Complex Analysis - II

DEMTH543

Edited by:
Dr. Kulwinder Singh

LOVELY PROFESSIONAL UNIVERSITY

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Purpose and Objectives:

An analytic continuation is a mathematical approach used to widen the scope of a given analytic function in the field of complex analysis. Analytic continuation frequently succeeds in defining further values of a function, for instance in a new region when the initial definition's infinite series representation becomes divergent.

The stepwise continuation method might, however, run into problems. These could be fundamentally topological, which would produce contradictions (defining more than one value). Alternatively, they might be related to the existence of singularities. The situation involving many complex variables is somewhat different because singularities need not be separate places in this case. Sheaf cohomology was largely developed because of research into this situation. In this unit first we will discuss the pre-requisite concepts for analytic continuation and then the definition of analytic continuation. After this unit students can be able to-

1. Understand the convergence analysis of a complex valued function.
2. Understand the definition of analytical continuation.
3. Solve some problems of analytical continuation.

Introduction

The Riemann hypothesis, which is closely related to the distribution of prime numbers, is perhaps the most important open topic in pure mathematics today. Analytic continuation is one of the fundamental methods required to comprehend the issue. An approach from the field of mathematics known as complex analysis called analytical continuation is employed to enlarge the domain of a complex analytic function. We will quickly go over some essential mathematics concepts prior to introducing the approach.

1.1 Taylor Series

Consider the case where we want to find a polynomial approximation to a function f(x). Polynomials are mathematical expressions made up of coefficients and variables. The variables are multiplied, subtracted, and added using only non-negative integer exponents. With one variable, x, a polynomial of degree *n* can be expressed as follows:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots, a_1 x^1 + a_0 x^0 \quad (1.1)$$

If we consider a 3-degree polynomial with $a_3 = \frac{1}{4}, a_2 = \frac{3}{4}, a_1 = -3,$ and $a_0 = -2$ then.

$$f(x) = \frac{x^3}{4} + \frac{3}{4}x^2 - 3x - 2 \quad (1.2)$$

And the graph of $f(x)$ is.

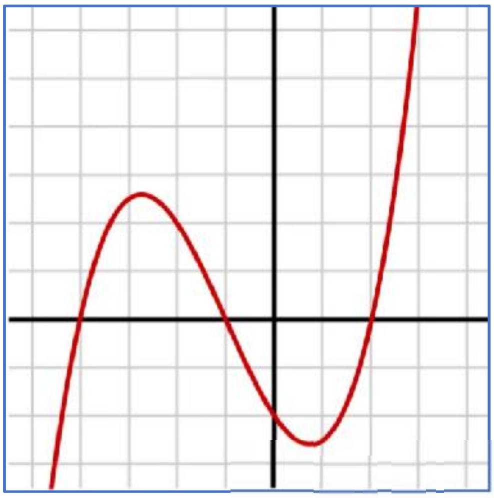

Figure 1.1: The graph of $f(x) = \frac{x^3}{4} + \frac{3}{4}x^2 - 3x - 2$

Imagine that the polynomial has infinite degrees now (it is given by an infinite sum of terms). These polynomials are referred to as Taylor series (or Taylor expansions). Polynomial representations of functions as infinite sums of terms are called Taylor series.

Every term in the series is calculated using the derivative values of $f(x)$ at a particular point (around which the series is centered). A formal Taylor series centered on a certain number and is given by:

$$f(x) = f^{(0)}(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \quad (1.3)$$

where the upper indices (0), (1), … indicate the order of the derivative of $f(x)$ $as\ x = a$. One can approximate a function using a polynomial with only a finite number of terms of the corresponding Taylor series. Such polynomials are called Taylor polynomials.

The Taylor polynomials for $f(x) = \frac{x^3}{4} + \frac{3}{4}x^2 - 3x - 2$ around $a = 0$ are given by:

$$f(x) = -2 - \frac{3}{1!}x + \frac{3}{2.2!}x^2 + \frac{3}{2.3!}x^3 + \cdots \quad (1.4)$$

Were $f^{(0)}(0) = -2, f^{(1)}(0) = -3, f^{(2)}(0) = \frac{3}{2}, f^{(2)}(0) = \frac{3}{2}$.

The equation (1.4) is same as the considered 3 degree polynomial equation (1.2).

1.2 <u>Convergence</u>

Our study of the analytic continuation will likewise heavily rely on the idea of convergence of infinite series. A list of items (or objects) having a specific order constitutes a mathematical sequence. The following S_n represents the n different sequences:

$$S_n = \{a_1, a_2, \ldots, a_n\} \quad (1.5)$$

A well-known example of a sequence is the Fibonacci sequence 0,1,1,2,3,5,8,13,21,34, 55, …where each number is the sum of the two preceding ones.

One builds a **series** by taking partial sums of the elements of a sequence. The series of partial sums can be represented by:

$$\{s_0, s_1, s_2 \dots, s_n\} \quad (1.6)$$

$$\text{where: } \{s_0 = a_0,\ s_1 = a_0 + a_1, s_2 = a_0 + a_1 + a_2, \dots,\} \quad (1.7)$$

An example of a series, the familiar geometric series, is shown below. In a geometric series, the common ratio between successive elements is constant. The geometric series with common ratio = 1/2we have:

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots, \qquad (1.8)$$

Fig. 1.2 shows pictorially that the geometric series above converges to twice the area of the largest square.

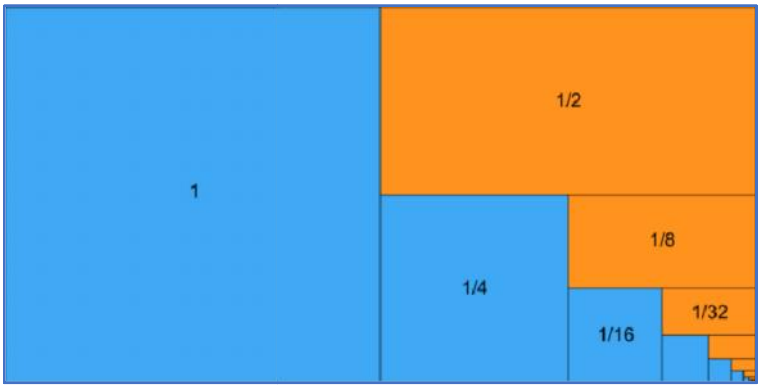

Figure 1.2: A pictorial demonstration of the convergence of the geometric series with common ration r=1/2 and first term a=1

A series such as in Eq. (1.7) is convergent if the sequence Eq. (1.6) of partial sums approaches some finite limit. Otherwise, the series is said to be divergent. An example of a convergent series is the geometric series in Eq.(1.8). An example of a divergent series is:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \to \infty \quad (1.9)$$

1.3 Analytic Functions, Poles, and Convergence Discs

Until now, our analysis was restricted to real numbers. Now we will extend it to complex numbers. The complex plane is a geometric representation of the complex numbers, as shown in Fig.1.3 .

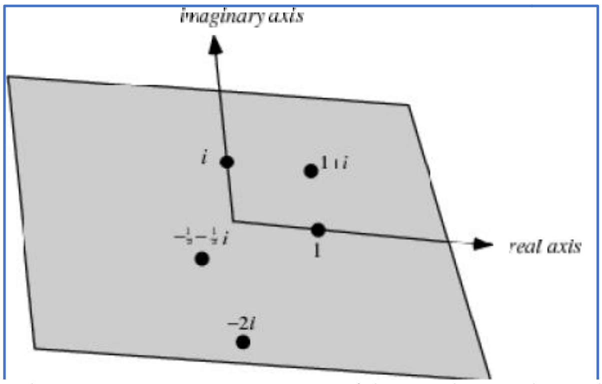

Figure 1.3: The complex plane, a geometric representation of the complex numbers. The figure shows the real and the (perpendicular) imaginary axis.

Let us consider an expansion of an analytic complex function $f(z)$. By definition, an analytic function is a function locally given by a convergent power series. If $f(z)$ is analytic at z_0 , the power series reads:

$$f(z_0 + z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \quad (1.10)$$

Equation (1.10) shows the Taylor expansion of an analytic function f(z) into a power series about a complex value z_0 .

In analogy with the case of the geometric series, where convergence was restricted to an interval with radius 1 on the real line, this series will converge only over a circular region of the complex plane centered on the complex number z_0 .

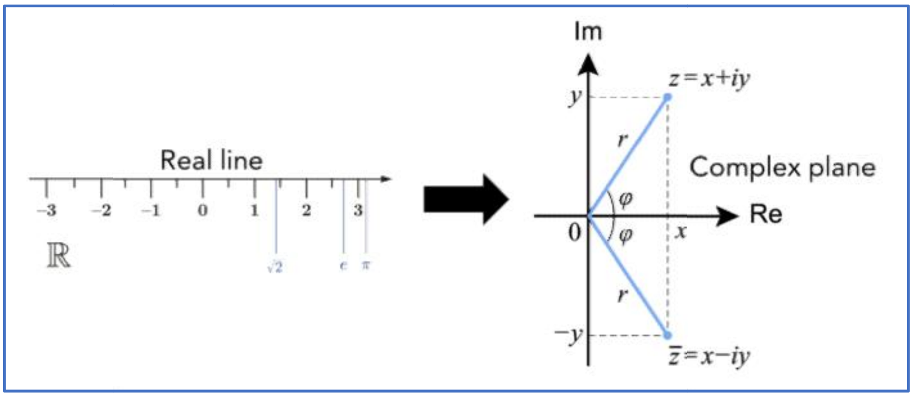

Figure 1.4: Going from the real line to the complex plane.

The convergence region of $f(z)$ is a circular region centered on z_0 extending to the closest pole, where $f(z)$ goes to infinity.

Fig. 1.5 shows the convergence region (bounded by the white circle) of the function $1/(1+z^2)$.

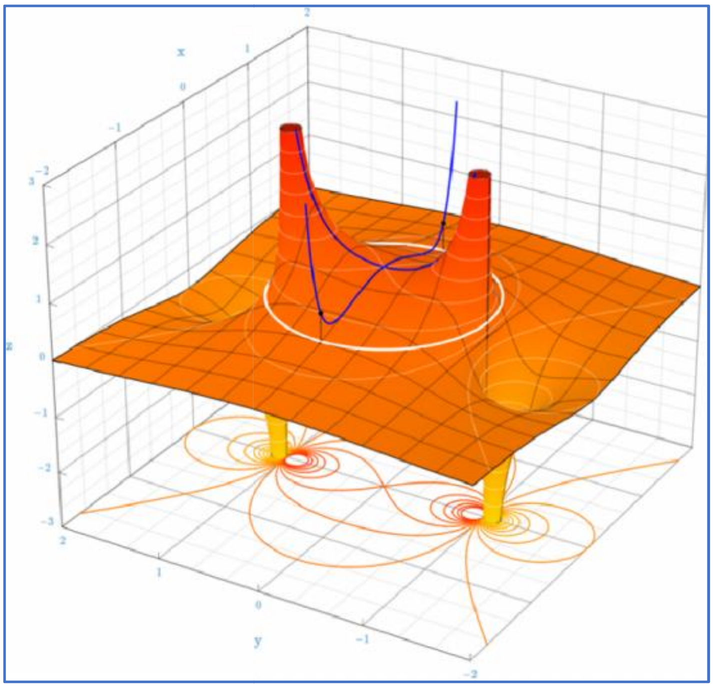

Figure 1.5:The white circle in the convergence disc of the function $1/(1+z^2)$.

A stronger criterion of convergence is called absolute convergence. We call the convergence we already discussed conditional convergence. Absolute convergence occurs when the following series converges:

$$s_0 = |a_0|,\ s_1 = |a_0| + |a_1|,\ s_2 = |a_0| + |a_1| + |a_2|, \ldots \quad (1.11)$$

When a series is absolutely convergent it is also conditionally convergent. There are a few tests of absolute convergence, one of them is the ratio test. Consider the general infinite series:

$$S = \sum_{n=0}^{\infty} a_n \quad (1.12)$$

Now define the following ratio: $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (1.13)

The ratio r in the equation (1.13) used in the ratio test of absolute convergence.

The series equation (1.13) converges absolutely if $r<1$ and diverges if $r>1$. If $r=1$, no conclusion can be taken.

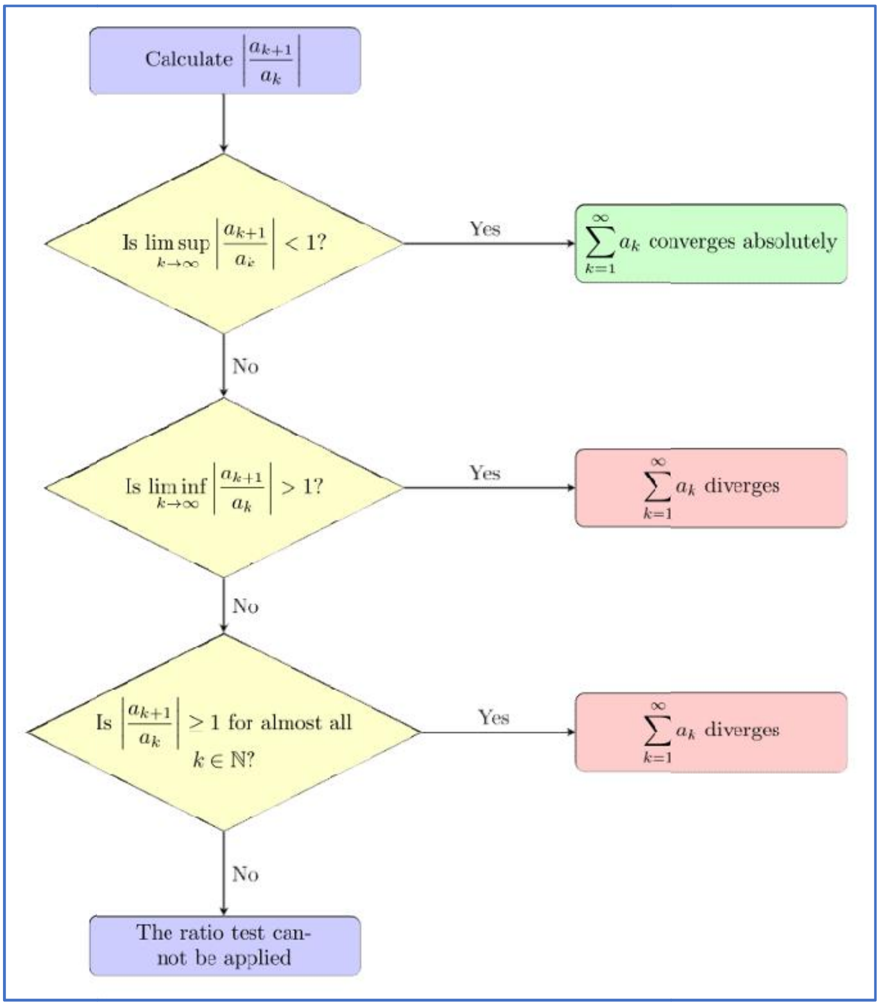

Figure 1.6: A decision diagram for the ratio test.

It is straightforward to apply the ratio test (or any other convergence test) to show the following important result:

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \text{ converges for any } k \geq 2. \quad (1.14)$$

1.4 Analytic Continuation

From the results regarding zeros of an analytic function, it follows that if two functions are regular in a domain D and if they coincide in a neighborhood, however small, of any point a of D, or only along a path-segment, however small, terminating in a point a of D, or only at an infinite number of distinct points with a limit-point a in D, then the two functions are identically the same in D. Thus, it emerges that a regular function defined in a domain D is completely determined by its values over any such sets of points.

This is a very great restraint in the behavior of analytic functions. One of the remarkable consequences of this feature of analytic functions, which is extremely helpful in studying them, is known as analytic continuation. Analytic continuation is a process of extending the definition of a domain of an analytic function in which it is originally defined i.e., it is a concept which is utilized for making the domain of definition of an analytic function as large as possible.

Letus supposethattwo functionsf_1(z) andf_2(z) aregiven, such thatf_1(z) isanalyticinthedomain D_1 and f_2(z) in a domain D_2We further assume that D_1 and D_2 have a common part $D_{12}(D_1\cap D_2)$.

If $f_1(z) = f_2(z)$ in the common part D_{12}, then we say that f_2(z) is the direct analyticcontinuationoff_1(z)fromD_1intoD_2viaD_{12}.

Conversely,f_1(z)isthedirectanalyticcontinuationof f_2(z) from D_2 into D_1 via D_{12}.Indeed f_1(z) and f_2(z) are analytic continuations of each other.

Both f_1(z) and f_2(z) may be regarded as partial representations or elements of one and the samefunction undertheconditionthat$f_1(z)=f_2(z)$ataninfinitesetofpointswithalimit-pointinD_{12}.

Itisobservedthatforthepurposeofanalyticcontinuation,itissufficientthatthedomains$D_1$and D_2haveonlyasmallarcin common.

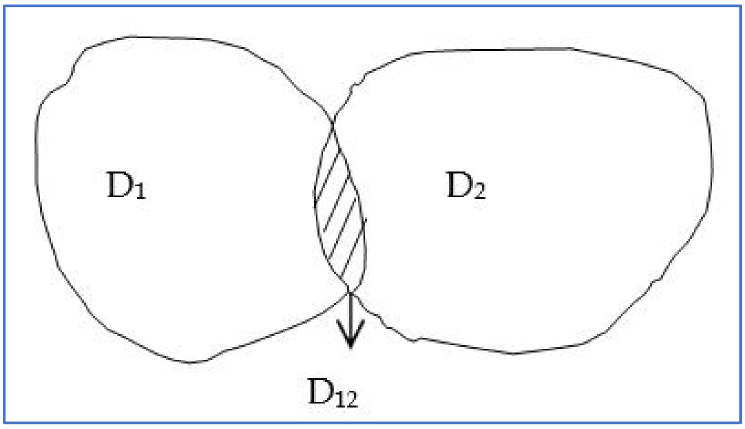

Figure 1.7: Analytic Continuation common domain.

Definition. An analytic function *f*(z) with its domain of definition D is called a functionelement and is denoted by (*f*, D).

If z∈D, then (*f*, D) is called a function element of z.Using thisnotation,wemaysaythat(f_1,D_1)and(f_2,D_2)areinanalyticcontinuationsofeachotheriff $D_1\cap D_2\neq\phi$and$f_1(z) = f_2(z)$ forallz∈$D_1\cap D_2$.

It can be further simplified as Suppose f_1(z) is analytical on a region D_1. Now suppose that D_1 is contained in a region f_2(z). The function *f(z)* can be analytically continued from D_1 *to* D_2 if there exists a function f_2(z) such that: f_2(z) is analytic on *S*, $f_2(z) = f_1(z)$ for all $z \in D_1$

Example 1.1:

Let us consider $f(z) = \sum_{n=0}^{\infty} z^n$, $\emptyset(z) = \frac{1}{1-z}$.

Then $f(z)$is analytic at all the points within the circle $|z| = 1$ and $\emptyset(z)$ is analytic all the points except $z = 1$.

Also $f(z) = \emptyset(z)$ within $|z| = 1$

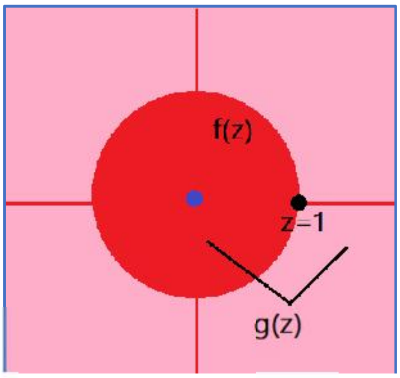

Figure 1.8: $\emptyset(z)$ gives the continuation of $f(z)$ over the rest of the plane.

Hence $\emptyset(z)$ gives the continuation of $f(z)$ over the rest of the plane.

Question:

Show that the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$

Solution:

Given that:

$$f_1(z) = \sum_{n=0}^{\infty} z^n \quad (1.15)$$

First, we will consider the convergent analysis for $f_1(z)$

The series equation (1.15) can be written as:

$$f_1(z) = 1 + z + z^2 + z^3 + \cdots, +z^n + \cdots,$$

$$\Rightarrow f_1(z) = (1-z)^{-1}$$

$$\Rightarrow f_1(z) = \frac{1}{1-z}$$

Hence it is observed that the $f_1(z)$has the sum $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and the nth sequence of the series is z^n

Now we apply the ratio test for convergent analysis.

Here, $U_k = z^k$

And $U_{k+1} = z^{k+1}$

If $\sum_{n=0}^{\infty} U_k$ is absolutely convergent then $\left|\frac{U_{k+1}}{U_k}\right| < 1$

$$\Rightarrow \left|\frac{z^{k+1}}{z^k}\right| < 1$$

$$\Rightarrow \left|\frac{z^k . z}{z^k}\right| < 1$$

$$\Rightarrow |z| < 1$$

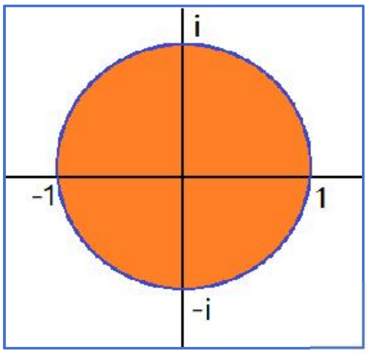

Figure 1.9: The area of unit disc $|z| < 1$

Hence $f_1(z)$ is convergent inside the region $|z| < 1$(see the Figure 1.9)

Now let us consider the second series.

$$f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n \quad (1.16)$$

$$\Rightarrow f_2(z) = \frac{1}{2} + \frac{(1+z)}{2.2} + \frac{(1+z)^2}{2.2^2} + \frac{(1+z)^3}{2.2^3} + \cdots, \frac{(1+z)^n}{2.2^n} + \cdots,$$

Hence the nth term of the series is $U_n = \frac{(1+z)^n}{2.2^n}$.

And $U_{n+1} = \frac{(1+z)^{n+1}}{2.2^{n+1}}$.

If $\sum_{n=0}^{\infty} U_n$ is absolutely convergent then $\left|\frac{U_{n+1}}{U_n}\right| < 1$

$$\Rightarrow \left|\frac{\frac{(1+z)^{n+1}}{2.2^{n+1}}}{\frac{(1+z)^n}{2.2^n}}\right| < 1$$

$$\Rightarrow \left|\frac{(1+z)^{n+1}.\,2.2^n}{2.2^{n+1}.\,(1+z)^n}\right| < 1$$

$$\Rightarrow \left|\frac{(1+z)^n.\,2.2^n.\,(1+z)}{2.2^n.\,(1+z)^n.\,2}\right| < 1$$

$$\Rightarrow \left|\frac{(1+z)}{2}\right| < 1$$

$$\Rightarrow |z+1| < 2$$

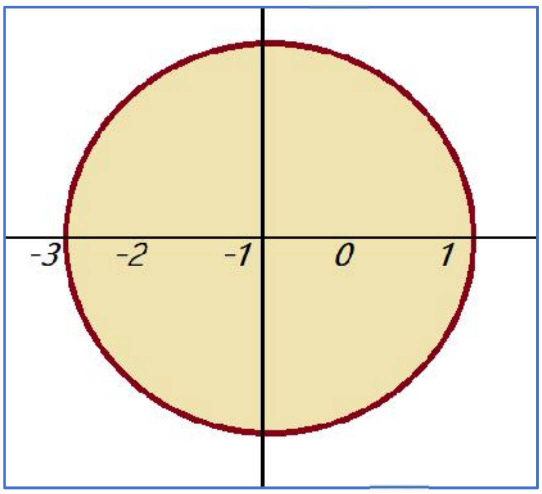

Figure 1.10: The area of disc $|z+1| < 2$

Hence $f_2(z)$ is convergent inside the region $|z+1| < 2(With\ the\ center\ z = -1, and\ r = 2)$.

Till now we have observed that the $f_1(z) = \frac{1}{1-z}$ is analytic inside the domain $D_1: |z| < 1$ and $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ is analytic inside the domain $D_2: |z+1| < 2$. It can be clearly seen from Fig.1.10 and Fig.1.9 that $f_2(z)$ and $f_1(z)$ share some common region.

Now we will show that $f_1(z) = f_2(z)$ for all $z \in D_1$.

As we have $f_1(z) = \frac{1}{1-z}$ and

$$f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n = \frac{1}{2} + \frac{(1+z)}{2.2} + \frac{(1+z)^2}{2.2^2} + \frac{(1+z)^3}{2.2^3} + \cdots, \frac{(1+z)^n}{2.2^n} + \cdots,$$

$$\Rightarrow f_2(z) = \frac{1}{2} + \frac{(1+z)}{2.2} + \frac{(1+z)^2}{2.2^2} + \frac{(1+z)^3}{2.2^3} + \cdots, \frac{(1+z)^n}{2.2^n} + \cdots,$$

Let $\frac{1+z}{2} = p$ then.

$$f_2(z) = \frac{1}{2} + \frac{p}{2} + \frac{p^2}{2} + \frac{p^3}{2} + \cdots, \frac{p^n}{2} + \cdots,$$

$$\Rightarrow f_2(z) = \frac{1}{2}(1 + p + p^2 + p^3 + \cdots, p^n + \cdots,)$$

$$\Rightarrow f_2(z) = \frac{1}{2}(1-p)^{-1}$$

$$\Rightarrow f_2(z) = \frac{1}{2(1-p)}$$

$$\Rightarrow f_2(z) = \frac{1}{2(1-\frac{1+z}{2})}$$

$$\Rightarrow f_2(z) = \frac{1}{(1-z)}$$

$$\Rightarrow f_2(z) = f_1(z)$$

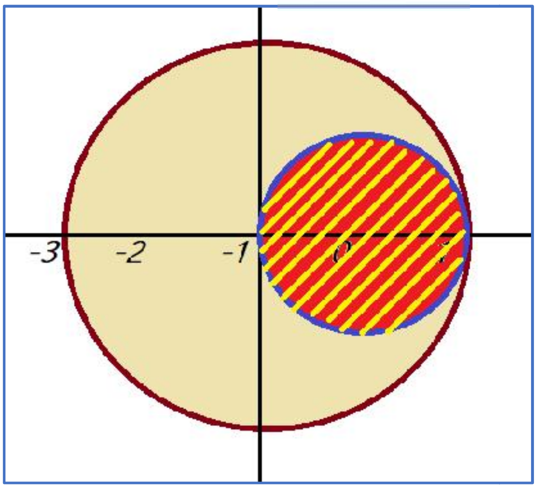

*Figure 1.11: The common region for z*ČD_1ą D_2.

As $f_1(z)$is analytic inside the domain $D_1: |z| < 1$ and $f_2(z)$is analytic inside the domain $D_2: |z+1| < 2$ .Thus $f_2(z)$extends the domain of an analytical function $f_1(z)$ to larger domain D_2. Hence the function $f_1(z)$be analytically continued from D_1 *to* D_2 as there exists a function $f_2(z)$such that: $f_2(z)$is analytic on D_2: $f_2(z) = f_1(z)$, for all $z \in D_1$.

1.5 Review questions

1. Explain how it is possible to continue analytically the function $f(z) = 1 + z + z^2 + \cdots + z^n + \cdots$ outside the circle of convergence of the power series.
2. Show the series $\sum_{n=0}^{\infty} z^{3n}$ cannotbecontinuedanalyticallybeyondthe circle |z|= 1
3. Show that the series $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuations of each other.
4. Prove that the series $1 + \sum_{n=0}^{\infty} z^{2n}$ cannot be continued analytically beyond |z|= 1
5. Prove that the function defined by $F_1(z) = z - z^2 + z^3 - z^4 + \cdots$, is analytic in the region $|z| < 1$ .And then find a function that represents all possible analytic continuations of $F_1(z)$.

1.6 Self-assessment

1. The function f(z) = $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ is convergent inside the
   A. Region $|z + 1| < 2$
   B. Region $|z - 1| < 2$
   C. Region $|z + 1| < 6$
   D. Region $|z| < 2$

2. The function f(z) = $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ is convergent inside the
   A. Region $|z + 1| < 2$
   B. Region $|z - 1| < 2$
   C. Region $|z + 1| < 6$
   D. Region $|z| < 2$

3. The function f(z) = $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ is convergent inside the
   A. Region $|z + 1| < 2$
   B. Region $|z - i| < \sqrt{5}$
   C. Region $|z| < 5$
   D. Region $|z| < \sqrt{5}$

4. The function f(z) = $\sum_{n=0}^{\infty} z^n$ is convergent inside the
   A. Region $|z| < 2$
   B. Region $|z - 1| < 2$
   C. Region $|z + 1| < 6$
   D. Region $|z| < 1$

5. The function f(z) = $\sum_{n=0}^{\infty} \frac{(2+z)^{n-1}}{(n+1)^3 . 4^n}$is convergent inside the
   A. Region $|z + 1| < 4$
   B. Region $|z - 1| < 2$
   C. Region $|z + 1| < 6$
   D. Region $|z| < 2$

6. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$from the domain *D₁ to D₂*then the which one of the points does lies in $D_1 \cap D_2$
   A. i
   B. 5

C. $-5i$
D. -20

7. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$from the domain D_1 *to* D_2then the which one of the points does lies in $D_1 \cap D_2$
   A. $10i$
   B. $\frac{1}{5} + \frac{1}{5}i$
   C. $-5.5i$
   D. -10

8. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$from the domain D_1 *to* D_2then the which one of the points does NOT lies in $D_1 \cap D_2$
   A. $\sqrt{5}$
   B. 0
   C. i
   D. $i/2$

9. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$from the domain D_1 *to* D_2then the which one of the points does not lies in $D_1 \cap D_2$
   A. $0.2i$
   B. 5
   C. $-0.1i$
   D. $0.1 + i$

10. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$from the domain D_1 *to* D_2then the which one of the points does lies in $D_1 \cap D_2$
    A. $0.1 + 0.1i$
    B. 50
    C. $-15i$
    D. -12

11. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$from the domain D_1 *to* D_2then the which one of the point lies in $D_1 \cap D_2$
A. $\frac{1}{2} + \frac{1}{4}i$
B. $\frac{9}{2} + \frac{1}{2}i$
C. $-3 + \frac{1}{2}i$
D. $-3 - 3i$

12. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$from the domain D_1 *to* D_2then the which one of the point lies in $D_1 \cap D_2$
    A. $\frac{1}{8} + \frac{1}{8}i$
    B. $\frac{10}{2} + \frac{1}{2}i$
    C. $-13 + \frac{1}{2}i$
    D. $-3 - 13i$

13. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain D_1 to D_2then the which one of the point does not lies in $D_1 \cap D_2$
    A. $\frac{1}{2} + \frac{1}{4}i$
    B. $\frac{1}{9} + \frac{1}{5}i$
    C. $-8 + \frac{1}{2}i$
    D. -2

14. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain D_1 to D_2then the which one of the point does not lies in $D_1 \cap D_2$
    A. -1
    B. -1.5
    C. $-13i$
    D. -2.5

15. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain D_1 to D_2then the which one of the point does not lies in $D_1 \cap D_2$
    A. 10
    B. 0
    C. $-i$
    D. -2.8

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	D
3	B
4	D
5	A
6	A
7	B
8	A
9	B
10	**A**
11	**A**
12	A
13	C

14	D
15	A

1.7 Summary

- An analytic function *f*(z) with its domain of definition D is called a functionelement and is denoted by (*f*, D).
- The series equation $S = \sum_{n=0}^{\infty} a_n$ converges absolutely if $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$<1 and diverges if *r*>1. If *r* =1, no conclusion can be taken.
- Suppose $f_1(z)$ is analytical on a region D_1. Now suppose that D_1 is contained in a region $f_2(z)$. The function *f(z)* can be analytically continued from D_1 *to* D_2 if there exists a function $f_2(z)$ such that: $f_2(z)$ is analytic on *S*, $f_2(z) = f_1(z)$ for all $z \in D_1$

1.8 Keywords

Absolute convergence: The series equation $S = \sum_{n=0}^{\infty} a_n$ converges absolutely if $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$<1 and diverges if *r*>1. If *r* =1, no conclusion can be taken.

Analytic continuation: If $z\in D$, (*f*, D) is a function element of z, then $(f_1,D_1)$and(f_2,D_2)areinanalyticcontinuationsofeachother*iff* $D_1\cap D_2\neq\phi$and$f_1(z) = f_2(z)$ forallz$\in D_1\cap D_2$.

1.9 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

Table of Contents

Purpose and Objectives:

After this unit students can be able to-

1. Understand the uniqueness of analytic continuation.
2. Solve the problem based on the power series method of analytic continuation.
3. Learn natural boundary of complete analytic function.

Introduction

Analytic continuation is a method used to extend the domain of definition of a function that is known to be analytic (i.e., holomorphic) in a certain region, to a larger region. This can be achieved by representing the function as a power series and finding the appropriate coefficients to ensure that the function satisfies the same differential equations in the extended region as it does in the original region.

For example, consider the complex function f(z) that is known to be analytic in a disk D centered at the origin. By expanding f(z) in a power series about the origin, we can obtain its Taylor series representation:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

Using this power series representation, we can extend the definition of f(z) to points outside the disk D by assuming that the series converges to the correct value at those points. This process is called analytic continuation, and it allows us to extend the domain of definition of f(z) to a larger region in the complex plane. In this unit first we will understand the uniqueness of analytic continuation, then the power series method of analytic continuation, and then the natural boundary of complete analytic function.

2.1 Uniqueness of Analytic Continuation by Direct Method

Theorem 2.1: Uniqueness of Analytic Continuation

There cannot be more than one continuation of analytic $f_2(z)$ into the same domain.

Proof:

Let $f_1(z)$ be analytic in the domain D_1.

Let $f_2(z)$ and $g_2(z)$ be analytic continuations of same function $f_1(z)$ from D_1 into the domain D_2 via D_{12} which is common in to both D_1 and D_2 .

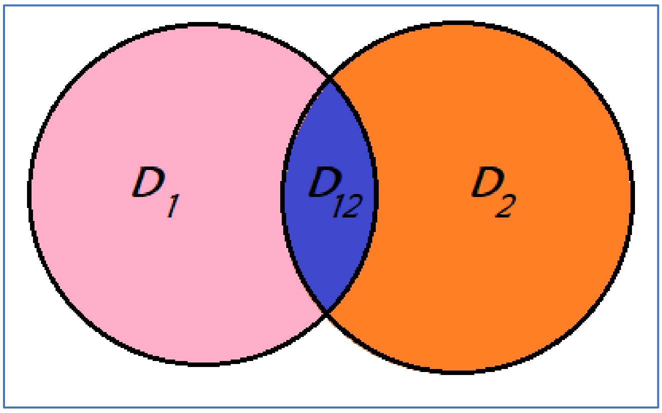

Figure 2.1: Analytical Continuation domains

If we show that $f_2(z) = g_2(z)$ throughout D_2, the result is followed by the this proof.

By the definition of analytic continuation.

$$f_1(z) = f_2(z), \forall z \in D_{12} \tag{2.1}$$

And $f_2(z)$ is analytic in D_2.

$$f_1(z) = g_2(z), \forall z \in D_{12} \tag{2.2}$$

And $g_2(z)$ is analytic in D_2.

From the equation (2.1) and (2.2) we can conclude that

$f_1(z) = f_2(z) = g_2(z), \forall z \in D_{12}$

Or

$f_2(z) = g_2(z), \forall z \in D_{12}$

Or

$(f_2 - g_2)(z) = 0, \forall z \in D_{12}$

f_2 and g_2 are analytic in D_2

$\Rightarrow \quad f_2 - g_2$ is analytic in D_2.

Thus, we see that $(f_2 - g_2)(z)$ vanishes in D_{12} which is a part of D_2.Also the function is analytic in D_2.

Hence, we must have

$(f_2 - g_2)(z) = 0, \forall z \in D_2$

$\Rightarrow \quad f_2(z) = g_2(z)\ \forall z \in D_2$.

So, there cannot be more than one continuation of analytic $f_2(z)$ into the same domain.

Remark

The uniqueness property requires the domains of the two analytic continuations to be the same. It is not generally true that if

$F_1 : D_1 \to C$ *and* $F_2 : D_1 \to C$ are two analytic continuations of $f : D \to C$ to different domains D_1, D_2, that they must agree on $D_1 \cap D_2$. A slightly more complicated example is the power series with Fibonacci coefficients:

$f(z) = f_0 + f_1 z + f_2 z^2 + \ldots$, which we considered a few lectures ago.

Initially we observed that this converges and thereby defines an analytic function in some neighborhood D of zero.

By applying the recurrence $f_{n+1} = f_n + f_{n-1}$, we were able to obtain the functional equation:

$(1 - z - z^2)f(z) = z \Rightarrow f(z) = \frac{z}{1-z-z^2}\ z \in D.$

We then used the right hand side as a definition of f in a much larger domain $D' = C\setminus\{\varphi, \psi\}$.

Formally, $F(z) = \frac{z}{1-z-z^2}$ is an analytic continuation of $f\ to\ D'$.

We didn't explicitly use a different name to distinguish between the continuation and the original function (since they agree where they are both defined) and we will sometimes follow this convention in the future.

In any case, we were then able to use the properties of F in the much larger domain D' (by applying the Residue theorem) to get a good handle on what is happening at zero, and thereby extract a formula for the coefficients.

A functional equation is not the only way to obtain an analytic equation, but it is often the best one. In general, what one is looking for is an alternate representation of the same function which makes sense in a larger region; this alternate description is then used as a definition in the larger region.

2.2 Power Series Method of Analytic Continuation

The Power Series Method of Analytic Continuation is a method used to extend the domain of a complex power series beyond its radius of convergence. It is based on the idea of representing a function as an infinite sum of powers and using this representation to extend the function to a larger domain.

The method works by finding the coefficients of the power series for a given function using Cauchy's Integral Formula, and then using these coefficients to analytically continue the function to a larger domain. This method is useful for finding the values of a function in complex domains, where it is not possible to use real analysis techniques.

Let the initial function $f_1(z)$ is represented by the Taylor's series

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n \tag{2.3}$$

where $a_n = \frac{f_1^{(n)}(z_1)}{n!}$

This series is convergent inside a circle C_1(see the figure 2.2)defined by

$$|z - z_1| = R_1 \tag{2.4}$$

Here $R_1 = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

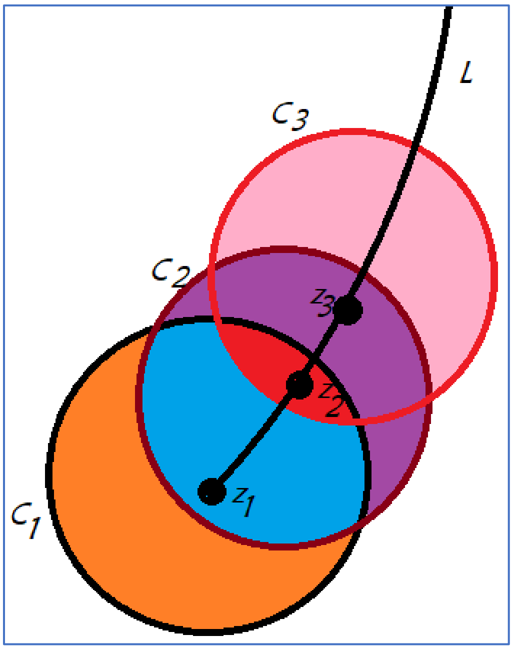

Figure 2.2: The curve L from z_1 and perform analytic continuation

We draw a curve L from $z_1$and perform analytic continuation along this path as follows

Take a point z_2 on L such that $z_2$lies inside the C_1 .

With this help of equation (2.3), we can find the $f_1(z_2), f_1'(z_2), f_1''(z_2) \dots, f_1^{(n)}(z_2)$ by repeated differentiation of (2.3).

Write

$$f_2(z) = \sum_{n=0}^{\infty} b_n (z - z_2)^n \tag{2.5}$$

where $b_n = \frac{f_2^{(n)}(z_2)}{n!}$

The power series (2.5) is convergent inside a circle C_2 defined by

$$|z - z_2| = R_2 \tag{2.6}$$

Here $R_2 = \lim_{n\to\infty} |b_n|^{\frac{1}{n}}$

Also $f_1(z) = f_2(z), \forall z \in C_{12}(The\ common\ part\ of\ C_1\ and\ C_2)$.

Hence $f_2(z)$ is an analytic continuation of $f_1(z)$ from $C_1\ to\ C_2$.

Now take a point z_3 on L such that $z_3$lies inside the C_2 .

With this help of equation (2.5), we can find the $f_2(z_3), f_2'(z_3), f_2''(z_3) \dots, f_2^{(n)}(z_3)$ by repeated differentiation of (2.5).

Write

$$f_3(z) = \sum_{n=0}^{\infty} c_n (z - z_3)^n \tag{2.7}$$

where $c_n = \frac{f_3^{(n)}(z_3)}{n!}$

The power series (2.7)(2.5) is convergent inside a circle C_3 defined by

$$|z - z_3| = R_3 \tag{2.8}$$

Here $R_3 = \lim_{n\to\infty} |c_n|^{\frac{1}{n}}$

Also $f_2(z) = f_3(z), \forall z \in C_{23}(The\ common\ part\ of\ C_2\ and\ C_3)$.

Hence $f_3(z)$ is an analytic continuation of $f_2(z)$ from $C_2\ to\ C_3$.

Now $f_3(z)$ is an analytic continuation of $f_1(z)$ from $C_2\ to\ C_3$.

Repeating this process, we get as continuations several different power series analytic in their respective domains D_1, D_2 ..., where D_1, D_2 ..., are respectively interiors of C_1, C_2 ...,.

Question:

Show that the power series $z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$may be analytically continued to a wider range by means of the series $log2 - \frac{1-z}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \cdots$

Solution:

Let $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ (2.9)

And $f_2(z) = log2 - \frac{1-z}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \cdots$ (2.10)

Here $f_1(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1}$ using the ratio test, if a_n is convergent then $\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$a_n = (-1)^n \frac{z^{n+1}}{n+1}$$

$$a_{n+1} = (-1)^{n+1} \frac{z^{n+2}}{n+2}$$

Now $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\frac{z^{n+2}}{n+2}}{(-1)^n\frac{z^{n+1}}{n+1}}\right| < 1$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n . \frac{-1}{(-1)^n} \frac{z^{n+1}.z.(n+1)}{(n+2)\, z^{n+1}}\right| < 1$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|-1 \frac{.z.(1+1/n)}{(1+2/n)}\right| < 1$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|-1 \frac{.z.(1+1/n)}{(1+2/n)}\right| < 1$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |z| < 1$

Hence $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots = \log(1+z)$ which is convergent for the $|z| < 1$. (2.11)

Thus $f_1(z)$ is analytic inside the circle C_1defined by $|z| = 1$(See the figure 2.3).

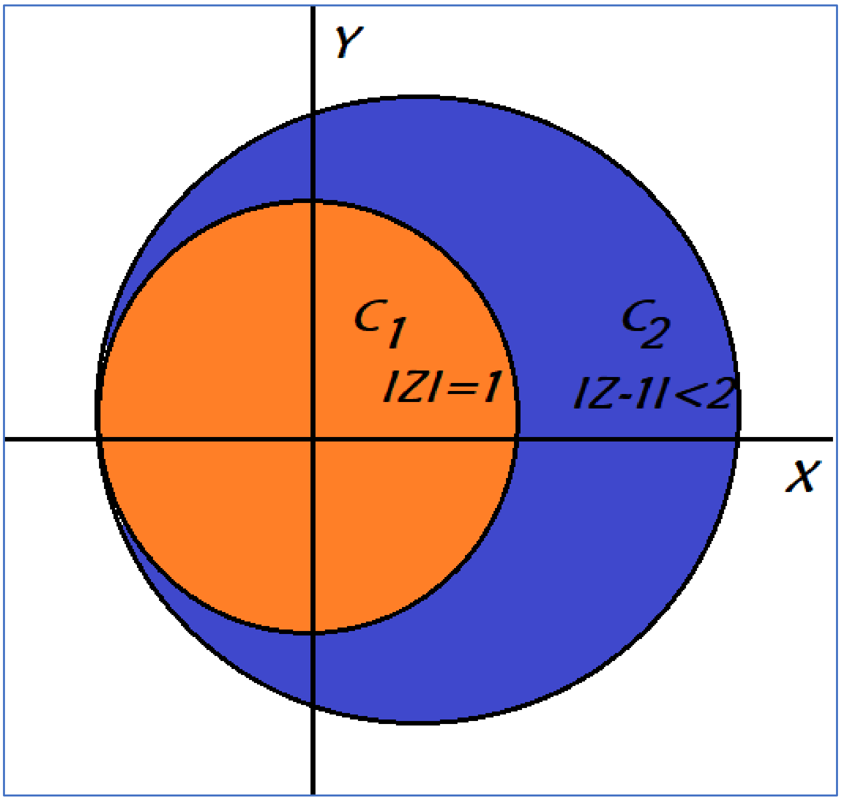

Figure 2.3: the domains of C_1 and C_2

Now we will show that $f_2(z)$ is analytic inside a domain and will also find the convergent analysis of $f_2(z)$.

$f_2(z)$ can be expressed as

$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$

$$\Rightarrow f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\left(\frac{1-z}{2}\right)^2 + \frac{1}{3}\left(\frac{1-z}{2}\right)^3 + \cdots\right]$$

Let the nth term of $f_2(z)$ is $b_n = (-1)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}$ and

$$b_{n+1} = (-1)\frac{1}{n+2}\left(\frac{1-z}{2}\right)^{n+2}$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(-1)\frac{1}{n+2}\left(\frac{1-z}{2}\right)^{n+2}}{(-1)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}}\right| < 1$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{n+1}{n+2}\left(\frac{1-z}{2}\right)^{n+2}\left(\frac{2}{1-z}\right)^{n+1}\right| < 1$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\left(\frac{1-z}{2}\right)\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\left(\frac{1-z}{2}\right)\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \left|\left(\frac{1-z}{2}\right)\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |1-z| < 2$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |z-1| < 2$$

Hence $f_2(z)$ is convergent for the $|z-1| < 2$.

Thus $f_2(z)$ is analytic inside the circle C_2defined by $|z-1| = 2$.

As we know that $\log\left[1-\left(\frac{1-z}{2}\right)\right] = -\left[\frac{1-z}{2}+\frac{1}{2}\left(\frac{1-z}{2}\right)^2 + \frac{1}{3}\left(\frac{1-z}{2}\right)^3 + \cdots\right]$

Then

$$f_2(z) = log2 + \log\left[1-\left(\frac{1-z}{2}\right)\right]$$

$$f_2(z) = log2 + \log\left[\frac{2-1+z}{2}\right]$$

$$f_2(z) = log2 + log(1+z) - log2$$

$$f_2(z) = log(1+z)$$

By (2.11),

$f_2(z) = f_1(z)$ in the area common to both C_1 and C_2.

Hence, we can say that $f_2(z)$ is analytic continuation of $f_1(z)$ from the interior of C_1 to the interior of C_2. Moreover C_2 is a larger range in comparison to C_1 as shown in the figure 2.3.

2.3 Natural Boundary

In complex analysis, a natural boundary of a complex-valued function is a boundary of its domain that is not a removable singularity. This means that the function cannot be extended analytically across the boundary, and its behavior there is determined by the behavior of the function on the boundary. A classic example of a natural boundary is the boundary of the unit disk in the complex plane, which is the unit circle. Functions defined on the unit disk have essential singularities on the boundary, which means that they cannot be extended analytically to the outside of the unit disk.

Definition 2.1: Function Element

An analytic function f with domain D is called a function element and is denoted by (f, D).

Definition 2.2: Complete Analytic Function

Suppose that $f(z)$ is analytic in a domain D. Let us form all possible analytic continuations of (f, D) and then all possible analytic continuations $(f_1, D_1), (f_2, D_2), \ldots, (f_n, D_n)$ of these continuations such that:

$$F(z) = \begin{cases} f_1(z)\ if\ z \in D_1 \\ f_2(z)\ if\ z \in D_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(z)\ if\ z \in D_n \end{cases} \tag{2.12}$$

Such a function $F(z)$ is called complete analytic function.

In this process of continuation, we may arrive at a closed curve beyond which it is not possible to take analytic continuation. Such a closed curve is known as the **natural boundary** of the complete analytic function. A point lying outside the natural boundary is known as the singularity of the complete analytic function. If no analytic continuation of $f(z)$ is possible to a point z_0, then z_0 is a singularity of $f(z)$.

Obviously, the singularity of $f(z)$ is also a singularity of the corresponding complete analytic function $F(z)$.

Example

Show that the circle of convergence of the power series $f(z) = 1 + z + z^2 + z^4 + z^8$ ..., is a natural boundary of its sum function.

Solution:

The circle of convergence of a power series is the largest circle centered at the origin within which the series converges to a function.

The sum function of the series $f(z) = 1 + z + z^2 + z^4 + z^8$ ..., is not defined at z = 1, so the circle of convergence of the series must include the origin and exclude z = 1.

Since the sum of the series diverges for z = 1, it is a natural boundary for the sum function. The circle of convergence for the series is the largest circle centered at the origin within which the sum function is defined and analytic, and thus it serves as a natural boundary for the sum function.

Example

Show that $f\ (z\ ) = \sum_{n=0}^{\infty} \frac{z^{2^{n+1}}}{1-z^{2^{n+1}}}$ is analytic in the domain |z | < 1 and the domain |z | > 1, and that |z | = 1 is a natural boundary for the function in each domain.

Solution:

For a function to be analytic, it must be complex differentiable at every point in its domain.

Let us first consider the domain |z| < 1.

For $|z| < 1, |z^{2^{n+1}}| < 1$ and thus $1 - |z^{2^{n+1}}| > 0$. Hence, the denominator $1 - z^{2^{n+1}}$is never 0 and the function is well-defined in this domain.

Next, we can apply the Cauchy-Riemann equations to show that the function is complex differentiable in this domain, and therefore analytic.

For |z| > 1, the same argument can be made: the denominator $1 - z^{2^{n+1}}$ is never 0 for |z| > 1, and the function is well-defined in this domain.

Finally, for |z| = 1, the function is not complex differentiable, which means that |z| = 1 is a natural boundary for the function in both domains.

Therefore, the function $\sum_{n=0}^{\infty} \frac{z^{2^{n+1}}}{1-z^{2^{n+1}}}$is analytic in the domain |z| < 1 and the domain |z| > 1, and |z| = 1 is a natural boundary for the function in each domain.

2.4 Review questions

1. Prove that the series $z^{1!} + z^{2!} + z^{3!} + \cdots$ has the natural boundary $|Z| = 1$.

2. Prove that |z| = 1 is a natural boundary for the series $\sum_{n=0}^{\infty} 2^{-n} * z^{3n}$

3. Let $F_1(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{3^n}$ Find an analytic continuation of $F_1\ (z)$, which converges for $z = 3 - 4i$

4. State and prove the uniqueness theorem of analytic continuation
5. Show that the series $1 + z + z^2 + z^4 + z^8 + \ldots$, can not be analytically continued beyond the $|z| = 1$

2.5 Self-assessment

1. The power series $f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ is convergent inside the
   A. Region $|z + 1| < 2$
   B. Region $|z - 1| < 2$
   C. Region $|z| < 1$
   D. Region $|z| < 2$

2. There cannot be more than one continuation of analytic $f(z)$ into the same domain.
   A. True
   B. *False*

3. The nth term of the series $z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$
   A. $a_n = (-1)^n \frac{z^{n+1}}{n+1}$
   B. $a_n = (-2)^n \frac{z^{n+1}}{n+1}$
   C. $a_n = (-1)^{n+1} \frac{z^{n+1}}{n+1}$
   D. $a_n = (-1)^n \frac{z^{n+1}}{n+2}$

4. An analytic function f with domain D is called a function element and is denoted by (f, D).

   A. True
   B. *False*

5. The nth term of the series $\left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$ *is*
   A. $(-2)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}$
   B. $(-1)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}$
   C. $(-1)\frac{1}{n+2}\left(\frac{1-z}{2}\right)^{n+1}$
   D. $(-1)\frac{1}{n+1}\left(\frac{1-z}{3}\right)^{n+1}$

6. The power series $\left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$ is convergent inside the
   A. $|z - 1| < 1$
   B. $|z| < 1$
   C. $|z| < 2$
   D. $|z - 1| < 2$

7. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
   $$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
   from the domain *D₁ to D₂* then the which one of the domain does lies in $D_1 \cap D_2$
   A. $|z - 1| < 1$
   B. $|z| < 1$

C. $|z| < 2$
D. $|z - 1| < 2$

8. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain D_1 *to* D_2 then the which one of the points does not lies in $D_1 \cap D_2$
A. $0.1 + 0.1i$
B. 0.5
C. $0.5i$
D. $2 + 5i$

9. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain D_1 *to* D_2 then the which one of the points does lies in $D_1 \cap D_2$
A. $0.1 + 0.2i$
B. 5
C. $5i$
D. $2 + 5i$

10. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain D_1 *to* D_2 then the which one of the points does not lies in $D_1 \cap D_2$
A. $0.2i$
B. 0.3
C. $0.4i$
D. $4i$

11. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is a natural boundary for its sum function then the circle of convergent is
A. $|z - 1| < 1$
B. $|z| < 1$
C. $|z| < 2$
D. $|z - 1| < 2$

12. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain D_1 *to* D_2 then the which one of the point lies in $D_1 \cap D_2$
A. $\frac{1}{8} + \frac{1}{8}i$
B. $\frac{10}{2} + \frac{1}{2}i$
C. $-13 + \frac{1}{2}i$
D. $-3 - 13i$

13. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain D_1 *to* D_2 then the which one of the points does not lies in $D_1 \cap D_2$
A. $0.3i$

B. 0.3
C. $0i$
D. $12 + i$

14. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to

$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$

from the domain D_1 *to* D_2 then the which one of the points does lies in $D_1 \cap D_2$
A. $0.01 + 0.02i$
B. 50
C. $50i$
D. $20 + 5i$

15. In complex analysis, a natural boundary of a complex-valued function is a boundary of its domain that is not a removable singularity.
A. $True$
B. $False$

Table 1: Answers of self-assessment

Question number	Correct answer
1	C
2	A
3	A
4	A
5	B
6	D
7	B
8	D
9	A
10	**A**
11	**C**
12	A
13	D
14	A
15	A

2.6 Summary

- The Power Series Method of Analytic Continuation is a method used to extend the domain of a complex power series beyond its radius of convergence.
- An analytic function f with domain D is called a function element and is denoted by (f, D).
- Uniqueness of Analytic Continuation: There cannot be more than one continuation of analytic $f_2(z)$ into the same domain.

2.7 Keywords

Complete analytic function: Suppose that $f(z)$ is analytic in a domain D. Let us form all possible analytic continuations of (f, D) and then all possible analytic continuations $(f_1, D_1), (f_2, D_2), \dots, (f_n, D_n)$ of these continuations such that:

$$F(z) = \begin{cases} f_1(z)\ if\ z \in D_1 \\ f_2(z)\ if\ z \in D_2 \\ \dots\dots\dots\dots\dots\dots\dots \\ f_n(z)\ if\ z \in D_n \end{cases}$$

Such a function $F(z)$ is called complete analytic function.

2.8 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

Table of Contents

Purpose and Objectives:

The Monodromy Theorem in complex analysis states that given a non-constant holomorphic function on a simply connected domain, its set of singularities is invariant under any loop in the domain. In other words, it characterizes the behavior of the function near its singularities and provides a way to study the topological structure of complex functions. The theorem is useful for solving certain types of differential equations, as well as for constructing complex functions with prescribed singularities.

Similarly, the Poisson integral formula has several applications in various fields, including:

1. Harmonic Analysis: The Poisson Integral Formula provides a tool for solving boundary value problems for harmonic functions and has applications in potential theory and boundary value problems.
2. Image Processing: The Poisson Integral Formula is used in image processing to restore images that have been degraded or to smooth out noise in images.
3. Numerical Analysis: The Poisson Integral Formula is used in numerical analysis to solve partial differential equations, especially in areas like electrostatics and heat transfer.
4. Complex Analysis: The Poisson Integral Formula is used in complex analysis to study conformal mappings, potential theory, and complex dynamics.
5. Signal Processing: The Poisson Integral Formula is used in signal processing to solve problems in signal restoration, noise reduction, and boundary value problems for signals.

After this unit students can be able to-

1. State and prove the Monodromy theorem.
2. Learn Poisson Integral Formula for analytic function.
3. Understand the Poisson Kernel, and Conjugate Poisson Kernel for analytic function.
4. Solve the problem based on the Poisson Integral Formula.

Introduction

The number of independent loops or paths around a singular point of an analytic function can be understood by sheets of the multi-valued analytic function.

In other words, if an analytic function has a singularity at a point, then the number of independent loops that can be taken around this point is equal to the number of branches of the function that can be defined in a neighborhood of the singularity.

The Monodromy Theorem is an important result in complex analysis and is used to study the behavior of multi-valued analytic functions near singular points. If a function f is analytic in the unit disk of the complex plane and continuous on its boundary, then it can be represented by the Poisson integral formula. In this unit first we will understand the Monodromy theorem then learn Poisson Integral Formula for analytic function. After that we will explore the concept of the Poisson Kernel, and Conjugate Poisson Kernel for analytic function. Finally solve the problem based on the Poisson Integral Formula.

3.1 Monodromy Theorem

Let D be a simply connected domain and suppose $f_0\,(z)$ is analytic in a domain $D_0 \subset D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

Proof:

Suppose the conclusion is false. Then there exist points $z_0 \in D_0, z_1 \in D$, and curves C_1, C_2

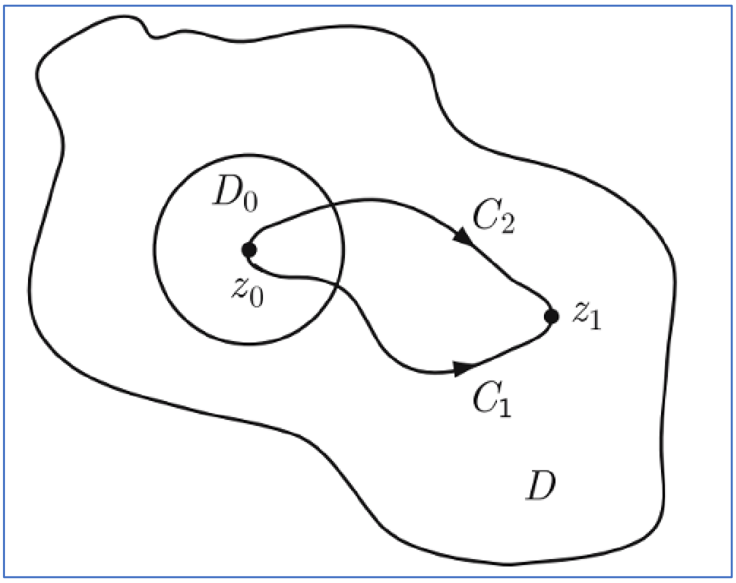

Figure 3.1: D be a simply connected domain and the points $z_0 \in D_0$, $z_1 \in D$, and $D_0 \subset D$.

both having initial point z_0 and terminal point z_1 such that (f_0, D_0) leads to a different function element in a neighborhood of z_1 when analytically continued along C_1 than when analytically continued along C_2 (see Figure3.1).

This means that (f_0, D_0) does not return to the same function element when analytically continued along the closed curve $C_1 - C_2$.

Lemma 3.1.

Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n \to \infty$, then there is exactly one point in common to all the S_n.

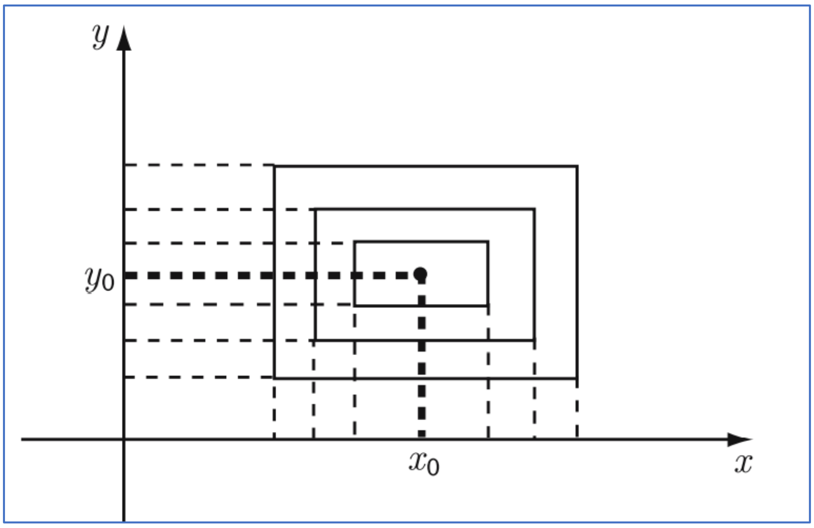

Figure 3.2: sequence of closed and bounded rectangles in the plane.

To prove the theorem, it thus suffices to show that the function element $(f_0, D_0), D_0 \subset D$, can be continued along any closed curve lying in D and return to the same value. In the special case that the closed curve C is a rectangle

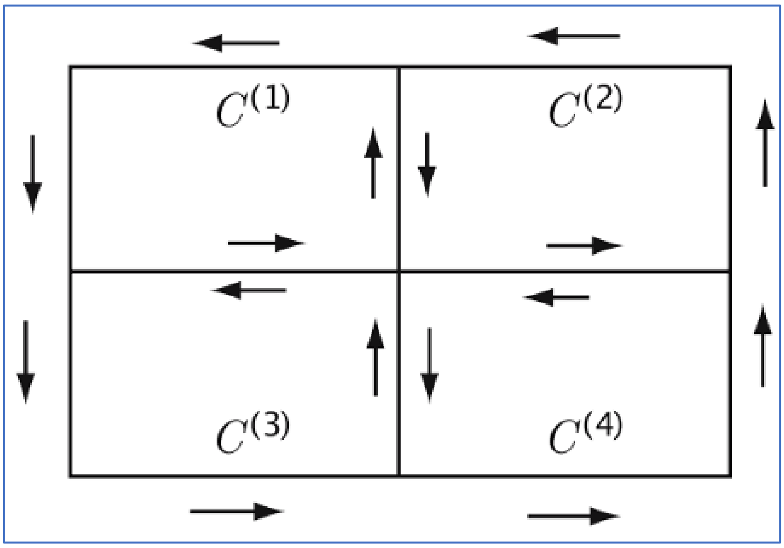

Figure 3.3: Rectangle C into four congruent rectangles

Divide the rectangle C into four congruent rectangles, as illustrated in Figure 3.3 continuation along C produces the same effect as continuation along these four rectangles taken together.

If the conclusion is false for C, then it must be false for one of the four sub-rectangles, which we denote by C_1. We then divide C_1 into four congruent rectangles, for one of which the conclusion is false.

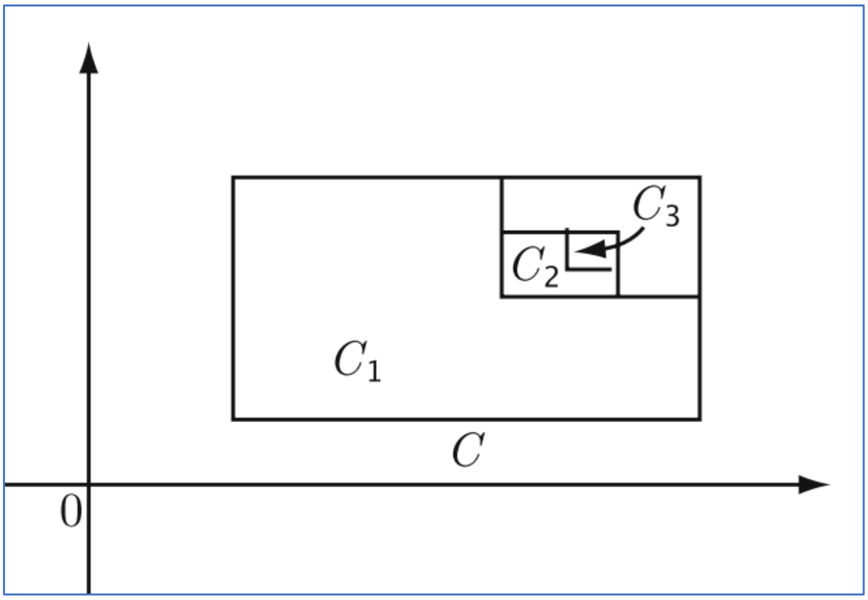

Figure 3.4:C_1 into four congruent rectangles

Continuing the process, we obtain a nested sequence of rectangles for which the conclusion is false.

According to Lemma 1, there is exactly one point, call it z_*, belonging to all the rectangles in the nest. Since $z_* \in D$, there exists a function element (f_*, D_*) with $z_* \in D_* \subset D$.

For n sufficiently large, the rectangle C_n of the nested sequence is contained in D_*

But this means that $f_*(z)$ is analytic in a domain containing C_n, contrary to the way C_n was defined. This contradiction concludes the proof in the special case in which the curve is a rectangle.

3.2 Poisson Integral Formula, Poisson Kernel, and Conjugate Poisson Kernel

If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within, C, then

$$f(z) = \frac{1}{2\pi i}\int_c \frac{(R^2 - a\bar{a})f(z)}{(z-a)(R^2 - z\bar{a})}dz$$

$$\Rightarrow f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\emptyset})}{R^2 - 2RrCos(\theta - \emptyset) + r^2}d\emptyset$$

Where $a = re^{i\theta}$is any point inside the circle $|z| = R$.

Proof:

Suppose f(z) is analytic within and on the circle C defined $|z| = R$.

Let $a = re^{i\theta}$is any point inside the circle $|z| = R$ so that $0 < r < R$.

Let the inverse of $A(a)$ is $A'(a')$ with respect to the circle C is given by $a' = R^2/\bar{a}$ which lies outside the circle C(See the Figure 3.5).

By Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz \tag{3.1}$$

Since $f(z)$ is analytic within and upon the circle C and so $\frac{f(z)}{(z-a')}$ is analytic within and on the circle.

By Cauchy's integral theorem

$$\int_c \frac{f(z)}{(z-a')}dz = 0 \tag{3.2}$$

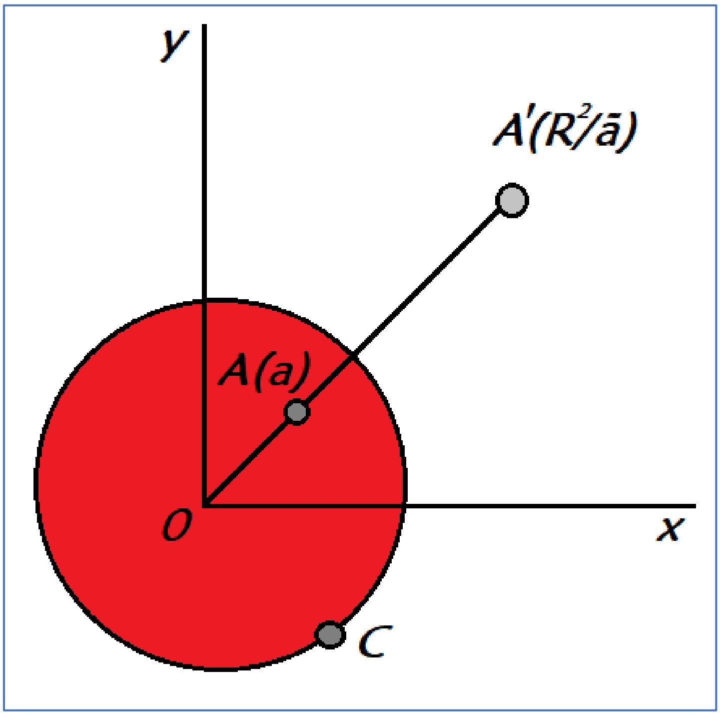

Figure 3.5: Inverse of $A(a)$ is $A'(a')$ with respect to the circle C is given by $a' = R^2/\bar{a}$

Note that $\frac{f(z)}{(z-a)}$ is not analytic within C

Now

(3.1)-(3.2) gives

$$f(a) - 0 = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz - \int_c \frac{f(z)}{(z-a')}dz \quad (3.3)$$

$$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{1}{(z-a)} - \frac{1}{(z-a')}\right] f(z)dz \quad (3.4)$$

$$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(z-a')-(z-a)}{(z-a)(z-a')}\right] f(z)dz \quad (3.5)$$

$$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(a-a')}{(z-a)(z-a')}\right] f(z)dz \quad (3.6)$$

$$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{\left(a-\frac{R^2}{\bar{a}}\right)}{(z-a)(z-\frac{R^2}{\bar{a}})}\right] f(z)dz \quad (3.7)$$

$$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(a\bar{a}-R^2)}{(z-a)(z\bar{a}-R^2)}\right] f(z)dz \quad (3.8)$$

$$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(R^2-a\bar{a})}{(z-a)(R^2-z\bar{a})}\right] f(z)dz \quad (3.9)$$

This proves the first required result.

Any point z on $|z| = R$ is expressible as $z = R\, e^{i\phi}$

Also $a = re^{i\theta}$ so that $\bar{a} = re^{-i\theta}$

Now $R^2 - a\bar{a} = R^2 - re^{i\theta}.re^{-i\theta}$ (3.10)

$$\Rightarrow R^2 - a\bar{a} = R^2 - r^2$$

Now

$$(z-a)(R^2 - z\bar{a}) = (R\, e^{i\phi} - re^{i\theta})(R^2 - Re^{i\phi}\, re^{-i\theta}) \quad (3.11)$$

$$\Rightarrow (z-a)(R^2 - z\bar{a}) = R\, e^{i\phi}(R\ - re^{i(\theta-\phi)})(R - re^{-i(\theta-\phi)})$$

$$\Rightarrow (z-a)(R^2 - z\bar{a}) = R\, e^{i\phi}[(R^2 + r^2 - rR(e^{i(\theta-\phi)} - e^{-i(\theta-\phi)})]$$

$$\Rightarrow (z-a)(R^2 - z\bar{a}) = R\, e^{i\phi}[(R^2 + r^2 - 2rRCos(\theta - \phi)] \tag{3.12}$$

$$dz = d(Re^{i\phi}) = Rie^{i\phi}d\phi \tag{3.13}$$

Writing (3.9) with the help of (3.11) and (3.13),

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2 - r^2)f(z)f(Re^{i\emptyset}).i}{[R^2 - 2RrCos(\emptyset - \theta) + r^2]Re^{i\emptyset}} d\emptyset$$

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]} d\emptyset \tag{3.14}$$

This proves the second result.

Here $\frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]}$ is known as the *Poisson Kernal* for the disk $|z\ | < R$.

Note that the *Poisson, kernel* is bounded above by $\frac{(R^2-r^2)}{[R^2-2Rr+r^2]} = \frac{R+r}{R-r}$.

The conjugate Poisson kernel is a mathematical function used in complex analysis and potential theory. It is defined as the conjugate of the *Poisson kernel*, which is a function that maps points in the complex plane to the unit disk. The conjugate Poisson kernel is given by the formula:

$$P^*(z)\ =\ P\left(\frac{1}{z^*}\right)$$

where $P(z)$ is the Poisson kernel and z^* is the complex conjugate of z.

Question:

Using Poisson's integral formula for the circle, show that:

$$\int_0^{2\pi} \frac{e^{cos\phi}.Cos(Sin\phi)d\phi}{5-4Cos(\theta-\phi)} = \frac{2\pi}{3}e^{cos\theta}\cos(sin\theta) \tag{3.15}$$

Solution:

By the Poisson's integral formula,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]} d\emptyset \tag{3.16}$$

If we compare R.H.S. of (3.15) with the given integral, then we find

$$R^2 + r^2 = 5 \tag{3.17}$$

$$rR = 2 \tag{3.18}$$

$$f(Re^{i\phi}) = e^{cos\phi}.Cos(Sin\phi) \tag{3.19}$$

Using (3.17) and (3.18)

$$R = 2, r = 1 \text{ and so } R^2\ - r^2 = 3 \tag{3.20}$$

Now (3.19) $\Rightarrow$

$$f(re^{i\theta}) = e^{cos\theta}\cos(sin\theta) \tag{3.21}$$

Putting value from (3.17), (3.18), (3.20), and (3.21) in the equation (3.16), we get

$$e^{cos\theta}\cos(sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{3e^{cos\phi}.Cos(Sin\phi)}{5 - 4Cos(\theta-\phi)} d\emptyset$$

$$\frac{2\pi}{3}e^{Cos\theta}\,\mathrm{Cos}(Sin\theta) = \int_0^{2\pi} \frac{e^{cos\phi}.Cos(Sin\phi)}{5 - 4Cos(\theta-\phi)} d\emptyset$$

Hence proved.

Question:

Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{621f\left(25e^{\frac{i\pi}{4}}\right)}{629-100Cos\left(\pi-\frac{\pi}{4}\right)} d\emptyset$

Solution:

We know that using the Poisson integral formula,

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f\left(Re^{i\emptyset}\right)}{[R^2-2RrCos(\theta-\phi)+r^2]} d\emptyset$$

Here

(R^2-r^2) =621

$f\left(Re^{i\emptyset}\right) = f(25e^{\frac{\pi}{4}i})$

$2Rr = 100$

$\theta - \phi = \pi - \frac{\pi}{4}$

(R^2+r^2) =629

Hence, we can conclude that.

$R = 25$

$r = 2$

$\theta = \pi$

$\phi = \frac{\pi}{4}$

$f\left(re^{i\theta}\right) = f(2e^{i\pi})$

$$\int_0^{2\pi} \frac{621f\left(25e^{\frac{i\pi}{4}}\right)}{629-100Cos\left(\pi-\frac{\pi}{4}\right)} d\emptyset = 2\pi f(2e^{i\pi})$$

3.3 <u>Review questions</u>

1. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{75f\left(10e^{\frac{i\pi}{4}}\right)}{125-100Cos\left(\frac{\pi}{2}-\frac{\pi}{4}\right)} d\emptyset$

2. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{64f\left(10e^{\frac{i\pi}{10}}\right)}{136-120Cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right)} d\emptyset$

3. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{9f\left(5e^{\frac{i\pi}{10}}\right)}{41-40Cos\left(\pi-\frac{\pi}{10}\right)} d\emptyset$

4. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{80f\left(9e^{\frac{i\pi}{2}}\right)}{82-18Cos\left(\pi-\frac{\pi}{2}\right)} d\emptyset$

5. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{99f\left(10e^{\frac{i\pi}{10}}\right)}{101-20Cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right)} d\emptyset$

3.4 Self-assessment

1. Let D be a simply connected domain and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

   A. True
   B. *False*

2. Let D be a simply connected domain and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there does not exists any single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

   A. True
   B. *False*

3. Let D be a simply connected domain and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is not analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

   A. True
   B. *False*

4. Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n\ \to\ \infty$, then there is exactly one point in common to all the S_n.

   A. True
   B. *False*

5. Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n\ \to\ \infty$, then there are two points in common to all the S_n.

   A. True
   B. *False*

6. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-1)f(Re^{i\emptyset})}{[R^2-2RCos(\pi-\phi)+1]}\,d\emptyset =$

   A. $f(e^{i\pi})$
   B. $f(2e^{i\pi})$
   C. $f(3e^{i\pi})$
   D. $2\,f(e^{i\pi})$

7. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{(R^2-4)f(Re^{i\emptyset})}{[R^2-4RCos(\pi-\phi)+4]}\,d\emptyset =$

   A. $\pi f(e^{i\pi})$
   B. $2\pi f(2e^{i\pi})$
   C. $3\pi f(3e^{i\pi})$
   D. $f(e^{i\pi})$

8. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-9)f(Re^{i\emptyset})}{[R^2-6RCos(\pi-\phi)+9]}d\emptyset =$
   - A. $f(3e^{i\pi})$
   - B. $f(2e^{i\pi})$
   - C. $f(3e^{i\pi})$
   - D. $2f(e^{i\pi})$

9. Using the Poisson integral formula, the $\int_0^{2\pi}\frac{(R^2-16)f(Re^{i\emptyset})}{[R^2-8RCos(\pi-\phi)+16]}d\emptyset =$
   - A. $\pi f(e^{i\pi})$
   - B. $2\pi f(4e^{i\pi})$
   - C. $3\pi f(3e^{i\pi})$
   - D. $f(5e^{i\pi})$

10. Using the Poisson integral formula, the $\int_0^{2\pi}\frac{21f(5e^{i\emptyset})}{29-20Cos(\pi-\phi)}d\emptyset =$
    - A. $\pi f(e^{i\pi})$
    - B. $2\pi f(2e^{i\pi})$
    - C. $3\pi f(3e^{i\pi})$
    - D. $f(e^{i\pi})$

11. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi}\frac{27f(6e^{i\emptyset})}{45-36Cos(\pi-\phi)]}d\emptyset =$
    - A. $f(3e^{i\pi})$
    - B. $f(2e^{i\pi})$
    - C. $f(3e^{i\pi})$
    - D. $2f(e^{i\pi})$

12. Using the Poisson integral formula, the $\int_0^{2\pi}\frac{20f(6e^{i\emptyset})}{52-48Cos(\pi-\phi)}d\emptyset =$
    - A. $\pi f(e^{i\pi})$
    - B. $2\pi f(4e^{i\pi})$
    - C. $3\pi f(3e^{i\pi})$
    - D. $f(5e^{i\pi})$

13. Using the Poisson integral formula, the $\int_0^{2\pi}\frac{99f(10e^{i\emptyset})}{101-20Cos(\pi-\phi)}d\emptyset =$
    - A. $\pi f(e^{i\pi})$
    - B. $2\pi f(e^{i\pi})$
    - C. $3\pi f(3e^{i\pi})$
    - D. $f(e^{i\pi})$

14. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi}\frac{91f(10e^{i\emptyset})}{109-60Cos(\pi-\phi)]}d\emptyset =$
    - A. $f(3e^{i\pi})$
    - B. $f(2e^{i\pi})$
    - C. $f(3e^{i\pi})$
    - D. $2f(e^{i\pi})$

15. Using the Poisson integral formula, the $\int_0^{2\pi}\frac{4f(5e^{i\emptyset})}{41-40Cos(\pi-\phi)}d\emptyset =$
    - A. $\pi f(e^{i\pi})$
    - B. $2\pi f(4e^{i\pi})$
    - C. $3\pi f(3e^{i\pi})$
    - D. $f(5e^{i\pi})$

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	B
3	B
4	A
5	B
6	A
7	B
8	D
9	B
10	**B**
11	**A**
12	B
13	B
14	A
15	B

3.5 Summary

- If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within, C, then

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{(R^2 - r^2)f(Re^{i\emptyset})}{R^2 - 2RrCos(\theta - \emptyset) + r^2} d\emptyset$$

Where $a = re^{i\theta}$is any point inside the circle $|z| = R$.

- $\frac{(R^2-r^2)f\left(Re^{i\emptyset}\right)}{[R^2-2RrCos(\theta-\phi)+r^2]}$ is known as the ***Poisson Kernal*** for the disk $|z\,| < R$.
- The *Poisson, kernel* is bounded above by $\frac{(R^2-r^2)}{[R^2-2Rr+r^2]} = \frac{R+r}{R-r}$.
- The conjugate Poisson kernel is a mathematical function used in complex analysis and potential theory. It is defined as the conjugate of the *Poisson kernel*, which is a function that maps points in the complex plane to the unit disk. The conjugate Poisson kernel is given by the formula:$P^*(z) \;=\; P\left(\frac{1}{z^*}\right)$ where $P(z)$ is the Poisson kernel and z^* is the complex conjugate of z.

3.6 Keywords

Monodromy Theorem:

Let D be a simply connected domain, and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

3.7 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
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Rajesh Kumar Chandrawat, Lovely Professional University

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Purpose and Objectives:

The Mean Value Property states that for a harmonic function in a domain, the average value of the function over a ball is equal to the value of the function at the center of the ball. Harmonic functions are a type of function that satisfy the mean value property. Hence, the mean value property is a necessary condition for a function to be harmonic.

In other words, harmonic functions are functions that have the property that the mean of their values over a small region is equal to the value of the function at a point in the interior of that region. The mean value property is a fundamental property of harmonic functions, and it plays a key role in various applications, such as potential theory and partial differential equations.

Harnack's inequality is a fundamental result in mathematics with various applications in several areas, including partial differential equations, geometry, and potential theory. It provides a relationship between the values of a harmonic function on a small ball and on a large one, which is useful in the study of the regularity and behavior of solutions to elliptic equations. Additionally, Harnack's inequality is also crucial in the study of the asymptotic behavior of Markov processes, stochastic differential equations, and other areas in probability theory.

The Dirichlet problem is a well-known problem in mathematics, specifically in the field of partial differential equations. It asks to find a solution to a partial differential equation that satisfies certain boundary conditions on a given domain. The problem is named after the German mathematician Peter Gustav Lejeune Dirichlet and has numerous applications in physics, engineering, and mathematics. It provides a way to model various physical phenomena such as heat conduction, diffusion, and potential flow.

After this unit students can be able to-

1. State and prove the Harnack's inequality.
2. Learn the mean value property of harmonic functions.
3. Solve the problem based on the Dirichlet problem.

Introduction

The Harnack's inequality is a result in mathematical analysis, which states that for a non-negative solution $u(x)$ of a linear elliptic partial differential equation in a domain, the maximum value of u in a ball centered at a point is bounded above by the average value of u over the same ball. Before embarking the concept of Harnack's inequality, first we discuss the relationship between mean value property and harmonic function. This result has important applications in the study of heat diffusion and potential theory we will understand the Dirichlet problem to find solutions to boundary value problems in these areas.

4.1 Relation Between Mean Value Property and Harmonic Functions

4.1.1 Harmonic function

A harmonic function is a real-valued function that satisfies Laplace's equation, which states that the sum of the second partial derivatives with respect to x and y is equal to zero.

Mathematically, for a function $u(x, y)$, the Laplace's equation can be expressed as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.1}$$

Harmonic functions have several important properties, including being analytic and continuous, having no local maxima or minima, and having a unique mean value over any region in which they are defined. These properties make harmonic functions useful in a variety of mathematical and scientific applications, such as solving boundary value problems and modeling physical phenomena.

4.1.2 The harmonic conjugate of a harmonic function

The harmonic conjugate of a harmonic function u is another harmonic function v that satisfies the condition $u + iv$ is analytic (i.e., it has continuous first and second partial derivatives). In other words, u and v together form a complex function that is analytic in the region where u is defined.

The harmonic conjugate of u is unique up to an additive constant, and it can be found by integrating the derivative of u with respect to y (or x, if u is expressed in terms of x).

For example, if $u(x, y) = f(x) + g(y)$, then its harmonic conjugate is given by $v(x, y) = -g(x) + f(y) + C$, where C is an arbitrary constant.

In conclusion, the harmonic conjugate of a harmonic function is a unique function that helps to form an analytic complex function in the region where the harmonic function is defined.

The harmonic conjugate of an analytic function is another function that, when added to the original function, forms a harmonic function. A harmonic function is a function that satisfies Laplace's equation, which states that the sum of the second partial derivatives with respect to x and y is zero.

Let $f(x, y)$ be an analytic function. Its harmonic conjugate, denoted by $g(x, y)$, is defined as:

$$g(x, y) = \partial y\, u(x, y) - \partial x\, v(x, y)$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(x, y)$, respectively.

The two functions, $f(x, y)$ *and* $g(x, y)$, are called Cauchy-Riemann partners, and their sum is a harmonic function.

4.1.3 Mean value property.

The mean value property of harmonic functions states that, for any point in a ball in a harmonic function, the value of the function at that point is equal to the average of the function's values over the boundary of the ball.

In mathematical terms, if $u(x)$ is a harmonic function in a ball $B(x_0, R)$ centered at x_0 with radius R, then:

$$u(x_0) = \left(\frac{1}{|B(x_0, r)|}\right) \int_{\{B(x_0, r)\}} u(x)\, dx \tag{4.2}$$

where $|B(x_0, r)|$ is the measure (area or volume, depending on the dimension) of the ball.

In other words, continuous function $u : G \to \mathbb{R}$ has the Mean Value Property (MVP) if whenever.

$$\bar{B}\ (a = x_0;\ R) \subset G, u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta \tag{4.3}$$

This property provides a useful tool for solving partial differential equations and finding potential functions in physics.

Proof:

Let $u: G \to \mathbb{R}$ be a harmonic function and let $\bar{B}\ (a : R)$ be a closed disk contained in G. If C is the circle, $|z - a| = R$ then then $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta$

The proof of the Mean Value Theorem is since a harmonic function is equal to its mean over any region. This means that the average value of the function over the boundary of a disk is equal to the value of the function at the center of the disk.

To prove this, we start by noting that a harmonic function is analytic, meaning it satisfies the Cauchy-Riemann equations and can be represented by a power series. Using this representation, we can write the function as:

$$u(z) = u(a) + \sum_{n=1}^{\infty}(z - a)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) \tag{4.4}$$

where $\frac{d^n u(a)}{dz^n}$ is the nth derivative of u evaluated at a.

Next, we consider the value of the function at a point on the boundary of the disk, given by $a + Re^{i\theta}$. Using this, we can rewrite the above power series as:

$$u\left(a + Re^{i\theta}\right) = u(a) + \sum_{n=1}^{\infty}\left(a + Re^{i\theta} - a\right)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) \tag{4.5}$$

$$\Rightarrow u\left(a + Re^{i\theta}\right) = u(a) + \sum_{n=1}^{\infty}\left(Re^{i\theta}\right)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) \tag{4.6}$$

Now, we integrate both sides over the interval $[0, 2\pi]$ to obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta = u(a) + \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=1}^{\infty}\left(Re^{i\theta}\right)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) d\theta \tag{4.7}$$

Since the function u is harmonic, it follows that all its derivatives are also harmonic.

This means that the second term on the right side is equal to zero. Therefore, we can simplify the above expression to:

$$\frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta = u(a) \tag{4.8}$$

Thus, the average value of the function over the boundary of the disk is equal to the value of the function at the center of the disk, proving the Mean Value Theorem.

4.2 Harnack's inequality

Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z : |z - z_0| < R\}$,with $u(z) \geq 0$

for all $z \in \Delta(z_0\ ;\ R)$, then for every z in this disk, we have

$$u(z_0)\frac{R - |z - z_0|}{R + |z - z_0|} \leq u(z) \leq u(z_0)\frac{R + |z - z_0|}{R - |z - z_0|}$$

Proof:

First, we consider the average value of the function u on the circle centered at z0 with radius $|z - z_0|$. By definition, this average value is given by

$A(|z - z_0|) = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} u\left(z_0 + |z - z_0| e^{it}\right) dt$. (4.9)

Next, we apply the Mean Value Property for Harmonic Functions to the function $u - u(z_0)$, which states that for any positive real number r such that $0 < r < R$, there exists a point θ in the interval $[0, 2\pi)$ such that

$u\left(z_0 + re^{i\theta}\right) - u(z_0) = r\,\partial_r u\left(z_0 + re^{i\theta}\right)$ (4.10)

Substituting this expression into the formula for $A(|z - z_0|)$ and interchanging the order of integration and differentiation, we obtain

$$\begin{aligned} A(|z - z_0|) &= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} u\left(z_0 + |z - z_0| e^{it}\right) dt \\ &= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \left[u(z_0) + |z - z_0| \partial_r\, u\left(z_0 + |z - z_0| e^{it}\right)\right] dt \\ &= u(z_0) + |z - z_0| \partial_r\, u(z_0). \end{aligned}$$

Finally, using the definition of partial derivative with respect to the radial coordinate, we have

$\partial_r\, u(z_0) = \left(\frac{1}{2}\right) \frac{u(z_0 + R) - u(z_0 - R)}{R} = \frac{A(R)}{R}$. (4.11)

Substituting this expression into the formula for $A(|z - z_0|)$, we obtain

$A(|z - z_0|) = u(z_0) + \frac{|z - z_0| A(R)}{R}$. (4.12)

Dividing both sides by $|z - z_0|$ and rearranging, we find that

$\frac{u(z_0)(R - |z - z_0|)}{R + |z - z_0|} \leq A(|z - z_0|) \leq \frac{u(z_0)(R + |z - z_0|)}{R - |z - z_0|}$. (4.13)

Since the average value of u on the circle centered at z_0 with radius $|z - z_0|$ provides an upper bound for the function u, we conclude that

$\frac{u(z_0)(R - |z - z_0|)}{R + |z - z_0|} \leq u(z) \leq \frac{u(z_0)(R + |z - z_0|)}{R - |z - z_0|}$ (4.14)

for every z in the disk $\Delta(z_0, R)$.

4.3 Dirichlet Problem

The Dirichlet problem in complex analysis is a boundary value problem that seeks to find a complex valued function that is analytic within a given domain and takes on specified boundary values on the boundary of that domain.

For example, consider the unit disk centered at the origin in the complex plane. The Dirichlet problem asks us to find a complex valued function $f(z)$ that is analytic within the unit disk and takes on the specified boundary value $f(e^{it}) = g(t)$ for all t in the interval $[0,2\pi]$, where $g(t)$ is a given function.

One possible solution to this problem is to use the theory of complex analysis and the representation of analytic functions using power series.

By using the Cauchy-Riemann equations, it can be shown that any complex valued function that is analytic within the unit disk can be represented by a power series of the form.

$f(z) = \Sigma\, a_n (z - z_0)^n$, where z_0 is the center of the disk.

The boundary values of the function can then be used to determine the coefficients of the power series.

For example, if $g(t) = \cos(t)$,

then $f\left(e^{it}\right) = \cos(t)$

$= \Sigma\, a_n\, e^{int}$,

where the coefficients can be calculated by matching the real and imaginary parts of both sides of the equation.

Initially, the problem was to determine the equilibrium temperature distribution on a disk from measurements taken along the boundary.

The temperature at points inside the disk must satisfy a partial differential equation called Laplace's equation corresponding to the physical condition that the total heat energy contained in the disk shall be a minimum.

A slight variation of this problem occurs when there are points inside the disk at which heat is added (sources) or removed (sinks) as long as the temperature remains constant at each point (stationary flow), in which case Poisson's equation is satisfied.

How to construct a harmonic function in a given domain when its values are prescribed on the boundary of the domain is the key problem is known as Dirichlet problem.

Boundary value problems associated to Laplace equation.

The Poisson equation is a second-order partial differential equation of the form

$$\nabla^2 u(x) = f(x)$$

where $u(x)$ is an unknown function and $f(x)$ is a given function. The equation states that the Laplacian of $u(x)$ is equal to $f(x)$.

The solution to the Poisson equation depends on the boundary conditions for the unknown function $u(x)$.

There are several methods for solving the Poisson equation, including numerical methods, analytical methods, and Green's function methods.

One common analytical method is to use the method of separation of variables.

Suppose that $u(x) = X(x)Y(y)$, then the Laplacian of $u(x)$ becomes.

$$\nabla^2 u(x) = \frac{\partial^2 u(x)}{\partial x^2} + \frac{\partial^2 u(x)}{\partial y^2}$$

$$= X''(x)Y(y) + X(x)Y''(y)$$

where $X''(x)$ *and* $Y''(y)$ denote the second derivatives with respect to x and y, respectively. Setting the right-hand side equal to $f(x)$, we have

$$X''(x)Y(y) + X(x)Y''(y) = f(x)$$

Dividing both sides by XY, we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{f(x)}{XY}$$

This equation is equal to a constant λ, so we have

$$\frac{X''(x)}{X(x)} = -\lambda \frac{Y''(y)}{Y(y)} = -\lambda + \frac{f(x)}{XY}$$

Solving the above two differential equations, we obtain the general solution.

$$u(x) = \sum C_n X_n(x) Y_n(y)$$

where C_n are constants and $X_n(x)$ *and* $Y_n(y)$ are the eigenfunctions corresponding to the eigenvalue λ_n

The final solution depends on the specific boundary conditions and the values of the constants C_n.

Note that this method is only applicable when the equation can be separated into two independent ordinary differential equations. In general, the Poisson equation requires numerical methods or Green's function methods to solve.

Another of the generic partial differential equations is Laplace's equation, $\nabla^2 u = 0$

This equation first appeared in the unit on complex variables when we discussed harmonic functions. Another example is the electric potential for electrostatics. As we described for static electromagnetic fields,

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, E = \nabla\phi.$$

In regions devoid of charge, these equations yield the Laplace equation $\nabla^2\phi = 0$.

Another example comes from studying temperature distributions.

Consider a thin rectangular plate with the boundaries set at fixed temperatures. Temperature changes of the plate are governed by the heat equation. The solution of the heat equation subject to these boundary conditions is time dependent.

In fact, after a long period of time the plate will reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature.

Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, which is another Laplace equation, $\nabla^2 u = 0$

Example

Equilibrium temperature distribution for a rectangular plate

Let us consider Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < H \tag{4.15}$$

with the boundary conditions

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(\pi,y) &= 0 \\ u(x,0) = f(x) &= Sinx \\ u(x,H=1) &= \frac{Sinx}{e} \end{aligned}\right\} \tag{4.16}$$

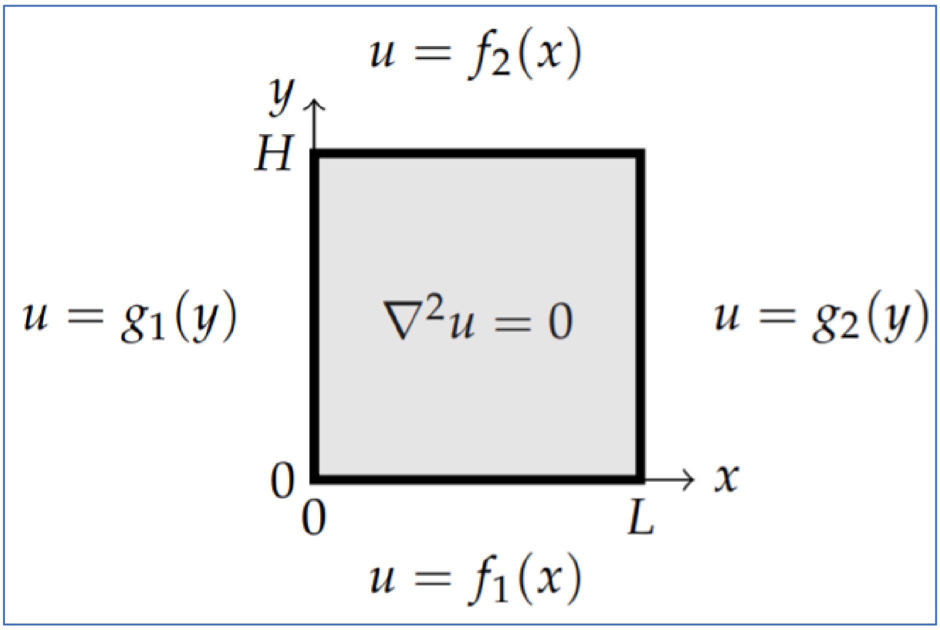

Figure 4.1: The boundary condition for the heat distribution problem.

Solution:

This is a partial differential equation for Laplace's equation, which describes the distribution of heat in a 2D space. To solve this equation, we can use separation of variables method.

Assume that the solution can be written as:

$$u(x,y) \;=\; X(x)\,Y(y)$$

Substituting this into the equation, we have:

$$(X(x)\,Y(y))'' \;+\; (X(x)\,Y(y))'' \;=\; 0$$

Dividing both sides by $X(x)\,Y(y)$, we get:

$$X''(x)\,Y(y) \;+\; X(x)\,Y''(y) \;=\; 0$$

Since this must be true for all x and y, we can divide both sides by $X(x)\,Y(y)$, to get:

$$\left(\frac{X''(x)}{X(x)}\right) + \left(\frac{Y''(y)}{Y(y)}\right) = \;0$$

This can be simplified to:

$$\lambda^2 \;=\; -\frac{X''(x)}{X(x)} \;=\frac{Y''(y)}{Y(y)}$$

where λ^2 is a constant.

Solving for $X(x)$ *and* $Y(y)$, we have:

$$X''(x) \;+\; \lambda^2\,X(x) \;=\; 0$$

$$Y''(y) \;-\; \lambda^2\,Y(y) \;=\; 0$$

The solutions for $X(x)$ *and* $Y(y)$ can be written as:

$X(x) \;=\; A\,cos(\lambda\,x) \;+\; B\,sin(\lambda\,x)$

$Y(y) \;=\; C\,e^{-\lambda\,y} \;+\; D\,e^{\lambda\,y}$

where $A, B, C,$ *and* D are constants.

Hence $u(x,y) = \;A\,cos(\lambda\,x) + \;B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} \;+\; D\,e^{\lambda\,y}]$ (4.17)

Using the boundary conditions, we can find the values of λ and the coefficients $A, B, C,$ *and* D.

$$\Rightarrow u(0,y) = \;A\,cos(\lambda * 0) + \;B\,sin(\lambda * 0).\left[C\,e^{-\lambda\,y} \;+\; D\,e^{\lambda\,y}\right] = 0$$

$$\Rightarrow u(0,y) = \;A\,.\left[C\,e^{-\lambda\,y} \;+\; D\,e^{\lambda\,y}\right] = 0$$

$$\Rightarrow A = 0$$

Now put $A = 0$ in (4.17)

$$u(x,y) = \;\;B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} \;+\; D\,e^{\lambda\,y}] \qquad (4.18)$$

Now

$$u(\pi,y) = B\,sin(\lambda\,\pi).\left[C\,e^{-\lambda\,y} \;+\; D\,e^{\lambda\,y}\right] = 0$$

$$\Rightarrow sin(\lambda\,\pi) = 0$$

$$\Rightarrow sin(\lambda\,\pi) = \sin(n\pi)$$

$\Rightarrow \lambda \;= \mathrm{n}, \mathrm{n} = \pm 1, \pm 2, \ldots,$

So

$$u(x,y) = \;\;B\,sin(n\,x).[C\,e^{-n\,y} \;+\; D\,e^{n\,y}] \qquad (4.19)$$

Now apply $u(x,0) \;=\; Sinx$

$$u(x,0) = \ B\ sin(n\,x).\,[C\ +\ D\,] = sinx$$

$$\Longrightarrow BC + BD = 1, n = 1$$

Now update the (4.19) Hence

$$u(x,y) = \ B\ sinx.\,[C\ e^{-y}\ +\ D\ e^{y}] \tag{4.20}$$

Now apply $u(x,1) = \frac{Sinx}{e}$

$$u(x,1) = \ B\ sinx.\,[C\ e^{-1}\ +\ D\ e^{1}] = \frac{sinx}{e}$$

$$\Longrightarrow BC\frac{sinx}{e} + BD.sinx.e = \frac{sinx}{e} + 0$$

$$\Longrightarrow BC = 1, BD = 0$$

Now update the (4.20)(4.19) Hence

$u(x,y) = \ \ sinx.\,[\ e^{-y}\ ]$ is the final solution of the given temperature distribution for a rectangular plate.

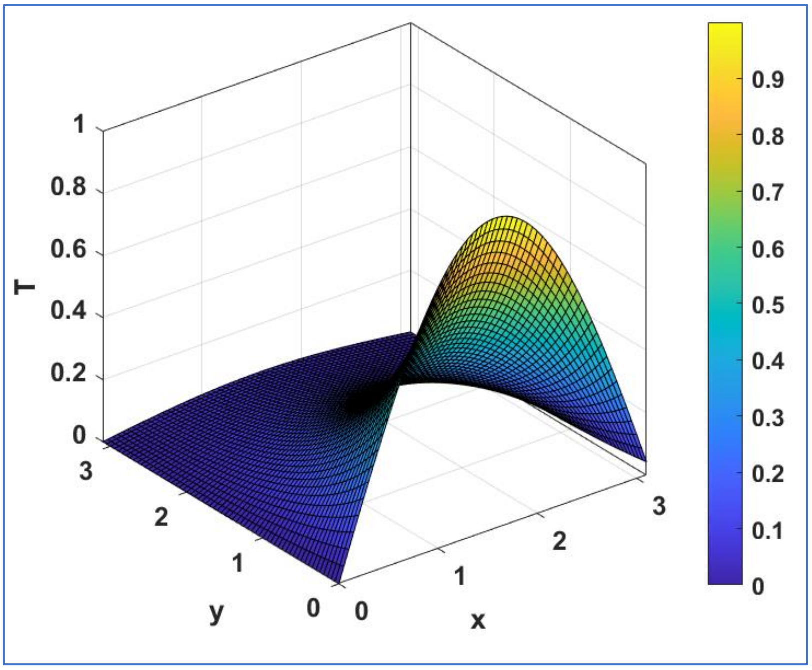

Figure4.2: The temperature distribution in x and y direction.

4.4 <u>Review questions</u>

1. Explain the Mean value property of harmonic function?
2. State and prove the Harnack's inequality for harmonic function in the closed disc?
3. Suppose $u(z\,)$ is harmonic in the disk $\Delta(1\ ;\ 1) = \{z :\ |z\ -\ 1\,| < 1\}$,with $u(z\,)\ \geq\ 0$ for all $z\ \in\ \Delta(1\ ;\ R)$, then for every z in this disk, then show that

$$u(1\,)\frac{1 - |z\ -\ 1\,|}{1 + |z\ -\ 1\,|}\ \leq\ u(z\,)\ \leq\ u(1\,)\frac{1 + |z\ -\ 1\,|}{1 - |z\ -\ 1\,|}$$

4. Solve $\frac{\partial^2 u}{\partial x^2}\ + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < H$,under the boundary conditions

$$\left.\begin{array}{r} u(0,y) = 0 \\ u(\pi,y) = 0 \\ u(x,0) = f(x) = 2Sinx \\ u(x,1) = \frac{Sinx}{e} \end{array}\right\}$$

5. Solve, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < 10, 0 < y < H$, with the boundary conditions

$$\left.\begin{array}{r} u(0,y) = 0 \\ u(\pi,y) = 0 \\ u(x,0) = 10Sinx \\ u(x,10) = 10\frac{Sinx}{e} \end{array}\right\}$$

4.5 Self-assessment

1. Using the Mean Value property, the integral $\frac{1}{2\pi}\int_0^{2\pi} u\left(5 + Re^{i\theta}\right)d\theta =$

   A. $u(5)$
   B. $u\left(\frac{5}{2}\right)$
   C. $u(25)$
   D. $u(10)$

2. Using the Mean Value property, the integral $\frac{1}{2\pi}\int_0^{2\pi} u\left(15 + Re^{i\theta}\right)d\theta =$

   A. $u(15)$
   B. $u\left(\frac{15}{2}\right)$
   C. $u(225)$
   D. $u(30)$

3. Using the Mean Value property, the integral $\frac{1}{2\pi}\int_0^{2\pi} u\left(1 + i + Re^{i\theta}\right)d\theta =$

   A. $u(1)$
   B. $u(i)$
   C. $u(1+i)$
   D. $u\left(\frac{1}{1+i}\right)$

4. Using the Mean Value property, which one of the following is true

   A. $\frac{1}{2\pi}\int_0^{2\pi} u\left(10 + Re^{i\theta}\right)d\theta = u(10)$
   B. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5 + Re^{i\theta}\right)d\theta = u(5/2)$
   C. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5/2 + Re^{i\theta}\right)d\theta = u(5)$
   D. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5 + Re^{i\theta}\right)d\theta = u(10)$

5. Using the Mean Value property, which one of the following is true

   A. $\frac{1}{2\pi}\int_0^{2\pi} u\left(1 + Re^{i\theta}\right)d\theta = u(10)$
   B. $\frac{1}{2\pi}\int_0^{2\pi} u\left(25 + Re^{i\theta}\right)d\theta = u(5/2)$
   C. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5/2 + Re^{i\theta}\right)d\theta = u(5)$
   D. $\frac{1}{2\pi}\int_0^{2\pi} u\left(2 + i + Re^{i\theta}\right)d\theta = u(2+i)$

6. Suppose $u(z\ )$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z:\ |z\ \ | < 1\}$,with $u(z\ )\ \geq\ 0$ for all $z\ \in\ \Delta(0\ ;\ 1)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0\ )\frac{1-|z\ \ |}{1+|z\ \ |} \leq\ u(z\ )$$

$$II: u(z\ )\ \leq\ u(0\ )\frac{1+|z\ \ |}{1-|z\ \ |}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

7. Suppose $u(z\ )$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z:\ |z\ \ | < 2\}$,with $u(z\ )\ \geq\ 0$ for all $z\ \in\ \Delta(0\ ;\ 2)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0\ )\frac{2-|z\ \ |}{2+|z\ \ |} \leq\ u(z\ )$$

$$II: u(z\ ) \geq\ u(0\ )\frac{2+|z\ \ |}{2-|z\ \ |}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

8. Suppose $u(z\ )$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z:\ |z-i\ | < 1\}$,with $u(z\ )\ \geq\ 0$ for all $z\ \in\ \Delta(i\ ;\ 1)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(i\ )\frac{1-|z\ \ |}{1+|z\ \ |} \leq\ u(z\ )$$

$$II: u(z\ )\ \leq\ u(i\ )\frac{1+|z\ \ |}{1-|z\ \ |}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

9. Suppose $u(z\ )$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z:\ |z\ \ | < 5\}$,with $u(z\ )\ \geq\ 0$ for all $z\ \in\ \Delta(0\ ;\ 5)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0\ )\frac{5-|z\ \ |}{5+|z\ \ |} >\ u(z\ )$$

$$II: u(z\ )\ \leq\ u(0\ )\frac{5+|z\ \ |}{5-|z\ \ |}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

10. Suppose $u(z\ )$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z :\ |z - 2\ \ | < 1\}$,with $u(z\ )\ \geq\ 0$ for all $z\ \in\ \Delta(0\ ;\ 1)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0\ )\frac{1 - |z - 2\ \ |}{1 + |z - 2\ \ |} \leq\ u(z\ )$$

$$II: u(z\ )\ \leq\ u(0\ )\frac{1 + |z - 2\ \ |}{1 - |z - 2\ |}$$

   A. $Only\ I$
   B. $Only\ II$
   C. $Both$
   D. $Neither\ I\ nor\ II$

11. Consider the $u(x, y) = A\ cos(\lambda\ x) + B\ sin(\lambda\ x).[C\ e^{-\lambda y} + D\ e^{\lambda y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of B using $u(0, y) = 0$.
   A. 0
   B. 2π
   C. 3
   D. 1

with the boundary conditions $\left.\begin{array}{r} u(0, y) = 0. \\ u(\pi, y) = 0 \\ u(x, 0) = f(x) = Sinx \\ u(x, H = 1) = \frac{Sinx}{e} \end{array}\right\}$

12. Consider the $u(x, y) = A\ cos(\lambda\ x) + B\ sin(\lambda\ x).[C\ e^{-\lambda y} + D\ e^{\lambda y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of λ using $u(0, y) = 0, , u(\pi, y) = 0$.
   A. $\lambda = 0.5$
   B. $\lambda\ = \text{n}, \text{n} = \pm 1, \pm 2, \ldots,$
   C. $\lambda\ = \frac{3}{2}$
   D. $Can\ not\ be\ determined$

13. Consider the $u(x, y) = A\ cos(\lambda\ x) + B\ sin(\lambda\ x).[C\ e^{-\lambda y} + D\ e^{\lambda y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of BC with the boundary conditions $\left.\begin{array}{r} u(0, y) = 0. \\ u(\pi, y) = 0 \\ u(x, 0) = f(x) = Sinx \\ u(x, H = 1) = \frac{Sinx}{e} \end{array}\right\}$
   A. 0.5
   B. 1
   C. $\frac{3}{2}$
   D. $Can\ not\ be\ determined$
   E.

14. Consider the $u(x, y) = A\ cos(\lambda\ x) + B\ sin(\lambda\ x).[C\ e^{-\lambda y} + D\ e^{\lambda y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of BD with the boundary conditions $\left.\begin{array}{r} u(0, y) = 0. \\ u(\pi, y) = 0 \\ u(x, 0) = f(x) = Sinx \\ u(x, H = 1) = \frac{Sinx}{e} \end{array}\right\}$

A. 0.5
B. 1
C. $\frac{3}{2}$
D. 0

15. Consider the $u(x,y) = A\cos(\lambda x) + B\sin(\lambda x).[C e^{-\lambda y} + D e^{\lambda y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the general solution of with the boundary conditions $\left.\begin{array}{r} u(0,y) = 0. \\ u(\pi,y) = 0 \\ u(x,0) = f(x) = Sinx \\ u(x,H=1) = \frac{Sinx}{e} \end{array}\right\}$

A. $u(x,y) = \ Sinx.[\,e^{-y}\,]$
B. $u(x,y) = \ Cosx.[\,e^{-y}\,]$
C. $u(x,y) = \ Sinx.[\,e^{y}\,]$
D. $u(x,y) = \ Cosx.[\,e^{y}\,]$

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	A
3	C
4	A
5	D
6	C
7	A
8	C
9	B
10	**C**
11	**A**
12	B
13	C
14	D
15	A

4.6 Summary

- **The mean value property of harmonic functions**:

For any point in a ball in a harmonic function, the value of the function at that point is equal to the average of the function's values over the boundary of the ball.

In mathematical terms, if $u(x)$ is a harmonic function in a ball $B(x_0, R)$ centered at x_0 with radius R, then: $u(x_0) = \left(\frac{1}{|B(x_0,r)|}\right) \int_{\{B(x_0,r)\}} u(x)\, dx$

where $|B(x_0, r)|$ is the measure (area or volume, depending on the dimension) of the ball.

- The average value of the function over the boundary of the disk is equal to the value of the function at the center of the disk, proving the Mean Value Theorem.

4.7 Keywords

Harnack's inequality

Suppose $u(z\ )$ is harmonic in the disk $\Delta(z_0\ ;\ R) = \{z : |z - z_0\ | < R\}$,with $u(z\ ) \geq 0$

for all $z \in \Delta(z_0\ ;\ R)$, then for every z in this disk, we have

$$u(z_0\ )\frac{R - |z - z_0\ |}{R + |z - z_0\ |} \leq u(z\ ) \leq u(z_0\ )\frac{R + |z - z_0\ |}{R - |z - z_0\ |}$$

4.8 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

Table of Contents

Purpose and Objectives:

After this unit students can be able to-

1. Understand the Schwarz Reflection Principle for analytic functions?
2. Prove the Schwarz Reflection Principle for analytic functions?
3. Learn the consequences of the Schwarz Reflection Principle

Introduction

If a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane. In other words, if a function f(z) is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

The Schwarz Reflection Principle has several important applications in complex analysis, such as proving the analyticity of functions, constructing entire functions with prescribed properties, and solving boundary value problems. In this unit we will explore the Schwarz Reflection Principle for analytic function.

5.1 Schwarz Reflection Principle for Analytic Functions

Statement:

The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.

or

The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane. In other words, if a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re(f(z)) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

Proof 1:

The proof of the Schwarz Reflection Principle relies on the fact that if a function is analytic in the upper half plane and its real part is non-negative on the boundary, then it can be represented as a real part of another analytic function.

Let $f(z)$ be analytic in the upper half plane and $Re(f(z)) \geq 0$ for all real z.

Then the function $g(z) = f(z) + i(-f(z))$ is analytic in the upper half plane and satisfies $Re(g(z)) = 0$ for all real z.

The proof also uses the maximum modulus principle and Liouville's theorem.

Liouville's Theorem

Liouville's Theorem states that a bounded holomorphic function on the entire complex plane must be constant. It is named after Joseph Liouville.

Statement:

Let $f(z): C \rightarrow C$ be an entire function. Suppose there exists some real number$M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

Or

If a function $f(z)$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof:

It is given that

i. A function $f(z)$ is analytic in the entire complex plane
ii. A function $f(z)$ is bounded, that $|f(z)| \leq M$.

Let us consider two points $a\ and\ b$ inside a particular domain(See the figure 5.1).

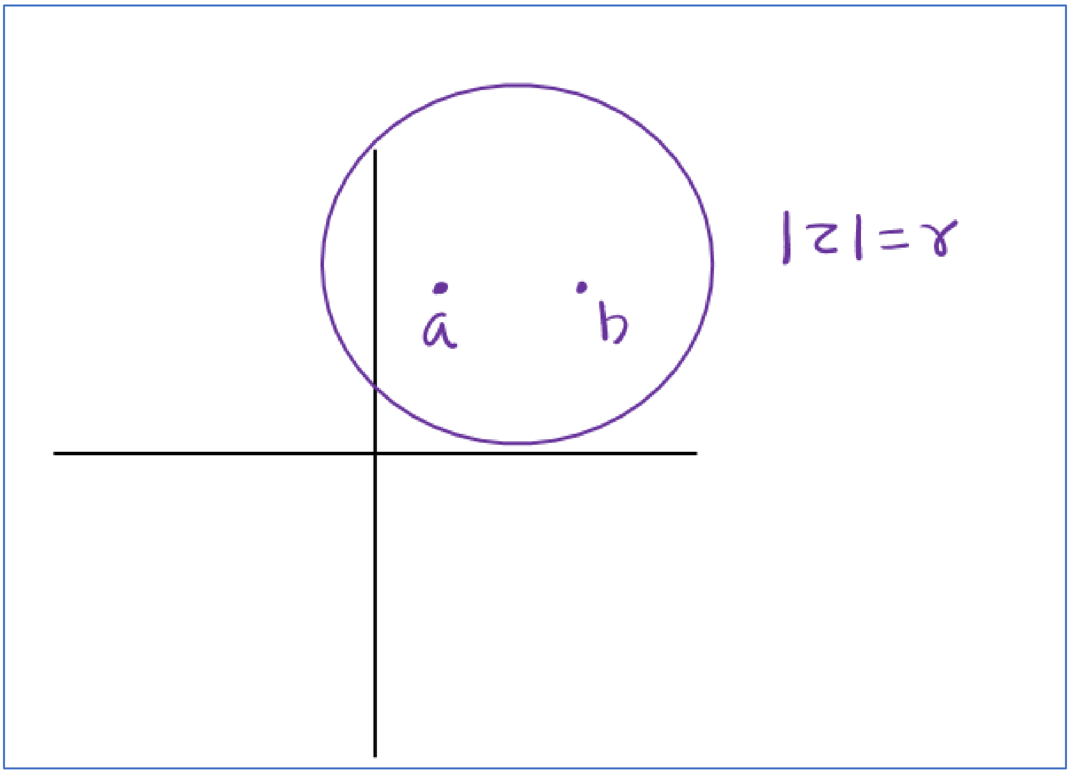

Figure 5.1: Two points a and b inside a particular domain

Then using Cauchy integral formula

$$\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-a}\right)dz = f(a)$$

$$\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-b}\right)dz = f(b)$$

If $f(z)$ is constant throughout the domain, then $f(a) = f(b)$.

Now let's prove $f(a) - f(b) = 0$.

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-a}\right)dz - \frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-b}\right)dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-a} - \frac{f(z)}{z-b}\right)dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c f(z)\left(\frac{1}{z-a} - \frac{1}{z-b}\right)dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c f(z)\left(\frac{z-b-z+a}{(z-a)(z-b)}\right)dz$$

$$f(a) - f(b) = \frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz$$

$$|f(a) - f(b)| = \left|\frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz\right|$$

$$|f(a) - f(b)| \le \left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{|(z-a)(z-b)|}\right)|dz|$$

$$|f(a) - f(b)| \le \left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{(|z|-|a|)(|z|-|b|)}\right)|dz|$$

Let

$z = re^{i\theta}$

$dz = re^{i\theta}.i.d\theta$

$|dz| = |re^{i\theta}.i.d\theta|$

$|dz| = |r|.|e^{i\theta}|.|i|.|d\theta|$

Here $|r| = r$

$|e^{i\theta}| = |cos\theta + isin\theta| = 1,$

$|i| = 1,$

$|dz| = r.|d\theta|$

$$|f(a) - f(b)| \le \frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(r-a)(r-b)}\right) r.|d\theta|$$

$$|f(a) - f(b)| \le \frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(1-a/r)(1-b/r)}\right).|d\theta|$$

If $f(z)$ is analytic in the entire complex plane, then $|z| = r \to \infty$. So

$|f(a) - f(b)| \le 0$

$f(a) - f(b) = 0$

Hence, we can say that $f(a) = f(b)$. It means that $f(z)$ is a constant.

Liouville's Theorem proof using Cauchy integral formula for derivatives.

If f(z) is analytic in a simply connected region then at any interior point of the region, z_0 inside *C*. Then say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$$\oint_c \left(\frac{f(z)}{(z-z_0)^{n+1}}\right) dz = 2\pi i \frac{f^n(z_0)}{n!}.$$

where C is any simple closed curve, in the region, which encloses z_0. Note the case $n = 1$:

$$\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{(z-z_0)^2}\right) dz = f'(z_0).$$

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{(z-z_0)^2}\right) dz\right|.$$

$$|f'(z_0)| \le \left|\frac{1}{2\pi i}\right| \oint_c \left(\frac{|f(z)|}{|(z-z_0)^2|}\right) |dz|.$$

Here $z = re^{i\theta}$

$dz = re^{i\theta}.i.d\theta.$

$|dz| = |re^{i\theta}.i.d\theta|.$

$|dz| = |r|.|e^{i\theta}|.|i|.|d\theta|.$

Here $|z - z_0| = r$

$|e^{i\theta}| = |cos\theta + isin\theta| = 1,$

$|i| = 1,$

$|dz| = r.|d\theta|.$

$$|f'(z_0)| \le \frac{1}{2\pi}\oint_c \left(\frac{M}{r^2}\right) r.|d\theta|.$$

$$|f'(z_0)| \le \frac{1}{2\pi}\oint_c \left(\frac{M}{r}\right).d\theta.$$

If $f(z)$ is analytic in the entire complex plane then $r \to \infty$. So

$|f'(z_0)| \le 0$

$f'(z_0) = 0$

$f(z) = \text{constant}.$

By Liouville's theorem, the imaginary part of $g(z)$ is constant on the boundary, say c. Then the function $h(z) = g(z) + ic$ is analytic in the entire plane and has the same real part as $f(z)$.

Proof 2:

Let $f(z)$ be a complex valued function that is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$).

Consider a point z in the lower half plane ($Im(z) < 0$)[See figure 5.2]

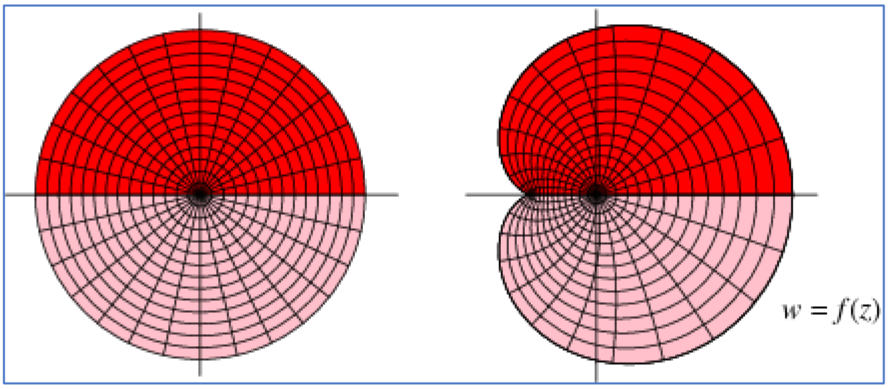

Figure 5.2: $w = f(z)$

Let $z = x + iy$, where x is real and y is negative.

Let's define a new point, $conj(z)$=$\bar{z}$, which is equal to the complex conjugate of z.

That is, $conj(z) = x - iy$.

Since $f(z)$ is continuous on the boundary of the upper half plane, it follows that $f(conj(z))$ is continuous in the lower half plane.

Also, since $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, we have:

$f(z) = \overline{f((conj(z)))}$

$= \overline{f(x - iy)}$

$= \overline{f(x + i(-y))}$

Thus, we can define a new function, $g(z)$, in the lower half plane as follows:

$g(z) = \overline{f(x + i(-y))}$

Since $g(z) = \overline{f(z)}$ is continuous in the lower half plane, and the conjugate of a continuous function is continuous, it follows that $g(z)$ is continuous in the lower half plane.

We now show that $g(z)$ is also analytic in the lower half plane.

Let $z = x + iy$, where x is real and y is negative.

Consider the derivative of $g(z)$ at the point z:

$g'(z) = \left(\frac{d}{dz}\right) \overline{f(x + i(-y))}$

$= \left(\frac{d}{dz}\right) \overline{f(z)}$

$= \overline{\left(\frac{d}{dz}\right) f(z)}$

Since $f(z)$ is analytic in the upper half plane, it follows that $\left(\frac{d}{dz}\right) f(z)$ is also analytic in the upper half plane.

Therefore, conjugate of $\left(\frac{d}{dz}\right) f(z)$ is also analytic in the lower half plane, and so is $g'(z)$.

Since $g(z)$ is continuous and its derivative is analytic in the lower half plane, it follows that $g(z)$ is analytic in the lower half plane.

Thus, we have shown that if $f(z)$ is a complex valued function that is analytic in the upper half plane and continuous on the boundary of the upper half plane, and if $f(z)$ satisfies:

$f(z) = \overline{f(conj(z))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane by defining a new function, $g(z)$, in the lower half plane.

5.2 consequences of the Schwarz Reflection Principle

1. One consequence of the Schwarz Reflection Principle is that if a function is analytic in the upper half plane and its real part is non-negative on the boundary, then it can be extended to an entire function that is real valued on the real axis.
2. Another consequence is that a function that is analytic in the upper half plane and satisfies a certain growth condition on the boundary (such as the Riemann mapping theorem) can be extended to an entire function with similar growth behavior.
3. Additionally, the Schwarz Reflection Principle can be used to construct solutions to boundary value problems, such as the Dirichlet problem, by reflecting solutions from one half plane to the other.

5.3 Different proofs of Schwartz Reflection Principle

The Schwartz Reflection Principle can be proved by various methods

1. Complex Analysis Proof: The Schwartz Reflection Principle can be proven using complex analysis by considering the analytic continuation of the function from the upper half plane to the lower half plane. The proof involves showing that the function, extended to the lower half plane, is a reflection of the function in the upper half plane across the real axis.

2. Harmonic Functions Proof: The Schwartz Reflection Principle can also be proven using the theory of harmonic functions. A function is considered harmonic if it satisfies Laplace's equation. By assuming that the function is harmonic in the upper half plane, it can be shown that its extension to the lower half plane is also harmonic, and therefore satisfies Laplace's equation, meaning it must be a reflection of the function in the upper half plane across the real axis.

3. Integral Transform Proof: The Schwartz Reflection Principle can be proven using the Fourier Transform by showing that the Fourier Transform of a function in the upper half plane, after being reflected across the real axis, is equal to the negative Fourier Transform of the original function in the lower half plane.

4. Paley-Wiener Theorem Proof: The Schwartz Reflection Principle can also be proven using the Paley-Wiener theorem, which states that the Fourier Transform of a function with compact support is a function that is entire and decays rapidly. By assuming that the function in question is the Fourier Transform of a function with compact support in the upper half plane, it can be shown that the function, after being reflected across the real axis, is the Fourier Transform of a function with compact support in the lower half plane.

5. Bochner's Theorem Proof: The Schwartz Reflection Principle can also be proven using Bochner's theorem, which states that a positive definite function is the Fourier Transform of a positive measure. By assuming that the function in question is positive definite in the upper half plane, it can be shown that the function, after being reflected across the real axis, is positive definite in the lower half plane, implying that it is the Fourier Transform of a positive measure.

5.4 Applications

- The main application of the Schwartz Reflection Principle is in the study of distributions and their derivatives. It provides a means to extend the definitions of distributions and derivatives to unbounded functions.
- The Schwartz Reflection Principle is a generalization of the Hahn-Banach Theorem. The Hahn-Banach Theorem states that a linear functional on a linear subspace can be extended to the entire space while preserving its norm. The Schwartz Reflection Principle extends this result to the case of distributions.
- The Schwartz Reflection Principle is an important tool in mathematical physics for defining distributions and derivatives of functions. In particular, it allows for the extension of the definitions of distributions and derivatives to unbounded functions, which is particularly useful in quantum field theory and quantum mechanics.

5.5 Review questions

1. What is the main application of the Schwartz Reflection Principle?
2. How does the Schwartz Reflection Principle relate to the Hahn-Banach Theorem?
3. What is the significance of the Schwartz Reflection Principle in mathematical physics?
4. State and prove the Schwartz Reflection Principle using Liouville's Theorem?
5. State and prove the Schwartz Reflection Principle without Liouville's Theorem ?

5.6 Self-assessment

1. Which one the following statement is true for the reference of Schwarz reflection principle?

   I: The principle that a complex function which is holomorphic in the upper half plane can be extended to a holomorphic function in the whole plane.

   II: The principle that states that a real-valued function cannot be analytically extended across a branch cut.

   A. Only I
   B. Only II
   C. Both I and II
   D. Neither I nor II

2. Which one the following statement is true for the reference of Schwarz reflection principle?

I: The principle that a holomorphic function in the unit disc can be extended to a holomorphic function in the whole plane.

II: The principle that states that the maximum value of a subharmonic function is achieved on the boundary of its domain.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

3. Which one the following statement is true for the necessary condition for the Schwarz reflection principle?

I: The function must be bounded in the upper half plane.

II: The function must be continuous in the upper half plane

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

4. Which one the following statement is true for the necessary condition for the Schwarz reflection principle?

I: The function must be holomorphic in the upper half plane.

II: The function must be continuous in the upper half plane

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

5. The Schwarz reflection principle can be used to extend which type of functions to the whole plane?
A. Only real-valued functions
B. Only harmonic functions
C. Both real valued and harmonic
D. Neither real nor harmonic

6. The Schwarz reflection principle can be used to extend which type of functions to the whole plane?
A. Holomorphic functions
B. Harmonic functions
C. Subharmonic functions
D. Neither real nor harmonic

7. The Schwarz reflection principle states that the Fourier transform of the product of two signals is equal to the convolution of their Fourier transforms?
A. True

B. False

8. The principle that states that the reflection of a Schwartz function across the x-axis is also a Schwartz function?
   A. True
   B. False

9. The principle that states that the Laplace transform of a signal is equivalent to its Fourier transform?
   A. True
   B. False

10. The principle that states that the derivative of a Schwartz function is also a Schwartz function.
    A. True
    B. False

11. What is the Schwartz Reflection Principle in mathematics?

    A. The principle that every polynomial function has a unique root
    B. The principle that states that the boundary values of an analytic function on the upper half-plane can be extended to an analytic function on the whole complex plane
    C. The principle that states that the roots of a polynomial equation occur in conjugate pairs.
    D. The principle that the value of a holomorphic function at a point is equal to its average value over any small circle centered at that point.

12. Which of the following is the best definition of the Schwartz Reflection Principle?

    I: A mathematical theorem that states that the reflection of a function across a vertical line always results in an even function.
    II: A mathematical theorem that states that the reflection of a function across a vertical line always results in an odd function.
    A. Only I
    B. Only II
    C. Both I and II
    D. Neither I nor II

13. Which of the following is the best definition of the Schwartz Reflection Principle?
    I: A mathematical theorem that states that the reflection of a function across a vertical line always results in a function with the same parity.

    II: A mathematical theorem that states that the reflection of a function across a vertical line always results in a different function.
    A. Only I
    B. Only II
    C. Both I and II
    D. Neither I nor II

14. The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.
    A. True
    B. False

15. Which one the following statement is true for the reference of Schwarz reflection principle?

    I: The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane.

    II: If a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re(f(z)) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

    A. Only I
    B. Only II
    C. Both I and II
    D. Neither I nor II

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	D
3	D
4	A
5	D
6	A
7	B
8	A
9	B
10	**B**
11	**B**
12	D
13	A

14	D
15	C

5.7 Summary

- The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane $(Im(z) > 0)$, and continuous on the boundary of the upper half plane $(Im(z) = 0)$, and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.

- The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane.

- If a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

- If a function $f(z)$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

5.8 Keywords

Liouville's Theorem

Let $f(z): C \to C$ be an entire function. Suppose there exists some real number $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

5.9 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

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Purpose and Objectives:

Meromorphic functions are an important class of functions studied in complex analysis. They are defined as functions that are holomorphic (analytic) everywhere except at a finite number of isolated singularities. Meromorphic functions are useful in studying the behavior of complex functions near singularities, and they provide a representation of any meromorphic function in terms of its poles and their residues. After this unit students can be able to-

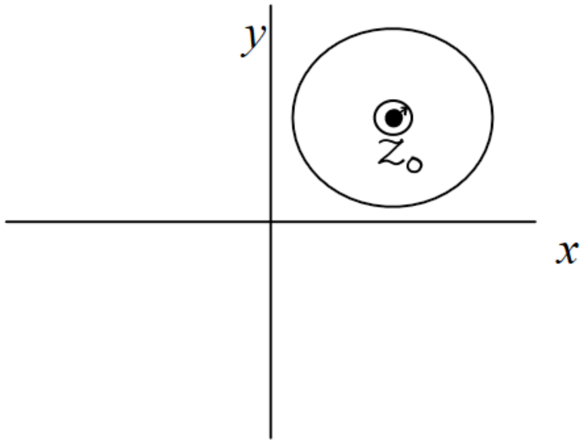
1. Understand the Meromorphic functions
2. State and prove the Mittag-Leffler theorem
3. Learn the infinite product of complex Numbers

Introduction

In this unit first we will understand the concept of singularities and poles for meromorphic function then the we will use the mesomorphic function to prove the Mittag-Leffler theorem. Last we will focus on the infinite product of complex Numbers.

6.1 Singularities

A point z_0 is called a singular point of a function $f(z)$ if $f(z)$ fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.



Example:

Behavior of following functions at $z = 0$.

$$f(z) = \frac{1}{z^9}$$

$$f(z) = \frac{Sinz}{z}$$

$$f(z) = \frac{e^z - 1}{z}$$

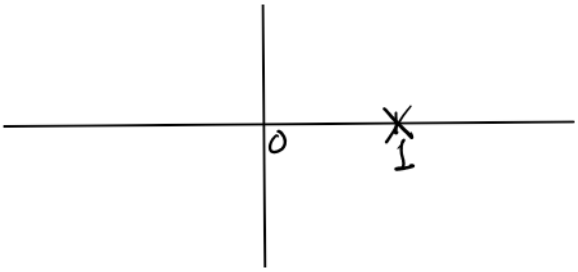
$$f(z) = \frac{1}{\sin(1/z)}$$

We observed that all the functions mentioned above are not analytic at $z = 0$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic.

Example:

Behavior of following function at $z = 1$.

$$f(z) = \frac{z}{1 - z}$$



We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

Example:

$f(z) = z^2$ is analytic everywhere so it has no singular point.

Example:

Behavior of following function in the entire z plane

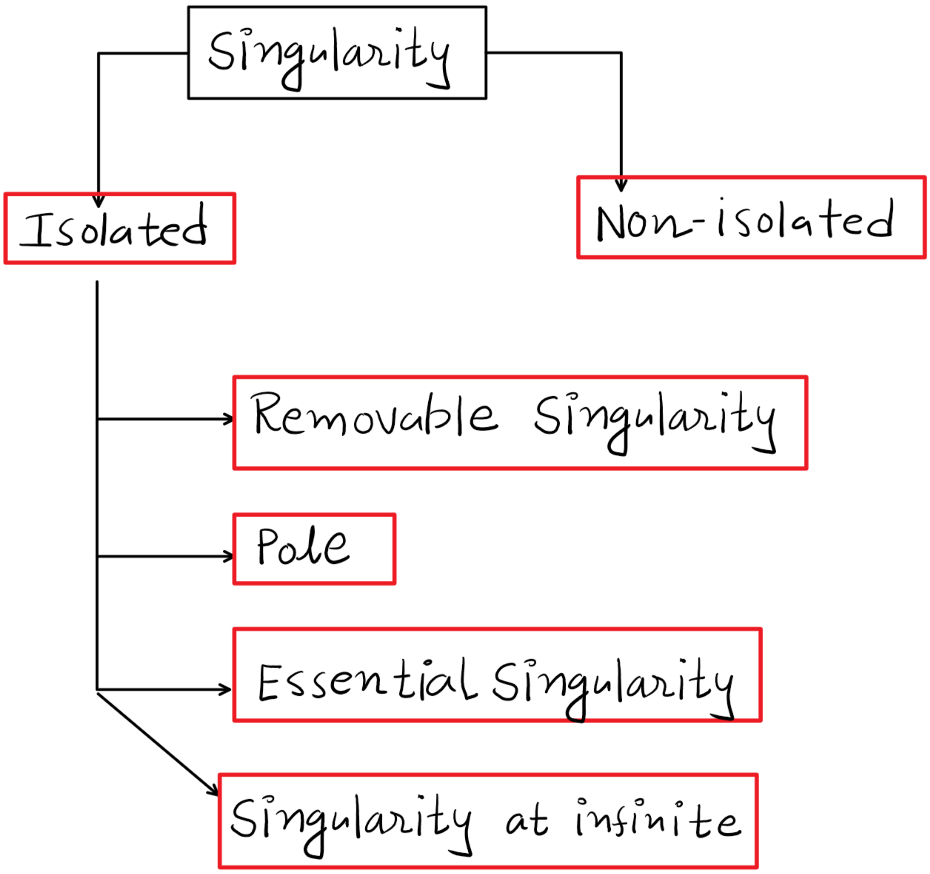
$f(z) = |z|^2$

We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

6.2 Classification of singularity

The singularity of a complex function can be classified into two groups, isolated and non-isolated. It can be done via Laurent series expension, but we can also classify the singularity without the Laurent series expension. In the forthcoming units we will consider the classification using the Laurent series.

The isolated singularity further can be classified into different type. The following diagram shows the different types of the singularities.



6.2.1 Isolated singularity

A point a is called an isolated singularity for $f(z)$ if $f(z)$ is not analytic at $z = a$ and there exist $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$. The neighbourhood $|z - a| < r$ contains no singularity of $f(z)$ except a.

Example:

$\boldsymbol{f(z) = \frac{z+1}{z^2(z^2+1)}}$ ***has three isolated singularities*** $\boldsymbol{z = 0, i, -i.}$

Example:

$f(z) = \frac{1}{sinz}$ has three isolated singularities $z = 0, \pm\pi, \pm 2\pi, \ldots,$

6.2.2 Removable singularity

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim\limits_{z\to z_0} f(z)$ exists then z_0 is the removable singularity.

Example:

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to 0} \frac{sinz}{z} = 1.$

$\lim\limits_{z\to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

Example:

Let $f(z) = \frac{z-sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to 0} \frac{z-sinz}{z^3}$

$\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to 0} \frac{1-cosz}{3z^2}$ [L-Hosptital rule]

$\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to 0} \frac{0+sinz}{6z^1}$ [L-Hosptital rule]

$\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to 0} \frac{cosz}{6}$ [L-Hosptital rule]

$\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to 0} \frac{1}{6}$

$\lim\limits_{z\to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

6.2.3 Pole

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim\limits_{z\to z_0} (z - z_0)^k f(z) = \lambda$, where $\lambda \neq 0$, then z_0 is the pole of order *k*.

If $k = 1$, then z_0 is the simple pole.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = \lim_{z \to 0} z \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{cosz}{z^2}$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{cosz}{z^2}.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order 2.

Example:

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1-e^{2z}}{z^3}$$

$$\lim_{z \to z_0} (z - 0)^3 f(z) = \lim_{z \to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{(1-e^{2z})}{z^3}.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} \frac{(1-e^{2z})}{z}.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$$

So $z_0 = 0$ is the pole of order 2.

6.2.4 Essential singularity

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z \to z_0} (z - z_0)^n f(z) = \infty$, then z_0 is essential singularity.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} e^{1/z}$$

$$\lim_{z \to z_0} (z - 0)^n f(z) = \lim_{z \to 0} z^n e^{1/z} .$$

$\lim_{z \to z_0} (z-0)^n f(z) = \lim_{z \to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

$\lim_{z \to z_0} (z-0)^n f(z) = \infty$.

So $z_0 = 0$ is an essential singularity.

6.2.5 Singularity at infinity

We classify the types of singularities at infinity by letting w = $1/z$ and analyzing the resulting function at w = 0.

Example:

$f(z) = z^3$.

$f(z) = g(w) = 1/w^3$.

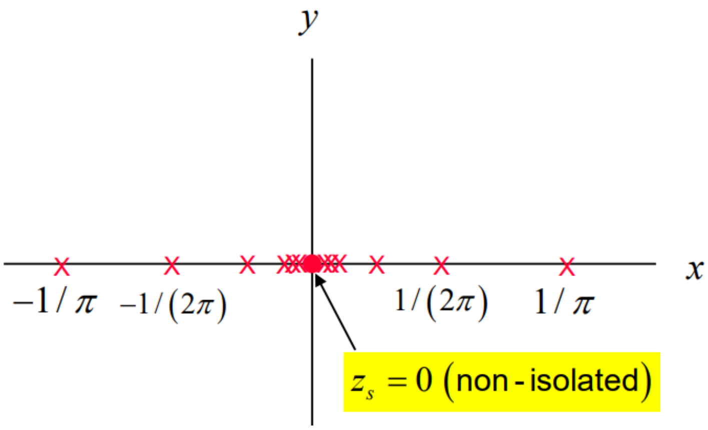
$g(w)$ has a pole of order 3 at w = 0 The function f(z) has a pole of order 3 at infinity.

6.2.5 Non-isolated singularity

A point a is called a non-isolated singularity for $f(z)$ if $f(z)$ is not is not isolated at $z = a$.

Example:

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$



The function is not analytic in any region 0 < |z| < δ.

6.3 <u>Classification of singularity by Laurent series expansion</u>

It is also possible to classify the singularity using the Laurent series expansion.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z-a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n$$

Were

$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta-a)^{-n+1}}\, d\zeta$

$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta$

The series consisting of the negative powers of $z-a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the principal part or singular part $of\ f(z)\ at\ z = a$.

The singular part of $f(z)\ at\ z = a$ determines the character of the singularity.

6.9.1 Removable singularity by Laurent series expansion

Let $\boldsymbol{a}$ be an isolated singularity for $\boldsymbol{f(z)}$. Then $\boldsymbol{a}$ is called a removable singularity if the principal part of $\boldsymbol{f(z)\ at\ z = a}$ has no terms.

If $\boldsymbol{a}$ is a removable singularity for $\boldsymbol{f(z)}$ then the Laurent's series expansion of $\boldsymbol{f(z)\ about\ z = a}$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Hence as $\boldsymbol{z \to a}$, $\boldsymbol{f(z) = a_0}$ Hence by defining $\boldsymbol{f(a) = a_0}$ the function $\boldsymbol{f(z)}$ becomes analytic at $\boldsymbol{a}$.

Example:

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

Now $f(z) = \frac{sinz}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.)$

$f(z) = \frac{sinz}{z} = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots.)$

Here the principal part of $f(z)\ at\ z = 0$ has no terms. Hence z = 0 is a removable singularity.

$\lim_{z\to z_0} f(z)$ also exists then $z_0 = 0$ is the removable singularity.

Example:

Let $f(z) = \frac{z-sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots,)}{z^3}$$

$$f(z) = \frac{\frac{z^3}{3!} - \frac{z^5}{5!} - \ldots,)}{z^3}$$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} - \ldots,$$

$z = 0$ is a removable singularity. By defining $f(0) = 1/6$ the function becomes analytic $at\ z = 0$.Also $\lim_{z\to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

6.9.2 Pole by Laurent series expansion

Let a be an isolated singularity of $f(z)$. The point a is called a pole if the principal part of $f(z)\ at\ z = a$ has a finite number of terms.

If the principal part of $f(z) at\ z = a$ is given by

$\frac{b1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_r}{(z-a)^r}$, . where $b_r \neq 0$.

We say that a is a pole of order r for $f(z)$. Note: A pole of order 1 is called a simple pole and a pole of order 2 is called double pole.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$f(z) = \frac{e^z}{z} = \frac{1}{z}(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots,)$

$f(z) = \frac{e^z}{z} = (1/z + 1 + \frac{z}{2} + \frac{z^2}{6} + \ldots,)$

Here the principal part of $f(z)\ at\ z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$. Also

$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0$. So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{cosz}{z^2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,}{z^2}$$

The principal part of $f(z)\ at\ z = 0$ contains the term $1/z^2$. Hence z=0 is a double pole of f(z).

Also $\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0$.

So $z_0 = 0$ is the pole of order 2.

6.9.3 Essential singularity

Let a be an isolated singularity of $\boldsymbol{f(z)}$. The point a is called an essential singularity of $\boldsymbol{f(z)}$ at $\boldsymbol{z = a}$ if the principal part of $\boldsymbol{f(z)\ at\ z = a}$ has an infinite number of terms.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = e^{1/z}$$

$f(z) = [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

The principal part of $f(z)$ has infinite number of terms. Hence $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Also $\lim_{z \to z_0} (z - 0)^n f(z) = \infty$.

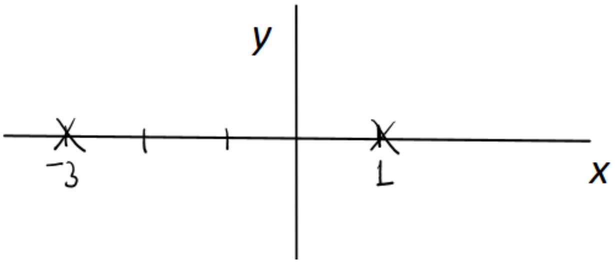
So $z_0 = 0$ is an essential singularity.

6.4 <u>Meromorphic Functions</u>

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example:

$$f(z) = \frac{z}{(z-1)(z+3)^2}$$



$f(z)$ is analytic everywhere in the complex plane except $z = 1$ *and* $z = -3$.Here $z = 1$ is a simple pile and $z = -3$ is the pole of order 3. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{e^z}{z}$$

$$\lim_{z\to z_0} (z-0)^1 f(z) = \lim_{z\to 0} z\frac{e^z}{z}$$

$$\lim_{z\to z_0} (z-0)^1 f(z) = e^0 = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order *1* or simple pole. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{cosz}{z^2}$$

$$\lim_{z\to z_0} (z-0)^2 f(z) = \lim_{z\to 0} z^2 \frac{cosz}{z^2}.$$

$$\lim_{z\to z_0} (z-0)^2 f(z) = \lim_{z\to 0} cosz = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order 2. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1-e^{2z}}{z^3}$

$\lim_{z \to z_0} (z-0)^3 f(z) = \lim_{z \to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$

$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} z^2 \frac{(1-e^{2z})}{z^3}.$

$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} \frac{(1-e^{2z})}{z}.$

$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$

So $z_0 = 0$ is the pole of order 2. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} e^{1/z}$

$\lim_{z \to z_0} (z-0)^n f(z) = \lim_{z \to 0} z^n e^{1/z}$.

$\lim_{z \to z_0} (z-0)^n f(z) = \lim_{z \to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

$\lim_{z \to z_0} (z-0)^n f(z) = \infty.$

So $z_0 = 0$ is an essential singularity.Thus this function is not meromorphic in the whole complex plane.

6.5 Mittag-Leffler theorem

The Mittag-Leffler theorem is a fundamental result in complex analysis that deals with the existence of meromorphic functions with prescribed poles and residues. Specifically, it states that for any sequence of distinct points in the complex plane and any sequence of complex numbers, there exists a meromorphic function with poles precisely at the given points and residues equal to the corresponding complex numbers.

More formally, let $z_1, z_2, \ldots$ be a sequence of distinct complex numbers, and let $c_1, c_2, \ldots$ be a sequence of complex numbers. Then there exists a meromorphic function $f(z)$ on the complex plane such that the only poles of $f(z)$ are at the points $z_1, z_2, \ldots$, and the residue of $f(z)$ at z_i is c_i, for $i = 1, 2, \ldots$.

Proof:

To prove the Mittag-Leffler theorem, we will construct the desired meromorphic function $f(z)$ using a standard technique known as the Weierstrass product formula.

This involves expressing $f(z)$ as an infinite product of simple functions, each of which has a single pole at one of the given points and the prescribed residue.

Let D_n be the disc centered at z_n with radius r_n such that D_n is disjoint from all other discs, and let $C_n$be the circle bounding D_n.

Then we define the function $g_n(z)$ as:

$$g_n(z) = (z - z_n)^{-1} e^{(p_n(z - z_n))}$$

where p_n is chosen so that the Laurent series of $g_n(z)$ at z_n has a constant term of c_n. Specifically, we set:

$$p_n = \frac{c_n}{r_n}$$

Using the Cauchy integral formula, we can express $g_n(z)$ as an integral over C_n:

$$g_n(z) = \frac{1}{2\pi i}\int_{C_n}\frac{g_n(w)}{w - z}\,dw$$

Now we define the function $F(z)$ as:

$$F(z) = \prod_{n=1}^{\infty} g_n(z)$$

This product converges absolutely and uniformly on compact sets, since the discs D_n are disjoint and the radii r_n are chosen appropriately. Moreover, $F(z)$ is meromorphic on the complex plane, since each $g_n(z)$ has a single pole at z_n and no other poles.

To see that $F(z)$ has the desired poles and residues, we consider the partial products:

$$F_{N(z)} = \prod_{n=1}^{N} g_n(z)$$

These are meromorphic functions with poles only at the points $z_1, z_2, \ldots, z_N$. Moreover, the residue of $F_{N(z)}\ at\ z_n\ is\ c_n$, by construction. Finally, we note that $F_{N(z)}$ converges to $F(z)$ as N goes to infinity, since the product converges absolutely and uniformly on compact sets.

Therefore, we have constructed a meromorphic function $f(z)$ with the desired poles and residues, namely:

$$f(z) = F(z)$$

This completes the proof of the Mittag-Leffler theorem.

Question:

Prove that $cotz = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2\pi^2}$ using Mittage Laffer's theorem

Proof:

To prove that $cotz - \frac{1}{z} = 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2\pi^2}$, we can use the Mittag-Leffler theorem.

To prove this identity using the Mittag-Leffler theorem, we need to first identify the poles and their residues of the function $cot(z)$.

We know that $cot(z)$ is periodic with period π, and has simple poles at $z = n\pi$ for all integers n.

Recall that the cotangent function can be expressed as the ratio of the cosine and sine functions:

$$cotz = \frac{\cos z}{sinz}$$

The poles of the cotangent function are the zeros of the sine function, which occur at $z = n\pi$ for all integers n. Thus, we can write:

$$cotz = \frac{cosz}{z - n\pi}$$

To prove this identity using Mittag-Leffler theorem, we need to find the poles and residues of the function cot(z) and the infinite sum in the equation.

First, we know that $cot(z)$ has simple poles at $z = n\pi$ for all integers n.

The residues at these poles are ± 1, depending on the sign of $sin(n\pi)$.

Next, we consider the infinite sum in the equation.

Let $f(z) = \sum \frac{1}{z^2 - n^2\pi^2}$.

This function has poles at $z = \pm n\pi$ for all integers n. The residues at these poles are given by

$$Res[f(z), z = n\pi] = lim_{z\to n\pi}\frac{(z - n\pi)1}{z^2 - n^2\pi^2} = \frac{1}{2n\pi}$$

and

$$Res[f(z), z = -n\pi] = lim_{z\to -n\pi}\frac{(z+n\pi)1}{z^2 - n^2\pi^2} = -\frac{1}{2n\pi}.$$

Now, using the Mittag-Leffler theorem, we can write

$$cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right)$$

$=\sum_{n=1}^{\infty} \frac{1}{2n\pi}\left(\frac{1}{z-n\pi} - \frac{1}{z+n\pi}\right)$

$= 2\sum_{n=1}^{\infty} \frac{1}{2n\pi}\frac{1}{z^2 - n^2\pi^2}$

$= 2f(z)$

Therefore, we have

$cot(z) = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{1}{2n\pi}\frac{1}{z^2 - n^2\pi^2}$ as desired.

6.6 Infinite Product of Complex Numbers

An infinite product of complex numbers is given by:

$z = z_1 . z_2 . z_3 \ldots, . z_n \ldots$

The notation for infinite product is $\prod_{i=1}^{n} z_i$

where $z_1, z_2, z_3, \ldots, z_n$ are complex numbers.

If the infinite product converges, then we can define it as follows:

$$z = \lim_{n\to\infty}(z_1 . z_2 . z_3 \ldots, z_n)$$

In general, an infinite product of complex numbers is said to converge if and only if the limit of the sequence of partial products (i.e., the product of the first n terms) exists and is nonzero.

Some important results related to infinite products of complex numbers are:

1. If the infinite product $|z|$ converges, then the infinite product z also converges.
2. The infinite product $(1 + z)$ converges if and only if the infinite product $(1 - |z|)$ converges.
3. The infinite product $(1 - z)$ converges if and only if the infinite product $(1 + |z|)$ converges.
4. The infinite product $\left(1 + \frac{z}{n}\right)^n$ converges to e^z as n approaches infinity, for any complex number z.
5. The infinite product $sin\left(\frac{z}{n}\right)$ converges to zero for any non-zero complex number z.

Question

Suppose an infinite product is absolutely convergent. Prove that it is convergent

Solution

Suppose the infinite product is given by:

$$P = a_1 . a_2 . a_3 \ldots$$

where ai are non-negative real numbers.

By the absolute convergence of P, we have that the series:

$$S = log(a_1) + log(a_2) + log(a_3) + \ldots$$

converges.

Since the logarithm function is continuous, we can take the exponential of both sides to obtain:

$$e^S = e^{\log(a_1)} e^{log(a2)} e^{log(a3)} \ldots$$

which simplifies to:

$$P = a_1 a_2 a_3 \ldots$$

Thus, the absolute convergence of P implies that the series S converges, which in turn implies that P converges as well.

Therefore, we have shown that if an infinite product is absolutely convergent, then it is also convergent.

6.7 Review questions

1. **$f(z) = \frac{z}{(z-1)^2}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**
2. **$f(z) = \frac{z}{(z-1)(z-5)(z-2)^3}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**
3. Check whether the following functions is meromorphic?

   $f(z) = \frac{e^z}{z}$
4. Check whether the following functions is meromorphic?

   $g(z) = \frac{sinz}{(z-1)^2}$.
5. State and prove the Mittag-Leffler theorem

6.8 Self-assessment

1. $f(z) = \frac{z}{(z-1)}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?

A. 5
B. 1
C. 2
D. 3

2. $f(z) = \frac{z}{(z-1)(z-2)^3}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?

A. 5
B. 1
C. 2
D. 3

3. Which one of the following functions is meromorphic?

$\boldsymbol{f(z) = \frac{e^z}{z}}$**, and** $\boldsymbol{g(z) = \frac{sinz}{(z-1)^2}}$.

A. Only $f(z)$

B. Only $g(z)$

C. Both $f(z)$ and $g(z)$

D. Neither $f(z)$ nor $g(z)$

4. Consider the $\boldsymbol{f(z) = \frac{z}{1-z}}$ **then**

A. $z_0 = 1$ is the singular point of $f(z)$

B. $z_0 = 0$ is the singular point of $f(z)$

C. $z_0 = 10$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

5. Consider the $\boldsymbol{f(z) = z^2}$ **then**

A. $z_0 = 1$ is the singular point of $f(z)$

B. $z_0 = 0$ is the singular point of $f(z)$

C. $z_0 = 10$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

6. Consider the $\boldsymbol{f(z) = \frac{z^2-9}{z^2(z-1)(z-1-2i)}}$ **then**

A. $z_0 = 1$ is one of the singular points of $f(z)$

B. $z_0 = 3$ is the singular point of $f(z)$

C. $z_0 = -3$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

7. What is Mittag-Leffler's theorem?

A. A theorem on the convergence of infinite series.

B. A theorem on the analytic continuation of meromorphic functions.

C. A theorem on the existence of a holomorphic function with prescribed singularities

8. What does the theorem say about meromorphic functions?

A. They can be extended to the whole complex plane.

B. They can be extended to a neighborhood of their poles.

C. They can be approximated by polynomials

9. What are the conditions for the Mittag-Leffler's theorem to hold?

A. The function must have isolated singularities and a certain growth condition.

B. The function must be holomorphic and bounded on a compact set.

C. The function must be a polynomial

10. What is the significance of the theorem in complex analysis?

A. It provides a method for approximating meromorphic functions

B. It is a fundamental tool for studying the Riemann zeta function

C. It allows us to construct meromorphic functions with prescribed singularities

11. What is the value of the infinite product (1+i)(1-i)(1+i)(1-i)...?

A. 1
B. -1
C. i
D. -i

12. What is the value of the infinite product (1+2i)(1-2i)(1+2i)(1-2i)...?

A. 1
B. -1
C. 2i
D. -2i

13. What is the value of the infinite product (1+i/2)(1-i/2)(1+i/2)(1-i/2)...?

A. 1
B. -1/2
C. i/2
D. -i/2

14. What is the value of the infinite product (1+i/3)(1-i/3)(1+i/3)(1-i/3)...?

A. 1
B. -1/3
C. i/3
D. -i/3

15. What is the value of the infinite product (1+3i)(1-3i)(1+3i)(1-3i)...?

A. 1
B. -1
C. 3i
D. -3i

Table 1: Answers of self-assessment

Question number	Correct answer
1	B
2	C
3	C
4	A
5	D
6	A
7	B
8	A

9	A
10	**C**
11	**A**
12	B
13	B
14	B
15	B

6.9 Summary

- The A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.
- The An infinite product of complex numbers is given by:

$z = z_1 . z_2 . z_3 \ldots, . z_n \ldots$

The notation for infinite product is $\prod_{i=1}^{n} z_i$

- If the infinite product $|z|$ converges, then the infinite product z also converges.
- The infinite product $(1+z)$ converges if and only if the infinite product $(1-|z|)$ converges.
- The infinite product $(1-z)$ converges if and only if the infinite product $(1+|z|)$ converges.

6.10 Keywords

Meromorphic function:

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Mittag-Leffler theorem :

Let $z_1, z_2, \ldots$ be a sequence of distinct complex numbers, and let $c_1, c_2, \ldots$ be a sequence of complex numbers. Then there exists a meromorphic function $f(z)$ on the complex plane such that the only poles of $f(z)$ are at the points $z_1, z_2, \ldots$, and the residue of $f(z)$ at z_i is c_i, for $i = 1, 2, \ldots$.

6.11 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Unit 07 - Infinite product of analytic functions

Purpose and Objectives:

An analytic continuation is a mathematical approach used to widen the scope of a given analytic function in the field of complex analysis. Analytic continuation frequently succeeds in defining further values of a function, for instance in a new region when the initial definition's infinite series representation becomes divergent.

The stepwise continuation method might, however, run into problems. These could be fundamentally topological, which would produce contradictions (defining more than one value). Alternatively, they might be related to the existence of singularities. The situation involving many complex variables is somewhat different because singularities need not be separate places in this case. Sheaf cohomology was largely developed because of research into this situation. In this unit first we will discuss the pre-requisite concepts for analytic continuation and then the definition of analytic continuation. After this unit students can be able to-

1. Understand the convergence analysis of a complex valued function.
2. Understand the definition of analytical continuation.
3. Solve some problems of analytical continuation.

Introduction

The Riemann hypothesis, which is closely related to the distribution of prime numbers, is perhaps the most important open topic in pure mathematics today. Analytic continuation is one of the fundamental methods required to comprehend the issue. An approach from the field of mathematics known as complex analysis called analytical continuation is employed to enlarge the domain of a complex analytic function. We will quickly go over some essential mathematics concepts prior to introducing the approach.

7.1 Taylor Series

Consider the case where we want to find a polynomial approximation to a function f(x). Polynomials are mathematical expressions made up of coefficients and variables. The variables are multiplied, subtracted, and added using only non-negative integer exponents. With one variable, x, a polynomial of degree n can be expressed as follows:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots, a_1 x^1 + a_0 x^0 \quad (1.1)$$

If we consider a 3-degree polynomial with $a_3 = \frac{1}{4}$, $a_2 = \frac{3}{4}$, $a_1 = -3$, and $a_0 = -2$ then.

$$f(x) = \frac{x^3}{4} + \frac{3}{4}x^2 - 3x - 2 \quad (1.2)$$

And the graph of $f(x)$ is.

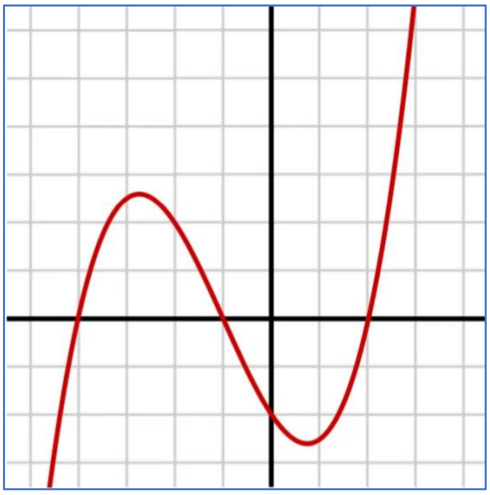


Figure 1.1: The graph of $f(x) = \frac{x^3}{4} + \frac{3}{4}x^2 - 3x - 2$

Imagine that the polynomial has infinite degrees now (it is given by an infinite sum of terms). These polynomials are referred to as Taylor series (or Taylor expansions). Polynomial representations of functions as infinite sums of terms are called Taylor series.

Every term in the series is calculated using the derivative values of $f(x)$ at a particular point (around which the series is centered). A formal Taylor series centered on a certain number and is given by:

$$f(x) = f^{(0)}(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \quad (1.3)$$

where the upper indices (0), (1), … indicate the order of the derivative of $f(x)$ $as\ x = a$. One can approximate a function using a polynomial with only a finite number of terms of the corresponding Taylor series. Such polynomials are called Taylor polynomials.

The Taylor polynomials for $f(x) = \frac{x^3}{4} + \frac{3}{4}x^2 - 3x - 2$ around $a = 0$ are given by:

$$f(x) = -2 - \frac{3}{1!}x + \frac{3}{2.2!}x^2 + \frac{3}{2.3!}x^3 + \cdots \quad (1.4)$$

Were $f^{(0)}(0) = -2$, $f^{(1)}(0) = -3$, $f^{(2)}(0) = \frac{3}{2}$, $f^{(2)}(0) = \frac{3}{2}$.

The equation (1.4) is same as the considered 3 degree polynomial equation (1.2).

7.2 <u>Convergence</u>

Our study of the analytic continuation will likewise heavily rely on the idea of convergence of infinite series. A list of items (or objects) having a specific order constitutes a mathematical sequence. The following S_n represents the n different sequences:

$$S_n = \{a_1, a_2, \ldots, a_n\} \quad (1.5)$$

A well-known example of a sequence is the Fibonacci sequence 0,1,1,2,3,5,8,13,21,34, 55, …where each number is the sum of the two preceding ones.

One builds a **series** by taking partial sums of the elements of a sequence. The series of partial sums can be represented by:

$$\{s_0, s_1, s_2 \ldots, s_n\} \quad (1.6)$$

$$\text{where: } \{s_0 = a_0,\ s_1 = a_0 + a_1, s_2 = a_0 + a_1 + a_2, \ldots,\} \quad (1.7)$$

An example of a series, the familiar geometric series, is shown below. In a geometric series, the common ratio between successive elements is constant. The geometric series with common ratio = 1/2 we have:

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots, \tag{1.8}$$

Fig. 1.2 shows pictorially that the geometric series above converges to twice the area of the largest square.

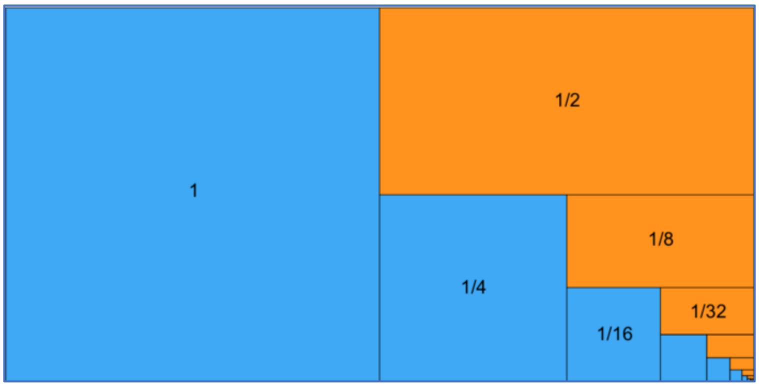


Figure 1.2: A pictorial demonstration of the convergence of the geometric series with common ration r=1/2 and first term a=1

A series such as in Eq. (1.7) is convergent if the sequence Eq. (1.6) of partial sums approaches some finite limit. Otherwise, the series is said to be divergent. An example of a convergent series is the geometric series in Eq.(1.8). An example of a divergent series is:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \to \infty \tag{1.9}$$

7.3 <u>Analytic Functions, Poles, and Convergence Discs</u>

Until now, our analysis was restricted to real numbers. Now we will extend it to complex numbers. The complex plane is a geometric representation of the complex numbers, as shown in Fig.1.3 .

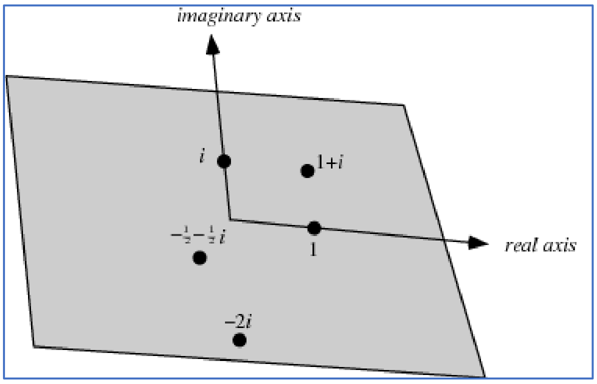


Figure 1.3: The complex plane, a geometric representation of the complex numbers. The figure shows the real and the (perpendicular) imaginary axis.

Let us consider an expansion of an analytic complex function $f(z)$. By definition, an analytic function is a function locally given by a convergent power series. If $f(z)$ is analytic at z_0, the power series reads:

$$f(z_0 + z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \tag{1.10}$$

Equation (1.10) shows the Taylor expansion of an analytic function f(z) into a power series about a complex value z_0.

In analogy with the case of the geometric series, where convergence was restricted to an interval with radius 1 on the real line, this series will converge only over a circular region of the complex plane centered on the complex number z_0.

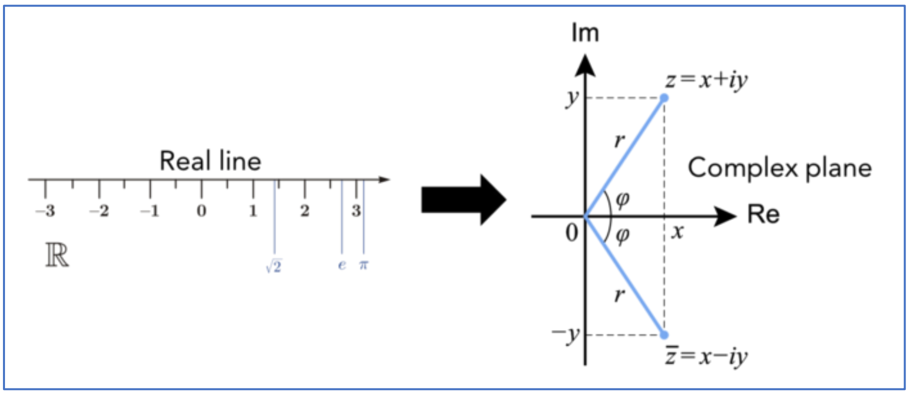


Figure 1.4: Going from the real line to the complex plane.

The convergence region of *f*(*z*) is a circular region centered on z_0 extending to the closest pole, where *f*(*z*) goes to infinity.

Fig. 1.5 shows the convergence region (bounded by the white circle) of the function $1/(1+z^2)$.

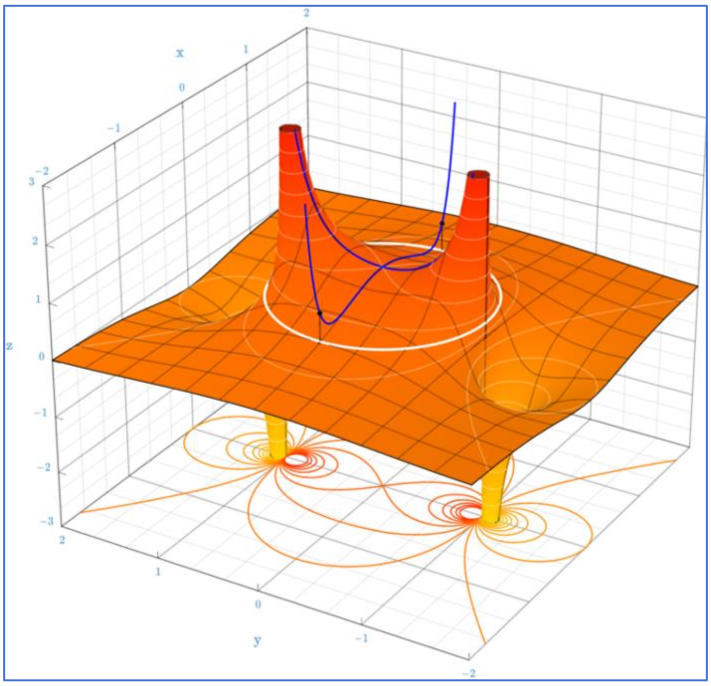


Figure 1.5:The white circle in the convergence disc of the function $1/(1+z^2)$.

A stronger criterion of convergence is called absolute convergence. We call the convergence we already discussed conditional convergence. Absolute convergence occurs when the following series converges:

$$s_0 = |a_0|,\ s_1 = |a_0| + |a_1|, s_2 = |a_0| + |a_1| + |a_2|, \ldots, \tag{1.11}$$

When a series is absolutely convergent it is also conditionally convergent. There are a few tests of absolute convergence, one of them is the ratio test. Consider the general infinite series:

$$S = \sum_{n=0}^{\infty} a_n \tag{1.12}$$

Now define the following ratio: $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (1.13)

The ratio r in the equation (1.13) used in the ratio test of absolute convergence.

The series equation (1.13) converges absolutely if *r*<1 and diverges if *r*>1. If *r*=1, no conclusion can be taken.

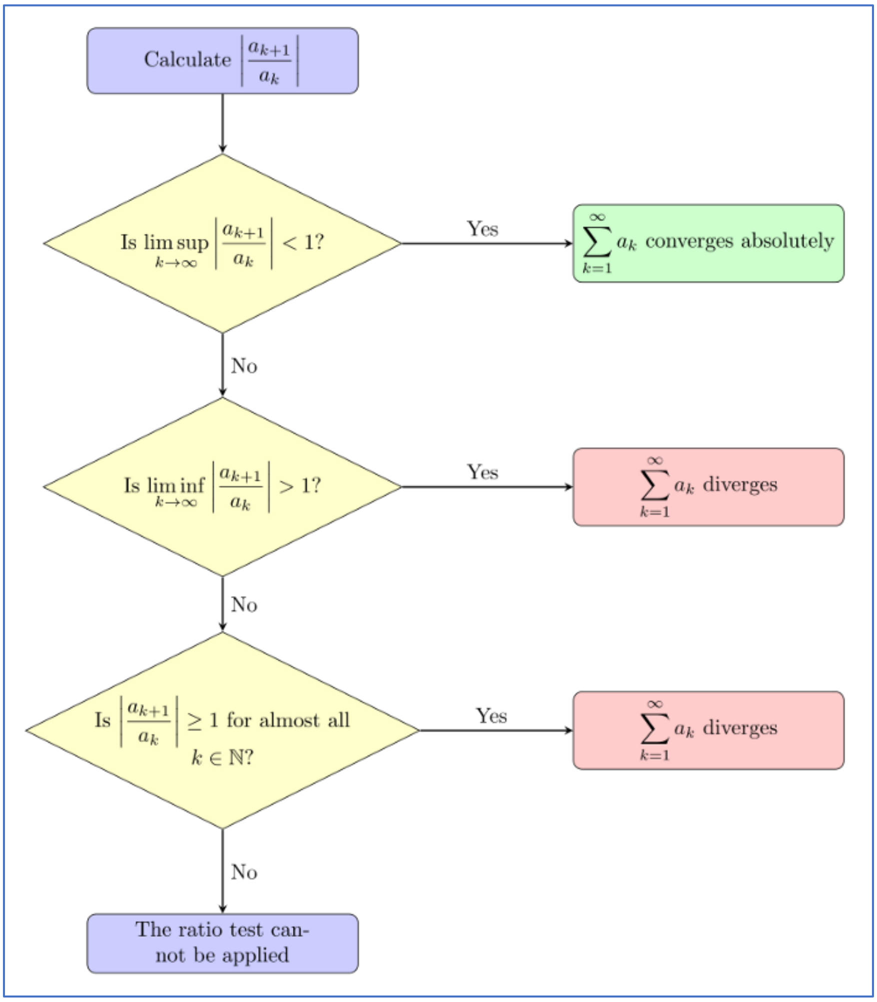


Figure 1.6: A decision diagram for the ratio test.

It is straightforward to apply the ratio test (or any other convergence test) to show the following important result:

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \text{ converges for any } k \geq 2. \tag{1.14}$$

7.4 Analytic Continuation

From the results regarding zeros of an analytic function, it follows that if two functions are regular in a domain D and if they coincide in a neighborhood, however small, of any point a of D, or only along a path-segment, however small, terminating in a point a of D, or only at an infinite number of distinct points with a limit-point a in D, then the two functions are identically the same in D. Thus, it emerges that a regular function defined in a domain D is completely determined by its values over any such sets of points.

This is a very great restraint in the behavior of analytic functions. One of the remarkable consequences of this feature of analytic functions, which is extremely helpful in studying them, is known as analytic continuation. Analytic continuation is a process of extending the definition of a domain of an analytic function in which it is originally defined i.e., it is a concept which is utilized for making the domain of definition of an analytic function as large as possible.

Let us suppose that two functions $f_1(z)$ and $f_2(z)$ are given, such that $f_1(z)$ is analytic in the domain D_1 and $f_2(z)$ in a domain D_2 We further assume that D_1 and D_2 have a common part D_{12} ($D_1 \cap D_2$).

If $f_1(z) = f_2(z)$ in the common part D_{12}, then we say that $f_2(z)$ is the direct analytic continuation of $f_1(z)$ from D_1 into D_2 via D_{12}.

Conversely, $f_1(z)$ is the direct analytic continuationof $f_2(z)$ from D_2 into D_1 via D_{12}. Indeed $f_1(z)$ and $f_2(z)$ are analytic continuations of each other.

Both $f_1(z)$ and $f_2(z)$ may be regarded as partial representations or elements of one and the same function under the condition that $f_1(z) = f_2(z)$ at an infinite set of points with a limit-point in D_{12}.

It is observed that for the purpose of analytic continuation, it is sufficient that the domains D_1 and D_2 have only a small arc in common.

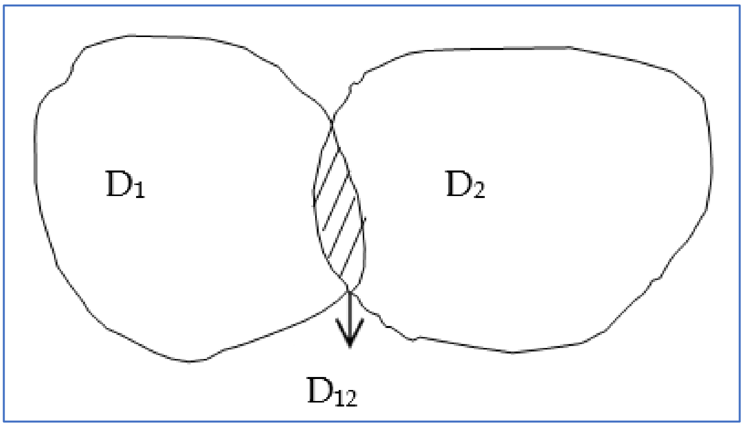


Figure 1.7: Analytic Continuation common domain.

Definition. An analytic function $f(z)$ with its domain of definition D is called a function element and is denoted by (f, D).

If $z \in D$, then (f, D) is called a function element of z. Using this notation, we may say that (f_1, D_1) and (f_2, D_2) are in analytic continuations of each other *iff* $D_1 \cap D_2 \neq \phi$ and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$.

It can be further simplified as Suppose $f_1(z)$ is analytical on a region D_1. Now suppose that D_1 is contained in a region $f_2(z)$. The function $f(z)$ can be analytically continued from D_1 *to* D_2 if there exists a function $f_2(z)$ such that: $f_2(z)$ is analytic on *S*, $f_2(z) = f_1(z)$ for all $z \in D_1$

Example 1.1:

Let us consider $f(z) = \sum_{n=0}^{\infty} z^n$, $\emptyset(z) = \frac{1}{1-z}$.

Then $f(z)$is analytic at all the points within the circle $|z| = 1$ and $\emptyset(z)$ is analytic all the points except $z = 1$.

Also $f(z) = \emptyset(z)$ within $|z| = 1$

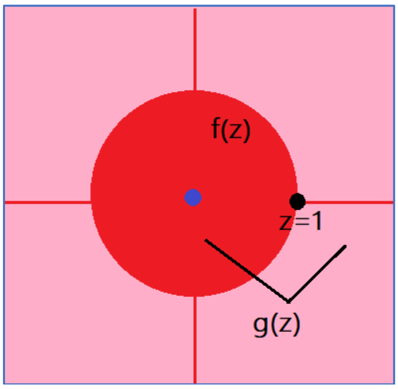


Figure 1.8: $\emptyset(z)$ gives the continuation of $f(z)$ over the rest of the plane.

Hence $\emptyset(z)$ gives the continuation of $f(z)$ over the rest of the plane.

Question:

Show that the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$

Solution:

Given that:

$$f_1(z) = \sum_{n=0}^{\infty} z^n \tag{1.15}$$

First, we will consider the convergent analysis for $f_1(z)$

The series equation (1.15) can be written as:

$f_1(z) = 1 + z + z^2 + z^3 + \cdots, +z^n + \cdots,$

$\Rightarrow f_1(z) = (1 - z)^{-1}$

$\Rightarrow f_1(z) = \frac{1}{1-z}$

Hence it is observed that the $f_1(z)$ has the sum $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and the nth sequence of the series is z^n

Now we apply the ratio test for convergent analysis.

Here, $U_k = z^k$

And $U_{k+1} = z^{k+1}$

If $\sum_{n=0}^{\infty} U_k$ is absolutely convergent then $\left|\frac{U_{k+1}}{U_k}\right| < 1$

$\Rightarrow \left|\frac{z^{k+1}}{z^k}\right| < 1$

$\Rightarrow \left|\frac{z^k.z}{z^k}\right| < 1$

$\Rightarrow |z| < 1$

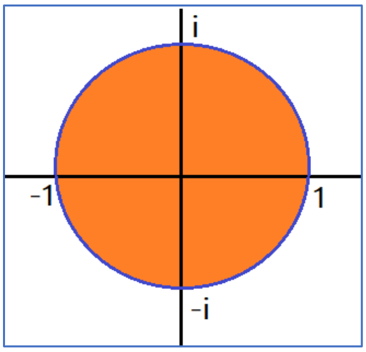


Figure 1.9: The area of unit disc $|z| < 1$

Hence $f_1(z)$ is convergent inside the region $|z| < 1$(see the Figure 1.9)

Now let us consider the second series.

$$f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n \qquad (1.16)$$

$$\Rightarrow \; f_2(z) = \frac{1}{2} + \frac{(1+z)}{2.2} + \frac{(1+z)^2}{2.2^2} + \frac{(1+z)^3}{2.2^3} + \cdots, \frac{(1+z)^n}{2.2^n} + \cdots,$$

Hence the nth term of the series is $U_n = \frac{(1+z)^n}{2.2^n}$.

And $U_{n+1} = \frac{(1+z)^{n+1}}{2.2^{n+1}}$.

If $\sum_{n=0}^{\infty} U_n$ is absolutely convergent then $\left|\frac{U_{n+1}}{U_n}\right| < 1$

$$\Rightarrow \left|\frac{\frac{(1+z)^{n+1}}{2.2^{n+1}}}{\frac{(1+z)^n}{2.2^n}}\right| < 1$$

$$\Rightarrow \left|\frac{(1+z)^{n+1}.2.2^n}{2.2^{n+1}.(1+z)^n}\right| < 1$$

$$\Rightarrow \left|\frac{(1+z)^n.2.2^n.(1+z)}{2.2^n.(1+z)^n.2}\right| < 1$$

$$\Rightarrow \left|\frac{(1+z)}{2}\right| < 1$$

$$\Rightarrow |z+1| < 2$$

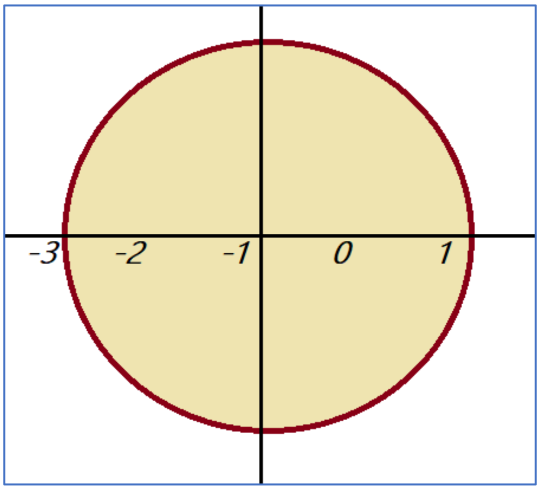


Figure 1.10: The area of disc $|z+1| < 2$

Hence $f_2(z)$ is convergent inside the region $|z+1| < 2$ $(With\ the\ center\ z = -1, and\ r = 2)$.

Till now we have observed that the $f_1(z) = \frac{1}{1-z}$ is analytic inside the domain $D_1 : |z| < 1$ and $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ is analytic inside the domain $D_2 : |z+1| < 2$. It can be clearly seen from Fig.1.10 and Fig.1.9 that $f_2(z)$ and $f_1(z)$ share some common region.

Now we will show that $f_1(z) = f_2(z)$ for all $z \in D_1$.

As we have $f_1(z) = \frac{1}{1-z}$ and

$$f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n = \frac{1}{2} + \frac{(1+z)}{2.2} + \frac{(1+z)^2}{2.2^2} + \frac{(1+z)^3}{2.2^3} + \cdots, \frac{(1+z)^n}{2.2^n} + \cdots,$$

$\Rightarrow f_2(z) = \frac{1}{2} + \frac{(1+z)}{2.2} + \frac{(1+z)^2}{2.2^2} + \frac{(1+z)^3}{2.2^3} + \cdots, \frac{(1+z)^n}{2.2^n} + \cdots,$

Let $\frac{1+z}{2} = p$ then.

$f_2(z) = \frac{1}{2} + \frac{p}{2} + \frac{p^2}{2} + \frac{p^3}{2} + \cdots, \frac{p^n}{2} + \cdots,$

$\Rightarrow f_2(z) = \frac{1}{2}(1 + p + p^2 + p^3 + \cdots, p^n + \cdots,)$

$\Rightarrow f_2(z) = \frac{1}{2}(1-p)^{-1}$

$\Rightarrow f_2(z) = \frac{1}{2(1-p)}$

$\Rightarrow f_2(z) = \frac{1}{2(1-\frac{1+z}{2})}$

$\Rightarrow f_2(z) = \frac{1}{(1-z)}$

$\Rightarrow f_2(z) = f_1(z)$

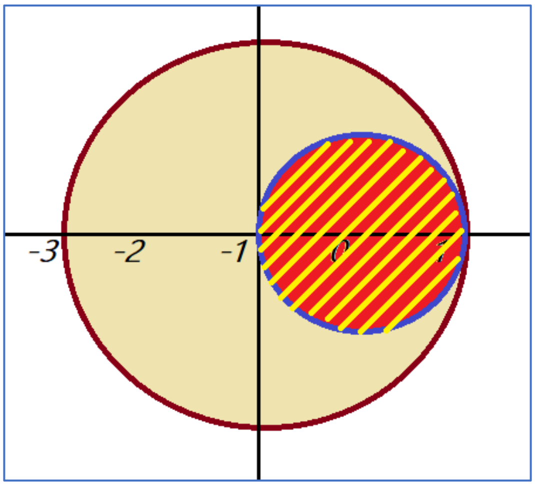


Figure 1.11: The common region for $z \in D_1 \cap D_2$.

As $f_1(z)$ is analytic inside the domain $D_1 : |z| < 1$ and $f_2(z)$ is analytic inside the domain $D_2 : |z+1| < 2$ .Thus $f_2(z)$ extends the domain of an analytical function $f_1(z)$ to larger domain D_2. Hence the function $f_1(z)$ be analytically continued from D_1 *to* D_2 as there exists a function $f_2(z)$ such that: $f_2(z)$ is analytic on D_2: $f_2(z) = f_1(z)$, for all $z \in D_1$.

7.5 Review questions

1. Explain how it is possible to continue analytically the function $f(z) = 1 + z + z^2 + \cdots + z^n + \cdots$ outside the circle of convergence of the power series.
2. Show the series $\sum_{n=0}^{\infty} z^{3n}$ cannot be continued analytically beyond the circle $|z| = 1$
3. Show that the series $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuations of each other.
4. Prove that the series $1 + \sum_{n=0}^{\infty} z^{2n}$ cannot be continued analytically beyond $|z| = 1$
5. Prove that the function defined by $F_1(z) = z - z^2 + z^3 - z^4 + \cdots$, is analytic in the region $|z| < 1$ .And then find a function that represents all possible analytic continuations of $F_1(z)$.

7.6 Self-assessment

1. The function $\mathrm{f}(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ is convergent inside the
   A. Region $|z+1| < 2$
   B. Region $|z-1| < 2$
   C. Region $|z+1| < 6$
   D. Region $|z| < 2$

2. The function $\mathrm{f}(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ is convergent inside the
   A. Region $|z+1| < 2$
   B. Region $|z-1| < 2$
   C. Region $|z+1| < 6$
   D. Region $|z| < 2$

3. The function $\mathrm{f}(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ is convergent inside the
   A. Region $|z+1| < 2$
   B. Region $|z-i| < \sqrt{5}$
   C. Region $|z| < 5$
   D. Region $|z| < \sqrt{5}$

4. The function $\mathrm{f}(z) = \sum_{n=0}^{\infty} z^n$ is convergent inside the
   A. Region $|z| < 2$
   B. Region $|z-1| < 2$
   C. Region $|z+1| < 6$
   D. Region $|z| < 1$

5. The function $\mathrm{f}(z) = \sum_{n=0}^{\infty} \frac{(2+z)^{n-1}}{(n+1)^3 . 4^n}$ is convergent inside the
   A. Region $|z+1| < 4$
   B. Region $|z-1| < 2$
   C. Region $|z+1| < 6$
   D. Region $|z| < 2$

6. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ from the domain *D_1 to D_2* then the which one of the points does lies in $D_1 \cap D_2$
   A. i
   B. 5
   C. $-5i$
   D. -20

7. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ from the domain *D₁ to D₂* then the which one of the points does lies in $D_1 \cap D_2$
   A. $10i$
   B. $\frac{1}{5} + \frac{1}{5}i$
   C. $-5.5i$
   D. -10

8. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ from the domain *D_1 to D_2* then the which one of the points does NOT lies in $D_1 \cap D_2$
   A. $\sqrt{5}$
   B. 0
   C. i
   D. $i/2$

9. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ from the domain *D_1 to D_2* then the which one of the points does not lies in $D_1 \cap D_2$
   A. $0.2i$
   B. 5
   C. $-0.1i$
   D. $0.1 + i$

10. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n/2^{n+1}$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ from the domain *D_1 to D_2* then the which one of the points does lies in $D_1 \cap D_2$
    A. $0.1 + 0.1i$
    B. 50
    C. $-15i$
    D. -12

11. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain *D_1 to D_2* then the which one of the point lies in $D_1 \cap D_2$
    A. $\frac{1}{2} + \frac{1}{4}i$
    B. $\frac{9}{2} + \frac{1}{2}i$
    C. $-3 + \frac{1}{2}i$
    D. $-3 - 3i$

12. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain *D_1 to D_2* then the which one of the point lies in $D_1 \cap D_2$
    A. $\frac{1}{8} + \frac{1}{8}i$
    B. $\frac{10}{2} + \frac{1}{2}i$
    C. $-13 + \frac{1}{2}i$
    D. $-3 - 13i$

13. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain *D_1 to D_2* then the which one of the point does not lies in $D_1 \cap D_2$
    A. $\frac{1}{2} + \frac{1}{4}i$

B. $\frac{1}{9}+\frac{1}{5}i$
C. $-8+\frac{1}{2}i$
D. -2

14. If the function $f_1(z)=\sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z)=\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain D_1 *to* D_2 then the which one of the point does not lies in $D_1 \cap D_2$
   A. -1
   B. -1.5
   C. $-13i$
   D. -2.5

15. If the function $f_1(z)=\sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z)=\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain D_1 *to* D_2 then the which one of the point does not lies in $D_1 \cap D_2$
   A. 10
   B. 0
   C. $-i$
   D. -2.8

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	D
3	B
4	D
5	A
6	A
7	B
8	A
9	B
10	**A**
11	**A**
12	A
13	C
14	D
15	A

7.7 Summary

- An analytic function *f*(z) with its domain of definition D is called a function element and is denoted by (*f*, D).
- The series equation $S = \sum_{n=0}^{\infty} a_n$ converges absolutely if $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$<1 and diverges if *r*>1. If *r* =1, no conclusion can be taken.
- Suppose $f_1(z)$ is analytical on a region D_1. Now suppose that D_1 is contained in a region $f_2(z)$. The function *f(z)* can be analytically continued from D_1 *to* D_2 if there exists a function $f_2(z)$ such that: $f_2(z)$ is analytic on *S*, $f_2(z) = f_1(z)$ for all $z \in D_1$

7.8 Keywords

Absolute convergence: The series equation $S = \sum_{n=0}^{\infty} a_n$ converges absolutely if $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$<1 and diverges if *r*>1. If *r* =1, no conclusion can be taken.

Analytic continuation: If z∈D, (*f*, D) is a function element of z, then (f_1, D_1) and (f_2, D_2) are in analytic continuations of each other iff $D_1 \cap D_2 \neq \phi$ and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$.

7.9 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

Unit 08 - Gamma function and its properties

Purpose and Objectives:

After this unit students can be able to-

1. Understand the uniqueness of analytic continuation.
2. Solve the problem based on the power series method of analytic continuation.
3. Learn natural boundary of complete analytic function.

Introduction

Analytic continuation is a method used to extend the domain of definition of a function that is known to be analytic (i.e., holomorphic) in a certain region, to a larger region. This can be achieved by representing the function as a power series and finding the appropriate coefficients to ensure that the function satisfies the same differential equations in the extended region as it does in the original region.

For example, consider the complex function f(z) that is known to be analytic in a disk D centered at the origin. By expanding f(z) in a power series about the origin, we can obtain its Taylor series representation:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

Using this power series representation, we can extend the definition of f(z) to points outside the disk D by assuming that the series converges to the correct value at those points. This process is called analytic continuation, and it allows us to extend the domain of definition of f(z) to a larger region in the complex plane. In this unit first we will understand the uniqueness of analytic continuation, then the power series method of analytic continuation, and then the natural boundary of complete analytic function.

8.1 Uniqueness of Analytic Continuation by Direct Method

Theorem 2.1: Uniqueness of Analytic Continuation

There cannot be more than one continuation of analytic $f_2(z)$ into the same domain.

Proof:

Let $f_1(z)$ be analytic in the domain D_1.

Let $f_2(z)$ and $g_2(z)$ be analytic continuations of same function $f_1(z)$ from D_1 into the domain D_2 via D_{12} which is common in to both D_1 and D_2 .

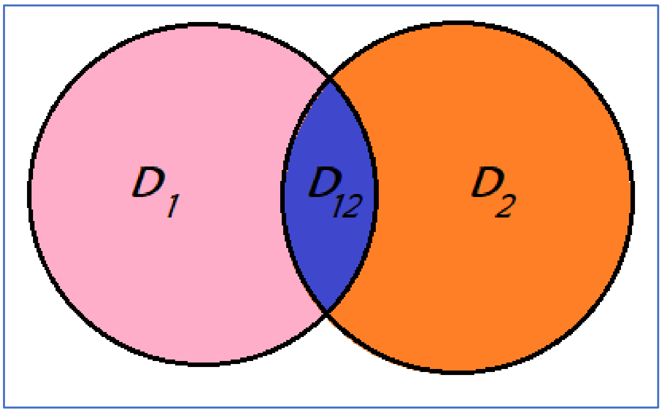


Figure 2.1: Analytical Continuation domains

If we show that $f_2(z) = g_2(z)$ throughout D_2, the result is followed by the this proof.

By the definition of analytic continuation.

$$f_1(z) = f_2(z), \forall z \in D_{12} \tag{2.1}$$

And $f_2(z)$ is analytic in D_2.

$$f_1(z) = g_2(z), \forall z \in D_{12} \tag{2.2}$$

And $g_2(z)$ is analytic in D_2.

From the equation (2.1) and (2.2) we can conclude that

$f_1(z) = f_2(z) = g_2(z), \forall z \in D_{12}$

Or

$f_2(z) = g_2(z), \forall z \in D_{12}$

Or

$(f_2 - g_2)(z) = 0, \forall z \in D_{12}$

f_2 and g_2 are analytic in D_2

$\Longrightarrow \quad f_2 - g_2$ is analytic in D_2.

Thus, we see that $(f_2 - g_2)(z)$ vanishes in D_{12} which is a part of D_2.Also the function is analytic in D_2.

Hence, we must have

$(f_2 - g_2)(z) = 0, \forall z \in D_2$

$\Longrightarrow \quad f_2(z) = g_2(z)\ \forall z \in D_2$.

So, there cannot be more than one continuation of analytic $f_2(z)$ into the same domain.

Remark

The uniqueness property requires the domains of the two analytic continuations to be the same. It is not generally true that if

$F_1 : D_1 \rightarrow C$ *and* $F_2 : D_1 \rightarrow C$ are two analytic continuations of $f : D \rightarrow C$ to different domains D_1, D_2, that they must agree on $D_1 \cap D_2$. A slightly more complicated example is the power series with Fibonacci coefficients:

$f(z) = f_0 + f_1 z + f_2 z^2 + \ldots$, which we considered a few lectures ago.

Initially we observed that this converges and thereby defines an analytic function in some neighborhood D of zero.

By applying the recurrence $f_{n+1} = f_n + f_{n-1}$, we were able to obtain the functional equation:

$(1 - z - z^2)f(z) = z \Rightarrow f(z) = \frac{z}{1 - z - z^2}\ z \in D.$

We then used the right hand side as a definition of f in a much larger domain $D' = C\backslash\{\varphi, \psi\}$.

Formally, $F(z) = \frac{z}{1-z-z^2}$ is an analytic continuation of $f\ to\ D'$.

We didn't explicitly use a different name to distinguish between the continuation and the original function (since they agree where they are both defined) and we will sometimes follow this convention in the future.

In any case, we were then able to use the properties of F in the much larger domain D' (by applying the Residue theorem) to get a good handle on what is happening at zero, and thereby extract a formula for the coefficients.

A functional equation is not the only way to obtain an analytic equation, but it is often the best one. In general, what one is looking for is an alternate representation of the same function which makes sense in a larger region; this alternate description is then used as a definition in the larger region.

8.2 Power Series Method of Analytic Continuation

The Power Series Method of Analytic Continuation is a method used to extend the domain of a complex power series beyond its radius of convergence. It is based on the idea of representing a function as an infinite sum of powers and using this representation to extend the function to a larger domain.

The method works by finding the coefficients of the power series for a given function using Cauchy's Integral Formula, and then using these coefficients to analytically continue the function to a larger domain. This method is useful for finding the values of a function in complex domains, where it is not possible to use real analysis techniques.

Let the initial function $f_1(z)$ is represented by the Taylor's series

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n \tag{2.3}$$

where $a_n = \frac{f_1^{(n)}(z_1)}{n!}$

This series is convergent inside a circle C_1(see the figure 2.2)defined by

$$|z - z_1| = R_1 \tag{2.4}$$

Here $R_1 = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

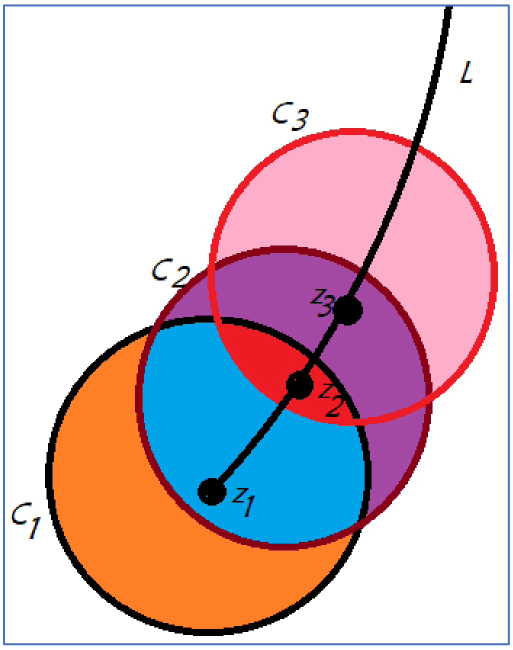


Figure 2.2: The curve L from z_1 and perform analytic continuation

We draw a curve L from $z_1$and perform analytic continuation along this path as follows

Take a point z_2 on L such that $z_2$lies inside the C_1 .

With this help of equation (2.3), we can find the $f_1(z_2), f_1'(z_2), f_1''(z_2) \dots, f_1^{(n)}(z_2)$ by repeated differentiation of (2.3).

Write

$$f_2(z) = \sum_{n=0}^{\infty} b_n (z - z_2)^n \tag{2.5}$$

where $b_n = \frac{f_2^{(n)}(z_2)}{n!}$

The power series (2.5) is convergent inside a circle C_2 defined by

$$|z - z_2| = R_2 \tag{2.6}$$

Here $R_2 = \lim_{n\to\infty} |b_n|^{\frac{1}{n}}$

Also $f_1(z) = f_2(z), \forall z \in C_{12}(The\ common\ part\ of\ C_1\ and\ C_2)$.

Hence $f_2(z)$ is an analytic continuation of $f_1(z)$ from $C_1\ to\ C_2$.

Now take a point z_3 on L such that $z_3$lies inside the C_2 .

With this help of equation (2.5), we can find the $f_2(z_3), f_2'(z_3), f_2''(z_3) \dots, f_2^{(n)}(z_3)$ by repeated differentiation of (2.5).

Write

$$f_3(z) = \sum_{n=0}^{\infty} c_n (z - z_3)^n \tag{2.7}$$

where $c_n = \frac{f_3^{(n)}(z_3)}{n!}$

The power series (2.7)(2.5) is convergent inside a circle C_3 defined by

$$|z - z_3| = R_3 \tag{2.8}$$

Here $R_3 = \lim_{n\to\infty} |c_n|^{\frac{1}{n}}$

Also $f_2(z) = f_3(z), \forall z \in C_{23}(The\ common\ part\ of\ C_2\ and\ C_3)$.

Hence $f_3(z)$ is an analytic continuation of $f_2(z)$ from $C_2\ to\ C_3$.

Now $f_3(z)$ is an analytic continuation of $f_1(z)$ from $C_2\ to\ C_3$.

Repeating this process, we get as continuations several different power series analytic in their respective domains D_1, D_2 ..., where D_1, D_2 ..., are respectively interiors of C_1, C_2 ...,.

Question:

Show that the power series $z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$may be analytically continued to a wider range by means of the series $log2 - \frac{1-z}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \cdots$

Solution:

Let $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ (2.9)

And $f_2(z) = log2 - \frac{1-z}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \cdots$ (2.10)

Here $f_1(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1}$ using the ratio test, if a_n is convergent then $\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$a_n = (-1)^n \frac{z^{n+1}}{n+1}$$

$$a_{n+1} = (-1)^{n+1} \frac{z^{n+2}}{n+2}$$

Now $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\frac{z^{n+2}}{n+2}}{(-1)^n\frac{z^{n+1}}{n+1}}\right| < 1$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n.\frac{-1}{(-1)^n}\frac{z^{n+1}.z.(n+1)}{(n+2)\,z^{n+1}}\right| < 1$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|-1\frac{.z.(1+1/n)}{(1+2/n)}\right| < 1$

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|-1\frac{.z.(1+1/n)}{(1+2/n)}\right| < 1$

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |z| < 1$

Hence $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots = \log(1+z)$ which is convergent for the $|z| < 1$. (2.11)

Thus $f_1(z)$ is analytic inside the circle C_1defined by $|z| = 1$(See the figure 2.3).

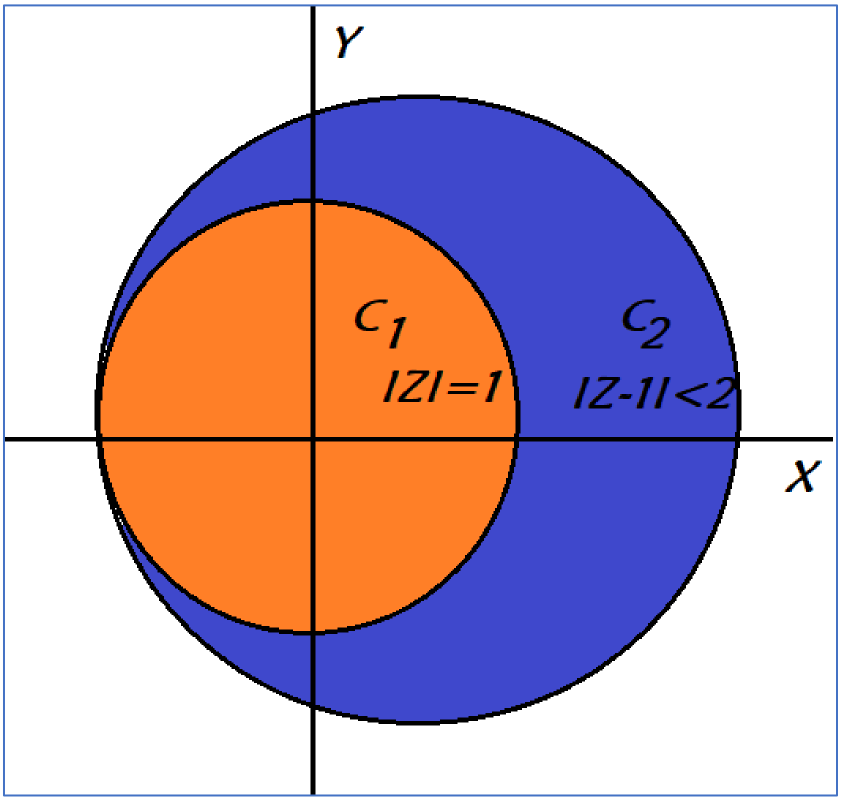


Figure 2.3: the domains of C_1 and C_2

Now we will show that $f_2(z)$ is analytic inside a domain and will also find the convergent analysis of $f_2(z)$.

$f_2(z)$ can be expressed as

$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$

$$\Rightarrow f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\left(\frac{1-z}{2}\right)^2 + \frac{1}{3}\left(\frac{1-z}{2}\right)^3 + \cdots\right]$$

Let the nth term of $f_2(z)$ is $b_n = (-1)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}$ and

$$b_{n+1} = (-1)\frac{1}{n+2}\left(\frac{1-z}{2}\right)^{n+2}$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(-1)\frac{1}{n+2}\left(\frac{1-z}{2}\right)^{n+2}}{(-1)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}}\right| < 1$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{n+1}{n+2}\left(\frac{1-z}{2}\right)^{n+2}\left(\frac{2}{1-z}\right)^{n+1}\right| < 1$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\left(\frac{1-z}{2}\right)\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\left(\frac{1-z}{2}\right)\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \left|\left(\frac{1-z}{2}\right)\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |1-z| < 2$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |z-1| < 2$$

Hence $f_2(z)$ is convergent for the $|z-1| < 2$.

Thus $f_2(z)$ is analytic inside the circle C_2defined by $|z-1| = 2$.

As we know that $\log\left[1-\left(\frac{1-z}{2}\right)\right] = -\left[\frac{1-z}{2}+\frac{1}{2}\left(\frac{1-z}{2}\right)^2+\frac{1}{3}\left(\frac{1-z}{2}\right)^3+\cdots\right]$

Then

$$f_2(z) = log2 + \log\left[1-\left(\frac{1-z}{2}\right)\right]$$

$$f_2(z) = log2 + \log\left[\frac{2-1+z}{2}\right]$$

$$f_2(z) = log2 + log(1+z) - log2$$

$$f_2(z) = log(1+z)$$

By (2.11),

$f_2(z) = f_1(z)$ in the area common to both C_1 and C_2.

Hence, we can say that $f_2(z)$ is analytic continuation of $f_1(z)$ from the interior of C_1 to the interior of C_2. Moreover C_2 is a larger range in comparison to C_1 as shown in the figure 2.3.

8.3 Natural Boundary

In complex analysis, a natural boundary of a complex-valued function is a boundary of its domain that is not a removable singularity. This means that the function cannot be extended analytically across the boundary, and its behavior there is determined by the behavior of the function on the boundary. A classic example of a natural boundary is the boundary of the unit disk in the complex plane, which is the unit circle. Functions defined on the unit disk have essential singularities on the boundary, which means that they cannot be extended analytically to the outside of the unit disk.

Definition 2.1: Function Element

An analytic function f with domain D is called a function element and is denoted by (f, D).

Definition 2.2: Complete Analytic Function

Suppose that $f(z)$ is analytic in a domain D. Let us form all possible analytic continuations of (f, D) and then all possible analytic continuations $(f_1, D_1), (f_2, D_2), \dots, (f_n, D_n)$ of these continuations such that:

$$F(z) = \begin{cases} f_1(z)\ if\ z \in D_1 \\ f_2(z)\ if\ z \in D_2 \\ \dots\dots\dots\dots\dots\dots \\ f_n(z)\ if\ z \in D_n \end{cases} \tag{2.12}$$

Such a function $F(z)$ is called complete analytic function.

In this process of continuation, we may arrive at a closed curve beyond which it is not possible to take analytic continuation. Such a closed curve is known as the **natural boundary** of the complete analytic function. A point lying outside the natural boundary is known as the singularity of the complete analytic function. If no analytic continuation of $f(z)$ is possible to a point z_0, then z_0 is a singularity of $f(z)$.

Obviously, the singularity of $f(z)$ is also a singularity of the corresponding complete analytic function $F(z)$.

Example

Show that the circle of convergence of the power series $f(z) = 1 + z + z^2 + z^4 + z^8$ ..., is a natural boundary of its sum function.

Solution:

The circle of convergence of a power series is the largest circle centered at the origin within which the series converges to a function.

The sum function of the series $f(z) = 1 + z + z^2 + z^4 + z^8$ ..., is not defined at z = 1, so the circle of convergence of the series must include the origin and exclude z = 1.

Since the sum of the series diverges for z = 1, it is a natural boundary for the sum function. The circle of convergence for the series is the largest circle centered at the origin within which the sum function is defined and analytic, and thus it serves as a natural boundary for the sum function.

Example

Show that $f(z) = \sum_{n=0}^{\infty} \frac{z^{2^{n+1}}}{1-z^{2^{n+1}}}$ is analytic in the domain |z | < 1 and the domain |z | > 1, and that |z | = 1 is a natural boundary for the function in each domain.

Solution:

For a function to be analytic, it must be complex differentiable at every point in its domain.

Let us first consider the domain |z| < 1.

For $|z| < 1, |z^{2^{n+1}}| < 1$ and thus $1 - |z^{2^{n+1}}| > 0$. Hence, the denominator $1 - z^{2^{n+1}}$ is never 0 and the function is well-defined in this domain.

Next, we can apply the Cauchy-Riemann equations to show that the function is complex differentiable in this domain, and therefore analytic.

For |z| > 1, the same argument can be made: the denominator $1 - z^{2^{n+1}}$ is never 0 for |z| > 1, and the function is well-defined in this domain.

Finally, for |z| = 1, the function is not complex differentiable, which means that |z| = 1 is a natural boundary for the function in both domains.

Therefore, the function $\sum_{n=0}^{\infty} \frac{z^{2^{n+1}}}{1-z^{2^{n+1}}}$ is analytic in the domain |z| < 1 and the domain |z| > 1, and |z| = 1 is a natural boundary for the function in each domain.

8.4 Review questions

1. Prove that the series $z^{1!} + z^{2!} + z^{3!} + \cdots$ has the natural boundary $|Z| = 1$.

2. Prove that |z| = 1 is a natural boundary for the series $\sum_{n=0}^{\infty} 2^{-n} * z^{3n}$

3. Let $F_1(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{3^n}$ Find an analytic continuation of $F_1\ (z)$, which converges for $z\ =\ 3\ -\ 4i$

4. State and prove the uniqueness theorem of analytic continuation

5. Show that the series $1 + z + z^2 + z^4 + z^8 + \ldots$, can not be analytically continued beyond the $|z| = 1$

8.5 Self-assessment

1. The power series $f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ is convergent inside the
   A. Region $|z+1| < 2$
   B. Region $|z-1| < 2$
   C. Region $|z| < 1$
   D. Region $|z| < 2$

2. There cannot be more than one continuation of analytic $f(z)$ into the same domain.
   A. True
   B. $False$

3. The nth term of the series $z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$

A. $a_n = (-1)^n \frac{z^{n+1}}{n+1}$

B. $a_n = (-2)^n \frac{z^{n+1}}{n+1}$

C. $a_n = (-1)^{n+1} \frac{z^{n+1}}{n+1}$

D. $a_n = (-1)^n \frac{z^{n+1}}{n+2}$

4. An analytic function f with domain D is called a function element and is denoted by (f, D).

   A. True
   B. $False$

5. The nth term of the series $\left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]\ is$
   A. $(-2)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}$
   B. $(-1)\frac{1}{n+1}\left(\frac{1-z}{2}\right)^{n+1}$
   C. $(-1)\frac{1}{n+2}\left(\frac{1-z}{2}\right)^{n+1}$
   D. $(-1)\frac{1}{n+1}\left(\frac{1-z}{3}\right)^{n+1}$

6. The power series $\left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$ is convergent inside the
   A. $|z-1| < 1$
   B. $|z| < 1$
   C. $|z| < 2$
   D. $|z-1| < 2$

7. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to

$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$

from the domain D_1 *to* D_2 then the which one of the domain does lies in $D_1 \cap D_2$

A. $|z-1|<1$
B. $|z|<1$
C. $|z|<2$
D. $|z-1|<2$

8. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain *D1 to D2* then the which one of the points does not lies in $D_1 \cap D_2$
A. $0.1 + 0.1i$
B. 0.5
C. $0.5i$
D. $2 + 5i$

9. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain *D1 to D2* then the which one of the points does lies in $D_1 \cap D_2$
A. $0.1 + 0.2i$
B. 5
C. $5i$
D. $2 + 5i$

10. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$
from the domain *D1 to D2* then the which one of the points does not lies in $D_1 \cap D_2$
A. $0.2i$
B. 0.3
C. $0.4i$
D. $4i$

11. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is a natural boundary for its sum function then the circle of convergent is
A. $|z-1|<1$
B. $|z|<1$
C. $|z|<2$
D. $|z-1|<2$

12. If the function $f_1(z) = \sum_{n=0}^{\infty} z^n$ is in analytic continuation to $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ from the domain *D1 to D2* then the which one of the point lies in $D_1 \cap D_2$
A. $\frac{1}{8} + \frac{1}{8}i$
B. $\frac{10}{2} + \frac{1}{2}i$
C. $-13 + \frac{1}{2}i$
D. $-3 - 13i$

13. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to
$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$

from the domain D_1 *to* D_2 then the which one of the points does not lies in $D_1 \cap D_2$

A. $0.3i$
B. 0.3
C. $0i$
D. $12 + i$

14. If the function $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is in analytic continuation to

$$f_2(z) = log2 - \left[\frac{1-z}{2} + \frac{1}{2}\frac{(1-z)^2}{2^2} + \frac{1}{3}\frac{(1-z)^3}{2^3} + \cdots\right]$$

from the domain D_1 *to* D_2 then the which one of the points does lies in $D_1 \cap D_2$

A. $0.01 + 0.02i$
B. 50
C. $50i$
D. $20 + 5i$

15. In complex analysis, a natural boundary of a complex-valued function is a boundary of its domain that is not a removable singularity.

A. $True$
B. $False$

Table 1: Answers of self-assessment

Question number	Correct answer
1	C
2	A
3	A
4	A
5	B
6	D
7	B
8	D
9	A
10	**A**
11	**C**
12	A
13	D
14	A

15	A

8.6 Summary

- The Power Series Method of Analytic Continuation is a method used to extend the domain of a complex power series beyond its radius of convergence.
- An analytic function f with domain D is called a function element and is denoted by (f, D).
- Uniqueness of Analytic Continuation: There cannot be more than one continuation of analytic $f_2(z)$ into the same domain.

8.7 Keywords

Complete analytic function: Suppose that $f(z)$ is analytic in a domain D. Let us form all possible analytic continuations of (f, D) and then all possible analytic continuations $(f_1, D_1), (f_2, D_2), \dots, (f_n, D_n)$ of these continuations such that:

$$F(z) = \begin{cases} f_1(z) \; if \; z \in D_1 \\ f_2(z) \; if \; z \in D_2 \\ \dots \dots \dots \dots \dots \dots \\ f_n(z) \; if \; z \in D_n \end{cases}$$

Such a function $F(z)$ is called complete analytic function.

8.8 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

Unit 09 - Riemann zeta function

Purpose and Objectives:

The Monodromy Theorem in complex analysis states that given a non-constant holomorphic function on a simply connected domain, its set of singularities is invariant under any loop in the domain. In other words, it characterizes the behavior of the function near its singularities and provides a way to study the topological structure of complex functions. The theorem is useful for solving certain types of differential equations, as well as for constructing complex functions with prescribed singularities.

Similarly, the Poisson integral formula has several applications in various fields, including:

1. Harmonic Analysis: The Poisson Integral Formula provides a tool for solving boundary value problems for harmonic functions and has applications in potential theory and boundary value problems.
2. Image Processing: The Poisson Integral Formula is used in image processing to restore images that have been degraded or to smooth out noise in images.
3. Numerical Analysis: The Poisson Integral Formula is used in numerical analysis to solve partial differential equations, especially in areas like electrostatics and heat transfer.
4. Complex Analysis: The Poisson Integral Formula is used in complex analysis to study conformal mappings, potential theory, and complex dynamics.
5. Signal Processing: The Poisson Integral Formula is used in signal processing to solve problems in signal restoration, noise reduction, and boundary value problems for signals.

After this unit students can be able to-

1. State and prove the Monodromy theorem.
2. Learn Poisson Integral Formula for analytic function.
3. Understand the Poisson Kernel, and Conjugate Poisson Kernel for analytic function.
4. Solve the problem based on the Poisson Integral Formula.

Introduction

The number of independent loops or paths around a singular point of an analytic function can be understood by sheets of the multi-valued analytic function.

In other words, if an analytic function has a singularity at a point, then the number of independent loops that can be taken around this point is equal to the number of branches of the function that can be defined in a neighborhood of the singularity.

The Monodromy Theorem is an important result in complex analysis and is used to study the behavior of multi-valued analytic functions near singular points. If a function f is analytic in the unit disk of the complex plane and continuous on its boundary, then it can be represented by the Poisson integral formula. In this unit first we will understand the Monodromy theorem then learn Poisson Integral Formula for analytic function. After that we will explore the concept of the Poisson Kernel, and Conjugate Poisson Kernel for analytic function. Finally solve the problem based on the Poisson Integral Formula.

9.1 Monodromy Theorem

Let D be a simply connected domain and suppose $f_0\,(z)$ is analytic in a domain $D_0 \subset D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

Proof:

Suppose the conclusion is false. Then there exist points $z_0 \in D_0, z_1 \in D$, and curves C_1, C_2

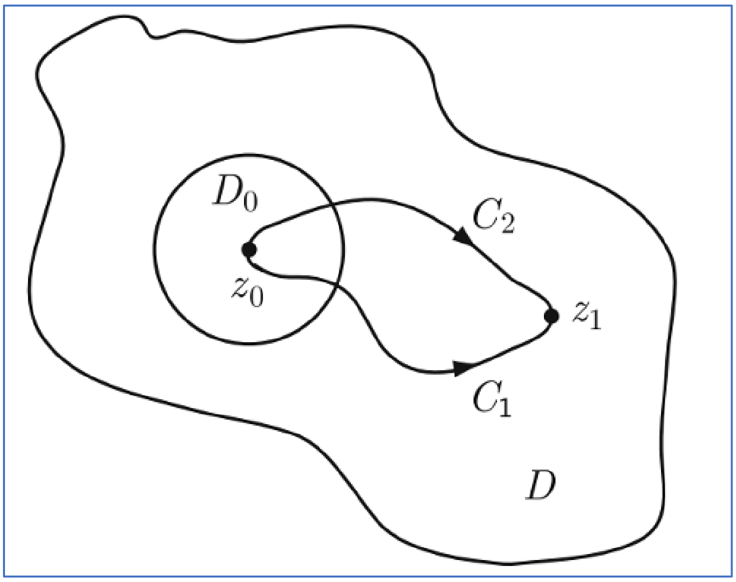


Figure 3.1: D be a simply connected domain and the points $z_0 \in D_0$, $z_1 \in D$, and $D_0 \subset D$.

both having initial point z_0 and terminal point z_1 such that (f_0, D_0) leads to a different function element in a neighborhood of z_1 when analytically continued along C_1 than when analytically continued along C_2 (see Figure3.1).

This means that (f_0, D_0) does not return to the same function element when analytically continued along the closed curve $C_1 - C_2$.

Lemma 3.1.

Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n \to \infty$, then there is exactly one point in common to all the S_n.

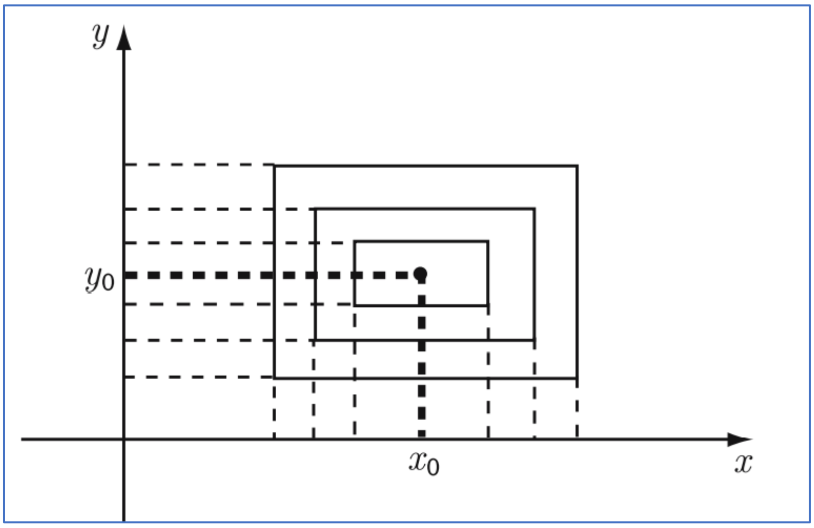


Figure 3.2: sequence of closed and bounded rectangles in the plane.

To prove the theorem, it thus suffices to show that the function element $(f_0, D_0), D_0 \subset D$, can be continued along any closed curve lying in D and return to the same value. In the special case that the closed curve C is a rectangle

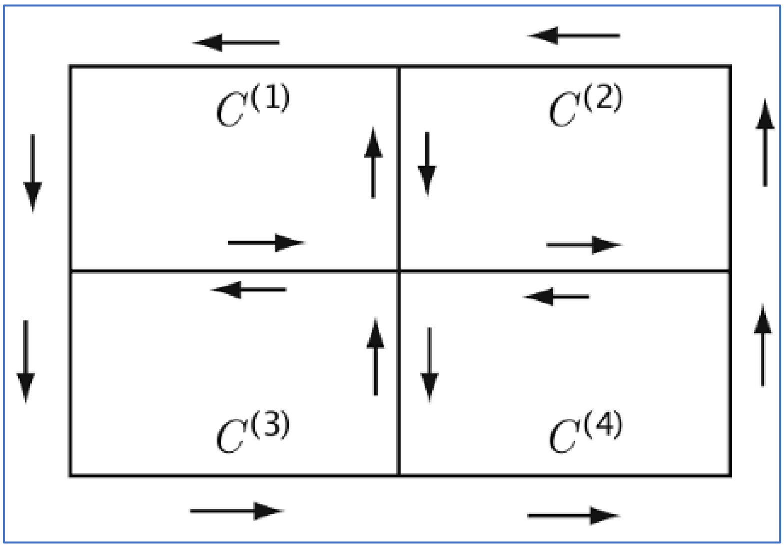


Figure 3.3: Rectangle C into four congruent rectangles

Divide the rectangle C into four congruent rectangles, as illustrated in Figure 3.3 continuation along C produces the same effect as continuation along these four rectangles taken together.

If the conclusion is false for C, then it must be false for one of the four sub-rectangles, which we denote by C_1. We then divide C_1 into four congruent rectangles, for one of which the conclusion is false.

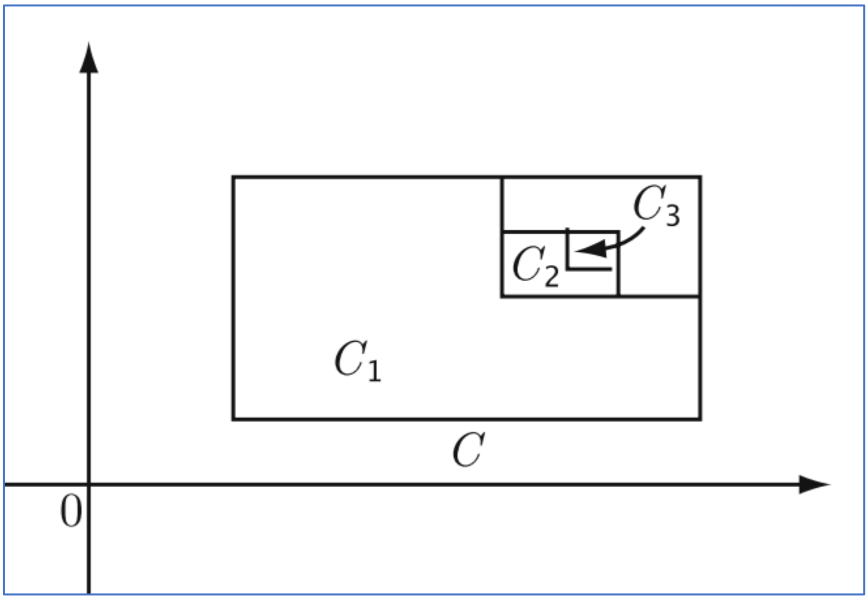


Figure 3.4: C_1 into four congruent rectangles

Continuing the process, we obtain a nested sequence of rectangles for which the conclusion is false.

According to Lemma 1, there is exactly one point, call it z_*, belonging to all the rectangles in the nest. Since $z_* \in D$, there exists a function element (f_*, D_*) with $z_* \in D_* \subset D$.

For n sufficiently large, the rectangle C_n of the nested sequence is contained in D_*

But this means that $f_*(z)$ is analytic in a domain containing C_n, contrary to the way C_n was defined. This contradiction concludes the proof in the special case in which the curve is a rectangle.

9.2 Poisson Integral Formula, Poisson Kernel, and Conjugate Poisson Kernel

If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within, C, then

$$f(z) = \frac{1}{2\pi i}\int_c \frac{(R^2 - a\bar{a})f(z)}{(z-a)(R^2 - z\bar{a})}dz$$

$$\Rightarrow f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\emptyset})}{R^2 - 2rRCos(\theta - \emptyset) + r^2}d\emptyset$$

Where $a = re^{i\theta}$is any point inside the circle $|z| = R$.

Proof:

Suppose f(z) is analytic within and on the circle C defined $|z| = R$.

Let $a = re^{i\theta}$is any point inside the circle $|z| = R$ so that $0 < r < R$.

Let the inverse of $A(a)$ is $A'(a')$ with respect to the circle C is given by $a' = R^2/\bar{a}$ which lies outside the circle C(See the Figure 3.5).

By Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz \tag{3.1}$$

Since $f(z)$ is analytic within and upon the circle C and so $\frac{f(z)}{(z-a')}$ is analytic within and on the circle.

By Cauchy's integral theorem

$$\int_c \frac{f(z)}{(z-a')}dz=0 \tag{3.2}$$

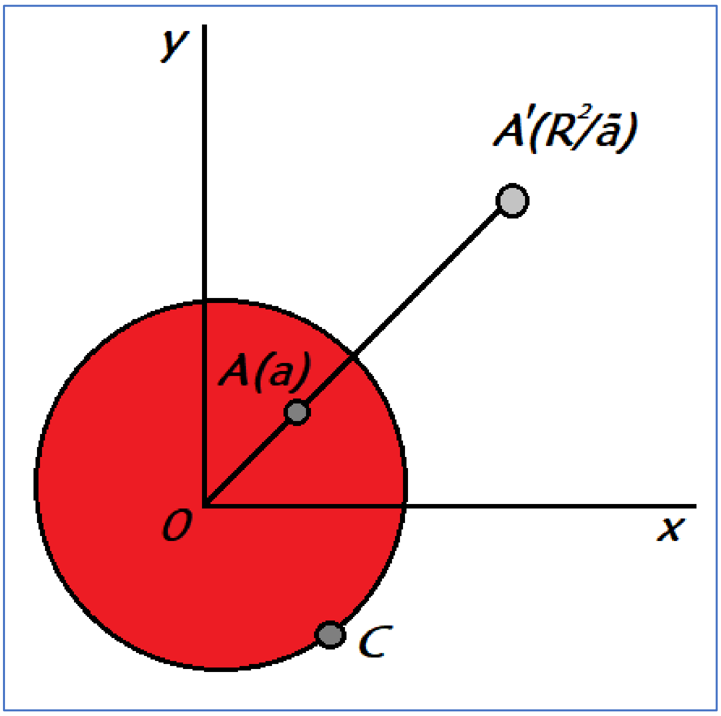


Figure 3.5: Inverse of $A(a)$ is $A'(a')$ with respect to the circle C is given by $a' = R^2/\bar{a}$

Note that $\frac{f(z)}{(z-a)}$ is not analytic within C

Now

(3.1)-(3.2) gives

$f(a) - 0 = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz - \int_c \frac{f(z)}{(z-a')}dz$ (3.3)

$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{1}{(z-a)} - \frac{1}{(z-a')}\right] f(z)dz$ (3.4)

$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(z-a')-(z-a)}{(z-a)(z-a')}\right] f(z)dz$ (3.5)

$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(a-a')}{(z-a)(z-a')}\right] f(z)dz$ (3.6)

$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{\left(a-\frac{R^2}{\bar{a}}\right)}{(z-a)(z-\frac{R^2}{\bar{a}})}\right] f(z)dz$ (3.7)

$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(a\bar{a}-R^2)}{(z-a)(z\bar{a}-R^2)}\right] f(z)dz$ (3.8)

$f(a) = \frac{1}{2\pi i}\int_c \left[\frac{(R^2-a\bar{a})}{(z-a)(R^2-z\bar{a})}\right] f(z)dz$ (3.9)

This proves the first required result.

Any point z on $|z| = R$ is expressible as $z = R\, e^{i\phi}$

Also $a = re^{i\theta}$ so that $\bar{a} = re^{-i\theta}$

Now $R^2 - a\bar{a} = R^2 - re^{i\theta}.re^{-i\theta}$ (3.10)

$$\Rightarrow R^2 - a\bar{a} = R^2 - r^2$$

Now

$(z-a)(R^2 - z\bar{a}) = (R\, e^{i\phi} - re^{i\theta})(R^2 - Re^{i\phi}\, re^{-i\theta})$ (3.11)

$$\Rightarrow (z-a)(R^2 - z\bar{a}) = R\, e^{i\phi}(R\; - re^{i(\theta-\phi)})(R - re^{-i(\theta-\phi)})$$

$$\Rightarrow (z-a)(R^2 - z\bar{a}) = R\, e^{i\phi}[(R^2 + r^2 - rR(e^{i(\theta-\phi)} - e^{-i(\theta-\phi)})]$$

$\Rightarrow (z-a)(R^2 - z\bar{a}) = R\, e^{i\phi}[(R^2 + r^2 - 2rRCos(\theta - \phi)]$ (3.12)

$dz = d(Re^{i\phi}) = Rie^{i\phi}d\phi$ (3.13)

Writing (3.9) with the help of (3.11) and (3.13),

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2)f(z)f(Re^{i\emptyset}).i}{[R^2 - 2RrCos(\emptyset - \theta) + r^2]Re^{i\emptyset}}d\emptyset$$

$f(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]}d\emptyset$ (3.14)

This proves the second result.

Here $\frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]}$ is known as the *Poisson Kernal* for the disk $|z\,| < R$.

Note that the *Poisson, kernel* is bounded above by $\frac{(R^2-r^2)}{[R^2-2Rr+r^2]} = \frac{R+r}{R-r}$.

The conjugate Poisson kernel is a mathematical function used in complex analysis and potential theory. It is defined as the conjugate of the *Poisson kernel*, which is a function that maps points in the complex plane to the unit disk. The conjugate Poisson kernel is given by the formula:

$$P^*(z) \;=\; P\left(\frac{1}{z^*}\right)$$

where $P(z)$ is the Poisson kernel and z^* is the complex conjugate of z.

Question:

Using Poisson's integral formula for the circle, show that:

$$\int_0^{2\pi} \frac{e^{cos\phi}.Cos(Sin\phi)d\phi}{5-4Cos(\theta-\phi)} = \frac{2\pi}{3}e^{cos\theta}\cos(sin\theta) \quad (3.15)$$

Solution:

By the Poisson's integral formula,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]}d\emptyset \quad (3.16)$$

If we compare R.H.S. of (3.15) with the given integral, then we find

$$R^2+r^2 = 5 \quad (3.17)$$

$$rR = 2 \quad (3.18)$$

$$f(Re^{i\phi}) = e^{cos\phi}.Cos(Sin\phi) \quad (3.19)$$

Using (3.17) and (3.18)

$R = 2, r = 1$ and so $R^2 \ -r^2 = 3$ (3.20)

Now (3.19) $\Longrightarrow$

$$f(re^{i\theta}) = e^{cos\theta}\cos(sin\theta) \quad (3.21)$$

Putting value from (3.17), (3.18), (3.20), and (3.21) in the equation (3.16), we get

$$e^{cos\theta}\cos(sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{3e^{cos\phi}.Cos(Sin\phi)}{5-4Cos(\theta-\phi)}d\emptyset$$

$$\frac{2\pi}{3}e^{Cos\theta}\,\mathrm{Cos}(Sin\theta) = \int_0^{2\pi} \frac{e^{cos\phi}.Cos(Sin\phi)}{5-4Cos(\theta-\phi)}d\emptyset$$

Hence proved.

Question:

Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{621f\left(25e^{\frac{i\pi}{4}}\right)}{629-100Cos\left(\pi-\frac{\pi}{4}\right)}d\emptyset$

Solution:

We know that using the Poisson integral formula,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]}d\emptyset$$

Here

(R^2-r^2) =621

$f(Re^{i\emptyset}) = f(25e^{\frac{\pi}{4}i})$

$2Rr = 100$

$\theta-\phi = \pi - \frac{\pi}{4}$

(R^2+r^2) =629

Hence, we can conclude that.

$R = 25$

$r = 2$

$\theta = \pi$

$\phi = \frac{\pi}{4}$

$f(re^{i\theta}) = f(2e^{i\pi})$

$$\int_0^{2\pi} \frac{621 f\left(25 e^{\frac{i\pi}{4}}\right)}{629 - 100 Cos\left(\pi - \frac{\pi}{4}\right)} d\emptyset = 2\pi f(2e^{i\pi})$$

9.3 Review questions

1. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{75 f\left(10 e^{\frac{i\pi}{4}}\right)}{125 - 100 Cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)} d\emptyset$

2. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{64 f\left(10 e^{\frac{i\pi}{10}}\right)}{136 - 120 Cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right)} d\emptyset$

3. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{9 f\left(5 e^{\frac{i\pi}{10}}\right)}{41 - 40 Cos\left(\pi - \frac{\pi}{10}\right)} d\emptyset$

4. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{80 f\left(9 e^{\frac{i\pi}{2}}\right)}{82 - 18 Cos\left(\pi - \frac{\pi}{2}\right)} d\emptyset$

5. Using the Poisson integral formula, find the value of $\int_0^{2\pi} \frac{99 f\left(10 e^{\frac{i\pi}{10}}\right)}{101 - 20 Cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right)} d\emptyset$

9.4 Self-assessment

1. Let D be a simply connected domain and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

   A. True
   B. *False*

2. Let D be a simply connected domain and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there does not exists any single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

   A. True
   B. *False*

3. Let D be a simply connected domain and suppose $f_0\ (z)$ is analytic in a domain $D_0\ \subset\ D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is not analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

   A. True
   B. *False*

4. Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n \to \infty$, then there is exactly one point in common to all the S_n.

   A. True
   B. $False$

5. Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n \to \infty$, then there are two points in common to all the S_n.

   A. True
   B. $False$

6. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-1)f(Re^{i\emptyset})}{[R^2-2RCos(\pi-\phi)+1]} d\emptyset =$
   A. $f(e^{i\pi})$
   B. $f(2e^{i\pi})$
   C. $f(3e^{i\pi})$
   D. $2f(e^{i\pi})$

7. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{(R^2-4)f(Re^{i\emptyset})}{[R^2-4RCos(\pi-\phi)+4]} d\emptyset =$
   A. $\pi f(e^{i\pi})$
   B. $2\pi f(2e^{i\pi})$
   C. $3\pi f(3e^{i\pi})$
   D. $f(e^{i\pi})$

8. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-9)f(Re^{i\emptyset})}{[R^2-6RCos(\pi-\phi)+9]} d\emptyset =$
   A. $f(3e^{i\pi})$
   B. $f(2e^{i\pi})$
   C. $f(3e^{i\pi})$
   D. $2f(e^{i\pi})$

9. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{(R^2-16)f(Re^{i\emptyset})}{[R^2-8RCos(\pi-\phi)+16]} d\emptyset =$
   A. $\pi f(e^{i\pi})$
   B. $2\pi f(4e^{i\pi})$
   C. $3\pi f(3e^{i\pi})$
   D. $f(5e^{i\pi})$

10. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{21f(5e^{i\emptyset})}{29-20Cos(\pi-\phi)} d\emptyset =$
    A. $\pi f(e^{i\pi})$
    B. $2\pi f(2e^{i\pi})$
    C. $3\pi f(3e^{i\pi})$
    D. $f(e^{i\pi})$

11. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi} \frac{27f(6e^{i\emptyset})}{45-36Cos(\pi-\phi)]} d\emptyset =$
    A. $f(3e^{i\pi})$
    B. $f(2e^{i\pi})$
    C. $f(3e^{i\pi})$
    D. $2f(e^{i\pi})$

12. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{20f(6e^{i\emptyset})}{52-48Cos(\pi-\phi)} d\emptyset =$
    A. $\pi f(e^{i\pi})$
    B. $2\pi f(4e^{i\pi})$
    C. $3\pi f(3e^{i\pi})$
    D. $f(5e^{i\pi})$

13. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{99f(10e^{i\emptyset})}{101-20Cos(\pi-\phi)} d\emptyset =$
    A. $\pi f(e^{i\pi})$
    B. $2\pi f(e^{i\pi})$
    C. $3\pi f(3e^{i\pi})$
    D. $f(e^{i\pi})$

14. Using the Poisson integral formula, the $\frac{1}{2\pi}\int_0^{2\pi} \frac{91f(10e^{i\emptyset})}{109-60Cos(\pi-\phi)]} d\emptyset =$
    A. $f(3e^{i\pi})$
    B. $f(2e^{i\pi})$
    C. $f(3e^{i\pi})$
    D. $2\,f(e^{i\pi})$

15. Using the Poisson integral formula, the $\int_0^{2\pi} \frac{4f(5e^{i\emptyset})}{41-40Cos(\pi-\phi)} d\emptyset =$
    A. $\pi f(e^{i\pi})$
    B. $2\pi f(4e^{i\pi})$
    C. $3\pi f(3e^{i\pi})$
    D. $f(5e^{i\pi})$

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	B
3	B
4	A
5	B
6	A
7	B
8	D
9	B
10	**B**
11	**A**

12	B
13	B
14	A
15	B

9.5 Summary

- If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within, C, then

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{(R^2 - r^2)f(Re^{i\emptyset})}{R^2 - 2RrCos(\theta - \emptyset) + r^2} d\emptyset$$

  Where $a = re^{i\theta}$is any point inside the circle $|z| = R$.

- $\frac{(R^2-r^2)f(Re^{i\emptyset})}{[R^2-2RrCos(\theta-\phi)+r^2]}$ is known as the ***Poisson Kernal*** for the disk $|z\,| < R$.

- The *Poisson, kernel* is bounded above by $\frac{(R^2-r^2)}{[R^2-2Rr+r^2]} = \frac{R+r}{R-r}$.

- The conjugate Poisson kernel is a mathematical function used in complex analysis and potential theory. It is defined as the conjugate of the *Poisson kernel*, which is a function that maps points in the complex plane to the unit disk. The conjugate Poisson kernel is given by the formula:$P^*(z) \;=\; P\left(\frac{1}{z^*}\right)$ where $P(z)$ is the Poisson kernel and z^* is the complex conjugate of z.

9.6 Keywords

Monodromy Theorem:

Let D be a simply connected domain, and suppose $f_0\,(z)$ is analytic in a domain $D_0 \subset D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.

9.7 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Unit 10 - Order of entire function

Purpose and Objectives:

The Mean Value Property states that for a harmonic function in a domain, the average value of the function over a ball is equal to the value of the function at the center of the ball. Harmonic functions are a type of function that satisfy the mean value property. Hence, the mean value property is a necessary condition for a function to be harmonic.

In other words, harmonic functions are functions that have the property that the mean of their values over a small region is equal to the value of the function at a point in the interior of that region. The mean value property is a fundamental property of harmonic functions, and it plays a key role in various applications, such as potential theory and partial differential equations.

Harnack's inequality is a fundamental result in mathematics with various applications in several areas, including partial differential equations, geometry, and potential theory. It provides a relationship between the values of a harmonic function on a small ball and on a large one, which is useful in the study of the regularity and behavior of solutions to elliptic equations. Additionally, Harnack's inequality is also crucial in the study of the asymptotic behavior of Markov processes, stochastic differential equations, and other areas in probability theory.

The Dirichlet problem is a well-known problem in mathematics, specifically in the field of partial differential equations. It asks to find a solution to a partial differential equation that satisfies certain boundary conditions on a given domain. The problem is named after the German mathematician Peter Gustav Lejeune Dirichlet and has numerous applications in physics, engineering, and mathematics. It provides a way to model various physical phenomena such as heat conduction, diffusion, and potential flow.

After this unit students can be able to-

1. State and prove the Harnack's inequality.
2. Learn the mean value property of harmonic functions.
3. Solve the problem based on the Dirichlet problem.

Introduction

The Harnack's inequality is a result in mathematical analysis, which states that for a non-negative solution $u(x)$ of a linear elliptic partial differential equation in a domain, the maximum value of u in a ball centered at a point is bounded above by the average value of u over the same ball. Before embarking the concept of Harnack's inequality, first we discuss the relationship between mean value property and harmonic function. This result has important applications in the study of heat diffusion and potential theory we will understand the Dirichlet problem to find solutions to boundary value problems in these areas.

10.1 Relation Between Mean Value Property and Harmonic Functions

Harmonic function

A harmonic function is a real-valued function that satisfies Laplace's equation, which states that the sum of the second partial derivatives with respect to x and y is equal to zero.

Mathematically, for a function $u(x, y)$, the Laplace's equation can be expressed as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.1}$$

Harmonic functions have several important properties, including being analytic and continuous, having no local maxima or minima, and having a unique mean value over any region in which they are defined. These properties make harmonic functions useful in a variety of mathematical and scientific applications, such as solving boundary value problems and modeling physical phenomena.

The harmonic conjugate of a harmonic function

The harmonic conjugate of a harmonic function u is another harmonic function v that satisfies the condition $u + iv$ is analytic (i.e., it has continuous first and second partial derivatives). In other words, u and v together form a complex function that is analytic in the region where u is defined.

The harmonic conjugate of u is unique up to an additive constant, and it can be found by integrating the derivative of u with respect to y (or x, if u is expressed in terms of x).

For example, if $u(x, y) = f(x) + g(y)$, then its harmonic conjugate is given by $v(x, y) = -g(x) + f(y) + C$, where C is an arbitrary constant.

In conclusion, the harmonic conjugate of a harmonic function is a unique function that helps to form an analytic complex function in the region where the harmonic function is defined.

The harmonic conjugate of an analytic function is another function that, when added to the original function, forms a harmonic function. A harmonic function is a function that satisfies Laplace's equation, which states that the sum of the second partial derivatives with respect to x and y is zero.

Let $f(x, y)$ be an analytic function. Its harmonic conjugate, denoted by $g(x, y)$, is defined as:

$$g(x, y) = \partial y\, u(x, y) - \partial x\, v(x, y)$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(x, y)$, respectively.

The two functions, $f(x, y)\ and\ g(x, y)$, are called Cauchy-Riemann partners, and their sum is a harmonic function.

Mean value property.

The mean value property of harmonic functions states that, for any point in a ball in a harmonic function, the value of the function at that point is equal to the average of the function's values over the boundary of the ball.

In mathematical terms, if $u(x)$ is a harmonic function in a ball $B(x_0, R)$ centered at x_0 with radius R, then:

$$u(x_0) = \left(\frac{1}{|B(x_0, r)|}\right) \int_{\{B(x_0, r)\}} u(x)\, dx \tag{4.2}$$

where $|B(x_0, r)|$ is the measure (area or volume, depending on the dimension) of the ball.

In other words, continuous function $u : G \to \mathbb{R}$ has the Mean Value Property (MVP) if whenever.

$$\bar{B}\,(a = x_0;\, R) \subset G, u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right) d\theta \tag{4.3}$$

This property provides a useful tool for solving partial differential equations and finding potential functions in physics.

Proof:

Let $u: G \to \mathbb{R}$ be a harmonic function and let $\bar{B}\,(a : R)$ be a closed disk contained in G. If C is the circle, $|z - a| = R$ then then $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right) d\theta$

The proof of the Mean Value Theorem is since a harmonic function is equal to its mean over any region. This means that the average value of the function over the boundary of a disk is equal to the value of the function at the center of the disk.

To prove this, we start by noting that a harmonic function is analytic, meaning it satisfies the Cauchy-Riemann equations and can be represented by a power series. Using this representation, we can write the function as:

$$u(z) = u(a) + \sum_{n=1}^{\infty}(z-a)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) \tag{4.4}$$

where $\frac{d^n u(a)}{dz^n}$ is the nth derivative of u evaluated at a.

Next, we consider the value of the function at a point on the boundary of the disk, given by $a + Re^{i\theta}$. Using this, we can rewrite the above power series as:

$$u\left(a + Re^{i\theta}\right) = u(a) + \sum_{n=1}^{\infty}\left(a + Re^{i\theta} - a\right)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) \tag{4.5}$$

$$\Rightarrow u\left(a + Re^{i\theta}\right) = u(a) + \sum_{n=1}^{\infty}\left(Re^{i\theta}\right)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right) \tag{4.6}$$

Now, we integrate both sides over the interval $[0, 2\pi]$ to obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta = u(a) + \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(Re^{i\theta}\right)^n * \frac{1}{n!}\left(\frac{d^n u(a)}{dz^n}\right)d\theta \tag{4.7}$$

Since the function u is harmonic, it follows that all its derivatives are also harmonic.

This means that the second term on the right side is equal to zero. Therefore, we can simplify the above expression to:

$$\frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta = u(a) \tag{4.8}$$

Thus, the average value of the function over the boundary of the disk is equal to the value of the function at the center of the disk, proving the Mean Value Theorem.

10.2 Harnack's inequality

Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,; R) = \{z : |z - z_0| < R\}$,with $u(z) \geq 0$

for all $z \in \Delta(z_0\,; R)$, then for every z in this disk, we have

$$u(z_0)\frac{R - |z - z_0|}{R + |z - z_0|} \leq u(z) \leq u(z_0)\frac{R + |z - z_0|}{R - |z - z_0|}$$

Proof:

First, we consider the average value of the function u on the circle centered at z0 with radius $|z - z_0|$. By definition, this average value is given by

$$A(|z - z_0|) = \left(\frac{1}{2\pi}\right)\int_0^{2\pi} u\left(z_0 + |z - z_0|e^{it}\right)dt\,. \tag{4.9}$$

Next, we apply the Mean Value Property for Harmonic Functions to the function $u - u(z_0)$, which states that for any positive real number r such that $0 < r < R$, there exists a point θ in the interval $[0, 2\pi)$ such that

$$u\left(z_0 + re^{i\theta}\right) - u(z_0) = r\,\partial_r u\left(z_0 + re^{i\theta}\right) \tag{4.10}$$

Substituting this expression into the formula for $A(|z - z_0|)$ and interchanging the order of integration and differentiation, we obtain

$$\begin{aligned} A(|z - z_0|) &= \left(\frac{1}{2\pi}\right)\int_0^{2\pi} u\left(z_0 + |z - z_0|e^{it}\right)dt \\ &= \left(\frac{1}{2\pi}\right)\int_0^{2\pi}\left[u(z_0) + |z - z_0|\partial_r\, u\left(z_0 + |z - z_0|e^{it}\right)\right]dt \\ &= u(z_0) + |z - z_0|\partial_r\, u(z_0). \end{aligned}$$

Finally, using the definition of partial derivative with respect to the radial coordinate, we have

$$\partial_r\, u(z_0) = \left(\frac{1}{2}\right)\frac{u(z_0 + R) - u(z_0 - R)}{R} = \frac{A(R)}{R}. \tag{4.11}$$

Substituting this expression into the formula for $A(|z - z_0|)$, we obtain

$$A(|z - z_0|) = u(z_0) + \frac{|z - z_0| A(R)}{R}. \tag{4.12}$$

Dividing both sides by $|z - z_0|$ and rearranging, we find that

$$\frac{u(z_0)(R - |z - z_0|)}{R + |z - z_0|} \leq A(|z - z_0|) \leq \frac{u(z_0)(R + |z - z_0|)}{R - |z - z_0|}. \tag{4.13}$$

Since the average value of u on the circle centered at z_0 with radius $|z - z_0|$ provides an upper bound for the function u, we conclude that

$$\frac{u(z_0)(R - |z - z_0|)}{R + |z - z_0|} \leq u(z) \leq \frac{u(z_0)(R + |z - z_0|)}{R - |z - z_0|} \tag{4.14}$$

for every z in the disk $\Delta(z_0, R)$.

10.3 Dirichlet Problem

The Dirichlet problem in complex analysis is a boundary value problem that seeks to find a complex valued function that is analytic within a given domain and takes on specified boundary values on the boundary of that domain.

For example, consider the unit disk centered at the origin in the complex plane. The Dirichlet problem asks us to find a complex valued function $f(z)$ that is analytic within the unit disk and takes on the specified boundary value $f(e^{it}) = g(t)$ for all t in the interval $[0,2\pi]$, where $g(t)$ is a given function.

One possible solution to this problem is to use the theory of complex analysis and the representation of analytic functions using power series.

By using the Cauchy-Riemann equations, it can be shown that any complex valued function that is analytic within the unit disk can be represented by a power series of the form.

$f(z) = \Sigma\, a_n (z - z_0)^n$, where z_0 is the center of the disk.

The boundary values of the function can then be used to determine the coefficients of the power series.

For example, if $g(t) = \cos(t)$,

then $f\left(e^{it}\right) = \cos(t)$

$= \Sigma\, a_n\, e^{int}$,

where the coefficients can be calculated by matching the real and imaginary parts of both sides of the equation.

Initially, the problem was to determine the equilibrium temperature distribution on a disk from measurements taken along the boundary.

The temperature at points inside the disk must satisfy a partial differential equation called Laplace's equation corresponding to the physical condition that the total heat energy contained in the disk shall be a minimum.

A slight variation of this problem occurs when there are points inside the disk at which heat is added (sources) or removed (sinks) as long as the temperature remains constant at each point (stationary flow), in which case Poisson's equation is satisfied.

How to construct a harmonic function in a given domain when its values are prescribed on the boundary of the domain is the key problem is known as Dirichlet problem.

Boundary value problems associated to Laplace equation.

The Poisson equation is a second-order partial differential equation of the form

$$\nabla^2 u(x) = f(x)$$

where $u(x)$ is an unknown function and $f(x)$ is a given function. The equation states that the Laplacian of $u(x)$ is equal to $f(x)$.

The solution to the Poisson equation depends on the boundary conditions for the unknown function $u(x)$.

There are several methods for solving the Poisson equation, including numerical methods, analytical methods, and Green's function methods.

One common analytical method is to use the method of separation of variables.

Suppose that $u(x) = X(x)Y(y)$, then the Laplacian of $u(x)$ becomes.

$$\nabla^2 u(x) = \frac{\partial^2 u(x)}{\partial x^2} + \frac{\partial^2 u(x)}{\partial y^2}$$

$$= X''(x)Y(y) + X(x)Y''(y)$$

where $X''(x)$ *and* $Y''(y)$ denote the second derivatives with respect to x and y, respectively. Setting the right-hand side equal to $f(x)$, we have

$$X''(x)Y(y) + X(x)Y''(y) = f(x)$$

Dividing both sides by XY, we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{f(x)}{XY}$$

This equation is equal to a constant λ, so we have

$$\frac{X''(x)}{X(x)} = -\lambda \frac{Y''(y)}{Y(y)} = -\lambda + \frac{f(x)}{XY}$$

Solving the above two differential equations, we obtain the general solution.

$$u(x) = \sum C_n X_n(x) Y_n(y)$$

where C_n are constants and $X_n(x)$ *and* $Y_n(y)$ are the eigenfunctions corresponding to the eigenvalue λ_n

The final solution depends on the specific boundary conditions and the values of the constants C_n.

Note that this method is only applicable when the equation can be separated into two independent ordinary differential equations. In general, the Poisson equation requires numerical methods or Green's function methods to solve.

Another of the generic partial differential equations is Laplace's equation, $\nabla^2 u = 0$

This equation first appeared in the unit on complex variables when we discussed harmonic functions. Another example is the electric potential for electrostatics. As we described for static electromagnetic fields,

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, E = \nabla\phi.$$

In regions devoid of charge, these equations yield the Laplace equation $\nabla^2\phi = 0$.

Another example comes from studying temperature distributions.

Consider a thin rectangular plate with the boundaries set at fixed temperatures. Temperature changes of the plate are governed by the heat equation. The solution of the heat equation subject to these boundary conditions is time dependent.

In fact, after a long period of time the plate will reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature.

Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, which is another Laplace equation, $\nabla^2 u = 0$

Example

Equilibrium temperature distribution for a rectangular plate

Let us consider Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < H \tag{4.15}$$

with the boundary conditions

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(\pi,y) &= 0 \\ u(x,0) = f(x) &= Sinx \\ u(x,H=1) &= \frac{Sinx}{e} \end{aligned}\right\} \tag{4.16}$$

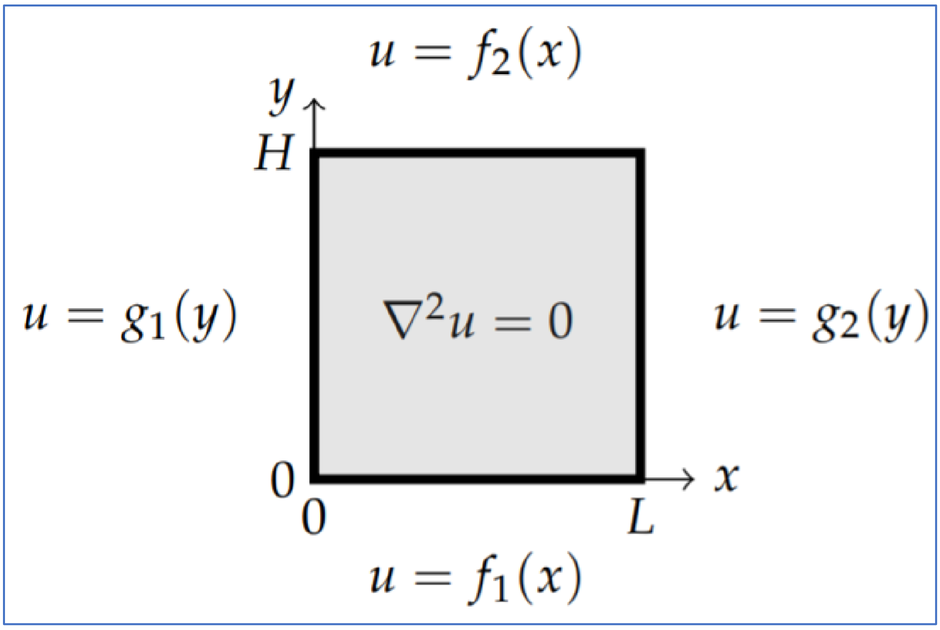


Figure 4.1: The boundary condition for the heat distribution problem.

Solution:

This is a partial differential equation for Laplace's equation, which describes the distribution of heat in a 2D space. To solve this equation, we can use separation of variables method.

Assume that the solution can be written as:

$$u(x,y) = X(x)\,Y(y)$$

Substituting this into the equation, we have:

$$(X(x)\,Y(y))'' + (X(x)\,Y(y))'' = 0$$

Dividing both sides by $X(x)\,Y(y)$, we get:

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

Since this must be true for all x and y, we can divide both sides by $X(x)\,Y(y)$, to get:

$$\left(\frac{X''(x)}{X(x)}\right) + \left(\frac{Y''(y)}{Y(y)}\right) = 0$$

This can be simplified to:

$$\lambda^2 = -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)}$$

where λ^2 is a constant.

Solving for $X(x)$ and $Y(y)$, we have:

$$X''(x) + \lambda^2 X(x) = 0$$
$$Y''(y) - \lambda^2 Y(y) = 0$$

The solutions for $X(x)$ and $Y(y)$ can be written as:

$X(x) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x)$

$Y(y) = C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}$

where $A, B, C, and\ D$ are constants.

Hence $u(x, y) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}]$ (4.17)

Using the boundary conditions, we can find the values of λ and the coefficients $A, B, C, and\ D$.

$$\Rightarrow u(0, y) = A\,cos(\lambda * 0) + B\,sin(\lambda * 0).\left[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}\right] = 0$$
$$\Rightarrow u(0, y) = A\,.\left[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}\right] = 0$$
$$\Rightarrow A = 0$$

Now put $A = 0$ in (4.17)

$$u(x, y) = B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}] \tag{4.18}$$

Now

$$u(\pi, y) = B\,sin(\lambda\,\pi).\left[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}\right] = 0$$
$$\Rightarrow sin(\lambda\,\pi) = 0$$
$$\Rightarrow sin(\lambda\,\pi) = \sin(n\pi)$$

$\Rightarrow \lambda = \mathrm{n}, \mathrm{n} = \pm 1, \pm 2, \ldots,$

So

$$u(x, y) = B\,sin(n\,x).[C\,e^{-n\,y} + D\,e^{n\,y}] \tag{4.19}$$

Now apply $u(x, 0) = Sinx$

$$u(x, 0) = B\,sin(n\,x).[C + D] = sinx$$
$$\Rightarrow BC + BD = 1, n = 1$$

Now update the (4.19) Hence

$$u(x, y) = B\,sinx.[C\,e^{-y} + D\,e^{y}] \tag{4.20}$$

Now apply $u(x, 1) = \frac{Sinx}{e}$

$$u(x, 1) = B\,sinx.[C\,e^{-1} + D\,e^{1}] = \frac{sinx}{e}$$
$$\Rightarrow BC\frac{sinx}{e} + BD.sinx.e = \frac{sinx}{e} + 0$$
$$\Rightarrow BC = 1, BD = 0$$

Now update the (4.20)(4.19) Hence

$u(x, y) = sinx.[\,e^{-y}\,]$ is the final solution of the given temperature distribution for a rectangular plate.

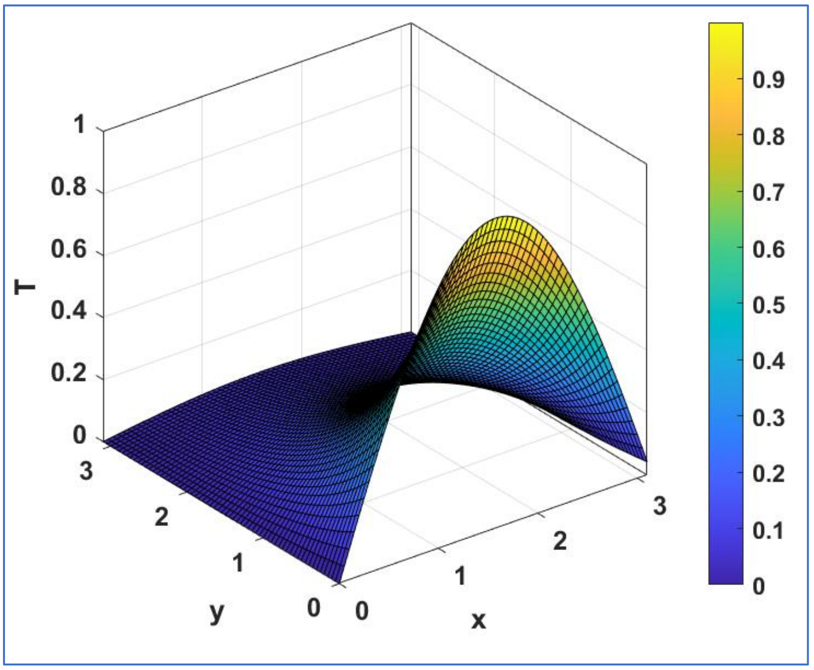


Figure4.2: The temperature distribution in x and y direction.

10.4 Review questions

1. Explain the Mean value property of harmonic function?
2. State and prove the Harnack's inequality for harmonic function in the closed disc?
3. Suppose $u(z)$ is harmonic in the disk $\Delta(1\,;\,1) = \{z : |z-1| < 1\}$,with $u(z) \geq 0$ for all $z \in \Delta(1\,;\,R)$, then for every z in this disk, then show that

$$u(1)\frac{1-|z-1|}{1+|z-1|} \leq u(z) \leq u(1)\frac{1+|z-1|}{1-|z-1|}$$

4. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < H$,under the boundary conditions

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(\pi,y) &= 0 \\ u(x,0) = f(x) &= 2Sinx \\ u(x,1) &= \frac{Sinx}{e} \end{aligned}\right\}$$

5. Solve, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < 10, 0 < y < H$, with the boundary conditions

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(\pi,y) &= 0 \\ u(x,0) &= 10Sinx \\ u(x,10) &= 10\frac{Sinx}{e} \end{aligned}\right\}$$

10.5 Self-assessment

1. Using the Mean Value property, the integral $\frac{1}{2\pi}\int_0^{2\pi} u\left(5 + Re^{i\theta}\right)d\theta =$

   A. $u(5)$
   B. $u\left(\frac{5}{2}\right)$
   C. $u(25)$

D. $u(10)$

2. Using the Mean Value property, the integral $\frac{1}{2\pi}\int_0^{2\pi} u\left(15 + Re^{i\theta}\right)d\theta =$

A. $u(15)$
B. $u\left(\frac{15}{2}\right)$
C. $u(225)$
D. $u(30)$

3. Using the Mean Value property, the integral $\frac{1}{2\pi}\int_0^{2\pi} u\left(1 + i + Re^{i\theta}\right)d\theta =$

A. $u(1)$
B. $u(i)$
C. $u(1 + i)$
D. $u\left(\frac{1}{1+i}\right)$

4. Using the Mean Value property, which one of the following is true

A. $\frac{1}{2\pi}\int_0^{2\pi} u\left(10 + Re^{i\theta}\right)d\theta = u(10)$
B. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5 + Re^{i\theta}\right)d\theta = u(5/2)$
C. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5/2 + Re^{i\theta}\right)d\theta = u(5)$
D. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5 + Re^{i\theta}\right)d\theta = u(10)$

5. Using the Mean Value property, which one of the following is true

A. $\frac{1}{2\pi}\int_0^{2\pi} u\left(1 + Re^{i\theta}\right)d\theta = u(10)$
B. $\frac{1}{2\pi}\int_0^{2\pi} u\left(25 + Re^{i\theta}\right)d\theta = u(5/2)$
C. $\frac{1}{2\pi}\int_0^{2\pi} u\left(5/2 + Re^{i\theta}\right)d\theta = u(5)$
D. $\frac{1}{2\pi}\int_0^{2\pi} u\left(2 + i + Re^{i\theta}\right)d\theta = u(2 + i)$

6. Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,;\,R) = \{z : |z| < 1\}$,with $u(z) \geq 0$ for all $z \in \Delta(0\,;\,1)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0)\frac{1 - |z|}{1 + |z|} \leq u(z)$$

$$II: u(z) \leq u(0)\frac{1 + |z|}{1 - |z|}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

7. Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,;\,R) = \{z : |z| < 2\}$,with $u(z) \geq 0$ for all $z \in \Delta(0\,;\,2)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0)\frac{2 - |z|}{2 + |z|} \leq u(z)$$

$$II: u(z) \geq u(0)\frac{2+|z|}{2-|z|}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

8. Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,;\,R) = \{z : |z-i| < 1\}$,with $u(z) \geq 0$ for all $z \in \Delta(i\,;\,1)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(i)\frac{1-|z|}{1+|z|} \leq u(z)$$

$$II: u(z) \leq u(i)\frac{1+|z|}{1-|z|}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

9. Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,;\,R) = \{z : |z| < 5\}$,with $u(z) \geq 0$ for all $z \in \Delta(0\,;\,5)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0)\frac{5-|z|}{5+|z|} > u(z)$$

$$II: u(z) \leq u(0)\frac{5+|z|}{5-|z|}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

10. Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,;\,R) = \{z : |z-2| < 1\}$,with $u(z) \geq 0$ for all $z \in \Delta(0\,;\,1)$, then which one of the following statements are true using the Harnack's inequality for every z in this disk,

$$I: u(0)\frac{1-|z-2|}{1+|z-2|} \leq u(z)$$

$$II: u(z) \leq u(0)\frac{1+|z-2|}{1-|z-2|}$$

A. $Only\ I$
B. $Only\ II$
C. $Both$
D. $Neither\ I\ nor\ II$

11. Consider the $u(x,y) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of B using $u(0,y) = 0$.
A. 0
B. 2π
C. 3

D. 1

with the boundary conditions $\left.\begin{array}{r} u(0,y)=0. \\ u(\pi,y)=0 \\ u(x,0)=f(x)=Sinx \\ u(x,H=1)=\frac{Sinx}{e} \end{array}\right\}$

12. Consider the $u(x,y) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of λ using $u(0,y) = 0,, u(\pi,y) = 0$.
   A. $\lambda = 0.5$
   B. $\lambda = \mathrm{n}, \mathrm{n} = \pm 1, \pm 2, \ldots,$
   C. $\lambda = \frac{3}{2}$
   D. $Can\ not\ be\ determined$

13. Consider the $u(x,y) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of BC with the boundary conditions $\left.\begin{array}{r} u(0,y)=0. \\ u(\pi,y)=0 \\ u(x,0)=f(x)=Sinx \\ u(x,H=1)=\frac{Sinx}{e} \end{array}\right\}$
   A. 0.5
   B. 1
   C. $\frac{3}{2}$
   D. $Can\ not\ be\ determined$
   E.

14. Consider the $u(x,y) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the value of BD with the boundary conditions $\left.\begin{array}{r} u(0,y)=0. \\ u(\pi,y)=0 \\ u(x,0)=f(x)=Sinx \\ u(x,H=1)=\frac{Sinx}{e} \end{array}\right\}$
   A. 0.5
   B. 1
   C. $\frac{3}{2}$
   D. 0

15. Consider the $u(x,y) = A\,cos(\lambda\,x) + B\,sin(\lambda\,x).[C\,e^{-\lambda\,y} + D\,e^{\lambda\,y}]$ be the solution of Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < L, 0 < y < 1$ in Cartesian coordinates then what is the general solution of with the boundary conditions $\left.\begin{array}{r} u(0,y)=0. \\ u(\pi,y)=0 \\ u(x,0)=f(x)=Sinx \\ u(x,H=1)=\frac{Sinx}{e} \end{array}\right\}$
   A. $u(x,y) = \ Sinx.[\,e^{-y}\,]$
   B. $u(x,y) = \ Cosx.[\,e^{-y}\,]$
   C. $u(x,y) = \ Sinx.[\,e^{y}\,]$
   D. $u(x,y) = \ Cosx.[\,e^{y}\,]$

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	A
3	C
4	A
5	D
6	C
7	A
8	C
9	B
10	**10.1C**
11	**10.2A**
12	B
13	C
14	D
15	A

10.6 Summary

- **The mean value property of harmonic functions**:
  For any point in a ball in a harmonic function, the value of the function at that point is equal to the average of the function's values over the boundary of the ball.

In mathematical terms, if $u(x)$ is a harmonic function in a ball $B(x_0, R)$ centered at x_0 with radius R, then: $u(x_0) = \left(\frac{1}{|B(x_0,r)|}\right) \int_{\{B(x_0,r)\}} u(x)\, dx$

where $|B(x_0, r)|$ is the measure (area or volume, depending on the dimension) of the ball.

- The average value of the function over the boundary of the disk is equal to the value of the function at the center of the disk, proving the Mean Value Theorem.

10.7 Keywords

Harnack's inequality

Suppose $u(z)$ is harmonic in the disk $\Delta(z_0\,;\,R) = \{z : |z - z_0| < R\}$,with $u(z) \geq 0$ for all $z \in \Delta(z_0\,;\,R)$, then for every z in this disk, we have

$$u(z_0)\frac{R-|z - z_0|}{R+|z - z_0|} \le u(z) \le u(z_0)\frac{R+|z - z_0|}{R-|z - z_0|}$$

10.8 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Unit 11 - Open mapping theorem

Purpose and Objectives:

After this unit students can be able to-

1. Understand the Schwarz Reflection Principle for analytic functions?
2. Prove the Schwarz Reflection Principle for analytic functions?
3. Learn the consequences of the Schwarz Reflection Principle

Introduction

If a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane. In other words, if a function f(z) is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

The Schwarz Reflection Principle has several important applications in complex analysis, such as proving the analyticity of functions, constructing entire functions with prescribed properties, and solving boundary value problems. In this unit we will explore the Schwarz Reflection Principle for analytic function.

11.1 Schwarz Reflection Principle for Analytic Functions

Statement:

The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\overline{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.

or

The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane. In other words, if a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

Proof 1:

The proof of the Schwarz Reflection Principle relies on the fact that if a function is analytic in the upper half plane and its real part is non-negative on the boundary, then it can be represented as a real part of another analytic function.

Let $f(z)$ be analytic in the upper half plane and $Re\big(f(z)\big) \geq 0$ for all real z.

Then the function $g(z) = f(z) + i(-f(z))$ is analytic in the upper half plane and satisfies $Re(g(z)) = 0$ for all real z.

The proof also uses the maximum modulus principle and Liouville's theorem.

Liouville's Theorem

Liouville's Theorem states that a bounded holomorphic function on the entire complex plane must be constant. It is named after Joseph Liouville.

Statement:

Let $f(z): C \to C$ be an entire function. Suppose there exists some real number$M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

Or

If a function $f(z)$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof:

It is given that

i. A function $f(z)$ is analytic in the entire complex plane
ii. A function $f(z)$ is bounded, that $|f(z)| \leq M$.

Let us consider two points $a \; and \; b$ inside a particular domain(See the figure 5.1).

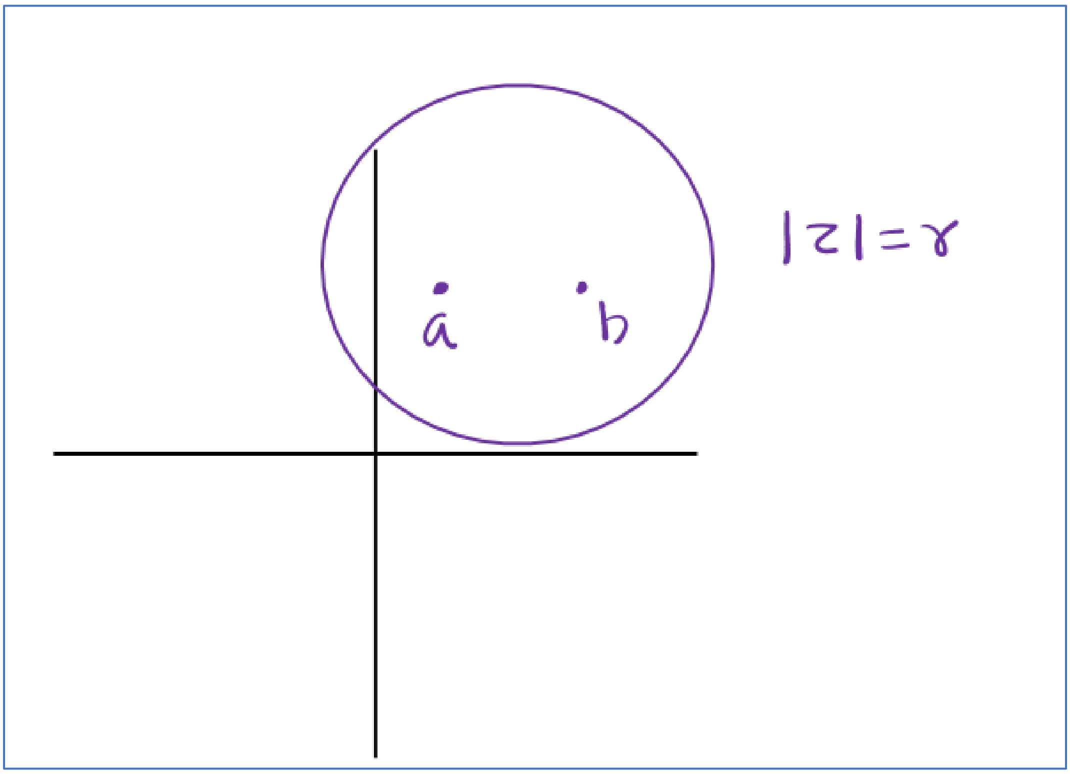


Figure 5.1: Two points a and b inside a particular domain

Then using Cauchy integral formula

$$\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-a}\right) dz = f(a)$$

$$\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-b}\right) dz = f(b)$$

If $f(z)$ is constant throughout the domain, then $f(a) = f(b)$.

Now let's prove $f(a) - f(b) = 0$.

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-a}\right) dz - \frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-b}\right) dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-a} - \frac{f(z)}{z-b}\right) dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c f(z)\left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c f(z)\left(\frac{z-b-z+a}{(z-a)(z-b)}\right) dz$$

$$f(a)-f(b)=\frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz$$

$$|f(a)-f(b)|=\left|\frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz\right|$$

$$|f(a)-f(b)|\leq\left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{|(z-a)(z-b)|}\right)|dz|$$

$$|f(a)-f(b)|\leq\left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{(|z|-|a|)(|z|-|b|)}\right)|dz|$$

Let

$z=re^{i\theta}$

$dz=re^{i\theta}.i.d\theta$

$|dz|=|re^{i\theta}.i.d\theta|$

$|dz|=|r|.|e^{i\theta}|.|i|.|d\theta|$

Here $|r|=r$

$|e^{i\theta}|=|cos\theta+isin\theta|=1,$

$|i|=1,$

$|dz|=r.|d\theta|$

$$|f(a)-f(b)|\leq\frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(r-a)(r-b)}\right)r.|d\theta|$$

$$|f(a)-f(b)|\leq\frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(1-a/r)(1-b/r)}\right).|d\theta|$$

If $f(z)$ is analytic in the entire complex plane, then $|z|=r\to\infty$. So

$|f(a)-f(b)|\leq 0$

$f(a)-f(b)=0$

Hence, we can say that $f(a)=f(b)$. It means that $f(z)$ is a constant.

Liouville's Theorem proof using Cauchy integral formula for derivatives.

If f(z) is analytic in a simply connected region then at any interior point of the region, z_0 inside *C*. Then say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$\oint_c\left(\frac{f(z)}{(z-z_0)^{n+1}}\right)dz=2\pi i\frac{f^n(z_0)}{n!}.$

where C is any simple closed curve, in the region, which encloses z_0. Note the case $n=1$:

$\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{(z-z_0)^2}\right)dz=f'(z_0).$

$|f'(z_0)|=\left|\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{(z-z_0)^2}\right)dz\right|.$

$|f'(z_0)|\leq\left|\frac{1}{2\pi i}\right|\oint_c\left(\frac{|f(z)|}{|(z-z_0)^2|}\right)|dz|.$

Here $z=re^{i\theta}$

$dz=re^{i\theta}.i.d\theta.$

$|dz|=|re^{i\theta}.i.d\theta|.$

$|dz|=|r|.|e^{i\theta}|.|i|.|d\theta|.$

Here $|z-z_0|=r$

$|e^{i\theta}|=|cos\theta+isin\theta|=1,$

$|i|=1,$

$|dz| = r.|d\theta|.$

$|f'(z_0)| \leq \frac{1}{2\pi} \oint_c \left(\frac{M}{r^2}\right) r.|d\theta|.$

$|f'(z_0)| \leq \frac{1}{2\pi} \oint_c \left(\frac{M}{r}\right).d\theta.$

If $f(z)$ is analytic in the entire complex plane then $r \to \infty$. So

$|f'(z_0)| \leq 0$

$f'(z_0) = 0$

$f(z) =$ constant.

By Liouville's theorem, the imaginary part of $g(z)$ is constant on the boundary, say c. Then the function $h(z) = g(z) + ic$ is analytic in the entire plane and has the same real part as $f(z)$.

Proof 2:

Let $f(z)$ be a complex valued function that is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$).

Consider a point z in the lower half plane ($Im(z) < 0$)[See figure 5.2]

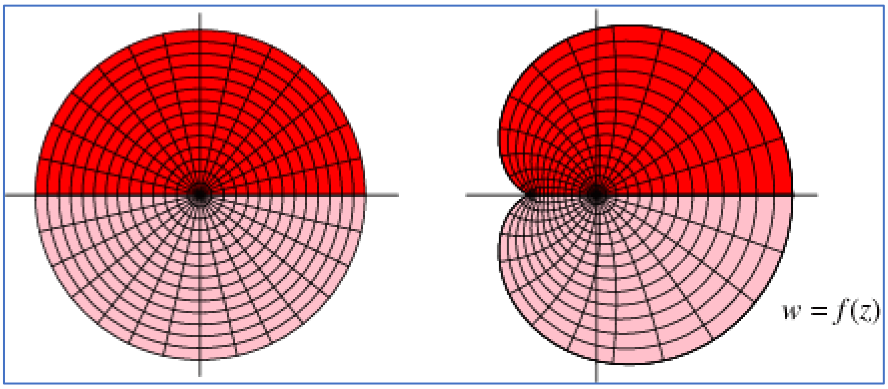


Figure 5.2: $w = f(z)$

Let $z = x + iy$, where x is real and y is negative.

Let's define a new point, $conj(z)$=$\bar{z}$, which is equal to the complex conjugate of z.

That is, $conj(z) = x - iy$.

Since $f(z)$ is continuous on the boundary of the upper half plane, it follows that $f(conj(z))$ is continuous in the lower half plane.

Also, since $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, we have:

$f(z) = \overline{f((conj(z)))}$

$= \overline{f(x - iy)}$

$= \overline{f(x + i(-y))}$

Thus, we can define a new function, $g(z)$, in the lower half plane as follows:

$g(z) = \overline{f(x + i(-y))}$

Since $g(z) = \overline{f(z)}$ is continuous in the lower half plane, and the conjugate of a continuous function is continuous, it follows that $g(z)$ is continuous in the lower half plane.

We now show that $g(z)$ is also analytic in the lower half plane.

Let $z = x + iy$, where x is real and y is negative.

Consider the derivative of $g(z)$ at the point z:

$g'(z) = \left(\frac{d}{dz}\right) \overline{f(x + \imath(-y))}$

$= \left(\frac{d}{dz}\right) \overline{f(z)}$

$= \overline{\left(\frac{d}{dz}\right) f(z)}$

Since $f(z)$ is analytic in the upper half plane, it follows that $\left(\frac{d}{dz}\right) f(z)$ is also analytic in the upper half plane.

Therefore, conjugate of $\left(\frac{d}{dz}\right) f(z)$ is also analytic in the lower half plane, and so is $g'(z)$.

Since $g(z)$ is continuous and its derivative is analytic in the lower half plane, it follows that $g(z)$ is analytic in the lower half plane.

Thus, we have shown that if $f(z)$ is a complex valued function that is analytic in the upper half plane and continuous on the boundary of the upper half plane, and if $f(z)$ satisfies:

$f(z) = \overline{f(conj(z))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane by defining a new function, $g(z)$, in the lower half plane.

11.2 consequences of the Schwarz Reflection Principle

1. One consequence of the Schwarz Reflection Principle is that if a function is analytic in the upper half plane and its real part is non-negative on the boundary, then it can be extended to an entire function that is real valued on the real axis.
2. Another consequence is that a function that is analytic in the upper half plane and satisfies a certain growth condition on the boundary (such as the Riemann mapping theorem) can be extended to an entire function with similar growth behavior.
3. Additionally, the Schwarz Reflection Principle can be used to construct solutions to boundary value problems, such as the Dirichlet problem, by reflecting solutions from one half plane to the other.

11.3 Different proofs of Schwartz Reflection Principle

The Schwartz Reflection Principle can be proved by various methods

1. Complex Analysis Proof: The Schwartz Reflection Principle can be proven using complex analysis by considering the analytic continuation of the function from the upper half plane to the lower half plane. The proof involves showing that the function, extended to the lower half plane, is a reflection of the function in the upper half plane across the real axis.

2. Harmonic Functions Proof: The Schwartz Reflection Principle can also be proven using the theory of harmonic functions. A function is considered harmonic if it satisfies Laplace's equation. By assuming that the function is harmonic in the upper half plane, it can be shown that its extension to the lower half plane is also harmonic, and therefore satisfies Laplace's equation, meaning it must be a reflection of the function in the upper half plane across the real axis.

3. Integral Transform Proof: The Schwartz Reflection Principle can be proven using the Fourier Transform by showing that the Fourier Transform of a function in the upper half plane, after being reflected across the real axis, is equal to the negative Fourier Transform of the original function in the lower half plane.

4. Paley-Wiener Theorem Proof: The Schwartz Reflection Principle can also be proven using the Paley-Wiener theorem, which states that the Fourier Transform of a function with compact support is a function that is entire and decays rapidly. By assuming that the function in question is the Fourier Transform of a function with compact support in the upper half plane, it can be shown that the function, after being reflected across the real axis, is the Fourier Transform of a function with compact support in the lower half plane.

5. Bochner's Theorem Proof: The Schwartz Reflection Principle can also be proven using Bochner's theorem, which states that a positive definite function is the Fourier Transform of a positive measure. By assuming that the function in question is positive definite in the upper half plane, it can be shown that the function, after being reflected across the real axis, is positive definite in the lower half plane, implying that it is the Fourier Transform of a positive measure.

11.4 Applications

- The main application of the Schwartz Reflection Principle is in the study of distributions and their derivatives. It provides a means to extend the definitions of distributions and derivatives to unbounded functions.
- The Schwartz Reflection Principle is a generalization of the Hahn-Banach Theorem. The Hahn-Banach Theorem states that a linear functional on a linear subspace can be extended to the entire space while preserving its norm. The Schwartz Reflection Principle extends this result to the case of distributions.
- The Schwartz Reflection Principle is an important tool in mathematical physics for defining distributions and derivatives of functions. In particular, it allows for the extension of the definitions of distributions and derivatives to unbounded functions, which is particularly useful in quantum field theory and quantum mechanics.

11.5 Review questions

1. What is the main application of the Schwartz Reflection Principle?
2. How does the Schwartz Reflection Principle relate to the Hahn-Banach Theorem?
3. What is the significance of the Schwartz Reflection Principle in mathematical physics?
4. State and prove the Schwartz Reflection Principle using Liouville's Theorem?
5. State and prove the Schwartz Reflection Principle without Liouville's Theorem ?

11.6 Self-assessment

1. Which one the following statement is true for the reference of Schwarz reflection principle?

   I: The principle that a complex function which is holomorphic in the upper half plane can be extended to a holomorphic function in the whole plane.

II: The principle that states that a real-valued function cannot be analytically extended across a branch cut.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

2. Which one the following statement is true for the reference of Schwarz reflection principle?

I: The principle that a holomorphic function in the unit disc can be extended to a holomorphic function in the whole plane.

II: The principle that states that the maximum value of a subharmonic function is achieved on the boundary of its domain.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

3. Which one the following statement is true for the necessary condition for the Schwarz reflection principle?

I: The function must be bounded in the upper half plane.

II: The function must be continuous in the upper half plane

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

4. Which one the following statement is true for the necessary condition for the Schwarz reflection principle?

I: The function must be holomorphic in the upper half plane.

II: The function must be continuous in the upper half plane

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

5. The Schwarz reflection principle can be used to extend which type of functions to the whole plane?

A. Only real-valued functions
B. Only harmonic functions
C. Both real valued and harmonic
D. Neither real nor harmonic

6. The Schwarz reflection principle can be used to extend which type of functions to the whole plane?
    A. Holomorphic functions
    B. Harmonic functions
    C. Subharmonic functions
    D. Neither real nor harmonic

7. The Schwarz reflection principle states that the Fourier transform of the product of two signals is equal to the convolution of their Fourier transforms?
    A. True
    B. False

8. The principle that states that the reflection of a Schwartz function across the x-axis is also a Schwartz function?
    A. True
    B. False

9. The principle that states that the Laplace transform of a signal is equivalent to its Fourier transform?
    A. True
    B. False

10. The principle that states that the derivative of a Schwartz function is also a Schwartz function.
    A. True
    B. False

11. What is the Schwartz Reflection Principle in mathematics?

    A. The principle that every polynomial function has a unique root
    B. The principle that states that the boundary values of an analytic function on the upper half-plane can be extended to an analytic function on the whole complex plane
    C. The principle that states that the roots of a polynomial equation occur in conjugate pairs.
    D. The principle that the value of a holomorphic function at a point is equal to its average value over any small circle centered at that point.

12. Which of the following is the best definition of the Schwartz Reflection Principle?

    I: A mathematical theorem that states that the reflection of a function across a vertical line always results in an even function.
    II: A mathematical theorem that states that the reflection of a function across a vertical line always results in an odd function.
    A. Only I
    B. Only II
    C. Both I and II
    D. Neither I nor II

13. Which of the following is the best definition of the Schwartz Reflection Principle?

I: A mathematical theorem that states that the reflection of a function across a vertical line always results in a function with the same parity.

II: A mathematical theorem that states that the reflection of a function across a vertical line always results in a different function.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

14. The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.
A. True
B. False

15. Which one the following statement is true for the reference of Schwarz reflection principle?

I: The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane.

II: If a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re(f(z)) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	D
3	D
4	A
5	D
6	A
7	B

8	A
9	B
10	**B**
11	**B**
12	D
13	A
14	D
15	C

11.7 Summary

- The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.

- The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane.

- If a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

- If a function $f(z)$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

11.8 Keywords

Liouville's Theorem

Let $f(z)\colon C \to C$ be an entire function. Suppose there exists some real number $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

11.9 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Rajesh Kumar Chandrawat, Lovely Professional University

Unit 12 - Normal families of analytic functions

Purpose and Objectives:

Meromorphic functions are an important class of functions studied in complex analysis. They are defined as functions that are holomorphic (analytic) everywhere except at a finite number of isolated singularities. Meromorphic functions are useful in studying the behavior of complex functions near singularities, and they provide a representation of any meromorphic function in terms of its poles and their residues. After this unit students can be able to-

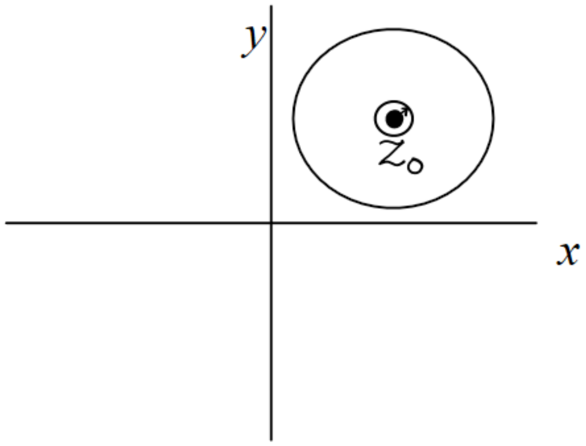
1. Understand the Meromorphic functions
2. State and prove the Mittag-Leffler theorem
3. Learn the infinite product of complex Numbers

Introduction

In this unit first we will understand the concept of singularities and poles for meromorphic function then the we will use the mesomorphic function to prove the Mittag-Leffler theorem. Last we will focus on the infinite product of complex Numbers.

12.1 Singularities

A point z_0 is called a singular point of a function $f(z)$ if $f(z)$ fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.



Example:

Behavior of following functions at $z = 0$.

$$f(z) = \frac{1}{z^9}$$

$$f(z) = \frac{Sinz}{z}$$

$$f(z) = \frac{e^z - 1}{z}$$

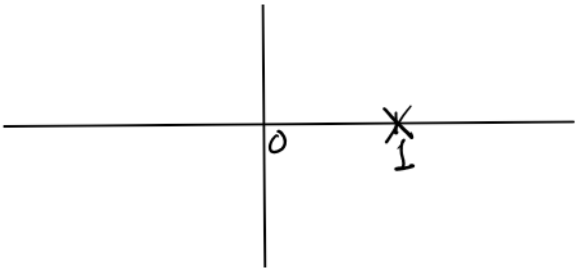
$$f(z) = \frac{1}{\sin(1/z)}$$

We observed that all the functions mentioned above are not analytic at $z = 0$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic.

Example:

Behavior of following function at $z = 1$.

$$f(z) = \frac{z}{1-z}$$



We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

Example:

$f(z) = z^2$ is analytic everywhere so it has no singular point.

Example:

Behavior of following function in the entire z plane

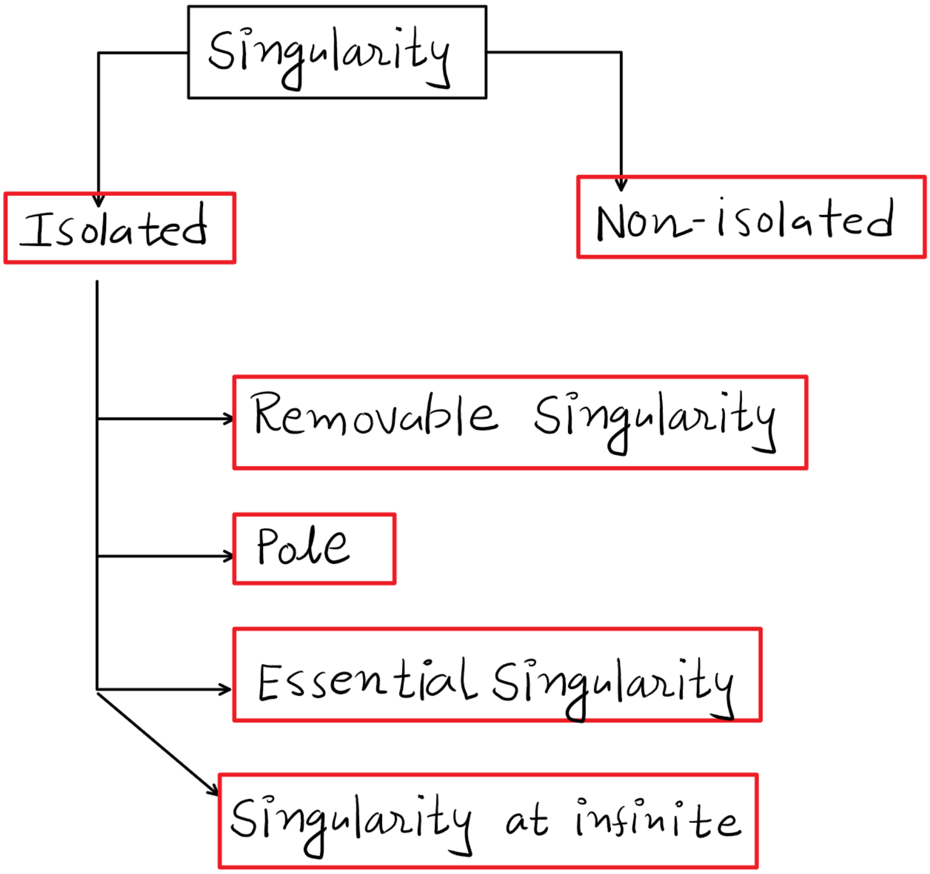
$f(z) = |z|^2$

We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

12.2 Classification of singularity

The singularity of a complex function can be classified into two groups, isolated and non-isolated. It can be done via Laurent series expension, but we can also classify the singularity without the Laurent series expension. In the forthcoming units we will consider the classification using the Laurent series.

The isolated singularity further can be classified into different type. The following diagram shows the different types of the singularities.



6.2.1 Isolated singularity

A point a is called an isolated singularity for $f(z)$ if $f(z)$ is not analytic at $z = a$ and there exist $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$. The neighbourhood $|z - a| < r$ contains no singularity of $f(z)$ except a.

Example:

$\boldsymbol{f(z) = \frac{z+1}{z^2(z^2+1)}}$ ***has three isolated singularities*** $\boldsymbol{z = 0, i, -i}$.

Example:

$f(z) = \frac{1}{sinz}$ has three isolated singularities $z = 0, \pm\pi, \pm 2\pi, \ldots,$

6.2.2 Removable singularity

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z \to z_0} f(z)$ exists then z_0 is the removable singularity.

Example:

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{sinz}{z} = 1.$

$\lim_{z \to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

Example:

Let $f(z) = \frac{z - sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{z - sinz}{z^3}$

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1 - cosz}{3z^2}$ [L-Hosptital rule]

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{0 + sinz}{6z^1}$ [L-Hosptital rule]

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{cosz}{6}$ [L-Hosptital rule]

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1}{6}$

$\lim_{z \to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

6.2.3 Pole

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z \to z_0} (z - z_0)^k f(z) = \lambda$, where $\lambda \neq 0$, then z_0 is the pole of order *k*.

If $k = 1$, then z_0 is the simple pole.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = \lim_{z \to 0} z \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{cosz}{z^2}$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{cosz}{z^2}.$$

$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

Example:

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1-e^{2z}}{z^3}$

$\lim_{z \to z_0} (z - 0)^3 f(z) = \lim_{z \to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$

$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{(1-e^{2z})}{z^3}.$

$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} \frac{(1-e^{2z})}{z}.$

$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

6.2.4 Essential singularity

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z \to z_0} (z - z_0)^n f(z) = \infty$, then z_0 is essential singularity.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} e^{1/z}$

$\lim_{z \to z_0} (z - 0)^n f(z) = \lim_{z \to 0} z^n e^{1/z}$.

$\lim_{z \to z_0} (z - 0)^n f(z) = \lim_{z \to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

$\lim_{z \to z_0} (z - 0)^n f(z) = \infty.$

So $z_0 = 0$ is an essential singularity.

6.2.5 Singularity at infinity

We classify the types of singularities at infinity by letting w = $1/z$ and analyzing the resulting function at w = 0.

Example:

$f(z) = z^3.$

$f(z) = g(w) = 1/w^3.$

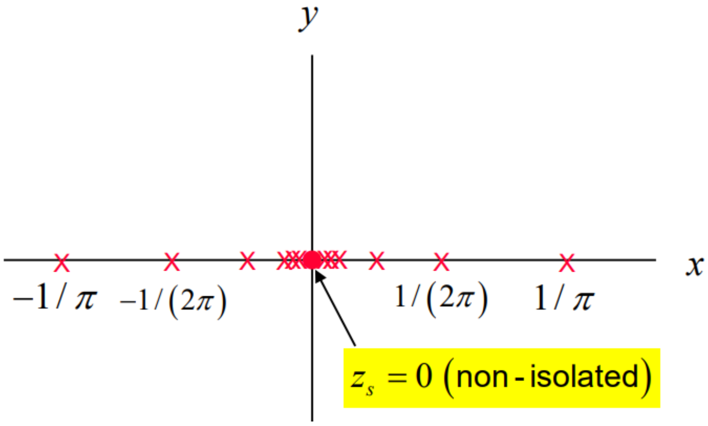
$g(w)$ has a pole of order 3 at w = 0 The function f(z) has a pole of order 3 at infinity.

6.2.5 Non-isolated singularity

A point a is called a non-isolated singularity for $f(z)$ if $f(z)$ is not is not isolated at $z = a$.

Example:

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$



The function is not analytic in any region $0 < |z| < \delta$.

12.3 Classification of singularity by Laurent series expansion

It is also possible to classify the singularity using the Laurent series expansion.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z - a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n$$

Were

$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - a)^{-n+1}} \, d\zeta$

$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$

The series consisting of the negative powers of $z - a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the principal part or singular part $of\ f(z)\ at\ z = a$.

The singular part of $f(z)\ at\ z = a$ determines the character of the singularity.

6.9.1 Removable singularity by Laurent series expansion

Let $\boldsymbol{a}$ be an isolated singularity for $\boldsymbol{f(z)}$. Then $\boldsymbol{a}$ is called a removable singularity if the principal part of $\boldsymbol{f(z)\ at\ z = a}$ has no terms.

If $\boldsymbol{a}$ is a removable singularity for $\boldsymbol{f(z)}$ then the Laurent's series expansion of $\boldsymbol{f(z)\ about\ z = a}$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Hence as $\boldsymbol{z \to a}$, $\boldsymbol{f(z) = a_0}$ Hence by defining $\boldsymbol{f(a) = a_0}$ the function $\boldsymbol{f(z)}$ becomes analytic at $\boldsymbol{a}$.

Example:

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

Now $f(z) = \frac{sinz}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.)$

$f(z) = \frac{sinz}{z} = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots.)$

Here the principal part of $f(z)\ at\ z = 0$ has no terms. Hence z = 0 is a removable singularity.

$\lim_{z \to z_0} f(z)$ also exists then $z_0 = 0$ is the removable singularity.

Example:

Let $f(z) = \frac{z - sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots,)}{z^3}$$

$$f(z) = \frac{\frac{z^3}{3!} - \frac{z^5}{5!} - \ldots,)}{z^3}$$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} - \ldots,$$

$z = 0$ is a removable singularity. By defining $f(0) = 1/6$ the function becomes analytic $at\ z = 0$.Also $\lim_{z \to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

6.9.2 Pole by Laurent series expansion

Let a be an isolated singularity of $f(z)$. The point a is called a pole if the principal part of $f(z)\ at\ z = a$ has a finite number of terms.

If the principal part of $f(z) at\ z = a$ is given by

$\frac{b1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_r}{(z-a)^r}$, . where $b_r \neq 0$.

We say that a is a pole of order r for $f(z)$. Note: A pole of order 1 is called a simple pole and a pole of order 2 is called double pole.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$f(z) = \frac{e^z}{z} = \frac{1}{z}(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots,)$

$f(z) = \frac{e^z}{z} = (1/z + 1 + \frac{z}{2} + \frac{z^2}{6} + \ldots,)$

Here the principal part of $f(z)\ at\ z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$. Also

$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0$. So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{cosz}{z^2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,}{z^2}$$

The principal part of $f(z)$ at $z = 0$ contains the term $1/z^2$. Hence z=0 is a double pole of f(z).

Also $\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

6.9.3 Essential singularity

Let a be an isolated singularity of $\boldsymbol{f(z)}$. The point a is called an essential singularity of $\boldsymbol{f(z)}$ at $\boldsymbol{z = a}$ if the principal part of $\boldsymbol{f(z)}$ $\boldsymbol{at\, z = a}$ has an infinite number of terms.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = e^{1/z}$$

$f(z) = [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,].$

The principal part of $f(z)$ has infinite number of terms. Hence $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Also $\lim_{z \to z_0} (z-0)^n f(z) = \infty.$

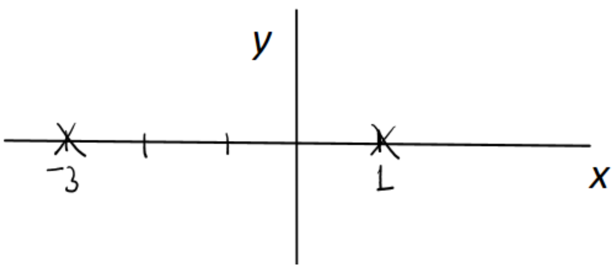
So $z_0 = 0$ is an essential singularity.

12.4 <u>Meromorphic Functions</u>

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example:

$$f(z) = \frac{z}{(z-1)(z+3)^2}$$



$f(z)$ is analytic everywhere in the complex plane except $z = 1$ and $z = -3$.Here $z = 1$ is a simple pile and $z = -3$ is the pole of order 3. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z-0)^1 f(z) = \lim_{z \to 0} z \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z-0)^1 f(z) = e^0 = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order *1* or simple pole. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{cosz}{z^2}$$

$$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} z^2 \frac{cosz}{z^2}.$$

$$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order 2. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1-e^{2z}}{z^3}$$

$$\lim_{z \to z_0} (z-0)^3 f(z) = \lim_{z \to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$$

$$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} z^2 \frac{(1-e^{2z})}{z^3}.$$

$$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} \frac{(1-e^{2z})}{z}.$$

$$\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$$

So $z_0 = 0$ is the pole of order 2. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} e^{1/z}$

$\lim_{z\to z_0} (z-0)^n f(z) = \lim_{z\to 0} z^n e^{1/z}$.

$\lim_{z\to z_0} (z-0)^n f(z) = \lim_{z\to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

$\lim_{z\to z_0} (z-0)^n f(z) = \infty.$

So $z_0 = 0$ is an essential singularity.Thus this function is not meromorphic in the whole complex plane.

12.5 Mittag-Leffler theorem

The Mittag-Leffler theorem is a fundamental result in complex analysis that deals with the existence of meromorphic functions with prescribed poles and residues. Specifically, it states that for any sequence of distinct points in the complex plane and any sequence of complex numbers, there exists a meromorphic function with poles precisely at the given points and residues equal to the corresponding complex numbers.

More formally, let $z_1, z_2, \ldots$ be a sequence of distinct complex numbers, and let $c_1, c_2, \ldots$ be a sequence of complex numbers. Then there exists a meromorphic function $f(z)$ on the complex plane such that the only poles of $f(z)$ are at the points $z_1, z_2, \ldots$, and the residue of $f(z)$ at z_i is c_i, for $i = 1, 2, \ldots$.

Proof:

To prove the Mittag-Leffler theorem, we will construct the desired meromorphic function $f(z)$ using a standard technique known as the Weierstrass product formula.

This involves expressing $f(z)$ as an infinite product of simple functions, each of which has a single pole at one of the given points and the prescribed residue.

Let D_n be the disc centered at z_n with radius r_n such that D_n is disjoint from all other discs, and let $C_n$be the circle bounding D_n.

Then we define the function $g_n(z)$ as:

$$g_n(z) = (z - z_n)^{-1} e^{(p_n(z - z_n))}$$

where p_n is chosen so that the Laurent series of $g_n(z)$ at z_n has a constant term of c_n. Specifically, we set:

$$p_n = \frac{c_n}{r_n}$$

Using the Cauchy integral formula, we can express $g_n(z)$ as an integral over C_n:

$$g_n(z) = \frac{1}{2\pi i} \int_{C_n} \frac{g_n(w)}{w - z} dw$$

Now we define the function $F(z)$ as:

$$F(z) = \prod_{n=1}^{\infty} g_n(z)$$

This product converges absolutely and uniformly on compact sets, since the discs D_n are disjoint and the radii r_n are chosen appropriately. Moreover, $F(z)$ is meromorphic on the complex plane, since each $g_n(z)$ has a single pole at z_n and no other poles.

To see that $F(z)$ has the desired poles and residues, we consider the partial products:

$$F_{N(z)} = \prod_{n=1}^{N} g_n(z)$$

These are meromorphic functions with poles only at the points $z_1, z_2, \ldots, z_N$. Moreover, the residue of $F_{N(z)}\ at\ z_n\ is\ c_n$, by construction. Finally, we note that $F_{N(z)}$ converges to $F(z)$ as N goes to infinity, since the product converges absolutely and uniformly on compact sets.

Therefore, we have constructed a meromorphic function $f(z)$ with the desired poles and residues, namely:

$$f(z) = F(z)$$

This completes the proof of the Mittag-Leffler theorem.

Question:

Prove that $cotz = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}$ using Mittage Laffer's theorem

Proof:

To prove that $cotz - \frac{1}{z} = 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}$, we can use the Mittag-Leffler theorem.

To prove this identity using the Mittag-Leffler theorem, we need to first identify the poles and their residues of the function $cot(z)$.

We know that $cot(z)$ is periodic with period π, and has simple poles at $z = n\pi$ for all integers n.

Recall that the cotangent function can be expressed as the ratio of the cosine and sine functions:

$$cotz = \frac{\cos z}{sinz}$$

The poles of the cotangent function are the zeros of the sine function, which occur at $z = n\pi$ for all integers n. Thus, we can write:

$$cotz = \frac{cosz}{z - n\pi}$$

To prove this identity using Mittag-Leffler theorem, we need to find the poles and residues of the function cot(z) and the infinite sum in the equation.

First, we know that $cot(z)$ has simple poles at $z = n\pi$ for all integers n.

The residues at these poles are ± 1, depending on the sign of $sin(n\pi)$.

Next, we consider the infinite sum in the equation.

Let $f(z) = \sum \frac{1}{z^2 - n^2\pi^2}$.

This function has poles at $z = \pm n\pi$ for all integers n. The residues at these poles are given by

$$Res[f(z), z = n\pi] = lim_{z \to n\pi} \frac{(z - n\pi)1}{z^2 - n^2\pi^2} = \frac{1}{2n\pi}$$

and

$$Res[f(z), z = -n\pi] = lim_{z \to -n\pi} \frac{(z + n\pi)1}{z^2 - n^2\pi^2} = -\frac{1}{2n\pi}.$$

Now, using the Mittag-Leffler theorem, we can write

$$cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right)$$

$= \sum_{n=1}^{\infty} \frac{1}{2n\pi}\left(\frac{1}{z-n\pi} - \frac{1}{z+n\pi}\right)$

$= 2\sum_{n=1}^{\infty} \frac{1}{2n\pi} \frac{1}{z^2 - n^2\pi^2}$

$= 2f(z)$

Therefore, we have

$cot(z) = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{1}{2n\pi} \frac{1}{z^2 - n^2\pi^2}$ as desired.

12.6 Infinite Product of Complex Numbers

An infinite product of complex numbers is given by:

$z = z_1.z_2.z_3 \ldots,.z_n \ldots$

The notation for infinite product is $\prod_{i=1}^{n} z_i$

where $z_1, z_2, z_3, \ldots, z_n$ are complex numbers.

If the infinite product converges, then we can define it as follows:

$$z = \lim_{n\to\infty}(z_1.z_2.z_3 \ldots, z_n)$$

In general, an infinite product of complex numbers is said to converge if and only if the limit of the sequence of partial products (i.e., the product of the first n terms) exists and is nonzero.

Some important results related to infinite products of complex numbers are:

1. If the infinite product $|z|$ converges, then the infinite product z also converges.
2. The infinite product $(1+z)$ converges if and only if the infinite product $(1-|z|)$ converges.
3. The infinite product $(1-z)$ converges if and only if the infinite product $(1+|z|)$ converges.
4. The infinite product $\left(1+\frac{z}{n}\right)^n$ converges to e^z as n approaches infinity, for any complex number z.
5. The infinite product $sin\left(\frac{z}{n}\right)$ converges to zero for any non-zero complex number z.

Question

Suppose an infinite product is absolutely convergent. Prove that it is convergent

Solution

Suppose the infinite product is given by:

$$P = a_1 . a_2 . a_3 \ldots$$

where ai are non-negative real numbers.

By the absolute convergence of P, we have that the series:

$$S = log(a_1) + log(a_2) + log(a_3) + \ldots$$

converges.

Since the logarithm function is continuous, we can take the exponential of both sides to obtain:

$$e^S = e^{\log(a_1)} e^{log(a2)} e^{log(a3)} \ldots$$

which simplifies to:

$$P = a_1 a_2 a_3 \ldots$$

Thus, the absolute convergence of P implies that the series S converges, which in turn implies that P converges as well.

Therefore, we have shown that if an infinite product is absolutely convergent, then it is also convergent.

12.7 Review questions

1. **$f(z) = \frac{z}{(z-1)^2}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

2. $\boldsymbol{f(z) = \frac{z}{(z-1)(z-5)(z-2)^3}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

3. Check whether the following functions is meromorphic?

   $\boldsymbol{f(z) = \frac{e^z}{z}}$

4. Check whether the following functions is meromorphic?

   $\boldsymbol{g(z) = \frac{sinz}{(z-1)^2}}$.

5. State and prove the Mittag-Leffler theorem

12.8 Self-assessment

1. $\boldsymbol{f(z) = \frac{z}{(z-1)}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

A. 5
B. 1
C. 2
D. 3

2. $\boldsymbol{f(z) = \frac{z}{(z-1)(z-2)^3}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

A. 5
B. 1
C. 2
D. 3

3. Which one of the following functions is meromorphic?

$\boldsymbol{f(z) = \frac{e^z}{z}}$ **, and** $\boldsymbol{g(z) = \frac{sinz}{(z-1)^2}}$.

A. Only $f(z)$
B. Only $g(z)$
C. Both $f(z)$ and $g(z)$
D. Neither $f(z)$ nor $g(z)$

4. Consider the $\boldsymbol{f(z) = \frac{z}{1-z}}$ **then**

A. $z_0 = 1$ is the singular point of $f(z)$
B. $z_0 = 0$ is the singular point of $f(z)$
C. $z_0 = 10$ is the singular point of $f(z)$
D. There is no singular point of $f(z)$

5. **Consider the $f(z) = z^2$ then**

A. $z_0 = 1$ is the singular point of $f(z)$

B. $z_0 = 0$ is the singular point of $f(z)$

C. $z_0 = 10$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

6. **Consider the $f(z) = \frac{z^2-9}{z^2(z-1)(z-1-2i)}$ then**

A. $z_0 = 1$ is one of the singular points of $f(z)$

B. $z_0 = 3$ is the singular point of $f(z)$

C. $z_0 = -3$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

7. What is Mittag-Leffler's theorem?

A. A theorem on the convergence of infinite series.

B. A theorem on the analytic continuation of meromorphic functions.

C. A theorem on the existence of a holomorphic function with prescribed singularities

8. What does the theorem say about meromorphic functions?

A. They can be extended to the whole complex plane.

B. They can be extended to a neighborhood of their poles.

C. They can be approximated by polynomials

9. What are the conditions for the Mittag-Leffler's theorem to hold?

A. The function must have isolated singularities and a certain growth condition.

B. The function must be holomorphic and bounded on a compact set.

C. The function must be a polynomial

10. What is the significance of the theorem in complex analysis?

A. It provides a method for approximating meromorphic functions

B. It is a fundamental tool for studying the Riemann zeta function

C. It allows us to construct meromorphic functions with prescribed singularities

11. What is the value of the infinite product (1+i)(1-i)(1+i)(1-i)...?

A. 1

B. -1

C. i

D. -i

12. What is the value of the infinite product (1+2i)(1-2i)(1+2i)(1-2i)...?

A. 1

B. -1

C. 2i

D. -2i

13. What is the value of the infinite product (1+i/2)(1-i/2)(1+i/2)(1-i/2)...?

A. 1

B. -1/2

C. i/2
D. -i/2

14. What is the value of the infinite product (1+i/3)(1-i/3)(1+i/3)(1-i/3)...?
A. 1
B. -1/3
C. i/3
D. -i/3

15. What is the value of the infinite product (1+3i)(1-3i)(1+3i)(1-3i)...?
A. 1
B. -1
C. 3i
D. -3i

Table 1: Answers of self-assessment

Question number	Correct answer
1	B
2	C
3	C
4	A
5	D
6	A
7	B
8	A
9	A
10	**C**
11	**A**
12	B
13	B
14	B
15	B

12.9 Summary

- The A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

- The An infinite product of complex numbers is given by:

$z = z_1.z_2.z_3 \ldots,.z_n \ldots$

The notation for infinite product is $\prod_{i=1}^{n} z_i$

- If the infinite product $|z|$ converges, then the infinite product z also converges.
- The infinite product $(1 + z)$ converges if and only if the infinite product $(1 - |z|)$ converges.
- The infinite product $(1 - z)$ converges if and only if the infinite product $(1 + |z|)$ converges.

12.10 Keywords

Meromorphic function:

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Mittag-Leffler theorem :

Let $z_1, z_2, \ldots$ be a sequence of distinct complex numbers, and let $c_1, c_2, \ldots$ be a sequence of complex numbers. Then there exists a meromorphic function $f(z)$ on the complex plane such that the only poles of $f(z)$ are at the points $z_1, z_2, \ldots$, and the residue of $f(z)$ at z_i is c_i, for $i = 1, 2, \ldots$.

12.11 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Unit 13 - Bieberbach's conjecture

Purpose and Objectives:

After this unit students can be able to-

1. Understand the Schwarz Reflection Principle for analytic functions?
2. Prove the Schwarz Reflection Principle for analytic functions?
3. Learn the consequences of the Schwarz Reflection Principle

Introduction

If a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane. In other words, if a function f(z) is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

The Schwarz Reflection Principle has several important applications in complex analysis, such as proving the analyticity of functions, constructing entire functions with prescribed properties, and solving boundary value problems. In this unit we will explore the Schwarz Reflection Principle for analytic function.

13.1 Schwarz Reflection Principle for Analytic Functions

Statement:

The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.

or

The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane. In other words, if a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

Proof 1:

The proof of the Schwarz Reflection Principle relies on the fact that if a function is analytic in the upper half plane and its real part is non-negative on the boundary, then it can be represented as a real part of another analytic function.

Let $f(z)$ be analytic in the upper half plane and $Re\big(f(z)\big) \geq 0$ for all real z.

Then the function $g(z) = f(z) + i(-f(z))$ is analytic in the upper half plane and satisfies $Re(g(z)) = 0$ for all real z.

The proof also uses the maximum modulus principle and Liouville's theorem.

Liouville's Theorem

Liouville's Theorem states that a bounded holomorphic function on the entire complex plane must be constant. It is named after Joseph Liouville.

Statement:

Let $f(z): C \to C$ be an entire function. Suppose there exists some real number $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

Or

If a function $f(z)$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof:

It is given that

i. A function $f(z)$ is analytic in the entire complex plane
ii. A function $f(z)$ is bounded, that $|f(z)| \leq M$.

Let us consider two points $a \; and \; b$ inside a particular domain(See the figure 5.1).

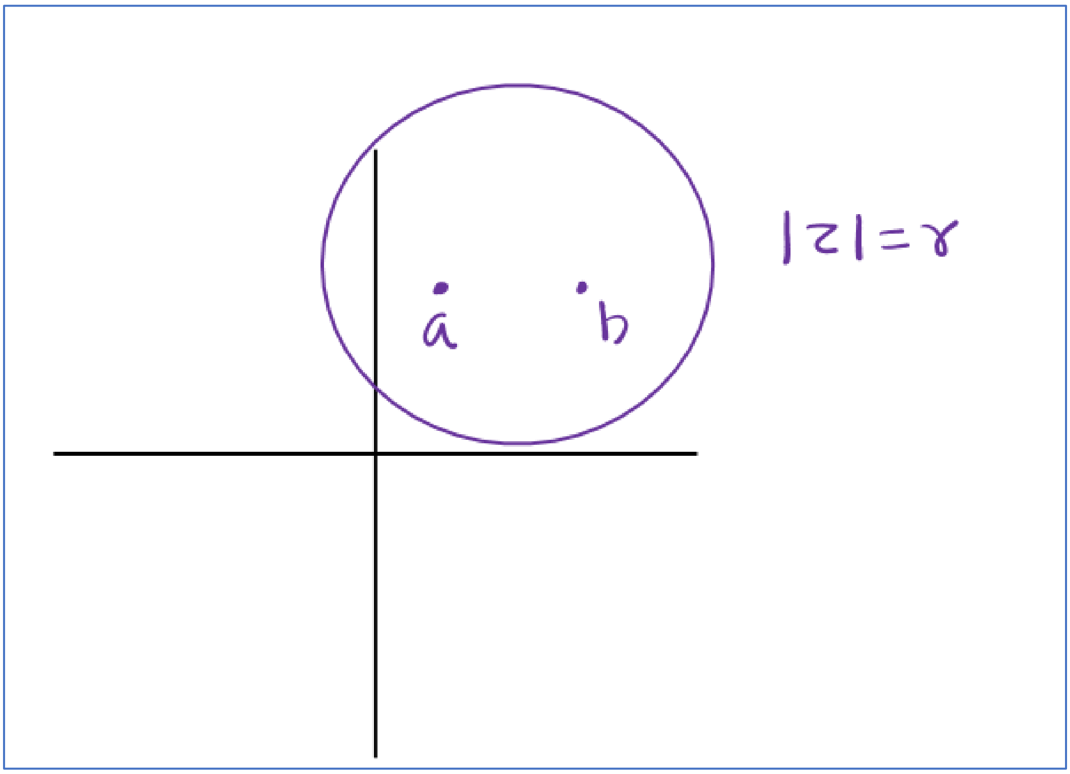


Figure 5.1: Two points $a \; and \; b$ inside a particular domain

Then using Cauchy integral formula

$$\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-a}\right) dz = f(a)$$

$$\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-b}\right) dz = f(b)$$

If $f(z)$ is constant throughout the domain, then $f(a) = f(b)$.

Now let's prove $f(a) - f(b) = 0$.

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-a}\right) dz - \frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-b}\right) dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{z-a} - \frac{f(z)}{z-b}\right) dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c f(z) \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz$$

$$f(a) - f(b) = \frac{1}{2\pi i}\oint_c f(z) \left(\frac{z-b-z+a}{(z-a)(z-b)}\right) dz$$

$f(a)-f(b)=\frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz$

$|f(a)-f(b)|=\left|\frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz\right|$

$|f(a)-f(b)|\leq\left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{|(z-a)(z-b)|}\right)|dz|$

$|f(a)-f(b)|\leq\left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{(|z|-|a|)(|z|-|b|)}\right)|dz|$

Let

$z=re^{i\theta}$

$dz=re^{i\theta}.i.d\theta$

$|dz|=\left|re^{i\theta}.i.d\theta\right|$

$|dz|=|r|.\left|e^{i\theta}\right|.|i|.|d\theta|$

Here $|r|=r$

$\left|e^{i\theta}\right|=|cos\theta+isin\theta|=1,$

$|i|=1,$

$|dz|=r.|d\theta|$

$|f(a)-f(b)|\leq\frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(r-a)(r-b)}\right)r.|d\theta|$

$|f(a)-f(b)|\leq\frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(1-a/r)(1-b/r)}\right).|d\theta|$

If $f(z)$ is analytic in the entire complex plane, then $|z|=r\to\infty$. So

$|f(a)-f(b)|\leq 0$

$f(a)-f(b)=0$

Hence, we can say that $f(a)=f(b)$. It means that $f(z)$ is a constant.

Liouville's Theorem proof using Cauchy integral formula for derivatives.

If f(z) is analytic in a simply connected region then at any interior point of the region, z_0 inside *C*. Then say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$\oint_c\left(\frac{f(z)}{(z-z_0)^{n+1}}\right)dz=2\pi i\frac{f^n(z_0)}{n!}.$

where C is any simple closed curve, in the region, which encloses z_0. Note the case $n = 1$:

$\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{(z-z_0)^2}\right)dz=f'(z_0).$

$|f'(z_0)|=\left|\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{(z-z_0)^2}\right)dz\right|.$

$|f'(z_0)|\leq\left|\frac{1}{2\pi i}\right|\oint_c\left(\frac{|f(z)|}{|(z-z_0)^2|}\right)|dz|.$

Here $z=re^{i\theta}$

$dz=re^{i\theta}.i.d\theta.$

$|dz|=\left|re^{i\theta}.i.d\theta\right|.$

$|dz|=|r|.\left|e^{i\theta}\right|.|i|.|d\theta|.$

Here $|z-z_0|=r$

$\left|e^{i\theta}\right|=|cos\theta+isin\theta|=1,$

$|i|=1,$

$|dz| = r.|d\theta|.$

$|f'(z_0)| \leq \frac{1}{2\pi} \oint_c \left(\frac{M}{r^2}\right) r.|d\theta|.$

$|f'(z_0)| \leq \frac{1}{2\pi} \oint_c \left(\frac{M}{r}\right).d\theta.$

If $f(z)$ is analytic in the entire complex plane then $r \to \infty$. So

$|f'(z_0)| \leq 0$

$f'(z_0) = 0$

$f(z) =$ constant.

By Liouville's theorem, the imaginary part of $g(z)$ is constant on the boundary, say c. Then the function $h(z) = g(z) + ic$ is analytic in the entire plane and has the same real part as $f(z)$.

Proof 2:

Let $f(z)$ be a complex valued function that is analytic in the upper half plane $(Im(z) > 0)$, and continuous on the boundary of the upper half plane $(Im(z) = 0)$.

Consider a point z in the lower half plane $(Im(z) < 0)$[See figure 5.2]

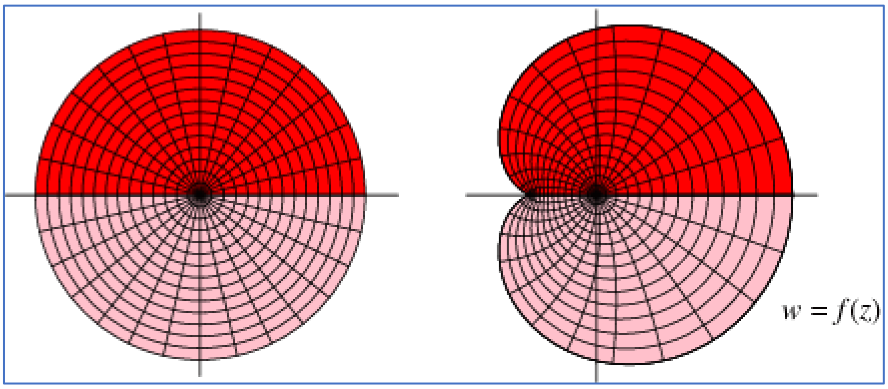


Figure 5.2: $w = f(z)$

Let $z = x + iy$, where x is real and y is negative.

Let's define a new point, $conj(z)$=$\bar{z}$, which is equal to the complex conjugate of z.

That is, $conj(z) = x - iy$.

Since $f(z)$ is continuous on the boundary of the upper half plane, it follows that $f(conj(z))$ is continuous in the lower half plane.

Also, since $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, we have:

$f(z) = \overline{f((conj(z)))}$

$= \overline{f(x - iy)}$

$= \overline{f(x + i(-y))}$

Thus, we can define a new function, $g(z)$, in the lower half plane as follows:

$g(z) = \overline{f(x + i(-y))}$

Since $g(z) = \overline{f(z)}$ is continuous in the lower half plane, and the conjugate of a continuous function is continuous, it follows that $g(z)$ is continuous in the lower half plane.

We now show that $g(z)$ is also analytic in the lower half plane.

Let $z = x + iy$, where x is real and y is negative.

Consider the derivative of $g(z)$ at the point z:

$g'(z) = \left(\frac{d}{dz}\right) \overline{f(x + \iota(-y))}$

$= \left(\frac{d}{dz}\right) \overline{f(z)}$

$= \overline{\left(\frac{d}{dz}\right) f(z)}$

Since $f(z)$ is analytic in the upper half plane, it follows that $\left(\frac{d}{dz}\right) f(z)$ is also analytic in the upper half plane.

Therefore, conjugate of $\left(\frac{d}{dz}\right) f(z)$ is also analytic in the lower half plane, and so is $g'(z)$.

Since $g(z)$ is continuous and its derivative is analytic in the lower half plane, it follows that $g(z)$ is analytic in the lower half plane.

Thus, we have shown that if $f(z)$ is a complex valued function that is analytic in the upper half plane and continuous on the boundary of the upper half plane, and if $f(z)$ satisfies:

$f(z) = \overline{f(conj(z))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane by defining a new function, $g(z)$, in the lower half plane.

13.2 consequences of the Schwarz Reflection Principle

1. One consequence of the Schwarz Reflection Principle is that if a function is analytic in the upper half plane and its real part is non-negative on the boundary, then it can be extended to an entire function that is real valued on the real axis.
2. Another consequence is that a function that is analytic in the upper half plane and satisfies a certain growth condition on the boundary (such as the Riemann mapping theorem) can be extended to an entire function with similar growth behavior.
3. Additionally, the Schwarz Reflection Principle can be used to construct solutions to boundary value problems, such as the Dirichlet problem, by reflecting solutions from one half plane to the other.

13.3 Different proofs of Schwartz Reflection Principle

The Schwartz Reflection Principle can be proved by various methods

1. Complex Analysis Proof: The Schwartz Reflection Principle can be proven using complex analysis by considering the analytic continuation of the function from the upper half plane to the lower half plane. The proof involves showing that the function, extended to the lower half plane, is a reflection of the function in the upper half plane across the real axis.

2. Harmonic Functions Proof: The Schwartz Reflection Principle can also be proven using the theory of harmonic functions. A function is considered harmonic if it satisfies Laplace's equation. By assuming that the function is harmonic in the upper half plane, it can be shown that its extension to the lower half plane is also harmonic, and therefore satisfies Laplace's equation, meaning it must be a reflection of the function in the upper half plane across the real axis.

3. Integral Transform Proof: The Schwartz Reflection Principle can be proven using the Fourier Transform by showing that the Fourier Transform of a function in the upper half plane, after being reflected across the real axis, is equal to the negative Fourier Transform of the original function in the lower half plane.

4. Paley-Wiener Theorem Proof: The Schwartz Reflection Principle can also be proven using the Paley-Wiener theorem, which states that the Fourier Transform of a function with compact support is a function that is entire and decays rapidly. By assuming that the function in question is the Fourier Transform of a function with compact support in the upper half plane, it can be shown that the function, after being reflected across the real axis, is the Fourier Transform of a function with compact support in the lower half plane.

5. Bochner's Theorem Proof: The Schwartz Reflection Principle can also be proven using Bochner's theorem, which states that a positive definite function is the Fourier Transform of a positive measure. By assuming that the function in question is positive definite in the upper half plane, it can be shown that the function, after being reflected across the real axis, is positive definite in the lower half plane, implying that it is the Fourier Transform of a positive measure.

13.4 Applications

- The main application of the Schwartz Reflection Principle is in the study of distributions and their derivatives. It provides a means to extend the definitions of distributions and derivatives to unbounded functions.
- The Schwartz Reflection Principle is a generalization of the Hahn-Banach Theorem. The Hahn-Banach Theorem states that a linear functional on a linear subspace can be extended to the entire space while preserving its norm. The Schwartz Reflection Principle extends this result to the case of distributions.
- The Schwartz Reflection Principle is an important tool in mathematical physics for defining distributions and derivatives of functions. In particular, it allows for the extension of the definitions of distributions and derivatives to unbounded functions, which is particularly useful in quantum field theory and quantum mechanics.

13.5 Review questions

1. What is the main application of the Schwartz Reflection Principle?
2. How does the Schwartz Reflection Principle relate to the Hahn-Banach Theorem?
3. What is the significance of the Schwartz Reflection Principle in mathematical physics?
4. State and prove the Schwartz Reflection Principle using Liouville's Theorem?
5. State and prove the Schwartz Reflection Principle without Liouville's Theorem ?

13.6 Self-assessment

1. Which one the following statement is true for the reference of Schwarz reflection principle?

   I: The principle that a complex function which is holomorphic in the upper half plane can be extended to a holomorphic function in the whole plane.

II: The principle that states that a real-valued function cannot be analytically extended across a branch cut.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

2. Which one the following statement is true for the reference of Schwarz reflection principle?

I: The principle that a holomorphic function in the unit disc can be extended to a holomorphic function in the whole plane.

II: The principle that states that the maximum value of a subharmonic function is achieved on the boundary of its domain.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

3. Which one the following statement is true for the necessary condition for the Schwarz reflection principle?

I: The function must be bounded in the upper half plane.

II: The function must be continuous in the upper half plane

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

4. Which one the following statement is true for the necessary condition for the Schwarz reflection principle?

I: The function must be holomorphic in the upper half plane.

II: The function must be continuous in the upper half plane

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

5. The Schwarz reflection principle can be used to extend which type of functions to the whole plane?

A. Only real-valued functions
B. Only harmonic functions
C. Both real valued and harmonic
D. Neither real nor harmonic

6. The Schwarz reflection principle can be used to extend which type of functions to the whole plane?
   A. Holomorphic functions
   B. Harmonic functions
   C. Subharmonic functions
   D. Neither real nor harmonic

7. The Schwarz reflection principle states that the Fourier transform of the product of two signals is equal to the convolution of their Fourier transforms?
   A. True
   B. False

8. The principle that states that the reflection of a Schwartz function across the x-axis is also a Schwartz function?
   A. True
   B. False

9. The principle that states that the Laplace transform of a signal is equivalent to its Fourier transform?
   A. True
   B. False

10. The principle that states that the derivative of a Schwartz function is also a Schwartz function.
    A. True
    B. False

11. What is the Schwartz Reflection Principle in mathematics?

    A. The principle that every polynomial function has a unique root
    B. The principle that states that the boundary values of an analytic function on the upper half-plane can be extended to an analytic function on the whole complex plane
    C. The principle that states that the roots of a polynomial equation occur in conjugate pairs.
    D. The principle that the value of a holomorphic function at a point is equal to its average value over any small circle centered at that point.

12. Which of the following is the best definition of the Schwartz Reflection Principle?

    I: A mathematical theorem that states that the reflection of a function across a vertical line always results in an even function.
    II: A mathematical theorem that states that the reflection of a function across a vertical line always results in an odd function.
    A. Only I
    B. Only II
    C. Both I and II
    D. Neither I nor II

13. Which of the following is the best definition of the Schwartz Reflection Principle?

I: A mathematical theorem that states that the reflection of a function across a vertical line always results in a function with the same parity.

II: A mathematical theorem that states that the reflection of a function across a vertical line always results in a different function.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

14. The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\bar{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.
    A. True
    B. False

15. Which one the following statement is true for the reference of Schwarz reflection principle?

I: The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane.

II: If a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

A. Only I
B. Only II
C. Both I and II
D. Neither I nor II

Table 1: Answers of self-assessment

Question number	Correct answer
1	A
2	D
3	D
4	A
5	D
6	A
7	B

8	A
9	B
10	**B**
11	**B**
12	D
13	A
14	D
15	C

13.7 Summary

- The Schwarz Reflection Principle states that if a complex valued function $f(z)$ is analytic in the upper half plane ($Im(z) > 0$), and continuous on the boundary of the upper half plane ($Im(z) = 0$), and if $f(z)$ satisfies $f(z) = \overline{f((\overline{z}))}$, then the function $f(z)$ can be extended to be analytic in the entire complex plane.

- The Schwarz Reflection Principle states that if a function is analytic in the upper half plane and its real part is non-negative on the boundary (the real axis), then it can be extended analytically to the entire plane.

- If a function $f(z)$ is analytic in the upper half plane and satisfies the condition $Re\big(f(z)\big) \geq 0$ for all real z, then $f(z)$ can be extended to the entire complex plane.

- If a function $f(z)$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

13.8 Keywords

Liouville's Theorem

Let $f(z): C \to C$ be an entire function. Suppose there exists some real number$M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

13.9 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

Unit 14 - Landau's theorem

Purpose and Objectives:

Meromorphic functions are an important class of functions studied in complex analysis. They are defined as functions that are holomorphic (analytic) everywhere except at a finite number of isolated singularities. Meromorphic functions are useful in studying the behavior of complex functions near singularities, and they provide a representation of any meromorphic function in terms of its poles and their residues. After this unit students can be able to-

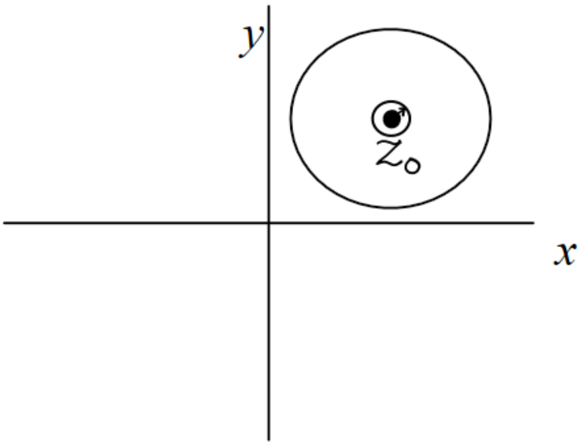
1. Understand the Meromorphic functions
2. State and prove the Mittag-Leffler theorem
3. Learn the infinite product of complex Numbers

Introduction

In this unit first we will understand the concept of singularities and poles for meromorphic function then the we will use the mesomorphic function to prove the Mittag-Leffler theorem. Last we will focus on the infinite product of complex Numbers.

14.1 Singularities

A point z_0 is called a singular point of a function f(z) if f(z) fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.



Example:

Behavior of following functions at $z = 0$.

$$f(z) = \frac{1}{z^9}$$

$$f(z) = \frac{Sinz}{z}$$

$$f(z) = \frac{e^z - 1}{z}$$

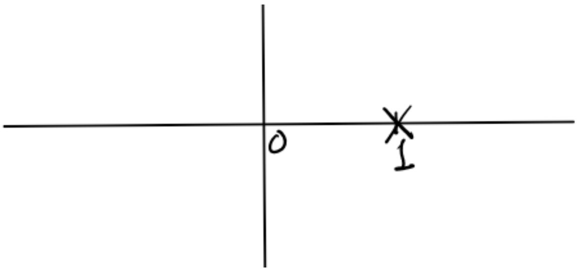
$$f(z) = \frac{1}{\sin(1/z)}$$

We observed that all the functions mentioned above are not analytic at $z = 0$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic.

Example:

Behavior of following function at $z = 1$.

$$f(z) = \frac{z}{1 - z}$$



We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

Example:

$f(z) = z^2$ is analytic everywhere so it has no singular point.

Example:

Behavior of following function in the entire z plane

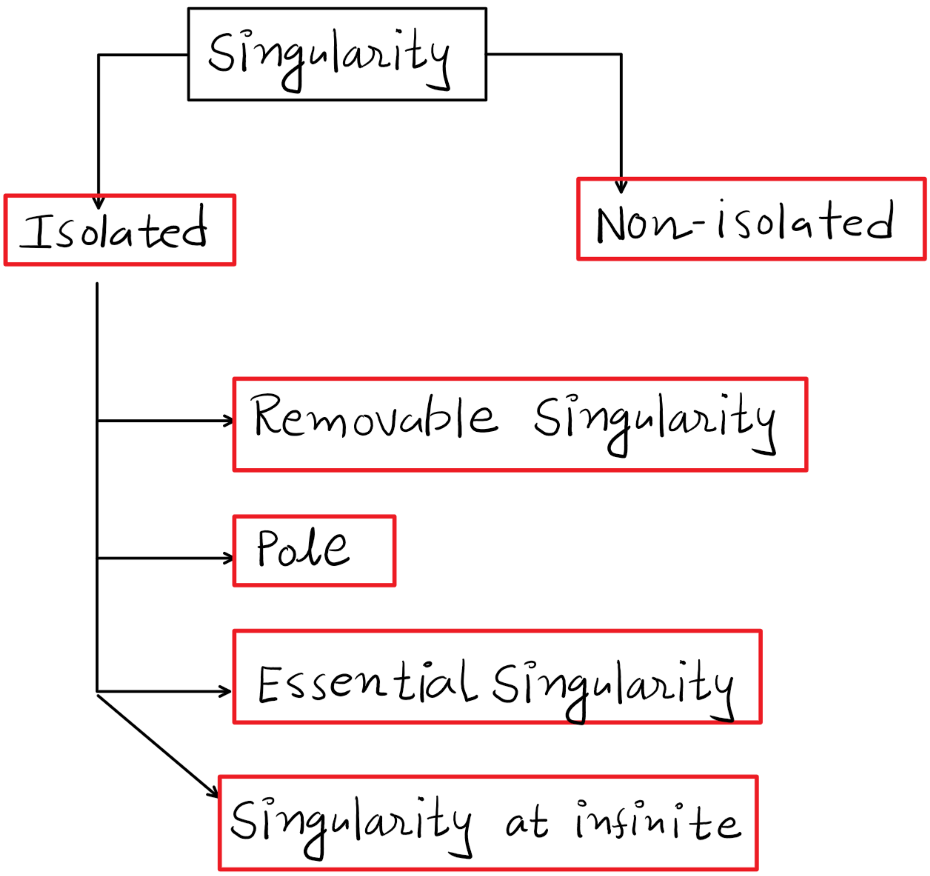
$f(z) = |z|^2$

We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

14.2 Classification of singularity

The singularity of a complex function can be classified into two groups, isolated and non-isolated. It can be done via Laurent series expension, but we can also classify the singularity without the Laurent series expension. In the forthcoming units we will consider the classification using the Laurent series.

The isolated singularity further can be classified into different type. The following diagram shows the different types of the singularities.



6.2.1 Isolated singularity

A point a is called an isolated singularity for $f(z)$ if $f(z)$ is not analytic at $z = a$ and there exist $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$. The neighbourhood $|z - a| < r$ contains no singularity of $f(z)$ except a.

Example:

$\boldsymbol{f(z) = \frac{z+1}{z^2(z^2+1)}}$ ***has three isolated singularities*** $\boldsymbol{z = 0, i, -i}$.

Example:

$f(z) = \frac{1}{sinz}$ has three isolated singularities $z = 0, \pm\pi, \pm2\pi, \ldots,$

6.2.2 Removable singularity

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z\to z_0} f(z)$ exists then z_0 is the removable singularity.

Example:

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to 0} \frac{sinz}{z} = 1.$

$\lim\limits_{z \to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

Example:

Let $f(z) = \frac{z-sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to 0} \frac{z-sinz}{z^3}$

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to 0} \frac{1-cosz}{3z^2}$ [L-Hosptital rule]

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to 0} \frac{0+sinz}{6z^1}$ [L-Hosptital rule]

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to 0} \frac{cosz}{6}$ [L-Hosptital rule]

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to 0} \frac{1}{6}$

$\lim\limits_{z \to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

6.2.3 Pole

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim\limits_{z \to z_0} (z - z_0)^k f(z) = \lambda$, where $\lambda \neq 0$, then z_0 is the pole of order *k*.

If $k = 1$, then z_0 is the simple pole.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = \lim_{z \to 0} z \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{cosz}{z^2}$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{cosz}{z^2}.$$

$\lim_{z\to z_0}(z-0)^2 f(z) = \lim_{z\to 0} cosz = 1 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

Example:

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{1-e^{2z}}{z^3}$

$\lim_{z\to z_0}(z-0)^3 f(z) = \lim_{z\to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$

$\lim_{z\to z_0}(z-0)^2 f(z) = \lim_{z\to 0} z^2 \frac{(1-e^{2z})}{z^3}.$

$\lim_{z\to z_0}(z-0)^2 f(z) = \lim_{z\to 0} \frac{(1-e^{2z})}{z}.$

$\lim_{z\to z_0}(z-0)^2 f(z) = \lim_{z\to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

6.2.4 Essential singularity

Let $f(z)$ is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z\to z_0}(z-z_0)^n f(z) = \infty$, then z_0 is essential singularity.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} e^{1/z}$

$\lim_{z\to z_0}(z-0)^n f(z) = \lim_{z\to 0} z^n e^{1/z}$.

$\lim_{z\to z_0}(z-0)^n f(z) = \lim_{z\to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

$\lim_{z\to z_0}(z-0)^n f(z) = \infty.$

So $z_0 = 0$ is an essential singularity.

6.2.5 Singularity at infinity

We classify the types of singularities at infinity by letting w = $1/z$ and analyzing the resulting function at w = 0.

Example:

$f(z) = z^3.$

$f(z) = g(w) = 1/w^3.$

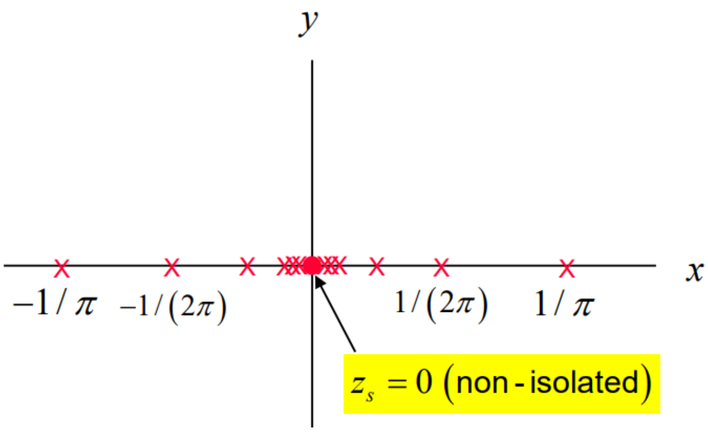
$g(w)$ has a pole of order 3 at w = 0 The function f(z) has a pole of order 3 at infinity.

6.2.5 Non-isolated singularity

A point a is called a non-isolated singularity for $f(z)$ if $f(z)$ is not is not isolated at $z = a$.

Example:

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$



The function is not analytic in any region $0 < |z| < \delta$.

14.3 <u>Classification of singularity by Laurent series expansion</u>

It is also possible to classify the singularity using the Laurent series expansion.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z - a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n$$

Were

$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - a)^{-n+1}}\, d\zeta$

$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta$

The series consisting of the negative powers of $z - a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the principal part or singular part $of\ f(z)\ at\ z = a$.

The singular part of $f(z)\ at\ z = a$ determines the character of the singularity.

6.9.1 Removable singularity by Laurent series expansion

Let $\boldsymbol{a}$ be an isolated singularity for $\boldsymbol{f}(\boldsymbol{z})$. Then $\boldsymbol{a}$ is called a removable singularity if the principal part of $\boldsymbol{f(z)\ at\ z = a}$ has no terms.

If $\boldsymbol{a}$ is a removable singularity for $\boldsymbol{f(z)}$ then the Laurent's series expansion of $\boldsymbol{f(z)\ about\ z = a}$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Hence as $z \to a, f(z) = a_0$ Hence by defining $f(a) = a_0$ the function $f(z)$ becomes analytic at a.

Example:

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

Now $f(z) = \frac{sinz}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots.)$

$f(z) = \frac{sinz}{z} = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots.)$

Here the principal part of $f(z)\ at\ z = 0$ has no terms. Hence z = 0 is a removable singularity.

$\lim_{z \to z_0} f(z)$ also exists then $z_0 = 0$ is the removable singularity.

Example:

Let $f(z) = \frac{z - sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots,)}{z^3}$$

$$f(z) = \frac{\frac{z^3}{3!} - \frac{z^5}{5!} - \dots,)}{z^3}$$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} - \dots,$$

$z = 0$ is a removable singularity. By defining $f(0) = 1/6$ the function becomes analytic $at\ z = 0$.Also $\lim_{z \to z_0} f(z)$ exists then $z_0 = 0$ is the removable singularity.

6.9.2 Pole by Laurent series expansion

Let a be an isolated singularity of $f(z)$. The point a is called a pole if the principal part of $f(z)\ at\ z = a$ has a finite number of terms.

If the principal part of $f(z) at\ z = a$ is given by

$\frac{b1}{z-a} + \frac{b_2}{(z-a)^2} + \dots + \frac{b_r}{(z-a)^r}$, . where $b_r \neq 0$.

We say that a is a pole of order r for $f(z)$. Note: A pole of order 1 is called a simple pole and a pole of order 2 is called double pole.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$f(z) = \frac{e^z}{z} = \frac{1}{z}(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots,)$

$f(z) = \frac{e^z}{z} = (1/z + 1 + \frac{z}{2} + \frac{z^2}{6} + \dots,)$

Here the principal part of $f(z)\ at\ z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$. Also

$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0$. So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{cosz}{z^2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,}{z^2}$$

The principal part of $f(z)\ at\ z = 0$ contains the term $1/z^2$. Hence z=0 is a double pole of f(z).

Also $\lim_{z \to z_0} (z-0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

6.9.3 Essential singularity

Let a be an isolated singularity of $\boldsymbol{f(z)}$. The point a is called an essential singularity of $\boldsymbol{f(z)}$ at $\boldsymbol{z = a}$ if the principal part of $\boldsymbol{f(z)\ at\ z = a}$ has an infinite number of terms.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = e^{1/z}$$

$f(z) = [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,].$

The principal part of $f(z)$ has infinite number of terms. Hence $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Also $\lim_{z \to z_0} (z-0)^n f(z) = \infty$.

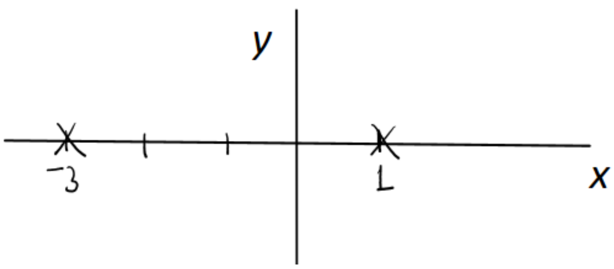
So $z_0 = 0$ is an essential singularity.

14.4 Meromorphic Functions

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example:

$$f(z) = \frac{z}{(z-1)(z+3)^2}$$



$f(z)$ is analytic everywhere in the complex plane except $z = 1$ and $z = -3$.Here $z = 1$ is a simple pile and $z = -3$ is the pole of order 3. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = \lim_{z \to 0} z \frac{e^z}{z}$$

$$\lim_{z \to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order *1* or simple pole. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{cosz}{z^2}$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{cosz}{z^2}.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0.$$

So $z_0 = 0$ is the pole of order 2. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{1-e^{2z}}{z^3}$$

$$\lim_{z \to z_0} (z - 0)^3 f(z) = \lim_{z \to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} z^2 \frac{(1-e^{2z})}{z^3}.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} \frac{(1-e^{2z})}{z}.$$

$$\lim_{z \to z_0} (z - 0)^2 f(z) = \lim_{z \to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$$

So $z_0 = 0$ is the pole of order 2. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

Example:

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} e^{1/z}$

$\lim_{z\to z_0} (z-0)^n f(z) = \lim_{z\to 0} z^n e^{1/z}$.

$\lim_{z\to z_0} (z-0)^n f(z) = \lim_{z\to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

$\lim_{z\to z_0} (z-0)^n f(z) = \infty.$

So $z_0 = 0$ is an essential singularity.Thus this function is not meromorphic in the whole complex plane.

14.5 Mittag-Leffler theorem

The Mittag-Leffler theorem is a fundamental result in complex analysis that deals with the existence of meromorphic functions with prescribed poles and residues. Specifically, it states that for any sequence of distinct points in the complex plane and any sequence of complex numbers, there exists a meromorphic function with poles precisely at the given points and residues equal to the corresponding complex numbers.

More formally, let $z_1, z_2, \ldots$ be a sequence of distinct complex numbers, and let $c_1, c_2, \ldots$ be a sequence of complex numbers. Then there exists a meromorphic function $f(z)$ on the complex plane such that the only poles of $f(z)$ are at the points $z_1, z_2, \ldots$, and the residue of $f(z)$ at z_i is c_i, for $i = 1, 2, \ldots$.

Proof:

To prove the Mittag-Leffler theorem, we will construct the desired meromorphic function $f(z)$ using a standard technique known as the Weierstrass product formula.

This involves expressing $f(z)$ as an infinite product of simple functions, each of which has a single pole at one of the given points and the prescribed residue.

Let D_n be the disc centered at z_n with radius r_n such that D_n is disjoint from all other discs, and let $C_n$be the circle bounding D_n.

Then we define the function $g_n(z)$ as:

$$g_n(z) = (z - z_n)^{-1} e^{(p_n(z - z_n))}$$

where p_n is chosen so that the Laurent series of $g_n(z)$ at z_n has a constant term of c_n. Specifically, we set:

$$p_n = \frac{c_n}{r_n}$$

Using the Cauchy integral formula, we can express $g_n(z)$ as an integral over C_n:

$$g_n(z) = \frac{1}{2\pi i} \int_{C_n} \frac{g_n(w)}{w - z} \, dw$$

Now we define the function $F(z)$ as:

$$F(z) = \prod_{n=1}^{\infty} g_n(z)$$

This product converges absolutely and uniformly on compact sets, since the discs D_n are disjoint and the radii r_n are chosen appropriately. Moreover, $F(z)$ is meromorphic on the complex plane, since each $g_n(z)$ has a single pole at z_n and no other poles.

To see that $F(z)$ has the desired poles and residues, we consider the partial products:

$$F_{N(z)} = \prod_{n=1}^{N} g_n(z)$$

These are meromorphic functions with poles only at the points $z_1, z_2, \ldots, z_N$. Moreover, the residue of $F_{N(z)}$ $at\ z_n\ is\ c_n$, by construction. Finally, we note that $F_{N(z)}$ converges to $F(z)$ as N goes to infinity, since the product converges absolutely and uniformly on compact sets.

Therefore, we have constructed a meromorphic function $f(z)$ with the desired poles and residues, namely:

$$f(z) = F(z)$$

This completes the proof of the Mittag-Leffler theorem.

Question:

Prove that $cotz = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}$ using Mittage Laffer's theorem

Proof:

To prove that $cotz - \frac{1}{z} = 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}$, we can use the Mittag-Leffler theorem.

To prove this identity using the Mittag-Leffler theorem, we need to first identify the poles and their residues of the function $cot(z)$.

We know that $cot(z)$ is periodic with period π, and has simple poles at $z = n\pi$ for all integers n.

Recall that the cotangent function can be expressed as the ratio of the cosine and sine functions:

$$cotz = \frac{\cos z}{sinz}$$

The poles of the cotangent function are the zeros of the sine function, which occur at $z = n\pi$ for all integers n. Thus, we can write:

$$cotz = \frac{cosz}{z - n\pi}$$

To prove this identity using Mittag-Leffler theorem, we need to find the poles and residues of the function cot(z) and the infinite sum in the equation.

First, we know that $cot(z)$ has simple poles at $z = n\pi$ for all integers n.

The residues at these poles are ± 1, depending on the sign of $sin(n\pi)$.

Next, we consider the infinite sum in the equation.

Let $f(z) = \sum\frac{1}{z^2 - n^2\pi^2}$.

This function has poles at $z = \pm n\pi$ for all integers n. The residues at these poles are given by

$$Res[f(z), z = n\pi] = lim_{z\to n\pi}\frac{(z-n\pi)1}{z^2 - n^2\pi^2} = \frac{1}{2n\pi}$$

and

$$Res[f(z), z = -n\pi] = lim_{z\to -n\pi}\frac{(z+n\pi)1}{z^2 - n^2\pi^2} = -\frac{1}{2n\pi}.$$

Now, using the Mittag-Leffler theorem, we can write

$$cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right)$$

$=\sum_{n=1}^{\infty}\frac{1}{2n\pi}\left(\frac{1}{z-n\pi} - \frac{1}{z+n\pi}\right)$

$= 2\sum_{n=1}^{\infty}\frac{1}{2n\pi}\frac{1}{z^2 - n^2\pi^2}$

$= 2f(z)$

Therefore, we have

$cot(z) = \frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{1}{2n\pi}\frac{1}{z^2 - n^2\pi^2}$ as desired.

14.6 Infinite Product of Complex Numbers

An infinite product of complex numbers is given by:

$z = z_1.z_2.z_3 \ldots,.z_n \ldots$

The notation for infinite product is $\prod_{i=1}^{n} z_i$

where $z_1, z_2, z_3, \ldots, z_n$ are complex numbers.

If the infinite product converges, then we can define it as follows:

$$z = \lim_{n\to\infty}(z_1.z_2.z_3 \ldots, z_n)$$

In general, an infinite product of complex numbers is said to converge if and only if the limit of the sequence of partial products (i.e., the product of the first n terms) exists and is nonzero.

Some important results related to infinite products of complex numbers are:

1. If the infinite product $|z|$ converges, then the infinite product z also converges.
2. The infinite product $(1+z)$ converges if and only if the infinite product $(1-|z|)$ converges.
3. The infinite product $(1-z)$ converges if and only if the infinite product $(1+|z|)$ converges.
4. The infinite product $\left(1+\frac{z}{n}\right)^n$ converges to e^z as n approaches infinity, for any complex number z.
5. The infinite product $sin\left(\frac{z}{n}\right)$ converges to zero for any non-zero complex number z.

Question

Suppose an infinite product is absolutely convergent. Prove that it is convergent

Solution

Suppose the infinite product is given by:

$$P = a_1 . a_2 . a_3 \ldots$$

where ai are non-negative real numbers.

By the absolute convergence of P, we have that the series:

$$S = log(a_1) + log(a_2) + log(a_3) + \ldots$$

converges.

Since the logarithm function is continuous, we can take the exponential of both sides to obtain:

$$e^S = e^{\log(a_1)} e^{log(a2)} e^{log(a3)} \ldots$$

which simplifies to:

$$P = a_1 a_2 a_3 \ldots$$

Thus, the absolute convergence of P implies that the series S converges, which in turn implies that P converges as well.

Therefore, we have shown that if an infinite product is absolutely convergent, then it is also convergent.

14.7 Review questions

1. $\boldsymbol{f(z) = \frac{z}{(z-1)^2}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

2. $\boldsymbol{f(z) = \frac{z}{(z-1)(z-5)(z-2)^3}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

3. Check whether the following functions is meromorphic?

   $\boldsymbol{f(z) = \frac{e^z}{z}}$

4. Check whether the following functions is meromorphic?

   $\boldsymbol{g(z) = \frac{sinz}{(z-1)^2}}$.

5. State and prove the Mittag-Leffler theorem

14.8 Self-assessment

1. $\boldsymbol{f(z) = \frac{z}{(z-1)}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

A. 5
B. 1
C. 2
D. 3

2. $\boldsymbol{f(z) = \frac{z}{(z-1)(z-2)^3}}$ **is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?**

A. 5
B. 1
C. 2
D. 3

3. Which one of the following functions is meromorphic?

$\boldsymbol{f(z) = \frac{e^z}{z}}$**, and** $\boldsymbol{g(z) = \frac{sinz}{(z-1)^2}}$.

A. Only $f(z)$
B. Only $g(z)$
C. Both $f(z)$ and $g(z)$
D. Neither $f(z)$ nor $g(z)$

4. Consider the $\boldsymbol{f(z) = \frac{z}{1-z}}$ **then**

A. $z_0 = 1$ is the singular point of $f(z)$
B. $z_0 = 0$ is the singular point of $f(z)$
C. $z_0 = 10$ is the singular point of $f(z)$
D. There is no singular point of $f(z)$

5. Consider the $f(z) = z^2$ then

A. $z_0 = 1$ is the singular point of $f(z)$

B. $z_0 = 0$ is the singular point of $f(z)$

C. $z_0 = 10$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

6. Consider the $f(z) = \frac{z^2-9}{z^2(z-1)(z-1-2i)}$ then

A. $z_0 = 1$ is one of the singular points of $f(z)$

B. $z_0 = 3$ is the singular point of $f(z)$

C. $z_0 = -3$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

7. What is Mittag-Leffler's theorem?

A. A theorem on the convergence of infinite series.

B. A theorem on the analytic continuation of meromorphic functions.

C. A theorem on the existence of a holomorphic function with prescribed singularities

8. What does the theorem say about meromorphic functions?

A. They can be extended to the whole complex plane.

B. They can be extended to a neighborhood of their poles.

C. They can be approximated by polynomials

9. What are the conditions for the Mittag-Leffler's theorem to hold?

A. The function must have isolated singularities and a certain growth condition.

B. The function must be holomorphic and bounded on a compact set.

C. The function must be a polynomial

10. What is the significance of the theorem in complex analysis?

A. It provides a method for approximating meromorphic functions

B. It is a fundamental tool for studying the Riemann zeta function

C. It allows us to construct meromorphic functions with prescribed singularities

11. What is the value of the infinite product (1+i)(1-i)(1+i)(1-i)...?

A. 1

B. -1

C. i

D. -i

12. What is the value of the infinite product (1+2i)(1-2i)(1+2i)(1-2i)...?

A. 1

B. -1

C. 2i

D. -2i

13. What is the value of the infinite product (1+i/2)(1-i/2)(1+i/2)(1-i/2)...?

A. 1

B. -1/2

C. i/2
D. -i/2

14. What is the value of the infinite product (1+i/3)(1-i/3)(1+i/3)(1-i/3)...?
A. 1
B. -1/3
C. i/3
D. -i/3

15. What is the value of the infinite product (1+3i)(1-3i)(1+3i)(1-3i)...?
A. 1
B. -1
C. 3i
D. -3i

Table 1: Answers of self-assessment

Question number	Correct answer
1	B
2	C
3	C
4	A
5	D
6	A
7	B
8	A
9	A
10	**C**
11	**A**
12	B
13	B
14	B
15	B

14.9 Summary

- The A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

- The An infinite product of complex numbers is given by:

$z = z_1.z_2.z_3 \ldots,.z_n \ldots$

The notation for infinite product is $\prod_{i=1}^{n} z_i$

- If the infinite product $|z|$ converges, then the infinite product z also converges.
- The infinite product $(1 + z)$ converges if and only if the infinite product $(1 - |z|)$ converges.
- The infinite product $(1 - z)$ converges if and only if the infinite product $(1 + |z|)$ converges.

14.10 Keywords

Meromorphic function:

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Mittag-Leffler theorem :

Let $z_1, z_2, \ldots$ be a sequence of distinct complex numbers, and let $c_1, c_2, \ldots$ be a sequence of complex numbers. Then there exists a meromorphic function $f(z)$ on the complex plane such that the only poles of $f(z)$ are at the points $z_1, z_2, \ldots$, and the residue of $f(z)$ at z_i is c_i, for $i = 1, 2, \ldots$.

14.11 Further Readings

Books

1. Complex Variables And Applications By Churchill, R. V. And Brown, J. W., Mcgraw Hill Education.
2. Foundations Of Complex Analysis By S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis By Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory And Applications By H. S. Kasana, Prentice Hall.

LOVELY PROFESSIONAL UNIVERSITY
Jalandhar-Delhi G.T. Road (NH-1)
Phagwara, Punjab (India)-144411
For Enquiry: +91-1824-521360
Fax.: +91-1824-506111
Email: odl@lpu.co.in