is injective, then $T'$ is injective.
B. If $T$ is surjective, then $T'$ is surjective.
C. If $T$ is compact, then $T'$ is also compact.
D. If $T'$ is compact, then $T$ is also compact.

4: Every bounded operator of finite rank is :

A. Compact.
B. Open.
C. has a non zero adjoint.
D. None of these.

5: Which of the following is the property of conjugate of the linear operator $T$.
A. $T'$ is linear.
B. $T'$ bounded.
C. Both (A) and (B).
D. None of the above.

6: Which of the following is a Banach space?
A. $P[a, b]$ with supremum norm.
B. C[a,b] with supremum norm.
C. Both (A) and (B).
D. None of the above.

7: Which of the following is true about Hahn-Banach theorem.
A. The Hahn-Banach theorem is an extension theorem for linear functionals.
B. The Hahn-Banach theorem is an extension theorem for linear functions.

C. Both (A) and (B)

D. None of the above

8: Consider the statements:

(i) Every compact operator is bounded.

(ii) Every bounded operator is compact.

A. Only (i) is true.

B. Only (ii) is true.

C. Both (i) and (ii) are true.

D. Neither (i) nor (ii) is true.

9: Which of the following statements is true regarding the Hahn-Banach theorem?

A. It guarantees the existence of a continuous linear functional on every vector space.

B. It ensures the existence of a bounded linear functional on every normed space.

C. It provides a way to extend a bounded linear functional from a subspace to the whole space.

D. It applies only to finite-dimensional vector spaces.

10: Which of the following is true?

A. If $A$,$B$ are invertible linear operators on $X$, then $A + B$ is invertible.

B. If $A$,$B$ are invertible linear operators on $X$, then $AB$ is invertible.

C. If $A$ is invertible linear operator on $X$, and $k$ is any scalar, then $kA$ is invertible.

D. If $A$,$B$ are invertible linear operators on $X$, then $A - B$ is invertible.

11: For any normed space $X$, the dual space $X^*$ is:

A. Always a Banach space.

B. Always a compact set.

C. Always finite dimensional.

D. Always an infinite dimensional.

12: Any bounded subset in $R^n$ is :

A. Compact.

B. Relatively compact.

C. open.

D. Closed.

13: Let $V$ be a linear space and $F$ is the field of real or complex numbers . A functional $p: V \to F$ is said to be finite

A. If $p(x)$ is finite for all $x \in V$.

B. If $p(x)$ is finite for some $x \in V$.

C. Both (A) and (B)

D. None of the above

14: A functional $p: V \to F$ is said to be convex functional if

A. $p(x) \geq 0 \; \forall \; x \in V$,

B. $p(ax) = ap(x) \; \forall \; x \in V$ and real $a \geq 0$

C. $p(x+y) \leq p(x) + p(y) \; \forall x, y \; \in V$

D. All of the above are true.

15: For what type of normed spaces does the Hahn-Banach theorem always hold?

A. Only for finite-dimensional normed spaces.

B. Only for infinite-dimensional normed spaces.

C. Only for Banach spaces.

D. For all normed spaces.

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | A | 3. | D | 4. | A | 5. | C |
| 6. | B | 7. | A | 8. | A | 9. | C | 10. | B |
| 11. | A | 12. | B | 13. | A | 14. | D | 15. | D |

## Review Questions

1. Define conjugate of an operator.
2. Define Convex functional.
3. State Hahn-Banach theorem for real spaces.
4. State Hahn- Banach theorem in Complex form.
5. State Hahn- Banach theorem for normed spaces.

## Further Readings


- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
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# Unit 05: Uniform Boundedness Principle and its Consequences

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Reflexive Spaces
- Baire's Category theorem
- Uniform Boundedness Principle

## Introduction

In this chapter, we discuss about Reflexive spaces . Further , we also recall some definitions like first category, Second Category and discuss Baire's Category theorem. Finally, we have proved Uniform Boundedness Principle.

## 5.1 Reflexive Spaces

Reflexive spaces play an important role in the general theory of locally convex topological vector space and in the theory of Banach spaces in particular. Hilbert spaces are prominent examples of reflexive Banach spaces. Reflexive Banach spaces are often characterized by their geometric properties. Now, we will define reflexive space.

**Definition.** Let $X$ be a Banach space and $J: X \to X^{**}$ be the canonical injection from $X$ into $X^{**}$ given by $\big(J(x)\big)(f) = \phi_x(f) = f(x) \quad \forall\, x \in X, f \in X^*$.

The space $X$ is reflexive if $J$ is surjective, i,e $J(X) = X^{**}$.

**Remark.** (i) Finite dimensional spaces are reflexive ( since $dimX = dimX^* = dimX^{**}$).

(ii) Every Hilbert space is reflexive.

(iii) $L^1, L^\infty, l^1$ and $l^\infty$ are not reflexive.

(iv) $C(K)$=space of continuous functions on an infinite compact metric space $K$ is not reflexive.

(v) If a normed space $X$ is reflexive, it is complete and hence a Banach space.

## 5.2 Category Theorem

Category Theorem is an important result in general topology and functional analysis. Baire's Category Theorem was first formulated by French mathematician René-Louis Baire in 1899. This theorem deals with the properties of complete metric spaces and provides a powerful tool for studying the nature of dense sets.

Before presenting the main theorem, it is essential to establish several important results. These results serve as foundational building blocks that will contribute to the proof of the main theorem. Hence, we now proceed to state these results.

Let $X$ be a metric , a subset $M \subseteq X$ is called rare ( or nowhere dense in $X$) if $\bar{M}$ has no interior point i,e $int(\bar{M}) = \phi$.

A subset $A$ of a metric space $X$ is said to be of the first category ( meager) if and only if $A$ can be covered by a countable union of its nowhere dense subsets. Otherwise $A$ is said to be of second category (Non- Meager).

A space $X$ is said to be of the first category if and only if $X$ as a subset of itself can be written as a countable union of nowhere dense subsets. Otherwise $X$ is said to be of the second category.

Thus a metric space $X$ is said to be of second category if and only if $X$ cannot be expressed as a countable union of nowhere dense subsets.

**Example.** Consider the set $Q$ of rationals as a subset of a real line . Let $\in Q$ , then $\{q\} = \overline{\{q\}}$ because $R - \{q\} = (-\infty, q) \cup (q, \infty)$ is open. Clearly $\{q\}$ contain no open ball. Hence $Q$ is nowhere dense in $R$ as well as in $Q$ . Also since $Q$ is countable, it is the countable union of subsets $\{q\}: q \in Q$ .

**Thus $Q$ is of the first category.**

Now we prove the main theorem.

**Theorem. ( Baire's Category Theorem).** If a metric space $X \neq \phi$ is complete then it is non-meager in itself. Hence if $X \neq \phi$ is complete and

$$X = \cup_{k=1}^{\infty} A_k,\ A_k \text{ is closed} \quad \ldots\ldots\ldots\ldots\ldots\textbf{(1)}$$

Then atleast one $A_k$ contains a nonempty open subset.

(**Or**)

A complete metric space is of second category**.**

**Proof.** Let $\boldsymbol{X}$ be a complete metric space. We show that for countable collection $\{A_n : n \in N\}$ of nowhere dense subsets, $X$ is their union . That is, there is a point of $X$ which is not in $\cup_{n\in N} A_n$.

Suppose, on the contrary that $X = \cup_{n\in N} A_n$ and each $A_n$ is nowhere dense subset.

Since $A_1$ is nowhere dense and $X$ is open with $X - \overline{A_1} \neq \phi$, there is an open ball $B_1$ Of radius $< \frac{1}{2}$ which is disjoint from $A_1$. Let $F_1$ be a concentric closed ball of radius half of the radius of $B_1$. Since $A_2$ is nowhere dense , $int(F_1)$ contains an open ball $B_2$ of radius $< \frac{1}{4}$ and disjoint from $A_2$. Let $F_2$ be the concentric closed ball of radius half that of $B_2$.

Likewise, since $A_3$ is nowhere dense, $int(F_2)$ contains an open ball $B_3$ of radius $< \frac{1}{8}$ and disjoint from $A_3$. Again choose a concentric closed ball $F_3$ of radius half of the radius of $B_3$.

Continuing in this way, we obtain a decreasing sequence of concentric closed balls $F_n$ of diameter $< \frac{1}{2^n+1}$ with each $F_n$ disjoint from $A_n$. By cantors intersection theorem there is a unique point $x \in \cap_{n\in N} F_n$ and so also in $X$ but not in any of the set $A_n$. Thus

$$X \neq \cup_{n\in N} A_n$$

Therefore $X$ is of the second category.

## 5.3 Uniform Boundedness Principle and its Consequences

Now, we prove another important result called the **Banach Steinhass theorem** which is commonly known as the uniform boundedness principle. It is concerned with a sequence of point-wise bounded sequences of linear operators. The uniform Boundedness Principle was obtained in its general form by S. Banach and Steinhauss in 1927. That is why it is also known as Banach – Steinhaus theorem. This theorem, like open mapping theorem and closed graph theorem, requires the completeness. Further, this theorem is derived from Baires Category theorem. The Principle of Uniform Boundedness asserts that if a sequence of bounded linear operators $T \in B(X,Y)$ where $X$ is a Banach space and $Y$ a normed space, is point wise bounded, then the sequence $\{T_n\}$ is uniformly bounded. In fact, it enables us to determine whether the norms of a given family of bounded linear operators have a finite least upper bound.

**Theorem.(Uniform boundedness Principle)** Let $X$ be Banach space and $Y$ a normed space. Let $T_n$ be a sequences of bounded linear opeartors from $X$ to $Y$ such that, for each $x \in X$,

$\{T_n x : n \in N\}$ is a bounded subset of $Y$. Then the sequence$\{\|T_n\|\}$ of norms of $T_n$ is also bounded.

**Proof.** Let $k$ be any natural number and

$$U_k = \{x \in N: \|T_n x\| \le k, n = 1,2,\dots,\}$$

Then $U_k$ is closed subset of $X$. For if $x \in \overline{U_k}$, then there is a sequence $\{x_n\}$ in $U_k$ which converges to $x$. So $\|T_n x_n\| \Rightarrow \lim_{n\to\infty} \|T_n x_n\| = \left\|T_n(\lim_{n\to\infty} x_n)\right\|$, as $\|.\|$ is continuous.

$$= \|T_n x\| \le k$$

Hence $x \in U_k$, so $U_k = \overline{U_k}$

Also, since each $x \in X$ is in some $U_k$ for some natural number $k$.

$$X \subseteq \cup_{k=1}^{\infty} U_k \subseteq X$$

So that

$$X = \cup_{k=1}^{\infty} U_k \dots\dots\dots..(1)$$

Now $X$ is a Banach space so is complete. Hence by Baire's Category theorem, atleast one of $U_k's$, say $U_{k_0}$, is not nowhere dense in $X$. So $U_{k_0}$ contains an open ball $(x_0, \epsilon)$ , that is

$$B(x_0, \epsilon) \subseteq U_{k_0} \dots\dots\dots..(2)$$

Next, let $0 \ne x$ be any arbitrary point of $X$. Take a point $x' \in X$, such that

$$x' = x_0 + \alpha x \dots\dots\dots..(3)$$

Where $\alpha = \frac{\epsilon}{2\|x\|}$. *Then*

$$\|x' - x_0\| = \frac{\epsilon}{2} < \epsilon$$

So that $x' \in B(X_0, \epsilon) \subseteq U_{k_0}$. Hence

$$\|T_{nx'}\| \le k_0 \dots\dots.(4)$$

Moreover,

$$\|T_n x_0\| \le k_0 \dots\dots.(5)$$

Hence, using (3), we have for all $n, n = 1,2,\dots$ and all $x \in X$,

$$\|T_n x\| = \left\|\frac{T_n(x' - x_0)}{\alpha}\right\| = \frac{1}{|\alpha|}\|T_n(x' - x_0)\|$$

$$\le \frac{1}{|\alpha|}(\|T_n x'\| + \|T_n x_0\|)$$

$$\le \frac{2}{|\alpha|} k_0$$

$$\le \frac{4k_0}{\epsilon}\|x\|$$

Hence,

$$\|T_n\| = \sup_{x \in X_{\|x\| \le 1}} \|T_n x\| \le \frac{4k_0}{\epsilon}$$

Thus,

$\{\|T_n\|\}$ is bounded.

**Theorem.** Let $X$ be a normed space and $S$ be a non empty subset of $X$. Then $S$ is bounded if and only if $f(S)$ is bounded for each bounded linear functional $f$ defined on $X$, i,e for each $f \in X^*$.

**Proof.** Suppose that $S$ is bounded subset of normed space $X$. Then, for some positive real number $k$,

$$\|x\| \le k\ \forall x\ \in S \ldots\ldots\ldots.(1)$$

Since $f: X \to F$ is bounded , there is a positive real number $k_1$ such that

$$|f(x)| \le k_1\|x\|\ \ \forall\ x \in S$$

$$\le k_1 k\ \ \forall\ x \in S$$

Hence, $f(S)$ is bounded.

Coversely suppose that , for each non-empty subset $S$ of $X$ and $f \in X^*$, $f(S)$ is bounded . That is

$$Sup\{|f(x)|: x \in S\} < \infty.$$

Let $X^{**}$ be the second dual of $X$ and for each $x \in X$, $g_x: X^* \to F$ be defined by

$$g_x(f) = f(x)$$

Then the mapping $\phi: X \to X^{**}$ defined by:

$$\phi(x) = g(x), x \in X$$

is the natural embedding of $X$ in $X^{**}$.

To see that $g_x$ is bounded for each $x \in X$ we note that

$$|g_x(f)| = |f(x)| \le \|f\|\|x\|\ \ \forall f \in N^*$$

$$\|g_x\| \le \|x\|$$

So, for each $x \in X$ there is an $f_0 \in X^*$ such that

$\|f_0\|$=1 and $f_0(x) = \|x\|$, (as proved in previous chapter corrolory of Hahn-Banach theorem)

So $\|g_x\| = \sup_{f \in X^*_{\|f\|=1}} |g_x(f)| \ge |g_x(f_0)| = |f_0(x)| = \|x\|$

Hence,

$$\|g_x\| = \|x\|$$

Now , consider the subset

$G = \{g_x: x \in S\}$ of $X^{**}$. Now $X^*$ is complete so is a Banach space.

Also for each $f \in X^*$,

$sup\{|g_x(f)| : g_x \in G\} = sup\{|f(x)|: x \in S\} < \infty$, by assumption,

Hence, by the uniform boundedness principle and using (3), we get

$$su\,p\{\|g_x\|: g_x \in G\} = sup\{\|x\|: x \in S\} < \infty$$

So $S$ is bounded.

## Summary

- Let $X$ be a Banach space and $J: X \to X^{**}$ be the canonical injection from $X$ into $X^{**}$ given by
  $$\big(J(x)\big)(f) = \phi_x(f) = f(x) \quad \forall\, x \in X, f \in X^*.$$
  The space $X$ is reflexive if $J$ is surjective, i,e $J(X) = X^{**}$.
- Finite dimensional spaces are reflexive.
- Every Hilbert space is reflexive.
- $L^1, L^\infty, l^1$ and $l^\infty$ are not reflexive
- $C(K)$=space of continuous functions on an infinite compact metric space $K$ is not reflexive.
- Let $X$ be a metric , a subset $M \subseteq X$ is called rare ( or nowhere dense in $X$) if $\bar{M}$ has no interior point i,e $int(\bar{M}) = \phi$.

- A subset $A$ of a metric space $X$ is said to be of the first category ( meager) if and only if $A$ can be covered by a countable union of its nowhere dense subsets. Otherwise $A$ is said to be of second category (Non- Meager).
- A space $X$ is said to be of the first category if and only if $X$ as a subset of itself can be written as a countable union of nowhere dense subsets. Otherwise $X$ is said to be of the second category.
- Every complete metric space is of second category.
- Let $X$ be Banach space and $Y$ a normed space. Let $T_n$ be a sequences of bounded linear opeartors from $X$ to $Y$ such that, for each $x \in X$, $\{T_n x : n \in N\}$ is a bounded subset of $Y$. Then the sequence$\{ \|T_n\|\}$ of norms of $T_n$ is also bounded.

## Keywords

- Banach Space
- Normed Space
- Reflexive
- Bounded
- First Category
- Second Category
- Interior point
- Linear functional

## Self Assessment

1: Which of the following is true about Reflexive spaces.

A. Finite dimensional spaces are reflexive.

B. Every Hilbert space is reflexive.
C. Both (A) and (B).

D. None of the above.

2: Pick out the correct statement.
A. $l_1$ is not reflexive.

B. $l_1$ is not separable.
C. Both (A) and (B).

D. None of the above.

3: Pick out the correct statement.

A. $l^\infty$ is not reflexive.

B. $l^\infty$ is not separable.
C. Both (A) and (B).

D. None of the above.

4: Baire's Category Theorem is applicable to which of the following spaces?

A. All metric spaces.
B. Only compact metric spaces.
C. Only finite metric spaces.
D. Only complete metric spaces.

5: Which of the following statement is true.

A. A space $X$ is said to be of the first category if and only if $X$ as a subset of itself can be written as a countable union of nowhere dense subsets.

B. Complete metric space is of second category.

C. Both (A) and (B).

D. None of the above.

6: Pick out the correct statement.

A. The set $Q$ of rationals is of First Category.
B. The set $Q$ of rationals is of Second Category.
C. The set $Q$ of rationals is uncountable.
D. All of the above are true.

7: Which of the following statements is not true regarding the Uniform Boundedness Principle?

A. It is also known as the Banach-Steinhaus Theorem.
B. It is applicable only to finite-dimensional normed spaces.
C. It guarantees pointwise convergence of a sequence of bounded linear operators.
D. It is a fundamental result in functional analysis.

8: Which of the following spaces is reflexive?
A. Euclidean space $R^n$.
B. $L^1, L^\infty$.
C. Hilbert space.
D. None of the above.

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | C | 2. | A | 3. | C | 4. | D | 5. | A |
| 6. | A | 7. | B | 8. | C | | | | . |

## Review Questions

1. Define a reflexive space.
2. Give an example of a Banach space that is not reflexive.
3. What is the concept of a "meager" or "nowhere dense" set in the context of Baire's Category Theorem?

4. State Baire's Category Theorem.
5. What is the concept of a "meager" or "nowhere dense" set in the context of Baire's Category Theorem.

## Further Readings


- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
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# Unit 06: Inner Product Space. Hilbert Space

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Inner Product Space
- Schwarz Inequality
- Parallelogram identity
- Polarization identity
- Continuity of an inner Product
- Hilbert Space.

## Introduction

In the preceding chapters, we studied normed and Banach spaces. These spaces enjoy linear properties as well as metric properties. Although the norm on a linear space generalizes the concept of length of a vector, but the main geometric concept, missing in abstract normed and Banach spaces, is the angle between two vectors. In fact these spaces are still too general to yield

a really rich theory of operators. In this chapter, we study linear spaces having an inner product, a generalization of usual dot product on finite dimensional linear spaces. The concept of an inner

product on a linear space leads to an inner product space and a complete inner product space

(Hilbert Space) is a special type of normed space (Banach space) which possesses an additional

Structure of an inner product.

The theory of Hilbert spaces was initiated in 1912 by a German mathematician, David Hilbert (1863–1943) in his work on quadratic forms in infinitely many variables, which he applied to the theory of integral equations. Years later John Von Neumann (1903–1957) first formulated an axiomatic theory of Hilbert spaces and developed the modern theory of operators. His remarkable contribution to this area has provided the mathematical foundation of quantum mechanics. His work provided a physical interpretation of quantum mechanics in terms of abstract relations in an infinite dimensional Hilbert spaces.

## 6.1 Inner Product Space

**Definition.** Let $X$ be a linear space over a field $F$ $(R\ or\ C)$. An inner product in $X$ is a function

$$\langle .,. \rangle : X \times X \to F$$

So that with each pair $x, y$ in $X$, a scalar to be denoted by $\langle x, y\rangle$ is associated satisfying the conditions

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0, x \in X$,

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle\ \forall\ x, y, z \in X$,

(iii) $\langle \alpha x, y \rangle = \alpha\langle x, y \rangle\ \forall\ x, y, \in X$ and $\alpha \in F$

(iv) $\langle x, y\rangle = \overline{\langle y, x \rangle}\ \forall\ x, y, \in X$

Where $\overline{\langle x, y \rangle}$ denotes the complex conjugate of $\langle x, y \rangle$.

The pair $(X, \langle .,. \rangle)$ is called an inner product space.

**Remark.** An inner product on $X$ defines a norm on $X$ given by

$\|x\| = \sqrt{\langle x, x\rangle}$, for all $x \in X$

And a metric on $X$ given by

$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}\ \forall\ x, y, \in X.$

Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

**Some Consequences of definition of inner product**

(a) For all $x, y, z \in X$ and $\alpha, \beta \in F$,

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

Also

$$\langle 0, z\rangle = \langle 0. x, z\rangle = 0. \langle x, z\rangle = 0\ \forall\ z \in X$$

(b) For all $x, y \in X$ and $\alpha \in F$,

$$\langle x, \alpha y\rangle = \overline{\alpha}\ \langle x, y\rangle$$

As

$$\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha\langle y, x\rangle}$$

$$= \overline{\alpha}\overline{\langle y, x \rangle}$$

$$= \overline{\alpha}\ \langle x, y\rangle.$$

(c) $\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$

**Remark.** If $X$ is a real inner product space , then

$$\langle x, y\rangle = \langle y, x\rangle\ \forall\ x, y \in X.$$

**Example:** The space $l_2^n$ with the inner product of two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i\ \overline{y_i}$$

is an inner product space.

**Example:** The space $l_2$ with the inner product of two vectors $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

is an inner product space.

**Remark.** An inner product is also called a pre-Hilbert space.

## 6.2 Further Properties of Inner Product Space

### Schwarz Inequality

If X is an inner product space, then

$$|\langle x, y\rangle| \le \sqrt{\langle x, x\rangle}.\ \sqrt{\langle y, y\rangle}\ \forall\ x, y \in X \text{.........}\ (1)$$

the equality holds Iff $x$ and y are linearly dependent.

Proof. If y = 0, then (1) holds because $\langle x, 0\rangle = 0$.

Also, if $x = 0$, then (1) holds because $\langle 0, y\rangle = 0$.

Now let $y \neq 0$. For every scalar α, we have

$\langle x - \alpha y, x - \alpha y\rangle \ge 0$

$$\Rightarrow \langle x, x\rangle + \langle x, -\alpha y\rangle + \langle -\alpha y,\ \rangle + \langle -\alpha y, -\alpha y\rangle \ge 0$$

$$\Rightarrow \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha\langle y,\ x\rangle + \alpha\bar{\alpha}\langle y, y\rangle \ge 0 \text{ ........... } (2)$$

Choose $\alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$ , we have $\bar{\alpha} = \frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}$

And so $\alpha\bar{\alpha} = \frac{\langle x,y\rangle}{\langle y,y\rangle}.\frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle} = \frac{|\langle x,y\rangle|^2}{|\langle y,y\rangle|^2}$

From equation (2), we obtain

$$\langle x, x\rangle - \frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}.\ \langle x, y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle}\langle y, x\ \rangle + \frac{|\langle x,y\rangle|^2}{|\langle y,y\rangle|^2}\langle y, y\rangle \ge 0$$

$$\Rightarrow \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} + \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} \ge 0$$

$$\Rightarrow \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} \ge 0$$

$$\Rightarrow \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} \le \langle x, x\rangle$$

$$\Rightarrow |\langle x, y\rangle|^2 \le \langle x, x\rangle\ \langle y, y\rangle$$

$$\Rightarrow |\langle x, y\rangle| \le \sqrt{\langle x, x\rangle}.\ \sqrt{\langle y, y\rangle}$$

Which is the required result.

Next, we see that the equality in (1) holds iff y = 0

From (2), we have

$|\langle x, y\rangle| \le \sqrt{\langle x, x\rangle}.\ \sqrt{\langle y, y\rangle}$ iff $\langle x - \alpha y, x - \alpha y\rangle \ge 0$

iff $x - \alpha y = 0$

iff $x = \alpha y$

iff $x$ and y are linearly dependent.

**Corollary.** Let X be an inner product space, then for any $x$ and y in X, we have

$$\sqrt{\langle x + y, x + y\rangle} \le \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}$$

**Proof.** We can write

$$\langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \overline{\langle x, y\rangle} + \langle y, y\rangle$$
$$= \langle x, x\rangle + 2\text{Re}\langle x, y\rangle + \langle y, y\rangle$$
$$\leq \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle$$
$$\leq \langle x, x\rangle + 2\left[\sqrt{\langle x, y\rangle}.\sqrt{\langle y, y\rangle}\right] + \langle y, y\rangle$$
$$= \left[\sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}\right]^2$$
$$\Rightarrow \langle x+y, x+y\rangle \leq \left[\sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}\right]^2$$

Taking Square root on both sides, we get

$$\sqrt{\langle x+y, x+y\rangle} \leq \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}.$$

**Theorem.** if X is an inner product space, then $\sqrt{\langle x, x\rangle}$ has the properties of norm.

(OR)

**Proof.** Let X be an inner product space . Define a map $\|\,.\,\| : X \to$ R by $\|x\| = \sqrt{\langle x, x\rangle}\ \forall\, x \in X.$

In order to show that $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on the inner product space X, we need to show that it satisfies all the conditions of a norm.

Now (i) For any x $\in X$, $\|x\| = \sqrt{\langle x, x\rangle}$ which gives

$$\|x\|^2 = \langle x, x\rangle \geq 0 \text{ (By definition)}$$
$$\Rightarrow \|x\|^2 \geq 0 \ \Rightarrow \|x\| \geq 0$$

Also $\|x\| = \sqrt{\langle x, x\rangle} \Rightarrow \|x\|^2 = \langle x, x\rangle = 0$ iff x = 0

$\Rightarrow \|x\|^2 = 0$ iff x = 0 ; i.e. $\|x\| = 0$ iff x = 0

(ii) By definition,

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle}$$
$$\Rightarrow \|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle$$
$$= \alpha\bar{\alpha}\langle x, x\rangle$$
$$= |\alpha|^2\|x\|^2$$
$$\Rightarrow \|\alpha x\| = |\alpha|\|x\|$$

(iii) For x, y $\in X$, we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$
$$= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|.\|y\| + \|y\|^2$$
$$\leq (\|x\| + \|y\|)^2$$

Hence

$$\|x\| + \|y\| \leq \|x\| + \|y\|$$

We see that all the conditions of a norm are satisfied .Thus $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on X and hence (X, $\|\,.\,\|$) is a norm linear space.

**Remark.** The Schwarz inequality can now be written in the form

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

## Parallelogram Law or Identity for Inner Product Spaces.

If X is an inner product space, then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for x and y in X.}$$

(OR)

In inner product spaces, parallelogram law holds.

**Proof.** We have

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y, x + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y, x + \langle y, y\rangle$$

$$= 2\langle x,x\rangle + 2\langle y,y\rangle$$

$$= 2\|x\|^2 + 2\|y\|^2$$

$$= 2(\|x\|^2 + \|y\|^2)$$

Hence,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

## Polarization Identity

If X is an inner product space, then for $x$, y in X, we have

$$\sum_{r=0}^{3} i^r \|x + i^r y\|^2 = 4\langle x,y\rangle.$$

**Proof.** We have

$$\sum_{r=0}^{3} i^r \|x + i^r y\|^2 = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

$$= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - [\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle]$$

$$+ i[\langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle + \langle iy,iy\rangle] - i[\langle x,x\rangle - \langle x,iy\rangle - \langle iy,x\rangle + \langle iy,iy\rangle]$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle$$

$$+ i\langle x,x\rangle + i\langle x,iy\rangle + i\langle iy,x\rangle + i\langle iy,iy\rangle - i\langle x,x\rangle + i\langle x,iy\rangle + i\langle iy,x\rangle - i\langle iy,iy\rangle$$

$$= 2\langle x,y\rangle + 2\langle y,x\rangle + 2i\langle x,iy\rangle + 2i\langle iy,x\rangle$$

$$= 2\langle x,y\rangle + 2\langle y,x\rangle + 2i\bar{i}\langle x,y\rangle - 2i\bar{i}\langle y,x\rangle$$

$$= 2\langle x,y\rangle + 2\langle y,x\rangle + 2\langle x,y\rangle - 2\langle y,x\rangle$$

$$= 4\langle x,y\rangle$$

Hence

$$\sum_{r=0}^{3} i^r \|x + i^r y\|^2 = 4\langle x,y\rangle.$$

**Remark.** If $X$ is a real inner product space, then the polarization identity becomes:

$4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 \quad$ for $x, y \in X$.

The Schwarz inequality is quite important and will be used in proofs over and over again . Another frequently used property is the continuity of inner product.

## Continuity of Inner Product Space

**Theorem.** Let $X$ be any inner product space and $\{x_n\}$ , $\{y_n\}$ be any sequences in $X$ such that $x_n \to x,\ y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$

**Proof.** For any natural number $n$, we have from definition of inner product spaces

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$$
$$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$$
$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$
$$\leq \|x_n\| \; \|y_n - y\| + \|x_n - x\| \; \|y\|$$

(by Cauchy Schwarz Inequality)

Thus, if $x_n \to x,\ y_n \to y$ then $\{x_n\}$ is bounded and

$$\|x_n - x\| \to 0, \|y_n - y\| \to 0 \text{ as } n \to \infty$$

So that

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0 \text{ as } n \to \infty$$

Hence

$$\langle x_n, y_n\rangle \to \langle x, y\rangle$$

**Theorem. .** Let $X$ be any inner product space and if $\{x_n\}$ , $\{y_n\}$ are Cauchy sequences in $X$, then $\langle x_n, y_n\rangle$ is a convergent sequence in $F$, where $F = R \text{ or } C$.

**Proof.** Suppose that $\{x_n\}$ , $\{y_n\}$ are Cauchy sequences in $X$. Then for all natural numbers $m, n$ we have

$$\|x_n - x_m\| \to 0 , \|y_n - y_m\| \to 0 \text{ as } m, n \to \infty$$

Hence , as above ,

$$|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y_m\rangle + \langle x_n, y_m\rangle - \langle x_m, y_m\rangle|$$
$$= |\langle x_n, y_n - y_m\rangle + \langle x_n - x_m, y_m\rangle|$$
$$\leq |\langle x_n, y_n - y_m\rangle| + |\langle x_n - x_m, y_m\rangle|$$
$$\leq \|x_n\| \; \|y_n - y\| + \|x_n - x\| \; \|y\|$$

(by Cauchy Schwarz Inequality)

Since every Cauchy sequence is bounded, the right hand side of above equation tends to 0 as $m, n \to \infty$. Hence

$\langle x_n, y_n\rangle$ is a Cauchy sequence in $F$. Since $F$ is $R \text{ or } C$ , this sequence converges in $F$.

## 6.3 <u>Hilbert Space</u>

**Definition.**
A complete inner product spac is called Hilbert space. Or an inner product space in which every Cauchy sequence converges is said to be Hilbert Space.

**Example:** Show that the Euclidean space $R^n$ is a Hilbert space with inner product defined by $\langle x, x\rangle = \|x\|^2 = \sum_{i=1}^{n} |x_i|^2$.

Solution:- Let $\{x_n\}$ be a Cauchy sequence in $R^n$ where $x_n = \{x_i^{(n)}\}_{i=1}^{\infty}$, then for any $\epsilon > 0$ there exists $n_0 \in N$ such that

$$\|x_n - x_m\| = \sqrt{\langle x_n - x_m, x_n - x_m\rangle} < \epsilon \,; \forall \mathrm{m}, \mathrm{n} \geq \mathrm{n}_0$$

$$\Rightarrow \sqrt{\sum_{\mathrm{i}=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^2} < \epsilon \,; \forall \mathrm{m}, \mathrm{n} \geq \mathrm{n}_0$$

$$\Rightarrow \left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon\,; \forall m, n \geq n_0$$

$\Rightarrow \{x_i^{(n)}\}$ is a Cauchy sequence in $R$ and since $R$ is complete therefore $x_i^{(n)} \to x_i \in R$, then there exists a natural number $n_i \in N$ such that $\left|x_i^{(n)} - x_i\right| < \frac{\epsilon}{\sqrt{p}}\,; \forall\, n \geq n_i$

$$\Rightarrow \left|x_1^{(n)} - x_1\right| < \frac{\epsilon}{\sqrt{p}}\,; \forall\, n \geq n_1$$

$$\left|x_2^{(n)} - x_2\right| < \frac{\epsilon}{\sqrt{p}}\,; \forall\, n \geq n_2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left|x_n^{(n)} - x_n\right| < \frac{\epsilon}{\sqrt{p}}\,; \forall n \geq n_n$$

If $x = (x_1, x_2, \dots, x_n)$ then $x \in R^n$.

Let $n' = max(n_1, n_2, \dots, n_n)$ then for the above expression we have

$$\|x_n - x\| = \sqrt{\sum_{i=1}^{n} \left|x_i^{(n)} - x_i\right|^2}$$

$$\Rightarrow \|x_n - x\| = \sqrt{\sum_{i=1}^{n} \left|x_1^{(n)} - x_1\right|^2 + \sum_{i=1}^{n} \left|x_2^{(n)} - x_2\right|^2 + \cdots \sum_{i=1}^{n} \left|x_n^{(n)} - x_n\right|^2}$$

$$\Rightarrow \|x_n - x\| < \sqrt{\frac{\epsilon^2}{n} + \frac{\epsilon^2}{n} + \cdots \frac{\epsilon^2}{n}} = \sqrt{\frac{n\epsilon^2}{n}}\,; n \geq n'$$

$$\Rightarrow \|x_n - x\| < \epsilon\,; n \geq n' \Rightarrow \sqrt{\langle x_n - x_m, x_n - x_m\rangle} < \epsilon\,; n \geq n'$$

This shows that $x_n$ converges in $R^n$. Hence $R^n$ is a Hilbert space.

Similarly we can show that $C^n$ is a Hilbert space with complex sequence.

**Example:** The space $l_2$ of all complex sequences $x = \{x_i\}$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ is an inner product space under inner product defined by $\langle x\,, y\rangle = \sum_{i=1}^{n} x_i\, \overline{y_i}\,; y = \{y_i\} \in l_2$.

We also know that $l_2$ is complete , hence $l_2$ is a Hilbert space.

**Example:** Every finite dimensional inner product space is a Hilbert Space.

Because every finite dimensional inner product space is a finite dimensional normed linear space and we know that every finite dimensional normed linear space is complete.

**Example:** Space $l^p$ with $p \neq 2$ is not an inner product space , hence not a Hilbert space.

**Example:** The space $C[a, b]$ is not an inner product space , hence not not a Hilbert space.

## Summary

Let $X$ be a linear space over a field $F$ $(R\ or\ C)$. An inner product in $X$ is a function

$$\langle .,. \rangle : X \times X \to F$$

So that with each pair $x, y$ in $X$, a scalar to be denoted by $\langle x, y \rangle$ is associated satisfying the conditions

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0, x \in X$,

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \; \forall \; x, y, z \in X$,

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \; \forall \; x, y \in X$ and $\alpha \in F$

(iv) $\langle x, y \rangle = \overline{\langle y, x \rangle} \; \forall \; x, y \in X$ .

- The space $l_2$ with the inner product of two vectors $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

is an inner product space.

- An inner product is also called a pre-Hilbert space.
- If X is an inner product space, then

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle}. \sqrt{\langle y, y \rangle} \; \forall \; x, y \in X.$$

(Cauchy Schwarx Inequality)

- If X is an inner product space, then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for } x \text{ and y in X.}$$

(Parallelogram law)

- if X is an inner product space, then $\sqrt{\langle x, x \rangle}$ has the properties of norm.
- The Schwarz inequality can also be be written in the form

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

- If X is an inner product space, then for $x$ , y in X, we have $\sum_{r=0}^{3} i^r \|x + i^r y\|^2 = 4\langle x, y \rangle$ (Polarization Identity)
- If $X$ is a real inner product space, then the polarization identity becomes: $4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2$ for $x, y \in X$.
- Let $X$ be any inner product space and $\{x_n\}$ , $\{y_n\}$ be any sequences in $X$ such that $x_n \to x$, $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$
- Let $X$ be any inner product space and if $\{x_n\}$ , $\{y_n\}$ are Cauchy sequences in $X$, then $\langle x_n, y_n \rangle$ is a convergent sequence in $F$, where $F = R \text{ or } C$.
- A complete inner product spac is called Hilbert space. Or an inner product space in which every Cauchy sequence converges is said to be Hilbert Space.
- Euclidean space $R^n$ is a Hilbert space with inner product defined by

$$\langle x, x \rangle = \|x\|^2 = \sum_{i=1}^{n} |x_i|^2.$$

## Keywords

- Inner product space
- Norm
- Cauchy-Schwarz inequality
- Polarization identity
- Parallelogram law
- Continuity
- Cauchy sequence
- Hilbert space

## Self Assessment

1: Pick the INCORRECT statement:

A. Every Hilbert space is a normed space.

B. Every Banach space is a Hilbert space.

C. Every Banach space is a topological space.

D. Every normed space is a metric space.

2**:** Which of the following is Cauchy-Schwartz inequality.

A. $|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}}. \langle y, y\rangle^{\frac{1}{2}}$

B. $|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} + \langle y, y\rangle^{\frac{1}{2}}$

C. $|\langle x, y\rangle| \geq \langle x, x\rangle^{\frac{1}{2}}. \langle y, y\rangle^{\frac{1}{2}}$

D. $|\langle x, y\rangle| \leq \langle x, x\rangle. \langle y, y\rangle$ .

3: Which of the following is known as Parallelogram law?

A. $\|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$

B. $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

C. $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

D. $\|x + y\|^2 - \|x - y\|^2 = 2\|x\|^2 + \|y\|^2$

4: If X is an inner product space, then for x , y in X, we have

$\sum_{r=0}^{3} i^r \|x + i^r y\|^2 =$

A. $5(x, y)$

B. $4(x, y)$

C. $3(x, y)$

D. None of these.

5: An inner product is also called a:

A. Pre-Hilbert space

B. Hilbert space

C. Complete normed space

D. None of these.

6: The term Hilbert space stands for a :

A. Compact linear space

B. Complete normed space

C. Complete metric space

D. Complete inner product space.

7 **:** In a complex inner product space , the conjugate symmetry property of the inner product is given as **.**

A. $\langle x, y \rangle = \langle y, x \rangle$

B. $\langle x, y \rangle = -\langle y, x \rangle$

C. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

D. None of these.

8: Let V be a real inner product space. Which of the following statements is true?

A. The inner product is always positive definite.

B. The inner product is always symmetric.

C. The inner product is always commutative.

D. The inner product is always associative.

9: Let V be a complex inner product space. Which of the following properties does the inner product satisfy?

A. Conjugate symmetry.

B. Distributive property.

C. Anticommutativity.

D. None of these

10: Let $X$ be a finite-dimensional inner product space. Which of the following statements is always true for any nonzero vector $x \in X$?

A. The norm of $x$ is always equal to 1

B. The norm of $x$ is always greater than or equal to zero.

C. The norm of $x$ is always less than or equal to zero.

D. The norm of $x$ is always positive.

11**:** Let $H$ be a Hilbert space over $R$ and $x, y \in H$, be such that $\|x\| = 4$, $\|y\| = 3$ and $\|x - y\| = 3$. Then $\langle x, y \rangle$ equals:

A. 6

B. 8

C. 10

D. 14

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | B | 2. | A | 3. | C | 4. | B | 5. | A |
| 6. | D | 7. | C | 8. | B | 9. | A | 10. | D |
| 11. | B | | | | | | | | |

## Review Questions

1. What is the definition of an inner product space?

2. Prove that the norm induced by an inner product satisfies the parallelogram law.
3. Give an example of a real inner product space.
4. Define a Hilbert space. How does it differ from a general inner product space?
5. State Cauchy Schwarz inequality for inner product space
6. State Parrallelogram identity for inner product space.

## Further Readings

1. Introductory Functional Analysis With Applications By Erwin Kreyszig.
2. Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
3. J. B Conway, A Course In Functional Analysis.
4. C. Goffman G Pedrick, A First Course In Functional Analysis.
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# Unit 07: Orthogonality of Inner Product Space

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Orthogonality of vectors
- Orthonormal sets
- Complete orthonormal set
- Pythagorean theorem
- Bessel's inequality
- Riesz- Fischer theorem

## Introduction

In this chapter , we introduce the idea of Orthogonality of inner product spaces and establish the basic terminology. We also discuss complete orthonormal sets, Pythagorean theorem, Bessel's inequality, Parseval's identity and Riesz- Fischer theorem. This chapter enables the students to carefully use the concepts of Orthogonality.

## 7.1 Orthogonality Of Vectors

Recall that the dot product of two vectors in the space $R^3$ is zero, the vectors are orthogonal or at least one of the vectors is the zero vector. We generalize this concept in an inner product space.

**Definition.** Let $X$ be an inner product space. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ if $\langle x, y \rangle = 0$.

Such vectors $x$ and $y$ are called orthogonal vectors, written $x \perp y$ (the symbol $\perp$ is pronounced as "per"). Similarly, for subsets $A, B \subset X$, we write $x \perp A$ if $x \perp a\ \forall a \in A$ and $A \perp B$ if $a \perp b, \forall a \in A$ and $b \in B$.

**Observations.**

(i) $x \perp y \Leftrightarrow y \perp x$

(ii) $x \perp 0, \forall x \in X$

(iii) 0 is the only vector in X orthogonal to itself.

(iv) For a subset A of an inner product space X, define the set

$$A^{\perp} = \{x \in X : x \perp A\}$$

We write $(A^{\perp})^{\perp} = A^{\perp\perp}$, $(A^{\perp\perp})^{\perp} = A^{\perp\perp\perp}$ and so on .

(v) $\{0\}^{\perp} = X$ and $X^{\perp} = \{0\}$ i,e 0 is the only vector orthogonal to every vector .

**Proof.** We have

$$\{0\}^{\perp} = \{x \in X : \langle x, 0 \rangle = 0\} = X$$

Since $\langle x, 0\rangle = 0, \forall x \in X$. Also if $x \neq 0$, then $\langle x, x\rangle \neq 0$. In other words , a non zero vector can not be orthogonal to the entore space X. Hence $X^{\perp} = \{0\}$.

(vi) If $A \neq \phi$ is subset of X, then the set $A^{\perp}$ is closed subspace of X. Furthermore, $A \cap A^{\perp}$ is either 0 or empty (when $0 \notin A$).

(vii) If A and B are subsets of X such that $A \subset B$, then $A^{\perp} \supset B^{\perp}$.

**Proof.** Let $x \in B^{\perp}$ then $\langle x, y\rangle = 0, \forall y \in B$ and in particular $\forall x \in A$ since $A \subset B$.This verifies that $x \in A^{\perp}$ . Hence $A^{\perp} \supset B^{\perp}$.

(viii) If A is a subset of X, then $A \subset A^{\perp\perp}$

**Proof.** Let $x \in A$. Then $x \perp A^{\perp}$, which means $x \in (A^{\perp})^{\perp}$

, thus $A \subset A^{\perp\perp}$.

(ix) If $A \neq \phi$ is a subset of X, then $A^{\perp} = A^{\perp\perp\perp}$.

**Example:** $R^n$ is an inner product space with inner product defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

Then the vectors $(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots (0,0,0,\ldots,1)$ are orthogonal, as the inner product of any two of the above vectors is zero.

**Example:** $l^2$ is an inner product space with the inner product defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

Then the vectors $e_1 = (1,0,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, $e_3 = (0,0,1,\ldots,0) \ldots e_i = (0,0,0,\ldots,1,0,0,\ldots), \ldots$ in $l^2$

are orthogonal because $e_i \perp e_j\ \forall i, j$ with $i \neq j$.

## 7.2 Orthonormal Sets

**Definition.** A set $S = \{x_i : i \in I\}$ in an inner product space $X$ is said to be orthonormal if

$$\langle x_i, x_j \rangle = \begin{cases} 0, & if\ i \neq j \\ 1, & if\ \ i = j \end{cases}$$

i,e $\langle x_i, x_j \rangle = \delta_{ij}$, the standard Kronecker delta. In other words the set $S$ is said to be orthonormal if it is orthogonal and $\|x\| = 1$ for every $x \in S$.

**Example:** Let $\{x_i : i \in I\}$ be an orthogonal set in an inner product space $X$, then the set

$$A = \left\{\frac{x_i}{\|x_i\|} : i \in I\right\} \text{ is orthonormal.}$$

**Solution.** Let $\frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|} \in A$, then

$$\left\langle \frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|} \right\rangle = \frac{1}{\|x_i\| \, \|x_j\|} \langle x_i, x_j \rangle$$

$$= \frac{1}{\|x_i\| \, \|x_j\|} \times 0 = 0$$

Thus the inner product of two different elements of $A$ is zero. So that $A$ is orthogonal .

Next, we show that norm of every element of $A$ is 1.

For this let $\frac{x_i}{\|x_i\|} \in A$, then

$$\left\| \frac{x_i}{\|x_i\|} \right\| = \frac{\|x_i\|}{\|x_i\|} = 1$$

This shows that A is orthonormal.

## 7.3 Complete Orthonormal Sets

**Definition.** An orthonormal set S in an inner product space X is said to be complete if there exists no orthonormal set in X of which S is a proper subset.

In other words, S is complete if it is maximal with respect to the property of being normal.

**Note: -** If S is complete orthonormal set, then there does not exist any non-zero vector such that $x \perp S$ and $\|x\| = 1$.

**Example.** In the space $l^2$, the orthonormal set composed of $e_1 = (1,0,0,\dots,)$, $e_2 = (0,1,0,\dots,)$, $e_3 = (0,0,1,\dots,)$ ...

Is a complete orthonormal set.

**Orthonormal sets in Hilbert Spaces:** An orthonormal set in a Hilbert space H is a non-empty subset of H which consists of mutually orthogonal unit vector : that is , it is non empty subset $\{e_i\}$ of H with the following property.

(i) $\langle e_i, e_j \rangle = 0, \text{ if } i \neq j$

(ii) $\langle e_i, e_j \rangle = 1, \text{ if } i = j$

**Note: -** See examples following the definition of orthonormal sets in Inner product spaces.

**Remark.** If $H = \{0\}$ i.e. H contains only the zero element, then it has no orthonormal set. If H contains a non-zero vector $x$, then we can construct $e$ by normalizing x, that is $e = \frac{x}{\|x\|}$, then the single element set $\{e\}$ is clearly an orthonormal set because $\langle e, e \rangle = \|e\|^2 = \left\|\frac{x}{\|x\|}\right\|^2 = \frac{\|x\|^2}{\|x\|^2} = 1$

Generally speaking if $\{x_i\}$ is a non empty set of mutually orthogonal non zero vectors in H, and if the $x_i$'s are normalized by replacing each of them by $e_i = \frac{x_i}{\|x_i\|}$, then the resulting set $\{e_i\}$ is orthonormal s

**Remark.** One of the simple geometric fact about orthogonal vectors is the Pythagorean theorem, which is given as follows.

## 7.4 Pythagorean Theorem

**Theorem.** If x and y are orthogonal vectors in an inner product space X, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2$$

Proof. We have

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \langle x,x\rangle + 0 + 0 + \langle y,y\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2\end{aligned}$$

Similarly, we can show that

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2$$

Hence,

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2.$$

## 7.5 Bessel's Inequality

**Theorem.** Let $S = \{e_1, e_2, \dots, e_n\}$ be an orthonormal set in a Hilbert space $H$. If $x$ is any vector in $H$, then

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2$$

(Bessel's inequality)………..(1)

And $x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i \perp e_j$ for each j

i.e. $x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i \perp S$.

**Proof.** We have : $0 \le \|x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i\|^2$

$$= \langle x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i,\ x - \sum_{j=1}^{n}\langle x, e_j\rangle e_j\rangle$$

$$= \langle x, x\rangle - \langle x, \sum_{j=1}^{n}\langle x, e_j\rangle e_j\rangle - \langle \sum_{i=1}^{n}\langle x, e_i\rangle e_i, x\rangle + \langle \sum_{i=1}^{n}\langle x, e_i\rangle e_i, \sum_{j=1}^{n}\langle x, e_j\rangle e_j\rangle$$

$$= \langle x, x\rangle - \sum_{j=1}^{n}\overline{\langle x, e_j\rangle}\langle x, e_j\rangle - \sum_{i=1}^{n}\langle x, e_i\rangle\langle e_i, x\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, e_i\rangle\overline{\langle x, e_j\rangle}\langle e_i, e_j\rangle$$

$$= \langle x, x\rangle - \sum_{j=1}^{n}|\langle x, e_j\rangle|^2 - \sum_{i=1}^{n}|\langle x, e_i\rangle|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, e_i\rangle\overline{\langle x, e_j\rangle}\langle e_i, e_j\rangle$$

$$= \langle x, x\rangle - \sum_{j=1}^{n}|\langle x, e_j\rangle|^2 - \sum_{i=1}^{n}|\langle x, e_i\rangle|^2 + \sum_{i=1}^{n}\langle x, e_i\rangle\overline{\langle x, e_i\rangle}\langle e_i, e_i\rangle$$

$$= \langle x, x\rangle - \sum_{j=1}^{n}|\langle x, e_j\rangle|^2 - \sum_{i=1}^{n}|\langle x, e_i\rangle|^2 + \sum_{i=1}^{n}|\langle x, e_i\rangle|^2$$

$$\Rightarrow 0 \le \langle x, x\rangle - \sum_{j=1}^{n}|\langle x, e_j\rangle|^2$$

$\Rightarrow \sum_{j=1}^{n}|\langle x, e_j\rangle|^2 \le \|x\|^2$, which is equivalent to (1)

In order to show that $x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i \perp S$, consider any $e_j$ *in* $S$ where $j = 1,2,3,\dots,n$

Then $\langle x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i, e_j\rangle = \langle x, e_j\rangle - \langle \sum_{i=1}^{n}\langle x, e_i\rangle e_i, e_j\rangle$

$$= \langle x, e_j\rangle - \sum_{i=1}^{n}\langle x, e_i\rangle\langle e_i, e_j\rangle$$

$= \langle x, e_j \rangle - \langle x, e_j \rangle \langle e_j, e_j \rangle$

$= \langle x, e_j \rangle - \langle x, e_j \rangle 1$

$= 0$

This shows that $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j$ for each $j$

$\Rightarrow x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp S$

This completes the proof.

**Theorem.** If $\{e_i\}$ is an orthonormal set in a Hilbert space $H$ and if $x$ is any vector in $H$, then the set

$S = \{e_i : \langle x, e_i \rangle \neq 0\}$ is either empty or countable.

**Theorem. (Generalization of Bessel's inequality).**

If $\{e_i\}$ is an orthonormal set in a Hilbert space $H$, then

$$\sum |\langle x, e_i \rangle|^2 \leq \|x\|^2 \quad \textbf{.........(1)}$$

for every vector $x \in H$ .

**Proof.** Let us define a set S as

$$S = \{e_i : \langle x, e_i \rangle \neq 0 \}$$

Then by the above theorem , S is either empty or countable .

If S is empty, then $\langle x, e_i \rangle = 0$, so $\sum |\langle x, e_i \rangle|^2$ is zero and so in this case (1) reduces to $0 \leq \|x\|^2$, which is obviously true.

If S is countable , then S is finite or countably infinite .

**When S is finite .** Let it can be written in the form $S = \{e_1, e_2, \ldots, e_n\}$ for some positive integer n. In this case, we denote $\sum |\langle x, e_i \rangle|^2$ to be $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2$ , which is clearly independent of the order in which the vectors of $S$ are arranged. So inequality (1) reduces to $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$ which is the Bessel's inequality when $\{e_i\}$ is finite orthonormal set as proved already.

**When S is countably infinite .** Let the vectors in S be arranged in some definite order i.e. $S = \{e_1, e_2, \ldots, e_n, \ldots\}$, as by the theory of "absolutely convergent series" we know that if $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$ converges , then every series obtained from this series by rearranging its terms also converges and all such series have the same sum. So we therefore can define $\sum |\langle x, e_i \rangle|^2$ to be $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2$ and it follows that $\sum |\langle x, e_i \rangle|^2$ is a non-negative extended real number , which depends only on S and not on the arrangement of vectors in S. So in this case (1) reduces to

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2 \quad .........(2)$$

Now from Bessel's inequality for finite case, we have:

$$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

It follows that no partial sum of the series on left side of (2) can exceed $\|x\|^2$ and so it is clear that (2) is true .

$$\Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

$$\Rightarrow \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

This completes the proof.

**Theorem.** If $\{e_i\}$ is an orthonormal set in a Hilbert space $H$ and if $x$ is any vector in $H$, then the set

$S = \{e_i : \langle x, e_i \rangle \neq 0\}$ is either empty or countable.

**Theorem.** Let $\{e_i\}$ be an orthonormal set in a Hilbert space H and let x be a vector in H, then

$$x - \sum \langle x, e_i \rangle e_i \perp \{e_i\}$$

**Theorem. (orthonormal bases)** Let $H$ be Hilbert space and let $\{e_i\}$ be an orthonormal set in H, then the following are equivalent

(i) $\{e_i\}$ is complete.

(ii) $x \perp \{e_i\} \Rightarrow x = 0$

(iii) If x is an arbitrary vector in H, then $x = \sum \langle x, e_i \rangle e_i$

(iv) If x is an arbitrary vector in H, then $\|x\|^2 = \sum |\langle x, e_i \rangle|^2$. (Parseval's identity).

**Proof.** (i) ⇒ (ii)

Suppose (i) is true i.e. $\{e_i\}$ is complete .

⇒ $\{e_i\}$ is maximal orthonormal set. On contrary suppose that (ii) is not true, then there exists a vector $x \neq 0$ such that $x \perp \{e_i\}$.

Define $e = \frac{x}{\|x\|}$ , then the set $\{e_i, e\}$ is an orthonormal set, which properly contains $\{e_i\}$, but this contradicts the completeness of $\{e_i\}$. Hence (ii) is true.

(ii) ⇒ (iii)

Suppose that (ii) is true i.e. $x \perp \{e_i\} \Rightarrow x = 0$. Now by above theorem, we have

$x - \sum \langle x, e_i \rangle e_i$ is orthogonal to $\{e_i\}$.

i, e.

$$x - \sum \langle x, e_i \rangle e_i \perp \{e_i\}$$

So by (ii), we get

$$x - \sum \langle x, e_i \rangle e_i = 0$$

Or

$$x = \sum \langle x, e_i \rangle e_i$$

for any vector $x \in H$. Hence (iii) is true.

(iii) ⇒ (iv)

Suppose that (iii) is true i.e. $x = \sum \langle x, e_i \rangle e_i$ for any vector $x \in H$.

Now $x = \sum \langle x, e_i \rangle e_i = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$

Then $\|x\|^2 = \langle x, x \rangle = \langle x, \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \rangle$

$$= \langle x, \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, e_i \rangle e_i \rangle$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \overline{\langle x, e_i \rangle} \langle x, e_i \rangle$$

$\Rightarrow \|x\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$

$= \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$

Using $\sum \langle x, e_i \rangle e_i$ in place of $\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$, we get

$$\|x\|^2 = \sum |\langle x, e_i \rangle|^2.$$

Hence (iv) is true.

Finally (iv) ⇒ (i)

Suppose that (iv) is true i.e. $\|x\|^2 = \|x\|^2 = \sum |\langle x, e_i \rangle|^2$.

We show that (i) is true . On the contrary assume that (i) is not true i.e. $\{e_i\}$ is not complete , then it is properly contained in an orthonormal set $\{e_i, e\}$. So by definition of orthonormal set, we can say that e is orthogonal to $e_i'$s.

Now $\|e\|^2 = \sum |\langle e, e_i \rangle|^2$ by (iv)

$= \sum \|0\|^2$

$= \|0\|$

$= 0$

i.e $\|e\| = 0$ and this contradicts the fact that $\|e\| = 1$.

So our supposition was wrong and hence $\{e_i\}$ is complete .

Hence (i) is true.

This completes the proof.

**Remark.** Let $\{e_i\}$ be a complete orthonormal set and let x be an arbitrary vector in a Hilbert space H. Then the numbers $\langle x, e_i\rangle$ are called the Fourier coefficients of x, the expression $\langle x, e_i\rangle e_i$ is called the Fourier expansion of x and the equation $\|x\|^2 = \sum|\langle x, e_i\rangle|^2$ is called Parseval's equation .

## 7.6 Riesz- Fischer Theorem

**Theorem.** Let $\{e_1, e_2, \dots, e_n, \dots\}$ be an orthonormal set in a Hilbert space $H$. Then, for any sequence $\{C_k\}$ of scalars, the following are equivalent.

(i) $\{c_k\} \in l^2$

(ii) $\sum_{k=1}^{\infty} c_k e_k$ converges in $H$

(iii) there is an element $x \in H$

$$\langle x, e_k\rangle = c_k, k = 1,2, \dots$$

**Proof.** Suppose that $(i)$ is true so that $\{c_k\} \in l^2$. Then

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty$$

For $n = 1,2, \dots$, let

$$s_n = \sum_{k=1}^{n} c_k e_k$$

We first show that $\{s_n\}$ is a Cauchy sequence in $H$. For this consider the expression $\|s_m - s_n\|, m > n$ ,

We have $\|s_m - s_n\|^2 = \langle s_m - s_n, s_m - s_n\rangle$

$= \langle \sum_{k=n+1}^{m} c_k e_k , \sum_{p=n+1}^{m} c_p e_p \rangle$

$= \sum_{k=n+1}^{m} |c_k|^2,$

Using the orthogonality of $e_{n+1}, e_{n+2}, \dots, e_m$. Since the series $\sum_{k=1}^{\infty} |c_k|^2$ converges in $F$, by Cauchy's criterion of convergence ,

$$\sum_{k=n+1}^{m} |c_k|^2 \to 0 \ as \ m, n \to \infty$$

So $\{s_n\}$ is a Cauchy sequence in $H$. As $H$ is complete , $s_n \to x \in H$

Thus

$\sum_{k=1}^{\infty} c_k e_k$

converges in $H$. So $(i) \Rightarrow (ii)$.

Next suppose that $(ii)$ is satisfied so that the series $\sum_{i=1}^{\infty} c_i e_i$ converges to $x \in H$.

Then

$x = \sum_{i=1}^{\infty} c_i e_i$

So that , for $k = 1,2, \dots$

$$\langle x, e_k\rangle = \langle \sum_{i=1}^{\infty} c_i e_i , e_k\rangle$$

$$= \sum_{i=1}^{\infty} c_i \langle e_i, e_k \rangle$$
$$= c_k$$

Hence $(iii)$ is satisfied .

Lastly suppose that $(iii)$ holds. Then , by Bessels inequality,

$$\sum_{k=1}^{\infty} |c_k|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\| < \infty$$

So that $\{c_k\} \in l^2$. Hence $(iii) \Rightarrow (i)$.

## Summary

- Let X be an inner product space. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ if $\langle x, y \rangle = 0$.

Such vectors x and y are called orthogonal vectors, written $x \perp y$ (the symbol $\perp$ is pronounced as "per"). Similarly, for subsets $A, B \subset X$, we write $x \perp A$ if $x \perp a \, \forall a \in A$ and $A \perp B$ if $a \perp b, \forall a \in A$ and $b \in B$.

- $x \perp y \Leftrightarrow y \perp x$.
- $x \perp 0, \forall x \in X$.
- 0 is the only vector in X orthogonal to itself.
- For a subset A of an inner product space X, define the set

$$A^{\perp} = \{x \in X : x \perp A\}$$

We write $(A^{\perp})^{\perp} = A^{\perp\perp}$, $(A^{\perp\perp})^{\perp} = A^{\perp\perp\perp}$ and so on .

- $\{0\}^{\perp} = X$ and $X^{\perp} = \{0\}$ i,e 0 is the only vector orthogonal to every vector .
- If $A \neq \phi$ is subset of X, then the set $A^{\perp}$ is closed subspace of X. Furthermore, $A \cap A^{\perp}$ is either 0 or empty (when $0 \notin A$).
- If A and B are subsets of X such that $A \subset B$, then $A^{\perp} \supset B^{\perp}$.
- If A is a subset of X, then $A \subset A^{\perp\perp}$.
- If $A \neq \phi$ is a subset of X, then $A^{\perp} = A^{\perp\perp\perp}$.
- $R^n$ is an inner product space with inner product defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

- Then the vectors $(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots (0,0,0,\ldots,1)$ are orthogonal, as the inner product of any two of the above vectors is zero.

- A set $S = \{x_i : i \in I\}$ in an inner product space $X$ is said to be orthonormal if

$$\langle x_i, x_j \rangle = \begin{cases} 0, & if \ i \neq j \\ 1, & if \ \ i = j \end{cases}$$

i,e $\langle x_i, x_j \rangle = \delta_{ij}$, the standard Kronecker delta. In other words the set $S$ is said to be orthonormal if it is orthogonal and $\|x\| = 1$ for every $x \in S$.

- An orthonormal set S in an inner product space X is said to be complete if there exists no orthonormal set in X of which S is a proper subset.
- In the space $l^2$, the orthonormal set composed of

$e_1 = (1,0,0,\ldots,), e_2 = (0,1,0,\ldots,), e_3 = (0,0,1,\ldots,) \ldots$ is a complete orthonormal set.

- If x and y are orthogonal vectors in an inner product space X, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2$$

(Pythagorean theorem)

- Let $S = \{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in a Hilbert space $H$. If $x$ is any vector in $H$, then

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2$$

(Bessel's inequality)

- **(**Generalization of Bessel's inequality).

  If $\{e_i\}$ is an orthonormal set in a Hilbert space $H$, then

$$\sum |\langle x, e_i\rangle|^2 \le \|x\|^2$$

- If $\{e_i\}$ is an orthonormal set in a Hilbert space $H$ and if $x$ is any vector in $H$, then the set $S = \{e_i : \langle x, e_i\rangle \neq 0\}$ is either empty or countable.
- Let $\{e_i\}$ be an orthonormal set in a Hilbert space H and let x be a vector in H, then

$$x - \sum \langle x, e_i\rangle e_i \perp \{e_i\}$$

## Keywords

- Orthogonality
- Inner product space
- Complete orthonormal set
- Hilbert space
- Pythagorean theorem
- Bessel's inequality
- Orthonormal bases
- Parseval's identity
- Riesz- Fischer theorem

## Self Assessment

**1:** Two Vectors $x, y$ in an inner product space are orthogonal if :

A. $\langle x, y\rangle \neq 0$

B. $\|x\| = \|y\| = 1$

C. $\langle x, y\rangle = 0$

D. None of these.

**2:** If Two vectors $x, y$ in an inner product space are orthogonal, then

A. $\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$

B. $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

C. $\|x+y\|^2 = 0$

D. None of these.

**3:** Let $M$ be a non empty subset of an inner product space $X$. Which of the following is not true.

A. $M^{\perp} = M^{\perp\perp\perp}$

B. $M \subset M^{\perp\perp}$

C. $M = M^{\perp\perp}$

D. If $\bar{M} = X$, then $M^{\perp} = \{0\}$

4 : In an orthonormal set of vectors, what is the inner product of any vector with itself?

A. 0

B. -1

C. 1

D. It depends on the vector.

5: If two vectors are orthogonal, what can be said about their inner product?

A. It is always zero.

B. It is always one.

C. It is undefined for orthogonal vectors.

D. None of these.

6: In a Hilbert space, what is the significance of a complete orthonormal set?

A. It forms a basis for the Hilbert space.

B. It is used for dimension reduction.

C. It only provides a partial basis,

D. None of these .

7: In the context of Hilbert spaces, what does Bessel's inequality state?

A. It provides a bound on the norm of a vector in a Hilbert space.

B. It states that the sum of the squared coefficients of a vector with respect to to an orthonormal set is bounded by the norm of the vector.

C. It defines the inner product between two vectors in a Hilbert space.

D. It establishes the existence of an orthonormal basis for any Hilbert space.

8: Let $S = \{e_1, e_2, \dots, e_n\}$ be an orthonormal set in a Hilbert space $H$. If $x$ is any vector in $H$, then which of the following is true.

A. $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \geq \|x\|^2$

B. $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 < \|x\|^2$

C. $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$

D. None of these .

9: If $\{e_i\}$ is an orthonormal set in a Hilbert space $H$ and if $x$ is any vector in $H$, then the set

$S = \{e_i : \langle x, e_i \rangle \neq 0\}$ is:

A. Non empty and uncountable

B. Non empty

C. Uncountable

D. Either empty or countable

10: The Riesz-Fischer theorem provides a characterization of which type of space?

A. Hilbert spaces
B. Normed spaces
C. Metric space
D. None of these

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | C | 2. | B | 3. | C | 4. | C | 5. | A |
| 6. | A | 7. | B | 8. | C | 9. | D | 10. | A |

## Review Questions

1. What does it mean for two vectors to be orthogonal?
2. Define a complete orthonormal set in a Hilbert space.
3. State Bessel's inequality in its general form, both for finite and countably infinite sets of orthogonal functions.
4. State Parseval’s Identity.
5. State Pythagorean theorem.
6. State Riesz- Fischer Theorem.

## Further Readings

1. Introductory Functional Analysis With Applications By Erwin Kreyszig.
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3. J. B Conway, A Course In Functional Analysis.
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# Unit 08: Open Mapping Theorem and Closed Graph Theorem

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Open mapping Theorem and its Applications
- Closed graph theorem
- Weak and strong convergence
- Convergence of Sequence of operators and functionals.

## Introduction

In this chapter, we discuss some very basic theorems of fundamental importance in functional analysis. These theorems include Open Mapping theorem , Closed graph theorem. Further we discuss about weak and strong convergence . Finally, we discuss about Convergence of sequences of operators and functionals.

## 8.1 Open Mapping Theorem and its Applications

We have discussed the Hahn-Banach theorem and the uniform boundedness theorem and shall now approach the third "big" theorem in this chapter, ***the Open mapping theorem.*** It will be concerned with open mappings. These are mappings such that the image of every open set is an open set .

More specifically, the open mapping theorem states conditions under which a bounded linear operator is an open mapping. As in the uniform boundedness theorem we again need completeness, and the present theorem exhibits another reason why Banach spaces are more satisfactory than incomplete normed spaces. The theorem also gives conditions under which the inverse of a hounded linear operator is bounded. The proof of the open mapping theorem will be based on Baire's category theorem.

Before proving the Open Mapping theorem , we first know the following definition and lemma’s .

1. A mapping $f: X \to Y$, where $X$ and $Y$ are topological spaces, is said to be open mapping if $f$ maps open subsets of $X$ into open open subsets of $Y$.

**Lemma.** Let $X$ be a normed linear space and $B(x_0, r)$ be an open ball in $X$. Then

$B(x_0;\ r)$=$x_0 + r\,B(0;1)$

**Proof.** By definition

$$B(x_0;r) = \{x \in X : \|x - x_0\| < r\}$$
$$= \{x \in X : \|z\| < r, where\ z = x - x_0\}$$
$$= \{x \in X : \|z\| < r, where\ x\ = z + x_0\}$$
$$= \{x_0 + z \in X: \|z\| < r\}$$
$$= x_0 + \{z \in X: \|z\| < r\}$$
$$= x_0 + \{z \in X : \left\|\frac{z}{r}\right\| < 1\}$$
$$= x_0 + \{z \in X : \|z'\| < 1\ where\ z' = \frac{z}{r}\}$$
$$= x_0 + \{z \in X : \|z'\| < 1\ where\ z = z'r\}$$
$$= x_0 + \{rz' \in X: \|z'\| < 1\}$$
$$= x_0 + r\{z' \in X: \|z'\| < 1\}$$
$$= x_0 + r\{z' \in X: \|z' - 0\| < 1\}$$
$$= x_0 + r\,B(0,1)$$

i.e. $B(x_0;r) = x_0 + rB(0,1)$

**Remark.** In particular $B(0;r) = rB(0;1)$

**Lemma.** Let $T$ be a bounded linear operator from a Banach space $X$ into a Banach space $Y$. Then for each open ball $B_0 = B(0,1) \subset X$, the image $T(B_0)$ contains an open ball in $Y$ with centre at origion.

**Theorem. ( The open mapping theorem)**

A bounded linear operator $T$ from a Banach space $X$ into a Banach space $Y$ is an open mapping.

**Proof.** Let $T: X \to Y$ be a bounded linear operator froam a Banach space $X$ into a Banach space $Y$. In order to show that $T$ is an open mapping , we need to show that for any open set $A \subseteq X$, the image of $A$ under $T$ is open in $Y$. For this let $y \in T(A)$: since $T$ is an operator , so there exists $x \in A$ such that $y = Tx \in T(A)$.

It is enough to show that $T(A)$contains an open ball around $y = Tx$.

Since A is open in $X$ ;and $X - A \leq 0$ by definition , it contains an open ball with centre $x$ and radius $r$

i.e.

$$B(x\,;r) \subseteq A.$$

We know by lemma (1) above that:

$$B(x;r) = x + r\,B(0;1)\text{………..(1)}$$

By lemma (2) above , for the open ball $B(0;1)$ in $X$, there is an open ball $B'(0,r')$ with centre at origin, in $Y$ such that

$$B'(0\,;r') \subseteq T(B(0;1))$$
$$\subseteq rT\big(B(0;1)\big)$$
$$= T(B(0;r))\text{……….(2)}$$
$$= y + r'B'(0,1)\ \text{by (1)}$$

Hence $B'(y;r') = y + B'(0;r')$

$= y + T(B(0;r))$ by (2)

$= Tx + T(B(0;r))$

$= T(x + B(0;r))$

$= T(x + rB(0,1))$

$\subseteq T(A)$

i.e. $B'(y;r') \subseteq T(A)$

This shows that $T(A)$ contains an open ball around $y = Tx$. Consequently $T(A)$ is open in $Y$ and hence $T$ is an open mapping.

**Corollary.** Let $T: X \to Y$ be a bijective bounded linear operator from a Banach space $X$ into a Banach space $Y$, then $T$ is homeomorphism.

**Proof.** We recall that $T$ is a homeomorphism if

(i) $T$ is continuous .

(ii) $T$ is bijective.

(iii) $T^{-1}$ is Continuous.

Since $T$ is continuous ( as $T$ is bounded ) and bijective, so $T^{-1}: Y \to X$ exists.

To show that $T^{-1}$ is continuous, let $u$ be an open set in $X$, then $(T^{-1})^{-1}u = Tu$, which is open in $Y$ because $T$ is open by open mapping theorem. So that the image of any open set in $Y$ is open in $X$ under $T^{-1}$, showing that $T^{-1}$ is continuous. Hence $T$ is a homeomorphism.

## 8.2 Closed Graph Theorem

The next important theorem which we shall prove is called the closed graph theorem. Before provimg this theorem we have some definitions.

Let $(X, \|.\|)$ and $(Y, \|.\|_0)$ be normed spaces

$$P = \{(x, y): x \in X, y \in Y\}$$

Define addition and scalar multiplication in $P$ by:

$$(x, y) + (x', y') = (x + x', y + y') \ldots\ldots\ldots..(1)$$

$$\alpha(x, y) = (\alpha x, \alpha y) \ldots\ldots\ldots..(2)$$

For all $x, x' \in X \; and \; y, y' \in Y \; and \; \alpha \in F = (R \; or \; C)$

Then $P$ is a linear space under the addition and scalar multiplication defined above. Next, we define norm on $P$ as follows:

For $(x, y) \in P$, we put

$$\|(x, y)\|_p = \left(\|x\|^p + \|y\|_0^p\right)^{\frac{1}{p}}, 1 \le p < \infty \ldots\ldots\ldots..(3)$$

Then obviously,

$$\|(x, y)\| \ge 0 \; and \; \|(x, y)\| = 0 \;\; \Leftrightarrow x = 0, y = 0$$

And $\|\alpha(x, y)\|_p^p = |\alpha| \;\; \|(x, y)\|_p^p$

$$\|(x, y) + (x', y')\|_p = \|(x + x', y + y')\|_p$$

$$= (\|x + x'\|^p + \|y + y'\|_0^p)^{\frac{1}{p}}$$

$$\le \left(\|x\|^p + \|x'\|^p + \|y\|_0^p + \|y'\|_0^p\right)^{\frac{1}{p}}$$

$$\le \left(\|x\|^p + \|y\|_0^p\right)^{\frac{1}{p}} + (\|x'\|^p + \|y'\|_0^p)^{\frac{1}{p}}$$

(By Minkowski's inequality)

$$\le \|(x, y)\|_p + \|x', y'\|_p$$

Hence $(P, \|.\|_p)$ is a normed space , called the product of the normed spaces $X$ and $Y$.

For $p = 1$ , (3) assumes the form

$$\|(x, y)\|_1 = \|x\| + \|y\|_0.$$

Another norm on $P$ is given as follows: For $(x, y) \in P$, we put

$$\|(x, y)\|_0 = \max(\|x\|, \|y\|_0)$$

It can be established that all these norms on $X \times Y$ are equivalent.

For $p = 2$, we have

$$\|(x, y)\|_2 = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$$

If $X$ and $Y$ are Banach spaces then so is their product $P$. This follows from the observation that

$(x_n, y_n) \to (x, y) \Leftrightarrow x_n \to x , y_n \to y$ .

For any two normed spaces $X$ and $Y$and a mapping $T: X \to Y$ , the set

$$G_T = \{(x, Tx): x \in X\}$$

is called the graph of $T$.

Since $X$ and $Y$ are metric spaces and so are Hausdroff spaces, their product $P = X \times Y$ , under the metric indued by the norm on $P$, is also metric space.

In general, for two topological spaces $X, Y$, the graph $G_T$ of a mapping $T: X \to Y$ may not be a closed subspace of $X \times Y$. Since every normed space is also a Hausdroff space , so in the case of normed spaces,the graph of continuous mapping $T: X \to Y$ is always closed .

**Theorem. (Closed Graph Theorem)**

Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ be a linear operator . Then $T$ is continuous if and only if the graph of $T$ is a closed subspace of $X \times Y$.

**Proof.** Suppose that $T: X \to Y$ is a continuous linear operator . we show that the graph

$$G_T = \{(x, Tx): x \in X\}$$

is closed in $X \times Y$. For this let $(x, y) \in \overline{G_T}$. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in $X$ and $Y$ respectively such that

$$x_n \to x, y_n \to y$$

Since $T$ is continuous and $Y_n = Tx_n$,

$$x_n \to x \Rightarrow Tx_n \to T_x = y$$

Hence $(x, y) = (x, T_x) \in G_T$, thus $G_T$ is closed.

Conversely suppose that, for a linear operator $T: X \to Y$, $G_T$ is closed.Then $G_T$ is a subspace of

$X \times Y$. Since $X$ and $Y$ are Banach spaces and $G_T$ is a closed subspace of the Banach space $X \times Y$, $G_T$ itself is complete and hence is a Banach space .

Consider the mapping $f: G_T \to X$ defined by

$$f(x, Tx) = x \ \forall \ x \in X.$$

Then $f$ is injective and linear . Also, since

$$\|f(x, Tx)\| = \|x\| \leq \|(x, Tx)\|$$

By definition of the product norm, $f$ is continuous. By the open mapping theorem , $f^{-1}$ is continuous and so bounded. Moreover

$$\|Tx\| \leq \|(x, Tx)\| = \|f^{-1}(x)\| \leq \|f^{-1}\|\|x\|$$

Hence $T$ is bounded and so is continuous .

## 8.3 Strong and Weak Convergence

We know that in calculus we define different types of convergence i.e ordinary, conditional, absolute and uniform convergence. This yields greater flexibility in the theory and applications of sequence and series. The situation is similar in functional analysis, and one has an even greater variety of possibilities that turn out to be of practical interest. Here we are concerned with weak convergence .This is basic concept . We present it now since its theory makes essential use of the uniform boundedness theorem which we have already discussed . In fact, this is one of the major application of that theorem.

**Definition. (Strong convergence)** A sequence $\{x_n\}$ in a normed space $X$ is said to be strongly convergent (or convergent in the norm) if there is an $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$

i,e.

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x.$$

$x$ is called the strong limit of $\{x_n\}$, and we say that $\{x_n\}$ converges strongly to $x$ .

Weak convergence is defined in terms of bounded linear functionals on $X$ as follows.

**Definition ( Weak convergence) .** A sequence $\{x_n\}$ in a normed space $X$ is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X$

$\lim_{n\to\infty} f(x_n) = f(x)$

This is written

$$x_n \to x$$

The element $x$ is called the weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to $x$.

Weak convergence has various applications throughout analysis for instance , in the calculus of variation, and general theory of differential equation.

For applying weak convergence one needs to know certain basic properties, which we state in the following lemma .

**Lemma.** Let $\{x_n\}$ be a weakly convergent sequence in a normed space $X$, say $x_n \to x$. Then:

(i) The weak limit $x$ of $\{x_n\}$ is unique.

(ii) Every subsequence of $\{x_n\}$ converges weakly to $x$.

(iii) The sequence $(\|x_n\|)$ is bounded.

**Proof.** (i) Suppose that $x_n \to x$ as well as $x_n \to y$ . Then

$$f(x_n) \to f(x) \text{ as well as } f(x_n) \to f(y).$$

Since $\{f(x_n)\}$ is a sequence of numbers , its limit is unique . Hence $f(x) = f(y)$, that is for every $f \in X'$ we have

$$f(x) - f(y) = f(x - y) = 0$$

$$\Rightarrow x - y = 0$$

This shows that the weak limit is unique.

(ii) follows from the fact that $\{f(x_n)\}$ is a convergent sequence of numbers, so that every subsequence of $\{f(x_n)\}$ converges and has the same limit as the sequence.

(iii) Since $\{f(x_n)\}$ is a convergent sequence of numbers , it is bounded, say $|f(x_n)| \le c_f\ \forall\ n$, where $c_f$ is a constant depending on $f$ but not on $n$ . Using the canonical mapping $C: X \to X''$, we can define $g_n \in X''$ by

$$g_n(f) = f(x_n)\ ,\ f \in X'$$

Then for all $n$,

$$|g_n(f)| = |f(x_n)| \le c_f,$$

that is, the sequence $\{|g_{n(f)}|\}$ is bounded for every $f \in X'$. As we know the dual space $X'$

of a normed space $X$ is a Banach space, so $X'$is complete , the Uniform boundedness theorem is applicable and implies that $\{|g_{n(f)}|\}$ is bounded . Now $\|g_n\| = \|x_n\|$ .

In finite dimensional normed spaces the distinction between strong and weak convergence disappears completely . Let us prove this fact and also justify the terms "strong" and "weak".

**Theorem. (Strong and weak convergence).** Let $\{x_n\}$ be a sequence in normed space $X$. Then

(i) Strong convergence implies weak convergence with the same limit.

(ii) The converse of (i) is not generally true.

(iii) If $\dim X < \infty$, then weak convergence implies srrong convergence.

**Proof.** By definition, $x_n \to x$ means $\|x_n - x\| \to 0$ and implies for every $f \in X'$,

$$|f(x_n) - f(x)| = f(x_n - x) \le \|f\|\|x_n - x\| \to 0$$

This shows that $x_n \to x$.

(ii) can be seen from an orthonormal sequence $\{e_n\}$ in a Hilbert space $H$. In fact , every $f \in H'$ has a Riesz representation $f(x) = \langle x, z \rangle$. Hence $f(e_n) = \langle e_n, z\rangle$. The Bessels inequality is as

$$\sum_{n=1}^{\infty} |\langle e_n, z\rangle|^2 \le \|z\|^2.$$

Hence the series on the left converges , so that its terms must approach zero as $n \to \infty$. This implies

$$f(e_n) = \langle e_n, z\rangle \to 0.$$

Since $f \in H'$ was arbitrary , we see that $e_n \to 0$. However, $\{e_n\}$ does not converge strongly because

$$\|e_m - e_n\|^2 = \langle e_m - e_n, e_m - e_n\rangle = 2 \qquad (m \ne n).$$

(iv) Suppose that $x_n \to x$ and $\dim X = k$. Let $\{e_1, e_2, \dots e_k\}$ be any basis for $X$ and say ,

$$x_n = \alpha_1^{(n)} e_1 + \cdots + \alpha_k^{(n)} e_k$$

And

$$x = \alpha_1 e_1 + \cdots + \alpha_k e_k$$

By assumption, $f(x_n) \to f(x)$ for every $f \in X'$. We take in particular $f_1, \dots, f_k$ defined by

$$f_j(e_j) = 1, \qquad f_j(e_m) = 0 \qquad (m \ne j)$$

Then

$$f_j(x_n) = \alpha_j^{(n)}, \qquad f_j(x) = \alpha_j.$$

Hence $f_j(x_n) \to f_j(x) \Rightarrow \alpha_j^{(n)} \to \alpha_j$. From this we readily obtain

$$\|x_n - x\| = \left\|\sum_{j=1}^{k} (\alpha_j^{(n)} - \alpha_j) e_j\right\|$$

$$\le \sum_{j=1}^{k} \left|\alpha_j^{(n)} - \alpha_j\right| \|e_j\| \to 0$$

As $n \to \infty$. This shows that $\{x_n\}$ converges strongly to $x$.

It is interesting to note that there also exist infinite dimensional spaces such that strong and weak convergence are equivalent concepts. As example is $l^1$. In conclusion let us take a look at weak convergence in two important types of spaces.

**Example. Hilbert space.** In a Hilbert space, $x_n \to x$ if and only if $\langle x_n, z\rangle \to \langle x, z\rangle \forall\ z$ in the space.

**Example.** In the space $l^p$, where $1 < p < +\infty$, we have $x_n \to x$ if and only if

(i) The sequence $\{\|x_n\|\}$ is bounded.

(ii) For every fixed $j$ we have $\xi_j^{(n)} \to \xi_j$ as $n \to \infty$; here, $x_n = \left(\xi_j^{(n)}\right)$ and $x = (\xi_j)$.

## 8.4 Convergence of Sequences of Operators and Functional

Sequences of bounded linear operators and functionals arise frequently in the abstract formulation of concrete situations, for instance in connection with convergence problems of Fourier series or sequences of interpolation polynomials or methods of numerical integration. In such cases one is usually concerned with the convergence of those sequences of operators or functionals with boundedness of corresponding sequences of norms or with similar properties.

Experience shows that for sequences of elements in a normed space, strong and weak convergence as defined in the above section are useful concepts. For sequences of operators $T_n \in B(X, Y)$ three types of convergence turn out to be of theoretical as well as practical value.These are

(i) Convergence in the norm on $B(X, Y)$,

(ii) Strong convergence of $\{T_n x\}$ in $Y$,

(iii) Weak convergence of $\{T_n x\}$ in $Y$,

The definition and terminology are as follows;

**Definition. (Convergence of sequence of operators)** Let $X$ and $Y$ be normed spaces. A sequence $\{T_n\}$ of operators $T_n \in B(X, Y)$ is said to be

(i ) **uniformly operator convergent** if $\{T_n\}$ converges in the norm on $B(X, Y)$

(ii) **Strongly operator convergent** if $\{T_n x\}$ converges strongly in $Y$ for every $x \in X$ ,

(iii ) **weakly operator convergent** if $\{T_n x\}$ converges weakly in $Y$ for every $x \in X$ .

In formulas this means that there is an operator $T: X \to Y$ such that

(i) $\|T_n - T\| \to 0$

(ii) $\|T_n x - Tx\| \to 0$ for all $x \in X$

(iii) $\|f(T_n x) - f(Tx)\| \to 0$ for all $x \in X$ and for all $f \in Y'$

respectively. $T$ is called the uniform, strong and weak operator limit of $\{T_n\}$, respectively.

**Definition. (Strong and weak convergence of a sequence of functional)** Let $\{f_n\}$ be a sequence of bounded linear functional on a normed space $X$. Then :

(a) Strong convergence of $\{f_n\}$ means that there is an $f \in X'$ such that $\|f_n - f\| \to 0$ . This is written

$$f_n \to f.$$

(b) Weak convergence of $\{f_n\}$ means that there is an $f \in X'$ such that $f_n(x) \to f(x)$ for all $x \in X$. This is written

$$f_n \to f .$$

$f$ in (a) and (b) is called the strong limit and weak limit of $\{f_n\}$, respectively.

**Lemma.** Let $T_n \in B(X, Y)$, where $X$ is a Banach space and $Y$ a normed space . If $\{T_n\}$ is strongly operator convergent with limit $T$, then $T \in B(X, Y)$.

**Proof.** Linearity of $T$ follows readily from that of $T_n$. Since $T_n x \to Tx$ for every $x \in X$, the sequence $\{T_n x\}$ is bounded for every $x$. Since $X$ is complete, $\|T_n\|$ is bounded by the uniform boundedness theorem , say $\|T_n\| \leq c \ \forall n$ . From this, it follows that

$$\|T_n x\| \leq \|T_n\| \|x\| \leq c\|x\|.$$

This implies

$$\|Tx\| \leq c\|x\|.$$

A useful criterion for strong operator convergence is

**Theorem.** A sequence $\{T_n\}$ of operators $T_n \in B(X, Y)$, where $X$ and $Y$ are Banach spaces , is strongly operator convergent if and only if:

(A) The sequence $\{\|T_n\|\}$ is bounded.

(B) The sequence $\{T_n x\}$ is Cauchy in $Y$ for every $x$ in a total subset $M$ of $X$.

**Proof.** If $T_n x \to Tx$ for every $x \in X$ , then (A) follows from the uniform boundedness theorem (since $X$ is complete) and (B) is trivial.

Conversely, suppose that (A) and (B) holds, so that, say $\|T_n\| \leq c \ \forall \ n$. We consider any $x \in X$ and show that $\{T_n x\}$ converges strongly in $Y$. Let $\epsilon > 0$ be given . Since span $M$ is dense in $X$, there is a $y \in span\ M$ such that

$$\|x - y\| < \frac{\epsilon}{3c}.$$

Since $y \in span\ M$, the sequence $\{T_n y\}$ is Cauchy by (B). Hence there is an $N$ such that

$$\|T_n y - T_m y\| < \frac{\epsilon}{3} \qquad (m, n > N)$$

Using these two inequalities and applying the triangle inequality, we see that $\{T_n x\}$ is Cauchy in $Y$ because for $m, n > N$ we obtain

$$\|T_n x - T_m x\| \leq \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\|$$

$$< \|T_n\| \, \|x - y\| + \frac{\epsilon}{3} + \|T_m\| \, \|x - y\|$$

$$< c\frac{\epsilon}{3c} + \frac{\epsilon}{3} + c\frac{\epsilon}{3c} = \epsilon$$

Since $Y$ is complete, $\{T_n x\}$ converges in $Y$. Since $x \in X$ was arbitrary, this proves strong operator convergence of $\{T_n\}$.

**Corollary. (Functionals)** A sequence $\{f_n\}$ of bounded linear functionals on a Banah space $X$ is weak convergent, the limit being a bounded linear functional on $X$, if and only if

(A) The sequence $\{\|f_n\|\}$ is bounded.

(B) The sequence $\{f_n(x)\}$ is Cauchy for every $x$ in a total subset $M$ of $X$.

## Summary

- A mapping $f: X \to Y$, where $X$ and $Y$ are topological spaces, is said to be open mapping if $f$ maps open subsets of $X$ into open open subsets of $Y$.
- Let $X$ be a normed linear space and $B(x_0, r)$ be an open ball in $X$. Then

$B(x_0;\ r) = x_0 + r\,B(0;1)$.

- Let $T$ be a bounded linear operator from a Banach space $X$ into a Banach space $Y$. Then for each open ball $B_0 = B(0,1) \subset X$, the image $T(B_0)$ contains an open ball in $Y$ with centre at origion.
- A bounded linear operator $T$ from a Banach space $X$ into a Banach space $Y$ is an open mapping. (Open mapping theorem)
- Let $T: X \to Y$ be a bijective bounded linear operator from a Banach space $X$ into a Banach space $Y$, then $T$ is homeomorphism.
- Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ be a linear operator . Then $T$ is continuous if and only if the graph of $T$ is a closed subspace of $X \times Y$. ( Closed graph theorem).
- A sequence $\{x_n\}$ in a normed space $X$ is said to be strongly convergent (or convergent in the norm) if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$
  i,e.
  $$\lim_{n\to\infty} x_n = x\ .$$
- A sequence $\{x_n\}$ in a normed space $X$ is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X$
  $$\lim_{n\to\infty} f(x_n) = f(x)$$
  This is written
  $$x_n \to x.$$

The element $x$ is called the weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to $x$.

- Let $T_n \in B(X, Y)$, where $X$ is a Banach space and $Y$ a normed space . If $\{T_n\}$ is strongly operator convergent with limit $T$, then $T \in B(X, Y)$.

## Keywords

- Open mappings
- Open set
- Bounded linear operator
- Incomplete normed spaces
- Open ball
- Homeomorphism
- Closed graph
- Closed subspace
- Weak convergence
- Strong convergence
- Hilbert space

## Self Assessment

1: If $X, Y$ are normed spaces and if $A : X \to Y$ is a bijective, bounded linear map, then:

A. $A$ is always an open map.

B. $A$ is an open map if $X$ is a Banach space.

C. $A$ is an open map if $Y$ is a Banach space.

D. $A$ is an open map if both $X$ and $Y$ are Banach spaces.

2: If $X$ and $Y$ are normed spaces, and if $T : X \to Y$ is a linear operator, then $T$ is bounded if and only if:

A. $T$ maps bounded subsets of $X$ into bounded subsets of $Y$.

B. $T$ maps open subsets of $X$ into open subsets of $Y$.

C. $T$ maps closed subsets of $X$ into closed subsets of $Y$.

D. $T$ is invertible.

3: Every bounded operator of finite rank is :

A. Open.

B. Compact.

C. Has a non zero adjoint.

D. None of these.

4: A bijective map $A : X \to Y$ is open if and only if :

A. $A : X \to Y$ is invertible.

B. $A : X \to Y$ is bounded.

C. $A^{-1} : Y \to X$ is bounded.

D. $A^{-1} : Y \to X$ is open.

5: If $\{A_n\}$ is a sequence of operators on a normed space $X$, then $A_n \to A$ strongly if and only if:

A. $A_n x \to Ax \; \forall \; x \in X$ .

B. $\|A_n - A\| \to 0$ as $n \to \infty$.

C. $f(A_n x) \to f(Ax) \forall \; x \in X$ and $\forall \; f \in X^*$.

D. None of these.

6: If $T$ is a bounded linear operator, then:

A. $\|Tx\| \geq \|T\|.\|x\|$.

B. $\|Tx\| \leq \|T\|.\|x\|$.

C. $\|Tx\| = \|T\|.\|x\|$.

D. None of these.

7: For $x, y$ in a normed space $X$, which of the following is not necessarily true?

A. $\|x + y\| \leq \|x\| + \|y\|$.

B. $|\|x\| - \|y\|| \leq \|x - y\|$

C. $|\|x\| - \|y\|| \leq \|x\| + \|y\|$.

D. $\|x - y\| \leq \|x\| + \|y\|$.

8: Let $X$ be a normed space and $f$ be a bounded, non-zero linear functional on $X$. Then, which of the following is not true?

A. $f$ is onto.

B. $f$ is continuous.

C. $Kerf$ is a closed subspace of $X$.

D. $f$ is an open map.

9 : Let $X$ be a normed space and $A, B$ be bounded linear operators on $X$ Then which of the following is true?

A. $\|AB\| \geq \|A\|.\|B\|$

B. $\|AB\| = \|A\|.\|B\|$

C. $\|AB\| \leq \|A\|.\|B\|$

D. None of these.

10: Which of the following theorems guarantees that a bounded linear operator between Banach spaces is an open map if it is onto.

A. Hahn-Banach Theorem.

B. Open Mapping theorem.

C. Baire's Category theorem.

D. None of these.

11**:** Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ be a bounded linear operator , if the range of $T$ is not closed, then which theorem can be used to find a closed subspace of $X$ on which $T$ is injective?

A. Hahn-Banach Theorem.

B. Open Mapping theorem.

C. Closed graph theorem.

D. None of these.

12: What is the main difference between strong and weak convergence?

A. Strong convergence requires convergence in norm, while weak convergence requires pointwise convergence.

B. Strong convergence requires pointwise convergence, while weak convergence requires convergence in norm.

C. Strong convergence and weak convergence are synonymous terms.

D. None of the above.

13: In a Hilbert space, which of the following statements is true?

A. Strongly convergent sequences are always weakly convergent.

B. Weakly convergent sequences are always strongly convergent.

C. Strong and weak convergence are equivalent.

D. None of these.

14: If a linear mapping between topological vector spaces is continuous, what can we conclude about its graph?

A. The graph is open.

B. The graph is compace.

C. The graph is closed.

D. None of these.

15: Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ be a linear map which is closed and surjective. Then $T$ is continuous and open. This is called

A. Closed graph theorem.

B. Hein-Borel theorem.

C. Open mapping theorem.

D. None of these.

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | D | 2. | A | 3. | B | 4. | C | 5. | A |
| 6. | B | 7. | D | 8. | D | 9. | C | 10. | B |
| 11. | C | 12. | A | 13. | A | 14. | C | 15. | C |

## Review Questions

1. State the Open Mapping Theorem in functional analysis.
2. State the Closed Graph Theorem.
3. Define the graph of a linear operator.
4. Under what conditions does the Closed Graph Theorem holds.
5. What is difference between strong and weak convergence.

## Further Readings

1. Introductory Functional Analysis With Applications By Erwin Kreyszig.
2. Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
3. J. B Conway, A Course In Functional Analysis.
4. C. Goffman G Pedrick, A First Course In Functional Analysis.
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# Unit 09: Decomposition Theorems in Hilbert Spaces

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Orthogonal complements and direct sums
- Pojection Theorem
- Convex sets in Hilbert Spaces
- The conjugate space of a Hilbert space

## Introduction

In this chapter, We discuss about Orthogonal complements and direct sums and its properties. Futher, we prove Projection theorem and convex sets in Hilbert space and discuss some important theorems. Finally we discuss about Conjugate space of a Hilbert space $H$.

## 9.1 Orthogonal Complements and Direct Sums

**Definition.** If $M$ is any subset of a Hilbert space $H$, then the orthogonal complement of $M$ denoted by $M^{\perp}$, is defined as

$$M^{\perp} = \{x \in H : \langle x, y\rangle = 0 \;\forall\; y \in M\}$$

$$= \{x \in H : x \perp M\}$$

And also $M^{\perp\perp} = (M^{\perp})^{\perp} = \{x \in H : \langle x, y\rangle = 0,\ \forall\, y \in M^{\perp}\}$

$$= \{x \in H\, x \perp M^{\perp}\}$$

**Remark.** From the above definitions, it is clear that

(i) $\{0\}^{\perp} = H$

(ii) $H^{\perp} = \{0\}$

**Theorem.** Let $M_1, M_2$ be subsets of a Hilbert space $H$, then prove the following:

(i) $M_1 \subseteq M_1^{\perp\perp}$, that is any subset of $H$ is contained in its double orthogonal complement .

(ii) If $M_1 \subseteq M_2$, then $M_2^{\perp} \subseteq M_1^{\perp}$.

(iii) $(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$ and $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.

(iv) $M_1^{\perp} = M_1^{\perp\perp\perp}$.

(v) $M_1 \cap M_1^{\perp} \subseteq \{0\}$.

(vi) $M_1^{\perp}$ is a closed subspace of $H$.

**Proof.**

(i) Let $x \in M_1$, then
$\langle x, y \rangle = 0 \; \forall \; y \in M_1^{\perp}$. Hence
$x \in M_1^{\perp\perp}$, that is $M_1 \subseteq M_1^{\perp\perp}$.

(ii) Suppose that $M_1 \subseteq M_2$. Let $x \in M_2^{\perp}$, then
$\langle x, y \rangle = 0 \; \forall \; y \in M_2$.
Since $M_1 \subseteq M_2$, we have
$\langle x, y \rangle = 0 \; \forall \; y \in M_1$.
Hence $x \perp M_1$,
implies $x \in M_1^{\perp}$
that is $M_2^{\perp} \subseteq M_1^{\perp}$.

(iii) Since $M_1 \subseteq M_1 \cup M_2$ and $M_2 \subseteq M_1 \cup M_2$
$\Rightarrow (M_1 \cup M_2)^{\perp} \subseteq M_1^{\perp}$ and $(M_1 \cup M_2)^{\perp} \subseteq M_2^{\perp}$
$\Rightarrow (M_1 \cup M_2)^{\perp} \subseteq M_1^{\perp} \cap M_2^{\perp}$………..(1)
Now, let $x \in M_1^{\perp} \cap M_2^{\perp}$
$\Rightarrow x \in M_1^{\perp}$ and $x \in M_2^{\perp}$
$\Rightarrow x \perp M_1$ and $x \perp M_2$
So by definition , $\langle x, u \rangle = 0$ for every $u \in M_1$ and
$\langle x, v \rangle = 0$ for every $v \in M_2$ .
And so $\langle x, u \rangle = 0$ for every $u \in M_1 \cup M_2$.
$\Rightarrow x \in (M_1 \cup M_2)^{\perp}$
So that $M_1^{\perp} \cap M_2^{\perp} \subseteq (M_1 \cup M_2)^{\perp}$………..(2)
From (1) and (2), we get
$(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$.
Next, we show that $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.
For this since , $M_1 \cap M_2 \subseteq M_1$ and $M_1 \cap M_2 \subseteq M_2$
$\Rightarrow (M_1)^{\perp} \subseteq (M_1 \cap M_2)^{\perp}$ and $M_2^{\perp} \subseteq (M_1 \cap M_2)^{\perp}$
$\Rightarrow (M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.

(iv) By (i) $M_1 \subseteq M_1^{\perp\perp}$
and so by part (ii) $(M_1^{\perp\perp})^{\perp} \subseteq M_1^{\perp}$
i.e. $M_1^{\perp\perp\perp} \subseteq M_1^{\perp}$...........(3)

Also by part (i), $M^{\perp} \subseteq (M^{\perp})^{\perp\perp} = M^{\perp\perp\perp}$………..(4)

From (3) and (4), we have

$M_1^{\perp} = M_1^{\perp\perp\perp}$.

(v) If $M_1 \cap M_1^{\perp} = \phi$, then clearly $M_1 \cap M_1^{\perp} = \phi \subseteq \{0\}$
i.e., $M_1 \cap M_1^{\perp} \subseteq \{0\}$
If $M_1 \cap M_1^{\perp} \neq \phi$, then let $x \in M_1 \cap M_1^{\perp}$

$\Rightarrow x \in M_1$ and $x \in M_1^{\perp}$.

Now since $x \in M_1^{\perp}$ and $x \in M_1$

$\Rightarrow \langle x, x \rangle = 0$

$\Rightarrow \|x\|^2 = 0,$

i.e. $\|x\| = 0$

i.e., $x = 0$

i.e., $x \in \{0\}$.

Hence,

$$M_1 \cap M_1^{\perp} \subseteq \{0\}.$$

(vi) Now we show $M_1^{\perp}$ is a closed subspace of $H$.

Let $x, y$ be any two elements in $M_1^{\perp}$ and $\alpha, \beta$ be any scalers, then for $u$ in $M_1$, we have:

$\langle x, u\rangle = 0$ and $\langle y, u\rangle = 0$ and therefore:

$$\langle \alpha x + \beta y, u\rangle = \langle \alpha x, u\rangle + \langle \beta y, u\rangle$$
$$= \alpha\langle x, u\rangle + \beta\langle y, u\rangle$$
$$= \alpha.0 + \beta.0$$
$$= 0$$

i.e., $\langle \alpha x + \beta y, u\rangle = 0$ for any $u$ in $M_1$.

$$\Rightarrow \alpha x + \beta y \in M_1^{\perp}$$

Which shows that $M_1^{\perp}$ is a subspace of $H$.

To complete the proof , it remains to show that $M_1^{\perp}$ is closed and in order to prove this, it is enough to show that if $\{x_n\}$ is any convergent sequence in $M_1^{\perp}$ converging to a point $x$ (say)

i.e., $x_n \to x$, then $x \in M_1^{\perp}$.

Now for any $u \in M_1$, we can write

$$\langle x, u\rangle = \langle \lim_{n\to\infty} x_n, u\rangle$$

$$= \lim_{n\to\infty}\langle x_n, u\rangle$$
$$= 0, \text{ because } x_n \in M_1^{\perp}$$

i.e., $\langle x, u\rangle = 0$ for any $u \in M_1^{\perp}$, $\Rightarrow x \perp M_1$

$\Rightarrow x \in M_1^{\perp}$. Thus $M_1^{\perp}$ is closed subspace of $H$.

**Theorem.** If $M$ is a closed linear subspace of a Hilbert space $H$, then $M \cap M^{\perp} = \{0\}$.

**Proof.** Let $x \in M \cap M^{\perp}$, then $x \in M$ and $x \in M^{\perp}$.

$\Rightarrow x \perp M$.

$\Rightarrow \langle x, y\rangle = 0$ for every $y$ in $M$.

$\Rightarrow \langle x, x\rangle = 0$, because $x \in M$.

$$\Rightarrow \|x\|^2 = 0$$

$$\Rightarrow \|x\| = 0 \Rightarrow x = 0.$$

This shows that $0 \in M \cap M^{\perp} \Rightarrow \{0\} \subseteq M \cap M^{\perp}$.

But we know that $M \cap M^{\perp} \subseteq \{0\}$, by part (v) of above theorem.

Hence $M \cap M^{\perp} = \{0\}$.

**Remark.** For sets $M$ and $M^{\perp}$, $M \cap M^{\perp} \subseteq \{0\}$ and for subspaces $M$ and $M^{\perp}$, $M \cap M^{\perp} = \{0\}$.

The reason is that it is not necessary for any subset to contain 0 but every subspace contains 0.

**Definition. (Direct sum).** A vector space $X$ is said to be the direct sum of two subspaces $Y$ and $Z$ of $X$, written

$$X = Y \oplus Z$$
,

if each $x \in X$ has a unique representation

$$x = y + z, y \in Y, z \in Z$$
.

Then $Z$ is called an algebraic complement of $Y$ in $X$ and vice versa, and $Y, Z$ is called a complementary pair of subspaces in $X$.

For example, $Y = R$ is a subspace of the Euclidean plane $R^2$. Clearly, $Y$ has infinitely many algebraic complements in $R^2$, each of which is a real line. But most convenient is a complement that is perpendicular. We make use of this fact when we choose a Cartesian coordinate system. In $R^3$ the situation is the same in principle.

**Theorem 1. (Minimizing vector)** Let $M$ be a non empty complete convex subset of an inner

Product space $X$ and $x \in X \backslash M$. Then there is a unique $y \in M$ such that

$$\|x - y\| = Inf_{y' \in M} \|x - y'\|$$

That is, there is a unique $y \in M$ which is closest to $x$.

**Proof.** We prove this theorem in the next section of this chapter.

**Theorem 2.** Let $M$ be a complete subspace of an inner product space $X$. Then there is a non zero vector $z \in X$ such that

$$z \perp M$$
.

**Theorem.** Let $M$ be a proper complete subspace of an inner product space $X$. Then

$$X = M \oplus M^{\perp}$$

**Proof.** Since $M$ is complete and being a subspace, is convex, by theorem 1 above there is a unique vector $y \in M$ such that

$\|x - y\| = Inf_{y' \in M} \|x - y'\|$, for each $x \in X \backslash M$.

Put $z = x - y$, by theorem 2, $z \perp M$ and so $z \in M^{\perp}$ which is a subspace of $X$. Hence

$$x = y + z, y \in M, z \in M^{\perp} \text{...........(1)}$$

To see that (1) is unique , suppose that

$$x = y_1 + z_1$$

Also then

$$y - y_1 = z_1 - z \in M \cap M^{\perp} = \{0\}$$

so $y = y_1, z = z_1$. Therefore

$$X = M \oplus M^{\perp}.$$

**Remark.** For any complete subspace $M$ of an inner product space $X$ the subspace $M^{\perp}$ of $X$ is called the orthogonal complement of $M$. In particular, if $M$ is closed subspace of a Hilbert space $H$, the $M^{\perp}$ is orthogonal complement of $M$ in $H$.

**Corollary.** Let $M$ be a closed subspace of a Hilbert space $H$. Then

$$H = M \oplus M^{\perp}.$$

**Proof.** Since $M$, as a closed subspace of a Hilbert space $H$ which is always a complete space , is complete.

**Corollary.** For any complete subspace $M$ of an inner product $X$,

$$M = M^{\perp\perp}.$$

**Remark.** A subspace $M$ of a Hilbert space $H$ is closed if and only if $M = M^{\perp\perp}$.

**Theorem.** Show that if $M$and $N$ are closed subspaces of a Hilbert space $H$ such that $M \perp N$. Then $M + N$ is closed subspace of $H$.

**Proof.** We know that if $M$ and $N$ are any subspaces, Then M+N is always a subspace . To show that M+N is aclosed subspace of $H$, let $z$ be a limit point of M+N, then there is a sequence $\{z_n\}$ in M+N such that

$$z = \lim_{n\to\infty} z_n$$

Now

$$z_n = x_n + y_n, x_n \in M , y_n \in N.$$

We show that $\{x_n\}$ and $\{y_n\}$ are Cauchy's sequences in $M$ and $N$ respectively.

Since by Pythagorean theorem,

$$\|z_m - z_n\|^2 = \|x_m + y_m - x_n - y_n\|^2$$

$$= \|x_m - x_n + y_m - y_n\|^2$$

$$= \|x_m - x_n\|^2 + \|y_m - y_n\|^2$$

and since $z_n$ is a Cauchy sequence, so also are $\{x_n\}, \{y_n\}$. Also as closed subspaces of $H$, both $M$ and $N$ are complete . So

$$\lim_{n\to\infty} x_n = x \in M$$

$$\lim_{n\to\infty} y_n = y \in N$$

Hence $z = \lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + y_n) = x + y$

Thus $z \in M + N$. That is , $M + N$ is closed.

**Theorem.** Let $M$ be a closed linear subspace of a Hilbert space $H$. Then $M \cap M^{\perp} = \{0\}$.

**Proof.** Since we know that if $M$ is a subset of a Hilbert space $H$. Then

$$M \cap M^{\perp} \subseteq \{0\}\text{………..(1)}$$

Given that $M$ is closed linear subspace of $H$ and we also know that $M^{\perp}$ is closed linear subspace of $H$. Let $x \in M \cap M^{\perp}$implies $x \in M$ and $x \in M^{\perp}$ and so $\langle x, x\rangle = 0$.

i.e., $\|x\|^2 = 0$

$$\Rightarrow x = 0 \Rightarrow 0 \in M \text{ and } 0 \in M^{\perp} \Rightarrow 0 \in M \cap M^{\perp}.$$

$$\Rightarrow \{0\} \in M \cap M^{\perp}\text{………..(2)}$$

Combining (1) and (2)

$$M \cap M^{\perp} = \{0\}.$$

**Projection Theorem.**

Let $M$ be any closed subspace of a Hilbert space $H$. Then

$$H = M \oplus M^{\perp}.$$

**Proof.** Suppose $M + M^{\perp}$ is proper subspaceof $H$ then there is a non-zero vector $z \in H$ such that

$z \perp (M + M^{\perp})$ i.e $z \in (M + M^{\perp})^{\perp}$.

Now $M \subseteq (M + M^{\perp})$ implies $(M + M^{\perp})^{\perp} \subseteq M^{\perp}$.

Also we know $M^{\perp} \subseteq (M + M^{\perp})$ implies $(M + M^{\perp})^{\perp} \subseteq M^{\perp\perp}$.

Thus $z \in (M + M^{\perp})^{\perp} \subseteq M^{\perp} \cap M^{\perp\perp} = \{0\} \Rightarrow z = 0$, a contradiction .

Hence $M + M^{\perp}$ is the whole of $H$. i.e. $H = M + M^{\perp}$ since $M \cap M^{\perp} = \{0\}$.

Thus

$$H = M \oplus M^{\perp}.$$

## 9.2 Convex Sets in Hilbert Spaces

Before defining the convex sets in Hilbert spaces we first recall the following.

In a metric space $X$, the distance $\delta$ from an element $x \in X$ to a nonempty subset $M \subset X$ is defined to be

$$\delta = Inf_{y' \in M} d(x, y').$$

In a normed space this becomes

$$\delta = Inf_{y' \in M} \|x - y'\|$$

The line segment joining two given elements $x$ and $y$ of a space $X$ is defined to be the set of all $z \in X$ of the form: $z = tx + (1 - t)y$ for every real number $t$ such that $0 \leq t \leq 1$.

**Definition.** A subset $M$ of $X$ is said to be convex if for every $x, y \in M$, the line segment joining $x$ and $y$ is contained in $M$,

i.e., $z = tx + (1 - t)y \in M$ for every $t$, where $0 \leq t \leq 1$.

Every subspace $Y$ of $X$ is convex, and the intersection of convex sets is a convex set.

We shall use the notion of convexity in the following theorem.

**Theorem. (Minimizing vector)** Let $M$ be a non empty complete convex subset of an inner product space $X$ and $x \in X \backslash M$. Then there is a unique $y \in M$ such that

$$\|x - y\| = Inf_{y' \in M} \|x - y'\|$$

That is, there is a unique $y \in M$ which is closest to $x$.

**Proof.** Let $d = Inf_{y' \in M} \|x - y'\|$

Then by definition of infimum , there is sequence $\{y_n\}$ in M such that

$$d = \lim_{n \to \infty} \|x - y_n\|$$

We show that $\{y_n\}$ is a Cauchy sequence in $M$.

Now by Parallelogram law, we have

$$\|x' - y'\|^2 = 2\|x'\|^2 + 2\|y'\|^2 - \|x' + y'\|^2 \text{...........(1)}$$

Replacing $x'$ by $y_m - x$ and $y'$ by $y_n - x$, we have

$$\|y_m - y_n\|^2 = 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - \|y_m + y_n - 2x\|^2$$

$$= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4\left\|\frac{1}{2}(y_m + y_n) - x\right\|^2 \text{...........(2)}$$

Since $M$ is convex , $\frac{1}{2}(y_m + y_n) \in M$, so we have from (2)

$$\|y_m - y_n\|^2 \leq 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4d^2$$

$$\to 0 \ as \ m, n \to \infty$$

As $\|y_m - x\| \to d$, $\|y_m - y\| \to 0$. Hence $\{y_n\}$ is a Cauchy sequence in $M$. Since $M$ is complete,

$y_n \to y \in M$ . So

$$d = \lim_{n\to\infty}\|x - y_n\|$$
$$= \left\|x - \lim_{n\to\infty} y_n\right\|$$
$$= \|x - y\| \text{ with } y \in M.$$

Next we prove the uniqueness of $y$. Suppose there is another $y_0 \in M$, such that

$$d = \|x - y_0\|$$

Then again, Using the parallelogram law as given in (1) and replacing $x'$ by $y - x$ and $y'$ by $y_0 - x$, we have :

$$\|y - y_0\|^2 = 2\|y - x\|^2 + 2\|y_0 - x\|^2 - \|y + y_0 - 2x\|^2$$
$$= \|x - y\|^2 = Inf_{y'\in M}\|x - y'\|$$

Since $M$ is convex and $\frac{1}{2}(y + y_0) \in M$, we have

$$\|y - y_0\|^2 \le 4d^2 - 4d^2$$
$$\le 0$$

But

$$\|y - y_0\| \ge 0.$$

Hence $\|y - y_0\| = 0$, that is $y = y_0$.

This proves the uniqueness of $y$.

## 9.3 The Conjugate Space of a Hilbert Space $H$

Let $H$ be a Hilbert space . Then a scalar valued fubction $f: H \to C$ is called a functional on $H$, if $f$ is linear and bounded (continuous). Set of all such functionals is denoted by $B(H, C)$ or simply $H^*$ and $H^*$ is called conjugate space of a Hilbert space $H$.

So if $f \in H^*$ implies $f: H \to C$ is a functional .

**(OR)**

Let $H$ be a Hilbert space . By $H^*$, we denote the conjugate space of $H$ (i.e. the set of all continuous linear transformations of $H$ into $C$). The elements of $H^*$are called continuous linear functionals or briefly functional.

One of the fundamental properties of a Hilbert space $H$ is the fact that there is a natural correspondence between the vectors in $H$ and functional in $H^*$.

**Theorem.** Let $y$ be a fixed vector in a Hilbert space $H$ and let $f_y$ be a function defined as

$f_y(x) = \langle x, y\rangle$ for every $x \in H$. Then $f_y$ is a functional on $H$ and $\|y\| = \|f_y\|$.

**Proof.** We prove that $f_y$ is linear and continuous so that it is a functional .

To prove $f_y$ is linear , let $x_1, x_2 \in H$ and $\alpha, \beta$ be any two scalars. Then for any fixed $y \in H$,

$$f_y(\alpha x_1 + \beta x_2) = \langle \alpha x_1 + \beta x_2, y\rangle$$
$$= \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$$
$$= \alpha f_y(x_1) + \beta f_y(x_2).$$

This shows that $f_y$ is linear . To prove that $f_y$ is continuous , for any $x \in H$

$$|f_y(x)| = |\langle x, y\rangle|$$

$$\le \|x\|\|y\| \ldots\ldots\ldots\ldots(1)$$

( by Schwarz inequality)

Suppose $\|y\| \le M$. Then for $M > 0$, $|f_y(x)| \le M\|x\|$.

Hence $f_y$ is bounded and hence it is continuous.

Now if $y = 0$, $\|y\| = 0$ and from the definition $f_y = 0$ so that $\|f_y\| \le \|y\|$ in this case.

Let $y \neq 0$.

From (1), we get

$$\sup \frac{|f_y(x)|}{\|x\|} \le \|y\|$$

Hence using the definition of the norm of a functional, we get

$$\|f_y\| \le \|y\| \ldots\ldots\ldots(2)$$

Further,

$$\|f_y\| = \sup\{|f_y(x)| : \|x\| \le 1\}. \ldots\ldots\ldots..(3)$$

Since
$y \neq 0, \left(\frac{y}{\|y\|}\right)$ is a unit vector .From (3), we get

$$\|f_y\| \ge \left|f_y\left(\frac{y}{\|y\|}\right)\right| \ldots\ldots\ldots..(4)$$

But

$$f_y\left(\frac{y}{\|y\|}\right) = \langle \frac{y}{\|y\|}, y \rangle$$

$$= \frac{1}{\|y\|}\langle y, y \rangle$$

$$= \|y\|.$$

Using this in (4), we get

$$\|f_y\| \ge \|y\| \ldots\ldots\ldots\ldots(5)$$

Combining (2) and (5), we get

$$\|f_y\| = \|y\|.$$

Thus we have proved that $T: H \to H^*$ is such that $T(y) = f_y$
is a norm preserving mappings.

**Theorem.** Show that the mapping $\phi: H \to H^*$ defined by $\phi(y) = f_y$ where $f_y(x) = \langle x, y \rangle$ for every $x \in H$ is an additive, one to one onto isometry but not linear.

**Proof.** First , we prove that $\phi$ is additive. For this we have to show that

$$\phi(y_1 + y_2) = \phi(y_1) + \phi(y_2) \text{ for } y_1, y_2 \in H.$$

Now we have

$$\phi(y_1 + y_2) = f_{y_1+y_2}.$$

Hence for every $x \in H$, we get

$$f_{y_1+y_2}(x) = \langle x, y_1 + y_2 \rangle$$

$$= \langle x, y_1 \rangle + \langle x, y_2 \rangle$$

$$= f_{y_1}(x) + f_{y_2}(x)$$

$$= (f_{y_1} + f_{y_2})(x).$$

Hence ,

$$f_{y_1+y_2} = f_{y_1} + f_{y_2}.$$

Which implies ,

$$\phi(y_1 + y_2) = \phi(y_1) + \phi(y_2).$$

This shows that $\phi$ is additive.

Now we show $\phi$ is one –one . Let $y_1, y_2 \in H$. Then

$\phi(y_1) = f_{y_1}$ and

$\phi(y_2) = f_{y_2}$, then

$\phi(y_1) = \phi(y_2)$, implies

$f_{y_1} = f_{y_2}$, which gives

$f_{y_1}(x) = f_{y_2}(x)\ \forall\ x \in H$………..(1)

$f_{y_1}(x) = \langle x, y_1 \rangle$ and

$f_{y_2}(x) = \langle x, y_2 \rangle$, so we get from (1)

$\langle x, y_1 \rangle = \langle x, y_2 \rangle$

i.e, $\langle x, y_1 - y_2 \rangle = 0\ \forall\ x \in H$………..(2)

Now choose $x = y_1 - y_2$, then (2) gives $\langle y_1 - y_2, y_1 - y_2 \rangle = \|y_1 - y_2\|^2 = 0$.

Which implies $y_1 = y_2$.

Therefore $\phi$ is one- one .

Now to prove $\phi$ is onto , let $f \in H^*$, then by Riesz representation theorem, there exists $y \in H$ such that

$$f(x) = \langle x, y \rangle.$$

Since $f(x) = \langle x, y \rangle$, we get $f = f_y$ so that $\phi(y) = f_y = f$. Hence for $f \in H^*$, there exists a pre image $y$ in $H$. There fore $\phi$ is onto.

To prove that $\phi$ is an isometry, let $y_1, y_2 \in H$. Then

$$\|\phi(y_1) - \phi(y_2)\| = \|f_{y_1} - f_{y_2}\|$$
$$= \|f_{y_1} + f_{(-y_2)}\|.$$

But $\|f_{y_1} + f_{-y_2}\| = \|f_{y_1 - y_2}\| = \|y_1 - y_2\|$ .

Hence $\|\phi(y_1) - \phi(y_2)\| = \|y_1 - y_2\|$.

Finally, we prove that $\phi$ is not linear, for this let $y \in H$ and $\alpha$ be any scalar. Then

$$\phi(\alpha y) = f_{\alpha y}.$$

Hence for any $x \in H$, we get $f_{\alpha y}(x) = \langle x, \alpha y \rangle$

$= \overline{\alpha}\, \langle x, y \rangle$

$= \overline{\alpha} f_y(x)$.

Which gives, $f_{\alpha y} = \overline{\alpha} f_y$.

So that,

$$\phi(\alpha y) = \overline{\alpha}\phi(y)$$

This shows that the mapping is not linear. Such a mapping is called conjugate linear. Thus $\phi$ is conjugate linear.

**Theorem.** If $H$ is a Hilbert space , then $H^*$ is also Hilbert space with the inner product defined by

$\langle f_x, f_y \rangle = \langle y, x \rangle \forall\ x, y \in H$………..(1)

**Proof.** Since $H$ is a Hilbert space , so $H$ is also a Banach space . We know that conjugate of a Banach space is also a Banach space . Therefore $H^*$ is also a Banach space.

To show $H^*$ is a Hilbert space, it is sufficient to show that $H^*$ is inner product space with respect to the inner product defined by

$$\langle f_x, f_y \rangle = \langle y, x \rangle \ \forall\ x, y \in H.$$

Let $x, y \in H$ and $\alpha, \beta$ be complex scalars , we have

1. $\langle f_x, f_x \rangle = \langle x, x \rangle$

   $= \|x\|^2$

   $= \|f_x\|^2$

   So that

   $\langle f_x, f_x \rangle \geq 0$ and $\|f_x\| = 0$ if and only if $f_x = 0$.

2. $\overline{\langle f_x, f_y \rangle} = \overline{\langle y, x \rangle}$

   $= \langle x, y \rangle$

   $= \langle f_y, f_x \rangle$.

3. In the above theorem, we have shown that then $f_{\alpha y} = \bar{\alpha} f_y$.

Hence

$$f_{\bar{\alpha} y} = \bar{\bar{\alpha}}\, f_y = \alpha f_y.$$

Now

$$\langle \alpha f_x + \beta f_y, z \rangle = \langle f_{\bar{\alpha} x} + f_{\bar{\beta} y}, f_z \rangle \ldots\ldots\ldots(2)$$

But,

$\langle f_{\bar{\alpha} x} + f_{\bar{\beta} y}, f_z \rangle = \langle z, \bar{\alpha} x + \bar{\beta} y \rangle$ by (1)

Now,

$\langle z, \bar{\alpha} x + \bar{\beta} y \rangle = \bar{\bar{\alpha}} \langle z, x \rangle + \bar{\bar{\beta}} \langle z, y \rangle$

$= \alpha \langle f_x, f_z \rangle + \beta \langle f_y, f_z \rangle \ldots\ldots\ldots..(3)$

From (2) and (3), we have

$$\langle \alpha f_x + \beta f_y, z \rangle = \alpha \langle f_x, f_z \rangle + \beta \langle f_y, f_z \rangle.$$

Which Completes the proof.

## Summary

- If $M$ is any subset of a Hilbert space $H$, then the orthogonal complement of $M$ denoted by $M^{\perp}$, is defined as

$$M^{\perp} = \{x \in H : \langle x, y \rangle = 0\ \forall\ y \in M\}$$
$$= \{x \in H : x \perp M\}.$$

- $M^{\perp\perp} = (M^{\perp})^{\perp} = \{x \in H : \langle x, y \rangle = 0,\ \forall\ y \in M^{\perp}\}$

  $= \{x \in H\ x\ \perp M^{\perp}\}$

- $\{0\}^{\perp} = H$
- $H^{\perp} = \{0\}$

- If $M_1, M_2$ be subsets of a Hilbert space $H$, then we have
  - ✓ $M_1 \subseteq M_1^{\perp\perp}$, that is any subset of $H$ is contained in its double orthogonal complement .
  - ✓ If $M_1 \subseteq M_2$, then $M_2^{\perp} \subseteq M_1^{\perp}$.
  - ✓ $(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$ and $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.
  - ✓ $M_1^{\perp} = M_1^{\perp\perp\perp}$.
  - ✓ $M_1 \cap M_1^{\perp} \subseteq \{0\}$.
  - ✓ $M_1^{\perp}$ is a closed subspace of $H$.
- If $M$ is a closed linear subspace of a Hilbert space $H$, then $M \cap M^{\perp} = \{0\}$.
- If $M$ be a complete subspace of an inner product space $X$. Then there is a non zero vector $z \in X$ such that
$$z \perp M$$
- If $M$ be a proper complete subspace of an inner product space $X$. Then
$$X = M \oplus M^{\perp}$$
- If $M$ be a closed subspace of a Hilbert space $H$. Then
$$H = M \oplus M^{\perp}.$$
- For any complete subspace $M$ of an inner product $X$,
$$M = M^{\perp\perp}.$$
- If $M$and $N$ are closed subspaces of a Hilbert space $H$ such that $M \perp N$. Then $M + N$ is closed subspace of $H$.
- Let $M$ be a closed linear subspace of a Hilbert space $H$. Then $M \cap M^{\perp} = \{0\}$.
- Let $M$ be any closed subspace of a Hilbert space $H$. Then
$$H = M \oplus M^{\perp} \text{( Projection Theorem)}$$
- The line segment joining two given elements $x$ and $y$ of a space $X$ is defined to be the set of all $z \in X$ of the form: $z = tx + (1-t)y$ for every real number $t$ such that $0 \le t \le 1$.
- A subset $M$ of $X$ is said to be convex if for every $x, y \in M$, the line segment joining $x$ and $y$ is contained in $M$,

  i.e., $z = tx + (1-t)y \in M$ for every $t$, where $0 \le t \le 1$.
- Every subspace $Y$ of $X$ is convex, and the intersection of convex sets is a convex set.
- If $M$ be a non empty complete convex subset of an inner product space $X$ and $x \in X \setminus M$. Then there is a unique $y \in M$ such that
$$\|x - y\| = Inf_{y' \in M} \|x - y'\|$$
That is, there is a unique $y \in M$ which is closest to $x$.
- Let $H$ be a Hilbert space . By $H^*$, we denote the conjugate space of $H$ (i.e. the set of all continuous linear transformations of $H$ into $C$). The elements of $H^*$are called continuous linear functionals or briefly functional.
- Let $y$ be a fixed vector in a Hilbert space $H$ and let $f_y$ be a function defined as $f_y(x) = \langle x, y \rangle$ for every $x \in H$. Then $f_y$ is a functional on $H$ and $\|y\| = \|f_y\|$.
- The mapping $\phi: H \to H^*$ defined by $\phi(y) = f_y$ where $f_y(x) = \langle x, y \rangle$ for every $x \in H$ is an additive, one to one onto isometry but not linear.
- If $H$ is a Hilbert space , then $H^*$ is also Hilbert space with the inner product defined by
$$\langle f_x, f_y \rangle = \langle y, x \rangle \; \forall \; x, y \in H$$

## Keywords

- Orthogonal complement
- Direct sum
- Convex set
- Conjugate space
- Closed subspace
- Hilbert space
- Inner product space
- Projection theorem

## Self Assessment

1: Let $H$ be a Hilbert space and $M$ be a subspace of $H$. Then which of the following is false?

A. $M^{\perp}$ is a subspace of $H$.

B. $M^{\perp}$ is a closed subspace of $H$.

C. $M \cap M^{\perp} = \{0\}$.

D. $M \cap M^{\perp} = \phi$

2: The distance between any two orthonormal vectors in an inner product space is:

A. 1

B. $\sqrt{2}$

C. 2

D. $\sqrt{5}$

3: Let $X$ be an inner product space. Then the orthogonal complement of $\{0\}$ is:

A. $X$

B. $\{0\}$

C. $X\{0\}$

D. $X^{\perp}$

4: What is a convex set in a Hilbert space?

A. A set that contains only a single point.

B. A set where every line segment between two points in the set lies entirely within the set.

C. A set of orthogonal vectors.

D. A set that is closed under addition but not under scalar multiplication.

5: Every perfectly convex set is:

A. Closed

B. Open

C. Half open

D. Convex

6: A convex set in a Banach space need not be:

A. Hausdorff

B. Convex

C. Perfectly convex

D. Closed

7: What is the orthogonal complement of a subspace?

A. The subspace itself.

B. The zero vector.

**C.** The set of all vectors orthogonal to the subspace**.**

D. None of the above.

8: In an inner product space, which of the following statement is true regarding the direct sum of two subspaces $U$ and $V$, denoted as $U \oplus V$?

A. $U \oplus V$ is always equal to $U + V$.

B. Every vector in $U \oplus V$ can be uniquely expressed as the sum of a vector from $U$ and a vector from $V$.

C. $U \oplus V$ contain only the zero vector.

D. None of the above

9: Which property is satisfied by the intersection of any number of convex sets in a Hilbert space?

A. The intersection is always a convex set.

B. The intersection is always a non –convex set.

C. The intersection is always a singleton set.

D. The intersection is always empty.

10: Let $M_1, M_2$ be subsets of a Hilbert space $H$, then which of the following is true.

A. $(M_1 \cup M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp}$

B. $(M_1 \cap M_2)^{\perp} \supseteq M_1^{\perp} \cup M_2^{\perp}$.

C. $M_1^{\perp} = M_1^{\perp\perp\perp}$.

D. All the above.

11**:** A subspace $M$ of a Hilbert space $H$ is closed if and only if:

A. $M = M^{\perp\perp}$.

B. $M = M^{\perp}$.

C. $M \subseteq M^{\perp}$.

D. All the above.

12: If $M$ be any closed subspace of a Hilbert space $H$. Then Which of the following is true?

A. $H = M \cap M^{\perp}$.

B. $H = M \oplus M^{\perp}$.

C. Both (A) and (B) are true.

D. None of the above.

13: Which of the following statement is true about conjugate space of a Hilbert Space?

A. It only contains real numbers.

B. It is always a finite-dimensional vector space.

C. It consists of continuous linear functionals on the Hilbert space.

D. None of the above.

14**:** If $M$and $N$ are closed subspaces of a Hilbert space $H$ such that $M \perp N$, the which of the following is true?

A. $M + N$ is closed subspace of $H$.

B. M+N is a subspace of $H$.

C. Both (A) and (B) are true.
D. None of the above.

15**:** Let $M_1, M_2$ be subsets of a Hilbert space $H$, then which of the following is true?

A. $M_1 \subseteq M_1^{\perp\perp}$, that is any subset of $H$ is contained in its double orthogonal complement .
B. If $M_1 \subseteq M_2$, then $M_2^{\perp} \subseteq M_1^{\perp}$.
C. Both (A) and (B).
D. None of the above.

16**:** If $M_1, M_2$ be subsets of a Hilbert space $H$, then which of the following is true?

A. $M_1 \cap M_1^{\perp} \subseteq \{0\}$.
B. $M_1^{\perp}$ is a closed subspace of $H$.
C. $M_1 \cap M_1^{\perp} = \{0\}$.
**D.** All the above.

17: **T**he mapping $\phi: H \to H^*$ defined by $\phi(y) = f_y$ where $f_y(x) = \langle x, y \rangle$ for every $x \in H$ is:

A. An additive mapping.
B. One to one
C. Not linear.
**D.** All the above.

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | D | 2 | B | 3 | A | 4 | B | 5 | D |
| 6 | C | 7 | C | 8 | B | 9 | A | 10 | D |
| 11 | A | 12 | B | 13 | C | 14 | C | 15 | C |
| 16 | D | 17 | D | | | | | | |

## Review Questions

1. Define the orthogonal complement of a subset S in an inner product space.
2. Define a convex set in a Hilbert space.
3. Provide an example of a convex set that is not a closed set.
4. State the Projection Theorem for a Hilbert space.
5. What is meant by the “ conjugate ” or “dual” of a Hilbert space ? How is the dual space constructed from the original Hilbert space?
6. Prove that If $M$ is a closed linear subspace of a Hilbert space $H$, then $M \cap M^{\perp} = \{0\}$.
7. Prove that If $M$and $N$ are closed subspaces of a Hilbert space $H$ such that $M \perp N$. Then $M + N$ is closed subspace of $H$.
8. Let $M_1, M_2$ be subsets of a Hilbert space $H$, then prove the following.
   I. $M_1 \subseteq M_1^{\perp\perp}$, that is any subset of $H$ is contained in its double orthogonal complement .
   II. If $M_1 \subseteq M_2$, then $M_2^{\perp} \subseteq M_1^{\perp}$.

III. $(M_1 \cup M_2)^\perp = M_1^\perp \cap M_2^\perp$ and $(M_1 \cap M_2)^\perp \supseteq M_1^\perp \cup M_2^\perp$.

IV. $M_1^\perp = M_1^{\perp\perp\perp}$.

V. $M_1 \cap M_1^\perp \subseteq \{0\}$.

9. Prove that If $M$ be a subset of a Hilbert space $H$, then $M^\perp$ is a closed subspace of $H$.

10. Show that the mapping $\phi: H \to H^*$ defined by $\phi(y) = f_y$ where $f_y(x) = \langle x, y \rangle$ for every $x \in H$ is an additive, one to one onto isometry but not linear.

## Further Readings


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# Unit 10: Riesz Representation Theorem and Operators on Hilbert Spaces

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Riesz representation theorem
- Hilbert adjoint operator
- Self adjoint operator
- Positive operator
- Normal operator
- Unitary operator
- Isometric operator

## Introduction

In this chapter, we discuss about Riesz representation theorem. Further, we shall introduce the operators on a Hilbert space like Hilbert adjoint operator, Self adjoint operator, positive operator, normal operator, unitary operator and isometric operators.

## 10.1 Riesz Representation Theorem

Let $H$ be a Hilbert space and let $f$ be any arbitrary functional in $H^*$, then there exists a unique vector $y$ in $H$ such that $f(x) = \langle x, y\rangle$ for every $x \in H$ and $\|f\| = \|y\|$.

**Proof.** Let $M$ be the null space (kernel) of $f$ , that is

$$M = \{x \in H: f(x) = 0\}$$

Since $f$ is continuous as $f$ is functional, so by the continuity of $f$, the null space $M$ of $f$ is closed subspace of $H$, as we know that the null space of a non-zero continuous linear operator is a closed subspace .

If $M = H$, the $f(x) = 0$ as by definition of $M$.

$$= \langle x, 0\rangle \text{ for all } x \in H.$$

If $M \neq H$, then $M$ is a proper closed subspace of $H$ and so there exists a non-zero vector $y_0 \in H$ which is orthogonal to $M$ i,e. $y_0 \perp M$.

So $y_0$ is not in $M$, thus $f(y_0) \neq 0$.

For any vector $x \in H$, the vector $z = x - \frac{f(x)}{f(y_0)} . y_0$ is in $M$,

Because $f(z) = f\left(x - \frac{f(x)}{f(y_0)} . y_0\right)$

$$= f(x) - \frac{f(x)}{f(y_0)} f(y_0)$$

$$= 0$$

.

Also since $y_0 \perp M$, so that $y_0 \perp z$ as $z \in M$

$\Rightarrow \langle z, y_0\rangle = 0$

$\Rightarrow \langle x - \frac{f(x)}{f(y_0)} . y_0, y_0\rangle = 0$

$\Rightarrow \langle x, y_0\rangle - \langle \frac{f(x)}{f(y_0)} y_0 , y_0\rangle = 0$

$\Rightarrow \langle x, y_0\rangle - \frac{f(x)}{f(y_0)} \langle y_0 , y_0\rangle = 0$

$\Rightarrow \frac{f(x)}{f(y_0)} \langle y_0 , y_0\rangle = \langle x, y_0\rangle$

$\Rightarrow f(x) = \frac{f(y_0)}{\langle y_0 , y_0\rangle} \langle x, y_0\rangle$

$\Rightarrow f(x) = \langle x, \frac{\overline{f(y_0)}}{\langle y_0 , y_0\rangle} y_0\rangle = \langle x, \frac{\overline{f(y_0)}}{\langle y_0 , y_0\rangle} y_0\rangle.$

Let $y = \frac{\overline{f(y_0)}}{\langle y_0 , y_0\rangle} y_0$ , then we have

$f(x) = \langle x, y\rangle$ for all $x \in H$ .

To complete the proof, it remains to show that $y$ is unique.

For if suppose $f(x) = \langle x, y'\rangle$ for all $x$, then

$\langle x, y\rangle = \langle x, y'\rangle$

$\Rightarrow \langle x, y\rangle - \langle x, y'\rangle = 0$

$\Rightarrow \langle x, y - y'\rangle = 0$ for all $x \in H$.

In particular $x = y - y'$, we get:

$\langle y - y', y - y'\rangle$

$\Rightarrow \|y - y'\|^2 = 0$

$\Rightarrow y - y' = 0$

$\Rightarrow y = y'$.

Hence $y$ is unique .

Next we show that $\|f\| = \|y\|$, we have

$$f(x) = \langle x, y\rangle$$

So $|f(x)| = |\langle x, y\rangle|$

$\leq \|x\|\|y\|$ ( By Schwarz inequality)

And thus it follows that

$\|f\| \leq \|y\|$ ( taking $Sup_{\|x\|=1}$ over both sides)

Also

$\|y\|^2 = \langle y, y\rangle = f(y)$

$\leq |f(y)|$

$\leq \|f\|\|y\|$

$\Rightarrow \|y\| \leq \|f\|$

Combining both the equations, we get

$$\|f\| = \|y\|.$$

## 10.2 Hilbert Adjoint Operator

Let $T: H_1 \to H_2$ be a bounded linear operator, where $H_1$ and $H_2$ are Hilbert spaces. Then the Hilbert adjoint operator $T^*$ of $T$ is the operator

$$T^*: H_2 \to H_1$$

Such that for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

We first show that this definition makes sense, that is we prove that for a given $T$ such a $T^*$ does exists.

**Theorem.** Show that the Hilbert adjoint operator $T^*$ of $T$ exists , is unique and is bounded linear operator with norm

$$\|T^*\| = \|T\|.$$

Before proving this theorem, we first give the another statement of Riesz representation.

**Theorem.** Let $H_1, H_2$ be Hilbert spaces and

$$h: H_1 \times H_2 \to K$$

a bounded sesquilinear form. Then $h$ has a representation

$$h(x, y) = \langle Sx, y\rangle$$

Where $S: H_1 \to H_2$ is a bounded linear operator. $S$ is uniquely determined by $h$ and has the norm

$$\|S\| = \|h\|.$$

### Proof of the Main Theorem

**The formula**

$$h(y, x) = \langle y, Tx\rangle \text{..........(1)}$$

defines a sesquilinear form on $H_2 \times H_1$ because the inner product is sesquilinear and $T$ is linear. In fact, conjugate linearity of the form is seen from

$$h(y, \alpha x_1 + \beta x_2) = \langle y, T(\alpha x_1 + \beta x_2)\rangle$$

$$= \langle y, \alpha T x_1 + \beta T x_2\rangle$$

$$= \bar{\alpha}\langle y, T x_1\rangle + \bar{\beta}\langle y, T x_2\rangle$$

$$= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2)$$

$h$ is bounded , by the Schwarz inequality

$$|h(y, x)| = |\langle y, Tx\rangle|$$

$$\leq \|y\|\|Tx\|$$
$$\leq \|T\|\|x\|\|y\|.$$

This implies,

$$\|h\| \leq \|T\|.$$

Moreover, we have

$$\|h\| \geq \|T\|.$$

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle y, Tx\rangle|}{\|y\|\|x\|} \geq \frac{|\langle Tx, Tx\rangle|}{\|Tx\|\|x\|}$$

$$= \|T\|$$

Combing the both equations, we get

$$\|h\| = \|T\|.$$

The above theorem gives a Riesz representation for $h$; writing $T^*$ for $S$, we have

$$h(x, y) = \langle T^* y, x\rangle, \ldots\ldots\ldots..(2)$$

and we know from the above theorem that $T^*: H_2 \to H_1$ is a uniquely determined bounded linear operator with norm

$$\|T^*\| = \|h\| = \|T\|$$

Also ,

$\langle y, Tx\rangle = \langle T^* y, x\rangle$ by comparing (1) and (2), so that we have $\langle Tx, y\rangle = \langle x, T^* y\rangle$ by taking conjugates, and we now see $T^*$ is the required operator.

For studying the properties of Hilbert adjoint opeartors , it will be convenient to make use of following lemma.

**Lemma. (Zero operator) .** Let $X$ and $Y$ be inner product spaces and $Q: X \to Y$ a bounded linear operator . Then:

(a) $Q = 0$ if and only if $\langle Qx, y\rangle = 0$ for all $x \in X$ and $y \in Y$.

(b) If $Q: X \to X$, where $X$ is complex, and $\langle Qx, x\rangle = 0$ for all $x \in X$, then $Q = 0$.

**Proof. (a )** $Q = 0$ means $Qx = 0$ for all $x$,

$$\Rightarrow \langle Qx, y\rangle = \langle 0, y\rangle = 0\langle w, y\rangle = 0.$$

Conversely suppose that $\langle Qx, y\rangle = 0$ for all $x$ and $y$,

$\Rightarrow Qx = 0$ for all $x$.

So that, $Q = 0$ by definition.

(b ) By assumption ,$\langle Qv, v\rangle = 0$ for every $v = \alpha x + y \in X$,

that is,

$$0 = \langle Q(\alpha x + y), \alpha x + y\rangle$$

$$= |\alpha|^2\langle Qx, x\rangle + \langle Qy, y\rangle + \alpha\langle Qx, y\rangle + \bar{\alpha}\langle Qy, x\rangle.$$

The first two terms on the right are zero by assumption. $\alpha = 1$ gives

$$\langle Qx, y\rangle + \langle Qy, x\rangle = 0.$$

$\alpha = i$ gives $\bar{\alpha} = -i$ and

$$\langle Qx, y\rangle - \langle Qy, x\rangle = 0.$$

By addition,

$\langle Qx, y \rangle = 0$ and $Q = 0$ follows from (a).

**Note:** In part (b) of this lemma, it is essential that $X$ be complex. Indeed , the conclusion may not hold if $X$ is real . A counterexample is a rotation $Q$ of the plane $R^2$ through a right angle. $Q$ is linear, and $Qx \perp x$, hence $\langle Qx, x \rangle = 0$ for all $x \in R^2$, but $Q \neq 0$.

We now prove some general properties of Hilbert- adjoint operators which one uses quite frequently in applying these operators.

**Properties of Hilbert- adjoint Operators**

**Theorem.** Show that the adjoint operator preserves addition, reverses the product and it is conjugate linear. That is if $T \to T^*$ is the adjoint operator on $\beta(H)$, then

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $(T_1T_2)^* = T_2^*T_1^*$

(iii) $(\alpha T)^* = \bar{\alpha}\, T^*$

**Proof.** For every $x, y \in H$, we have

$\langle x, (T_1 + T_2)^* y \rangle = \langle (T_1 + T_2)x, y \rangle$ .

But

$$\begin{aligned}\langle (T_1 + T_2)x, y \rangle &= \langle T_1x + T_2x, y \rangle \\ &= \langle T_1x, y \rangle + \langle T_2x, y \rangle \\ &= \langle x, T_1^* y \rangle + \langle x, T_2^* y \rangle \\ &= \langle x, T_1^* y + T_2^* y \rangle\end{aligned}$$

Hence,

$\langle x, (T_1 + T_2)^* y \rangle = \langle x, (T_1^* + T_2^*) y \rangle$.

From the uniqueness of the adjoint, we get

$$(T_1 + T_2)^* = T_1^* + T_2^*.$$

(ii) For every $x, y \in H$, we have $\langle x, (T_1T_2)^* y \rangle = \langle (T_1T_2)x, y \rangle$

$$= \langle T_1(T_2x), y \rangle.$$

But

$$\langle T_1(T_2x), y \rangle = \langle T_2(x), T_1^* y \rangle = \langle x, T_2^*T_1^* y \rangle$$

From the above two, we get

$$\langle x, (T_1T_2)^* y \rangle = \langle x, (T_2^*T_1^*) y \rangle \text{ for all } y \in H.$$

Therefore from uniqueness of adjoint

$$(T_1T_2)^* = T_2^*T_1^*.$$

(iii) For every $x, y \in H$, we have

$\langle x, (\alpha T)^* y \rangle = \langle (\alpha T)x, y \rangle = \alpha \langle Tx, y \rangle$.

But $\alpha \langle Tx, y \rangle = \alpha \langle x, T^* y \rangle$

$= \langle x, \bar{\alpha}\,(T^* y) \rangle$.

Therefore , by the uniqueness of the adjoint , we have

$$(\alpha T)^* = \bar{\alpha}\, T^*.$$

**Theorem.** Let $H$ be a Hilbert space. Then the adjoint operator $T \to T^*$ on $\beta(H)$ ( set of all bounded linear transformations on $H$ into $H$) has the following properties:

(i) $T^{**} = T$

(ii) $\|T^*\| = \|T\|$

(iii) $\|T^*T\| = \|T\|^2$

**Proof.** For every $x, y \in H$, we get

$$\langle x, T^{**}y\rangle = \langle x, (T^*)^*y$$
$$= \langle T^*x, y\rangle$$

But

$$\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle}$$
$$= \overline{\langle Ty, x\rangle}$$
$$= \langle x, Ty\rangle.$$

From the above two, we get

$$\langle x, T^{**}y\rangle = \langle x, Ty\rangle.$$

Which proves that

$T^{**} = T$ by the uniqueness of inverse.

(ii) For any vector $y \in H$, we have

$$\|T^*y\| \leq \|T\|\|y\|.$$

Hence we get

$$\sup_{y\neq 0} \frac{\|T^*y\|}{\|y\|} \leq \|T\|.$$

Using the definition of norm of the operator $T^*$, we get

$$\|T^*\| \leq \|T\|\text{………..(1)}$$

Now applying (i) to the operator $T^*$, we get

$$\| (T^*)^* \| \leq \|T^*\|$$

or

$$\| T^{**} \| \leq \|T^*\|\text{………..(2)}$$

But by (i), we have

$$T^{**} = T\text{…………(3)}$$

Using (3) in (2), we get

$$\|T\| \leq \|T^*\|\text{………..(4)}$$

From (1) and (4), we get

$$\|T\| = \|T^*\|.$$

(iii) To prove $\|T^*T\| = \|T\|^2$,

Let us consider,

$$\|T^*T\| \leq \|T^*\|\|T\|............(5)$$

By (ii) above, $\|T^*\| = \|T\|$ so that we get from (5)

$$\|T^*T\| \leq \|T\|^2...........(6)$$

To obtain reverse inequality, let us consider

$$\|Tx\|^2 \leq \langle Tx, Tx\rangle$$

$$= \langle T^*Tx, x\rangle. \dots \dots \dots . (7)$$

By using Schwarz inequality ,we have

$$\langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\|$$

$$\leq \|T^*T\| \, \|x\|\|x\|..........(8)$$

From (7) and (8) we get

$$\|Tx\|^2 \leq \|T^*T\| \, \|x\|^2 \text{ for every } x \in H. \, ..........(9)$$

But

$$\|T\|^2 = \sup_{x\neq 0} \left\{\frac{\|T(x)\|}{\|x\|}\right\}^2$$

$$= \sup_{x\neq 0} \left\{\frac{\|T(x)\|^2}{\|x\|^2}\right\}...........(10)$$

From (9), we get

$$\|T\|^2 \leq \|T^*T\|..........(11)$$

Therefore, we have from (6) and (11),

$$\|T^*T\| = \|T\|^2$$

Taking $T^*$ instead of $T$, we get as in the above

$$\|(T^*)^*T^*\| = \|T^*\|^2.$$

Using (i) and (ii) in the above , we get

$$\|TT^*\| = \|T\|^2.$$

**Note:** From the properties of $T^*$ as discussed above , we have the following corollary.

**Corollary.** If $\{T_n\}$ is a sequence of bounded linear operators on a Hilbert space $H$ and $T_n \to T$, then $T_n^* \to T^*$.

We have from the properties of $T^*$

$$\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$$

Since $T_n \to T$ as $n \to \infty, T_n^* \to T^*$ follows from the above.

**Note:** The adjoint operator on $\beta(H)$ is one to one and onto. If $T$ is non singular operator on $H$, then $T^*$ is also non-singular and $(T^*)^{-1} = (T^{-1})^*$ .

## 10.3 Self Adjoint Operators

The motivation for the introduction of the self adjoint operators is the properties of complex numbers with conjugate mapping $z \to \bar{z}$. This mapping $z \to \bar{z}$ of the complex plane into itself behaves like the adjoint operators .This operation $z \to \bar{z}$ has all the properties of the adjoint operators. As we know that complex number is real if and only if $z = \bar{z}$. Analogue of this characterisation in $\beta(H)$ leads to the notion of self adjoint operators in Hilbert spaces.

**Definition.** An operator $T$ on a Hilbert space $H$ is said to be self adjoint operator if $T^* = T$. From this definition we have the following simple operators.

(i) $0$ and $I$ are examples of self adjoint operators.

(ii) An operator $T$ on $H$ is self adjoint operator , then $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$ and conversely.

If $T^*$ is an adjoint opearator of $T$ on $H$, then we know from the definition

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \text{ for all } x, y \in H.$$

If $T$ is self adjoint then $T^* = T$, using this in above we get ,

$$\langle Tx, y\rangle = \langle x, Ty\rangle \text{ for every } x, y \in H.$$

To prove the converse , let us assume $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$…………(1)

We have to show that $T$ is self adjoint . If $T^*$ is the adjoint of $T$, then we have

$$\langle Tx, y \rangle = \langle x, T^* y\rangle \text{……….(2)}$$

From (1) and (2), we have

$$\langle x, Ty\rangle = \langle x, T^* y\rangle$$

Which gives ,

$$\langle x, (T - T^*)y\rangle = 0 \text{ for all } x, y \in H$$

Since $x \neq 0$, we have $(T - T^*)y = 0$ for all $x, y \in H$, we have

$$T = T^*$$

Proving that $T$ is self adjoint.

(iii) For any $T \in \beta(H)$, $T + T^*$ and $T^*T$ are self adjoint.

By the propert of Hilbert adjoint operators, we have

$(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$ so that we have

$$(T + T^*)^* = T + T^*,$$

also

$(T^*T)^* = T^*T^{**} = T^*T$, so that

$(T^*T)^* = T^*T$.

Hence $T + T^*$ and $T^*T$ are self adjoint.

**Theorem.** If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if they commute. That is $ST = TS$.

**Proof.** Given $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$. Then $S^* = S, T^* = T$.

Let us assume that $S$ and $T$ commute, we will prove that $ST$ is self adjoint.

Now,

$$(ST)^* = T^* S^*$$
$$= TS$$
$$= ST$$

implies that

$$(ST)^* = ST$$

Conversely, let us assume that $ST$ is self adjoint and we will show that $ST$ commute.

By hypothesis , we have

$$(ST)^* = ST \text{……….(1)}$$

But

$$(ST)^* = T^*S^* = TS, \text{……….(2)}$$

( by properties of adjoint operators)

From (1) and (2), we have

$$ST = TS.$$

Or in other words we can say that if $ST$ is self adjoint , then they commute.

**Theorem.** An operator $T$ on a complex Hilbert space $H$ is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x$.

**Proof.** Let us assume that $T$ is self adjoint operator on $H$.

i.e, $T = T^*$, then for every $x \in H$, we have

$$\langle Tx, x\rangle = \langle x, T^*x\rangle$$

$$= \langle x, Tx\rangle$$

$$= \overline{\langle Tx, x\rangle}\,.$$

Thus $\langle Tx, x\rangle$ is equal to its own conjugate and is therefore real.

To prove the converse part, let us assume that $\langle Tx, x\rangle$ is real for all $x \in H$ and we will show that $T$ is self adjoint. Since $\langle Tx, x\rangle$ is real for all $x \in H$, we have

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$$

$$= \overline{\langle x, T^*x\rangle}$$

$$= \langle T^*x, x\rangle$$

Where $T^*$ is the adjoint of $T$ which exists for every $x \in H$, from the above we get

$$\langle Tx, x\rangle - \langle T^*x, x\rangle = 0 \text{ for all } x \in H.$$

This gives $\langle Tx - T^*x, x\rangle$ for all $x \in H$.

Hence, we have

$$\langle (T - T^*)x, x\rangle = 0 \text{ for all } x \in H.$$

As we know that, if $T$ is an operator on a Hilbert space $H$, then $\langle Tx, x\rangle = 0$ for all $x \in H$ if and only if $T = 0$.

Thus, we have

$$T - T^* = 0$$

or,

$$T = T^*.$$

Therefore the operator $T$ is self adjoint.

## 10.4 Positive Operator

As we have seen in previous section that $\langle Tx, x\rangle$ is real for self adjoint operators, we can introduce the order relation among them and define positive operators by considering the real values which the self adjoint operators take.

**Definition.** If $S$ is the set of all self-adjoint operators, we can define an order relation denoted by $\leq$ on $S$ as follows.

If $T_1, T_2 \in S$, then we write

$$T_1 \leq T_2 \text{ if } \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle \text{ for all } x \in H.$$

**Definition.** A self adjoint operator $T$ on $H$ is said to be positive if $T \geq 0$ in the order relation. This means $\langle Tx, x\rangle \geq 0 \; \forall \; x \in H$.

From the definition, we have the following properties:

(i ) The identity operator $I$ and the zero operator 0 are positive operators.

As we know that the identity operator $I$ and the zero operator 0 are self adjoint .

Further,

$$\langle Ix, x\rangle = \langle x, x\rangle$$

$$= \|x\|^2$$

$$\geq 0.$$

Also,

$$\langle 0x, x\rangle = \langle 0, x\rangle$$

$$= 0.$$

Hence $I$ and 0 are positive operators.

(ii) For an arbitrary $T$ on $H$, then $TT^*$ and $T^*T$ are positive operators.

First we note that $TT^*$ and $T^*T$ are self-adjoint. Bu using properties of adjoint operators, we get

$$(TT^*)^* = (T^*)^*T^*$$

$$= T^{**}T^*$$

$$= TT^*$$

Also,

$$(T^*T)^* = T^*(T^*)^*$$

$$= T^*T^{**}$$

$$= T^*T$$

Now we prove that they are positive ,

$$\text{i.e, } \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle$$

$$= \|T^*x\|^2$$

$$\geq 0.$$

And

$$\langle T^*Tx, x\rangle = \langle Tx, T^{**}x\rangle$$

$$= \langle Tx, Tx\rangle$$

$$= \|Tx\|^2$$

$$\geq 0.$$

Hence $TT^*$ and $T^*T$ are positive operators.

**Note:** If $T$ is a positive operator on a Hilbert space $H$, then $I + T$ is non –singular.

**Note:** If two bounded self-adjoint linear operators $S$ and $T$ on a Hilbert space $H$ are positive and commute , then their product $ST$ is positive.

`

## 10.5 Normal Operators

**Definition.** Let $H$ be a Hilbert space and let $N \in \beta(H)$ and $N^*$ be the adjoint of $N$. Then $N$ is said to be normal operator if,

$$NN^* = N^*N.$$

That is $N$ is said to be normal if it commutes with its adjoint.

From the definition of normal operator, we get the following properties.

(i) Every self adjoint operator is normal.
As since $T$ is self adjoint, we have $T^* = T$.
Hence,
$TT^* = T^*T$ is true so that $T$ is normal operator.

**Note:** A normal operator need not be self adjoint.

**Note:** The limit $N$ of any convergent sequence $\{N_k\}$ of any normal operator is normal.

**Theorem.** If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:

(i) $N_1 + N_2$ is normal .

(ii) $N_1 . N_2$ is normal .

**Proof.** Since $N_1$ and $N_2$ are normal, we get

$$N_1N_1^* = N_1^*N_1 ...............(1)$$

and

$$N_2N_2^* = N_2^*N_2 ...............(2)$$

From hypothesis either commutes with adjoint of the other.

So,

$$N_1N_2^* = N_2^*N_1 ...............(3)$$

and

$$N_2N_1^* = N_1^*N_2 ...............(4)$$

To $prove$ $(i)$, we have to show that

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)^*(N_1 + N_2). ...............(5)$$

Using the fact that adjoint operators preserves addition, we get

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)(N_1^* + N_2^*).$$

$$= N_1N_1^* + N_1N_2^* + N_2N_1^* + N_2N_2^*. ...............(6)$$

*By using* (1) (2) (3) *and* (4) *in* (6), *we get*

$$= N_1^*N_1 + N_2^*N_1 + N_1^*N_2 + N_2^*N_2$$

$$= N_1^*(N_1 + N_2) + N_2^*(N_1 + N_2)$$

$$= (N_1^* + N_2^*)(N_1 + N_2)$$

$$= (N_1 + N_2)^*(N_1 + N_2)$$

Hence,

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)^*(N_1 + N_2).$$

Therefore $N_1 + N_2$ is normal.

Now to prove (ii), we have to prove

$$(N_1N_2)(N_1N_2)^* = (N_1N_2)^*(N_1N_2)$$

Now we have

$$(N_1N_2)(N_1N_2)^* = N_1N_2N_2^*N_1^*$$

$$= N_1(N_2N_2^*)N_1^*.$$

But,

$$N_1(N_2N_2^*)N_1^* = N_1(N_2^*N_2)N_1^*$$

$$= (N_1N_2^*)(N_2N_1^*)$$

$$= (N_2^*N_1)(N_1^*N_2).$$

But,

$$(N_2^*N_1)(N_1^*N_2) = N_2^*(N_1N_1^*)N_2$$

$$= (N_2^*N_1^*)(N_1N_2)$$

$$= (N_1N_2)^*(N_1N_2).$$

Thus,

$$(N_1N_2)(N_1N_2)^* = (N_1N_2)^*(N_1N_2).$$

So that,

$N_1N_2$ is Normal.

**Note:** An operator $N$ on a Hilbert space $H$ is normal if and only if

$$\|N^*x\| = \|Nx\| \text{ for every } x \in H.$$

**Note:** If $N$ is a normal operator on $H$, then

$$\|N^2\| = \|N\|^2.$$

## 10.6 Unitary and Isometric Operators

A special type of normal operators which are of considerable interest in applied mathematics is that of unitary operators.

**Definition.** An operator $U$ on a Hilbert space $H$ is said to be unitary if

$$UU^* = U^*U = I.$$

From the definition of unitary operator, we note down the following

(i ) If $U$ is unitary, then it is normal ,

(ii) $U^* = U^{-1}$.

Before characterizing an unitary operator on a Hilbert space, we first define **isometric operator** on H.

**Definition.** An operator $T$ on $H$ is said to be isometric if

$$\|Tx - Ty\| = \|x - y\| \ \forall\, x, y \in H$$

Since $T$ is linear , the condition is equivalent to $\|Tx\| = \|x\| \ \forall\, x, y \in H$ .

**Example:** Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for a separable Hilbert space $H$ and $T \in \beta(H)$ be defined as

$$T(x_1e_1 + x_2e_2 + \cdots) = x_1e_1 + x_2e_2 + \cdots \text{ where } x = \{x_n\}$$

Then

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2$$

so $T$ is an isometric operator. The operator $T$ defined is called the right shift operator given by

$$Te_n = e_{n+1}.$$

**Note:** If $T$ is an operator on a Hilbert space $H$, then the following conditions are equivalent to one another .

(i) $T^*T = I$

(ii) $\langle Tx, Ty\rangle = \langle x, y\rangle \forall\, x, y \in H.$

(iii) $\|Tx\| = \|x\| \ \forall\, x \in H.$

**Theorem.** An operator $T$ on a Hilbert space $H$ is unitary if and only if it is an isomorphism of $H$ onto itself.

**Proof.** Let $T$ be an unitary operator on $H$. Then from the definition of the unitary operator, it is invertible. Therefore it is onto . Further,

$$TT^* = I$$

Hence,

$$\|Tx\| = \|x\| \ \forall\, x \in H$$

This proves that $T$ is an isometric isomorphism of $H$ onto itself.

Now to prove the converse let us assume that $T$ is an isometric isomorphism of $H$ onto itself.

Then $T$ is one-one and onto . Therefore $T^{-1}$ exists. From our assumption

$$\|Tx\| = \|x\| \ \forall\, x \in H. \quad \text{..........(1)}$$

By the above note we have $T^*T = I$.

Hence ,

$$(T^*T)T^{-1} = IT^{-1},$$

Which gives,

$$T^*(TT^{-1}) = T^{-1}$$

so that

$$T^*I = T^{-1}.$$

Thus

$$T^* = T^{-1}$$

Premultiplying this by $T$ we have

$$TT^* = TT^{-1}$$

So that,

$$TT^* = I$$

Now postmultiplying by $T$, we have

$$T^*T = T^{-1}T$$

$$T^*T = I.$$

Hence $T^*T = TT^* = I$.

Which proves that $T$ is unitary.

**Note:** If $T$ is an unitary operator on $H$, then $\|Tx\| = 1$.

For an unitary operator, we have

$$\|Tx\| = \|x\|,$$

So that,

$$\|T\| = \sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|=1}\|x\| = 1.$$

**Note:** The range $U(H)$ of a unitary operator $U$ is a closed subspace of $H$.

## Summary

- If $H$ is a Hilbert space and $f$ be any arbitrary functional in $H^*$, then there exists a unique vector $y$ in $H$ such that $f(x) = \langle x, y\rangle$ for every $x \in H$ and $\|f\| = \|y\|$.
- If $T: H_1 \to H_2$ be a bounded linear operator, where $H_1$ and $H_2$ are Hilbert spaces. Then the Hilbert adjoint operator $T^*$ of $T$ is the operator

$$T^*: H_2 \to H_1$$

Such that for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

- The Hilbert adjoint operator $T^*$ of $T$ exists , is unique and is bounded linear operator with norm

$$\|T^*\| = \|T\|.$$

- If $X$ and $Y$ be inner product spaces and $Q: X \to Y$ a bounded linear operator . Then:

  (i) $Q = 0$ if and only if $\langle Qx, y\rangle = 0$ for all $x \in X$ and $y \in Y$.

  (ii) If $Q: X \to X$, where $X$ is complex, and $\langle Qx, x\rangle = 0$ for all $x \in X$, then $Q = 0$.
- The adjoint operator preserves addition, reverses the product and it is conjugate linear. That is if $T \to T^*$ is the adjoint operator on $\beta(H)$, then

  a. $(T_1 + T_2)^* = T_1^* + T_2^*$

  b. $(T_1T_2)^* = T_2^*T_1^*$

  c. $(\alpha T)^* = \bar{\alpha}\,T^*$.

- Let $H$ be a Hilbert space. Then the adjoint operator $T \to T^*$ on $\beta(H)$ has the following properties:
    a. $T^{**} = T$
    b. $\|T^*\| = \|T\|$
    c. $\|T^*T\| = \|T\|^2$.
- If $\{T_n\}$ is a sequence of bounded linear operators on a Hilbert space $H$ and $T_n \to T$, then $T_n^* \to T^*$.
- The adjoint operator on $\beta(H)$ is one to one and onto. If $T$ is non singular operator on $H$, then $T^*$ is also non-singular and $(T^*)^{-1} = (T^{-1})^*$.
- An operator $T$ on a Hilbert space $H$ is said to be self adjoint operator if $T^* = T$.
- $0$ and $I$ are examples of self adjoint operators.
- If an operator $T$ on $H$ is self adjoint operator , then $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$ and conversely.
- If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if they commute. That is $ST = TS$.
- An operator $T$ on a complex Hilbert space $H$ is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x$.
- A self adjoint operator $T$ on $H$ is said to be positive if $T \geq 0$ in the order relation. This means $\langle Tx, x\rangle \geq 0 \; \forall \; x \in H$.
- The identity operator $I$ and the zero operator $0$ are positive operators.
- For an arbitrary $T$ on $H$, then $TT^*$ and $T^*T$ are positive operators.
- If $T$ is a positive operator on a Hilbert space $H$, then $I + T$ is non –singular.
- If two bounded self-adjoint linear operators $S$ and $T$ on a Hilbert space $H$ are positive and commute , then their product $ST$ is positive.
- If $H$ be a Hilbert space and let $N \in \beta(H)$ and $N^*$ be the adjoint of $N$. Then $N$ is said to be normal operator if,
$$NN^* = N^*N.$$
- Every self adjoint operator is normal.
- A normal operator need not be self adjoint.
- The limit $N$ of any convergent sequence $\{N_k\}$ of any normal operator is normal.

- If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:

a. $N_1 + N_2$ is normal .
b. $N_1 . N_2$ is normal

- An opeator $N$ on a Hilbert space $H$ is normal if and only if
$$\|N^*x\| = \|Nx\| \text{ for every } x \in H.$$
- If $N$ is a normal operator on $H$, then
$$\|N^2\| = \|N\|^2.$$
- An operator $U$ on a Hilbert space $H$ is said to be unitary if
$$UU^* = U^*U = I.$$
- If $U$ is unitary, then it is normal and $U^* = U^{-1}$.

- An operator $T$ on $H$ is said to be isometric if
$$\|Tx - Ty\| = \|x - y\| \; \forall \; x, y \in H \; .$$
- An operator $T$ on a Hilbert space $H$ is unitary if and only if it is an isomorphism of $H$ onto itself.
- If $T$ is an unitary operator on $H$, then $\|Tx\| = 1$.
- The range $U(H)$ of a unitary operator $U$ is a closed subspace of $H$.

## Keywords

- Hilbert Space
- Hilbert adjoint operator
- Self adjoint operator
- Positive operator
- Normal operator
- Unitary operator
- Isometric operator
- Schwarz inequality
- Linear operator
- Zero operator
- Bounded linear transformation

## Self Assessment

1**:** Which of the following properties is true for Hilbert adjoint operator in a Hilbert space?

A. The adjoint operator preserves addition.

B. The adjoint operator reverses the product.

C. The adjoint operator is conjugate linear.

D. All of the above.

2: Let $H$ be a Hilbert space. Then the adjoint operator $T \to T^*$ on $\beta(H)$ ( set of all bounded linear transformations on $H$ into $H$) satisfies which of the following properties:

A. $T^{**} = T$.

B. $\|T^*\| = \|T\|$.

C. $\|T^*T\| = \|T\|^2$.

D. All of the above .

3: An operator $T$ on a Hilbert space $H$ is said to be self adjoint operator if:

A. $T^* = T$.

B. $T^{**} = T$.

C. $T^* < T$.

D. $T^* > T$.

4: Which of the following is/are self adjoint operator/operators.

A. The zero operator.

B. The identity operator.

C. Both (A) and (B).

D. None of (A) and (B).

5: If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H,$ then their product $ST$ is self adjoint if and only if .

A. $S + T = T + S$.

B. $ST = TS$.

C. $ST > TS$.

D. None of the above.

6: An operator $T$ on a complex Hilbert space $H$ is self adjoint if and only if $\langle Tx, x\rangle$ is:

A. Real for all $x$.

B. Real for some $x$.

C. Real for some $x > 0$.

D. All of the above are true.

7: if $T$ is an operator on a Hilbert space $H,$ then $\langle Tx, x\rangle = 0$ for all $x \in H$ if and only if:

A. $T = 0$.

B. $T > 0$.

C. $T < 0$.

D. $T \leq 0$.

8: Which of the following is/are positive operator/operators.

A. The zero operator.

B. The identity operator.

C. Both (A) and (B).

D. None of (A) and (B).

9: Let $H$ be a Hilbert space and let $N \in \beta(H)$ and $N^*$ be the adjoint of $N$. Then $N$ is said to be normal operator if:

A. $NN^* = N$.

B. $NN^* = N^*N$.

C. $NN^* = N^*$.

D. None of the above.

10: Which of the following is/are true?

A. Every self adjoint operator is normal.

B. A normal operator need not be self adjoint.

C. The limit $N$ of any convergent sequence $\{N_k\}$ of any normal operator is normal.

D. All of the above are true.

11: Which of the following is/are true about Unitary operator?

A. $UU^* = U^*U = I$.

B. If $U$ is unitary, then it is normal.

C. $U^* = U^{-1}$.

D. All of the above are true.

12**:** Which of the following is/are true about Unitary operator?

A. If $T$ is an unitary operator on $H$, then $\|Tx\| = 1$.
B. The range $U(H)$ of a unitary operator $U$ is a closed subspace of $H$.
C. Both (A) and (B).
D. None of (A) and (B).

13**:** Which of the following is/are true about Normal operator?

A. An operator $N$ on a Hilbert space $H$ is normal if and only if $\|N^*x\| = \|Nx\|$ for every $x \in H$.
B. If $N$ is a normal operator on $H$, then $\|N^2\| = \|N\|^2$.
C. If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:
$N_1 + N_2$ is normal and $N_1.N_2$ is normal.
D. All of the above are true.

14**:** An operator $T$ on a Hilbert space $H$ is said to be isometric if:

A. $\|Tx - Ty\| \geq \|x - y\| \; \forall \, x, y \in H$ .
B. $\|Tx - Ty\| = \|x - y\| \; \forall \, x, y \in H$ .
C. $\|Tx - Ty\| < \|x - y\| \; \forall \, x, y \in H$ .
D. $\|Tx - Ty\| \leq \|x - y\| \; \forall \, x, y \in H$ .

15**:** Which of the following is/are true?
A. The adjoint operator on $\beta(H)$ is one to one.
B. The adjoint operator on $\beta(H)$ is onto.
C. Both (A) and (B).
D. None of (A) and (B)

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | D | 2 | D | 3 | A | 4 | C | 5 | B |
| 6 | A | 7 | A | 8 | C | 9 | B | 10 | D |
| 11 | D | 12 | C | 13 | D | 14 | B | 15 | C |

## Review Questions

1. State Riesz representation theorem.
2. What is Hilbert adjoint operator.
3. Define Self adjoint operator.
4. Show that the Hilbert adjoint operator $T^*$ of $T$ exists , is unique and is bounded linear operator with norm

$$\|T^*\| = \|T\|.$$

5. Show that the adjoint operator preserves addition, reverses the product and it is conjugate linear.
6. Define self adjoint operator and give examples.

7. Show that If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if they commute. That is $ST = TS$.
8. Show that normal operator need not be self adjoint.
9. If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:
   I. $N_1 + N_2$ is normal .
   II. $N_1 . N_2$ is normal .
10. Define Unitary operator and isometric operator.

## Further Readings

1. Introductory Functional Analysis With Applications By Erwin Kreyszig.
2. Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
3. J. B Conway, A Course In Functional Analysis.
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# Unit 11: Unitary and Normal Operators

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Self adjoint operator and its properties
- Positive operator and its Properties
- Normal and Unitary operator and their properties.
- Isometric operator

## Introduction

In this chapter, we discuss about Self adjoint operator and its properties. Further, positive operators and its properties are discussed. Finally we discuss about normal operator, unitary operators and isometric operator.

## 11.1 Self Adjoint Operators

The motivation for the introduction of the self adjoint operators is the properties of complex numbers with conjugate mapping $z \to \bar{z}$. This mapping $z \to \bar{z}$ of the complex plane into itself behaves like the adjoint operators .This operation $z \to \bar{z}$ has all the properties of the adjoint operators. As we know that complex number is real if and only if $z = \bar{z}$. Analogue of this characterisation in $\beta(H)$ leads to the notion of self adjoint operators in Hilbert spaces.

**Definition.** An operator $T$ on a Hilbert space $H$ is said to be self adjoint operator if $T^* = T$. From this definition we have the following simple operators.

(i) $0$ and $I$ are examples of self adjoint operators.

(ii) An operator $T$ on $H$ is self adjoint operator , then $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$ and conversely.

If $T^*$ is an adjoint opearator of $T$ on $H$, then we know from the definition

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x, y \in H.$$

If $T$ is self adjoint then $T^* = T$, using this in above we get ,

$$\langle Tx, y\rangle = \langle x, Ty\rangle \text{ for every } x, y \in H.$$

To prove the converse , let us assume $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$…………(1)

We have to show that $T$ is self adjoint . If $T^*$ is the adjoint of $T$, then we have

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \text{……….(2)}$$

From (1) and (2), we have

$$\langle x, Ty\rangle = \langle x, T^* y\rangle$$

Which gives ,

$$\langle x, (T - T^*)y\rangle = 0 \text{ for all } x, y \in H$$

Since $x \neq 0$, we have $(T - T^*)y = 0$ for all $x, y \in H$, we have

$$T = T^*$$

Proving that $T$ is self adjoint.

(iii) For any $T \in \beta(H)$, $T + T^*$ and $T^*T$ are self adjoint.

By the propert of Hilbert adjoint operators, we have

$(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$ so that we have

$$(T + T^*)^* = T + T^*,$$

also

$(T^*T)^* = T^*T^{**} = T^*T$, so that

$(T^*T)^* = T^*T$.

Hence $T + T^*$ and $T^*T$ are self adjoint.

**Theorem.** If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if they commute. That is $ST = TS$.

**Proof.** Given $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$. Then $S^* = S, T^* = T$.

Let us assume that $S$ and $T$ commute, we will prove that $ST$ is self adjoint.

Now,

$$(ST)^* = T^* S^*$$
$$= TS$$
$$= ST$$

implies that

$$(ST)^* = ST$$

Conversely, let us assume that $ST$ is self adjoint and we will show that $ST$ commute.

By hypothesis , we have

$$(ST)^* = ST \text{………..(1)}$$

But

$$(ST)^* = T^*S^* = TS, \text{………..(2)}$$

( by properties of adjoint operators)

From (1) and (2), we have

$$ST = TS.$$

Or in other words we can say that if $ST$ is self adjoint , then they commute.

**Theorem.** An operator $T$ on a complex Hilbert space $H$ is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x$.

**Proof.** Let us assume that $T$ is self adjoint operator on $H$.

i.e, $T = T^*$, then for every $x \in H$, we have

$$\langle Tx, x\rangle = \langle x, T^*x\rangle$$

$$= \langle x, Tx\rangle$$

$$= \overline{\langle Tx, x\rangle}\,.$$

Thus $\langle Tx, x\rangle$ is equal to its own conjugate and is therefore real.

To prove the converse part, let us assume that $\langle Tx, x\rangle$ is real for all $x \in H$ and we will show that $T$ is self adjoint. Since $\langle Tx, x\rangle$ is real for all $x \in H$, we have

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$$

$$= \overline{\langle x, T^*x\rangle}$$

$$= \langle T^*x, x\rangle$$

Where $T^*$ is the adjoint of $T$ which exists for every $x \in H$, from the above we get

$$\langle Tx, x\rangle - \langle T^*x, x\rangle = 0 \text{ for all } x \in H.$$

This gives $\langle Tx - T^*x, x\rangle$ for all $x \in H$.

Hence, we have

$$\langle (T - T^*)x, x\rangle = 0 \text{ for all } x \in H.$$

As we know that, if $T$ is an operator on a Hilbert space $H$, then $\langle Tx, x\rangle = 0$ for all $x \in H$ if and only if $T = 0$.

Thus, we have

$$T - T^* = 0$$

or,

$$T = T^*.$$

Therefore the operator $T$ is self adjoint.

## 11.2 Positive Operator

As we have seen in previous section that $\langle Tx, x\rangle$ is real for self adjoint operators, we can introduce the order relation among them and define positive operators by considering the real values which the self adjoint operators take.

**Definition.** If $S$ is the set of all self-adjoint operators, we can define an order relation denoted by $\leq$ on $S$ as follows.

If $T_1, T_2 \in S$, then we write

$$T_1 \leq T_2 \text{ if } \langle T_1x, x\rangle \leq \langle T_2x, x\rangle \text{ for all } x \in H.$$

**Definition.** A self adjoint operator $T$ on $H$ is said to be positive if $T \geq 0$ in the order relation. This means $\langle Tx, x\rangle \geq 0\ \forall\ x \in H$.

From the definition, we have the following properties:

(i ) The identity operator $I$ and the zero operator 0 are positive operators.

As we know that the identity operator $I$ and the zero operator 0 are self adjoint .

Further,

$$\langle Ix, x\rangle = \langle x, x\rangle$$

$$= \|x\|^2$$

$$\geq 0.$$

Also,

$$\langle 0x, x\rangle = \langle 0, x\rangle$$

$$= 0.$$

Hence $I$ and 0 are positive operators.

(ii) For an arbitrary $T$ on $H$, then $TT^*$ and $T^*T$ are positive operators.

First we note that $TT^*$ and $T^*T$ are self-adjoint. Bu using properties of adjoint operators, we get

$$(TT^*)^* = (T^*)^*T^*$$

$$= T^{**}T^*$$

$$= TT^*$$

Also,

$$(T^*T)^* = T^*(T^*)^*$$

$$= T^*T^{**}$$

$$= T^*T$$

Now we prove that they are positive ,

$$\text{i.e, } \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle$$

$$= \|T^*x\|^2$$

$$\geq 0.$$

And

$$\langle T^*Tx, x\rangle = \langle Tx, T^{**}x\rangle$$

$$= \langle Tx, Tx\rangle$$

$$= \|Tx\|^2$$

$$\geq 0.$$

Hence $TT^*$ and $T^*T$ are positive operators.

**Note.** If $T$ is a positive operator on a Hilbert space $H$, then $I + T$ is non –singular.

**Note.** If two bounded self-adjoint linear operators $S$ and $T$ on a Hilbert space $H$ are positive and commute , then their product $ST$ is positive. `

## 11.3 Normal Operators

**Definition.** Let $H$ be a Hilbert space and let $N \in \beta(H)$ and $N^*$ be the adjoint of $N$. Then $N$ is said to be normal operator if,

$$NN^* = N^*N.$$

That is $N$ is said to be normal if it commutes with its adjoint.

From the definition of normal operator, we get the following properties.

(i) Every self adjoint operator is normal.

As since $T$ is self adjoint, we have $T^* = T$.

Hence,

$TT^* = T^*T$ is true so that $T$ is normal operator.

**Note:** A normal operator need not be self adjoint.

**Note:** The limit $N$ of any convergent sequence $\{N_k\}$ of any normal operator is normal.

**Theorem.** If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:

(i) $N_1 + N_2$ is normal .

(ii) $N_1 . N_2$ is normal .

**Proof.** Since $N_1$ and $N_2$ are normal, we get

$$N_1 N_1^* = N_1^* N_1 \text{...............}(1)$$

and

$$N_2 N_2^* = N_2^* N_2 \text{...............}(2)$$

From hypothesis either commutes with adjoint of the other.

So,

$$N_1 N_2^* = N_2^* N_1 \text{...............}(3)$$

and

$$N_2 N_1^* = N_1^* N_2 \text{...............}(4)$$

To $prove$ $(i)$, we have to show that

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)^*(N_1 + N_2). \text{ ...............}(5)$$

Using the fact that adjoint operators preserves addition, we get

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)(N_1^* + N_2^*).$$

$$= N_1 N_1^* + N_1 N_2^* + N_2 N_1^* + N_2 N_2^*. \text{ ...............}(6)$$

$$By\ using\ (1)\ (2)\ (3)\ and\ \ (4)\ \ in\ (6), we\ get$$

$$= N_1^* N_1 + N_2^* N_1 + N_1^* N_2 + N_2^* N_2$$

$$= N_1^*(N_1 + N_2) + N_2^*(N_1 + N_2)$$

$$= (N_1^* + N_2^*)(N_1 + N_2)$$

$$= (N_1 + N_2)^*(N_1 + N_2)$$

Hence,

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)^*(N_1 + N_2).$$

Therefore $N_1 + N_2$ is normal.

Now to prove (ii), we have to prove

$$(N_1N_2)(N_1N_2)^* = (N_1N_2)^*(N_1N_2)$$

Now we have

$$(N_1N_2)(N_1N_2)^* = N_1N_2N_2^*N_1^*$$

$$= N_1(N_2N_2^*)N_1^*.$$

But,

$$N_1(N_2N_2^*)N_1^* = N_1(N_2^*N_2)N_1^*$$

$$= (N_1N_2^*)(N_2N_1^*)$$

$$= (N_2^*N_1)(N_1^*N_2).$$

But,

$$(N_2^*N_1)(N_1^*N_2) = N_2^*(N_1N_1^*)N_2$$

$$= (N_2^*N_1^*)(N_1N_2)$$

$$= (N_1N_2)^*(N_1N_2).$$

Thus,

$$(N_1N_2)(N_1N_2)^* = (N_1N_2)^*(N_1N_2).$$

So that,

$N_1N_2$ is Normal.

**Note:** An operator $N$ on a Hilbert space $H$ is normal if and only if

$$\|N^*x\| = \|Nx\| \text{ for every } x \in H.$$

**Note:** If $N$ is a normal operator on $H$, then

$$\|N^2\| = \|N\|^2.$$

## 11.4 Unitary and Isometric Operators

A special type of normal operators which are of considerable interest in applied mathematics is that of unitary operators.

**Definition.** An operator $U$ on a Hilbert space $H$ is said to be unitary if

$$UU^* = U^*U = I.$$

From the definition of unitary operator, we note down the following

(i ) If $U$ is unitary, then it is normal ,

(ii) $U^* = U^{-1}$.

Before characterizing an unitary operator on a Hilbert space, we first define **isometric operator** on H.

**Definition.** An operator $T$ on $H$ is said to be isometric if

$$\|Tx - Ty\| = \|x - y\| \ \forall \ x, y \in H$$

Since $T$ is linear , the condition is equivalent to $\|Tx\| = \|x\| \ \forall \ x, y \in H$ .

**Example:** Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for a separable Hilbert space $H$ and $T \in \beta(H)$ be defined as

$$T(x_1e_1 + x_2e_2 + \cdots) = x_1e_1 + x_2e_2 + \cdots \text{ where } x = \{x_n\}$$

Then

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2$$

so $T$ is an isometric operator. The operator $T$ defined is called the right shift operator given by

$$Te_n = e_{n+1}.$$

**Note:** If $T$ is an operator on a Hilbert space $H$, then the following conditions are equivalent to one another .

(i) $T^*T = I$

(ii) $\langle Tx, Ty\rangle = \langle x, y\rangle \forall \ x, y \in H.$

(iii) $\|Tx\| = \|x\| \ \forall \ x \in H.$

**Theorem.** An operator $T$ on a Hilbert space $H$ is unitary if and only if it is an isomorphism of $H$ onto itself.

**Proof.** Let $T$ be an unitary operator on $H$. Then from the definition of the unitary operator, it is invertible. Therefore it is onto . Further,

$$TT^* = I$$

Hence,

$$\|Tx\| = \|x\| \ \forall \ x \in H$$

This proves that $T$ is an isometric isomorphism of $H$ onto itself.

Now to prove the converse let us assume that $T$ is an isometric isomorphism of $H$ onto itself.

Then $T$ is one-one and onto . Therefore $T^{-1}$ exists. From our assumption

$$\|Tx\| = \|x\| \ \forall \ x \in H. \ \ldots\ldots\ldots.(1)$$

By the above note we have $T^*T = I$.

Hence ,

$$(T^*T)T^{-1} = IT^{-1},$$

Which gives,

$$T^*(TT^{-1}) = T^{-1}$$

so that

$$T^*I = T^{-1}.$$

Thus

$$T^* = T^{-1}$$

Premultiplying this by $T$ we have

$$TT^* = TT^{-1}$$

So that,

$$TT^* = I$$

Now postmultiplying by $T$, we have

$$T^*T = T^{-1}T$$

$$T^*T = I.$$

Hence $T^*T = TT^* = I$.

Which proves that $T$ is unitary.

**Note:** If $T$ is an unitary operator on $H$, then $\|Tx\| = 1$.

For an unitary operator, we have

$$\|Tx\| = \|x\|,$$

So that,

$$\|T\| = \sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|=1}\|x\| = 1.$$

**Note:** The range $U(H)$ of a unitary operator $U$ is a closed subspace of $H$.

## Summary

- An operator $T$ on a Hilbert space $H$ is said to be self adjoint operator if $T^* = T$.
- $0$ and $I$ are examples of self adjoint operators.
- If an operator $T$ on $H$ is self adjoint operator , then $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$ and conversely.
- If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if they commute. That is $ST = TS$.
- An operator $T$ on a complex Hilbert space $H$ is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x$.
- A self adjoint operator $T$ on $H$ is said to be positive if $T \geq 0$ in the order relation. This means $\langle Tx, x\rangle \geq 0 \; \forall \; x \in H$.
- The identity operator $I$ and the zero operator $0$ are positive operators.
- For an arbitrary $T$ on $H$, then $TT^*$ and $T^*T$ are positive operators.
- If $T$ is a positive operator on a Hilbert space $H$, then $I + T$ is non –singular.
- If two bounded self-adjoint linear operators $S$ and $T$ on a Hilbert space $H$ are positive and commute , then their product $ST$ is positive.
- If $H$ be a Hilbert space and let $N \in \beta(H)$ and $N^*$ be the adjoint of $N$. Then $N$ is said to be normal operator if,

$$NN^* = N^*N.$$

- Every self adjoint operator is normal.
- A normal operator need not be self adjoint.
- The limit $N$ of any convergent sequence $\{N_k\}$ of any normal operator is normal.
- If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:

a. $N_1 + N_2$ is normal .

b. $N_1 . N_2$ is normal

- An opeator $N$ on a Hilbert space $H$ is normal if and only if

$$\|N^*x\| = \|Nx\| \text{ for every } x \in H.$$

- If $N$ is a normal operator on $H$, then

$$\|N^2\| = \|N\|^2.$$

- An operator $U$ on a Hilbert space $H$ is said to be unitary if

$$UU^* = U^*U = I.$$

- If $U$ is unitary, then it is normal and $U^* = U^{-1}$.
- An operator $T$ on $H$ is said to be isometric if

$$\|Tx - Ty\| = \|x - y\| \; \forall \; x, y \in H \; .$$

- An operator $T$ on a Hilbert space $H$ is unitary if and only if it is an isomorphism of $H$ onto itself.
- If $T$ is an unitary operator on $H$, then $\|Tx\| = 1$.
- The range $U(H)$ of a unitary operator $U$ is a closed subspace of $H$.

## Keywords

- Hilbert Space
- Self adjoint operator
- Positive operator
- Normal operator
- Unitary operator
- Isometric operator
- Linear operator
- Zero operator
- Bounded linear transformation

## Self Assessment

1**:** An operator $T$ on $H$ is said to be isometric if

$\|Tx - Ty\| = \|x - y\| \; \forall \; x, y \in H$ .

A. True

B. False

2: An operator $U$ on a Hilbert space $H$ is said to be unitary if

$UU^* = U^*U = I$.

A. True

B. False

3: An operator $T$ on a Hilbert space $H$ is said to be self adjoint operator if:

A. $T^* = T$.

B. $T^{**} = T$.

C. $T^* < T$.

D. $T^* > T$.

**4:** Which of the following is/are self adjoint operator/operators.

A. The zero operator.

B. The identity operator.

C. Both (A) and (B).

D. None of (A) and (B).

5**:** If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if .

A. $S + T = T + S$.

B. $ST = TS$.

C. $ST > TS$.

D. None of the above.

6**:** An operator $T$ on a complex Hilbert space $H$ is self adjoint if and only if $\langle Tx, x\rangle$ is:

A. Real for all $x$.

B. Real for some $x$.

C. Real for some $x > 0$.

D. All of the above are true.

7: if $T$ is an operator on a Hilbert space $H$, then $\langle Tx, x\rangle = 0$ for all $x \in H$ if and only if:

A. $T = 0$.

B. $T > 0$.

C. $T < 0$.

D. $T \leq 0$.

**8:** Which of the following is/are positive operator/operators.

A. The zero operator.

B. The identity operator.

C. Both (A) and (B).

D. None of (A) and (B).

**9:** Let $H$ be a Hilbert space and let $N \in \beta(H)$ and $N^*$ be the adjoint of $N$. Then $N$ is said to be normal operator if:

A. $NN^* = N$.

B. $NN^* = N^*N$.

C. $NN^* = N^*$.

D. None of the above.

10**:** Which of the following is/are true?

A. Every self adjoint operator is normal.

B. A normal operator need not be self adjoint.

C. The limit $N$ of any convergent sequence $\{N_k\}$ of any normal operator is normal.

D. All of the above are true.

11**:** Which of the following is/are true about Unitary operator?

A. $UU^* = U^*U = I$.

B. If $U$ is unitary, then it is normal.

C. $U^* = U^{-1}$.

D. All of the above are true.

12**:** Which of the following is/are true about Unitary operator?

A. If $T$ is an unitary operator on $H$, then $\|Tx\| = 1$.
B. The range $U(H)$ of a unitary operator $U$ is a closed subspace of $H$.
C. Both (A) and (B).
D. None of (A) and (B).

13**:** Which of the following is/are true about Normal operator?

A. An operator $N$ on a Hilbert space $H$ is normal if and only if $\|N^*x\| = \|Nx\|$ for every $x \in H$.
B. If $N$ is a normal operator on $H$, then $\|N^2\| = \|N\|^2$.
C. If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:
$N_1 + N_2$ is normal and $N_1.N_2$ is normal.
D. All of the above are true.

14**:** An operator $T$ on a Hilbert space $H$ is said to be isometric if:

A. $\|Tx - Ty\| \geq \|x - y\| \; \forall \, x, y \in H$ .
B. $\|Tx - Ty\| = \|x - y\| \; \forall \, x, y \in H$ .
C. $\|Tx - Ty\| < \|x - y\| \; \forall \, x, y \in H$ .
D. $\|Tx - Ty\| \leq \|x - y\| \; \forall \, x, y \in H$ .

15: Which of the following is/are true?
A. The adjoint operator on $\beta(H)$ is one to one.
B. The adjoint operator on $\beta(H)$ is onto.
C. Both (A) and (B).
D. None of (A) and (B).

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | A | 3. | A | 4. | C | 5. | B |
| 6. | A | 7. | A | 8. | C | 9. | B | 10. | D |
| 11. | D | 12. | C | 13. | D | 14. | B | 15. | C |

## Review Questions

1. Define self adjoint operator and give examples.
2. Show that If $S$ and $T$ are self adjoint opeartors on a Hilbert space $H$, then their product $ST$ is self adjoint if and only if they commute. That is $ST = TS$.
3. Show that normal operator need not be self adjoint.
4. If $N_1$ and $N_2$ are normal operators on a Hilbert space $H$ with property that either commutes with the adjoint of the other then:
   I. $N_1 + N_2$ is normal .
   II. $N_1.N_2$ is normal .
5. Define Unitary operator and isometric operator.

## Further Readings

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# Unit 12 : Reflexivity of Hilbert Space and Orthogonal Projection

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Reflexivity of Hilbert space
- Orthogonal Projection
- Properties of Orthogonal Projection.

## Introduction

In this chapter , we discuss about reflexivity of Hilbert space. Further, we discuss about Orthogonal projection and properties of orthogonal projection.

## 12.1 Reflexivity of Hilbert space

Recall that a normed space $X$ is reflexive if there is an isometric isomorphism between $X$ and its second dual $X''$. In the following theorem we establish the reflexivity of Hilbert Spaces. Thus if $H$ is a Hilbert space and $H''$ its second dual , then it will be shown that there is a bijective linear mapping $\psi$ between $H$ and $H''$ such that

$$\|y\| = \|\psi_y\| \ \forall y \in H.$$

**Theorem.** Show that every Hilbert space is reflexive.

**Proof.** Let $H$ be a Hilbert space , $H'$ its dual and $y$ an arbitrary element of $H$. As by Riesz Representation theorem that every bounded linear functional on $H$ is of the form $f_y$ given by

$$f_y(x) = \langle x, y\rangle, x \in H \ldots\ldots\ldots.(1)$$

And that the mapping $\phi: H \to H''$ given by

$$\phi_y = f_y, y \in H \ldots\ldots\ldots(2)$$

is an isometric isomorphism between $H$ and $H'$. Now define a mapping $\psi: H \to H''$ defined by

$$(\psi_y)(g) = g(y), g \in H' \ldots\ldots\ldots.(3)$$

We show that, provided $H$ is a Hilbert space, $\psi$ is surjective mapping from $H$ to $H''$. For this we have to prove that, given any element $h$ of $H''$, there is a $z \in H$ such that

$$\psi_z = H.$$

For this consider the mapping $\phi: H \to H''$ defined by (2), $g: H \to F$ as follows:

For any $y \in H$, $\phi(y) \in H'$. Also, for any $h \in H''$, $h(\phi(y))$ is in $F$.

So we put

$$\phi(y) = f_y, y \in H.$$

Let $y \in H$. Then under $y$, we let $\psi$ mapped onto $\psi_y$, where $\psi_y$ is an element of the dual space $H''$ of $H'$ defined by :

$$g(y) = \overline{h(\phi(y))} \text{..........(4)}$$

We first show that $g$ is linear.

For $y_1, y_2 \in H$,

$$g(y_1 + y_2) = \overline{h(\phi(y_1 + y_2))}$$

$$= \overline{h(\phi(y_1) + \phi(y_2))}$$

$$= \overline{h(\phi(y_1)} + \overline{h(\phi(2)}$$

$$= g(y_1) + g(y_2) \text{............(5)}$$

And,

$$g(\alpha y) = \overline{h(\phi(\alpha y))}$$

$$= \overline{h(\bar{\alpha}(\phi(y)))}$$

$$= \overline{\alpha h(\phi(y)})$$

$$(As\ \phi\ is\ conjugate\ linear)$$

$$= \alpha g(y) \text{ ..........(6)}$$

Now ,

$$|g(y)| = |\overline{h(\phi(y))}|$$

$$= |h(\phi(y))|$$

$$\leq \|h\|\|\phi(y)\|$$

$$\leq \|h\|\|y\| , \|\phi(y)\| = \|f(y)\| = \|y\|,$$

so that ,

$$\|g\| \leq \|h\|,$$

Where $\|h\|$ is finite because $h \in H''$. Hence $g$ is a bounded linear functional in $H'$. By Riesz represemtation theoremthere is a unique $z \in H$ such that

$$g(y) = \langle y, z\rangle = \overline{h(\phi(y))}$$

From (4) or equivalentely

$$h(\phi(y)) = \langle z, y\rangle.$$

But then , from (3) and the definition of $\phi$ we have:

$$(\psi_z)(\phi(y)) = f_y(z) = \langle z, y\rangle = h(\phi(y))$$

for all $y \in H, \phi(y) \in H'$. Hence

$$\psi_z = h \text{..........(7)}$$

Thus $\psi$ is surjective.

To see that $\psi$ is injective, suppose that, for $y_1, y_2 \in H$,

$$\psi_{y_1} = \psi_{y_2}$$

Then

$$\left(\psi_{y_1}\right)(g) = g(y_1) = g(y_2) = (\psi_{y_2})(g)$$

For all $g \in H'$. Hence , by (1)

$$\langle x, y_1\rangle = \langle x, y_2\rangle$$

For all $x \in H$ because each $g$ is given by such an expression.

Therefore,

$$y_1 = y_2.$$

To show that $\psi$ is linear, let $y_1, y_2 \in H$. Then

$$(\psi_{y_1})(g) = g(y_1)$$

$$(\psi_{y_2})(g) = g(y_2)$$

And

$$(\psi_{y_1+y_2})(g) = g(y_1 + y_2)$$

$$= g(y_1) + g(y_2)$$

$$= (\psi_{y_1})(g) + (\psi_{y_2})(g)$$

$$= (\psi_{y_1} + \psi_{y_2})(g) \qquad \text{..........(8)}$$

while

$$(\psi_{(\alpha y)})g = g(\alpha y)$$

$$= \alpha g(y)$$

$$= \alpha(\psi_y)g$$

For all $g \in H'$. Hence

$$\psi_{(y_1+y_2)} = \psi_{y_1} + \psi_{y_2}$$

and

$$\psi(\alpha y) = \alpha\psi_y.$$

Lastly, to see that $\psi$ is an isometry , let $y \in H$.

Then by Riesz representation theorem,

$$\|y\| = \|\phi(y)\|$$

Where $\phi(y)$ is in $H'$. Again by Riesz representation theorem, any $g \in H'$ is given by $g(z) = \langle z, x\rangle, z \in H, x \in H$ and $\phi(y) = g$.

$$\|y\| = \|g\| = \sup_{\|z\|=1_{z\in H}} |g(z)|$$

$$= \text{Sup}_{\|g\|=1_{g\in H'}} |(\psi_y)(g)|$$

$$= \|\psi_y\|.$$

Hence $\psi$ is an isometric isomorphism between $H$ and $H''$. So $H$ is reflexive.

## 12.2 Orthogonal Projection

Let $Y$ be a closed subspace of a Hilbert space $H$. Then we know that

$$H = Y \oplus Y^{\perp} \qquad \text{..........(1)}$$

$x = y + z$ , $\qquad (y \in Y, z \in Y^{\perp})$.

Since the sum is direct, $y$ is unique for any given $x \in H$. Hence (1) defines a linear operator

$$P: H \to H$$

$$x \longrightarrow y = Px$$

$P$ is called an orthogonal projection or projection on $H$. More specifically, $P$ is called the projection of $H$ onto $Y$. Hence a linear operator $P: H \to H$ is a projection on $H$ if there is a closed subspace $Y$ of $H$ such that $Y$ is the range of $P$ and $Y^{\perp}$ is the null space of $P$ and $P/Y$ is the identity operator on $Y$. From (1), we can now write

$$x = y + z$$

$$= Px + (I - P)x.$$

This shows that the projection of $H$ onto $Y^{\perp}$ is $I - P$.

There is another characterization of a projection on $H$, which is sometimes used as a definition.

**Theorem 1 .** A bounded linear operator $P: H \to H$ on a Hilbert space $H$ is a projection if and only if $P$ is self adjoint and idempotent ( $that\ is, P^2 = P$).

**Proof.** Suppose that $P$ is a projection on $H$ and denote $P(H)$ by $Y$. Then $P^2 = P$ because for every $x \in H$ and $Px = y \in Y$ we have

$$P^2x = Py = y = Px.$$

Furthermore, let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, where $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^{\perp}$.Then

$$\langle y_1, z_2\rangle = \langle y_2, z_1\rangle = 0$$

because $Y \perp Y^{\perp}$, and self adjointness of $P$ is seen from

$$\langle Px_1, x_2\rangle = \langle y_1, y_2 + z_2\rangle = \langle y_1, y_2\rangle = \langle y_1 + z_1, y_2\rangle = \langle x_1, Px_2\rangle$$

Conversely suppose that $P^2 = P = P^*$ and denote $P(H)$ by $Y$. Then for every $x \in H$

$$x = Px + (I - P)x.$$

Orthogonality $Y = P(H) \perp (I - P)(H)$ follows from

$$\langle Px, (I - P)v\rangle = \langle x, P(I - P)v\rangle = \langle x, Pv - P^2v\rangle = \langle x, 0\rangle = 0.$$

$Y$ is the null space $N(I - P)$ *of* $(I - P)$, because $Y \subset N(I - P)$ can be seen from

$$(I - P)Px = Px - P^2x = 0$$

and $Y \supset N(I - P)$ follows if we note that $(I - P)x = 0$ implies $x = Px$. Hence $Y$ is closed . Finally, $P\backslash Y$ is the identity operator on $Y$ since writing $y = Px$, we have $Py = P^2x = Px = y$.

**Theorem.** For any projection $P$ on a Hilbert space $H$,

$$\langle Px, x\rangle = \|Px\|^2 \text{..........(2)}$$

$$P \geq 0 \text{...............(3)}$$

$$\|P\| \leq 1; \quad \|P\| = 1 \text{ if } P(H) \neq \{0\} \text{..........(4)}$$

**Proof.** (2) and (3) follows from

$$\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0$$

.

By the Schwarz inequality

$$\|Px\|^2 = \langle Px, x\rangle \leq \|Px\|\|x\|$$

So that $\frac{\|Px\|}{\|x\|} \leq 1$ for every $x \neq 0$, and $\|P\| \leq 1$.

Also $\frac{\|Px\|}{\|x\|} = 1$ if $x \in P(H)$ and $x \neq 0$.

This proves (5).

**Note:** Every projection is linear. For if $x_1, x_2 \in H$, then

$$x_1 = y_1 + z_1, y_1 \in Y, z_1 \in Y^{\perp}$$

$$x_2 = y_2 + z_2, y_2 \in Y, z_1 \in Y^{\perp}$$

So that,

$$P(x_1 + x_2) = P(y_1 + y_2 + z_1 + z_2)$$

$$= y_1 + y_2$$

$$= P(x_1) + P(x_2)$$

Also for any $\alpha \in F$,

$$P(\alpha x_1) = P(\alpha y_1 + \alpha z_1)$$

$$= \alpha y_1$$

$$= \alpha P(x_1) \forall\, x_1 \in H$$

**Note:** The product of projections need not be a projection .

**Note:** The Product of two bounded self adjoint linear operators $S$ and $T$ on a Hilbert space $H$ is self adjoint if and only if the operators commute,

$ST = TS$. (Already proved in chapter 10)

**Theorem. ( Product of projections)**

In connection with product (composites ) of projections on a Hilbert space $H$, the following two statements hold.

(i) $P = P_1P_2$ is a projection on $H$ if and only if the projections $P_1$ and $P_2$ commute, that is , $P_1P_2 = P_2P_1$. Then $P$ projects $H$ onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H)$.

(ii) Two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_YP_V = 0$.

**Proof.** (i) Suppose that $P_1P_2 = P_2P_1$. Then $P$ is self adjoint , by above note. $P$ is idempotent since

$$P^2 = (P_1P_2)(P_1P_2)$$

$$= P_1^2P_2^2$$

$$= P_1P_2$$
$$= P.$$

Hence $P$ is a projection by theorem 1 above , and for every $x \in H$ we have

$$Px = P_1(P_2x) = P_2(P_1x).$$

Since $P_1$ projects $H$ onto $Y_1$,we must have $P_1(P_2x) \in Y_1$.

Similarly,

$P_2(P_1x) \in Y_2$.

Together, $Px \in Y_1 \cap Y_2$. Since $x \in H$ was arbitrary, this shows that $P$ projects $H$ into $Y = Y_1 \cap Y_2$.

Actually, $P$ projects $H$ onto $Y$. Indeed, if $y \in Y$, then $y \in Y_1, y \in Y_2$ and

$$Py = P_1P_2y = P_1y = y.$$

Conversely, if $P = P_1P_2$ is a projection defined on $H$, then $P$ is self adjoint by theorem 1 above and $P_1P_2 = P_2P_1$ follows by above note.

**Proof of (ii)** . if $Y \perp V$ then $Y \cap V = \{0\}$

and

$P_YP_Vx = 0 \; \forall \; x \in H$ by part (i), so that $P_YP_V = 0$.

Conversely, if $P_YP_V = 0$, then for every $y \in Y$ and $v \in V$ we obtain

$$\langle y, v\rangle = \langle P_Yy, P_Vv\rangle$$
$$= \langle y, P_YP_Vv\rangle$$
$$= \langle y, v\rangle$$
$$= 0.$$

Hence $Y \perp V$.

Theorem (**Sum of projections**). Let $P_1$ and $P_2$ be projections on a Hilbert space $H$. Then

(i) The sum $P = P_1 + P_2$ is a projection on $H$ if and only $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal .

(ii) If $P = P_1 + P_2$ is a projection , $P$ projects $H$ onto $Y = Y_1 \oplus Y_2$.

**Proof.** If $P = P_1 + P_2$ is a projection, $P = P^2$, by theorem 1 above, we have

$$P_1 + P_2 = (P_1 + P_2)^2$$
$$= P_1^2 + P_1P_2 + P_2P_1 + P_2^2.$$

By theorem 1 above , we have

$$P_1^2 = P_1$$

and

$$P_2^2 = P_2,$$

Therefore,

$$P_1P_2 + P_2P_1 = 0 \text{ ……(5)}$$

Multiplying by $P_2$ from the left, we obtain

$$P_2P_1P_2 + P_2P_1 = 0 \text{ ……(6)}$$

Multiplying this by $P_2$ from the right , we obtain

$$2P_2P_1P_2 = 0,$$

so that $P_2P_1 = 0$ by (6) and $Y_1 \perp Y_2$ by (ii) part of above theorem.

Conversely, if $Y_1 \perp Y_2$, then $P_1P_2 = P_2P_1 = 0$ again by (ii) part of above theorem.

This yields (5), which implies $P^2 = P$. Since $P_1$ and $P_2$ are self adjoint, so is

$P = P_1 + P_2$. Hence $P$ is a projection by Theorem 1.

**Proof of (ii).** We determine the closed subspace $Y \subset H$ onto which $P$ projects. Since $P = P_1 + P_2$, for every $x \in H$ we have

$$y = Px = P_1x + P_2x.$$

Here $P_1x \in Y_1$ and $P_2x \in Y_2$. Hence $y \in Y_1 \oplus Y_2$, so that $Y \subset Y_1 \oplus Y_2$.

We show that $Y \supset Y_1 \oplus Y_2$.

Let $v \in Y_1 \oplus Y_2$ be arbitrary. Then

$v = y_1 + y_2$. Here, $y_1 \in Y_1$ and $y_2 \in Y_2$,

Apply $P$ and using $Y_1 \perp Y_2$, we thus obtain

$$\begin{aligned} Pv &= P_1(y_1 + y_2) + P_2(y_1 + y_2) \\ &= P_1y_1 + P_2y_1 \\ &= y_1 + y_2 \\ &= v. \end{aligned}$$

Hence $v \in Y$ and $Y \supset Y_1 \oplus Y_2$.

Thus,

$$Y = Y_1 \oplus Y_2.$$

## 12.3 Further properties of Projections

We now discuss some further properties of Projections.

1) Let $P_1$ and $P_2$ be projections defined on a Hilbert space $H$. Denote by $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ the subspaces onto which $H$ is projected by $P_1$ and $P_2$, and let $N(P_1)$ and $N(P_2)$ be the null spaces of these projections . Then the following conditions are equivalent:

    a. $P_2P_1 = P_1P_2 = P_1$
    b. $Y_1 \subset Y_2$
    c. $N(P_1) \supset N(P_2)$
    d. $\|P_1x\| \le \|P_2x\| \;\forall\, x \in H$
    e. $P_1 \le P_2$.

2) Let $P_1$ and $P_2$ be projections defined on a Hilbert space $H$. Then

    (i) The difference $P = P_2 - P_1$ is a projection on $H$ if and only if $Y_1 \subset Y_2$, where $Y_j = P_j(H)$.

    (ii) If $P = P_2 - P_1$ is a projection, $P$ projects $H$ onto $Y$, where $Y$ is the orthogonal complement of $Y_1$ in $Y_2$.

From these two properties of projection we can now derive a basic result about the convergence of a monotonic increasing sequence of projections.

3) **(Monotone increasing sequence)**. Let $\{P_n\}$ be a monotone increasing sequence of projections $P_n$ defined on the Hilbert space $H$. Then

    (i) $\{P_n\}$ is strongly operator convergent, say $P_nx \to Px$ for every $x \in H$, and the limit operator $P$ is a projection defined on $H$.

    (ii) $P$ projects $H$ onto

$$P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}.$$

(iii) $P$ has the null space

$$N(P) = \bigcap_{n=1}^{\infty} N(P_n).$$

**Notes:** Two projections $P$ and $Q$ on a Hilbert space $H$ are said to be orthogonal if

$$PQ = 0.$$

**Notes:** If $P$ is the projection on the closed linear subspace $Y$ of $H$, then $x \in Y$ if and only if $Px = x$.

**Notes:** If $P$ is the projection on the closed linear subspace $Y$ of $H$, then $Px = x$. if and only if

$$\|Px\| = \|x\|.$$

**Notes:** If $P$ is the projection on a Hilbert space $H$, then

(i) $P$ is a positive operator on $H$.

(ii) $0 \leq P \leq 1$

(iii) $\|Px\| \leq \|x\|$ for every $x \in H$

(iv) $\|P\| \leq \|1\|$.

**Notes:** A projection on $H$ whose range and null spaces are orthogonal is called perpendicular projection.

**Notes:** If $P$ is the projection on a closed linear subspace $Y$ of $H$ if and only if $(I - P)$ is a projection on $M^{\perp}$.

## Summary

- Every Hilbert space is reflexive.
- If $Y$ be a closed subspace of a Hilbert space $H$. Then we know that $H = Y \oplus Y^{\perp}$.
- A bounded linear operator $P: H \to H$ on a Hilbert space $H$ is a projection if and only if $P$ is self adjoint and idempotent.
- Every projection is linear.
- The product of projections need not be a projection .
- The Product of two bounded self adjoint linear operators $S$ and $T$ on a Hilbert space $H$ is self adjoint if and only if the operators commute.
- The product of projections $P = P_1 P_2$ is a projection on $H$ if and only if the projections $P_1$ and $P_2$ commute, that is , $P_1 P_2 = P_2 P_1$. Then $P$ projects $H$ onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H)$.
- Two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$.
- If $P_1$ and $P_2$ be projections on a Hilbert space $H$. Then

  (i) The sum $P = P_1 + P_2$ is a projection on $H$ if and only

$Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal .

(ii) If $P = P_1 + P_2$ is a projection , $P$ projects $H$ onto $Y = Y_1 \oplus Y_2$.

- Two projections $P$ and $Q$ on a Hilbert space $H$ are said to be orthogonal if $PQ = 0$.
- If $P$ is the projection on the closed linear subspace $Y$ of $H$, then $x \in Y$ if and only if $Px = x$.
- If $P$ is the projection on the closed linear subspace $Y$ of $H$, then $Px = x$. if and only if $\|Px\| = \|x\|$.
- If $P$ is the projection on a Hilbert space $H$, then
  - (i) $P$ is a positive operator on $H$.
  - (ii) $0 \leq P \leq 1$
  - (iii) $\|Px\| \leq \|x\|$ for every $x \in H$
  - (iv) $\|P\| \leq \|1\|$.
- A projection on $H$ whose range and null spaces are orthogonal is called perpendicular projection.
- If $P$ is the projection on a closed linear subspace $Y$ of $H$ if and only if $(I - P)$ is a projection on $M^{\perp}$.

## Keywords

- Hilbert space
- Isometric isomorphism
- Reflexivity
- Dual
- Linear mapping
- Projection
- Orthogonal projection

## Self Assessment

1: Which of the following statements is/are true?

A. Every Hilbert space is reflexive.

B. Every Banach space is reflexive.

C. Both (A) and (B).

D. None of (A) and (B).

2: Which of the following statements is/are true?

A. A bounded linear operator $P: H \to H$ on a Hilbert space $H$ is a projection if and only if $P$ is self adjoint and idempotent.

B. Every projection is linear.

C. Both (A) and (B).

D. None of (A) and (B).

3: For any projection $P$ on a Hilbert space $H$,which of the following is/are true?

A. $\langle Px, x\rangle = \|Px\|^2$.

B. $P \geq 0$.

C. $\|P\| \leq 1$ ; $\|P\| = 1$ if $P(H) \neq \{0\}$.
D. All of the above are true.

4: Which of the following statement is/are true?
A. The product of projection is always a projection.
B. $P = P_1P_2$ is a projection on $H$ if and only if the projections $P_1$ and $P_2$ commute, that is , $P_1P_2 = P_2P_1$.
C. The product of projection is never a projection.
D. None of the above.

5: Which of the following statement is/are true?

A. Two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_YP_V = 0$.
B. Two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_YP_V > 0$.
C. Two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_YP_V < 0$.
D. All of the above are true.

6: Let $P_1$ and $P_2$ be projections on a Hilbert space $H$. Then Which of the following statement is/are true?

A. The sum $P = P_1 + P_2$ is a projection on $H$ if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal .
B. If $P = P_1 + P_2$ is a projection if $P$ projects $H$ onto $Y = Y_1 \oplus Y_2$.
C. Both (A) and (B) are true.
D. None of the above.

7: Two projections $P$ and $Q$ on a Hilbert space $H$ are said to be orthogonal if:

A. $PQ = 1$.
B. $PQ.1$.
C. $PQ = 0$.
D. None of the above.

8: If $P$ is the projection on the closed linear subspace $Y$ of $H$, then $Px = x$. if and only if:

A. $\|Px\| = x$.
B. $\|Px\| = \|x\|$.
C. $\|Px\| = 1$.
D. None of the above.

9: If $P$ is the projection on a Hilbert space $H$, then which of the following is/are true?

A. $P$ is a positive operator on $H$.
B. $0 \leq P \leq 1$
C. $\|P\| \leq \|1\|$.
D. All of the above are true.

10: Which of the following statement is/are true?

A. A projection on $H$ whose range and null spaces are orthogonal is called perpendicular projection.
B. If $P$ is the projection on a closed linear subspace $Y$ of $H$ if and only if $(I - P)$ is a projection on $M^{\perp}$.
C. If $P$ is the projection on the closed linear subspace $Y$ of $H$, then $x \in Y$ if and only if $Px = x$.
D. All of the above are true.

11. If $P$ is the projection on a closed linear subspace $Y$ of $H$ if and only if:

A. $(I - P)$ is a projection on $M^{\perp}$.
B. $(I - P)$ is a projection on $M^{\perp\perp}$.
C. $(I - P)$ is a projection on $M$.
D. None of the above .

12. Let $P_1$ and $P_2$ be projections defined on a Hilbert space $H$. Then

A. The difference $P = P_2 - P_1$ is a projection on $H$ if and only if $Y_1 \subset Y_2$, where $Y_j = P_j(H)$.
B. If $P = P_2 - P_1$ is a projection, if $P$ projects $H$ onto $Y$, where $Y$ is the orthogonal complement of $Y_1$ in $Y_2$.
C. Both (A) and (B) are true.
D. None of the above.

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | C | 3. | D | 4. | B | 5. | A |
| 6. | C | 7. | C | 8. | B | 9. | D | 10. | D |
| 11. | A | 12. | C | | | | | | |

## Review Questions

1. Show that every Hilbert space is reflexive.
2. Show that a bounded linear operator $P: H \rightarrow H$ on a Hilbert space $H$ is a projection if and only if $P$ is self adjoint and idempotent.
3. Show that every projection is linear.
4. Show that two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$.
5. Show that product $P = P_1 P_2$ is a projection on $H$ if and only if the projections $P_1$ and $P_2$ commute, that is , $P_1 P_2 = P_2 P_1$.

## Further Readings

- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.

- C. Goffman G Pedrick, A First Course In Functional Analysis.
- B.V. Limaya, Functional Analysis.

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# Unit 13: Spectral Theory of Linear Operators in Normal Spaces

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Spectrum of an operator
- Spectral properties of bounded linear operators

## Introduction

In this chapter, we discuss about Spectrum of an operator. Further, we discuss about spectral properties of bounded self-adjoint linear operator.

## 13.1 Spectrum of an Operator

The generalization of the matrix eigenvalue theory leads to the spectral theory of operators on a Banach space or Hilbert space. Before defining the spectrum of an operator, we first recall some definitions.

**Definition.** Let $T$ be an operator on a Hilbert space $H$. Then a scalar $\lambda$ is called an eigenvalue of $T$ if there exists a non zero vector $x$ in $H$ such that

$$Tx = \lambda x.$$

**Note:** Eigenvalue is also called characteristic value, proper value or spectral value.

**Definition.** If $\lambda$ is an eigenvalue of $T$, then any non zero vector $x$ in $H$ such that $Tx = \lambda x$ is called an eigenvector of $T$.

**Note:** Eigen vector is also called characteristic vector, proper vector or spectral vector.

**Definition.** The eigenvectors corresponding to eigenvalue $\lambda$ and the zero vector form a vector subspace, which is called the eigenspace of $T$ corresponding to eigenvalue $\lambda$.

From the definition of eigenvalues and eigenvectors, we have the following properties.

I. If $x$ is an eigen vector of $T$ corresponding to eigenvalue $\lambda$ and $\alpha$ is any nonzero scalar, then $\alpha x$ is also an eigenvector of $T$ corresponding to same eigen value.

Since $x$ is an eigenvector of $T$ corresponding to the eigen value $\lambda$ and $Tx = \lambda x$. Since $\alpha \neq 0$, we have

$$\alpha x \neq 0.$$

Hence (I) follows from

$$T(\alpha x) = \alpha T x$$

$$= \alpha \lambda x,$$

Which gives,

$$T(\alpha x) = \lambda(\alpha x).$$

**Note:** Thus (I) tells us that corresponding to single eigenvalue there may correspond more than one eigenvector.

**Note:** If $x$ is an eigenvector of $T$, then $x$ cannot correspond to more than one eigenvalue of $T$.

**Note:** If the Hilbert space has no non-zero vectors, then $T$ cannot have any eigenvectors and hence the whole theory reduces to triviality. So we shall assume throughtout this chapter $H \neq \{0\}$.

**Spectrum of an operator**

**Definition.** The set of all eigenvalues of $T$ is called spectrum of $T$ and is denoted by $\sigma(T)$. Its complement $\rho(T) = \boldsymbol{C} - \sigma(T)$ in the complex plane is called resolvent set of $T$.

**Example:** For a two dimensional Hilbert space $H$, let $B = \{e_1, e_2\}$ be a basis and $T$ be an operator on $H$ given by the matrix

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \ldots\ldots\ldots\ldots(1)$$

If $T$ is given by $Te_1 = e_2$ and $Te_2 = -e_1$, find the spectrum of $T$.

**Solution.** Using the matrix $A$ of the operator $T$, we have

$$Te_1 = \alpha_{11}e_1 + \alpha_{21}e_2$$

$$= e_2,$$

So that,

$\alpha_{11} = 0$ and $\alpha_{21} = 1$

$$Te_2 = \alpha_{12}e_1 + \alpha_{22}e_2$$

$$= -e_1,$$

So that,

$\alpha_{12} = -1$ and $\alpha_{22} = 0$

Hence the matrix representation of $T$ is

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For this matrix, the eigenvalues are given by the characteristic equation

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\Rightarrow \lambda = \pm i,$$

So that,

$$\sigma(T) = \{\pm i\}.$$

**Note:** An operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non zero vector $x$ in $H$ such that $Tx = 0$.

**Theorem.** If $T$ is an operator on a finite dimensional Hilbert space, then the following statements are true.

(i) $T$ is singular if and only if $0 \in \sigma(T)$.

(ii) If $T$ is non -singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.

(iii) If $A$ is non singular, then $\sigma(ATA^{-1}) = \sigma(T)$.

(iv) If $\lambda \in \sigma(T)$ and if $P$ is a polynomial , then $P(\lambda) \in \sigma\big(P(T)\big)$.

**Proof.** (i) We know that $T$ is singular if and only if there exists a non-zero vector $x \in H$ such that $Tx = 0$.

That is, $Tx = 0x$.

Hence $T$ is singular if and only if 0 is the eigenvalue of $T$.

That is $0 \in \sigma(T)$.

(ii) Let $T$ be non-singular and $\lambda \in \sigma(T)$. Hence $\lambda \neq 0$ by (i) so that$\lambda^{-1}$ exist . Since $\lambda$ is an eigen value of $T$, so there exixts a non –zero vector $x \in H$ such that

$$Tx = \lambda x.$$

Premultiplying by $T^{-1}$ we get

$$T^{-1}Tx = T^{-1}(\lambda x),$$

Which gives,

$$T^{-1}(x) = \frac{1}{\lambda}x \text{ for } x \neq 0.$$

Hence $\lambda^{-1} \in \sigma(T)$.

(iii) Let $S = ATA^{-1}$. Then we find $S - \lambda I$.

Now $S - \lambda I = ATA^{-1} - A(\lambda I)A^{-1}$

$$= A(T - \lambda I)A^{-1}.$$

Hence,

$\det(S - \lambda I) = \det(A(T - \lambda I)A^{-1})$.

But,

$$\det(A(T - \lambda I)A^{-1}) = \det(T - \lambda I)$$

This proves that $\det(S - \lambda I) = \det(T - \lambda I)$.

Thus $\lambda$ is an eigen value of $T$ if and only if $\det(T - \lambda I) = 0$.

Hence,

$\det(T-\lambda I)=0$ if and only if $\det(S-\lambda I)=0$.

This proves that $S$ and $T$ have the same eigenvalues so that

$$\sigma(ATA^{-1})=\sigma(T).$$

(iv) If $\lambda\in\sigma(T)$, $\lambda$ is an eigen value of $T$. Then there exists a non-zero vector $x$ such that

$$Tx=\lambda x.$$

Hence,

$$T(Tx)=T(\lambda x)$$

$$=\lambda Tx$$

$$=\lambda^2 x.$$

Hence if $\lambda$ is an eigenvalue of $T$, then $\lambda^2$ is an eigenvalue of $T^2$. Continuing in this way, we see that if $\lambda$ is an eigen value of $T$, then $\lambda^n$ is an eigenvalue of $T^n$ for any positive integer $n$.

Let $P(t)=a_0+a_1t+a_2t^2+\cdots a_mt^m$, where $a_0,a_1,a_2,\ldots,a_m$ are scalars. Then

$$[P(T)]x=(a_0I+a_1T+a_2T^2+\cdots a_mT^m)x$$

$$=a_0x+a_1(\lambda x)+\cdots a_m(\lambda^m x)$$

$$=[a_0+a_1(\lambda)+\cdots a_m(\lambda^m)]x.$$

Hence $P(\lambda)=a_0+a_1\lambda+\cdots+a_m\lambda^m$ is an eigen value of $P(T)$. Thus if $\lambda\in\sigma(T)$, then $P(\lambda)\in\sigma\big(P(T)\big)$.

**Note:** An operator on a Hilbert space $H$ need not necessarily posses an eigenvalue as illustrated by the following Example.

**Example:** Consider the Hilbert space $l_2$ and $T$ on $l_2$ defined by

$$T(x_1,x_2,\ldots,x_n)=\{0,x_1,x_2,\ldots\}.$$

If $\lambda$ is an eigenvalue of T, then there exists a non zero vector $(x_1,x_2,\ldots,x_n)$ such that

$$Tx=\lambda x$$

Which gives,

$$\{0,x_1,x_2,\ldots\}=(\lambda x_1,\lambda x_2,\ldots,\lambda x_n),$$

which implies,

$$x_1=0,\lambda x_2=x_1,\lambda x_n=x_{n-1}\ \ldots$$

By hypothesis $x=\{x_n\}\in l_2$ is non zero vector so that $x_n\neq 0$ for any $n$.

Hence $\lambda x_1=0$ implies $\lambda=0$ and $\lambda x_2=x_1$ implies $x_1=0$ contracdicting that $x$ is non-zero vector.

Hence $T$ cannot have eigen values.

**Theorem.** (Spectral mapping theorem for polynomials )

Let $T$ be an operator on a complex Banach space $B$ and let $p$ be a polynomial. Then

$$\sigma\big(p(T)\big)=p\big(\sigma(T)\big)=\{p(\lambda):\lambda\in\sigma(T)\}.$$

**Example.** Find the spectrum of the idempotent operator $T$ on a Banach space.

Since $T$ is idempotent operator, then

$$T^2 = T$$

or

$$T^2 - T = 0.$$

Let $p(T) = T^2 - T$. Then $p(T) = 0$ by hypothesis.

Hence

$$p\big(\sigma^2(T)\big) = \sigma^2(T) - \sigma(T)$$

$$= \sigma(T)(\sigma(T) - 1) = 0$$

So that $\sigma(T) = 1 \; or \; \sigma(T) = 0$

Hence

$$\sigma(T) = \{0,1\}.$$

## 13.2 Spectral Properties of Bounded Self-Adjoint Linear Operators

Throughout this section we shall consider bounded linear operators which are defined on a complex Hilbert space $H$ and map $H$ into itself. Furthermore, these operations will be self-adjoint. A bounded self-adjoint linear operator $T$ may not have eigenvalues, but if $T$ has eigenvalues, the Following basic facts can readily be established.

**Theorem.** Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H$. Then

i. All the eigen values of $T$ (if they exists) are real.
ii. Eigenvectors corresponding to numerically different eigenvalues of $T$ are orthogonal.

**Proof.** (i) Let $\lambda$ be an eigenvalue of $T$ and $x$ a corresponding eigenvector. Then $x \neq 0$ and $Tx = \lambda x$. Using the self-adjointness of $T$, we have

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle$$

$$= \langle x, Tx\rangle$$

$$= \langle x, \lambda x\rangle$$

$$= \bar{\lambda}\langle x, x\rangle .$$

Here $\langle x, x\rangle = \|x\|^2 \neq 0$, and division by $\langle x, x\rangle$ gives $\lambda = \bar{\lambda}$.

Hence $\lambda$ is real.

**Proof of (ii).** Let $\lambda$ and $\mu$ be eigen values of $T$, and let $x$ and $y$ be corresponding eigenvectors . Then

$$Tx = \lambda x$$

And

$$Ty = \mu y,$$

Since $T$ is self-adjoint and $\mu$ is real,

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle$$

$$= \langle x, Ty\rangle$$

$$= \langle x, \mu y\rangle$$

$$= \mu\langle x, y\rangle \;.$$

Since $\lambda \neq \mu$, we must have

$$\langle x, y\rangle = 0,$$

Which shows that $x$ and $y$ are orthogonal.

**Note** **(**Resolvent set) Let $T: H \rightarrow H$ be a bounded self adjoint linear operator on a complex Hilbert space $H$. Then a number $\lambda$ belongs to the resolvent set $\rho(T)$ of $T$ if and only if there exists a $c > 0$ such that for every $x \in H$,

$$\|T_\lambda x\| \geq c\|x\| \ldots\ldots\ldots\ldots(1)$$

**Theorem.** (Spectrum) The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \rightarrow H$ on a complex Hilbert space $H$ is real.

**Proof.** By above note , we show that $a\lambda = \alpha + i\beta$ $(\alpha, \beta \in R)$ with $\beta \neq 0$ must belong to $\rho(T)$, so that , $\sigma(T) \subset R$.

For every $x \neq 0$ in $H$ we have

$$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle$$

and, since $\langle x, x\rangle$ and $\langle Tx, x\rangle$ are real,

$$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \bar{\lambda}\langle x, x\rangle.$$

Here $\bar{\lambda} = \alpha - i\beta$. By subtraction,

$$\overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = (\lambda - \bar{\lambda})\langle x, x\rangle$$

$$= 2i\beta\|x\|^2.$$

The left side is $-2i\, Im\,\langle T_\lambda x, x\rangle$, where $Im$ denotes the imaginary part, the latter cannot exceed the absolute value, so that, dividing by 2, taking absolute values and applying the Schwarz inequality, we obtain

$$|\beta|\|x\|^2 = |Im\,\langle T_\lambda x, x\rangle| \leq |\,\langle T_\lambda x, x\rangle| \leq \|T_\lambda x\|\|x\|.$$

Division by $\|x\| \neq 0$ gives $|\beta|\|x\| \leq \|T_\lambda x\|$.

If $\beta \neq 0$, then $\lambda \in \rho(T)$ by above note . Hence for $\lambda \in \sigma(T)$, we must have $\beta = 0$, that is, $\lambda$ is real.

## Summary

- If $T$ be an operator on a Hilbert space $H$. Then a scalar $\lambda$ is called an eigenvalue of $T$ if there exists a non zero vector $x$ in $H$ such that
  $$Tx = \lambda x.$$
- Eigenvalue is also called characteristic value, proper value or spectral value.
- If $\lambda$ is an eigenvalue of $T$, then any non zero vector $x$ in $H$ such that $Tx = \lambda x$ is called an eigenvector of $T$.
- Eigen vector is also called characteristic vector, proper vector or spectral vector.
- The eigenvectors corresponding to eigenvalue $\lambda$ and the zero vector form a vector subspace, which is called the eigenspace of $T$ corresponding to eigenvalue $\lambda$.
- If $x$ is an eigen vector of $T$ corresponding to eigenvalue $\lambda$ and $\alpha$ is any nonzero scalar, then $\alpha x$ is also an eigenvector of $T$ corresponding to same eigen value.
- Corresponding to single eigenvalue there may correspond more than one eigenvector.
- If $x$ is an eigenvector of $T$, then $x$ cannot correspond to more than one eigenvalue of $T$.

- The set of all eigenvalues of $T$ is called spectrum of $T$ and is denoted by $\sigma(T)$. Its complement $\rho(T) = \boldsymbol{C} - \sigma(T)$ in the complex plane is called resolvent set of $T$.
- An operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non zero vector $x$ in $H$ such that $Tx = 0$.
- If $T$ is an operator on a finite dimensional Hilbert space, then the following statements are true.
    a) $T$ is singular if and only if $0 \in \sigma(T)$.
    b) If $T$ is non -singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.
    c) If $A$ is non singular, then $\sigma(ATA^{-1}) = \sigma(T)$.
    d) If $\lambda \in \sigma(T)$ and if $P$ is a polynomial , then $P(\lambda) \in \sigma\big(P(T)\big)$.
- An operator on a Hilbert space $H$ need not necessarily posses an eigenvalue .
- If $T$ be an operator on a complex Banach space $B$ and let $p$ be a polynomial. Then
$$\sigma\big(p(T)\big) = p\big(\sigma(T)\big) = \{p(\lambda): \lambda \in \sigma(T)\}.$$
- Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H$. Then

a) All the eigen values of $T$ (if they exists) are real.

b) Eigenvectors corresponding to numerically different eigenvalues of $T$ are orthogonal.

- If $T: H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space $H$. Then a number $\lambda$ belongs to the resolvent set $\rho(T)$ of $T$ if and only if there exists a $c > 0$ such that for every $x \in H$,
$$||T_\lambda x|| \geq c||x||$$
- The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space $H$ is real.
- If $T$ is an arbitrary operator on a finite dimensional Hilbert space $H$, then the spectrum of $T$ namely $\sigma(T)$ is a finite subset of the complex plane and the number of points in $\sigma(T)$ does not exceed the dimension $n$ of $H$.
- An operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non-zero vector $x \in H$ such that $Tx = 0$.

## Keywords

- Spectrum of an operator
- Bounded linear operator
- Eigen values
- Eigen vectors
- Eigen space
- Closed subspace
- Charteristic equation
- Hilbert space
- Banach space
- Idempotent operator
- Self adjoint operator

## Self Assessment

1: Let $T$ be an operator on a Hilbert space $H$. Then a scalar $\lambda$ is called the ………..of $T$ if there exists a non zero vector $x$ in $H$ such that $Tx = \lambda x$.

A. Eigenvalue
B. Proper value
C. Characteristic value
D. All of the above.

2: If $\lambda$ is an eigenvalue of $T$, then any non zero vector $x$ in $H$ such that $Tx = \lambda x$ is called the ……….of $T$.

A. Eigenvector
B. Proper vector
C. Both (A) and (B)
D. None of (A) and (B)

3: If $x$ is an eigen vector of $T$ corresponding to eigenvalue $\lambda$ and $\alpha$ is any nonzero scalar, then $\alpha x$ is also an eigenvector of $T$ corresponding to same eigen value.

A. True
B. False

4: If the Hilbert space has no non-zero vectors, then the operator $T$ cannot have any eigenvectors.

A. True
B. False

5: The set of all eigenvalues of an operator $T$ is called……………. of $T$.

A. Eigenvector
B. Proper vector
C. Spectrum
D. None of the above.

6: An operator $T$ on a finite dimensional Hilbert space $H$ is …………….if and only if there exists a non zero vector $x$ in $H$ such that $Tx = 0$.

A. Non-singular
B. Singular
C. Regular
D. None of the above.

7: If $T$ is an operator on a finite dimensional Hilbert space, then which of the following statements is/are true.

A. $T$ is singular if and only if $0 \in \sigma(T)$, where $\sigma(T)$ is the spectrum of $T$.
B. If $T$ is non -singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.
C. If $A$ is non singular, then $\sigma(ATA^{-1}) = \sigma(T)$.
D. All of the above.

8: An operator on a Hilbert space $H$ need not necessarily posses an eigenvalue.

A. True
B. False

9: Which of the following is the spectrum of the idempotent operator $T$ on a Banach space.

A. $\sigma(T) = \{0, -1\}$.

B. $\sigma(T) = \{0,0\}$.

C. $\sigma(T) = \{-1,1\}$.

D. $\sigma(T) = \{0,1\}$.

10: Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H$.

Then which of the following statement is/are true.

A. All the eigen values of $T$ (if they exists) are real.

B. Eigenvectors corresponding to numerically different eigenvalues of $T$ are orthogonal.

C. Both (A) and (B)

D. None of (A) and (B)

11: The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space $H$ is real.

A. True

B. False

12: The eigenvectors corresponding to eigenvalue $\lambda$ and the zero vector form a vector subspace, which is called the eigenspace of $T$ corresponding to eigenvalue $\lambda$.

A. True

B. False

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | D | 2. | C | 3. | A | 4. | A | 5. | C |
| 6. | B | 7. | D | 8. | A | 9. | D | 10. | C |
| 11. | A | 12. | A | | | | | | |

## Review Questions

1. Define Spectrum of an operator on a finite dimensional Hilbert space.
2. Define the resolvent set of an operator.
3. Show that an operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non zero vector $x$ in $H$ such that $Tx = 0$.
4. Find the spectrum of an idempotent operator $T$ on a Banach space.
5. Show that if $T: H \to H$ be a bounded self- adjoint linear operator on a complex Hilbert space $H$. Then all the eigen values of $T$, if they exists are real.
6. Define eigen values and eigen vectors of an operator.
7. Define eigenspace of an operator.
8. Show that eigenspace of an operator on a Hilbert space is a non zero closed linear Subspace of $H$.

## Further Readings

- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
- C. Goffman G Pedrick, A First Course In Functional Analysis.
- B.V. Limaya, Functional Analysis

# Unit 14 : Spectrum Of Normal Operators

**CONTENTS**



## Objectives

After studying this unit, you will be able to understand:

- Spectrum of Normal Operator
- Spectral Resolution and its Properties
- Non-emptiness of the Spectrum

## Introduction

In this chapter, we discuss about spectrum of Normal Operators. Further, we discuss about spectral resolution and some of its important properties. Finally, we discuss nin-emptiness of the spectrum.

## 14.1 Spectrum of Normal Operator

Below we shall give some properties of the spectra of a normal operator.

**Theorem.** If $T$ is a normal operator on a Hilbert space $H$, then $x$ is an eigen vector of $T$ with eigen value $\lambda$ iff $x$ is an eigen vector of $T^*$ with eigenvalue $\bar{\lambda}$.

**Proof.** Since $T$ is normal operator on $H$,therefore $T - \lambda I$ is also normal operator on $H$ where $\lambda$ is any scalar.

Now,

$$(T - \lambda I)^* = T^* - \bar{\lambda}\, I^*$$

$$= T^* - \bar{\lambda}\, I.$$

Since $T - \lambda I$ is normal, we know that an operator $T$ on a Hilbert space $H$ is normal iff

$\|T^* x\| = \|T\, x\|$ for every $x$, therefore we have

$$\|(T-\lambda I)x\| = \|(T-\lambda I)^* x\| \ \forall x \in H$$
$$\Leftrightarrow \|(T-\lambda I)x\| = \left\|(T^* - \bar{\lambda} I)x\right\| \ \forall x \in H$$

$$\Leftrightarrow \|Tx - \lambda x\| = \left\|T^* x - \bar{\lambda} x\right\| \ \forall x \in H \ldots\ldots\ldots(1)$$

From (1), we conclude that

$Tx - \lambda x = 0$ if and only if $T^* x - \bar{\lambda} x = 0$

Therefore $x$ is an eigen vector of $T$ with eigen value $\lambda$ if and only if it is an eigenvector of $T^*$ with eigen value $\bar{\lambda}$.

**Theorem.** If $T$ is a normal operator on a Hilbert space $H$, then the eigenspaces of $T$ are pairwise orthogonal.

**Proof.** Let $M_1, M_2$ be eigenspaces of a normal operator $T$ on $H$ corresponding to the distinct eigenvalues $\lambda_1$ and $\lambda_2$.

Then to prove that $M_1 \perp M_2$. Let $x_1$ be any vector in $M_1$ and $x_2$ be any vector in $M_2$.

Then ,

$$Tx_1 = \lambda_1 x_1 \text{ and } Tx_2 = \lambda_2 x_2.$$

We have,

$$\lambda_1 \langle x_1, x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle$$

$$= \langle Tx_1, x_2 \rangle$$

$$= \langle x_1, T^* x_2 \rangle$$

$$= \langle x_1, \overline{\lambda_2}\, x_2 \rangle$$

$$= \lambda_2 \langle x_1, x_2 \rangle.$$

Therefore,

$$(\lambda_1 - \lambda_2)\langle x_1, x_2 \rangle = 0$$

Implies,

$$\langle x_1, x_2 \rangle = 0 \text{ as } \lambda_1 \neq \lambda_2$$
$$\Rightarrow x_1 \perp x_2$$

Thus,

$$x_1 \perp x_2 \ \ \forall\, x_1 \in M_1 \ \text{ and } \forall\, x_2 \in M_2.$$

Hence,

$$M_1 \perp M_2.$$

**Theorem.** If $T$ is a normal operator on a Hilbert space $H$, then each eigenspace of $T$ reduces $T$.

**Proof.** Let $M$ be an eigen space of $T$ corresponding to the eigen value $\lambda$, in order to prove that $M$ reduces $T$ we have to show that $M$ is invariant both under $T$ and $T^*$, as we know that a closed linear subspace $M$ of a Hilbert space $H$ reduces an operator $T$ iff $M$ is invariant under both $T$ and $T^*$.

Now $M$ is invariant under $T$ because $M$ is eigenspace of $T$. To show that $M$ is also invariant under $T^*$, let us take any vector $x \in M$ . Then

$$Tx = \lambda x$$

Therefore $T^* x = \bar{\lambda}\, x$. Since $M$ is linear subspace of $H$, therefore $x \in M$ and $\bar{\lambda}$ is some scalar .

Implies ,

$\bar{\lambda}\, x \in M$. Thus $x \in M$ implies $T^{*}x = \bar{\lambda}\, x \in M$ .

Therefore $M$ is also invariant under $T^{*}$.

Hence $M$ reduces $T$.

**Theorem.** Let $T$ be an operator on a finite dimensional Hilbert space $H$. Let $\lambda_1, \lambda_2. \dots, \lambda_m$ be the distinct eigen values of $T$ and let $M_1, M_2, \dots, M_m$ be their corresponding eigenspaces , and let $P_1, P_2, \dots, P_m$ be the projections on these eigenspaces. Then the following statements are equivalent.

I. The $M_i's$ are pairwise orthogonal and span $H$.

II. The $P_i's$ are pairwise orthogonal , $P_1 + P_2 + \cdots + P_m = I$ and $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$.

III. $T$ is normal.

## 14.2 Spectral Resolution

**Definition.** Let $T$ be an operator on a Hilbert space $H$. If there exists distinct complex numbers

$$\lambda_1, \lambda_2, \dots, \lambda_m$$

and non zero pairwise orthogonal projections $P_1, P_2, \dots, P_m$ such that

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$$

and ……………(1)

$$P_1 + P_2 + \cdots + P_m = I,$$

Then the expression (1) for $T$ is called spectral resolution for $T$.

**Note:** The spectral theorem tells us that every normal operator $T$ on a non zero finite dimensional Hilbert space $H$ has a spectral resolution.

Now in the following theorem we shall prove that spectral resolution of a normal operator on a finite dimensional non zero Hilbert space is unique.

**Theorem.** The spectral resolution of a normal operator on a finite dimensional non zero Hilbert space is unique.

**Proof.** Let $T$ be a normal operator on a finite dimensional non zero Hilbert space $H$.

Let

$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$……….(1)

Be a spectral resolution of $T$.

Then

$\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct complex numbers and $P_i's$ are non-zero pairwise orthogonal projections such that

$$P_1 + P_2 + \cdots + P_m = I \quad \text{……….(2)}$$

First we show that the scalars $\lambda_1, \lambda_2, \dots, \lambda_m$ are precisely the distinct eigen values of $T$.

First we shall prove that the scalars $\lambda_1, \lambda_2, \dots, \lambda_m$ are precisely the distinct eigen values of $T$.

Let us first show that for each $i$, $\lambda_i$ is an eigen value of $T$.

Since $P_i \neq 0$, therefore there exists a non zero vector $x$ in the range of $P_i$. But $P_i$ is the projection. Therefore

$$P_i x = x.$$

Now

$$Tx = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)x$$

$$= (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m) P_i x$$

$$= \lambda_1 P_1 P_i x + \lambda_2 P_2 P_i x + \cdots + \lambda_m P_m P_i x$$

$$= \lambda_i P_i^2 x \qquad (P_i\, P_j = 0, if\ i \neq j)$$

$$= \lambda_i P_i x \qquad p_i^2 = p_i, p_i \text{ being a projection.}$$

$$= \lambda_i x.$$

Thus $x$ is a non zero vector such that $Tx = \lambda_i x$, therefore $\lambda_i$ is an eigen value of $T$.

Now we show that each eigen value of $T$ is an element of the set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$.

Since $T$ is an operator on a finite dimensional Hilbert space , therefore $T$ must posses an eigen value. Let $\lambda$ be an eigen value of $T$. Then there exists a non zero vector $x$ such that

$$Tx = \lambda x$$

$$\Rightarrow Tx = \lambda I x \qquad \text{as } Ix = x$$

$$\Rightarrow (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m) x = \lambda (P_1 + P_2 + \cdots + P_m) x$$

$$\Rightarrow (\lambda_1 - \lambda) P_1 x + (\lambda_2 - \lambda) P_2 x + \cdots + (\lambda_m - \lambda) P_m x = 0.$$

Operating on this with $p_i$ and remembering that $P_i^2 = P_i$ and $P_i\, P_j = 0, if\ i \neq j$, we get

$$(\lambda_i - \lambda) P_i x = 0 \text{ for } i = 1,2, \ldots, m$$

If $\lambda_i \neq \lambda$ for each $i$, then we have $P_i x = 0$ for each $i$. Then we have

$$P_i x = 0 \text{ for each } i.$$

Therefore,

$P_1 x + P_2 x + \cdots + P_m x = 0$

$$\Rightarrow (P_1 + P_2 + \cdots + P_m) x = 0$$

$$\Rightarrow Ix = 0$$

$$\Rightarrow x = 0$$

This contradicts the fact that $x \neq 0$.

Hence $\lambda$ must be equal to $\lambda_i$ for some $i$.

This we have proved that in the spectral resolution (1) of $T$ the scalars $\lambda_i' s$ are precisely the distinct eigen values of $T$.

Therefore if

$$T = \alpha_1 Q_1 + \alpha_2 Q_2 + \cdots + \alpha_k Q_k \ldots\ldots\ldots.(3)$$

is another spectral resolution of $T$, then the scalars $\alpha_i' s$ are precisely the distinct eigenvalues of $T$. Therefore remaining the projections $Q_i' s$, if necessary , we can write (3) in the form

$$T = \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_m Q_m.$$

Now we shall show that in the spectral resolution (1) of $T$ the $P_i' s$ are uniquely determined as specific polynomials in $T$.

We have

$$T^0 = I = P_1 + P_2 + \cdots + P_m$$

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$$

$$T^2 = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)\,(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)$$

$$= \lambda_1^2 P_1 + \cdots + \lambda_m^2 P_m$$

$$[P_i^2 = P_i\,, P_i\, P_j = 0, if\ i \neq j]$$

Similarly ,

$T^n = \lambda_1^n P_1 + \cdots + \lambda_m^n P_m$, where $n$ is any positive integer.

Therefore, if $g(t)$ is any polynomial with complex coefficients, in the complex variable $t$, then taking linear combination of the above relation ,we get

$$g(T) = g(\lambda_1)P_1 + g(\lambda_2)P_2 + \cdots + g(\lambda_m)P_m$$

$$= \textstyle\sum_{j=1}^{m} g(\lambda_j)P_j.$$

Now suppose that $p_i$ is a polynomial such that

$$p_i(\lambda_j) = \delta_{ij}$$

That is

$p_i(\lambda_i) = 1$ , if $i = j$

and

$p_i(\lambda_i) = 0$ , if $i \neq j$.

Then taking $p_i$ in place of $g$, we get

$$p_i(T) = \sum_{j=1}^{m} p_i(\lambda_j)P_j.$$

$$= \sum_{j=1}^{m} \delta_{ij}P_j.$$

$$= P_i.$$

Thus for each $i, p_i(T) = P_i$.

Which is a polynomial in $T$. But we must show the existence of such a polynomial $p_i$ over the field of complex numbers .

Obiviously

$$p_i(t) = \frac{(t-\lambda_1)\ldots(t-\lambda_{i-1})(t-\lambda_{i+1})\ldots(t-\lambda_m)}{(\lambda_i-\lambda_1)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_m)}$$

Serves the purpose .

That is

$p_i(\lambda_i) = 1$ , if $i = j$

and

$p_i(\lambda_i) = 0$ , if $i \neq j$.

If we apply the above discussion for $Q_i's$ then we shall get

$Q_i = p_i(T)$ for each $i$.

Therefore $P_i = Q_i$ for each $i$.

Hence, the two spectral resolutions are the same.

**Note:** If $T$ is a normal operator on a finite dimensional Hilbert space $H$, the there exists an orthonormal basis for $H$ relative to which the matrix of $T$ is diagonal matrix.

## 14.3 Non-Emptiness of the Spectrum

The following theorem establishes the non-emptiness of the spectrum of an operator on a finite dimensional Hilbert space $H$.

**Theorem.** If $T$ is an arbitrary operator on a finite dimensional Hilbert space $H$, then the spectrum of $T$ namely $\sigma(T)$ is a finite subset of the complex plane and the number of points in $\sigma(T)$ does not exceed the dimension $n$ of $H$.

For the proof of this theorem, we need the following lemma

**Lemma.** An operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non-zero vector $x \in H$ such that $Tx = 0$.

**Proof of lemma .** Suppose there exists a non zero vector $x \in H$ such that $Tx = 0$. We can write

$$Tx = 0 \text{ as } Tx = T0.$$

Since $x \neq 0$,

the two distinct elements $x, 0 \in H$ have the same image under $T$. Therefore the mapping $T$ is not one-one . Hence $T^{-1}$ does not exists . Hence it is singular.

To prove the converse assume that $T$ is singular. Suppose there exists no non-zero vector such that

$$Tx = 0.$$

This means that $Tx = 0$,

implies ,

$$x = 0.$$

Then $T$ must be one-one. Since $H$ is finite dimensional and $T$ is one-one, $T$ is onto so that $T$ is non-singular contracdicting the the hypothesis that $T$ is singular .

Hence there must be a non-zero vector $x$ such that $Tx = 0$.

**Proof of the theorem.** Let $T$ be an operator on a finite dimensional Hilbert space $H$ of dimension $n$. A scaler $\lambda \in \sigma(T)$, if there exists a non-zero vector $x$ such that $(T - \lambda I)x = 0$.

Now,

$(T - \lambda I)x = 0$ if and only if $(T - \lambda I)$ is singular by the above lemma .

But,

$(T - \lambda I)$ is singular if and only if $det\ (T - \lambda I) = 0$.

Thus,

$\lambda \in \sigma(T)$if and only if $\lambda$ satisfies the equation $de\ (T - \lambda I) = 0$.

Let $B$ be an ordered basis for $H$.Thus $det\ (T - \lambda I) = det\ ([T - \lambda I]_B)$.

But ,

$$\det([T - \lambda I]_B) = \det([T]_B - \lambda[I]_B).$$

Thus,

$$\det(T - \lambda I) = \det([T]_B - \lambda[\delta_{ij}]).$$

So,

$$det\ (T - \lambda I) = 0 \text{ implies } \det\big([T]_B - \lambda\big[\delta_{ij}\big]\big) = 0 \ldots\ldots\ldots\ldots(1)$$

If $[T]_B = [\alpha_{ij}]$ is the matrix of $T$, then (1) gives

$$\begin{bmatrix} \alpha_{11}-\lambda & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-\lambda & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \dots & \dots & \alpha_{nn}-\lambda \end{bmatrix} = 0 \text{......................(2)}$$

The expansion of the determinant of (2) gives a polynomial equation in $\lambda$ of degree $n$ with complex coefficients . So, by the fundamental theorem of algebra this equation must have atleast one root in the field of complex numbers. Hence every operator $T$ on $H$ has an eigenvalue so that $\sigma(T) \neq 0$. Further, this equation in $\lambda$ has exactely $n$ roots in the complex field. If the equation has repeated roots , then the number of distinct roots are less than $n$. So that $T$ has an eigenvalue and the number of distinct eigenvalues of $T$ is less than or equal to $n$. Hence the number of elements of $\sigma(T)$ is less than or equal to $n$.

**Note:** If the scalers associated with $H$ are complex, $\sigma(T)$ contains atleast one point. It may contain as many as n distinct points but not more than $n$ points. If the scaler field is real, it is possible that $\sigma(T)$ is empty. Hence in the spectral theory, we usually take the complex scalers so that we get a richer theory.

## Summary

- If $T$ is a normal operator on a Hilbert space $H$, then $x$ is an eigen vector of $T$ with eigen value $\lambda$ iff $x$ is an eigen vector of $T^*$ with eigenvalue $\bar{\lambda}$.
- If $T$ is a normal operator on a Hilbert space $H$, then the eigenspaces of $T$ are pairwise orthogonal.
- If $T$ is a normal operator on a Hilbert space $H$, then each eigenspace of $T$ reduces $T$.
- A closed linear subspace $M$ of a Hilbert space $H$ reduces an operator $T$ iff $M$ is invariant under both $T$ and $T^*$.
- An operator $T$ on a Hilbert space $H$ is normal iff

$$\|T^*x\| = \|T\,x\| \text{ for every } x.$$

- If $T$ be an operator on a finite dimensional Hilbert space $H$. Let $\lambda_1, \lambda_2. \dots, \lambda_m$ be the distinct eigen values of $T$ and let $M_1, M_2, \dots, M_m$ be their corresponding eigenspaces , and let $P_1, P_2, \dots, P_m$ be the projections on these eigenspaces. Then the following statements are equivalent.

a. The $M_i's$ are pairwise orthogonal and span $H$.

b. The $P_i's$ are pairwise orthogonal , $P_1 + P_2 + \dots + P_m = I$ and $T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m$.

c. $T$ is normal.

- If $T$ be an operator on a Hilbert space $H$. If there exists distinct complex numbers

$$\lambda_1, \lambda_2, \dots, \lambda_m$$

and non zero pairwise orthogonal projections $P_1, P_2, \dots, P_m$ such that

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m$$

and ..............(1)

$$P_1 + P_2 + \dots + P_m = I,$$

Then the expression (1) for $T$ is called spectral resolution for $T$.

- The spectral resolution of a normal operator on a finite dimensional non zero Hilbert space is unique.
- If $T$ is a normal operator on a finite dimensional Hilbert space $H$, the there exists an orthonormal basis for $H$ relative to which the matrix of $T$ is diagonal matrix.
- If $T$ is an arbitrary operator on a finite dimensional Hilbert space $H$, then the spectrum of $T$ namely $\sigma(T)$ is a finite subset of the complex plane and the number of points in $\sigma(T)$ does not exceed the dimension $n$ of $H$.

- An operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non-zero vector $x \in H$ such that $Tx = 0$.

## Keywords

- Spectrum of an operator
- Normal operator
- Spectral resolution
- Hilbert space
- Eigenvalue
- Eigenvector
- Eigenspace
- Linear space
- Invarience
- Finite dimensional non zero Hilbert space
- Diagonal matrix

## Self Assessment

1: If $T$ is a normal operator on a Hilbert space $H$, then $x$ is an eigen vector of $T$ with eigen value $\lambda$ iff $x$ is an eigen vector of $T^*$ with eigenvalue $\bar{\lambda}$.

A. True
B. False

2: Which of the following statement is /are true.

I. If $T$ is a normal operator on a Hilbert space $H$, then the eigenspaces of $T$ are pairwise orthogonal.
II. If $T$ is a normal operator on a Hilbert space $H$, then each eigenspace of $T$ reduces $T$.

.

A. Only I is true.
B. Only II is true.
C. Neither (I) nor (II)
D. Both (I) and (II).

3: Which of the following statement is /are true.

I. A closed linear subspace $M$ of a Hilbert space $H$ reduces an operator $T$ iff $M$ is invariant under both $T$ and $T^*$.
II. An operator $T$ on a Hilbert space $H$ is normal iff

$$\|T^*x\| = \|T\,x\| \text{ for every } x.$$

A. Only I .
B. Only II
C. Neither (I) nor (II)
D. Both (I) and (II).

4 : The spectral resolution of a normal operator on a finite dimensional non zero Hilbert space is unique.

A. True

B. False

5 : Which of the following statement is /are true.

I. Every self adjoint operator is normal.

II. A normal operator need not be self adjoint.

A. Only I .

B. Only II

C. Both (I) and (II).

D. Neither (I) nor (II)

6 : If $N$ is a normal operator on $H$, then

A. $\|N^2\| < \|N\|^2$.

B. $\|N^2\| = \|N\|^2$.

C. $\|N^2\| > \|N\|^2$.

D. None of the above.

7: An operator $T$ on a finite dimensional Hilbert space $H$ is singular if and only if there exists a non-zero vector $x \in H$ such that $Tx = 0$.

A. True

B. False

8. If $T$ is a normal operator on a finite dimensional Hilbert space $H$, the there exists an orthonormal basis for $H$ relative to which the matrix of $T$ is diagonal matrix.

A. True

B. False

## Answers for Self Assessment

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | D | 3. | D | 4. | A | 5. | C |
| 6. | B | 7. | A | 8. | A | | | | |

## Review Questions

Q1:- What is Spectrum of Normal operator.

Q2:- Show that if T is a normal operator on a Hilbert space H, then the eigen spaces of T are pairwise orthogonal.

Q3:- prove that the spectral resolution of a normal operator on a finite dimensional non zero Hilbert space is unique.

Q4:- Prove that an operator T on a finite dimensional Hilbert space H is singular if and only if there exists a non zero x in H such that Tx=0. Prove that the zero operator on any normed space is compact.

## Further Readings


- Introductory Functional Analysis With Applications By Erwin Kreyszig.
- Functional Analysis By Walter Ruddin, Mcgraw Hill Education.
- J. B Conway, A Course In Functional Analysis.
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