Partial Differential Equations

DEMTH530

Edited by:
Dr. Kulwinder Singh

LOVELY PROFESSIONAL UNIVERSITY

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Content

Dr. Preety Kalra, Lovely Professional University

Unit 01: Linear First Order Partial Differential Equation

CONTENTS

Objectives

After studying this unit, you will be able to

- understand about the different types of partial differential equations.
- analyze in the form of an explicit form, preferably in the form of elementary functions.
- find the qualitative property of the partial differential equation.
- understand the integral surfaces and orthogonal surfaces.

Introduction

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations.

1.1 Partial Differential Equation (P.D.E.)

Definition 1.1.1 An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation. For examples of partial differential equations we list the following:

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy \quad (1.1.1)$$

$$\left(\frac{\partial z}{\partial x}\right)^2+\frac{\partial^3 z}{\partial y^3}=2x\left(\frac{\partial z}{\partial x}\right) \quad (1.1.2)$$

$$z\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial y}=x \quad (1.1.3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz \quad (1.1.4)$$

$$\frac{\partial^2 z}{\partial x^2}=\left(1+\frac{\partial z}{\partial y}\right)^{\frac{1}{2}} \quad (1.1.5)$$

$$y\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}=z\left(\frac{\partial z}{\partial y}\right) \quad (1.1.6)$$

Definition 1.1.2 Order of a partial differential equation: The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation. In Art. 1.1.1, equations (1.1.1), (1.1.3), (1.1.4) and (1.1.6) are of the first order, (1.1.5) is of the second order and (1.1.2) is of the third order.

Definition 1.1.3 Degree of a partial differential equations: The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, i.e., made free from radicals and fractions so far as derivatives are concerned. In 1.1.1, equations (1.1.1), (1.1.2), (1.1.3) and (1.1.4) are of first degree while equations (1.1.5) and (1.1.6) are of second degree.

Definition 1.1.4. Linear and non-linear partial differential equations: A partial differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a non-linear partial differential equation. In Art. 1.1.1, equations (1.1.1) and (1.1.4) are linear while equations (1.1.2), (1.1.3), (1.1.5) and (1.1.6) are nonlinear.

Notes: When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$$p=\frac{\partial z}{\partial x},\qquad q=\frac{\partial z}{\partial y},\qquad r=\frac{\partial^2 z}{\partial x^2},\qquad s=\frac{\partial^2 z}{\partial x\partial y},\qquad \text{and}\qquad t=\frac{\partial^2 z}{\partial y^2}.$$

In case there are n independent variables, we take them to be x_1, x_2....., x_n and z is then regarded as the dependent variable. In this case we use the following notations :

$$p_1=\frac{\partial z}{\partial x_1},\qquad p_2=\frac{\partial z}{\partial x_2},\qquad p_3=\frac{\partial z}{\partial x_3},\qquad \ldots\ldots\ldots\ldots\ldots..,\qquad p_n=\frac{\partial z}{\partial x_n}$$

Caution: Sometimes the partial differentiations are also denoted by making use of suffixes. Thus we write $u_x=\frac{\partial u}{\partial x},\ u_y=\frac{\partial u}{\partial y},\ u_{xx}=\frac{\partial^2 u}{\partial x^2}, u_{xy}=\frac{\partial^2 u}{\partial x\partial y}$and so on.

1.2 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples

Definition 1.2.1 Linear equation: A first order equation $f\,(x,y,z,p,q)\ =\ 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x,y)\,p\ +\ Q(x,y)\,q\ =\ R(x,y)\,z\ +\ S(x,y)$.

Example 1.2.1:$yx^2p+xy^2q=xyz+x^2y^3$and $p+q=z+xy$ are both first order linear partial differential equations.

Definition 1.2.2 Semi-linear equation: A first order partial differential equation $f\,(x,y,z,p,q)\ =\ 0$ is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form $P(x,y)\,p\ +\ Q(x,y)\,q\ =\ R(x,y,z)$.

Example 1.2.2:$xyp + x^2yq = x^2y^2z^2$and $yp + xq = x^2z^2/y^2$ are both first order semi-linear partial differential equations.

Definition 1.2.3 Quasi-linear equation: A first order partial differential equation $f\ (x, y, z, p, q) = 0$ is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form $P(x, y, z)\ p + Q(x, y, z)\ q = R(x, y, z)$

Example 1.2.3:$x^2zp + y^2zq = xy$ and $(x^2 - yz)p + (y^2 - xz)q = z^2 - xy$ are first order quasi-linear partial differential equations.

Definition 1.2.4 Non-linear equation: A first order partial differential equation $f\ (x, y, z, p, q) = 0$ which does not come under the above three types, in known as a non-liner equation.

Example 1.2.3:$p^2 + q^2 = 1,\ pq = z$ and $x^2p^2 + y^2q^2 = z^2$ are all non-linear partial differential equations.

1.3 Linear Partial Differential Equations of Order One

LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as Lagrange equation. For Example xyp + yzq = zx is a Lagrange equation.

Lagrange's method of solving Pp + Qq = R, when P, Q and R are functions of x, y, z

Theorem 1.4.1: The general solution of Lagrange equation

$$Pp + Qq = R, \tag{1.4.1}$$

is $\phi(u, v) = 0$ (1.4.2)

where ϕ is an arbitrary function and

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ (1.4.3)

are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{1.4.4}$$

Here, $c_1$and $c_2$are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if u/v is not merely a constant.

Proof: Differentiating (1.4.2) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \tag{1.4.5}$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \tag{1.4.6}$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between (1.4.5) and (1.4.6), we have

$$\left[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right] = 0$$

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) =$$

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right)q = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{1.4.7}$$

Hence (1.4.2) is a solution of the equation (1.4.7) .

Taking the differentials of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, we get

$$\left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz = 0 \tag{1.4.8}$$

and

$$\left(\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\right)dy + \left(\frac{\partial v}{\partial z}\right)dz = 0 \tag{1.4.9}$$

Since u and v are independent functions, solving (1.4.8) and (1.4.9) for the ratios dx : dy : dz, gives

$$\frac{dx}{\frac{\partial u\partial v}{\partial z\partial y}-\frac{\partial u\partial v}{\partial y\partial z}} = \frac{dy}{\frac{\partial u\partial v}{\partial x\partial z}-\frac{\partial v\partial u}{\partial x\partial x}} = \frac{dz}{\frac{\partial u\partial v}{\partial x\partial y}-\frac{\partial u\partial v}{\partial y\partial x}} \qquad (1.4.10)$$

Comparing (1.4.4) and (1.4.10), we obtain

$$\frac{\frac{\partial u\partial v}{\partial z\partial y}-\frac{\partial u\partial v}{\partial y\partial z}}{P} = \frac{\frac{\partial u\partial v}{\partial x\partial z}-\frac{\partial v\partial u}{\partial x\partial x}}{Q} = \frac{\frac{\partial u\partial v}{\partial x\partial y}-\frac{\partial u\partial v}{\partial y\partial x}}{R} = k, (say)$$

$$\Rightarrow \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} = Pk, \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} = Qk,\ \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = Rk$$

Substituting these values in (1.4.7), we get $k(Pp + Qq) = Rk$, or $Pp + Qq = R$, which is the given equation (1.4.1).

Therefore, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the system of differential equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, then $\phi(u, v) = 0$ is a solution of $Pp + Qq = R$, being an arbitrary function. This is what we wished to prove.

Notes: Equations (1.4.4) are called Lagrange's auxiliary (or subsidiary) equations for (1.4.1).

Working Rule for solving $Pp + Qq = R$ by Lagrange's method.

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R \qquad (1.5.1)$$

Step 2. Write down Lagrange's auxiliary equations for (1.5.1) namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (1.5.2)$$

Step 3. Solve (1.5.2) by using the well known methods. Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (1.5.2).

Step 4. The general solution (or integral) of (1.5.1) is then written in one of the following three equivalent forms : $\phi(u, v) = 0$, u=$\phi(v)$ or v=$\phi(u)$, ϕ being an arbitrary function.

Examples based on working rule 1.5.

In what follows we shall discuss four rules for getting two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (1.6.1)$$

Accordingly, we have four types of problems based on $+Qq = R$.

1.4 Type 1 based on Rule I for solving (dx)/P = (dy)/Q = (dz)/R

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1.6.1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1.6.1).

Example 1.6.1: Solve $\left(\frac{y^2z}{x}\right)p + xzq = y^2$.

Solution: Given $\left(\frac{y^2z}{x}\right)p + xzq = y^2$ (1.6.2)

The Lagrange's auxiliary equations for (1.6.2) are

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \qquad (1.6.3)$$

Taking the first two fractions of (1.6.3), we have

$x^2zdx = y^2zdy$ or $3x^2dx - 3y^2dy = 0,$ (1.6.4)

Integrating (1.6.4), $x^3 - y^3 = c_1, c_1$ being an arbitrary constant. (1.6.5)

Next, taking the first and the last fractions of (1.6.3), we get

$xy^2dx = y^2zdz$ or $2xdx - 2zdz = 0$ (1.6.6)

Integrating (1.6.6), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant. (1.6.7)

From (1.6.5) and (1.6.7), the required general integral is $\Phi(x^3 - y^3, x^2 - z^2)$, ϕ being an arbitrary function.

Example 1.6.2: Solve $y^2p - xyq = x(z - 2y)$.

Solution: Here Lagrange's auxiliary equations are

$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$ (1.6.8)

Taking the first two fractions of (1.6.8) and re–writing, we get

$2xdx + 2ydy = 0$ so that $x^2 + y^2 = c_1$. (1.6.9)

Now, taking the last two fractions of (1.6.8) and re–writing, we get

$\frac{dz}{dy} = -\frac{z-2y}{y}$ or $\frac{dz}{dy} + \frac{z}{y} = 2$ (1.6.10)

which is linear in z and y. Its I.F. $= e^{\int \left(\frac{1}{y}\right)dy} = e^{loglogy} = y$. Hence solution of (1.6.10) is

$z.y = \int 2ydy + c_2$ or $zy - y^2 = c_2$,

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired solution, where ϕ is an arbitrary function.

1.5 Type 2 based on Rule II for solving (dx)/P = (dy)/Q = (dz)/R.

Suppose that one integral of (1.6.1) is known by using rule I explained in Art 2.5 and suppose also that another integral cannot be obtained by using rule I of Art. 2.5. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

Example 1.6.3:: Solve $p + 3q = 5z + tan(y - 3x)$.

Solution: Given $p + 3q = 5z + tan(y - 3x)$ (1.6.11)

The Lagrange's subsidiary equations for (1.6.11) are

$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+tan(y-3x)}$ (1.6.12)

Taking the first two fractions, $dy - 3dx = 0$ (1.6.13)

Integrating (1.6.13), $y - 3x = c_1$, $c_1$being an arbitrary constant (1.6.14)

Using (16.14), from (1.6.12) we get $\frac{dx}{1} = \frac{dz}{5z+tan(c_1)}$ (1.6.15)

Integrating (1.6.15), $x - \frac{1}{5}log(5z + tan\, tanc_1) = \frac{1}{5}c_2$, $c_2$being an arbitrary constant.

or $5x - log(5z + y - 3x)) = c_2$, using (1.6.14) (1.6.16)

From (1.6.14) and (1.6.16), the required general integral is $5x - log(5z + tan\, tan(y - 3x)) = \phi(y - 3x)$, where ϕ is an arbitrary function.

Example 1.6.4: Solve $xyp + y^2q = zxy - 2x^2$.

Solution: Given $xyp + y^2q = zxy - 2x^2$ (1.6.17)

The Lagrange's subsidiary equations for (1.6.17) are

$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$ (1.6.18)

Taking the first two fractions of (1.6.18), we have

$\frac{dx}{xy} = \frac{dy}{y^2}$ or $\frac{dx}{x} - \frac{dy}{y} = 0$ (1.6.19)

Integrating (1.6.19), $logx - logy = log\ logc_1$ or $\frac{x}{y} = c_1$ (1.6.20)
From (1.6.20), $x = c_1y$.
Hence from second and third fractions of (1.6.18), we get

$\frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$ or $c_1dy - \frac{dz}{z - 2c_1^2} = 0$ (1.6.21)
Integrating (1.6.21), $c_1y - log(z - 2c_1^2) = c_2$ or $x - log\ log\left(z - \frac{2x^2}{y^2}\right) = c_2$ (1.6.22)
From (1.6.20) and (1.6.22), the required general solution is

$x - log\ log\left(z - \frac{2x^2}{y^2}\right) = \phi\left(\frac{x}{y}\right)$, ϕ being an arbitrary function.

1.6 Type 3 based on Rule III for solving (dx)/P = (dy)/Q = (dz)/R

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well-known principle of algebra, each fraction in (1.6.1) will be equal to $\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$ (1.6.23)
If $PP_1 + QQ_1 + RR_1 = 0$, then we know that the numerator of (1.6.23) is also zero. This give $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u(x, y, z) = c_1$. This method may be repeated to get another integral $u(x, y, z) = c_2$. P_1, Q_1 and R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I or rule II as the case may be.

Example 1.6.5: Solve $z(x + y)p + z(x - y)q = x^2 + y^2$.

Solution: Given $z(x + y)p + z(x - y)q = x^2 + y^2$ (1.6.24)
The Langrange's subsidiary equations for (1) are $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ (1.6.25)
Choosing x, –y, –z, as multipliers, each fraction
$\frac{xdx - ydy - zdz}{xz(x+y) - zy(x-y) - z(x^2+y^2)} = \frac{xdx - ydy - zdz}{0}$
$\therefore xdx - ydy - zdz = 0$ or $2xdx - 2ydy - 2zdz = 0$
Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constant. (1.6.26)
Again, choosing y, x, –z as multipliers, each fraction
$\frac{ydx + xdy - zdz}{yz(x+y) + zx(x-y) - z(x^2+y^2)} = \frac{ydx + xdy - zdz}{0}$
$\therefore ydx + xdy - zdz = 0$
$2d(xy) - 2zdz = 0$
Integrating, $2xy - z^2 = c_2$, c_2 being an arbitrary constant. (1.6.27)
From (1.6.26) and (1.6.27), the required general solution is given by $\phi(x^2 - y^2 - z^2, 2xy - z^2)$, ϕ being an arbitrary function.

Example 1.6.6 : Solve $(x + 2z)p + (4zx - y)q = 2x^2 + y$.

Solution: Here Lagrange's auxiliary equations are $\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y}$ (1.6.28)
Choosing y, x, –2z as multipliers, each fraction of (1.6.28)
$\frac{ydx + xdy - 2zdz}{y(x+2z) + x(4zx-y) - 2z(2x^2+y)} = \frac{ydx + xdy - 2zdz}{0} = d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$ (1.6.29)
Choosing 2x, –1, –1 as multipliers, each fraction of (1.6.28)
$\frac{2xdx - dy - dz}{2x(x+2z) - (4zx-y) - (2x^2+y)} = \frac{2xdx - dy - dz}{0}$ or $2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$ (1.6.30)
$\therefore$ *From* (2) *and* (3), *solution is* $\phi(xy - z^2, x^2 - y - z) = 0$, ϕ *being an arbitrary function.*

1.7 Type 4 based on Rule IV for solving (dx)/P = (dy)/Q = (dz)/R

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well-known principle of algebra, each fraction of (1.6.1) will be equal to
$\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$ (1.6.31)

Suppose the numerator of (1.6.31) is the exact differential of the denominator of (1.6.31). Then (1.6.31) can be combined with a suitable fraction in (1.6.1) to give an integral.
However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$\frac{P_2dx+Q_2dy+R_2dz}{PP_2+QQ_2+RR_2} \quad (1.6.32)$$

is such that its numerator is exact differential of the denominator. Fractions (16.31) and (1.6.32) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 or rule 2 or rule 3.

Example 1.6.7 : Solve $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$.

Solution: Here the Lagrange's auxiliary equations for the given equation

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)} \quad (1.6.33)$$

Taking the first two fractions of (1.6.33), $x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$

Integrating, $x^3 + y^3 = c_2, c_2$ being an arbitrary as constant.

Choosing 1, –1, 0 as multipliers, each fraction of (1.6.33)

$$=\frac{dx-dy}{y^2(x-y)-x^2(y-x)} = \frac{dx-dy}{(x-y)(y^2+x^2)} \quad (1.6.34)$$

Combining the third fraction of (1.6.33) with fraction (1.6.34), we get

$$\frac{dx-dy}{(x-y)(y^2+x^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{(x-y)} - \frac{dz}{z} = 0.$$

Integrating, $log\ log(x-y) - log\ logz = log\ logc_2$ or $\frac{x-y}{z} = c_2$
(1.6.35)

From (1.6.34) and (1.6.35), solution is $\phi(x^3+y^3, \frac{x-y}{z}) = 0, \phi$ being an arbitrary function.

Example 1.6.8 : Find the general integral of $xzp + yzq = xy$.

Solution: Here the Lagrange's auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \quad (1.6.36)$$

From the first two fractions of (1.6.36), $\frac{dx}{xz} = \frac{dy}{yz}$.

Integrating, $log\ logx = log\ logy + log\ logc_1$ $\frac{x}{y} = c_1$
(1.6.37)

Choosing 1/x, 1/y, 0 as multipliers, each fraction of (1.6.36)

$$= \frac{\left(\frac{1}{x}\right)dx+(1/y)dy}{\left(\frac{1}{x}\right)xy+(1/y)yz} = \frac{ydx+xdy}{2xyz} \quad (1.6.38)$$

Combining the last fraction of (1.6.36) with fraction (1.6.37), we have

$$\frac{ydx+xdy}{2xyz} = \frac{dz}{xy} \quad \text{or} \quad ydx + xdy = 2zdz \quad \text{or} \quad d(xy) = 2zdz \quad \text{or} \quad d(xy) - 2zdz = 0$$

Integrating, $xy - z^2 = c_2,\ c_2$ being an arbitrary constant. (1.6.39)

From (1.6.37) and (1.6.39) solution is $\phi\left(\frac{x}{y}, xy - z^2\right) = 0,\ \phi$being an arbitrary function.

1.8 Integral Surfaces Passing through a given Curve

In the last article we obtained general integral of $Pp + Qq = R$. We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.

Let $Pp + Qq = R$. (1.7.1)

be the given equation. Let its auxiliary equations give the following two independent solutions $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ (1.7.2)

Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by x = x(t), y = (t), z = z(t), (1.6.3)

where t is a parameter. Then (1.7.2) may be expressed as

$u(x(t), y(t), z(t)) = c_1$ and $v(x(t), y(t), z(t)) = c_2$ (1.7.4)

We eliminate single parameter t from the equations of (1.7.4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (1.7.2) and obtain the required integral surface.

Example 1.7.1:Find the integral surface of the linear partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0, z = 1$.

Solution: Given $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ (1.7.5)

Lagrange's auxiliary equations of (1.7.5) are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad (1.7.6)$$

Choosing 1/x, 1/y, 1/z as multipliers, each fraction of (1.7.6)

$$\frac{\left(\frac{1}{x}\right)dx+\left(\frac{1}{y}\right)dy+\left(\frac{1}{z}\right)dz}{(y^2+z)-(x^2+z)+(x^2-y^2)} = \frac{\left(\frac{1}{x}\right)dx+\left(\frac{1}{y}\right)dy+\left(\frac{1}{z}\right)dz}{0} \Rightarrow log\ logx + log\ logy + loz\ z = log\ logc_1 \quad \text{or} \quad xyz = c_1 \quad (1.7.8)$$

Choosing x, y, –1 as multipliers, each fraction of (1.7.6)

$$\frac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{xdx + ydy - dz}{0}$$

$$\Rightarrow xdx + ydy - dz = 0 \quad or \quad x^2 + y^2 - z = c_2 \quad (1.7.9)$$

Taking t as a parameter, the given equation of the straight-line x + y = 0, z = 1 can be put in parametric form x = t, y = –t, z = 1. (1.7.10)

Using (1.7.10), (1.7.9) may be re–written as $t^2 = c_1$ and $2t^2 - 2 = c_2$. (1.7.11)

Eliminating t from the equations of (5), we have $2(c_1) - 2 = c_2$ or $2c_1 + c_2 + 2 = 0$. (1.7.12)

Putting values of c_1 and c_2 from (3) in (6), the desired integral surface is $2xyz + x^2 + y^2 - 2z + 2 = 0$.

1.9 Surfaces Orthogonal to a Given System of Surfaces

Let f(x, y, z) = C (1.8.1)

represents a system of surfaces where C is a parameter. Suppose we wish to obtain a system of surfaces which cut each of (1.8.1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1.8.1) which passes through that point are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

Let the surface $z = \phi(x,y)$ (1.8.2)

cuts each surface of (1.8.1) at right angles. Then the normal at (x, y, z) to (1.8.2) has direction ratios $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ i.e., p, q, –1. Since normals at (x, y, z) to (1.8.1) and (1.8.2) are at right angles, we have

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} = 0 \quad \text{or} \quad p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \quad (1.8.3)$$

which is of the form $Pp + Qq = R$.

Conversely, we easily verify that any solution of (1.8.3) is orthogonal to every surface of (1.8.1).

Example 1.8.1 : Find the surface which intersects the surfaces of the system z(x + y) = c(3z + 1) orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.

Solution: The given system of surfaces is $f(x,y,z) = \frac{z(x+y)}{3z+1} = c$ (1.8.4)

$\therefore \frac{\partial f}{\partial x} = \frac{z}{3z+1},$ $\frac{\partial f}{\partial y} = \frac{z}{3z+1}$ $\frac{\partial f}{\partial z} = (x+y)\frac{3z+1-z3}{(3z+1)^2} = \frac{(x+y)}{(3z+1)^2}$

The required orthogonal surface is solution of

$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$ or $\frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{(x+y)}{(3z+1)^2}$

$z(3z+1)p + z(3z+1)q = (x+y)$ (1.8.5)

Lagrange's auxiliary equations for (1.8.5) are $\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{(x+y)}$. (1.8.6)

Taking the first two fractions of (1.8.6), we get dx – dy = 0 so that x – y = c_1. (1.8.7)

Choosing x, y, –z(3z + 1) as multipliers, each fraction of (1.8.6) = $xdx + ydy - z(3z+1)dz/0$

$xdx + ydy - 3z^2\,dz - zdz = 0$ or $2xdx + 2ydy - 6z^2\,dz - 2zdz = 0$

Integrating, $x^2 + y^2 - 2z^3 - z = c_2$, c_2 being an arbitrary constant. (1.8.8)

Hence any surface which is orthogonal to () has equation of the form

$x^2 + y^2 - 2z^3 - z = \phi\,(x - y)$, ϕ being an arbitrary function ...(6)

In order to get the desired surface passing through the circle $x^2 + y^2 = 1$, z = 1 we must choose ϕ (x – y) = –2. Thus, the required particular surface is $x^2 + y^2 - 2z^3 - z = -2$.

1.10 Cauchy's Problem For First Order Equations

The aim of an existence theorem is to establish conditions under which we can decide whether or not a given partial differential equation has a solution at all; the next step of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first order partial differential equation are easily contained in the classic problem of Cauchy, which for the two independent variables can be stated as follows:

Cauchy's problem for first order partial differential equation

If (a) $x_0(\mu), y_0(\mu)$ and $z_0(\mu)$ are functions which, together with their first derivatives, are continuous in the interval I defined by $\mu_1 < \mu < \mu_2$.

(b) And if f(x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the $xyzpq$ space, then it is required to establish the existence of a function $\phi(x, y)$ with the following properties :

(i) $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

(ii) For all values of x and y lying in R, the point $\{x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)\}$ lies in U and $f\{x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)\} = 0$.

(iii) For all μ belonging to the interval I, the point $\{x_0(\mu), y_0(\mu)\}$ belongs to the region R, and $\phi\{x_0(\mu), y_0(\mu)\} = z_0$

Stated geometrically, what we wish to prove is that there exists a surface $z = \phi(x, y)$ which passes through the curve C whose parametric equations are given by $x = x_0(\mu), y = y_0(\mu)$ and $z = z_0(\mu)$ and at every point of which the direction (p, q, – 1) of the normal is such that f$(x, y, z, p, q) = 0$

Summary

- The first-order linear, quasi-linear and semi linear partial differential equations are defined.
- All the types of differential equations with examples are explained.
- Different kinds of solutions of Lagrange's equation are elaborated.
- Discussion to find the integral surface passing through a given curve.
- Surface orthogonalto a given system of surfaces determined.

Keywords

- Linear
- Quasi linear
- Semi linear
- Lagrange's method
- Integral surface
- Orthogonal surface

Self Assessment

1. The equation of $x^2 p + yq = (x-y)z^2 + x - y$ is

A. Quasi-linear
B. Semi-linear
C. Linear
D. Non-linear

2. The differential equation $p \tan y + q \tan x = \sec^2 z$ is of order

A. 1
B. 2
C. 0
D. None of these

3. The equation $\frac{\partial^2 z}{\partial x^2} - 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 = 0$ is of order

A. 1
B. 2
C. 3
D. None of these

4. The equation $(2x+3y)p + 4xq - 8pq = x + y$ is

A. Linear
B. Non-linear
C. Semi-linear
D. Quasi-linear

5. The equation $(x+y-z)(\partial z/\partial x) + (3x+2y)(\partial z/\partial y) + 2z = x + y$ is

A. Linear
B. Quasi-linear
C. Non-linear
D. Semi-linear

6. The partial differential equation $f(x,y,z)(\partial z/\partial x) + g(x,y,z)(\partial z/\partial y) = h(x,y,z)$ is

A. Quasi-linear

B. Semi-linear

C. Linear

D. Non-linear

7. The auxiliary equation of $p+q+1=0$ is

A. $dx = dy = dz$

B. $dx = dy = -dz$

C. $dx / p = dy / q = dz$

D. None of these

8. The general solution of partial differential equation $Pp + Qq = R$ is

A. $\phi(u,v) = 1$

B. $\phi(u,v) = -1$

C. $\phi(u,v) = 0$

D. $\phi(u,v) = c$

9. What is the nature of Lagrange's linear partial differential equation?

A. First-order,Third-degree

B. Second-order,First-degree

C. First-order,Second-degree

D. First-order, First-degree

10. The solution of the equation $xu_x + yu_y = 0$ is of the form

A. $f(y/x)$

B. $f(y+x)$

C. $f(y-x)$

D. $f(xy)$

11. The subsidiary equations for partial differential equation $y^2 z / x + zxy = y^2$ are

A. $dx / y^2 z = dy / zx = dz / y^2$

B. $dx / x^2 = dy / y^2 = dz / zx$

C. $dx / x^2 z = dy / y = dz / zx$

D. $dx /(1/x^2) = dy /(1/y^2) = dz /(1/zx)$

12. The general solution of partial differential equation $(y-z)p + (z-x)q = x - y$ is

A. $\phi(x+y+z, x^2+y^2+z^2) = 0$

B. $\phi(xyz, x^2+y^2+z^2) = 0$

C. $\phi(x+y+z, xyz) = 0$

D. $\phi(x-y-z, x^2-y^2-z^2) = 0$

13. The integral surface which passes through the given curve is taken as equation in

A. Parametric form

B. Hyperbolic form

C. Constants

D. None of these

14. Theintegralsurfacetothefirstorderpartialdifferentialequation $2y(z-3)p+(2x-z)q=y(2x-3)$ passing through the curve $x^2+y^2=2x,\ z=0$ is
A. $x^2+y^2-z^2-2x+4z=0$
B. $x^2+y^2-z^2+2x+8z=0$
C. $x^2+y^2+z^2-2x+8z=0$
D. None of these

15. The integral surface of the partial differential equation $(x^2-yz)p+(y^2-zx)q=z^2-xy$ which passes through the line $x=1, y=0$ is
A. $(x+y)(xy+yz+zx)+y+z=0$
B. $(x+y)(xy+yz+zx)+y-z=0$
C. $(x-y)(xy+yz+zx)+y-z=0$
D. None of these

16. The direction ratios of normal to the surface $z=\phi(x,y)at\ (x,y,z)$ are
A. (p,q,-1)
B. (p,q,1)
C. (-p,-q,1)
D. (-p,q,-1)

17. If the two surfaces are cuts orthogonal to each other, then the solutions of these equations are reduces to
A. Heat equation
B. Wave equation
C. Lagrange's linear equation
D. None of these

18. If the two surfaces $z=\phi(x,y)$ and $f(x,y,z)=c$ are orthogonal then it satisfy the condition
A. $p\,\frac{\partial f}{\partial x}+q\,\frac{\partial f}{\partial x}=\frac{\partial f}{\partial z}$
B. $p\,\frac{\partial f}{\partial x}+q\,\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}=0$
C. $p\,\frac{\partial f}{\partial x}-q\,\frac{\partial f}{\partial x}-\frac{\partial f}{\partial z}$=0
D. None of these

Answers for Self Assessment

1.	B	2.	A	3.	B	4.	B	5.	B
6.	A	7.	B	8.	C	9.	D	10.	A
11.	D	12.	A	13.	A	14.	A	15.	C

16. A 17. C 18. A

Review Questions

Q1. Find the equation of integral surface of the differential equation

$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ passes through the line $x = 1,\ y = 0$.

Q2. Solve $p + q = x + y + z$.

Q3. Find the integral surface of the partial differential equation

$2y(z-3)p + (2x - z)q = y(2x-3)$ which passes through the circle $z = 0,\ x^2 + y^2 = 2x.$

Q4. Find the general solution of the differential equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$.

Q5.Find the integral surface of the linear partial differential equation

$x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$ which contains the straight line $x - y = 0,\ z = 1.$

Further Readings

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Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

https://onlinecourses.nptel.ac.in/noc21_ma09/preview

https://onlinecourses.swayam2.ac.in/cec22_ma12/preview

Dr. Preety Kalra, Lovely Professional University

Unit 02: Non-Linear First Order Partial Differential Equations- I

CONTENTS

Objectives

After studying this unit, you will be able to

- understand about the nonlinear partial differential equation of first order.
- analyze in the form of Cauchy's characteristic curve functions.
- find the envelope of family of curves.
- understand the integral surfaces passing through the given curve.

Introduction

We turn now to the more difficult problem of finding the solutions of the partial differential equation $F(x, y, z, p, q) = 0$ (2.0.1)

in which the function F is not necessarily linear in p and q. The partial differential equation of the two - parameter system $f(x, y, z, a, b) = 0$ (2.0.2)

was of this form. It will be shown a little later that the converse is also true; i.e., that any partial differential equation of the type (2.0.1) has solutions of the type (2.0.2). Any envelope of the system (2.0.2) touches at each of its points a member of the system. It possesses therefore the same set of values (x, y, z, p, q) as the particular surface, so that it must also be a solution of the differential equation. In this way we are led to three classes of integrals of a partial differential equation of the type (1):

(a) Two -parameter systems of surfaces $f(x, y, z, a, b) = 0$. Such an integral is called a complete integral.

(b) If we take any one -parameter subsystem $f(x, y, z, a, \phi(a)) = 0$ of the system (2.0.2), and form its envelope, we obtain a solution of equation.

(1). When the function $\phi(a)$ which defines this subsystem is arbitrary, the solution obtained is called the general integral of (2.0.1) corresponding to the complete integral (2.0.2). When a definite function $\phi(a)$ is used, we obtain a particular case of the general integral.

(c) If the envelope of the two -parameter system (2.0.2) exists, it is also a solution of the equation (2.0.1); it is called the singular integral of the equation.

Example 2.0.1: We can illustrate these three kinds of solution with reference to the partial differential equation

$z^2(1+p^2+q^2)=1$ (2.0.3)

We can show that $(x-a)^2+(y-b)^2+z^2=1$ (2.0.4)

was a solution of this equation with arbitrary a and b. Since it contains two arbitrary constants, the solution (2.0.4) is thus a complete integral of the equation (2.0.3).

Putting b = a in equation (2.0.4), we obtain the one -parameter subsystem

$$(x-a)^2+(y-a)^2+z^2=1$$

whose envelope is obtained by eliminating a between this equation and

$$x+y-2a=0$$

So that it has equation $(x-y)^2+2z^2=0$ (2.0.5)

Differentiating both sides of this equation with respect to x and y, respectively, we obtain the relations

$$2zp=y-x, \qquad 2zq=x-y$$

from which it follows immediately that (2.0.5) is an integral surface of the equation (2.0.3). It is a solution of type (b); i.e., it is a general integral of the equation (2.0.3).

The envelope of the two-parameter system (2.0.3) is obtained by eliminating a and b from equation (2.0.4) and the two equations $x-a=0$, $y-b=0$ i.e., the envelope consists of the pair of planes z = ± I. It is readily verified that these planes are integral surfaces of the equation (2.0.3) ; since they are of type (c) they constitute the singular integral of the equation.

Notes: It should be noted that, theoretically, it is always possible to obtain different complete integrals which are not equivalent to each other, i.e., which cannot be obtained from one another merely by a change in the choice of arbitrary constants. When, however, one complete integral has been obtained, every other solution, including every other complete integral, appears among the solutions of type (b) and (c) corresponding to the complete integral we have found.

2.1 Few Important Definitions

Now we shall define few important terms which will help us to understand the Cauchy's method of characteristics for solving non-linear partial differential equation.

Definition 2.0.1: Plane Element:A plane passing through the point $P(x_0,y_0,z_0)$ with its normal parallel to the direction n defined by direction rations $(p_0,q_0,-1)$ is uniquely given by the set of five real numbers $D(x_0,y_0,z_0,p_0,q_0)$. Conversely any such set of five real numbers defines a plane in three-dimensional space. Thus a set of five real numbers $D(x,y,z,p,q)$ is called a plane element of the space i.e., a plane in three dimensional space.

Definition 2.0.2: Integral Element: Consider a partial differential equation of first order, i.e., $f(x,y,z,p,q)=0$ (2.0.6)

A particular plane element $D(x_0,y_0,z_0,p_0,q_0)$ whose components satisfy the equation (2.0.6)is called and integral element of equation (2.0.6) at the point (x_0,y_0,z_0).

Definition 2.0.3: Elementary Cone: We assume that it is possible to solve an equation of type (2.0.6) for q in terms of x,y,z and p i.e., from (2.0.6), we obtain an expression

$q=F(x,y,z,p)$ (2.0.7)

From (2.0.7) we can calculate the value q for given values of x,y,z and p.

Now keeping x_0,y_0 fixed z_0 and varying p only, we obtain a set of plane elements $\{x_0,y_0,z_0,p,F(x_0,y_0,z_0,p)\}$, which depend on the single parameter p only. Thus, varying p we obtain a set of plane elements all of which pass through the fixed point $P(x_0,y_0,z_0)$ i.e., all these plane elements envelope a cone with vertex at P. This cone is generated is called the elementary cone of equation (2.0.6) at the point $P(x_0,y_0,z_0)$.

Definition 2.0.4: Tangent Element: Consider a surface S whose equation is

$z=g(x,y)$ (2.0.8)

If the function $g(x,y)$ and its first order partial derivatives $g_x(x,y)$ and $g_y(x,y)$ are continuous in a certain region R of the xy plane, them the tangent plane at each point of the surface S gives a plane element of the type $\{x_0, y_0, g(x_0,y_0), g_x(x_0,y_0), g_y(x_0,y_0)\}$ which is called tangent element of the surface S at the point $\{x_0, y_0, g(x_0,y_0)\}$.

Thus, the surface (2.0.8) is an integral surface (i.e. solution) of the partial differential equation (2.0.6) such that at each point of the surface, its tangent element touches the elementary cone of the equation (2.0.6).

2.2 Cauchy's Method of Characteristics

We shall now consider methods of solving the nonlinear partial differential equation

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\right) = 0 \qquad (2.1.1)$$

In this section we shall consider a method, due to Cauchy, which is based largely on geometrical ideas. The plane passing through the point $P(x_0, y_0, z_0)$ with its normal parallel to the direction n defined by the direction ratios $(p_0, q_0, -1)$ is uniquely specified by the set of numbers $D(x_0, y_0, z_0, p_0, q_0)$. Conversely any such set of five real numbers defines a plane in three - dimensional space. For this reason a set of five numbers $D(x, y, z, p, q)$ is called a plane element of the space. In particular a plane element $(x_0, y_0, z_0, p_0, q_0)$ whose components satisfy an equation

$$F(x, y, z, p, q) = 0 \qquad (2.1.2)$$

is called an integral element of the equation (2.1.2) at the point (x_0, y_0, z_0).

It is theoretically possible to solve an equation of the type (2.1.2) to obtain an expression

$$q = G(x, y, z, p) \qquad (2.1.3)$$

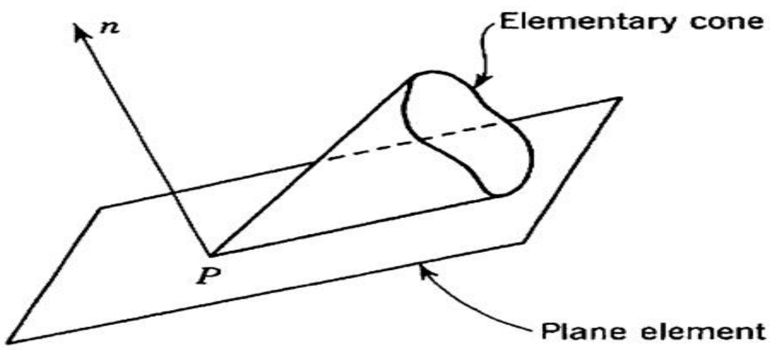

Figure 2.1

from which to calculate q when x, y, z and p are known. Keeping x_0, y_0 and z_0 fixed and varying p, we obtain a set of plane elements $\{x_0, y_0, z_0, p_0, G(\{x_0, y_0, z_0, p_0)\}$, which depend on the single parameter p. As p varies, we obtain a set of plane elements all of which pass through the point P and which therefore envelop a cone with vertex P; the cone so generated is called the elementary cone of equation (2) at the point P. (see Figure 2.1.)

Consider now a surface S whose equation is

$$z = g(x, y) \qquad (2.1.4)$$

If the function $g(x,y)$and its first partial derivatives $g_x(x,y), g_y(x,y)$ are continuous in a certain region R of the xy plane, then the tangent plane at each point of S determines a plane element of the type $\{x_0, y_0, z_0, p_0, G(\{x_0, y_0, z_0, p_0)\}$, (2.1.5)

which we shall call the tangent element of the surface S at the point $\{x_0, y_0, g(x_0, y_0)\}$. It is obvious on geometrical grounds that:

Theorem 2.1.1. A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.

Proof: A. curve C with parametric equations $x = x(t), y = y(t), z = z(t)$ (2.1.6) lies on the surface (4), $z(t) = g\{x(t), y(t)\}$ for all values of t in the appropriate interval I. If P_0 is a point on this curve determined by the parameters t_0, then the direction ratios of the tangent line P_0P_1(see Figure 2.2) are $\{x'(t_0), y'(t_0), z'(t_0)\}$where $x'(t_0)$ denotes the value of dx /dt when. t = t_0, etc. This direction will be perpendicular to the direction $(p_0, q_0, -1)$ if

$$z'(t_0) = p_0x_0'(t_0) + q_0y_0'(t_0)$$

For this reason we say that any set $\{x(t), y(t), z(t), p(t), q(t)\}$ (2.1.7)

of five real functions satisfying the condition

$z'(t) = p(t)x'(t) + q(t)y'(t)$ (2.1.8)

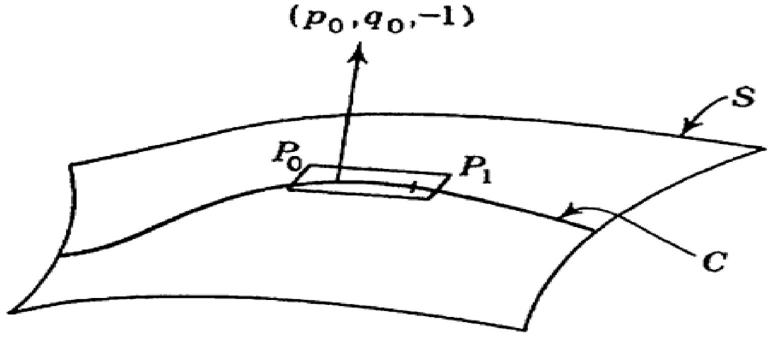

Figure: 2.2

defines a strip at the point (x,y,z) of the curve C. If such a strip is also an integral element of equation (2.1.2), we say that it is an integral strip of equation (2.1.2); i.e., the set of functions (2.1.7) is an integral strip of equation (2.1.2) provided they satisfy condition (2.1.8) and the further condition $F\{x(t), y(t), z(t), p(t), q(t)\} = 0$ (2.1.9)

for all t in I. If at each point the curve (2.1.6) touches a generator of the elementary cone, we say that the corresponding strip is a characteristic strip. We shall now derive the equations determining a characteristic strip. The point $(x + dx, y + dy, z + dz)$ lies in the tangent plane to the elementary cone at P if

$dz = pdx + qdy$ (2.1.10)

where p, q satisfy the relation (2.1.2). Differentiating (2.1.10) with respect to p, we obtain

$0 = dx + \frac{dq}{dp}dy$ (2.1.11)

where, from (2.1.2),

$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{dq}{dp} = 0$ (2.1.12)

Solving the equations (2.1.10), (2.1.11), and (2.1.12) for the ratios of dy, dz to dx, we obtain

$\frac{dx}{F_p} = \frac{dx}{F_q} = \frac{dz}{pF_p + qF_q}$ (2.1.13)

So that along a characteristic strip , $y'(t), z'(t)$ must be proportional to $F_p, F_q pF_p + qF_q$, respectively. If we choose the parameter t in such a way that

$x'(t) = F_p, y'(t) = F_q,$ (2.1.14)

And $z'(t) = pF_p + qF_q,$ (2.1.15)

Along a characteristic strip p is a function of t so that

$$p'(t) = \frac{\partial p}{\partial x}x'(t) + \frac{\partial p}{\partial y}y'$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q}$$

Since $\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}$. Differentiating equation (2) with respect to x, we find that

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0$$

so that on a characteristic strip

$p'(t)=-(F_x+pF_z)$ (2.1. 16)

and it can be shown similarly that $q'(t)=-(F_y+qF_z)$ (2.1.17)

Collecting equations (2.1.14) to (2.1.17) together, we see that we have the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t)=F_p,\quad y'(t)=F_q,\quad z'(t)=pF_p+qF_q,\quad p'(t)=-(F_x+pF_z),\qquad q'(t)=-(F_y+qF_z)$$

These equations are known as the characteristic equations of the differential equation (2.1.2).

Notes: The characteristic strip is determined uniquely by any initial element (x_0,y_0,z_0,p_0,q_0) and any initial value to of t. The main theorem about characteristic strips is:

Theorem 2.1.2: Along every characteristic strip of the equation F(x,y,z,p,q) = 0 the function F(x,y,z,p,q) is a constant.

Proof: The proof is a matter simply of calculation. Along a characteristic strip we have

$$\frac{d}{dt}F\{x(t),y(t),z(t),p(t),q(t)\}=$$

$$=F_x x'(t)+F_y y'(t)+F_z z'(t)+F_p p'(t)+F_q q'(t)$$

$$=F_xF_p+F_yF_q+F_z(pF_p+qF_q)-F_p\,(F_x+pF_z)-F_q\,(F_y+qF_z)$$

$$=0$$

so that F(x,y,z,p,q) = k, a constant along the strip.

Example 2.1.3: Find the characteristics of the equation $z=pq$, and determine the integral surface which passes through the parabola $x=0,y^2=z$.

Solution: Given equation is $z=pq$ (2.1.18)

We are to find its integral surface which passes through the given parabola given by

$x=0,y^2=z$ (2.1.19)

Re-writing (2.1.19) in parametric form, we have

$x=0,\qquad y=\lambda,\qquad z=\lambda^2,\qquad \lambda$ being a parameter. (2.1.20)

Let the initial values x_0,y_0,z_0,p_0,q_0 of x,y,z,p,q be taken as

$x_0=x_0(\lambda)=0,\qquad y_0=y_0(\lambda)=0,\qquad z_0=z_0(\lambda)=\lambda^2$ (2.1.21)

Let p_0,q_0 be the initial values of p,q corresponding to the initial values x_0,y_0,z_0. Since initial values (x_0,y_0,z_0,p_0,q_0) satisfy (2.1.18), we have

$p_0q_0=z_0,$ or $p_0q_0=\lambda^2$ by (2.1.21) (2.1.22)

Also, we have

$$z_0'(\lambda)=p_0x_0'(\lambda)+q_0{y_0}'(\lambda)$$

So that $2\lambda=p_0\times 0+q_0\times 1$ or $q_0=2\lambda$ (2.1.23)

Solving (2.1.22) and (2.1.23), $p_0=\frac{\lambda}{2}$ and $q_0=2\lambda$ (2.1.24)

Collecting relations (2.1.21) and (2.1.24) together, initial values of (x_0,y_0,z_0,p_0,q_0) are given by

$x_0=0,\quad y_0=\lambda,\quad z_0=\lambda^2,\quad p_0=\frac{\lambda}{2},\quad q_0=2\lambda\quad when\ \ t=t_0=0$ (2.1.25)

Re-writing (2.1.8), let $f(x,y,z,p,q)=pq-z=0$ (2.1.26)

The usual characteristic equations of (2.1.26) are given by

$x'(t)=f_p=q$ (2.1.27)

$y'(t) = f_q = p$ (2.1.28)

$z'(t) = pf_p + qf_q = 2pq$ (2.1.29)

$p'(t) = -f_x - pf_z = p$ (2.1.30)

$q'(t) = -f_y - qf_z = q$ (2.1.31)

From (2.1.27) and (2.1.31), $x'(t) - q'(t) = 0,$ so that $x - q = c_1,$ (2.1.32)

Where c_1 being an arbitrary constant. Using initial values (2.1.25), (2.1.31) gives

$$x_0 - q_0 = c_{1,} \qquad 0 - 2\lambda = c_1 \qquad or \qquad c_1 = -2\lambda.$$

Then (2.1.32) becomes $x - q = -2\lambda$ or $x = q - 2\lambda$ (2.1.33)

From (2.1.28) and (2.1.31), $y'(t) - p'(t) = 0,$ so that $y - p = c_2,$ (2.1.34)

Where c_2 being an arbitrary constant. Using initial values (2.1.25), (2.1.34) gives

$$y_0 - p_0 = c_{2,} \qquad \lambda - \frac{\lambda}{2} = c_2 \qquad or \qquad c_2 = \lambda/2.$$

Then (2.1.34) becomes $y - p = \frac{\lambda}{2}$ or $y = p + \frac{\lambda}{2}$ (2.1.35)

From (2.1.30) $p'(t) = p,$ $\frac{1}{p}dp = dt,$ $log\ logp = t + logc_3,$ $p = c_3e^t$ (2.1.36)

From (2.1.25) , (2.1.36) gives , $p_0 = c_3e^0,$ $c_3 = \frac{\lambda}{2}.$

Hence (2.1.36) reduces to $p = \frac{\lambda}{2}e^t.$ (2.1.37)

From (2.1.31) $q'(t) = q,$ $\frac{1}{q}dq = dt,$ $log\ logq = t + logc_4,$ $q = c_4e^t$ (2.1.38)

From (2.1.25) , (2.1.38) gives , $q_0 = c_4e^0,$ $c_4 = 2\lambda.$

Hence (2.1.38) reduces to $q = 2\lambda e^t.$ (2.1.39)

Using (2.1.33) and (2.1.39), $x = 2\lambda e^t - 2\lambda = 2\lambda(e^t - 1)$ (2.1.40)

Using (2.1.34) and (2.1.37), or $y = \frac{\lambda}{2}e^t + \frac{\lambda}{2} = \frac{\lambda}{2}(e^t + 1)$ (2.1.41)

Substituting values of p and q from (2.1.36) and (2.1.38) in (2.1.29), we get

$z'(t) = 2\frac{\lambda}{2}e^t.2\lambda e^t$ or $z'(t) = 2\lambda^2e^{2t}$ or $dz = 2\lambda^2e^{2t}dt$

Integrating, $z = \lambda^2e^{2t} + c_5,$ c_5 being an arbitrary constant. (2.1.42)

Using initial values (2.1.25), (2.1.42) gives $z_0 = \lambda^2e^0 + c_5$ or $\lambda^2 = \lambda^2e^0 + c_5$ or $c_5 = 0$

Then, (2.1.41) gives $z = \lambda^2e^{2t}.$ (2.1.43)

The required characteristics of (2.1.18) are given by (2.1.40), (2.1.41) and (2.1.43).

To find the required integral surface of (2.1.18), we now proceed to eliminate two parameters t and λ from three equations (2.1.40), (2.1.41) and (2.1.43). Solving (2.1.40) and (2.1.41) for $e^t$and λ, we have

$e^t = \frac{x+4y}{4y-x}$ and $\lambda = \frac{4y-x}{4}.$

Substituting these values of $e^t$and λ in (2.1.43), we have

$z = \left(\frac{4y-x}{4}\right)^2\left(\frac{x+4y}{4y-x}\right)^2$ or $16\,z = (x + 4y)^2$

which is the required integral surface of (2.1.18) passing through (2.1.19).

Example 2.1.4: Find the solutions of the equation $z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$which passes through the x -axis.

Solution: It is readily shown that the initial values are $x_0 = v, y_0 = 0, z_0 = 0, p_0 = 0, q_0 = 2v, t_0 = 0.$

The characteristic equations of this partial differential equation are

$\frac{dx}{dt} = p + q - y, \frac{dy}{dt} = p + q - x, \frac{dx}{dt} = p + q - y, \frac{dz}{dt} = p(p + q - y) + q(p + q - x),$

$$\frac{dx}{dt} = p + q - y, \frac{dx}{dt} = p + q - x$$

from which it follows immediately that

$$x = p + v, y = q - 2v$$

Also it is readily shown that

$$\frac{d}{dt}(p + q - x) = p + q - x$$

$$\frac{d}{dt}(p + q - y) = p + q - y$$

giving $p + q - x = ve^t, p + q - y = 2ve^t$

Hence we have $x = v(2e^t - 1), \quad y = v(e^t - 1), \quad p = 2v(e^t - 1), \quad q = v(e^t + 1)$ (2.1.45)

Substituting in the third of the characteristic equations, we have

$$\frac{dz}{dt} = 5v^2e^{2t} - 3v^2e^t$$

With solution

$z = \frac{5}{2}v^2(e^{2t} - 1) - 3v^2(e^t - 1)$ (2.1.46)

Now from the first pair of equations (2.1.45) we have

$$e^t = \frac{y - x}{2x - y}, v = x - 2y$$

so that substituting in (2.1.46), we obtain the solution

$z = \frac{1}{2}y(4x - 3y).$

Summary

- The first-order nonlinear partial differential equations are defined.
- All the types of differential equations solutions with examples are explained.
- One parameter and two parameter systems are elaborated.
- Cauchy's Method of Characteristic equations are derived
- Integral surfaces for given nonlinear equation determined.

Keywords

- Non Linear PDE
- One parameter solution
- Two parameter solution
- Plane element
- Tangent element
- Envelope
- Cauchy's Method of Characteristics

Self Assessment

1. If the characteristic strip contains at least one integral element of $f(x, y, z, p, q) = 0$, then

A. It is an integral strip of equation

B. The elementary curve of the equation

C. Line of the equation

D. Any curve to the equation

2. The characteristics strip corresponding partial differential equation $pq = z$ are

A. $x'(t) = -q, y'(t) = p$

B. $x'(t) = -q, y'(t) = -p$

C. $x'(t) = q, y'(t) = p$

D. None of these

3. The characteristics strip corresponding partial differential equation $z = (p^2 + q^2)/2 + (p - x)(q - y)$ are

A. $p'(t) = p + q + y,\ q'(t) = p + q + x$

B. $p'(t) = p + q - y,\ q'(t) = p + q - x$

C. $p'(t) = q - y,\ q'(t) = p - x$

D. $p'(t) = p + q,\ q'(t) = p + q$

4. The integral surface be an integral surface of a partial differential equation is that at each points its tangent element should touch to

A. The elementary cone of the equation

B. The elementary curve of the equation

C. Line of the equation

D. Any curve to the equation

5. The equations of characteristics strip corresponding partial differential equation $f(x, y, z, p, q) = 0$ are

A. $x'(t) = f_q, y'(t) = f_p$

B. $x'(t) = pf_p, y'(t) = qf_q$

C. $x'(t) = pf_q, y'(t) = qf_p$

D. $x'(t) = f_p, y'(t) = f_q$

6. Along every characteristic strip of equation $F(x, y, z, p, q) = 0$,

A. The function $F(x, y, z, p, q)$ is zero

B. The function $F(x, y, z, p, q)$ is positive only

C. The function $F(x, y, z, p, q)$ is a constant.

D. None of these

7. A solution of a partial differential equation of the first order that contains as many arbitrary constants as there are independent variables is called as

A. Particular integral

B. Singular solution

C. Complete solution

D. None of these

8. Any envelope of system $f(x, y, z, a, b) = 0$ touches

A. No point of its member system

B. Each point of its member system

C. Entirely within the system

D. None of these

9. An envelope is defined as the curve that is

A. Tangent to a given family of curves.

B. Passing through given family of curves.

C. Not tangent to a given family of curves.

D. None of these

Answers for Self Assessment

1.	A	2.	C	3.	B	4.	A	5.	A
6.	C	7.	C	8.	B	9.	A		

Review Questions

Q1. Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$.

Q2. Determine the characteristics of the equation $p^2 + q^2 = 4z$ and find the solution of this equation which reduces to $z = x^2 + 1$ when y = 0.

Q3. Find a complete integral of the partial differential $(p^2 + q^2)x = pz$ and deduce the surface solution which passes through the curve $x = 0, z^2 = 4y$.

Further Readings

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Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

https://onlinecourses.nptel.ac.in/noc21_ma09/preview

https://onlinecourses.swayam2.ac.in/cec22_ma12/preview

Dr. Preety Kalra, Lovely Professional University

Unit 03:Non-Linear First Order Partial Differential Equations-II

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept to solve nonlinear first order partial differential equations.
- understand the concept of Charpit's method
- know about the general solution of nonlinear partial differential equations.
- Applyspecial cases of Charpit's methodto solve nonlinear first order partial differential equations.
- find the condition of compatibility for systems of first order partial differential equations.

Introduction

In this chapter, more general method of solving partial differential equations of order one but of any degree and compatible system of first order partial differential equations will be discussed.

A method of solving the partial differential equation

$f(x,y,z,p,q)=0$ (3.0.1)

due to Charpit, is based on the considerations of the previous chapter. The fundamental idea in Charpit's method is the introduction of a second partial differential equation of the first order

$g(x,y,z,p,q,a)=0$ (3.0.2)

which contains an arbitrary constant a and which is such that:

(a) Equations (3.0.1) and (3.0.2) can be solved to give

$p=p(x,y,z,a),\ q=q(x,y,z,a)$

(b) The equation

$dz=p(x,y,z,a)dx+q(x,y,z,a)dy$ (3.0.3)

is integrable. When such a function g has been found, the solution of equation (3.0.3)

$F(x,y,z,a,b)=0$ (3.0.4)

containing two arbitrary constants a, b will be a solution of equation (3.0.1). Further, it will be seen that equation (3.0.4) is a complete integral of equation (3.0.1).

3.1 Charpit's Method

Let the given partial equation differential of first order and non–linear in p and q be

$$f(x,y,z,p,q)=0 \tag{3.1.1}$$

We know that

$$dz = pdx + qdy \tag{3.1.2}$$

The next step consists in finding another relation

$$F(x,y,z,p,q)=0 \tag{3.1.3}$$

such that when the values of p and q obtained by solving (3.1.1) and (3.1.3), are substituted in (3.1.2), it becomes integrable. The integration of (3.1.2) will give the complete integral of (3.1.1).

In order to obtain (3.1.3), differentiate partially (3.1.1) and (3.1.3) with respect to x and y and get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0 \tag{3.1.4}$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0 \tag{3.1.5}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0 \tag{3.1.6}$$

and

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0 \tag{3.1.7}$$

Eliminating $\frac{\partial p}{\partial x}$from (3.1.4) and (3.1.5), we get

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p}-\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p}=0$$

or$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0$ (3.1.8)

Similarly,eliminating $\frac{\partial q}{\partial y}$from (3.1.6) and (3.1.7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0 \tag{3.1.9}$$

Since$\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}$, the last term in (3.1.8) is the same as that in (3.1.9), except for a minus sign and hence they cancel on adding (3.1.8) and (3.1.9).

Therefore, adding (3.1.8) and (3.1.9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \tag{3.1.10}$$

This is a linear equation of the first order and integral of (3.1.10) is obtained by solving the auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p}=\frac{dq}{\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}} \tag{3.1.11}$$

Since any of the integrals of (3.1.11) will satisfy (3.1.10), an integral of (3.1.11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral

Remark 3.1.1: In what follows we shall use the following standard notations

$$\frac{\partial f}{\partial x}=f_x,\quad \frac{\partial f}{\partial y}=f_y,\quad \frac{\partial f}{\partial z}=f_z,\quad \frac{\partial f}{\partial p}=f_p,\quad \frac{\partial f}{\partial q}=f_q$$

Therefore, Charpit's auxiliary equations (3.1.11) may be re-written as

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dF}{0}.$$

3.2 Working Rule While Using Charpit's Method

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations (3.1.11)

Step 3. Using the value of f in step 1 write down the values of i.e.f_x, f_y, ... etc. occuring in step 2 and put these in Charpit's equations (3.1.11).

Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in dz = p dx + q dy which on integration gives the complete integral of the given equation. The Singular and General integrals may be obtained in the usual manner.

Cautions: Sometimes Charpit's equations give rise to p = a and q = b, where a and b are constants. In such cases, putting p = a and q = b in the given equation will give the required complete integral.

SOLVED–EXAMPLES

Example 3.2.1: Find a complete integral of $z=px+qy+p^2+q^2$.

Solution: Let $f(x,y,z,p,q)=z-px-qy-p^2-q^2=0$ (3.2.1.)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p}=\frac{dq}{\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}} \quad (3.2.2)$$

From (3.2.1), $f_x=-p,\ f_y=-q,\ f_z=0,\ \ f_p=-x-2p,\ f_q=-y-2q$ (3.2.3)

Using (3.2.3), (3.2.2) reduces to

$$\frac{dp}{0}=\frac{dq}{0}=\frac{dz}{p(x+2p)+q(y+2q)}=\frac{dx}{x+2p}=\frac{dy}{y+2q} \quad (3.2.4)$$

Taking the first fraction of (3.2.4), dp = 0 so that p = a (3.2.5)

Taking the second fraction of (3.2.4), dq = 0 so that q = b (3.2.6)

Putting p = a and q = b in (3.2.1), the required complete integral is $z=ax+by+a^2+b^2$, a, b being arbitrary constants.

Example 3.2.2: Find a complete integral of $q=3p^2$.

Solution: Here given equation is f(x,y,z,p,q) =$3p^2-q=0$. (3.2.7)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p}=\frac{dq}{\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or$$\frac{dp}{0+0.p}=\frac{dq}{0+0.q}=\frac{dz}{-6p^2+q}=\frac{dx}{-6p}=\frac{dy}{1} \quad (3.2.8)$$

Taking the first fraction of (3.2.7), dp = 0 so that p = a (3.2.9)

Substituting this value of p in (3.2.7), we get q = $3a^2$ (3.2.10)

Putting these values of p and q in dz = pdx + qdy, we get dz = adx + $3a^2$dy so that z = ax + 3a² y + b, which is a complete integral, a and b being arbitrary constants.

Example 3.2.3: Find a complete integral of $z^2(p^2z^2+q^2)=1$.

Solution: Here given equation is $f(x,y,z,p,q)=p^2z^4+q^2z^2-1=0$ (3.2.11)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p}=\frac{dq}{\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{p(4p^2z^3+2zq^2)}=\frac{dq}{q(4p^2z^3+2zq^2)}=\frac{dz}{-2p^2z^4-2q^2z^2}=\frac{dx}{-2pz^4}=\frac{dy}{-\frac{\partial f}{\partial q}} \quad (3.2.12)$$

Taking the first two fractions, (1/p)dp = (1/q)dq so that p = aq.

Solving (3.2.11) and (3.2.12) for p and q, $p=\frac{a}{z(a^2z^2+1)^{\frac{1}{2}}}$, $q=\frac{1}{z(a^2z^2+1)^{\frac{1}{2}}}$.

$\therefore dz=pdx+qdy=\frac{adx+dy}{z(a^2z^2+1)^{\frac{1}{2}}}$ or $adx+dy=z(a^2z^2+1)^{\frac{1}{2}}dz$.

Integrating $ax+y=\int(a^2z^2+1)^{\frac{1}{2}}z\,dz$ (3.2.13)

Putting $a^2z^2+1=t^2$ so that $2a^2z\,dz=2tdt$, (3.2.13) becomes

$ax+y=\int\frac{1}{a^2}tdt$ or $ax+y+b=\left(\frac{1}{3a^2}\right)t^3$, where $t=(a^2z^2+1)^{\frac{1}{2}}$

or $ax+y+b=\left(\frac{1}{3a^2}\right)(a^2z^2+1)^{\frac{3}{2}}$ or $9a^4(ax+y+b)^2=(a^2z^2+1)^3$

which is a complete integral, a and b being arbitrary constants.

3.3 Special Methods of Solutions Applicable to Certain Standard Forms

We now consider equations in which p and q occur other than in the first degree, that is non-linear equations. We have already discussed the general method. We now discuss four standard forms to which many equations can be reduced, and for which a complete integral can be obtained by inspection or by other shorter methods.

Standard Form I. Only p and q Present

Under this standard form, we consider equations of the form f(p, q) = 0 (3.3.1)

Charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

giving $\frac{dp}{0}=\frac{dq}{0}$, by (3.3.1)

Taking the first ratio, dp = 0 so that p = constant = a, say (3.3.2)

Substituting in (1), we get f(a, q) = 0, giving q = constant = b, say, (3.3.3)

where b is such that f(a, b) = 0. (3.3.4)

Then, dz = p dx + q dy = adx + bdy, using (3.3.2) and (3.3.3).

Integrating, z = ax + by + c, (3.3.5)

where c is an arbitrary constant. (3.3.5) together with (3.3.4) give the required solution.

Now solving (3.3.4) for b, suppose we obtain b = F(a), say.

Putting this value of b in (3.3.5), the complete integral of (3.3.1) is

z = ax + yF(a) + c, (3.5.6)

which contains two arbitrary constants a and c which are equal to the number of independent variables, namely x and y.

Remark 3.3.1: Sometimes change of variables can be employed to transform a given equation to standard form I.

Example 3.3.1: Solve pq = k, where k is a constant.

Solution: Given that pq = k. (3.3.7)

Since (3.3.7) is of the form f(p, q) = 0, its solution is

z = ax + by + c, (3.3.8)

whereab = k or b = k/a, on putting a for p and b for q in (3.3.7).

From (3.3.8), the complete integral is z = ax + (k/a)y + c, (3.3.9)

which contains two arbitrary constants a and c.

Example 3.3.2:$p^2 + q^2 = m^2$, where m is a constant.

Solution: Given that $p^2 + q^2 = m^2$ (3.3.10)

Since (3.3.10) is of the form f(b, q) = 0, its solution is z = ax + by + c, (3.3.11)

where $a^2 + b^2 = m^2$ or $b^2 = (m^2 - a^2)^{1/2}$, on putting a for p and b for b in (3.3.10).

From (3.3.11), the complete integral is $z = ax + (m^2 - a^2)^{1/2} + c$, (3.3.12)

which contains two arbitrary constants a and c.

Standard form II.Clairaut Equation.

A first order partial differential equation is said to be of Clariaut form if it can be written in the form

z = px + qy + f(p, q) (3.3.13)

Let F(x, y, z, p, q) (px + qy + f(p, q) – z (3.3.14)

Charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \text{ or}$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px-qy-pf_p-qf_q} = \frac{dx}{-x-f_p} = \frac{dy}{-y-f_q} \quad \text{by (3.3.13)} \qquad (3.3.15)$$

Then, first and second fractions (3.3.15), dp = 0 and dq = 0 this gives p = a and q = b.

Substituting these values in (3.3.13), the complete integral is z = ax + by + f(a, b)

Remark 3.3.2: Observe that the complete integral of (3.3.13) is obtained by merely replacing p and q by a and b respectively.

Example 3.3.3: Solve z = px + qy + pq.

Solution: The complete integral is z = ax + by + ab, a, b being arbitrary constants.

Example 3.3.4: Prove that complete integral of the equations $(px + qy - z)^2 = 1 + p^2 + q^2$is $ax + by + cz = (a^2 + b^2 + c^2)^{\frac{1}{2}}$.

Solution: Re-writting the given equation, we have

$px + qy - z = \pm\sqrt{1 + p^2 + q^2}$ or $z = px + qy \pm \sqrt{1 + p^2 + q^2}$ (3.3.16)

which is of standard form II and so its complete integral is

$z = Ax + By \pm \sqrt{1 + A^2 + B^2}$ (3.3.17)

To get the desired form of solution we take +ve sign in (3.3.17) and set A = –a/c and B = –b/c. Then (3.3.17) becomes

$$z = -(ax + by)/c \pm \frac{1}{c}\sqrt{c^2 + a^2 + b^2}$$

or $ax + by + cz = \sqrt{a^2 + b^2 + c^2}$.

Standard form III. Only p, q and z present.

Under this standard form we consider differential equation of the form

$f(p, q, z) = 0$ (3.3.18)

Charpit's auxiliary equations are

$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$ or

$\frac{dp}{pf_z} = \frac{dq}{qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$, using (3.3.18)

Taking the first two ratios, $\frac{dp}{p} = \frac{dq}{q}$.

Integrating, $q = ap$, a being an arbitrary constant. (3.3.19)

Now $dz = pdx + qdy = pdx + apdy$, Using (3.3.19)

$dz = p(dx + ady) = pd(x + ay) = pdu$, (3.3.20)

Where u=$x + ay$. (3.3.21)

Now, (3.3.20) ⇒ $p = dz/du$ and so by (3.3.19) $q = qp = a\left(\frac{dz}{du}\right)$.

Substituting these values of p and q in (3.3.18), we get

$f\left(\frac{dz}{du}, a\frac{dz}{du}, z\right) = 0$ (3.3.22)

which is an ordinary differential equation of first order. Solving (3.3.22), we get z as a function of u. Complete integral is then obtained by replacing u by (x + ay).

Working rule for solving equations of the form

f(p,q,z) = 0 (3.3.23)

Step I. Let u = x + ay, where a is an arbitrary constant. (3.3.24)

Step II. Replace p and q by dz/du and a(dz/du) respectively in (3.3.23) and solve the resulting ordinary differential equation of first order by usual methods.

Step III. Replace u by x + ay in the solution obtained in step II.

Example 3.3.5: Find a complete integral of $p^2 = qz$.

Solution: Given equation is $p^2 = qz$ (3.3.25)

which is of the form f(p, q, z) = 0. Let u = x + ay, where a is an arbitrary constant. Now, replacing p and q by dz/du and a(dz/du) respectively in (3.3.25), we get

$\left(\frac{dz}{du}\right)^2 = a\left(\frac{dz}{du}\right) z$ or $\frac{dz}{du} = az$ or $\frac{dz}{z} = au$.

Integrating, $\log z - \log b = au$ or $z = be^{au}$ or $z = be^{a(x+ay)}$, which is a complete integral containing two arbitrary constants a and b.

Example 3.3.6: Find a complete integral of $z = pq$.

Solution: Given equation is z = pq, (3.3.26)

which is of the form f(p, q, z) = 0.

Let u = x + ay, where a is an arbitrary constant.

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$respectively in (3.3.26), we get

$z = a\left(\frac{dz}{du}\right)^2$ or $\frac{dz}{du} == \pm\frac{\sqrt{z}}{\sqrt{a}}$ or $\pm\sqrt{a}z^{-1/2}dz = du.$

Integrating, $\pm 2\sqrt{az}$=u+b, or $4az = (x + ay + b)^2$ as u = x + ay.

Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$.

A form in which z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.

Let$F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0$ (3.3.27)

Then Charpit's auxiliary equations are

$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$ or

$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p\left(\frac{\partial f_1}{\partial p}\right)+q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}$, by (3.3.27)

Taking the first and the fourth ratios, we have

$\frac{\partial f_1}{\partial p}dp + \frac{\partial f_1}{\partial x}dx = 0$or $df_1 = 0$

Integrating, $f_1 = a$, a being an arbitrary constant.

(3.3.27)⇒ $f_1(x, p) = f_2(y, q) = a.$ (3.3.28)

Now, (3.3.28) ⇒ $f_1(x, p) = a$ and $f_2(y, q) = a.$ (3.3.29)

From (3.3.29), on solving for p and q respectively, we get

$p = F_1(x, a)$, say and $q = F_2(y, a), say$ (3.3.30)

Substituting these values in $dz = pdx + qdy$, we get $dz = F_1(x, a), dx + F_2(y, a)dy$.

Integrating, $z = \int F_1(x, a)\ dx + \int F_2(y, a)\, dy + b,$

which is a complete integral containing two arbitrary constants a and b.

Example 3.3.7: Find a complete integral of $x(1 + y)p = y(1 + x)q$.

Solution: Separating p and x from q and y, the given equation reduces to

$xp/(1 + x) = yq/(1 + y)$.

Equating each side to an arbitrary constant a, we have

$\frac{xp}{(1+x)} = a$and $\frac{yq}{1+y} = a$ so that $p = a\left(\frac{1+x}{x}\right)$ and $q = a\left(\frac{1+y}{y}\right)$.

Putting these values of p and q in dz = p dx + q dy, we get

$dz = a\left(\frac{1+x}{x}\right)dx + a\left(\frac{1+y}{y}\right)dy$ or $dz = a\left(\frac{1}{x} + 1\right)dx + a\left(\frac{1}{y} + 1\right)dy.$

Integrating, $z = a(logx + x) + a(logy + y) + b = a(logxy + x + y) + b,$

which is a complete integral containing two arbitrary constants a and b.

3.4 Compatible System of First–Order Equations

Consider first order partial differential equations

$f(x, y, z, p, q) = 0$ (3.4.1)

and$g(x, y, z, p, q) = 0$ (3.4.2)

Equations (3.4.1) and (3.4.2) are known as compatible when every solution of one is also a solution of the other.

To find condition for (3.4.1) and (3.4.2) to be compatible.

Let J = Jacobian of f and g = $\frac{\partial(f,g)}{\partial(p,q)} \neq 0$ (3.4.3)

Then (3.4.1) and (3.4.2) can be solved to obtain the explicit expressions for p and q given by

$p = \varphi(x, y, z)$and$q = \psi(x, y, z)$ (3.4.4)

The condition that the pair of equations (3.4.1) and (3.4.2) should be compatible reduces then to the condition that the system of equations (3.4.4) should be completely integrable, i.e., that the equation

$dz = pdx + qdy$ or $\phi dx + \psi dy - z = 0$,, using (3.4.4) (3.4.5)

should be integrable. (3.4.5) is integrable if

$$\phi\left(\frac{\partial\psi}{\partial z} - 0\right) + \psi\left(0 - \frac{\partial\phi}{\partial z}\right) + (-1)\left(\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}\right) = 0$$

which is equivalent to

$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z} = \frac{\partial\phi}{\partial y} + \psi\frac{\partial\phi}{\partial z}.$ (3.4.6)

Substituting from equations (3.4.4) in (3.4.1) and differentiating w.r.t. 'x' and 'z' respectively, we get

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial\psi}{\partial x} = 0$ (3.4.7)

$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z} + \frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z} = 0$ (3.4.8)

From (3.4.7) and (3.4.8), $\frac{\partial f}{\partial x} + \phi\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\left(\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi}{\partial z}\right) + \frac{\partial f}{\partial q}\left(\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z}\right) = 0$ (3.4.9)

Similarly (3.4.2) yields), $\frac{\partial g}{\partial x} + \phi\frac{\partial g}{\partial z} + \frac{\partial g}{\partial p}\left(\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi}{\partial z}\right) + \frac{\partial g}{\partial q}\left(\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z}\right) = 0$ (3.4.10)

Solving (3.4.9) and (3.4.10), $\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z} = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right\}$ (3.4.11)

Again, substituting from equations (3.4.4) in (3.4.1) and differentiating w.r.t. 'y' and 'z' and proceeding as before, we obtain

$\frac{\partial\phi}{\partial y} + \psi\frac{\partial\phi}{\partial z} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right\}$ (3.3.12)

Substituting from equations (3.4.11) and (3.4.12) in (3.4.1) and replacing ϕ, ψ by p, q respectively, we obtain

$\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)}\right\} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}\right\}$or $[f, g] = 0$ (3.4.13)

Where $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$ (3.4.14)

Example 3.4.1: Show that the equations $xp = yq$ and z$(xp + yq) = 2xy$ are compatible and solve them.

Solution: Let $f(x, y, z, p, q) = xp - yq = 0$ (3.4.15)

and$g(x, y, z, p, q) =$ z$(xp + yq) - 2xy = 0$ (3.4.16)

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix} = 2xy,$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2p - xyq,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy,$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = y^2q + xyp.$$

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 2xy - x^2p^2 - xypq - 2xy + xypq + y^2q^2$$

$= -xp(xp+yq) + yq(xp+yq) = -(xp-yq)(xp+yq) = 0$, using (3.4.15).

Hence (3.4.15) and (3.4.16) are compatible.

Solving (3.4.15) and (3.4.16) for p and q, $p = \frac{y}{z}$ and $q = \frac{x}{z}$. (3.4.17)

Using (3.4.17) in dz = pdx + qdy, we have dz = (y/z)dx + (x/z)dy or z dz = d(xy).

Integrating, $\frac{z^2}{2} = xy + \frac{c}{2}$ or $z^2 = 2xy + c$, where c is an arbitrary constant.

Summary

- The concept to solve nonlinear first order partial differential equation is discussed.
- Charpit method and its special cases are derived
- The compatibility of system of partial differential equation was discussed.
- The condition of compatibility is derived with examples.

Keywords

- Non linear
- Charpit'smethod
- Compatible system
- Clairaut
- Special cases

Self Assessment

1. The Charpit's auxiliary equation is

A. $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dx}{-pf_z} = \frac{dx}{-f_y}$

B. $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dx}{-f_x-pf_z} = \frac{dx}{-f_y-qf_z}$

C. $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{f_p+f_q} = \frac{dx}{-f_x-f_z} = \frac{dx}{-f_y-f_z}$

D. None of the above

2. The complete integral of equation $\mathbf{pxy + pq + qy = yz}$ is

A. $z + ax = be^{y}(a + y)^{a}$
B. $z - ax = be^{y}(a + y)^{a}$
C. $z - ax = be^{y}(a + y)^{-a}$
D. None of these

3. The general solution of the partial differential equation $\mathbf{2zx - px^2 - 2qxy + pq = 0}$ is

A. $z = ay(x^2 - a)$
B. $z = ay + b(x^2 - a)$
C. $z = ay - b(x^2 + a)$
D. None of these

4. The Clairaut's equation is

A. $z = px + qy + f(p,q)$
B. $z = px - q + f(p,q)$
C. $z = px^2 + qy^2 + f(p,q)$
D. $z = pqxy + f(p,q)$

5. The complete integral of the equation $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$ is

A. $z = ax - by - c\sqrt{(1 + a^2 + c^2)}$
B. $z = ax + by + c\sqrt{(1 + a^2 + b^2)}$
C. $z = ax - by + c$
D. $z = ax + by + c$

6. The complete integral of the equation $f(z, p, q) = 0$ is obtained by the relation

A. $p = aq$
B. $z = px + qy$
C. $ap + bq = c$
D. None of these

7. The first order partial differential equation is separable if it can be written in the form

A. $f(x, y) = g(p, q)$
B. $f(x, p) = g(y, q)$
C. $f(z, p) = g(z, q)$
D. None of these

8. The partial differential equations $f(x,y,z,p,q)=0, g(x,y,z,p,q)=0$ are compatible to each other if

A. $\frac{\partial(f,g)}{\partial(p,q)}=0$
B. $\frac{\partial(f,g)}{\partial(x,y)}=0$
C. $\frac{\partial(f,g)}{\partial(p,q)}\neq 0$
D. $\frac{\partial(f,g)}{\partial(x,y)}\neq 0$

9. If the partial differential equations $f(x,y,z,p,q)=0, g(x,y,z,p,q)=0$ are compatible to each other, then

A. $[f,g]=0$
B. $[f,g]\neq 0$
C. $[f,g]=a$, a is an arbitrary constant
D. None of these

10. If the partial differential equations $f(x,y,z,p,q)=0, g(x,y,z,p,q)=0$ are compatible to each other, then

A. $[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+\frac{\partial(f,g)}{\partial(z,q)}=0$
B. $[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0$
C. $[f,g]=\frac{\partial(f,g)}{\partial(x,p)}-p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}-q\frac{\partial(f,g)}{\partial(z,q)}=0$
D. None of these

11. The first order two partial differential equations are compatible if every solution of one is ……………………..

A. Also a solution of other
B. Not a solution of other
C. Constant only
D. None of these

Answers for Self Assessment

1.	A	2.	C	3.	B	4.	A	5.	B
6.	A	7.	B	8.	C	9.	A	10.	B
11.	A								

Review Questions

Q1. Find a complete integral of $yzp^2 - q = 0$.

Q2.Find a complete and singular integrals of $2xz - px^2 - 2q\,xy + pq = 0$.

Q3. Solve $p^2 + q^2 = 1$

Q4. Find the complete integral of the equation $z = px + qy + \log(pq)$.

Q5. Show that the equations $xp - yq = x$ and $x^2 p + q = xz$ are compatible.

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.
2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

https://onlinecourses.nptel.ac.in/noc21_ma09/preview

https://onlinecourses.swayam2.ac.in/cec22_ma12/preview

Unit 04: Linear Second Order Partial Differential Equations with Constant Coefficients - I

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of second order partial differential equations.
- understand origin of second order differential equation.
- Know about the classification of second order differential equations.
- find the solution of linear differential equations with constant coefficients.

Introduction

In the last chapters we considered the solution of partial differential equations of the first order. We shall now proceed to the discussion of equations of the second order. In this chapter we shall confine ourselves to a preliminary discussion of these equations, and then in the following two chapters we shall consider in more detail the three main types of linear partial differential equation of the second order. Though we are concerned mainly with second -order equations, we shall also have something to say about partial differential equations of order higher than the second.

4.1 The Origin of Second -Order Equations

Suppose that the function z is given by an expression of the type

$$z = f(u) + g(v) + w \qquad (4.1.1)$$

where f and g are arbitrary functions of u and y, respectively, and u, y, and w are prescribed functions of x and y. Then writing

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2} \qquad (4.1.2)$$

we find, on differentiating both sides of (4.1.1) with respect to x and y, respectively, that

$$p = f'(u)u_x + g'(v)v_x + w_x$$

$$q = f'(u)u_y + g'(v)v_y + w_y$$

and hence that

$$r = f''(u)u_x^2 + g''(v)v_x^2 + f'(u)u_{xx} + g'(v)v_{xx} + w_{xx}$$
$$s = f''(u)u_x u_y + g''(v)v_x v_y + f'(u)u_{xy} + g'(v)v_{xy} + w_{xy}$$
$$t = f''(u)u_y^2 + g''(v)v_y^2 + f'(u)u_{yy} + g'(v)v_{yy} + w_{yy}$$

We now have five equations involving the four arbitrary quantities f ' f '', g', g ''. If we eliminate these four quantities from the five equations, we obtain the relation

$|p - w_x u_x v_x q - w_y u_y v_y r - w_{xx} u_{xx} v_{xx} \quad 0000 u_x^2 v_x^2 \; s - w_{xy} u_{xy} v_{xy} t - w_{yy} u_{yy} v_{yy} \; u_x u_y v_x v_y u_y^2 v_y^2| = 0$ (4.1.3)

which involves only the derivatives p, q, r, s, t and known functions of x and y. It is therefore a partial differential equation of the second order. Furthermore if we expand the determinant on the left -hand side of equation (3) in terms of the elements of the first column, we obtain an equation of the form

$$Rr + Ss + Tt + Pp + Qq = W \qquad (4.1.4)$$

where R, S, T, P, Q, W are known functions of x and y. Therefore the relation (4.1.1) is a solution of the second -order linear partial differential equation (4). It should be noticed that the equation (4.1.4) is of a particular type: the dependent variable z does not occur in it.

As an example of the procedure of the last paragraph, suppose that

$$z = f(x + ay) + g(x - ay) \qquad (4.1.5)$$

where f and g are arbitrary functions and a is a constant. If we differentiate (4.1.5) twice with respect to x, we obtain the relation

$$r = f'' + g''$$

while if we differentiate it twice with regard to y, we obtain the relation

$$t = a^2 f'' + a^2 g''$$

so that functions z which can be expressed in the form (4.1.5) satisfy the partial differential equation

$$t = a^2 r \qquad (4.1.6)$$

Similar methods apply in the case of higher -order equations. It is readily shown that any relation of the type

$$z = \sum_{r=1}^{n} \quad f_r(v_r) \qquad (4.1.7)$$

where the functions f_r are arbitrary and the functions v_r are known, leads to a linear partial differential equation of the nth order.

Remarks 4.1.1: The partial differential equations we have so far considered in this section have been linear equations. Naturally it is not only linear equations in which we are interested. In fact, we have already encountered a nonlinear equation of the second order; that if the surface z = f(x,y) is a developable surface, the function f must be a solution of the second -order nonlinear equation

$$rt - s^2$$

4.2 Classification of Second Order Partial Differential Equation

We classify second-order equations of the type (4.1.4) by their canonical forms; we say that an equation of this type is:

(a) Hyperbolic if S^2 -- 4RT > 0,

(b) Parabolic if S^2 - 4RT = 0,

(c) Elliptic if S^2 - 4RT < 0.

Homogeneous and Non–homogeneous linear equations with constant coefficients.

A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a linear partial differential equation.

When all the derivatives appearing are of the same order, then the resulting equation is called a linear homogeneous partial differential equation with constant coefficients and it is then of the form $F(D,D') = f(x,y)$ where F(D,D') denotes a differential operator of the type

$$F(D,D') = \sum_r \sum_s c_{rs} D^r D'^s$$

On the other hand, when all the derivatives are not of the same order, then it is called a non–homogeneous linear partial differential equation with constant coefficients.

In this chapter we propose to study the various methods to find complementary functions for solving homogeneous linear partial differential equation with constant coefficients, namely.

4.3 Linear Partial Differential Equations with Constant Coefficients

We shall now consider the solution of a very special type of linear partial differential equation, that with constant coefficients. Such as equation can be written in the form

$F(D,D') = f(x,y)$ (4.3.1)

where F(D,D') denotes a differential operator of the type

$F(D,D') = \sum_r \sum_s c_{rs} D^r D'^s$ (4.3.2)

in which the quantities c_{rs} are constants, and $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$.

The most general solution, i.e., one containing the correct number of arbitrary elements, of the corresponding homogeneous linear partial differential equation

$F(D,D') = 0$ (4.3.3)

is called the complementary function of the equation (4.3.1), just as in the theory of ordinary differential equations. Similarly any solution of the equation (4.3.1) is called a particular integral of (4.3.1). As in the theory of linear ordinary differential equations, the basic theorem is :

Theorem 4.3.1: If u is the complementary function and z_1 a particular integral of a linear partial differential equation, then u +$z_1$is a general solution of the equation.

Proof: The proof of this theorem is obvious. Since the equations (4.3.1) and (4.3.3) are of the same kind, the solution u +z_1 will contain the correct number of arbitrary elements to qualify as a general solution of (4.3.1). Also

$F(D,D')u = 0, F(D,D')z_1 = f(x,y)$ so that

$$F(D,D')(u + z_1) = f(x,y)$$

showing that u+ z_1 is in fact a solution of equation (4.3.1). This completes the proof.

Another result which is used extensively in the solution of differential equations is:

Theorem 4.3.2: If$u_1, u_2, \dots\dots\dots, u_n$are solutions of the homogeneous linear partial differential equation $F(D,D')z = 0$, then

$$\sum_{r=1}^{n} c_r u_r$$

where the $c_r's$ are arbitrary constants, is also a solution.

Proof: The proof of this is immediate, since

$$F(D,D')(c_r u_r) = c_r F(D,D') u_r$$

and $$F(D,D') \sum_{r=1}^{n} v_r = \sum_{r=1}^{n} F(D,D') v_r$$

for any set of functions v_r. Therefore

$$F(D,D')\sum_{r=1}^{n} c_r u_r = \sum_{r=1}^{n} F(D,D')(c_r u_r)$$

$$= \sum_{r=1}^{n} c_r F(D,D') u_r = 0$$

We classify linear differential operators $F(D,D')$ into two main types, which we shall treat separately. We say that:

(a) $F(D,D')$is reducible if it can be written as the product of linear factors of the form $D + aD' + b$, with a, b constants;

(b) $F(D,D')$is irreducible if it cannot be so written.

For example, the operator

$$(D^2 - D'^2)$$

which can be written in the form

$$(D + D')(D - D')$$

is reducible, whereas the operator s

$$D^2 - D'$$

which cannot be decomposed into linear factors, is irreducible.

4.4 Reducible Equations

The starting point of the theory of reducible equations is the result:

Theorem 4.3.3: If the operator $F(D,D')$ is reducible, the order in which the linear factors occur is unimportant.

The theorem will be proved if we can show that

$$(\alpha_r D + \beta_r D' + \gamma_r)(\alpha_s D + \beta_s D' + \gamma_s) = (\alpha_s D + \beta_s D' + \gamma_s)(\alpha_r D + \beta_r D' + \gamma_r) \quad (4.3.4)$$

for any reducible operator can be written in the form

$$F(D,D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r) \quad (4.3.5)$$

and the theorem follows at once. The proof of (4.2.4) is immediate, since both sides are equal to

$\alpha_r\alpha_s D^2 + (\alpha_s\beta_r + \beta_s\alpha_r)DD' + \beta_r\beta_s D'^2 + (\gamma_s\alpha_r - \alpha_s\gamma_r)D + (\gamma_s\beta_r + \beta_s\gamma_r)D' + \gamma_r\gamma_s$.

Theorem 4.3.4: If $\alpha_r D + \beta_r D' + \gamma_r$ is a factor of $F(D,D')$ and $\phi_r(\xi)$is an arbitrary function of the single variable ξ, then if $\alpha_r \neq 0$,

$$u_r = exp\ exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)$$

is a solution of the equation $F(D,D') = 0$.

Proof: By direct differentiation we have

$$Du_r = -\frac{\gamma_r}{\alpha_r}u_r + \beta_r\ exp\ exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'_r(\beta_r x - \alpha_r y)$$

$$D'u_r = -\alpha_r\ exp\ exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'_r(\beta_r x - \alpha_r y)$$

so that

$$(\alpha_r D + \beta_r D' + \gamma_r)u_r = 0 \quad (4.3.6)$$

Now by Theorem 4.3.3

$$F(D,D')u_r = \{\prod_{s=1}^{n} (\alpha_s D + \beta_s D' + \gamma_s)\}(\alpha_r D + \beta_r D' + \gamma_r)u_r \quad (4.3.7)$$

the prime after the product denoting that the factor corresponding to s = r is omitted.

Combining equations (4.3.6) and (4.3.7), we see that $F(D,D')u_r = 0$ which proves the theorem.

By an exactly similar method. we can prove:

Remarks 4.3.5: If $\beta_r D' + \gamma_r$ is a factor of $F(D,D')$ and $\phi_r(\xi)$is an arbitrary function of the single variable ξ, then if $\beta_r \neq 0$,

$$u_r = exp\ exp\left(-\frac{\gamma_r y}{\beta_r}\right)\phi_r(\beta_r x)$$

is a solution of the equation $F(D,D') = 0$.

Remarks 4.3.6: In the decomposition of linear $F(D,D')$ into linear factor we may get multiplication factors of the type $(\alpha_r D + \beta_r D' + \gamma_r)^n$. Then the solution corresponding to factor of this type can be obtained by simple application of Theorem 4.3.4.

Example 4.3.7: If n=2, we wish to find the solutions of equation

$$(\alpha_r D + \beta_r D' + \gamma_r)^2 z = 0 \qquad (4.3.8)$$

Solution: If we let $Z = (\alpha_r D + \beta_r D' + \gamma_r)z$

then $(\alpha_r D + \beta_r D' + \gamma_r)Z = 0$

which according to Theorem 4.3.4 has solution

$$Z = exp\ exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)$$

if $\alpha_r \neq 0$. To find the corresponding function z we have therefore to solve the first order linear partial differential equation

$$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = exp\ exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y). \qquad (4.3.9)$$

We get the auxiliary equations are

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z + e^{-\frac{\gamma_r x}{\alpha_r}}\phi_r(\beta_r x - \alpha_r y)}$$

with solution

$$\beta_r x - \alpha_r y = c_1$$

Substituting this in auxiliary equations, we get the

$$\frac{dx}{\alpha_r} = \frac{dz}{-\gamma_r z + e^{-\frac{\gamma_r x}{\alpha_r}}\phi_r(c_1)}$$

which is a first order linear equation with solution

$$z = \frac{1}{\alpha_r}\{\phi_r(c_1)x + c_2\}e^{-\frac{\gamma_r x}{\alpha_r}}$$

Equation (4.3.9) and hence equation (4.3.8), therefore has a solution

$$z = \frac{1}{\alpha_r}\{\phi_r(\beta_r x - \alpha_r y)x + \psi_r(\beta_r x - \alpha_r y)\}e^{-\frac{\gamma_r x}{\alpha_r}}$$

Where the functions ϕ_r, ψ_r are arbitrary.

This result readily generalized (by induction) to give

Remarks 4.3.8: If $(\alpha_r D + \beta_r D' + \gamma_r)^n (\alpha_r \neq 0)$ is a factor of $F(D,D')$ and if the functions $\phi_{r1}, \phi_{r2}, \dots\dots\dots \phi_{rn}$ are arbitrary, then

$$e^{-\frac{\gamma_r x}{\alpha_r}}\sum_{s=1}^{n} x^{s-1}\phi_{rs}(\beta_r x-\alpha_r y)$$

is a solution of $F(D,D')=0$.

Similarly the generalization, If $(\beta_r D'+\gamma_r)^m(\alpha_r\neq 0)$ is a factor of $F(D,D')$ and if the functions $\phi_{r1},\phi_{r2},\dots\dots.\phi_{rm}$ are arbitrary, then

$$e^{-\frac{\gamma_r y}{\beta_r}}\sum_{s=1}^{m} x^{s-1}\phi_{rs}(\beta_r x)$$

is a solution of $F(D,D')=0$.

We are now in a position to state the complementary function of the equation (1) when the operator F(D,D) is reducible. As a result, we see that if

$$F(D,D')=\prod_{r=1}^{n}\ (\alpha_r D+\beta_r D'+\gamma_r)^{mr} \qquad (4.3.10)$$

and if none of the $\alpha_r's$ is zero, then the corresponding complementary function is

$$u=\sum_{r=1}^{n}\ exp\ exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\sum_{s=1}^{m_r}\ x^{s-1}\phi_{rs}(\beta_r x-\alpha_r y) \qquad (4.3.11)$$

where the functions $\phi_{rs}(s=1,\dots,nr;\ r=1,\dots,n)$ are arbitrary. If some of the $\alpha_r's$ are zero, the necessary modifications to the expression (4.3.11) can be made. From equation (4.3.10) we see that the order of equation (4.3.3) is $m_1+m_2++m_n$; since the solution (4.3.11) contains the same number of arbitrary functions, it has the correct number and is thus the complete complementary function. To illustrate the procedure we consider a simple special case:

Example 4.3.9: Solve the equation $\frac{\partial^4 z}{\partial x^4}+\frac{\partial^4 z}{\partial y^4}=2\frac{\partial^4 z}{\partial x^2\partial y^2}$

Solution: In the notation of this section this equation can be written in the form

$$(D+D')^2(D-D')^2=0$$

so that by the rule (4.3.11) the solution of it is

$$z=x\phi_1(x-y)+\phi_2(x-y)+x\psi_1(x+y)+\psi_2(x+y)$$

where the functions $\phi_1,\phi_2,\psi_1,\psi_2$are arbitrary.

4.5 Irreducible Equations

When the operator $F(D,D')$ is irreducible, it is not always possible to find a solution with the full number of arbitrary functions, but it is possible to construct solutions which contain as many arbitrary constants as we wish. The equation can be homogenous or non-homogeneous which cannot reduced to linear factors.

We can present C.F. of irreducible equation

$$F(D,D')=0 \qquad (4.3.12)$$

in the following manner .

C.F.= $\sum\ Ae^{hx+ky}$

where A, h, k are arbitrary constants such that $F(h,k)=0$.

Example 4.3.10: Solve$(D-D'^2)z=0$

Solution: Here $D-D'^2$ is not a linear factor in D and D′.

Let $z=\sum\ Ae^{hx+ky}$ be a trial solution of the given equation. Then

$Dz = Ahe^{hx+ky}$ and $D'^2 z = Ak^2 e^{hx+ky}$.

Putting these values in the given differential equation, we get

$A(h-k^2)e^{hx+ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$

Replacing h by k^2, the most general solution of the given equation is

$z = \sum \quad Ae^{k^2x+ky}$, where A and k are arbitrary constants.

Summary

- The origin of second order differential equation is defined.
- The concept of classification second order differential equation is discussed.
- Linear partial differential equation with constant coefficients is elaborated.
- The solutions of reducible and irreducible equations are derived with examples.

Keywords

- Linear second-order partial differential equation
- Origin of second-order equations
- Classification
- Reducible
- Irreducible

Self Assessment

1. The partial differential equation $5\frac{\partial^2 z}{\partial x^2} + 6\frac{\partial^2 z}{\partial y^2} = xy$ is classified as

A. Elliptic

B. Parabolic

C. Hyperbolic

D. None of the above

2. Using substitution, which of the following equations are solutions to the partial differential equation?

$$\frac{\partial^2 u}{\partial x^2} = 9\frac{\partial^2 u}{\partial y^2}$$

A. $\cos(3x - y)$

B. $\sin(3x - 3y)$

C. $e^{-3\pi x}\sin(\pi y)$

D. $x^2 + y^2$

3. The partial differential equation $xy\frac{\partial z}{\partial x} = 5\frac{\partial^2 z}{\partial y^2}$ is classified as

A. Elliptic
B. Parabolic
C. Hyperbolic
D. None of the above

4. The partial differential equation $5\frac{\partial^2 z}{\partial x^2} + 6\frac{\partial^2 z}{\partial y^2} = xy$ is classified as

A. Elliptic
B. Parabolic
C. Hyperbolic
D. None of the above

5. The partial differential equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ is a

A. linear equation of order 2
B. non-linear equation of order 2
C. linear equation of order 1
D. non-linear equation of order 1

6. What is the general form of second order non-linear partial differential equations (x and y being independent variables and z being a dependent variable)?
   A. $F(x,y,z,\partial z/\partial x,\partial z/\partial y,\partial^2 z/\partial x^2,\partial^2 z/\partial y^2,\partial^2 z/\partial x\partial y)=0$
   B. $F(x,z,\partial z/\partial x,\partial z/\partial y,\partial^2 z/\partial x^2,\partial^2 z/\partial y^2)=0$
   C. $F(y,z,\partial z\partial x,\partial z\partial y)=0$
   D. $F(x,y)=0$

7. The partial differential equation of n order is requires

A. Only one independent variable
B. Two or more independent variables
C. More than three independent variables
D. Equal number of dependent and independent variables

8. In partial differential equation $f(D,D')z = f(x,y), DD'$ means

A. $\frac{\partial}{\partial x}\frac{\partial}{\partial y}$

B. $\frac{\partial^2}{\partial x\partial y}$

C. $\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y}$

D. $\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}$

9. In the solution of differential equation $(D+2D')(D+D')z = x+2y$, the CF is

A. $\phi_1(2x-y)+\phi_2(x-y)$

B. $\phi_1(2y+x)+\phi_2(2y+x)$

C. $\phi_1(x-y)+\phi_2(y+x)$

D. $\phi_1(y+2x)+\phi_2(y+x)$

10. Point out the correct homogeneous linear partial differential equation

A. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}=\sin x$

B. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=x+y$

C. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial^2 z}{\partial y^2}+xy=0$

D. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}=0$

11. Solution of the differential equation $D^3-6D^2D'+11DD'^2-6D'^2=0$ is

A. $z=\phi_1(y+x)+\phi_2(y+2x)+\phi_3(y-3x)$

B. $z=\phi_1(y-x)+\phi_2(y+x)+\phi_3(y+3x)$

C. $z=\phi_1(y-x)+\phi_2(2y-x)+\phi_3(2y-3x)$

D. $z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

12. The solution of differential equation $(D^3-3D^2D'+3DD'^2-D'^3)z=0$ is

A. $\phi_1(2x-y)+\phi_2(x-y)+x\phi_3(x-y)$

B. $\phi_1(y+x)+x\phi_2(y+x)+x^2\phi_3(y+x)$

C. $\phi_1(2x-y)+x\phi_2(x-y)+x^2\phi_3(x-y)$

D. None of these

13. In the solution of differential equation $2r + 5s + 2t = x + 2y$, the CF is

A. $\phi_1(2x - y) + \phi_2(x - y)$

B. $\phi_1(2y - x) + \phi_2(y - 2x)$

C. $\phi_1(x - y) + \phi_2(y + x)$

D. $\phi_1(y + 2x) + \phi_2(y + x)$

14. Solution of the differential equation $(D - 2D' - 1)(D - 2D'^2 - 1)Z = 0$ is

A. $z = e^x \phi_1(y - 2x) + \sum Ae^{(2k^2+1)x+ky}$

B. $z = e^x \phi_1(2y + 2x) + \sum Ae^{(2k^2-1)x+ky}$

C. $z = e^x \phi_1(y + 2x) + \sum Ae^{(2k^2+1)x+ky}$

D. $z = e^x \phi_1(y - x) + \sum Ae^{(2k^2)x+ky}$

15. Solution of the differential equation $(D - D'^2)z = 0$ is

A. $z = \sum Ae^{kx+y}$

B. $z = \sum Ae^{k^2x+ky}$

C. $z = \sum Ae^{kx-y}$

D. None of these

Answers for Self Assessment

1.	C	2.	A	3.	B	4.	D	5.	B
6.	A	7.	B	8.	A	9.	A	10.	D
11.	A	12.	B	13.	B	14.	C	15.	B

Review Questions

Q1. Solve $(D^3 - 3D^2D' + 2DD'^2)z = 0$.

Q2. Solve $(D^2 + DD' - 6D'^2)z = 0$.

Q3. . Solve $25r - 40s + 16t = 0$

Q4. Solve $\left(D^3 - 4DD'^2 + 4DD'^2\right)z = 0$

Q5. Solve $(2D^4 - 3D^2D' + D'^2)z = 0$

Q6. Solve $(D + 2D' - 3)(D^2 + D')z = 0$

Further Readings

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Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

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https://onlinecourses.swayam2.ac.in/cec22_ma12/preview

Unit 05: Linear Second Order Partial Differential Equations with Constant Coefficients - II

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of need of particular integral.
- understand the concept of homogenous and non-homogenous differential equation solutions.
- know about the conditions to find particular integral for second order and higher order.
- apply appropriate method to find the complete solution.

Introduction

In in this chapter, how to find the particular integral for both homogeneous and non-homogeneous higher order linear partial differential equations will be discussed with different all its type.

5.1 Particular Integral (P.I.) of Homogeneous Linear Partial Differential Equation

$F(D, D')z = f(x, y)$ (5.1.1)

The inverse operator $\frac{1}{F(D,D')}$ of the operator $F(D, D')$ is defined by the following identity

$$F(D, D')\left(\frac{1}{F(D, D')}\right) f(x, y) = f(x, y))$$

Particular integral (P.I.) of (5.1.1) is $\left(\frac{1}{F(D,D')}\right) f(x, y)$.

In what follows we shall treat the symbolic functions of D and D' as we do for the symbolic functions of D alone in ordinary differential equations. Thus it will be factorized and resolved into partial fractions or expanded in an infinite series as the case may be. The reader is advised to note carefully the following results

(i) D, D^2 , ... will stand for differentiating partially with respect to x once, twice and so on.

For examples, $Dx^4y^5 = \frac{\partial}{\partial x}x^4y^5 = 4x^3y^5$, $D^2x^4y^5 = \frac{\partial^2}{\partial x^2}x^4y^5 = 12x^2y^5$.

(ii) D', D'^2 ... will stand for differentiating partially with respect to y once, twice and so on.

For example, $D'x^4y^5 = \frac{\partial}{\partial y}x^4y^5 = 5x^4y^4$, $D'^2x^4y^5 = \frac{\partial^2}{\partial y^2}x^4y^5 = 20x^4y^3$.

(iii) 1/D, 1/D^2 , ... will stand for integrating partially with respect to x once, twice and so on.

For example, $\frac{1}{D}x^4y^5 = \int x^4y^5dx = \frac{x^5y^5}{5}$, $\frac{1}{D^2}x^4y^5 = \int\int x^4\,y^5dx = \frac{x^6y^5}{30}$

(iv) 1/D', 1/D'^2 , ... will stand for integrating partially with respect to y once, twice and so on.

For example, $\frac{1}{D'}x^4y^5 = \int x^4y^5dy = \frac{x^4y^6}{6}$, $\frac{1}{D'^2}x^4y^5 = \int\int x^4\,y^5dx = \frac{x^4y^7}{42}$

5.2 Short methods of finding the P.I. in certain cases

Before taking up the general method for finding P.I. of F(D, D')z = f(x, y) we begin with cases when f(x, y) is in two special forms. The methods corresponding to these forms are much shorter than the general methods.

A Short Method I. When f(x, y) is of the form f(ax + by).

The method under consideration is based on the following theorem.

Theorem 5.2.1: If F(D, D') be homogeneous function of D and D' of degree n, then

$$\frac{1}{F(D,D')}\phi^{(n)}(ax+by) = \frac{1}{F(a,b)}\phi(ax+by)$$

provided $F(a,b) \neq 0, \phi^{(n)}$ being the nth derivative of ϕ w.r.t. ax + by as a whole.

Proof: By direct differentiation, we have $D^r\phi(ax+by) = a^r\phi^{(r)}(ax+by)$,

$D'^s\phi(ax+by) = b^s\phi^{(s)}(ax+by)$, and $D^rD'^s\phi(ax+by) = a^rb^s\phi^{(r+s)}(ax+by)$.

Since F(D, D') is homogeneous function of degree n, so we have

$$F(D,D')\phi(ax+by) = F(a,b)\phi^{(n)}(ax+by) \quad (5.2.1)$$

Operating both sides of (5.2.1) by 1/F(D, D'), we have

$$\phi(ax+by) = F(a,b)\frac{1}{F(D,D')}\phi^{(n)}(ax+by). \quad (5.2.2)$$

Since $F(a,b) \neq 0$,dividing both sides of (5.2.2) by F(a, b), we get

$$\frac{1}{F(D,D')}\phi^{(n)}(ax+by) = \frac{1}{F(a,b)}\phi(ax+by). \quad (5.2.3)$$

An important deduction from result (3): Putting ax + by = v, (5.2.3) gives

$$\frac{1}{F(D,D')}\phi^{(n)}(v) = \frac{1}{F(a,b)}\phi(v) \quad (5.2.4)$$

Integrating both sides of (5.2.4) n times w.r.t. 'v', we have

$$\frac{1}{F(D,D')}\phi(v) = \frac{1}{F(a,b)}\int\int \dots\dots\dots \int \phi(v)\,dvdv\ \dots\dots\dots.dv \text{ where } v = ax+by.$$

Exceptional case when F(a, b) = 0. When F(a, b) = 0, then the above theorem does not hold good. In such a case the new method is based on the following theorem. Note that F(a, b) = 0 if and only if (bD – aD') is a factor F(D, D').

Theorem 5.2.2: $\frac{1}{(bD-aD')^n}\phi(ax+by)=\frac{x^n}{b^n n!}\phi(ax+by)$.

Proof: Consider the equation $(bD-aD')z=\phi(ax+by)$ (5.2.5)

or

$bp-aq=x^r\phi(ax+by)$ (5.2.6)

Lagrange's subsidiary equations for (5.2.6) are $\frac{dx}{b}=\frac{dy}{-a}=\frac{dz}{x^r\phi(ax+by)}$ (5.2.7)

Taking the first two fractions of (5.2.7), $adx+bdy=0$ so that $ax+by=c_1$ (5.2.8)

Taking the first and third members of (5.2.7) and using (5.2.8), we get

$\frac{dx}{b}=\frac{dz}{x^r\phi(c_1)}$ or $dz=\frac{x^r\phi(c_1)}{b}dx$

Integrating, $z=\frac{x^{r+1}\phi(c_1)}{b(r+1)}=\frac{x^{r+1}\phi(ax+by)}{b(r+1)}$ (5.2.9)

(5.2.9) is a solution of (5.2.5).

Now, from (5.2.5), $z=\frac{1}{bD-aD'}x^r\phi(ax+by)$ (5.2.10)

From (5.2.9) and (5.2.10)

$\frac{x^{r+1}\phi(ax+by)}{b(r+1)}=\frac{1}{bD-aD'}x^r\phi(ax+by)$ by (5.2.8) (5.2.11)

Hence, if $z=\frac{1}{(bD-aD')^n}\phi(ax+by)$, then we have

$z=\frac{1}{(bD-aD')^{n-1}}\left[\frac{1}{bD-aD'}x^0\phi(ax+by)\right]$ as $x^0=1$.

$=\frac{1}{(bD-aD')^{n-1}}\frac{x}{b}\phi(ax+by)$using (5.2.11) for r = 0

$$=\frac{1}{b}\frac{1}{(bD-aD')^{n-2}}\left[\frac{1}{bD-aD'}x\phi(ax+by)\right]$$

$=\frac{1}{b}\frac{1}{(bD-aD')^{n-2}}\frac{x^2}{2b}\phi(ax+by)$ using (5.2.11) for r = 1

$=\frac{1}{(bD-aD')^{n-2}}\frac{x^2}{2b^2}\phi(ax+by)$

after repeated use of (5.2.11) for n – 2 times more

$$=\frac{x^n}{n!\,b^n}\phi(ax+by)$$

Solved Examples based on Short Method I

Example 5.2.1: Solve$(D^2+3DD'+2D'^2)z=x+y$.

Solution: The auxiliary equation of the given equation is

$$D^2+3DD'+2D'^2=0$$

$(D+2D')(D+D')=0$

C.F is $\phi_1(x-y)+\phi_2(2x-y)$, ϕ_1 and $\phi_2$being arbitrary functions.

Now, P.I = $\frac{1}{D^2+3DD'+2D'^2}(x+y)$

$\frac{1}{1^2+3.1.1+2.1^2}\int\int v\,dvdv$ where $v=x+y$

$=\frac{1}{6}\int\frac{v^2}{2}dv=\frac{v^3}{36}=\frac{1}{36}(x+y)^3$.

Hence required general solution is $z=C.F+P.I$

$z=$ is $\phi_1(x-y)+\phi_2(2x-y)+\frac{1}{36}(x+y)^3$.

Example 5.2.2: Solve$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}$

Solution: The auxiliary equation of the given equation is $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3 = 0$

$$(D - D')(D - 2D')(D - 3D') = 0$$

C.F is $\phi_1(x + y) + \phi_2(2x + y) + \phi_3(3x + y), \phi_1,\ \phi_2$ and $\phi_3$being arbitrary functions.

Now, P.I = $\frac{1}{(D-D')(D-2D')(D-3D')} e^{5x+6y}$

= $\frac{1}{(5-6)(5-12)(5-18)} \int\int\int e^v\, dvdvdv$ where $v = 5x + 6y$.

$$= -\frac{1}{91}\int\int e^v\, dvdv == -\frac{1}{91}e^v == -\frac{1}{91}e^{5x+6y}$$

Hence the required solution is $z = \phi_1(x + y) + \phi_2(2x + y) + \phi_3(3x + y) - \frac{1}{91}e^{5x+6y}$

Example 5.2.3: Solve $(D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x + 2y)$

Solution: The auxiliary equation of the given equation is

$$D(D^2 - 4DD' + 4D'^2) = 0$$

$$D(D - 2D')^2 = 0$$

C.F is $\phi_1(y) + \phi_2(2x + y) + x\phi_3(2x + y), \phi_1,\ \phi_2$ and $\phi_3$being arbitrary functions.

Now, P.I = $\frac{1}{D(D-2D')^2} 2\sin(3x + 2y)$

$= 2\frac{1}{3(3-2\,.2)^2}\int\int\int Sin\, v\, dvdvdv$ where v = $(3x + 2y)$

$$= \frac{2}{3}\int\int -\cos v\ dvdv = -\frac{2}{3}\int \sin v\, dv\ \ = \frac{2}{3}\cos v$$

$$= \frac{2}{3}cos(3x + 2y)$$

Hence the required solution is $z = \phi_1(y) + \phi_2(2x + y) + x\phi_3(2x + y) + \frac{2}{3}cos(3x + 2y)$

Example 5.2.4: Solve $(D^2 - 6DD' + 9D'^2)z = tan(3x + y)$

Solution: Here auxiliary equation is $(D^2 - 6DD' + 9D'^2 = 0$

$$(D - 3D')^2 = 0$$

C.F is $\phi_1(y + 3x) + x\phi_2(y + 3x), \phi_1$ and $\phi_2$being arbitrary functions.

Now, P.I = $\frac{1}{(D-3D')^2} tan(3x + y)$

= $\frac{x^2}{1^2\,.\,2!} ta\,n(3x + y) = \frac{x^2}{2}\tan(3x + y)$

The required solution is $z = \phi_1(y + 3x) + x\phi_2(y + 3x) + \frac{x^2}{2}\tan(3x + y)$, ϕ_1 and $\phi_2$being arbitrary functions.

5.3 Short Method II. When f(x, y) is of the form xmyn or a Rational Integral Algebraic Function of x and y

Then the particular integral (P.I.) is evaluated by expanding the symbolic function 1/f (D, D') in an infinite series of ascending powers of D or D'. In solved examples 1 and 2 of Art. 4.11, we have shown that P.I. obtained on expanding 1/f(D, D') in ascending powers of D is different from that obtained on expanding 1/f(D, D') in ascending powers of D'. Since to get the required general solution of given differential equation any P.I. is required, any of the two methods can be used. The difference in the two answers of P.I. is not material as it can be incorporated in the arbitrary functions occuring in C.F. of that given differential equation.

Remarks 5.3.1: If n < m, 1/f(D, D') should be expanded in powers of D'/D whereas if m < n, 1/f(D, D') should be expanded in powers of D/D'.

Solved Examples based on Short Method II

Example 5.3.1: Solve$(D^2 - a^2D'^2)z = x$

Solution: Here auxiliary equation is $D^2 - a^2D'^2 = 0$

$$(D - aD')(D + aD') = 0$$

C.F is $\phi_1(y + ax) + \phi_2(y - ax)$, ϕ_1 and $\phi_2$being arbitrary functions.

Now, P.I = $\frac{1}{D^2 - a^2D'^2}\ x = \frac{1}{D^2\left[1 - a^2\frac{D'^2}{D^2}\right]}\ x$

$= \frac{1}{D^2}\left[1 - a^2\frac{D'^2}{D^2}\right]^{-1}\ x\ = \frac{1}{D^2}\left[1 + a^2\frac{D'^2}{D^2} + \cdots.\right]\ x = \frac{1}{D^2}x = \frac{x^3}{6}.$

The required solution is $z = \phi_1(y + ax) + \phi_2(y - ax) + \frac{x^3}{6}$, ϕ_1 and $\phi_2$being arbitrary functions.

Example 5.3.2: Solve$(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy.$

Solution: Here auxiliary equation is $(D^2 - 6DD' + 9D'^2 = 0$

$$(D - 3D')^2 = 0$$

C.F is $\phi_1(y + 3x) + x\phi_2(y + 3x)$, ϕ_1 and $\phi_2$being arbitrary functions.

Now, P.I = $\frac{1}{(D-3D')^2}12(x^2 + 3xy) = 12\frac{1}{D^2\left(1 - \frac{3D'}{D}\right)^2}(x^2 + 3xy)$

$= \frac{12}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2}(x^2 + 3xy) = \frac{12}{D^2}\left(1 + 6\frac{D'}{D} + \cdots\ldots..\right)(x^2 + 3xy)$

[Retain upto D' as maximum power of y in $(x^2 + 3xy)$ is one]

$$\frac{12}{D^2}\left(x^2 + 3xy + 6\frac{D'}{D}(x^2 + 3xy)\right) = \frac{12}{D^2}\left(x^2 + 3xy + 6\frac{1}{D}(3x)\right)$$

$$\frac{12}{D^2}\left(x^2 + 3xy + 18\frac{x^2}{2}\right) = \frac{12}{D^2}(10x^2 + 3xy) = 120\left(\frac{x^4}{3\,.\,4}\right) + 36y\left(\frac{x^3}{2\,.\,3}\right) = 10\,x^4 + 6x^3y.$$

5.4 A General Method of Finding the Particular Integral of Linear Homogeneous Equation with Constant Coefficients

Working rule for finding P.I. (General method) of $F(D, D')z = f(x, y)$ (5.4.1)

$P.I = \frac{1}{(D-m_1D')(D-m_2D')(D-m_3D')\ldots\ldots\ldots\ldots(D-m_nD')}f(x, y)$ (5.4.2)

We shall use one of the following formulas :

Formula I : $\frac{1}{(D-mD')}f(x, y) = \int f(x, c - mx)dx$, where c = y + mx (5.4.3)

Formula II : : $\frac{1}{(D+mD')}f(x, y) = \int f(x, c + mx)dx$, where c = y - mx (5.4.4)

Hence in order to evaluate P.I. (5.4.2), we apply (5.4.3) or (5.4.4) depending on the factor $D - mD'$ and $D + mD'$. Note that result (5.3.4) can be obtained from (5.3.3) by replacing m by –m.

Example 5.4.1: Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution: Here given $(D^2 - DD' - 2D'^2)z = (y-1)e^x$or $(D+D')(D-2D')z = (y-1)e^x$

Its auxiliary equation is $(D+D')(D-2D') = 0$

The C.F is $\phi_1(y-x) + \phi_2(y+2x)$, ϕ_1 and $\phi_2$being arbitrary functions

P.I. = $\frac{1}{(D+D')(D-2D')}(y-1)e^x = \frac{1}{(D+D')}\left\{\frac{1}{(D-2D')}(y-1)e^x\right\}$

$= \frac{1}{(D+D')}\int(c-2x-1)e^x dx$ taking $c = y+2x$.

$= \frac{1}{(D+D')}\{(c-2x-1)e^x - \int(-2)e^x dx\}$, integrating by parts

$$= \frac{1}{(D+D')}\{(c-2x-1)e^x + 2e^x\} = \frac{1}{(D+D')}\{(c-2x+1)e^x\}$$

$$= \frac{1}{(D+D')}\{(y+2x-2x+1)e^x\} == \frac{1}{(D+D')}(y+1)e^x$$

$= \int(c'+x+1)e^x dx$ and taking $c' = y-x$

$= (c'+x+1)e^x - e^x = ye^x$ since as $c' = y-x$

Hence the required general solution is $z = \phi_1(y-x) + \phi_2(y+2x) + ye^x$

Example 5.4.2: Solve$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = sinx$.

Solution: Rewriting, the given equation is $(D+D')z = sinx$.

Its auxiliary equation is $(D+D') = 0$.

The C.F is $\phi(y-x)$, where ϕ is an arbitrary function.

P.I. = $\frac{1}{(D+D')}sinx = \int sinx\, dx = -\cos x$

Hence the required solution is . $z = C.F + P.I = \phi(y-x) - \cos x$.

5.5 <u>Particular Integral of Non–Homogeneous Linear Partial Differential Equation</u>

$F(D,D')z = f(x,y)$ (5.5.1)

The inverse operator $\frac{1}{F(D,D')}$ of the operator $F(D,D')$ is defined by the following identity:

$$F(D,D')\left(\frac{1}{F(D,D')}f(x,y)\right) = f(x,y)$$

Particular integral (P.I.) = $\left(\frac{1}{F(D,D')}f(x,y)\right)$.

Determination particular integral of non–homogeneous linear partial differential equations (reducible or irreducible), namely

$F(D,D')z = f(x,y)$ (5.5.2)

The methods of finding particular integrals of non–homogeneous partial differential equations are very similar to those of ordinary linear differential equation with constant coefficients. We now give a list of some cases of finding P.I. of (5.5.2).

Case I. When $f(x,y) = e^{ax+by}$ and $F(a,b) \neq 0$.

Then P.I = $\frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}$.

Thus in this case we replace D by a and D′ by b.

Case II. When $f(x, y) = \text{Sin}(ax + by)$ or $\text{Cos}(ax + by)$

Then P.I = $\frac{1}{F(D,D')}\sin(ax + by)$ or $\frac{1}{F(D,D')}\cos(ax + by)$

Which is evaluated by putting $D^2 = -a^2$, $D^2 = -b^2$, $DD' = -ab$, provided the denominator is non–zero.

Case III. When $f(x, y) = x^m y^n$.

Then, P.I = $\frac{1}{F(D,D')}x^m y^n = [F(D, D')]^{-1}x^m y^n$

which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D′/D or D/D′ or D or D′ as the case may be. In practice, we shall expand in ascending powers of D′/D. However note that if we expand in ascending powers of D/D′, we shall get a P.I. of apparently different form. In this connection remember that both forms of P.I. are correct because the two could be transformed into each other with the help of C.F. of the given equation.

Case IV. When $f(x, y) = Ve^{ax+by}$,when V is a function of x and y.

Then, P.I = $\frac{1}{F(D,D')}Ve^{ax+by} = e^{ax+by}\,\frac{1}{F(D+a,D'+b)}V$

Example 5.5.1: Solve$(D^2 - D'^2 + D - D')z = e^{2x+3y}$

Solution: The given equation can be re–written as

$((D - D')(D + D') + D - D')z = e^{2x+3y}$or $(D - D')(D + D' + 1)z = e^{2x+3y}$

∴ C.F is $\phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1 and ϕ_2 being an arbitrary function.

and P.I = $\frac{1}{(D-D')(D+D'+1)}e^{2x+3y} = \frac{1}{(2-3)(2+3+1)}e^{2x+3y} = -\frac{1}{6}e^{2x+3y}$.

Hence the required general solution is $z = \phi_1(y + x) + e^{-x}\phi_2(y - x) - \frac{1}{6}e^{2x+3y}$.

Example 5.5.2: Solve$(D^2 - DD' + D' - 1)z = \cos(x + 2y)$

Solution: The given equation can be re–written as

$(D^2 - DD' + D' - 1)z = \cos(x + 2y)$ or $(D - 1)(D - D' + 1)z = \cos(x + 2y)$.

∴ C.F is $e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1 and ϕ_2 being an arbitrary function.

P.I. =$\frac{1}{(D-1)(D-D'+1)}\cos(x + 2y) = \frac{1}{-1+1\,.2+D'-1}\cos(x + 2y) = \frac{1}{D'}\cos(x + 2y)$

$= \left(\frac{1}{2}\right)\sin(x + 2y)$, as 1/D′ stands for integration w.r.t. y keeping x as constant

Hence the required solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + \left(\frac{1}{2}\right)\sin(x + 2y)$.

Example 5.5.3: Solve$(D^2 - D'^2 - 3D + 3D')z = xy$

Solution: Re–writing, given equation is

$$(D - D')(D + D' + 3)z = xy$$

Its C.F. is $\phi_1(y + x) + e^{3x}\phi_2(x - y), \phi_1$ and $\phi_2$being arbitrary functions

P.I.=$\frac{1}{(D-D')(D+D'+3)}xy = \frac{1}{D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D+D'}{3}\right)xy$

$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \cdots.\right)\left(1 + \frac{D+D'}{3} + \frac{(D+D')^2}{9} + \cdots \ldots \ldots \ldots\right)xy$

$= -\frac{1}{3D}\left(1 + \frac{1}{3}D + \frac{1}{3}D' + \frac{2}{9}DD' + \frac{D'}{D} + \frac{1}{3}D' + \cdots..\right)xy$

$$=-\frac{1}{3D}\left(xy+\frac{1}{3}y+\frac{2}{3}x+\frac{2}{9}+\frac{x^2}{2}\right)=-\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2}{9}x+\frac{x^3}{6}\right)$$

$$\therefore z=\phi_1(y+x)+e^{3x}\phi_2(x-y)-\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2}{9}x+\frac{x^3}{6}\right)$$

Summary

- The particular integral is defined.
- Particular method derived for homogenous differential equations.
- Different kind of function with their P.I elaborated with examples.
- The non-homogenous differential equation with all kinds of functions are discussed.

Keywords

- Homogeneous
- Non-homogenous
- Second order
- Higher order
- Particular Integral
- Reducible
- Irreducible

Self Assessment

1. In the solution of differential equation $(D^2-2DD'+D'^2)z=e^{x+2y}$ the PI is

A. $x+2y$
B. e^{x+2y}
C. e^{x-2y}
D. e^{2x+y}

2. P.I of the equation is $r-2s+t=\cos(2x+3y)$ is

A. –cos (2x+3y)
B. cos (2x+3y)
C. sin (2x+3y)
D. –sin(2x+3y)

3. In The PI of the partial differential equation $\left(D^2+2DD'+D'^2\right)Z=e^{2x+3y}$

A. e^{2x+3y}
B. $\frac{1}{25}e^{2x+3y}$
C. $-e^{2x+3y}$
D. $-\frac{1}{25}e^{2x+3y}$

4. The general solution of partial differential equation $(D-D')^2z=\tan(y+x)$ **is**

A. $z = \phi_1(y+x) + x\phi_2(y+x) + \frac{x^2}{2}\tan(y+x)$

B. $z = \phi_1(y-x) + x\phi_2(y-x) + \frac{y^2}{2}\tan(y+x)$

C. $z = \phi_1(y+x) + \phi_2(y+x) + \frac{y^2}{2}\tan(y+x)$

D. $z = \phi_1(y-x) + \phi_2(y-x) + \frac{x^2}{2}\tan(y+x)$

5. Solution of the differential equation $(D-2D'-1)(D-2D'^2+1)Z=0$ is

A. $z = e^x\phi_1(y-2x) + \sum Ae^{(2k^2+1)x+ky}$

B. $z = e^x\phi_1(2y+2x) + \sum Ae^{(2k^2-1)x+ky}$

C. $z = e^x\phi_1(y+2x) + \sum Ae^{(2k^2+1)x+ky}$

D. $z = e^x\phi_1(y-x) + \sum Ae^{(2k^2)x+ky}$

6. Solution of the differential equation $(D-D'^2)z = e^{2x+y}$ is

A. $z = \sum Ae^{kx+y}$

B. $z = \sum Ae^{k^2x+ky} + e^{2x+y}$

C. $z = \sum Ae^{kx-y} - e^{2x+y}$

D. None of these

7. The particular integral of the equation $(D^2-D')z = 2y - x^2$ is

A. x^2y

B. x^2y^2

C. xy

D. x^2y^3

8. The solution of In the solution of differential equation $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{x+2y}$ the PI is

A. $x+2y$

B. $-\frac{1}{3}e^{x+2y}$

C. e^{x-2y}

D. $\frac{1}{3}e^{2x+y}$

9. The solution of differential equation $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$ is

A. $e^{-x}\phi_1(y) + e^x\phi_2(y - x) - \frac{1}{10}[\cos(x + 2y) + 2\sin(x + 2y)]$

B. $e^{-x}\phi_1(y) + e^x\phi_2(y - x) + \frac{1}{10}[\cos(x + 2y) - 2\sin(x + 2y)]$

C. $e^{-x}\phi_1(y) + e^x\phi_2(y + x) + \frac{1}{10}[\cos(x + 2y) + 3\sin(x + 2y)]$

D. None of these

10. In the solution of differential equation $3s - 2t - q = \sin(2x + 3y)$, the CF is

A. $\phi_1(x) + e^{-x/3}\phi_2(2x - 3y)$

B. $\phi_1(x) + e^{x/3}\phi_2(2x + 3y)$

C. $\phi_1(x) + e^{x/3}\phi_2(2x - 3y)$

D. $\phi_1(x) + e^{-x/3}\phi_2(2x + 3y)$

11. The solution of the differential equation $s + p - q = z + xy$ is

A. $z = e^x\phi_1(y) + e^y\phi_2(x) - xy + y + x + 1$

B. $z = e^x\phi_1(y) + e^{-y}\phi_2(x) - xy - y + x + 1$

C. $z = e^x\phi_1(y) + e^{-y}\phi_2(x) + xy + y + x + 1$

D. $z = e^x\phi_1(y) + e^{-y}\phi_2(x) + xy + x + 1$

12. Solution of the differential equation $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x + y)$ is

A. $z = \sum Ae^{hx-ky} - \frac{4}{3}e^{x+y}\sin(x + y)$

B. $z = \sum Ae^{hx-ky} + \frac{1}{3}e^{x+y}\sin(x + y)$

C. $z = \sum Ae^{hx+ky} + \frac{4}{3}e^{x+y}\sin(x + y)$

D. None of these

13. The PI of the differential equation is $(D^2 - D')z = e^{x+y}$

A. $-ye^{x+y}$
B. e^{x+y}
C. xye^{x+y}
D. None of these

14. The PI of the differential equation is $(D^2 - D')z = xe^{x+y}$

A. $\frac{x}{4}(x-1)e^{x+y}$
B. $x(x+1)e^{x+y}$ e^{x+y}
C. $\frac{x}{4}(x+1)e^{x+y}$
D. None of these

15. The particular integral of the equation $(D^2 - D')z = e^{x+y}$ is
A. $\frac{1}{2}xe^{x+y}$
B. xe^{x+y}
C. $\frac{1}{2}ye^{x+y}$
$-xye^{x+y}$
D.

Answers for Self Assessment

1.	B	2.	B	3.	B	4.	A	5.	B
6.	B	7.	A	8.	B	9.	A	10.	B
11.	B	12.	C	13.	A	14.	A	15.	B

Review Questions

Q1. Solve $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$.

Q2. Solve $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$.

Q3. . Solve $r - 5s + 4t = \sin(4x + y)$.

Q4. Solve $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y + x)^{1/2}$.

Q5. Solve $(D^2 - 2DD' + D'^2)z = e^{x+y} + x^3$.

Q6. Solve $(D^2 + DD' - 6D'^2)z = ysinx$.

Q7. Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$.

Q8. Solve $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x + y)$.

Further Readings

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Dr. Preety Kalra, Lovely Professional University

Unit 06: Monge's Method and Method of Separation of Variables

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of need of Monge's method.
- understand the concept of method of separation of variables.
- know the properties of Monge's method.
- apply appropriate methods to find the solutions of second order PDE.

Introduction

The most general form of partial differential equation of order two is

$$f(x, y, z, p, q, r, s, t) = 0 \qquad (6.0.1)$$

It is only in special cases that (6.0.1) can be integrated.

Some well known methods of solutions were given by Monge. His methods are applicable to a wide class (but not all) of equations of the form (6.0.1).

Monge's methods consists in finding one or two first integrals of the form

$$u = \varphi(v), \qquad (6.0.2)$$

where u and v are known functions of x, y, z, p and q and φ is an arbitrary function. In other words, Monge's methods consists in obtaining relations of the form (6.0.2) such that equation (6.0.1) can be derived from (6.0.2) by eliminating the arbitrary function. A relation of the form (6.0.2) is known as an intermediate integral of (6.0.1). Every equation of the form (6.0.1) need not possess an intermediate integral. However, it has been shown that most general partial differential equations having (6.0.2) as an intermediate integral are of the following forms

Rr + Ss + Tt = V and Rr + Ss + Tt + U(rt – s^2) = V, (6.0.3)

where R, S, T, U and V are functions of x, y, z, p and q. Even equations (6.0.3) need not always possess an intermediate integral. In what follows we shall assume that an intermediate integral of (6.0.3) exists.

6.1 Monge's Method Of Integrating Rr + Ss + Tt = V

Given Rr + Ss + Tt = V, (6.1.1)

where R, S, T and V are functions of x, y, z, p and q.

We know that $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$,

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$, $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}$

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x}$, and $s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y}$ (6.1.2)

Now, $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy$ (6.1.3)

and $dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy$ (6.1.4)

From (6.1.3) and (6.1.4), $r = (dp - sdy)/dx$ and $t = (dq - sdx)/dy$ (6.1.5)

Substituting the values of r and s given by (6.1.5) in (6.1.1), we get

$\frac{R(dp - sdy)}{dx} + Ss + T\frac{(dq - sdx)}{dy} = V$ or $R(dp - sdy)dy + Ssdxdy + T(dq - sdx)dx = Vdxdy$

or $\{Rdpdy + Tdqdx - Vdxdy\} - s\{R(dy)^2 - Sdxdy + T(dx)^2\} = 0$ (6.1.6)

Clearly any relation between x, y, z, p and q which satisfies (6.1.6) must also satisfy the following two simultaneous equations

$Rdpdy + Tdqdx - Vdxdy = 0$ (6.1.7)

$R(dy)^2 - Sdxdy + T(dx)^2$ (6.1.8)

The equations (6.1.7) and (6.1.8) are called Monge's subsidiary equations and the relations which satisfy these equations are called intermediate integrals.

Equation (6.1.8) being a quadratic, in general, it can be resolved into two equations, say

$dy - m_1 dx = 0$ (6.1.9)

$dy - m_2 dx = 0$ (6.1.10)

Now the following two cases arise:

Case I. When m_1 and m_2 are distinct in (6.1.9) and (6.1.10).

In this case (6.1.7) and (6.1.9), if necessary by using well known result

dz = pdx + qdy,

will give two integrals u_1 = a and v_1 = b, where a and b are arbitrary constants. These give

$u_1 = f(v_1)$, (6.1.11)

where f_1 is an arbitrary function. It is called an intermediate integral of (6.1.1).

Next, taking (6.1.7) and (6.1.10) as before, we get another intermediate integral of (6.1.1), say

$u_2 = f(v_2)$, (6.1.11)

where f_2 is an arbitrary function.Thus we have in this case two distinct intermediate integrals (6.1.11) and (6.1.12). Solving (6.1.11) and (6.1.12), we obtain values of p and q in terms of x, y and z. Now substituting these values of p and q in well-known relation

dz = pdx + qdy (6.1.13)

and then integrating (6.1.13), we get the required complete integral of (6.1.1).

Case II . When $m_1 = m_2$ i.e., (6.1.8) is a perfect square.

As before, in this we get only one intermediate integral which is in Lagrange's form

$Pp + Qq = R$ (6.1.14)

Solving (6.1.14) with help of Lagrange's method, we get the required complete integral of (6.1.1).

Remark 6.1.1: Usually while dealing with case I, we obtain second intermediate integral directly by using symmetry. However sometimes in absence of any symmetry, we find the complete integral with help of only one intermediate integral. This is done with help of using Lagrange's method.

Remark 6.1.2: While obtaining an intermediate integral, remember to use the relation dz = pdx + qdy as explained below :

(i) pdx + qdy + 2xdx = 0 can be re–written as dz + 2xdx = 0 so that $z + x^2 = c$.

(ii) xdp + ydq = dx can be re–written as xdp + ydq + pdx + qdy = dx + pdx + qdy or d(xp) + d(yq) = dx + dz so that xp + yq = x + z + c, on integration

Remark 6.1.3:**Important Note**. For sake of convenience, we have divided all questions based on Rr + Ss + Tt = V in four types. We shall now discuss them one by one.

6.2 Type 1. When the given equation Rr + Ss + Tt = V leads to two distinct intermediate integrals and both of them are used to get the desired solution

Working rule for solving problems of type 1.

Step 1. Write the given equation in the standard form$Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations:

$Rdpdy + Tdqdx - Vdxdy = 0$ (6.2.1)

$R(dy)^2 - Sdxdy + T(dx)^2$ (6.2.2)

Step 3. Factorise (6.2.1) into two distinct factors.

Step 4. Using one of the factors obtained in (6.2.1), (6.2.2) will lead to an intermediate integral. In general, the second intermediate integral can be obtained from the first one by inspection, taking advantage of symmetry. In absence of any symmetry, the second factor obtained in step 3 is used in (6.2.2) to arrive at second intermediate integral.

Step 5. Solve the two intermediate integrals obtained in step 4 and get the values of p and q.

Step 6. Substitute the values of p and q in dz = pdx + qdy and integrate to arrive at the required general solution by integrating dz = pdx + qdy.

SOLVED EXAMPLES

Example 6.2.1: Solve $r = a^2t$

Solution: Given equation is $r - a^2t = 0$.

Comparing it with Rr + Ss + Tt = V, we have R = 1, S = 0, T = – a2 , V = 0.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become$dpdy - a^2dqdx = 0$ (6.2.3)

and $(dy)^2 - a^2(dx)^2 = 0$ (6.2.4)

Equation (6.2.4) may be factorized as $(dy - adx)(dy + adx) = 0$.

Hence two systems of equations to be considered are

$dpdy - a^2dqdx = 0, \quad dy - adx = 0$ (6.2.5)

and $dpdy - a^2dqdx = 0, \quad dy + adx = 0$ (6.2.6)

Integrating the second equation of (6.2.5), we get $y - ax = c_1$ (6.2.7)

Eliminating dy/dx between the equations of (6.2.5), we get

$dp - adq = 0$ so that $p - aq = c_2$ (6.2.8)

Hence the intermediate integral corresponding to (6.2.5) is $p - aq = \phi_1(y - ax)$ (6.2.9)

Similarly another intermediate integral corresponding to (6.2.6) is

$p - aq = \phi_2(y + ax)$ (6.2.10)

Here ϕ_1 and $\phi_2$are arbitrary functions.

Solving (6.2.9) and (6.2.10) for p and q, we have

$p = \frac{1}{2}\{\phi_2(y + ax) + \phi_1(y - ax)\}$ and $q = \frac{1}{2a}\{\phi_2(y + ax) - \phi_1(y - ax)\}$.

Substituting these values of p and q in dz = pdx + qdy, we get

$$dz = \frac{1}{2}\{\phi_2(y + ax) + \phi_1(y - ax)\}dx + \frac{1}{2a}\{\phi_2(y + ax) - \phi_1(y - ax)\}dy$$

$$= \frac{1}{2a}\phi_2(y + ax)(dy + adx) - \frac{1}{2a}\phi_1(y - ax)(dy - adx)$$

Integrating, $z = \psi_2(y - ax) + \psi_1(y + ax), \psi_1, \psi_2$ being arbitrary functions.

6.3 Type 2. When the given equation Rr + Ss + Tt = V leads to two distinct intermediate integrals and only one is employed to get the desired solution

Working rule for solving problems of type 2.

Step 1. Write the given equation in the standard form Rr + Ss + Tt = V.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ (6.3.1)

$R(dy)^2 - Sdxdy + T(dx)^2$ (6.3.2)

Step 3. Factorize (6.3.1) into two distinct factors.

Step 4. Take one of the factors of step 3 and use (6.3.2) to get an intermediate integral. Don't find second intermediate integral as we did in type 1.

Step 5. Re-write the intermediate integral of the step 4 in the form of Lagrange equation, namely, Pp + Qq = R. Using the well-known Lagrange's method we arrive at the desired general solution of the given equation.

SOLVED EXAMPLES

Example 6.3.1: Solve (r – s)y + (s – t)x + q – p = 0.

Solution: The given can be written as yr + s(x − y) − tx = p − q (6.3.3)

Comparing (1) with Rr + Ss + Tt = V, R = y, S = x – y, T = –x and V = p – q.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $ydpdy - xdqdx - (p-q)dxdy = 0$ (6.3.4)

$y(dy)^2 - (x-y)dxdy - x(dx)^2 = 0$ (6.3.5)

Rewriting (6.3.5), $(dy+dx)(ydy - xdx) = 0$.

so that $dy + dx = 0$ or $dy = -dx$ (6.3.6)

and $ydy - xdx = 0$.

Using (6.3.6), (6.3.4) becomes

$-ydpdx - xdqdx + qdx(-dx) - pdxdy = 0$ or $ydp + xdq + qdx + pdy = 0$

or $ydp + pdy + xdq + qdx = 0$ or $d(yp) + d(xq) = 0$ so that $yp + xq = c_1$. (6.3.7)

Integrating (6.3.6), $x + y = c_2$, c_2 being an arbitrary constant. (6.3.8)

From (6.3.7) and (6.3.8), one intermediate integral is

$yp + xq = f(x+y)$, (6.3.9)

which is of the Lagrange's form and so its subsidiary equations are

$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x+y)}$ (6.3.10)

From first and second fractions of (6.3.10),

$$2xdx - 2ydy = 0.$$

Integrating, $x^2 - y^2 = a$, $a$being an arbitrary constant (6.3.11)

Taking first and third fractions of (6.3.10), we get

$\frac{dx}{y} = \frac{dz}{f(x+y)}$ or $\frac{dx}{(x^2-a)^{1/2}} = \frac{dz}{f(x+(x^2-a)^{1/2})}$ or

$dz = f(x + (x^2-a)^{1/2})(x^2-a)^{1/2}dx$ (6.3.12)

Put $x + (x^2-a)^{1/2} = v$ so that $\left[1 + \frac{x}{(x^2-a)^{\frac{1}{2}}}\right]dx = dv$ (6.3.13)

or $\frac{x+(x^2-a)^{\frac{1}{2}}}{(x^2-a)^{\frac{1}{2}}}dx = dv$ or $\frac{dx}{(x^2-a)^{\frac{1}{2}}} = \frac{dv}{v}$

Then, (6.3.12) reduces to $dz - \frac{1}{v}f(v)dv = 0$.

Integrating, $z - F(v) = b$ or $z - F\left(x + (x^2-a)^{\frac{1}{2}}\right) = b$.

$z - F(x+y) = b$ as $y = (x^2-a)^{\frac{1}{2}}$ (6.3.14)

From (6.3.11) and (6.3.14), the required general solution is

$z - F(x+y) = G(x^2 - y^2)$,

$z = F(x+y) + G(x^2-y^2)$, where F and G are arbitrary functions.

6.4 Type 3. When the given equation Rr + Ss + Tt = V leads to two Identical Intermediate Integrals

Working rule for solving problems of type 3

Step 1. Write the given equation in the standard form Rr + Ss + Tt = V.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ (6.4.1)

$R(dy)^2 - Sdxdy + T(dx)^2$ (6.4.2)

Step 3. R.H.S. of (2) reduces to a perfect square and hence it gives only one distinct factor in place of two as in type 1 and type 2.

Step 4. Start with the only one factor of step 3 and use (2) to get an intermediate integral.

Step 5. Re–write the intermediate integral of the step 4 in the form of Pp + Qq = R and use Lagrange's method to obtain the required general solution of the given equation.

SOLVED EXAMPLES

Example 6.2.1: Solve $(1 + q)^2 r - 2(1 + p + q + pq)s + (1 + p)^2 t = 0$

Solution: Comparing the given equation with Rr + Ss + Tt = V, (6.4.3)

$R = (1 + q)^2$, $S = -2(1 + p + q + pq)$, $T = (1 + p)^2$, $V = 0$ (6.4.4)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ (6.4.5)

and $R(dy)^2 - Sdxdy + T(dx)^2=0$ (6.4.6)

Using (6.4.4), (6.4.5) and (6.4.6) become

$(1 + q)^2dpdy + (1 + p)^2dqdx = 0$ (6.4.7)

and $(1 + q)^2(dy)^2 + 2(1 + p + q + pq)dxdy + (1 + p)^2(dx)^2 = 0$ (6.4.8)

Since $1 + p + q + pq = (1 + p)(1 + q)$, (6.4.8) becomes $[(1 + q)dy + (1 + p)dx]^2 = 0$

so that $(1 + q)dy + (1 + p)dx = 0$ or $(1 + q)dy = -(1 + p)dx$. (6.4.9)

Keeping (6.4.9) in view, (6.4.7) may be re–written as

$(1 + q)dp\{(1 + q)dy\} - (1 + p)dq\{-(1 + p)dx\} = 0.$ (6.4.10)

Dividing each term of (6.4.10) by (1 + q)dy, or its equivalent –(1 + p)dx, we get

$(1 + q)dp - (1 + p)dq = 0$ or $\frac{dp}{1+p} - \frac{dq}{1+q} = 0$

Integrating it, $\frac{(1+p)}{1+q} = c_1$, c_1 being an arbitrary constant. (6.4.11)

From (6.4.9), $dx + dy + pdx + qdy = 0$, or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_2$, c_2 being an arbitrary constant (6.4.12)

From (6.4.11) and (6.4.12), one intermediate integral of (6.4.3) is

$\frac{1+p}{1+q} = F(x + y + z)$ or $1 + p = (1 + q)F(x + y + z)$ or

$p - qF(x + y + z) = F(x + y + z) - 1.$ (6.4.13)

which is of the form Pp + Qq = R. So Lagrange's auxiliary equations are

$\frac{dx}{1} = \frac{dy}{-F(x+y+z)} = \frac{dz}{F(x+y+z)-1}$ (6.4.14)

Choosing 1, 1, 1 as multipliers, each fraction of (6.4.14)= $dx + dy + dz/0$

so that $dx + dy + dz = 0$ giving $x + y + z = c_2$ (6.4.15)

Using (6.4.15) and taking the first two fractions of (6.4.14),

we have $dx = -dy/F(c_2)$ or $dy + dx\, F(c_2)=0$

Integrating it, $y+xF(c_2) = c_3$ or $y+xF(x + y + z) = c_3$. (6.4.16)

From (6.4.15) and (6.4.16), the required general solution is y + x F(x + y + z) = G(x + y + z), F, G being arbitrary functions.

6.5 Type 4. When the given equation Rr + Ss + Tt = V fails to yield an intermediate integral as in cases 1, 2 and 3

Working rule for solving problems of type 4

Suppose the R.H.S. of $R(dy)^2 - Sdxdy + T(dx)^2$=0 neither gives two factors nor a perfect square (as in Types 1, 2 and 3 above). In such cases factors dx, dy, p, 1 + p etc. are cancelled as the case may be and an integral of given equation is obtained as usual.

SOLVED EXAMPLES

Example 6.5.1: Solve $pq = x(ps - qr)$.

Solution: Give $xqr - xps + 0.t = -pq$ (6.5.1)

Comparing (1) with Rr + Ss + Tt = V, R = xq, S = xp, T = o and V = –pq

Monge's subsidiary equations $Rdpdy + Tdqdx - Vdxdy = 0$

and $R(dy)^2 - Sdxdy + T(dx)^2$=0

become $xqdpdy + pqdxdy = 0$ (6.5.2)

and $xq(dy)^2 - xpdxdy$=0 (6.5.3)

Dividing (6.5.2) by qdy we get xdp + pdx = 0 (6.5.4)

and dividing (6.5.3) by xdy, we get qdy + pdx = 0 (6.5.5)

Using dz = pdx + qdy, (6.5.5) gives dz = 0 so that z = c_1 (6.5.6)

Integrating (6.5.4), xp = c_2, c_2 being an arbitrary constant (6.5.7)

From (6.5.6) and (6.5.7), one integral of (6.5.1) is xp = f(z) or$x\frac{\partial z}{\partial x} = f(z)$ or $\frac{1}{f(z)}\frac{\partial z}{\partial x} = \frac{1}{x}$

Integrating it partially w.r.t. x, F(z) = log x + G(y), F, G being arbitrary functions.

6.6 Separation of Variables

A powerful method of finding solutions of second -order linear partial differential equations is applicable in certain circumstances. If, when we assume a solution of the form

z = X(x) Y(y) (6.6.1)

for the partial differential equation Rr + Ss + Tt + Pp + Qq + Zz = F (6.6.2)

it is possible to write the equation (6.6.2) in the form

$\frac{1}{X}f(D)X = \frac{1}{Y}g(D')Y$ (6.6.3)

where $f(D), g(D')$ are quadratic functions of $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$ respectively, we say that the equation (6.6.2) is separable in the variables x, y. The derivation of a solution of the equation is then immediate. For the left -hand side of (6.6.3) is a function of x alone, and the right -hand side is a function of y alone, and the two can be equal only if each is equal to a constant, A. say. The problem of finding solutions of the form (6.6.1) of the partial differential equation (6.6.2) therefore reduces to solving the pair of second -order linear ordinary differential equations

$f(D)X = \lambda X,\ \ g(D)Y = \lambda Y$ (6.6.4)

The method is best illustrated by means of a particular example. Consider the one -dimensional diffusion equation

$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t}$ (6.6.5)

If we write z = X(x)T(t)

we find that $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{kT}\frac{dT}{dt}$

so that the pair of ordinary equations corresponding to (6.6.4) is

$$\frac{d^2X}{dx^2} = \lambda X, \qquad \frac{dT}{dt} = k\lambda T$$

so that if we are looking for a solution which tends to zero as $t \to \infty$ we may take

X = A cos (nx +ϵ), T= Be^{-kn2t}

where we have written $-n^2$ for A. Thus

$$z(x,t) = c_n \cos(nx + \epsilon_n)e^{-kn^2 t}$$

where c_n is a constant, is a solution of the partial differential (6.6.5) for all values of n.

Hence expressions formed by summing over all values of n

$$z(x,t) = \sum_{n=0}^{\infty} c_n \cos(nx + \epsilon_n)e^{-kn^2 t} \tag{6.6.6}$$

are, formally at least, solutions of equation (6.6.5). It should be noted that the solutions (6.6.6.) have the property that $z \to 0$ as $t \to \infty$ co and that

$$z(x,0) = \sum_{n=0}^{\infty} c_n \cos(nx + \epsilon_n) \tag{6.6.7}$$

For example, if we wish to find solutions of the form

$$z = X(x)\, Y(y) T(t) \tag{6.6.8}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{k}\frac{\partial z}{\partial t} \tag{6.6.9}$$

we note that for such a solution equation (6.6.9) can be written as

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{kT}\frac{dT}{dt}$$

so that we may take

$$\frac{dT}{dt} = -n^2Kt\,, \quad \frac{d^2X}{dx^2} = -l^2X, \quad \frac{d^2Y}{dy^2} = -m^2Y$$

provided that

$$l^2 + m^2 = n^2.$$

Hence we have solutions of equation (6.6.9) of the form

$$z(x,y,t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} c_{lm} \cos(lx + \epsilon_l)\cos(my + \epsilon_m)\, e^{-k(l^2+m^2)t} \tag{6.6.10}$$

Summary

- The concept of the Monge's method is discussed.
- The types of Monge's method with their solution are derived.
- The properties of Monge's method were discussed.
- The method of separation of variable is elaborated.

Keywords

- Second order PDE
- Monge's Method
- Subsidiary Equations
- Intermediate Integrals
- Method of Separation of Variable.

Self Assessment

1. Which of the following equation satisfied the partial differential equation $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$, where $y(0,t) = e^{-t}$,

   A. $\frac{X' - X}{X} = -\frac{T'}{T}$

B. $\frac{X^{'} - X}{X} = \frac{T^{'}}{T}$

C. $\frac{X^{'} - X}{2X} = -\frac{T^{'}}{T}$

D. $\frac{X^{'} + X}{X} = \frac{T^{'}}{T}$

2. For $\frac{X^{''}}{X} = \frac{1}{c^2}\frac{T^{'}}{T} = k$ when k is zero then solution for X is

A. $X = c_1 \cos px + c_2 \sin\ px$

B. $X = c_1 \sin\ px$

C. $X = c_1 x + c_2$

D. $X = c_1 x + c_2 x^2$

3.If $\frac{X^{''}}{X} = \frac{1}{c^2}\frac{T^{''}}{T} = k$, when k is negative, solution for T is

A. $T = c_1 e^{pt} + c_2 e^{-pt}$

B. $T = c_1 e^{p^2 t} + c_2 e^{-p^2 t}$

C. $T = c_1 \cos pt + c_2 \sin\ pt$

D. $T = c_1 \cos cpt + c_2 \sin cpt$

4. Which of the following is a wave equation?

A. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

B. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

C. $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$

D. None of these

5.The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is ___ in nature

A. elliptic

B. Hyperbolic

C. Parabolic

D. None of these

6. The equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial y^2}$ is ____ in nature

A. Hyperbolic
B. Parabolic
C. elliptic
D. None of these

7.Solving $\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k$ if k is negative and $\mathbf{k = -p^2}$ then solution for T is

A. $T = c_1 e^{cp^2 t}$
B. $T = c_1 e^{-c^2 p^2 t}$
C. $T = c_1 e^{cpt}$
D. $T = c_1 e^{-cpt}$

8. Solving $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ with the help of separation of variable and if $\frac{X' - X}{2X} = \frac{T'}{T} = k$.
The auxiliary equationsatisfied of X is

A. $m - (1 + 2k) = 0$
B. $m + (1 + 2k) = 0$
C. $m - 1 + 2k = 0$
D. $m + 1 - 2k = 0$

9. If by using method of separation of variables on $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$, then which of the following satisfied

A. $\frac{X'' - 2X'}{X} = -\frac{Y'}{Y}$

B. $\frac{X'' + 2X'}{X} = -\frac{Y'}{Y}$

C. $\frac{X'' + 2X}{X} = -\frac{Y'}{Y}$

D. $\frac{X''-2X}{X}=\frac{Y'}{Y}$

10. The equation $\boldsymbol{Rdpdy+Tdqds-Vdxdxy=0}$ is called

A. Auxiliary equation
B. Monge's subsidiary equation
C. Ordinary differential equation
D. None of these

11. Monge's method is used to solve a partial differential equation of

A. nth order
B. third order
C. second order
D. None of these

12. The equation $\boldsymbol{R(dy)^2-Sdxdxy+T(dx)^2=0}$ is called

A. Auxiliary equation
B. Differential equation
C. Monge's subsidiary equation
D. None of these

13. Is case of repeating roots from $\boldsymbol{R(dy)^2-Sdxdxy+T(dx)^2=0}$, then intermediate integral solved by

A. Lagrange's Method
B. Cauchy's Method
C. Charpit's Method
D. None of these

14. The Monge's subsidiary equation for pde$\boldsymbol{(x-y)(xr-xs-ys+yt)=(x+y)(p-q)}$ is

A. $x(x-y)dpdy+y(x-y)dqdx-(x+y)(p-q)dxdy=0$

$B.\ x(x-y)dpdx+y(x-y)dqdy-(x+y)(p-q)dxdy=0$

$C.\ x(x-y)dpdy+y(x-y)dqx-(x+y)(p-q)dpdq=0$

D.None of these

15.The Monge's subsidiary equation for pde$\boldsymbol{(x-y)(xr-xs-ys+yt)=(x+y)(p-q)}$ is

$A.\ x(x-y)(dy)^2-(x^2-y^2)dydx+y(x-y)(dx)^2=0$

$B.\ x(x-y)(dy)^2+(x^2-y^2)+y(x-y)(dx)^2=0$

$C.x(x-y)(dy)^2-(x^2+y^2)+y(x-y)(dx)^2=0$

D.None of these

Answers for Self Assessment

1.	C	2.	C	3.	D	4.	B	5.	A
6.	B	7.	B	8.	A	9.	A	10.	B
11.	C	12.	C	13.	A	14.	A	15.	A

Review Questions

Q1. By separating the variables, show that the one-dimensional wave equation $\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$ has solutions of the form $A\exp(\pm inx \pm inct)$, where A and n are constants. Hence show that functions of the form $z(x,t)=\sum_r\left\{A_r\cos\frac{r\pi ct}{a}+B_r\sin\frac{r\pi ct}{a}\right\}\sin\frac{r\pi x}{a}$ where the A_r's and B_r's are constants, and satisfy the wave equation and the boundary conditions $z(0,t)=0,\ z(a,t)=0$ for all t.

Q2. Solve by Monge's method $\frac{\partial^2 z}{\partial x^2}-\cos^2 x\frac{\partial^2 z}{\partial y^2}+\tan x\frac{\partial z}{\partial x}=0.$

Q3. Solve $(x-y)(xr-xs-ys+yt)=(x+y)(p-q)$.

Q4. Solve by Monge's method $r-t\cos^2 x+p\tan x=0.$

Q5. Using method of separation of variable, solve $\partial u/\partial x=2(\partial u/\partial t)+u,\ where\ u(x,0)=6e^{-3x}$

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.

2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

https://onlinecourses.nptel.ac.in/noc21_ma09/preview

https://onlinecourses.swayam2.ac.in/cec22_ma12/preview

Unit 07: Laplace Transforms

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of Laplace transform.
- understand the concept of inverse Laplace transform.
- determine the properties of Laplace transform.
- find the solution of PDE using Laplace transform.

Introduction

The method of Laplace transform provided an effective and easy means for the solutions of many problems in engineering and science. Thus the knowledge of Laplace transform has become an essential part of mathematical background required for engineers and scientists. The method of Laplace transform gives directly the solution of differential equations with given boundary conditions without first finding the solution and then evaluating constants by using given boundary conditions. Moreover, the ready tables of Laplace transform reduce the problem of solving differential equations to algebraic manipulation.

7.1 Laplace Transform

In this section we introduce the concept of Laplace transform and discusssome of its properties.

Definition

The Laplace transform is defined in the following way. Let f (t) be defined for t ≥ 0. Then the Laplace transform off, which is denoted by $\mathcal{L}[f(t)]$ or by F(s), is defined by the following equations

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty e^{-st}dt \qquad (7.1.1)$$

The integral which defined a Laplace transform is an improper integral. An improper integral may converge or diverge, depending on the integrand. When the improper integral in convergent then we say that the function f (t) possesses a Laplace transform. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.

Inverse Laplace Transform

If $\mathcal{L}[f(t)] = F(s)$, then we can write it as $\mathcal{L}^{-1}[F(s)] = f(t)$. Here $f(t)$ is called the inverse Laplace transform of F(s). The symbol L, which transforms $f(t)$ to F(s)is called Laplace transformation operator.

Laplace Transforms of Elementary Functions

(i) $\mathcal{L}[1] = \frac{1}{s}$

(ii) $\mathcal{L}[t^n] = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}} \\ \frac{n!}{s^{n+1}}, for\ n = 0,1,2,3, \ldots \ldots \end{cases}$

(iii) $\mathcal{L}[e^{at}] = \frac{1}{s-a}, (s > a)$

(iv) $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}, (s > 0)$

(v) $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}, (s > 0)$

(vi) $\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}, (s > |a|)$

(vii) $\mathcal{L}[\cosh at] = \frac{s}{s^2-a^2}, (s > |a|)$

These formulas are proved below in terms of following examples by using the definition (7.1.1).

Example 7.1.1: Find the Laplace transform, if it exists, of each of the following functions

(a) $f(t) = e^{at}$ (b) $f(t) = 1$ (c) $f(t) = t$

Solution: (a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t} dt = \lim_{T\to\infty} \int_0^T e^{-(s-a)t} dt.$$

But $\int_0^T e^{-(s-a)t} dt = \begin{cases} T & if\ s = a \\ \frac{1-e^{-(s-a)T}}{s-a} & if\ s \neq a \end{cases}$

For the improper integral to converge we need s > a. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt = \frac{1}{s}, s > 0.$$

(c) We have

$$\mathcal{L}[1] = \int_0^\infty te^{-st} dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}, s > 0.$$

Example 7.1.2: Find the Laplace transform, if it exists, of each of the following functions

(a) $f(t) = \sin at$(b) $f(t) = \cos at$ (c) $f(t) = \sinh at$ (d) $f(t) = \cosh at$

Solution: (a) $\mathcal{L}[sin\ at] = \int_0^\infty e^{-st} \sin at\, dt = \lim_{T\to\infty} \int_0^T e^{-st} \sin at\, dt$

$== \lim_{T\to\infty} \left[\frac{1}{s^2+a^2} e^{-st}(-s\, sinat - a\cos at)\right]_0^T$

$= \frac{1}{s^2+a^2}[0 - e^0(0 - a.1)] = \frac{a}{s^2+a^2}$.

(b) $\mathcal{L}[cos\ at] = \int_0^\infty e^{-st} \cos at\, dt = \lim_{T\to\infty} \int_0^T e^{-st} \cos at\ \ dt$

$== \lim_{T\to\infty} \left[\frac{1}{s^2+a^2} e^{-st}(-s\, cos\ at + a\sin at)\right]_0^T$

$= \frac{1}{s^2+a^2}[0 - e^0(-s + 0)] = \frac{s}{s^2+a^2}$.

(c) $\mathcal{L}[sinh\ at] = \int_0^\infty e^{-st} \sinh at\, dt$

$$= \int_0^\infty e^{-st} \frac{(e^{at} - e^{-at})}{2} dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$\frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)-(s-a)}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

(c) $\mathcal{L}[cosh\ at] = \int_0^\infty e^{-st} \cosh at\, dt$

$$= \int_0^\infty e^{-st} \frac{(e^{at} + e^{-at})}{2} dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt + \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$\frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)+(s-a)}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

Inverse Laplace Transforms of Elementary Functions

(i) $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

(ii) $\mathcal{L}[t^n] = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}} \\ \frac{n!}{s^{n+1}}, for\ n = 0,1,2,3,\dots\dots \end{cases}$

(iii) $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

(iv) $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{\mathrm{a}} \sin at$

(v) $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

(vi) $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

(vii) $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

Properties of Laplace Transform

1. **Linearity property**: Ifa, b, c are constants and f, ϕ, ψ are any functions of t, then $\mathcal{L}[af(t) + b\phi(t) - c\psi(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[\phi(t)] - c\mathcal{L}[\psi(t)]$

   Poof: Now $\mathcal{L}[af(t) + b\phi(t) - c\psi(t)] = \int_0^\infty e^{-st}[af(t) + b\phi(t) - c\psi(t)]dt$

   $$= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st}\phi(t)dt - c\int_0^\infty e^{-st}\psi(t)dt$$

   $$= a\mathcal{L}[f(t)] + b\mathcal{L}[\phi(t)] - c\mathcal{L}[\psi(t)]$$

Remarks 7.1.1: (i)The above property can be generalised to any number of functions.

(ii) Due to above property, L is called linear operator.

2. **First shifting property**: If $\mathcal{L}[f(t)] = F(s)$, then If $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

   Proof: $\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}(e^{at}f(t)dt$

   $= \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-kt}f(t)dt$ where $k = s - a > 0$

   $$= F(k) = F(s-a)$$

   Hence $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

Remarks 7.1.2: (i) If the Laplace transform of $f(t)$ if $F(s)$, then the Laplace transform of $e^{at}f(t)$ is obtained simply by replacing s by $s - a$. Now by applying first shifting property, we have the following list of useful results.

(i) $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

(ii) $\mathcal{L}[e^{at}t^n] = \begin{cases} \frac{\Gamma(n+1)}{(s-a)^{n+1}} \\ \frac{n!}{(s-a)^{n+1}}, for\ n = 0,1,2,3, \ldots \ldots \end{cases}$

(iii) $\mathcal{L}[e^{at}\sin bt] = \frac{a}{(s-a)^2+b^2}, (s > 0)$

(iv) $\mathcal{L}[e^{at}\cos bt] = \frac{s}{(s-a)^2+b^2}, (s > 0)$

(v) $\mathcal{L}[e^{at}\sinh bt] = \frac{a}{(s-a)^2-b^2}, (s > |a|)$

(vi) $\mathcal{L}[e^{at}\cosh bt] = \frac{a}{(s-a)^2-b^2}, (s > |a|)$

Where in each case s>a.

Example 7.1.3: Find the Laplace transform of

(a) $f(t) = \sin 2t \cos 3t$(b) $f(t) = \sin^2 3t$ (c) $f(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$

Solution: (a) Here $f(t) = \frac{1}{2}(2\sin 2t\cos 3t) = \frac{1}{2}[\sin 5t - sint]$

$$\mathcal{L}(f(t)) = \left\{\frac{1}{2}(\sin 5t - sint)\right\} = \frac{1}{2}[\mathcal{L}(\sin 5t) - \mathcal{L}(sint)]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1}\right]$$

(b) $f(t) = \sin^2 3t = \frac{1}{2}(1 - cos6t)$

$$\mathcal{L}(f(t)) = \mathcal{L}\left[\frac{1}{2}(1 - cos6t)\right]$$

$$=\frac{1}{2}[\mathcal{L}(1)-\mathcal{L}(cos6t)]$$

$=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+36}\right]=\frac{1}{2}\left[\frac{s^2+36-s^2}{s(s^2+36)}\right]=\frac{18}{s(s^2+36)}.$

(c) $f(t)=\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3=t^{3/2}-\frac{1}{t^{\frac{3}{2}}}-3\sqrt{t}\,\frac{1}{\sqrt{t}}(\sqrt{t}-1/\sqrt{t})$

$$=t^{3/2}-t^{-\frac{3}{2}}-3t^{\frac{1}{2}}+3t^{-1/2}$$

$\mathcal{L}(f(t))=\mathcal{L}(t^{3/2}-t^{-\frac{3}{2}}-3t^{\frac{1}{2}}+3t^{-1/2})$

$$=\frac{1}{s^{\frac{5}{2}}}\Gamma\left(\frac{5}{2}\right)-\frac{1}{s^{\frac{-1}{2}}}\Gamma\left(-\frac{1}{2}\right)-3\frac{1}{s^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)+\frac{3}{s^{1/2}}\Gamma\left(\frac{1}{2}\right)$$

We know that $\Gamma(n+1)=n\Gamma(\mathrm{n})=\sqrt{\pi}$.

$$\mathcal{L}(f(t))=\frac{1}{s^{\frac{5}{2}}}\frac{3}{4}\sqrt{\pi}-\frac{1}{s^{\frac{-1}{2}}}(-2\sqrt{\pi})-3\frac{1}{s^{\frac{3}{2}}}\left(\frac{1}{2}\sqrt{\pi}\right)+\frac{3}{s^{\frac{1}{2}}}\sqrt{\pi}.$$

$$=\frac{\sqrt{\pi}}{4}\left[\frac{3}{s^{5/2}}+\frac{8}{s^{-1/2}}-\frac{6}{s^{3/2}}+\frac{12}{s^{1/2}}\right]=\frac{\sqrt{\pi}}{4s^{5/2}}[3+8s^3-6s+12s^2]$$

Example 7.1.4: Find the Laplace transform of

(a) $f(t)=\mathrm{e}^{2\mathrm{t}}(\sin 2t\cos 3t)$ (b) $f(t)=t^2e^{-2t}$

Solution: (a) $f(t)=\frac{1}{2}(2\sin 2t\cos 3t)=\frac{1}{2}[\sin 5t-sint]$

$$\mathcal{L}(f(t))=\left\{\frac{1}{2}(\sin 5t-sint)\right\}=\frac{1}{2}[\mathcal{L}(\sin 5t)-\mathcal{L}(sint)]$$

$$=\frac{1}{2}\left[\frac{5}{s^2+5^2}-\frac{1}{s^2+1}\right]$$

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}}(\sin 2t\cos 3t)]=\frac{1}{2}\left[\frac{5}{(s-2)^2+5^2}-\frac{1}{(s-2)^2+1}\right]$$

(b) $\mathcal{L}(t^2)=\frac{2}{s^3}$

By applying the first shifting property

$$\mathcal{L}(e^{-2t}t^2)=\frac{2}{(s+2)^3}$$

3. **Change of scale property: If** $\mathcal{L}[f(t)]=F(s)$, then $\mathcal{L}[f(at)]=\frac{1}{a}F\left(\frac{s}{a}\right)$.

$$\mathcal{L}[f(at)]=\int_0^\infty e^{-st}f(at)dt=\int_0^\infty e^{-su/a}f(u)\frac{1}{a}du$$

[Putting at=u, dt=du/a, when t = 0, u = 0, when t → ∞, u → ∞]

$$=\frac{1}{a}\int_0^\infty e^{-su/a}f(u)\frac{1}{a}du=\frac{1}{a}F\left(\frac{s}{a}\right).$$

Example 7.1.5: If) $\mathcal{L}(sint\ t)=\frac{1}{s^2+1}$, find the Laplace transform of $\mathcal{L}(sintat)$by using change scale property

Solution: Given $\mathcal{L}(sint\ t)=\frac{1}{s^2+1}$.

By change scale property

$\mathcal{L}(sint\ at)=\frac{1}{a}\frac{1}{(\frac{s}{a})^2+1}=\frac{1}{a}\frac{a^2}{s^2+a^2}=\frac{a}{s^2+a^2}.$

Laplace Transforms of Derivative

Let $f(t)$ be real, continuous functions for $t \geq 0$ and exponential order. Also $f'(t)$ is continuous. Then,

(i) $\mathcal{L}[f'(t)] = sF(s) - f(0)$

(ii) $\mathcal{L}[f^n(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'^{(0)} - \cdots \ldots \ldots \ldots - sf^{n-2}(0) - f^{n-1}(0)$

Laplace Transforms of Integrals

If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{1}{s}F(s)$.

Multiplication by t^n

If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[t^n f(t)\,] = (-1)^n \frac{d^n}{ds^n}\, F(s)$.

Example 7.1.4: Find the Laplace transform of

(a) $t^3 \sin at$(b) $f(t) = t^2 e^{-3t}$

Solution: (a) $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

: $\mathcal{L}[t^3 \sin at] = \frac{(-1)^3 d^3}{ds^3}\left(\frac{a}{s^2+a^2}\right) = -a\frac{d^2}{ds^2}\left[\frac{d}{ds}(s^2+a^2)^{-1}\right] = -a\frac{d^2}{ds^2}[(-1)(s^2+a^2)^{-2}(2s)]$

$$= 2a\frac{d^2}{ds^2}\left[\frac{s}{(s^2+a^2)^2}\right] = 2a\frac{d}{ds}\left[\frac{(s^2+a^2)^2.1 - s2(s^2+a^2)2s}{(s^2+a^2)^2}\right] = 2a\frac{d}{ds}\left[\frac{(s^2+a^2)[s^2+a^2-4s^2]}{(s^2+a^2)^4}\right]$$

$$= 2a\frac{d}{ds}\left[\frac{a^2-3s^2}{(s^2+a^2)^3}\right] = 2a\frac{d}{ds}\left[\frac{(s^2+a^2)^3(-6s) - (a^2-3s^2).3(s^2+a^2)^2.2s}{(s^2+a^2)^6}\right]$$

$= 2a\frac{d}{ds}\left[\frac{(s^2+a^2)^2(-6s)[s^2+a^2+a^2-3s^2]}{(s^2+a^2)^6}\right] = -\frac{12as}{(s^2+a^2)^4}[2a^2-2s^2] = \frac{24as(s^2-a^2)}{(s^2+a^2)^4}$.

(b) $\mathcal{L}[e^{-3t}] = \frac{1}{s+3}$

$$\mathcal{L}[t^2 e^{-3t}] = \frac{(-1)^2 d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{2}{(s+3)^3}$$

7.2 Solution of Partial Differential Equation Using Laplace Transforms

The Laplace transforms is very useful in solving various partial differential equations subject to the given boundary conditions:

Laplace Transform of Some Partial derivatives

(1) If $\mathcal{L}\left[\frac{\partial y}{\partial t}\right] = s\bar{y}(x,s) - y(x,0)$

(2) If $\mathcal{L}\left[\frac{\partial^2 y}{\partial t^2}\right] = s^2\bar{y}(x,s) - sy(x,0)$

(3) If $\mathcal{L}\left[\frac{\partial y}{\partial x}\right] = \frac{d\bar{y}}{dx}$

(4) If $\mathcal{L}\left[\frac{\partial^2 y}{\partial x^2}\right] + \frac{d^2\bar{y}}{dx^2} = 0$

Example 7.2.1: Solve $\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}$, where $y(0,t) = 0 = y(5,t)$ and $y(x,0) = 10sin4\pi x$

Solution: Taking the Laplace transform of both the sides of the given equation, we get

$$\mathcal{L}\left[\frac{\partial y}{\partial t}\right] = 2\mathcal{L}\left[\frac{\partial^2 y}{\partial x^2}\right]$$

$$s\bar{y}(x,s) - y(x,0) = 2\frac{d^2\bar{y}}{dx^2}$$

$\frac{d^2\bar{y}}{dx^2} - \frac{s}{2}\bar{y}(x,s) = -5sin4\pi x$ (7.2.1)

Taking the general solution of (7.2.1) is given by

$$\bar{y} = C_1 e^{\sqrt{s/2}x} + C_2 e^{-\sqrt{s/2}x} - \frac{5sin4\pi x}{-(4\pi)^2 - \frac{s}{2}}$$

$$\bar{y} = C_1 e^{\sqrt{s/2}x} + C_2 e^{-\sqrt{s/2}x} + \frac{10sin4\pi x}{32\pi^2 + s}$$

Given that $y(0,t) = 0 = y(5,t)$. Therefore

$\bar{y}(0,s) = 0 = \bar{y}(5,s)$.

Putting these values in (7.2.1), we get

$0 = C_1 + C_2$ (7.2.2)

And $0 = C_1 e^{5\sqrt{s/2}} + C_2 e^{-5\sqrt{s/2}} + \frac{10}{32\pi^2+s} sin20\pi$

$C_1 e^{5\sqrt{s/2}} + C_2 e^{-5\sqrt{s/2}} = 0$ (7.2.3)

Solving (7.2.2) and (7.2.3), we get $C_1 = C_2 = 0$.

Therefore from (7.2.1), we have

$$\bar{y} = \frac{10}{32\pi^2 + s} sin4\pi x$$

$$y = \mathcal{L}^{-1}\left[\frac{10}{32\pi^2 + s} sin4\pi x\right] = 10e^{-32\pi^2 t}\sin 4\pi x$$

Example 7.2.2: A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t > 0$, a constant temperature $V_0 > 0$ is applied and maintained at the force $x = 0$. Find the temperature at any point of the solid at any time $t > 0$.

Solution: We know that the temperature $u(x,t)$ at any point of the solid at any time $t > 0$ is governed by one dimensional heat equation

$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}, (x > 0, t > 0)$ (7.2.4)

With boundary condition Solve $u(0,t) = V_0, u(x,0) = 0$.

Taking Laplace transform of both sides of (7.2.4), we get

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = C^2\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$s\bar{u}(x,s) - u(x,0) = C^2\frac{d^2\bar{u}}{dx^2}$$

$\frac{d^2\bar{u}}{dx^2} - \frac{s}{C^2}\bar{u} = 0$ (7.2.6)

The solution of (7.2.6) is given by

$\bar{u}(x,s) = Ae^{\sqrt{s/C^2}x} + Be^{-\sqrt{s/C^2}x}$.

Since u is finite when $x \to \infty$, therefore, $\bar{u}$ is also finite when $x \to \infty$.

Therefore form (7.2.6), $A = 0$, otherwise $\bar{u} \to \infty$ as $x \to \infty$. Now, taking the Laplace transforms of the condition $u(0,t) = V_0$, we have

$\bar{u}(0,s) = \int_{t=0}^{\infty} V_0\, e^{-st} dt = \frac{V_0}{s}$.

Therefore, from (7.2.6), we have

$\bar{u}(0,s) = B = \frac{V_0}{s}$.

Hence $\bar{u}(x,s) = \frac{V_0}{s} e^{-\sqrt{s/C^2}x}$.

$$u(x,t) = \mathcal{L}^{-1}\left\{\frac{V_0}{s} e^{-\sqrt{s/C^2}x}\right\}$$

$u(x,t) = \frac{V_0}{s} \operatorname{erf}\left\{\frac{x}{2c\sqrt{t}}\right\}$.

Summary

- The concept of the Laplace transform is discussed.
- The properties of Laplace transform were elaborated.
- The formula of Laplace transform with their solution are derived.
- The Solution of PDE with Laplace transform method is elaborated.

Keywords

- Laplace transform
- Inverse Laplace transform
- Linearity property
- First shifting property
- Derivative formula
- Integral Formula
- Partial differential equation

Self Assessment

1. Laplace transform of $e^{-20t} \cosh 5t$ is

A. $\dfrac{s+20}{(s+20)^2 - 25}$

B. $\dfrac{s-20}{(s-20)^2 - 25}$

C. $\dfrac{s-25}{(s-25)^2 + 400}$

D. $\dfrac{s+25}{(s+25)^2 - 400}$

2. Laplace inverse of $\frac{3}{s+7}$ is

A. $7e^{3t}$

B. $3e^{7t}$

C. $3e^{-7t}$

D. $7e^{-3t}$

3. Laplace transform of $(t^2+2)^2$ is

A. $\frac{24+8s^2+4s^4}{s^5}$

B. $\frac{24+8s+4s^3}{s^4}$

C. $\frac{6+2s^2+s^4}{s^5}$

D. None of these

4. Laplace inverse of $\frac{1}{s-\frac{1}{2}}$ is

A. e^{-2t}

B. $e^{\frac{1}{2}t}$

C. $e^{\frac{-1}{2}t}$

D. e^{2t}

5. Laplace inverse of $\frac{\beta}{(s-\alpha)^2+\beta^2}$ is

A. $e^{\alpha t}\sin\beta t$

B. $e^{\beta t} \sin \alpha t$

C. $e^{\beta t} \sinh \alpha t$

D. $e^{\alpha t} \sinh \beta t$

6. Laplace transform of $t^{\frac{-1}{2}}$ is

A. $\sqrt{\frac{\pi}{s}}$

B. $\frac{\sqrt{\pi}}{s}$

C. $\frac{\pi}{s}$

D. None of these

7. $L(\cos^2 2t) = ...$

A. $\frac{s^2}{s(s^2+16)}$

B. $\frac{8}{s(s^2+16)}$

C. $\frac{s^2-8}{s(s^2+16)}$

D. $\frac{s^2+8}{s(s^2+16)}$

8. The Laplace inverse $\frac{s}{s^2-49}$ is equal to

A. $\sin 7t$

B. $\cos 7t$

C. $\sinh 7t$

D. $\cosh 7t$

9. The Laplace transform $e^{-2t}\,t^3$ is equal to

$\dfrac{3\,!}{(s-2)^{n+1}}$

A.

$\dfrac{1}{(s+2)^4}$

B.

$\dfrac{3\,!}{(s+2)^4}$

C.

$\dfrac{1}{(s-2)^4}$

D.

10. Inverse Laplace transform of $\dfrac{2}{(s+2)^3}$ is

A. $t^2 e^{-t}$

B. $t^2 e^{2t}$

C. $t^2 e^{t}$

D. $t^2 e^{-2t}$

11. Laplace transform of $(\sin t + \cos t)^2$ is

$\dfrac{s^2+2s+4}{s\left(s^2-4\right)}$

A.

$\dfrac{s^2-2s+4}{s\left(s^2-4\right)}$

B.

$\dfrac{s^2+2s+4}{s\left(s^2+4\right)}$

C.

$\dfrac{s^2-2s+4}{s\left(s^2+4\right)}$

D.

Answers for Self Assessment

1.	A	2.	C	3.	A	4.	B	5.	A
6.	A	7.	D	8.	C	9.	D	10.	D
11.	C								

Review Questions

1. By using Laplace transforms, find the temperature $u(x,t)$ in a slab whose ends $x=0$ and $x=a$ are kept at temperature zero and whose initial temperature is $\sin\left(\pi x/a\right)$.
2. Solve $\frac{\partial y}{\partial t}=\frac{\partial^2 y}{\partial x^2}, x>0, t>0$, where $y(0,t)=1$, and $y(x,0)=0$.
3. Solve $\frac{\partial y}{\partial t}=\frac{\partial^2 y}{\partial x^2}, x>0, t>0$, where $y\left(\frac{\pi}{2},t\right)=0, \left(\frac{\partial y}{\partial x}\right)_{x=0}=0$ and $y(x,0)=cos5x$.
4. An infinite long string having one end $x=0$ is initially at on the x-axis. The end $x=0$under goes a periodic transverse displacement given by $A_0 \sin nt, t>0$, find the displacement of any point on the string at $t>0$.

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.

2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

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Unit 08 : Fourier Transform

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of Fourier transform.
- understand the properties ofFourier transform.
- know about the sine and cosine Fourier transform.
- apply Fourier transform to solve partial differential equation.

Introduction

Fourier Transform

If a function f(x) defined on the interval]-∞, ∞ [, and piecewise continuous in each finite partial interval and absolutely integrable in]-∞, ∞ [, then

$F(f(x)) = \bar{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$ (8.0.1)

is defined as Fourier transform of f(x). The inverse formula for Fourier transform is given by

$$F^{-1}(\bar{f}(p)) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\bar{f}(p)dp \qquad (8.0.2)$$

Remark 8.0.1:We can also define

$F\big(f(x)\big) = \bar{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx$ and

$$F^{-1}(\bar{f}(p)) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\bar{f}(p)dp$$

8.1 Fourier Sine and Cosine Transform

Definition (8.0.1): The infinite Fourier sine transform of the function f(x), 0<x<∞ is denoted by $F_s\big(f(x)\big)$ or $\bar{f}_s(p)$and defined by

$$F_s(f(x)) = \bar{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, sinpx \, dx$$

The inverse formula for infinite Fourier sine transform is given by

$$f(x) = F_s^{-1}(\bar{f}_s(p)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_s(p) \, sinpx \, dp$$

Definition (8.0.2): The infinite Fourier cosine transform of f(x), 0<x<∞ is denoted by $F_c(f(x))$ or $\bar{f}_c(p)$ and defined by

$$F_c(f(x)) = \bar{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cospx \, dx$$

The inverse formula for infinite Fourier cosine transform is given by

$$f(x) = F_c^{-1}(\bar{f}_c(p)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(p) \, cospx \, dp$$

8.2 Linearity Property of Fourier Transform

Let $\bar{f}(p)$ and $\bar{g}(p)$are Fourier transforms of f(x) and g(x) respectively. Then

F{af(x) + bg(x)} = a$\bar{f}(p)$+b $\bar{g}(p)$, where a and b are constants.

Change of Scale Property

Theorem 8.2.1.(For Complex Fourier Transform). If $\bar{f}(p)$is the complex Fourier transform of f(x), the complex Fourier transform of f(ax) is given by

$$\frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$$

Proof. By definition, we have

$$\bar{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x) dx$$

Consider

$$\bar{f}(ap) = \int_{-\infty}^{\infty} e^{ipx} f(ax) dx$$

Putting
$ax = t \Rightarrow adx = dt$, we get

$$\bar{f}(ap) = \frac{1}{a} \int_{-\infty}^{\infty} e^{ip\left(\frac{t}{a}\right)} f(t) dt = \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{p}{a}\right)t} f(x) dx = \frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$$

Remark 8.2.1: In a similar way, we can prove that:

(a) If $\bar{f}_s(p)$ is the Fourier sine transform of f(x), then Fourier sine transform of f(ax) is given by

$$\frac{1}{a}\bar{f}_s\left(\frac{p}{a}\right)$$

(b) If $\bar{f}_c(p)$ is the Fourier cosine transform of f(x), then Fourier cosine transform of f(ax) is given by

$$\frac{1}{a}\bar{f}_c\left(\frac{p}{a}\right)$$

Application to Partial Differential Equation

Theorem 8.2.2 (Shifting Property). If$\bar{f}(p)$ is the complex Fourier transform of f(x), then complex Fourier transform of f(x-a) is e^{ipa} f(p).

Proof. By definition, we have

$$\bar{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$$

Consider

$$\bar{f}(x-a) = \int_{-\infty}^{\infty} e^{ipx} f(x-a)dx$$

Putting x-a=t, we have

$$\bar{f}(t) = \int_{-\infty}^{\infty} e^{ip(t+a)} f(t)dt = e^{ipa}\int_{-\infty}^{\infty} e^{ipt} f(t)dt = e^{ipa}\bar{f}(p)$$

Some Important Integrals (To be Used Directly)

(1) $\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - bcos\, bx)$

(2) $\int e^{ax} cosbx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + bsin\, bx)$

(3) $\int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$

(4) $\int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$

(5) $\frac{d^n}{dx^n}\left(\frac{x}{a^2+x^2}\right) = \frac{(-1)^n n!}{(a^2+x^2)^{(n+1)/2}} \cos\left[(n+1)\, tan^{-1}\left(\frac{a}{x}\right)\right]$

(6) $\frac{d^n}{dx^n}\left(\frac{a}{a^2+x^2}\right) = \frac{(-1)^n n!}{(a^2+x^2)^{(n+1)/2}} \sin\left[(n+1)\, tan^{-1}\left(\frac{a}{x}\right)\right]$

(7) $\int_0^{\infty} \frac{\sin px}{x} dx = \begin{cases} \frac{\pi}{2}; if\ \ p > 0 \\ -\frac{\pi}{2}; if\ p < 0 \end{cases}$

(8) $\int_{-\infty}^{\infty} e^{-x^2}$ dx $= \sqrt{\pi}$, $\int_0^{\infty} e^{-x^2}$ dx $= \frac{\sqrt{\pi}}{2}$

Theorem 8.2.3 (Modulation Theorem) If $\bar{f}(p)$is the complex Fourier transform of f(x), then, the Fourier transform of

f(x) cos ax is $\frac{1}{2}\left(\bar{f}(p-a) + \bar{f}(p+a)\right)$.

Proof. By definition, we haves

$$\bar{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$$

Now,

$$F(f(x)\cos ax) = \int_{-\infty}^{\infty} e^{ipx} f(x) \cos ax\, dx$$

$$= \int_{-\infty}^{\infty} e^{ipx} f(x) \frac{e^{iax} + e^{-iax}}{2}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(p-a)x} f(x)dx$$

$$= \frac{1}{2}\left(\bar{f}(p-a) + \bar{f}(p+a)\right)$$

8.3 Application of Fourier Transform to Boundary Value Problem

The infinite sine and cosine transforms can be applied when the range of the variable selected for exclusion is 0 to ∞. The choice of sine and cosine transform is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. Hence, we have

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2}\, sinpx\, dx$$

$$= \left[\frac{\partial u}{\partial x}\sin px\right]_0^{\infty} - p\int_0^{\infty} \frac{\partial u}{\partial x} cospx\, dx$$

$$= -p\int_0^{\infty} \frac{\partial u}{\partial x}\, cospxdx$$

if $\frac{\partial u}{\partial x} \to 0\ as\ x \to \infty$.

$$= -p\left\{[ucos\, px]_0^{\infty} + p\int_0^{\infty} usinpx\, dx\right\}$$

$$= p(u)_{x=0} - p^2\bar{u}_s$$

[By assuming u→ 0 as x → ∞]

Therefore, $F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = pu(0,t) - p^2\bar{u}_s(p,t)$.

where u(x, t) is a function of two variables x and t and $\bar{u}_s(p,t)$ is the Fourier sine transform of u(x, t) with respect to x.

Further

$$F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2}\, cospx\, dx$$

$$= \left[\frac{\partial u}{\partial x}\cos px\right]_0^{\infty} - p\int_0^{\infty} \frac{\partial u}{\partial x} sinpx\, dx$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + p\int_0^{\infty} \frac{\partial u}{\partial x}\, sin\, pxdx$$

assuming $\frac{\partial u}{\partial x} \to 0\ as\ x \to \infty$.

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + p\left\{[usin\, px]_0^{\infty} - p\int_0^{\infty} ucospx\, dx\right\}$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - p^2\int_0^{\infty} u(x,t)\cos px\, dx$$

Then, $F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - p^2\bar{u}_c(p,t)$.

where, $\bar{u}_c(p,t)$ is the Fourier cosine transform of u(x, t) with respect to x.

Remark 8.3.1: It must be noted that the successful use of a sine transform in removing a term $\frac{\partial^2 u}{\partial x^2}$ required u(0, t), i.e., u at x = 0, while the use of a cosine transform for the same purpose requires $u_x(0, t)$, i.e., $\frac{\partial u}{\partial x}$ at x=0.

The term $\frac{\partial u}{\partial x}$ or any partial derivative of odd order cannot be removed with the help of sine or cosine transforms.

When one of the variables in a differential equation ranges from -∞ to ∞ then that variable can be excluded with the help of complex Fourier transforms.

SOLVED EXAMPLES

Example 8.3.1: Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ if u(0, t) = 0, u(x, 0) = e^{-x}, x>0, u(x, t) is bounded where x > 0, t> 0.

Solution. As per given $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ (8.3.1)

subject to the boundary conditions

u(0, t) = 0, u(x, t) is bounded. (8.3.2)

and initial condition

u(x, 0) = e^{-x}, x>0 (8.3.3)

Since, u(0, t) is given, taking the Fourier sine transform of both sides of (8.3.1), we get

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\, sinpx\, dx$$

$$\frac{d}{dt}\int_0^\infty u(x,t)\sin px\, dx = 2\left\{\left(\frac{\partial u}{\partial x} sinpx\right)_0^\infty - \int_0^\infty \frac{\partial u}{\partial x} pcos\, px\, dx\right\}$$

$\frac{d\bar{u}_s}{dt} = -2p\int_0^\infty \frac{\partial u}{\partial x} cos\, px\, dx$ if $\frac{du}{dx} \to 0$ as $x \to \infty$

Assume ū$_s$(p, t) =$\int_0^\infty usinpxdx$

$$= -2p\left\{(u(x,t)\, cospx)_0^\infty - \int_0^\infty u(x,t)(-psin\, px)\, dx\right\}$$

$$= -2p\left\{0 - u(0,t) + p\int_0^\infty u(x,t) sin\, pxdx\right\}$$

$$= 2pu(0,t) - 2p^2\bar{u}_s$$

$$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s$$

On separating the variables, we get

$$\frac{d\bar{u}_s}{\bar{u}_s} = -2p^2 dt \Rightarrow \log\bar{u}_s - \log C = -2p^2 t$$

$\Rightarrow \log\frac{\bar{u}_s}{C} = -2p^2 t \quad \Rightarrow \bar{u}_s = Ce^{-2p^2 t}$ (8.3.4)

Now, taking the Fourier sine transform of both sides of (8.3.3), we get

$$\int_0^\infty u(x,0) sin\, pxdx = \int_0^\infty e^{-x}\, sin\, pxdx$$

$\bar{u}_s(p,0) = \left[\frac{e^{-x}}{1+p^2}(-sinpx - pcos\, px)\right]_0^\infty = \frac{p}{1+p^2}$ (8.3.5)

Putting t=0 in (8.3.4) and (8.3.5), we get

$$\frac{p}{1+p^2} = C$$

$$\bar{u}_s(p,t) = \frac{p}{1+p^2} e^{-2p^2 t}$$

Taking the inverse Fourier sine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2} e^{-2p^2 t}\, sinpx\, dx$$

Example 8.3.2: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$ subject to the conditions u(0, t) = 0,

$u(x,0) = \begin{cases} 1; & 1 < x < 1 \\ 0\,; & x > 1 \end{cases}$, u(x, t) is bounded where x > 0, t> 0.

Solution. Taking the Fourier sine transform of both the sides of given PDE, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\, sinpx\, dx$$

$\frac{d}{dt}\int_0^\infty u(x,t)\sin px\, dx = \left\{\left(\frac{\partial u}{\partial x} sinpx\right)_0^\infty - p\int_0^\infty \frac{\partial u}{\partial x} cos\, px\, dx\right\} \frac{d\bar{u}_s}{dt} = -p\int_0^\infty \frac{\partial u}{\partial x} cos\, px\, dx$ if $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

Assume $\bar{u}_s$(p, t) = $\int_0^\infty usinpxdx$

$$= -p\left\{(u(x,t)\, cospx)_0^\infty + p\int_0^\infty u(x,t) sin\, px\, dx\right\}$$

$$= p\left\{0 - u(0,t) + p\int_0^\infty u(x,t) sin\, pxdx\right\}$$

$= pu(0,t) - p^2\bar{u}_s$ if u→ 0 as x →∞

On separating the variables, we get

$$\frac{d\bar{u}_s}{\bar{u}_s} = -p^2 dt \Rightarrow \log \bar{u}_s - \log C = -p^2 t$$

$$\Rightarrow \log \frac{\bar{u}_s}{C} = -p^2 t \;\; \Rightarrow \bar{u}_s = Ce^{-p^2 t} \qquad (8.3.6)$$

Putting t=0, we get

$$\bar{u}_s(p,0) = C \qquad (8.3.7)$$

Now, $\bar{u}_s(p,0) = \int_0^\infty u(x,0)\sin px\, dx$

$$= \int_0^1 u(x,0) sinpx\, dx + \int_1^\infty u(x,0) sinpx\, dx = \int_0^1 \sin px\, dx$$

Now, from (8.3.6)

$$C = \int_0^1 \sin px\, dx = \left[\frac{\cos px}{-p}\right]_0^1 = \frac{1 - cosp}{p}$$

Thus, (8.3.6) gives $\bar{u}_s(p,t) = \left[\frac{1-cosp}{p}\right] e^{-p^2 t}$

Finally, taking the inverse Fourier sine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-cosp}{p} e^{-p^2 t} sinpx\, dp$$

which is the required solution.

Summary

- The Fourier transforms and its integral formula is defined.
- The properties of Fourier transform are discussed.
- Fourier sine and cosine formula is derived.
- Solution of PDE by using Fourier transform elaboratedwith an examples.

Keywords

- Fourier transform
- Change scale property
- Shifting property
- Fourier sine transform
- Fourier cosine transform
- PDE

Self Assessment

Choose the most suitable answer from the options given with each question.

1. The integral formula $F(f(x)) = \bar{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$ is known as
   A. Laplace transform
   B. Inverse Laplace transform
   C. Fourier transform
   D. None of these

2. The inverse Fourier formula is given by

A. $F^{-1}(\bar{f}(p)) = f(x) = \frac{1}{2\pi}\int_{0}^{\infty} e^{px}\bar{f}(p)dp$

B. $F^{-1}(\bar{f}(p)) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\bar{f}(p)dp$

C. $F^{-1}(\bar{f}(p)) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{px}\bar{f}(p)dp$

D. None of these

3. Which of the properties are followed by Fourier transfer?

   A. Linearity

   B. Change scale

   C. Both (a) and (b)

   D. None of these

4. The infinite Fourier sine transform of the function f(x), 0<x<∞ is
   A. $F_s(f(x)) = \bar{f}_s(p) = \int_{0}^{\infty} f(x)\ cospx\ dx$

B. $F_s(f(x)) = \overline{f_s}(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\, sinpx\, dx$

C. $F_s(f(x)) = \overline{f_s}(p) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)\, sinpx\, dx$

D. None of these

5. The infinite Fourier cosine transform of the function f(x), 0<x<∞ is

A. $F_c(f(x)) = \overline{f_c}(p) = \int_0^{\infty} f(x)\, cospx\, dx$

B. $F_c(f(x)) = \overline{f_c}(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \cos px\, dx$

C. $F_s(f(x)) = \overline{f_s}(p) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)\, sinpx\, dx$

D. None of these

6. The use of a sine transform in removing a term $\frac{\partial^2 u}{\partial x^2}$is required

A. u(0, t), i.e., u at x = 0,

B. $u_x(0, t)$, i.e., $\frac{\partial u}{\partial x}$ at x=0.

C. Both (a) and (b)

D. None of these

7. The use of a cosine transform in removing a term $\frac{\partial u}{\partial x}$ is required

A. u(0, t), i.e., u at x = 0,

B. $u_x(0, t)$, i.e., $\frac{\partial u}{\partial x}$ at x=0.

C. Both (a) and (b)

D. None of these

Answer forSelfAssessment

1.	C	2.	B	3.	C	4.	B	5.	B
6.	A	7.	B						

Review Questions

1. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$ subject to conditions

(i) $u = 0$ when $x = 0, t > 0$

(ii) $u = \begin{cases} 1, 0 < x < 1 \\ 0, x \geq 1 \end{cases}$ and

(iii) $u(x,t)$ is bounded

2. Solve the boundary value problem $\frac{\partial^2 u}{\partial^2 t} = 9\frac{\partial^2 u}{\partial x^2}$ subject to boundary conditions $u(0,t) = 0, u(2,t) = 0, u(x,0) = (0.05)x(2-x)$

   $u_t(x,0) = 0$, where $0<x<2$, $t>0$.

3. Use finite Fourier transform to solve $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, $0<x<6$, $t>0$ and $v_x(0,t), v_x(6,t), v(x,0) = 2x$.

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.

2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

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Unit 09: Other Transforms

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of transforms require to solve partial differential equation.
- understand the more techniques for partial differential equation.
- know about the Hopf-Cole transform for quadratic nonlinear partial differential equation.
- apply Hodograph for nonlinear system of differential equation.
- find the condition through Legendre transformfor system of differential equations.

Introduction

In this chapter, we describe several techniques or more transforms like Hodograph, Hopf-Cole and Legendre to solve linear, nonlinear, quasi linear partial differential equations by converting nonlinear to linear partial differential equation or by converting system of nonlinear to linear partial differential equations.

9.1 Hopf-Cole Transformation

a. A parabolic PDE with quadratic nonlinearity

We consider first of all an initial-value problem for a quasilinear parabolic equation:

$$\begin{cases} u_t - a\Delta u + b|Du|^2 = 0 & in \quad R^n \times (0,\infty) \\ u = g & on \quad R^n \times \{t=0\} \end{cases} \qquad (9.1.1)$$

where $a > 0$. This sort of nonlinear PDE arises in stochastic optimal control theory.

Assuming for the moment u is a smooth solution of (9.1.1), we get

$w := \phi(u),$

where $\phi: R \to R$ is a smooth function, as yet unspecified. We will try to choose ϕ so that w solves a linear equation. We have

$$w_t = \phi'(u)u_t,\ \Delta w = \phi'(u)\Delta u + \phi''(u)|Du|^2;$$

And consequently (9.1.1) implies

$$\begin{aligned} w_t = \phi'(u)u_t &= \phi'(u)[a\Delta u - b|Du|^2] \\ &= a\Delta w - [a\phi''(u) + b\phi'(u)]|Du|^2 \\ &= a\Delta w, \end{aligned}$$

Provided we choose ϕ to satisfy $a\phi'' + b\phi' = 0$. We solve this differential equation by setting $\phi = e^{-\frac{bz}{a}}$. Thus we see that if u solves (9.1.1), then

$$w = e^{-\frac{bu}{a}} \tag{9.1.2}$$

Solves this initial-value problem for the heat equation (with conductivity a):

$$\begin{cases} w_t - a\Delta w = 0 & in \quad R^n \times (0,\infty) \\ w = e^{-bg/a} & on \quad R^n \times \{t = 0\} \end{cases} \tag{9.1.3}$$

Formula (9.1.2) is the Hopf-Cole transformation.

b. Burgers' equation with viscosity.

As a further application, we examine now for $n = 1$ the initial-value problem for the viscous Burgers' equation:

$$\begin{cases} u_t - au_{xx} + uu_x = 0 & in \quad R^n \times (0,\infty) \\ u = g & on \quad R^n \times \{t = 0\} \end{cases} \tag{9.1.4}$$

If we set

$$w(x,t) := \int_{-\infty}^{x} u(y,t)dy$$

(9.1.5)

and

$$h(x) := \int_{-\infty}^{x} g(y)dy$$

$$\begin{cases} w_t - aw_{xx} + \frac{1}{2}w_x^2 = 0 & in \quad R^n \times (0,\infty) \\ w = h & on \quad R^n \times \{t = 0\} \end{cases}$$

This is an equation of the form (9.1.1) for $n = 1, b = \frac{1}{2}$.

9.2 Hodograph Transforms

The hodograph transform is a technique for converting certain quasilinear systems of PDE into linear systems, by reversing the roles of the dependent and independent variables. As this method is most easily understood by an example, we investigate here the equations of steady, two-dimensional, irrotational fluid flow:

$$\begin{cases} (a)\ (\sigma^2(u) - (u^1)^2)u^1_{x_1} - u^1u^2(u^1_{x_2} + u^2_{x_1}) + (\sigma^2(u) - (u^2)^2)u^2_{x_2} = 0 \\ \qquad (b) u^1_{x_2} + u^2_{x_1} = 0 \end{cases} \tag{9.2.1}$$

in R^2. The unknown is the velocity field $u = (u^1, u^2)$, and the function $\sigma(.): R^2 \to R$, the local sound speed, is given.

The system (9.2.1) is quasilinear. Let us now, however, no longer regard u^1 and u^2 as functions of x_1 and x_2:

$$u^1 = u^1(x_1, x_2), u^2 = u^2(x_1, x_2), \tag{9.2.2}$$

But rather regard x^1 and x^2as functions of u_1 and u_2:

$$x^1 = x^1(u_1, u_2), x^2 = x^2(u_1, u_2). \tag{9.2.3}$$

We have exchanged sub- and superscripts in the notation to emphasize the interchange between independent and dependent variables.

According to the Inverse Function Theorem we can, locally atleast, invert equations (9.2.2) to yield (9.2.3) provided

$$J = \frac{\partial(u^1, u^2)}{\partial(x_1, x_2)} = u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1} \neq 0 \tag{9.2.4}$$

in some region of R^2. Assuming now (9.2.4) holds, we calculate

$$\begin{cases} u^2_{x_2} = J x^1_{u_1}, u^2_{x_1} = -J x^2_{u_1} \\ u^1_{x_2} = -J x^1_{u_2}, u^1_{x_1} = J x^2_{u_2}. \end{cases} \tag{9.2.5}$$

We insert (9.2.5) in (9.2.1), to discover

$$\begin{cases} (a)\ (\sigma^2(u) - u_1^2) x^2_{u_2} + u_1 u_2 (x^1_{u_2} + x^2_{u_1}) + (\sigma^2(u) - u_2^2) x^1_{u_1} = 0 \\ \qquad (b) x^1_{u_2} - x^2_{u_1} = 0. \end{cases} \tag{9.2.6}$$

This is a linear system for $x = (x^1, x^2)$, as function of $u = (u_1, u_2)$.

Remarks 9.2.1: We can utilize the method of potential functions to simplify further. Indeed, equation (9.2.6)(b) suggests that we look for a single function $z = z(u)$ such that

$$\begin{cases} x^1 = z_{u_1} \\ x^2 = z_{u_2}. \end{cases}$$

Then (9.2.6)(a) transforms into the linear, second-order PDE

$$(\sigma^2(u) - u_1^2) z_{u_2 u_2} + 2u_1 u_2 z_{u_1 u_2} + (\sigma^2(u) - u_2^2) z_{u_1 u_1} = 0. \tag{9.2.7}$$

9.3 Legendre Transform

A technique closely related to the hodograph transform is the classical Legendre transform, a version of which we have already encountered before. The idea is to regard the components of the gradient of a solution as new independent variables.

Once again an example is instructive. We investigate the minimal surface equation

$$div\left(\frac{Du}{(1 + |Du|^2)^{\frac{1}{2}}}\right) = 0,$$

For which $n = 2$ may be rewritten as

$$(1 + u^2_{x_2}) u_{x_1 x_1} - 2u_{x_1} u_{x_2} u_{x_1 x_2} + (1 + u^2_{x_1}) u_{x_2 x_2} = 0. \tag{9.2.8}$$

Let us now assume that at least in some region of R^2, we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), p^2 = u_{x_2}(x_1, x_2), \tag{9.2.9}$$

to solve for

$$x^1 = x^1(p_1, p_2), x^2 = x^2(p_1, p_2). \tag{9.2.10}$$

The inverse function theorem assures us we can do so in a neighborhood of any point where

$$J = \det D^2 u \neq 0. \tag{9.2.11}$$

Now define

$$v(p) := x(p).p - u(x(p)), \tag{9.2.12}$$

Where $x = (x^1, x^2)$ is given by (9.2.8), $p = (p_1, p_2)$. We discover after some calculations that

$$\begin{cases} u_{x_1x_1} = Jv_{p_2p_2} \\ u_{x_1x_2} = -Jv_{p_1p_2} \\ u_{x_2x_2} = Jv_{p_1p_1}. \end{cases} \tag{9.2.13}$$

Upon substituting the identities (9.2.13) into (9.2.8), we derive for v the linear equation

$$(1 + p_2^2)v_{p_2p_2} + 2p_1p_2v_{p_1p_2} + (1 + p_1^2)v_{p_1p_1} = 0.$$

Remarks 9.3.1: The hodograph and Legendre techniques for obtaining linear out of nonlinear PDE are in practice tricky to use, as it is usually not possible to transform given boundary conditions very easily.

Summary

- The more methods to solve nonlinear and quasilinear partial differential equations are discussed.
- Hopf-cole transformation is derived and applied to convert nonlinear to linear PDE.
- The Hodograph technique is explained with an example.
- The relation between the hodograph and Legendreis discussed.

Keywords

- Non Linear PDE
- Quasi Linear PDE
- Hodograph
- Hope-Cole
- Legendre

Self Assessment

1. The Hopf-Cole transformation converts a nonlinear partial differential equation to

A. Linear pde

B. Nonlinear pde

C. System of pde

D. None of these

2. The transform which convert a parabolic with quadratic nonlinearity to linear is
A. Laplace transform
B. Fourier transform
C. Hopf-cole
D. None of these

3. The Hopf-cole transformation for the pde $\boldsymbol{u_t - au_{xx} + b|Du|^2 = 0}$ is given by
A. $W = e^{-\frac{bu}{a}}$
B. $W = e^{\frac{bu}{a}}$
C. $W = e^{\frac{au}{b}}$
D. None of these

4. The partial differential equation $\boldsymbol{w_t - w_{xx} = 0}$ is known as
A. Laplace equation
B. Wave equation
C. Heat equation
D. None of these

5. The Hodograph transformation converts certain quasi linear system of partial differential equation to
A. Linear system of pde
B. Nonlinear pde system
C. System of equations
D. None of these

6. The Hodograph transformation convert nonlinear to linear system of pde
A. By converting both in single variables
B. By reversing the roleof the dependent and independent variables.
C. By taking inverse of both the variables
D. None of these

7. The transformation which convert a nonlinear pde into linear by interchanging the variables is called
A. Hopf-cole transformation
B. Hodograph transformation
C. Legendre transformation
D. None of these

Answer forself Assessment

1.	A	2.	C	3.	A	4.	C	5.	A
6.	B	7.	C						

Review Questions

1. By using a transformation convert a parabolic PDE with quadratic nonlinearity to linear partial differential equation.
2. Describe the method to convert a quasi-linear partial differential equation to linear partial differential equation.
3. Describe the method to convert a parabolic partial differential equation with quadratic non linearity to linear partial differential equation.
4. Discuss the method to convert nonlinear system of differential to linear differential equations.

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.

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Unit 10 : Laplace Equation I

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of physical interpretation of Laplace equation.
- understand the fundamental solution of Laplace equation
- know about the elementary solution.
- applyEnergy method to find the minimizers solution

Introduction

Laplace equation: Among the most important of all partial differential equations are undoubtedly Laplace's equations

$$\Delta u = 0 \tag{10.0.1}$$

In equation (10.0.1), $x \in U$ and the unknown is $u: \overline{U} \to R, u = u(x)$, where $U \subset R^n$ is a given open set.

Definition 10.0.1: A C^2function u satisfying (10.0.1) is called a harmonic function.

Physical Interpretation: Laplace's equation comes up in a wide variety of physical contexts. In a typical interpretation u denotes the density of some quantity (e.g. chemical concentration) in equilibrium. Then if V is any smooth sub region within U, the net flux of u through ∂V is zero:

$$\int_{\partial V} F.vds = 0 \tag{10.0.2}$$

F denoting the flux density and v the unit outer normal field. In view of the Gauss-Green theorem, we have

$$\int_V div\,Fdx = \int_{\partial V} F.vdS = 0, \tag{10.0.3}$$

and so

$div\,F = 0 \ \ in\, U,$ (10.0.4)

Since V was arbitrary. In many instances it is physically reasonable to assume the flux F is proportional to the gradient Du, but points in the opposite direction (since the flow is from region of higher to lower concentration). Thus,

$F = -aDu\ (a > 0).$ (10.0.5)

Substituting into (10.0.4), we obtain Laplace's equation

$$div(Du) = \Delta u = 0.$$

If u denotes the $\begin{cases} chemical\ concentration \\ temperature \\ electrostatic\ potential \end{cases}$

equation (10.0.5) is $\begin{cases} Fix's\ law\ of\ diffusion \\ Fourier's\ law\ of\ heat\ conduction \\ Ohm's\ law\ of\ electrical\ conduction. \end{cases}$

Laplace's equation arises a well in the study of analytic functions and the probabilistic investigation of Brownian motion.

10.1 Fundamental Solution

One good strategy for investigating any partial differential equation is first to identify some explicit solutions and then, provided the PDE is linear to assemble more complicated solutions out of the specific ones previously noted. Furthermore, in looking for explicit solutions it is often wise to restrict attention to classes of functions with certain symmetry properties. Since Laplace's equation in invariant under rotations, it consequently seems advisable to search first for radial solutions, that is, functions of $r = |x|$.

Let us therefore attempt to find a solution u of Laplace's equation (10.0.1) in $U = R^n$, having the form

$$u(x) = v(r),$$

Where $r = |x| = (x_1^2 + x_2^2 + \cdots \ldots + x_2^n)^{1/2}$

and v is to be selected (if possible) so that $\Delta u = 0$ holds. First note for $i = 1,2, \ldots \ldots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + x_2^2 + \cdots \ldots + x_2^n)^{-1/2} 2x_i = \frac{x_i}{r}(x \neq 0).$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

for $i = 1,2, \ldots \ldots, n$ and so

$$\Delta u = v''(r) + \frac{n-1}{r}v'.$$

Hence $\Delta u = 0$ if and only if

$$v''(r) + \frac{n-1}{r}v' = 0.$$

If $v' \neq 0$, we deduce

$\log(v')' = \frac{v''}{v'} = \frac{1-n}{r},$

And hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if $r > 0$, we have

$$v(r) = \begin{cases} b\ logr + c, (n = 2) \\ \dfrac{b}{r^{n-2}} + c \quad (n \geq 3) \end{cases}$$

where b and c are constants.

These considerations motivate the following

Definition: The function

$$\phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, (n=2) \\ \frac{1}{n(n-2)\alpha(n)|x|^{n-2}} \quad (n \geq 3) \end{cases},$$

defined for $x \in R^n, x \neq 0$, is the fundamental solution of Laplace's equation.

10.2 Elementary Solutions of Laplace's Equation

If we take the function p to be given by the equation

$$\psi = \frac{q}{|r-r'|} = \frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \quad (10.2.1)$$

where q is a constant and (x',y',z') are the coordinates of a fixed point, then since

$$\frac{\partial\psi}{\partial x} = -\frac{q(x-x')}{|r-r'|^3}, etc.$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{q}{|r-r'|^3} + \frac{3q(x-x')^2}{|r-r'|^5}, etc$$

It follows that

$$\nabla^2\psi = 0$$

showing that the function (10.2.1) is a solution of Laplace's equation except possibly at the point (x',y',z'), where it is not defined. From what it follows that the function given by equation (10.2.1) is a possible form for the electrostatic potential corresponding to a space which, apart from the isolated point (x',y',z'), is empty of electric charge. To find the charge at this singular point we make use of Gauss' theorem. If S is any sphere with center (x',y',z'), then it is easily shown that

$$\int_S \frac{\partial\psi}{\partial n} dS = -4\pi q$$

from which it follows, by Gauss' theorem, that equation (1) gives the solution of Laplace's equation corresponding to an electric charge$+q$.

By a simple superposition procedure it follows immediately that

$$\psi = \sum_{i=1}^{n} \frac{q_i}{|r-r_i|} \quad (10.2.2)$$

is the solution of Laplace's equation corresponding to n charges q_i situated at points with position vectors r_i (i = 1, 2, . . . , n). In electrical problems we encounter the situation where two charges +q and -q are situated very close together, say at points r' and $r' + \delta r'$, where $r' = (l, m, n)a$.The solution of Laplace's equation corresponding to this distribution of charge is

$$\psi = \frac{-q}{|r-r'|} + \frac{q}{|r-r'-\delta r'|}$$

Now

$$\frac{1}{|r-r'-\delta r'|} = \frac{1}{|r-r'|} + \frac{l(x-x')+m(y-y')+n(z-z')}{|r-r'|^3}a + O(a^2)$$

so that if$a \to 0, q \to \infty$ in such a way that $qa \to \mu$, i.e dipole is formed, it follows that the corresponding Laplace's equation is

$$\psi = \mu\frac{l(x-x')+m(y-y')+n(z-z')}{|r-r'|^3} \quad (10.2.3)$$

a result which may be written in other ways : If we introduce a vector $v = \mu(l, m, n)$, then

$$\psi = \frac{v.(r-r')}{|r-r'|^3} \quad (10.2.4)$$

Also since

$$\frac{\partial}{\partial x'}\frac{1}{|r-r'|} = \frac{(x-x')}{|r-r'|^3}, etc.$$

it follows that (3) may be written in the form

$$\psi = (v.grad')\frac{1}{|r-r'|} = \mu\left(l\frac{\partial}{\partial x'} + m\frac{\partial}{\partial y'} + n\frac{\partial}{\partial z'}\right)\frac{1}{|r-r'|} \quad (10.2.5)$$

In reality we usually have to deal with continuous distributions of charge rather than with point charges or dipoles. By analogy with equation (10.2.2) we should therefore expect that when a continuous distribution of charge fills a region V of space, the corresponding form of the function v is given by the Stieltjes integral

$$\psi = \int_V \frac{dq}{|r-r'|} (10.2.6)$$

where q is the Stieltjes measure of the charge at the point r', or if p denotes the charge density, by

$$\psi(r) = \int_V \frac{\rho(r')d\tau'}{|r-r'|} (10.2.7)$$

By a similar argument it can be shown that the solution corresponding to a surface S carrying an electric charge of density is

$$\psi(r) = \int_S \frac{\sigma\,(r')dS'}{|r-r'|} (10.2.8)$$

10.3 Energy Method

Most of our analysis of harmonic functions thus far has depended upon fairly explicit representation formulas entailing the fundamentals solution and elementary solution. In this section we illustrate some 'energy methods' which is to say techniques involving the L^2- norms of various expressions. These ideas foreshadow latter theoretical developments in Parts.

a. Uniqueness.

Consider first the boundary value problem

$$\begin{cases} -\Delta u = f \ in \ U \\ u = g \ on \ \partial U \end{cases} \quad (10.3.1)$$

We have already employed the maximum principle to show uniqueness, but now set forth a simple alternative proof. Assume U is open, bounded, and ∂U is C^1.

Theorem 10.3.1(Uniqueness): there exists at most one solution $u \in C^2(\bar{U})$ of (10.3.1).

Proof: Assume$\tilde{u}$is another solution and set $w = u - \tilde{u}$. Then $\Delta w = 0$ in U, and so an integration by parts shows

$$0 = -\int_U w\Delta w dx = \int_U |Dw|^2 dx.$$

Thus

$$Dw = 0 \ in \ U,$$

And since $w = 0 \ on \ \partial U$, we deduce $w = u - \tilde{u} = 0$ in U.

b. **Dirichlet's Principle**

Next let us demonstratethat a solution of the boundary-value problem (10.3.1) can be Poisson's equation can be characterized has the minimizers of an appropriate functional. For this, we define the energy functional

$$I(w) = \int_U \frac{1}{2}|Dw|^2 - wf dx,$$

w belonging to the admissible set

$A = \{w \in C^2(\bar{U}) | w = g \ on \ \partial U\}$.

Theorem 10.3.1: (Dirichlets principal). Assume$\boldsymbol{u} \in C^2(\bar{U})$ solves (10.3.1). Then

$$I[u] = \min_{w \in A} I[w]. \quad (10.3.2)$$

Conversely, if $\boldsymbol{u} \in A$ satisfies (10.3.2)then$u$ solves the boundary-value problem (10.3.1)

In other words if $u \in A$, the PDE $\Delta u = f$is equivalent to the statement that$u$ minimizes the energy I[.].

Proof: 1. Choose if $w \in A$. Then (10.3.2) implies.

$$0 = \int_U (-\Delta u - f)(u - w) dx.$$

An integration by parts yields

$$0 = \int_U Du.D(u - w) - f(u - w) dx,$$

and there is no boundary term since $u - w = g - g = 0$ on ∂U. Hence

$$\int_U |Du|^2 - uf\ dx = \int_U Du.Dw - wf\ dx$$

$$\leq \int_U \frac{1}{2}|Du|^2 dx + \int_U \frac{1}{2}|Dw|^2 - wf\ dx,$$

Where we employed the estimates

$$|Du.Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy-Schwarz and Cauchy inequalities. Rearranging, we conclude

$$I[u] \leq I[w] (w \in A) \quad (10.3.3)$$

Since $u \in A$, (10.3.2) follows from (10.3.3)

2. Now, conversely, suppose (10.3.2) holds, Fix any $v \in C_c^\infty(U)$ and write

$i(\tau) := I[u + \tau v] (\tau \in R)$.

Since $u + \tau v \in A$ for each τ, the scalar function $i(.)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left(' = \frac{d}{d\tau}\right),$$

Provided this derivative exists. But

$$i(\tau) = \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v) f dx$$

$$= \int_U \frac{1}{2}|Du|^2 + \tau Du.Dv + \tau^2 |Dv|^2 - (u + \tau v) f dx.$$

Consequently

$$0 = i'(0) = \int_U Du.Dv - vf dx = \int_U (-\Delta u - f) v dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U.

Summary

- The physical interpretation of Laplace equation is discussed.
- The radial solution in terms of fundamental solution is determined.

- Theelementary function for Laplace equation is solved.
- Determined the energy function for Laplace equation.

Keywords

- Laplace equation
- Fundamental solution
- Elementary solution
- Energy method

Self Assessment

1. If the function **φ** is harmonic in a circle S and continuous of S, then the value of **φ**at the center of S is equal to

A. Arithmetic mean of its value on the circumference of S.

B. Geometric mean of its value on the circumference of S.

C. Arithmetic mean of its value everywhere on S.

D. Geometric mean of its value everywhere on S.

2. Which of function defined below for $x \in R^n, x \neq 0,$ is the fundamental solution of Laplace's Equation?

A. $\varphi(x) = \frac{1}{2\pi} log|x|,\ n = 2$

B. $\varphi(x) = \frac{1}{2} log|x|,\ n = 2$

C. $\varphi(x) = \frac{1}{2} log|x|,\ n = 2$

D. $\varphi(x) = -\frac{1}{2\pi} log|x|,\ n = 2$

3. What is true for the fundamental solutions of Laplace equation?

A. $\varphi(x) = \frac{1}{\alpha(n)n(n-2)}\frac{1}{|x|^{n-1}},\ n \geq 3$

B. $\varphi(x) = \frac{1}{\alpha(n)n(n-1)}\frac{1}{|x|^{n-1}},\ n \geq 3$

C. $\varphi(x) = \frac{1}{\alpha(n)n(n-2)}\frac{1}{|x|^{n-2}},\ n \geq 3$

D. $\varphi(x) = \frac{1}{\alpha(n)n(n-1)}\frac{1}{|x|^{n-2}},\ n \geq 3$

4. Laplace equation will be invariant

A. Under rotation with respect to radial function

B. Dilation with respect to radial function

C. Magnification with respect to radial function

D. None of these

5. The elementary solution of the Laplace equation governed by the

A. Charges situated at different point.

B. Motion of the particles.
C. Energy function response
D. None of these

6. The elementary solution of the Laplace equation over the volume V with boundary surface S is

A. $\psi(r) = \int \frac{\rho(r')d\tau'}{|r-r'|}$

B. $\psi(r) = \int \frac{\rho(r')d\tau'}{|r+r'|}$

C. $\psi(r) = -\int \frac{\rho(r')d\tau'}{|r-r'|}$

D. $\psi(r) = -\int \frac{\rho(r')d\tau'}{|r+r'|}$

7. The elementary solution of the Laplace equation is derived by using
A. Green theorem
B. Gauss′ theorem
C. Both Green and Gauss
D. None of these

8. For the boundary value problem $u \in C^2(\tilde{U})$ of $\begin{cases} -\Delta u = f \ in \ U \\ u = g \ on \ \partial U \end{cases}$

A. There exists at most one solution
B. There exists more than one solution
C. There exists no solution
D. There exists infinite many solution

9. The energy functional for w which is belonging the admissible set is

A. $\boldsymbol{I(w) = \int \frac{1}{2}|DW|^2 - wfdx}$

B. $I(w) = \int \frac{1}{2}|DW|^2 + wfdx$

C. $I(w) = \int |DW|^2 - wfdx$

D. $I(w) = \int |DW|^2 + wfdx$

10. For the boundary value problem for Poisson's equation which can be characterized as
A. The maximize of the appropriate energy function
B. The minimize of the appropriate energy function
C. Neither minimize or maximize of the appropriate energy function
D. None on these

AnswersforSelf Assessment

1.	A	2.	D	3.	D	4.	A	5.	A
6.	A	7.	A	8.	A	9.	A	10.	A

Review Questions

1. Derive the fundamental solution using radial function for Laplace equation.
2. Find the elementary function or solution for Laplace equation.
3. Prove that solution of Laplace equation is minimizing the energy function.

Further Readings

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Unit 11: Laplace Equation II

CONTENTS

Objectives

After studying this unit, you will be able to

- understand about the harmonic function in term of Laplace equation.
- know about the spherical mean and mean value theorem.
- Apply potential function to solve system of PDE.
- Determine Green's function using harmonic function.

Introduction

In this chapter, we are going to discuss about harmonic function and its properties in terms of mean value formula. Further the Green's function for harmonic function and potential function will be discussed.

11.1 The Spherical Mean

Let $\mathbb{R}$ be a region bounded ∂ $\mathbb{R}$and let P(x,y,z) be any point in $\mathbb{R}$. Also, let S(P,r) represents a sphere with centre at P and radius r such that it lies entirely within the domain $\mathbb{R}$. Let u be continuous function in $\mathbb{R}$. Then the spherical mean of u denoted by $\bar{u}$ is defined as

$$\bar{u}(r) = \frac{1}{4\pi r^2}\int_{S(P,r)}\int u(Q)\ dS \qquad (11.1.1)$$

Where $Q(\xi,\eta,\zeta)$ is any variable point on the surface of the sphere S(P,r) and dS is the surface element of integration. For a fixed radius r, the value$\bar{u}(r)$ is the average of the values of u taken over the sphere $S(P,r)$, and hence it is called the spherical mean. Taking the origin at P, in terms of spherical polar coordinates, we have

$$\xi = x + r\sin\theta cos\phi$$

$$\eta = y + r\ sin\theta sin\phi$$

$$\zeta = z + r\ cos\theta$$

Then, the spherical mean can be written as

$$u(r) = \frac{1}{4\pi r^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} u(\ x + r\sin\theta cos\phi\, , y + r\ sin\theta sin\phi, z + r\ cos\theta) r^2 \sin\theta d\theta d\phi.$$

Also, since u is continuous on $S(P,r)$, $\bar{u}$too is a continuous function of r on some interval $0 < r \leq R$, which can be verified as follows:

$$\bar{u}(r) = \frac{1}{4\pi}\int\int u(Q)\sin\theta d\theta d\phi = \frac{u(Q)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = u(Q)\,.$$

Now, taking the limit as $r \to 0, Q \to P$, we have

$$\lim_{r\to 0}\bar{u} = u(P).$$

Hence, $\bar{u}$ is continuous in $0 \leq r \leq R$.

11.2 Mean Value Theorem of Harmonic Functions

Theorem 11.2.1: Let$u$ be a harmonic in a region $\mathbb{R}$. Also, let $P(x,y,z)$ be a given point in $\mathbb{R}$ and $S(P,r)$ be a sphere with centre at P such that $S(P,r)$ is completely contained in the domain of harmonicity of u. Then

$$u(P) = \bar{u}(r) = \frac{1}{4\pi r^2}\int\int_{S(P,r)} u(Q)dS.$$

Proof: Since u is harmonic in $\mathbb{R}$, its spherical mean $\bar{u}(r)$ is continuous in $\mathbb{R}$ and is given by

$$\bar{u}(r) = \frac{1}{4\pi r^2}\int\int_{S(P,r)} u(Q)dS = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(\xi,\eta,\zeta) r^2 \sin\theta d\theta d\phi$$

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(u_\xi \xi_r + u_{\eta\eta_r} + u_\zeta \zeta_r\right)\sin\theta d\theta d\phi$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(u_\xi sin\theta\cos\phi + u_\eta sin\theta sin\phi + u_\zeta\cos\theta\right)\sin\theta d\theta d\phi. \qquad (11.2.1)$$

Noting that $sin\theta cos\phi$, $sin\theta sin\phi$ and $cos\theta$ are the direction cosines of the normal $\hat{n}$ on $S(P,r)$,

$$\nabla u = iu_\xi + ju_\eta + ku_\zeta, \hat{n} = (in_1, jn_2, kn_3),$$

The expression within the parentheses of the integrand of eq. (11.2.1) can be written as $\nabla u.\hat{n}$. Thus

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi r^2}\int\int_{S(P,r)} \nabla u.\hat{n} r^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi r^2}\int\int_{S(P,r)} \nabla u.\hat{n}\ dS$$

$$= \frac{1}{4\pi r^2}\int\int\int_{V(P,r)} \nabla.\nabla u\ dV$$

(by divergence theorem)

As u is harmonic,

$$\frac{1}{4\pi r^2}\int\int\int_{V(P,r)} \nabla^2 u\ dV = 0$$

Therefore, $\frac{d\bar{u}}{dr} = 0$, implying $\bar{u}$ $is\ constant$.

Now the continuity of $\bar{u}$ at $r = 0$ gives,

$$\bar{u}(r) = u(P) = \frac{1}{4\pi r^2}\int\int_{S(P,r)} u(Q)dS.$$

11.3 Properties of Harmonic Functions

Solution of Laplace equation is called harmonic functions which possess a number of interesting properties, and they are presented in the following theorems.

Theorem 11.3.1: If a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere.

Proof: If ϕ is a harmonic function, then $\nabla^2\phi = 0$ in $\mathbb{R}$. Also, if $\phi = 0$ on $\partial\mathbb{R}$, we shall show that $\phi = 0$ in $\overline{\mathbb{R}} = \mathbb{R}U\partial\mathbb{R}$. Recalling Green's first identity, we get

$$\int\int\int_R (\nabla\phi)^2 dV = \int\int_{\partial R} \phi\frac{\partial\phi}{\partial n}\,dS - \int\int\int_R \phi\nabla^2\phi\,dV$$

and using the above facts we have, at once, the relation

$$\int\int\int_R (\nabla\phi)^2\,dV = 0.$$

Since $\int_R(\nabla\phi)^2$ is positive, it follows that the integral will be satisfied only if $\nabla\phi = 0$. This implies that ϕ is a constant in $\mathbb{R}$. Since ϕ is continuous in $\overline{\mathbb{R}}$ and ϕ is zero in $\partial\mathbb{R}$, it follows that $\phi = 0$ in $\mathbb{R}$.

Theorem 11.3.2: If ϕ is harmonic function in $\mathbb{R}$ and $\frac{\partial\phi}{\partial n} = 0$ on $\partial\mathbb{R}$, then ϕ is a constant in $\overline{\mathbb{R}}$.

Proof: Using Green's first identity and the data of the theorem , we arrive at

$$\int\int\int_R (\nabla\phi)^2\,dV = 0$$

implying $\nabla\phi = 0$, i.e. ϕ is constant in $\mathbb{R}$. Since the value of ϕ is not known on the boundary $\partial\mathbb{R}$ while $\frac{\partial\phi}{\partial n} = 0$, it is implied that ϕ is a constant on $\partial\mathbb{R}$and hence on $\overline{\mathbb{R}}$.

11.4 Maximum-Minimum Principal

Theorem 11.4.1: Let$\mathbb{R}$ be the region bounded by $\partial\mathbb{R}$. Also, let u be a function which is continuous in a closed region $\overline{\mathbb{R}}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$. Further, if u is not constant everywhere on$\overline{\mathbb{R}}$, then the maximum and minimum values of u must occur on the boundary $\partial\mathbb{R}$.

Proof: Suppose u is a harmonic function but not constant everywhere on $\overline{\mathbb{R}}$. If possible, let u attain its maximum value M at some interior point P in $\mathbb{R}$. Since M is the maximum of u which is not constant, there should exists a sphere $S(P,r)$ about P such that some of the values of u on $S(P,r)$ must be less than M. But by the mean value property, the value of u at Pis the average of the values of u on $S(P,r)$ and hence it is less than M. This contradicts the assumptions that u is M at P. Thus the u must be constant over the entire sphere $S(P,r)$.

Let Q be any other point inside R which can be connected to P by an arc lying entirely within the domain R. By covering this arc with sphere and using the Heine-Borel theorem to choose a finite number of covering spheres and repeating the argument given above, we can arrive at the conclusion that u will have the same constant value at Q as at P. Thus u cannot attain a maximum value at any point in side the region R. Therefore, u can attain its maximum value only on the boundary $\partial\mathbb{R}$. A similar argument will lead to the conclusion that u can attain its minimum value only on the boundary $\partial\mathbb{R}$.

11.5 Potential Function

Another technique is to utilize a potential function to convert a nonlinear system of PDE into a single linear PDE. We consider as an example Euler's equations for inviscid, incompressible fluid flow:

$$\begin{cases} (a)\ u_t + u.Du = -Dp + f\ in\ R^3 \times (0,\infty) \\ (b)\quad div\, u = 0 \qquad in\ R^3 \times (0,\infty) \\ (c)\qquad u = g \qquad in\ R^3 \times (t=0) \end{cases} \tag{11.5.1}$$

Here the unknowns are velocity field $u = (u^1, u^2, u^3)$ and the scalar pressure p; external force $f = (f^1, f^2, f^3)$ and the initial velocity $g = (g^1, g^2, g^3)$ are given. Here D as usual denotes the gradient in the spatial variables $x = (x_1, x_2, x_3)$. The vector equation (11.5.1(a)) means

$$u_t^i + \sum_{j=1}^{3} u_j u_{x_j}^i = -p_{x_i} + f^i (i = 1,2,3).$$

We will assume

$$div\ g = 0 \tag{11.5.2}$$

If furthermore there exists a scalar function $h: R^3 \times (0,\infty) \to R$ such that

$$f = Dh \tag{11.5.3}$$

We say that the external force is derived from the potential h.

We will try to find a solution (u, p) of (11.5.1) for which the velocity field u is also derived from a potential, say

$$u = Dv \tag{11.5.4}$$

Our flow will then be irrotational, as $curl\ u \equiv 0$. Now equation (11.5.1(b)) says

$$0 = div\ u = \Delta v \tag{11.5.5}$$

and so v must be harmonic as a function of x, for ach time $t > 0$. Thus if we can find a smooth function v satisfying (11.5.5) and $Dv(.,0) = g$, we can then recover u from v by (11.5.4).

How do we compute the pressure p?

Let us observe that if

$u = Dv$, then $u.Du = \frac{1}{2} D(|Dv|^2)$.

Consequently (11.5.1(a)) reads

$$D\left(v_t + \frac{1}{2}|Dv|^2\right) = D(-p + h),$$

In view of (11.5.3). Therefore we can take

$$v_t + \frac{1}{2}|Dv|^2 + p = h. \tag{11.5.6}$$

This is Bernoulli's law. But now we can employ (11.5.5) to compute p, since v and h are already known.

11.6 Green's Function for Laplace Equation

We return now to the consideration of the interior Dirichlet problem formulated. Suppose, in the first instance, that the values of ψ and $\partial\psi/\partial n$ are known at every point of the boundary S of a finite region V and that $\nabla^2\psi = 0$ within V. We can then determine ψ by a simple application of Green's theorem in the form.

$$\int_\Omega (\psi\nabla^2\psi' - \psi'\nabla^2\psi)d\tau = \int_\Sigma (\psi\frac{\partial\psi'}{\partial n} - \psi'\frac{\partial\psi}{\partial n})\, dS \tag{11.6.1}$$

Where $\sum$ denotes the boundary of the region Ω.

If we are interested in determining the solution $\psi(r)$ of our problem at a point P with position vector r, then we surround P by a sphere C which has its center at P and has radius $\epsilon$and take $\sum$ to be the region which is exterior to C and interior to S. Putting

$$\psi' = \frac{1}{|r' - r|}$$

and noting that $\nabla^2\psi' = \nabla^2\psi = 0$, within Ω, we see that

$$\int_C \left\{\psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|} - \frac{1}{|r'-r|}\frac{\partial\psi}{\partial n}\right\}dS' + \int_S \left\{\psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|} - \frac{1}{|r'-r|}\frac{\partial\psi}{\partial n}\right\}dS' = 0 \qquad (11.6.2)$$

where the normal n are in the directions shown in Fig. . Now, on the surface of the sphere C,

$$\frac{1}{|r'-r|} = \frac{1}{\varepsilon}, \quad \frac{\partial}{\partial n}\frac{1}{|r'-r|} = \frac{1}{\varepsilon^2},$$

$$dS' = \varepsilon^2 \sin\theta d\theta d\phi$$

and

$$\psi(r') = \psi(r) + \varepsilon\left\{\sin\theta\cos\phi\frac{\partial\psi}{\partial x} + \sin\theta\sin\phi\frac{\partial\psi}{\partial y} + \cos\theta\frac{\partial\psi}{\partial z}\right\}$$

$$\frac{\partial\psi}{\partial n} = \left(\frac{\partial\psi}{\partial n}\right)_P + O(\varepsilon)$$

so that

$$\int_C \left\{\psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|}\right\}dS' = 4\pi\psi(r) + O(\varepsilon)$$

and

$$\int_C \frac{1}{|r'-r|}\frac{\partial\psi}{\partial n}dS' = O(\varepsilon)$$

Substituting these results into equation (11.6.2) and letting s tend to zero, we find that

$$\psi(r) = \frac{1}{4\pi}\int_S \left\{\frac{1}{|r'-r|}\frac{\partial\psi(r')}{\partial n} - \psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|}\right\}dS' \qquad (11.6.3)$$

so that the value of ψ , at an interior point of the region V can be determined in terms of the values of ψ, and $\frac{\partial\psi}{\partial n}$ on the boundary S.

A similar result holds in the case of the exterior Dirichlet problem. In this case we take the region Ωoccurring in equation (11.6.1) to be the region bounded by S, a small sphere C surrounding P, and $\sum$a sphere with center the origin and large radius R. Taking the directions of the normals to be as indicated in Fig. 24 and proceeding as above, we find, in this instance, that

$$4\pi\psi(r) + O(\varepsilon) + \int_\Sigma \left\{\frac{1}{R}\frac{\partial\psi}{\partial n} + \frac{\psi}{R^2}\right\}dS' + \int_S \left\{\frac{1}{|r'-r|}\frac{\partial\psi}{\partial n} - \psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|}\right\}dS' = 0 \qquad (11.6.4)$$

Letting $\varepsilon \to 0$ and $R \to \infty$, we see that the solution (11.6.3) is valid in the case of the exterior Dirichlet problem provided that $R\psi$ and $R^2\frac{\partial\psi}{\partial n}$ remain finite as$R \to \infty$.

Equation (11.6.3) would seem at first sight to indicate that to obtain a solution of Dirichlet's problem we need to know not only the value of the function ψ but also the value of $\frac{\partial\psi}{\partial n}$. That this is not in fact so can be shown by the introduction of the concept of a Green's function. We define a Green's function G(r,r') by the equation

$$G(r,r') = H(r,r') + \frac{1}{|r'-r|} \qquad (11.6.5)$$

where the function H(r,r') satisfies the relations

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)H(r,r') = 0 \qquad (11.6.6)$$

and

$$H(r,r') + \frac{1}{|r'-r|} = 0 \quad on\, S \tag{11.6.7}$$

Then since, just as in the derivation of equation (11.6.3), we can show that

$$\psi(r) = \frac{1}{4\pi}\int_S \left\{G(r,r')\frac{\partial\psi(r')}{\partial n} - \psi(r')\frac{\partial}{\partial n}G(r,r')\right\} dS'$$

it follows that if we have found a function G(r,r') satisfying equations (11.6.5), (11.6.6), and (11.6.7), then the solution of the Dirichlet problem is given by the relation

$$\psi(r) = -\frac{1}{4\pi}\int_S \left\{\psi(r')\frac{\partial}{\partial n}G(r,r')\right\} dS'$$

The solution of the Dirichlet problem is thus reduced to the determination of the Green's function G(r,r').

Summary

- The spherical mean isderived for harmonic function.
- The mean value theorem for harmonic function and its properties arediscussed.
- The potential function for solution of system of PDE elaborated.
- Determined the Green function for Laplace equation.

Keywords

- Laplace equation
- Spherical Mean
- Mean value theorem
- Harmonic function and its properties
- Potential function
- Green function

Self Assessment

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1. If $u \in C^2(U)$ is harmonic, then for each ball $B(x,r) \subset U$.

A. $u(x) = -\oint_{B(x,r)} udS = \oint_{B(x,r)} udy$

B. $u(s) = \oint_{B(x,r)} udS = -\oint_{B(x,r)} udy$

C. $u(x) = \oint_{\partial B(x,r)} udS = \oint_{B(x,r)} udy$

D. $u(x) = -\oint_{\partial B(x,r)} udS = \oint_{B(x,r)} udy$

2. If the function φ is harmonic in a sphere S and continuous on S, then the value of φ at the center of S is equal to

A. Arithmetic mean of its value on the circumference of S.

B. Geometric mean of its value on the circumference of S.

C. Spherical mean of its value on the circumference of S.

D. None of these

3. If u be harmonic in a Region R. Also, let P(x,y,z) be a given point in R and S(P,r) be a sphere with centre at P such that S(P,r) is completely contained in the harmonicity of u. Then

A. $u(P) = \bar{u}(r) = -\frac{1}{4\pi r^2}\iint_{S(P.r)} u(Q)dS$

B. $u(P) = \bar{u}(r) = \frac{1}{4\pi}\iint_{S(P.r)} u(Q)dS$

C. $u(P) = \bar{u}(r) = \frac{1}{4\pi r^2}\iint_{S(P.r)} u(Q)dS$

D. None of these

4. If a harmonic function vanishes everywhere on the boundary, then

A. It is identically zero everywhere.

B. It is identically nonzero everywhere.

C. It is identically zero only inside the domain.

D. It is identically nonzero inside the domain.

5. Maximum principle of Laplace equation is the

A. Strong maximum principle

B. Strong minimum principle

C. Weak maximum principle

D. Weak minimum principle

6. If a harmonic function is not constant everywhere then the maximum value must occur

A. Only on the boundary

B. Inside the boundary

C. Outside the boundary

D. Anywhere on the domain.

7. A function which is harmonic satisfying the Laplace equation

A. Will not be a smooth function

B. Not analytic anywhere in

C. Will satisfy the mean value theorem

D. None of these

8. Potential functions helps to convert a nonlinear system of partial differential equation into

A. A linear system of pde.

B. A single linear pde.

C. A semi linear pde

D. None of these

9. The equation $\boldsymbol{v_t} + \frac{1}{2}|\boldsymbol{Dv}|^2 + \boldsymbol{p} = \boldsymbol{h}$ is known as

A. Strong maximum value

B. Newton Law

C. Bernoulli's law

D. None of these

10. The Euler equation for an inviscid, incompressible fluid flow is given by

A. $u_t + \text{u}.\text{Du} = -\text{D}p + f$

B. $u_t + \text{u}.\text{Du} = \text{D}p + f$

C. $u_t + \text{Du} = -\text{D}p + f$

D. None of these

Answer forSelf Assessment

l.	C	2.	C	3.	C	4.	A	5.	A
6.	A	7.	C	8.	B	9.	C	10.	A

Review Questions

1) State and prove mean value theorem for harmonic function.
2) Discuss the method to convert system of nonlinear PDE to linear PDE.
3) Derive the Green function for Laplace equation.
4) State and prove the maximum principle for harmonic function.

Further Readings

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2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

Web Links

https://onlinecourses.nptel.ac.in/noc22_ma73/preview

https://onlinecourses.nptel.ac.in/noc21_ma09/preview

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Unit 12: Wave Equation

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of wave equation occurrence in other fields.
- understand the application of wave equation
- know about the elementary solution of wave equation.
- determine the uniqueness of solution by energy method.

Introduction

In this chapter we shall consider the wave equation

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \tag{12.0.1}$$

which is a typical hyperbolic equation. This equation is sometimes written in the form

$$\Box^2 \psi = 0$$

where $\Box^2$ denotes the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

If we assume a solution of the wave equation of the form

$$\psi = \Psi(x, y, z)e^{\pm ikct}$$

then the function Ψ must satisfy the equation

$$(\nabla^2 + k^2)\Psi = 0$$

which is called the space form of the wave equation or Helmholtz's equation.

12.1 The Occurrence of the Wave Equation in Physics

We shall begin this chapter by listing several kinds of situations in physics which can be discussed by means of the theory of the wave equation.

(a) **Transverse Vibrations of a String**. If a string of uniform linear density p is stretched to a uniform tension T, and if, in the equilibrium position, the string coincides with the x axis, then when the string is disturbed slightly from its equilibrium position, the transverse displacement y(x,t) satisfies the one -dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad (12.1.1)$$

where $c^2 = T/\rho$. At any point $x = a$ of the string which is fixed $y(a,t) = 0$ for all values of t.

(b) **Longitudinal Vibrations in a Bar**. If a uniform bar of elastic material of uniform cross section whose axis coincides with Ox is stressed in such a way that each point of a typical cross section of the bar takes the same displacement $\xi(x,t)$ then

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \qquad (12.1.2)$$

where$c^2 = E/\rho$, E being the Young's modulus and ρ the density of the material of the bar. The stress at any point in the bar is

$$\sigma = E\frac{\partial \xi}{\partial x} \qquad (12.1.3)$$

For instance, suppose that the velocity of the end x = 0 of the bar 0 ≤ x ≤ a is prescribed to be v(t), say, and that the other end x =ais free from stress. Suppose further that at that time t = 0 the bar is at rest. Then the longitudinal displacement of sections of the bar are determined by the partial differential equation (12.1.2) and the boundary and initial conditions

(i) $\frac{\partial \xi}{\partial t} = v(t)$ for $x = 0$

(ii) $\frac{\partial \xi}{\partial x} = 0$ for $x = a$

(iii) $\xi = \frac{\partial \xi}{\partial t} = 0$ at $t = 0, 0 \le x \le 0$

(c) **Longitudinal Sound Waves**. If plane waves of sound are being propagated in a horn whose cross section for the section with abscissa x is A(x) in such a way that every point of that section has the same longitudinal displacement ξ(x,t), thenξ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left\{\frac{1}{A}\frac{\partial (A\xi)}{\partial x}\right\} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \qquad (12.1.4)$$

which reduces to the one -dimensional wave equation (12.1.2) in the case in which the cross section is uniform. In equation (12.1.4)

$$c^2 = \left(\frac{dP}{d\rho}\right)_0$$

where the suffix 0 denotes that we take the value of $\frac{dP}{d\rho}$ in the equilibrium state. The change in pressure in the gas from the equilibrium value P_0 is given by the formula

$$P - P_0 = -c^2 \rho_0 \frac{\partial \xi}{\partial x}$$

where ρ_0 is the density of the gas in the equilibrium state. For instance, if we are considering the motion of the gas when a sound wave passes along a tube which is free at each of the ends x = 0, x = a, then we must determine solutions of equation (12.1.4) which are such that

$\frac{\partial \xi}{\partial x} = 0$ at $x = 0$ and at $x = a$.

(d) **Electric Signals in Cables**. We have already remarked that if the resistance per unit length R, and the leakage parameter G are both zero, the voltage V(x,t) and the current z(x,t) both satisfy the one- dimensional wave equation, with wave velocity c defined by the equation

$$c^2 = \frac{1}{LC} \qquad (12.1.5)$$

where L is the inductance. and C the capacity per unit length.

(e) **Transverse Vibrations of a Membrane**. If a thin elastic membrane of uniform areal density σ is stretched to a uniform tension T, and if, in the equilibrium position, the membrane coincides with the xy plane, then the small transverse vibrations of the membrane are governed by the wave equation

$$\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \qquad (12.1.6)$$

where z(x,y,t) is the transverse displacement (assumed small) at time t of the point (x,y) of the membrane. The wave velocity c is defined by the equation

$$c^2 = \frac{T}{\sigma} \qquad (12.1.6)$$

If the membrane is held fixed at its boundaryΓ, then we must have $z = 0$ on Γ for all values of t.

(f) **Electromagnetic Waves**. If we write

$$H = curl\, A, \qquad E = -\frac{1}{c}\frac{\partial A}{\partial t} - grad\, \phi$$

then Maxwell's equations

$$div\, E = 4\pi q, \qquad div\, H = 0$$

$$curl\, E = -\frac{1}{c}\frac{\partial H}{\partial t}, \qquad curl\, H = \frac{4\pi i}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

are satisfied identically provided that A and ϕ satisfy the equations

$$\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \frac{4\pi}{c} i, \quad \nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - 4\pi\rho$$

Therefore in the absence of charges or currents and the components of A satisfy the wave equation.

12.2 Elementary Solutions of the One -dimensional Wave Equation

A general solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{12.2.1}$$

is

$$y = f(x+ct) + g(x-ct) \tag{12.2.2}$$

where the functions f and g are arbitrary. In this section we shall show how this solution may be used to describe the motion of a string. In the first instance we shall assume that the string is of infinite extent and that at time $t = 0$ the displacement and the velocity of the string are both prescribed so that

$$y = \eta(x), \frac{\partial y}{\partial t} = v(x) \;\; at \;\; t = 0 \tag{12.2.3}$$

Our problem then is to solve equation (12.2.1) subject to the initial conditions (12.2.3). Substituting from (12.2.3) into (12.2.2), we obtain the relations

$$\eta(x) = f(x) + g(x), \;\; v(x) = cf'(x) - cg'(x) \tag{12.2.4}$$

Integrating the second of these relations, we have

$$f(x) - g(x) = \frac{1}{c}\int_b^x v(\xi)d\xi,$$

where b is arbitrary. From this equation and the first of the equations (12.2.4) we obtain the formulas

$$f(x) = \frac{1}{2}\eta(x) + \frac{1}{2c}\int_b^x v(\xi)d\xi$$

$$g(x) = \frac{1}{2}\eta(x) - \frac{1}{2c}\int_b^x v(\xi)d\xi$$

Substituting these expressions in equation (12.2.2), we obtain the solution

$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi \tag{12.2.5}$$

The solution (5) is known as d'Alembert's solution of the one -dimensional wave equation. If the string is released from rest, y_0, so that equation (12.2.5) becomes

$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} \tag{12.2.6}$$

showing that the subsequent displacement of the string is produced by two pulses of "shape" $y = \frac{1}{2}\eta(x)$, each moving with velocity c, one to the right and the other to the left.

12.3 Energy Method

This suggests that perhaps some other way of measuring the size and smoothness of functions may be appropriate. Indeed we will see in this section that the wave equation is nicely behaved with respect to certain integral 'energy' norm.

Uniqueness

Let $U \subset R^n$ be a bounded, open set with a smooth boundary ∂U, and as usual set $U_T = U \times (0,T], \Gamma = \bar{U}_T - U_T, where\ T > 0$.

We are interested in initial/boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f \; in \; U_T \\ u = g \; on \; \Gamma_T \\ u_t = h \; on \; U \times \{t = 0\}. \end{cases} \tag{12.3.1}$$

Theorem 12.3.1: (Uniqueness for Wave equation). There exists at most $u \in C^2(\bar{U}_T)$solvinf (12.3.1)

Proof: If $\bar{u}$is another such solution, then $w = u - \bar{u}$ solves

$$\begin{cases} w_{tt} - \Delta w = 0 \; in \; U_T \\ w = 0 \; on \; \Gamma_T \\ w_t = 0 \; on \; U \times \{t = 0\}. \end{cases}$$

Define the ''energy''

$$e(t) = \frac{1}{2}\int_U w_t^2(x,t) + |Dw(x,t)|^2 \, dx \quad (0 \le t \le T).$$

We compute

$$\dot{e}(t) = \int_U w_t w_{tt} + Dw.Dw_t \, dx \qquad \left[. = \frac{d}{dt} \right]$$

$$= \int_U w_t(w_{tt} - \Delta w)dx = 0$$

There is no boundary term since $w = 0$, and hence $w_t = 0$, $on \; \partial U \times [0, T]$.

Thus for all $0 \le t \le T, e(t) = e(0) = 0$, and so $w_t, Dw = 0$ within U_T.Since $w = 0$ on $U \times \{t = 0\}$, we conclude $w = u - \bar{u} = 0 \; in \; U_T$.

Summary

- The wave equation and its occurrence in Physics is discussed.
- Thespherical solution for wave equation is derived.
- Theboundary value and initial value problem is defined.
- The unique solution using energy method is determined.

Keywords

- Wave equation
- Elementary solution
- Unique solution
- Energy method

Self Assessment

1. The elementary solution of the one dimensional wave equation is called also as

A. D′ Alembert solution
B. Helmholtz's solution
C. Riemann-Volterra solution
D. Weber's solution

2. Which of the following is correct solution of wave equation?

A. $y = \frac{1}{2}\{\eta(x + ct) + \eta(x - ct)\} - \frac{1}{2}\int_{x-ct}^{x+ct} v(\xi)d\xi$

B. $y = \frac{1}{2}\{\eta(x + ct) - \eta(x - ct)\} + \frac{1}{2}\int_{x+ct}^{x-ct} v(\xi)d\xi$

C. $y = \frac{1}{2}\{\eta(x + ct) + \eta(x - ct)\} + \frac{1}{2}\int_{x-ct}^{x+ct} v(\xi)d\xi$

D. $y = \frac{1}{2}\{\eta(x + ct) + \eta(x - ct)\} - \frac{1}{2}\int_{x+ct}^{x-ct} v(\xi)d\xi$

3. The energy function for the wave equation over the domain U is given by

A. $e(t) = \frac{1}{2}\int w_t^2(x,t)+|Dw|^2 dx,\ \ (0\le t<T)$

B. $e(t) = \int w^2(x,t)+|Dw|^2 dx,\ \ (0\le t<T)$

C. $e(t) = \frac{1}{2}\int w_t^2(x,t)+|Dw|^4 dx,\ \ (0\le t<T)$

D. $e(t) = \frac{1}{2}\int w_t^2(x,t)-|Dw|^2 dx,\ \ (0\le t<T)$

4. The uniqueness of the solution of boundary problem of wave equation is given by

A. Elementary method

B. Energy method

C. Fundamental method

D. None of these

5. A general solution of the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ is

A. $y = f(x+ct) + g(x-ct)$

B. $y = f(x+ct) - g(x-ct)$

C. $y = f(x+ct)$

D. None of these

6. If the string is released from rest, then the elementary solution becomes

A. $\{\eta(x+ct) - \eta(x-ct)\}$

B. $2\{\eta(x+ct) - \eta(x-ct)\}$

C. $\frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\}$

D. $None\ of\ these$

7. The displacement of the string released from rest is produced by

A. Two pulses

B. Straight curve

C. Arc

D. None of these

8. The equation is $\begin{cases} u_{tt} - \Delta u = f\ in\ U_T \\ u = g\ on\ \Gamma_T \\ u_t = h\ on\ U \times \{t=0\}. \end{cases}$ governed as

A. Initial value problem

B. Boundary Value problem

C. Both (a) and (b)

D. None of these

Answersforself Assessment

1. A 2. C 3. A 4. B 5. A

6. C 7. A 8. C

Review Questions

1. Derive d'Alembert's formulas for one dimensional wave equation.
2. State and prove uniqueness by the energy methods for wave equation.

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.

2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

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Unit 13: Similarity Solutions

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept similarity solution.
- understand about the plane wave and traveling wave solution
- know about theexponential solution corresponding to number of equations.
- determine soliton by using traveling wave function.
- apply the similarity under scaling function for porous medium.

Introduction

Similarity Solutions

When investigating partial differential equations it is often profitable to look for specific solutions u, the form of which reflects various symmetries in the structure of the PDE. We have already seen this idea in our derivation of the fundamental solutions for Laplace's equation, and our discovery of rarefaction waves for conservation laws. Following are some other applications of this important method.

13.1 Plane and Traveling Waves

Consider first a partial differential equation involving the two variables $x \in \mathbb{R}, t \in \mathbb{R}$.

A solution u of the form

$u(x,t) = v(x - \sigma t), x \in \mathbb{R}, t \in \mathbb{R}$ (13.1.1)

is called a traveling wave (with speedσ and profile v).

More generally, a solution u of a PDE in the n + 1 variables$x = (x_1, x_2, \dots \dots, x_n) \in R^n, t \in R$ having the form

$u(x,t) = v(y.x - \sigma t), x \in \mathbb{R}^n, t \in \mathbb{R}$ (13.1.2)

is called a plane wave (with wavefront normal to y ∈ R", velocity$\frac{\sigma}{|y|}$, andprofile u).

Exponential Solutions

In view of the Fourier transform, it is particularly enlightening when studying linear partial differential equations toconsider complex-valued plane wave solutions of the form

$u(x,t) = e^{i(y.x+\omega t)}$ (13.1.3)

where $\omega \in \mathbb{C}$ and $y = (y_1, y_2, \ldots\ldots y_n) \in R^n$, ωbeing the frequency and $\{y_i\}_{i=1}^{n}$,the wave numbers.

We will next substitute trial solutions of the form (13.1.3) into various linear PDE, paying particular attention to the relationship between y and ωforced by the structure of the equation.

(i) Heat Equation. If u is given by (13.1.3), we compute

$$u_t - \Delta u = (i\omega + |y|^2)u = 0,$$

provided$\omega = i|y|^2$. Hence

$$u = e^{iy.x - |y|^2 t}$$

solves the heat equation for each $y \in R^n$. Taking real and imaginary parts,we discover further that

$e^{-|y|^2 t}\cos(y.x)$ and $e^{-|y|^2 t}\sin(y.x)$ are the solution as well. Notice in this example that since ω is purely imaginary, there resulta a real, negative exponential term $e^{-|y|^2 t}$ in the formulas, which corresponds to dissipation.

(ii) Wave Equation. Upon our substituting (13.1.3) into the wave equation, we discover

$$u_{tt} - \Delta u = (-\omega^2 + |y|^2)u = 0,$$

provided $\omega = \pm|y|$. Consequently

$$u = e^{i(y.x \pm |y|)t}$$

solves the wave equation, as do the pair of functions $\cos(y.x \pm |y|)t$ and $\sin(y.x \pm |y|)t)$. Since ω is real, there are no dissipation effects in these solutions.

(iii) Dispersive Equations. We now let n = 1 and substitute $u(x,t) = e^{i(y.x+\omega t)}$intoAiry's equation

$$u_t + u_{xxx} = 0$$

We calculate

$$u_t + u_{xxx} = i(\omega - y^3)u = 0$$

whenever $\omega = y^3$. Thus

$$u = e^{i(y.x \pm y^3)t}$$

solves Airy's equation, and once again as ω is real there is no dissipation.

Notice: however that the velocity of propagation is y^2, which depends non- linearly upon the frequency of the initial value e^{iyx}. Thus waves of different frequencies propagate at different velocities: the PDE creates dispersion.

(iv) Schrödinger's Wave Equation

Likewise, if n ≥ 1 and we substitute

$$u(x,t) = e^{i(y.x+\omega t)}$$

into Schrödinger's equation,

$$iu_t + \Delta u = 0$$

we compute

$$iu_t + \Delta u - (\omega + |y|^2)u = 0.$$

Consequently$\omega = -|y|^2$, and

$$u = e^{i(y.x - |y|^2 t)}$$

Again, the solution displays dispersion.

13.2 Solitons

We consider next the Korteweg-De-Vries (KdV) equation in the form

$$u_t + 6uu_x + u_{xxx} = 0 \quad in\ \mathbb{R} \times (0, \infty). \qquad (13.2.1)$$

This nonlinear dispersive equation being model for surface waves in water.

We seek a traveling wave solution having structure

$$u(x,t) = v(x - \sigma t)(x \in R, t > 0) \qquad (13.2.2)$$

Then u solves the KdV equation (13.2.1), provided v satisfies the ODE

$$-\sigma v' + 6vv' + v''' = 0 \qquad \left(' = \frac{d}{ds}\right) \qquad (13.2.3)$$

We integrate (13.3.3.) by first noting

$$-\sigma v + 3v^2 + v'' = a \qquad (1323.4)$$

a denoting some constant. Multiply this equality by v' to obtain

$$-\sigma vv' + 3v^2 v' + v'v'' = av' \qquad (13.2.5)$$

so deduce

$$\frac{(v')^2}{2} = -v^3 + \frac{\sigma}{2}v^2 + av + b \qquad (13.2.6)$$

where b is another arbitrary constant.

We investigate (13.2.6) by looking now only for solutions $v$which satisfy $v, v', v'' \to 0$ as $s \to \pm\infty$. (In which case the function u having the form (13.2.2) is called a solitary wave.) Then (13.2.6), (13.2.5) imply a = b = 0. Equation (13.2.6) thereupon simplifies to read

$$\frac{(v')^2}{2} = v^2\left(-v + \frac{\sigma}{2}\right).$$

Hence $v' = \pm v(\sigma - 2v)^{1/2}$.

We take the minus sign above for computational convenience, and obtain then this implicit formula for v:

$$s = -\int_0^{v(s)} \frac{dz}{z(\sigma - 2z)^{1/2}} + c \qquad (13.2.7)$$

For some constant c.

Now substitute $z = \frac{\sigma}{2}\operatorname{sech}^2\theta$.

It follows that $\frac{dz}{d\theta} = -\sigma Sech^2\theta\ tanh\theta$ and $z(\sigma - 2z)^{1/2} = \frac{\sigma^{\frac{3}{2}}}{2} Sech^2\theta\ tanh\theta$. Hence (13.2.7) becomes

$$s = \frac{2}{\sqrt{\sigma}}\theta + c \qquad (13.2.8)$$

where θ implicitly given by the relation

$$\frac{\sigma}{2}\operatorname{sech}^2\theta = v(s)(13.2.9)$$

We lastly combine (13.2.8) and (13.2.9), to compute

$$v(s) = \frac{\sigma}{2}\text{sech}^2\left(\frac{\sqrt{\sigma}}{2}(s-c)\right) \qquad (s \in R)$$

Conversely, it is routine to check $v$so defined actually solves the ODE (13.2.3.

The upshot is that

$$u(x,t) = \frac{\sigma}{2}\text{sech}^2\left(\frac{\sqrt{\sigma}}{2}(x-\sigma t-c)\right) \qquad x \in R, t \geq 0$$

is a solution of KdV equation for each $c \in R, \sigma > 0$. A solution of this form is called a soliton . Note the velocity of the soliton depends upon its height.

Notes: The KdV equation is in factutterly remarkable, in that it is completely integrable, which mean that in principle that the exact solution can be computed for essentially arbitrary initial data.

13.3 Similarity Under Scaling

We next illustrate the possibility of findings the other types of 'similarity' solutions to PDE.

A scaling invariant solution Consider the porous medium equation

$$\boldsymbol{u_t - \nabla(u^\gamma) = 0 \quad in\ R^n \times (0,\infty)} \qquad \textbf{(13.3.1)}$$

where $u \geq 0$ and $\gamma > 1$ is constant.

As in our later derivation of the fundamental solution od heat equation, let us look for a solution u having the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right), \qquad x \in R^n, t > 0, \qquad (13.3.2)$$

Where the constants α, β and the function $v: R^n \to R$ must be determined.

Remember that we come upon (13.3.2) if we seek a solution u of (13.3.1) invariant under dilation scaling

$$u(x,t) \to \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

so that $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$

for all $\lambda > 0, x \in R^n, t > 0$. Setting $\lambda = t^{-1}$, we obtain (13.3.2) for $v(y) = u(y,1)$.

We insert (13.3.2) into (13.3.1), and discover

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y.Dv(y) + t^{-(\alpha\gamma+2\beta)} \Delta(v^\gamma)(y) = 0 \qquad (13.3.3)$$

for $y = t^{-\beta}x$.

In order to convert (13.3.3) into an expression involving the variable y alone, let us require

$$\alpha + 1 = \alpha\gamma + 2\beta \qquad (13.3.4)$$

Then (13.3.3) reduces to

$$\alpha v + \beta y.Dv + \Delta(v^\gamma) = 0 \qquad (13.3.5)$$

At this point we have effected a reduction from n + 1 to n variables. We simplify further by supposing v is radial; that is, $v(y) = w(|y|)$for some $w: R \to R$. Then (13.3.5) becomes

$$\alpha w + \beta r w' + (w^\gamma)'' + \frac{n-1}{r}(w^\gamma)' = 0 \qquad (13.3.6)$$

Where $= |y|, \quad ' = \frac{d}{dr}$. Now if we set

$$\alpha = n\beta, \qquad (13.3.7)$$

(13.3.6) thereupon simplifies to read

$$(r^{n-1}(w^\gamma)')' + \beta(r^n w)' = 0$$

Thus

$$r^{n-1}(w^{\gamma})' + \beta r^{n} w = a$$

for some constant a.

Assuming $\lim_{r\to\infty} w, w' = 0,$ we conclude a = 0; whence

$$(w^{\gamma})' = -\beta r w.$$

But then

$$(w^{\gamma-1})' = -\frac{\gamma-1}{\gamma}\beta r$$

Consequently

$$w^{\gamma-1} = b - \frac{\gamma-1}{2\gamma}\beta r^{2},$$

b is constant; and so

$$w = \left(b - \frac{\gamma-1}{2\gamma}\beta r^{2}\right)_{+}^{\frac{1}{\gamma-1}} \tag{13.3.8}$$

where we look the positive part of right hand side of (13.3.8) to ensure $w \geq 0$. Recalling $v(y) = w(r)$ and (13.3.2), we obtain

$$u(x,t) = \frac{1}{t^{\alpha}}\left(b - \frac{\gamma-1}{2\gamma}\beta\frac{|x|^{2}}{t^{2\beta}}\right)_{+}^{\frac{1}{\gamma-1}}, \qquad (\, x \in R^{n}, t > 0\,) \tag{13.3.9}$$

where from (13.3.4) and (13.3.7),

$$\alpha = \frac{n}{n(\gamma-1)+2}, \beta = \frac{1}{n(\gamma-1)+2} \tag{13.3.10}$$

The formula (13.3.9) and (13.3.10) are Barenblatt's solution to the porous medium equation.

Summary

- The plane wave and traveling wave solutions are discussed.
- The exponential solution are determined for different equation.
- The dissipation effect using exponential is discussed.
- The similarity solution under scaling are explained.
- The soliton equation are elaborated with examples.

Keywords

- Plane wave
- Traveling wave
- Soliton
- Similarity solutions
- Porous medium
- Scaling

Self Assessment

1. The u of the form $u(x,t) = v(x - \sigma t), x \epsilon R, t \epsilon R$ with speed σ and profile v is called as
A. Traveling wave
B. Plane wave
C. Transverse wave
D. Longitudinal wave

2. The u of the form $u(x,t) = v(y.x - \sigma t), x \epsilon R, t \epsilon R$ with speed σ and profile v is called as
A. Traveling wave
B. Plane wave
C. Transverse wave
D. Longitudinal wave

3. Which of the equation has no dissipation effect?
A. Heat equation
B. Wave equation
C. Dispersive equation
D. None of these

4. Which is not the kind of wave equation solutions?
A. Traveling wave
B. Plane wave
C. Solitons
D. None of these

5. The nonlinear dispersive equation $u_t + 6uu_x + u_{xxx} = 0$ is also known as
A. Korteweg-de Vrise equation
B. Alembert equation
C. Helmholtz's equation
D. Riemann-Volterra equation

6. The wave equation which is used to represent the surface of water represent as
A. $u_t + 6uu_{xx} + u_{xxx} = 0$
B. $u_t + 6uu_x^2 + u_{xxx} = 0$
C. $u_t + 6uu_x + u_{xxx} = 0$
D. $u_t + 6uu_{xx}^2 + u_{xxx} = 0$

7. The wave function represents below is the type of

$$u(x,t) = \frac{\sigma}{2}\operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2}(x - \sigma t - c)\right) \quad (x \in \mathbb{R},\ t \geq 0)$$

A. Stationary wave
B. Travelling wave
C. Vibrating wave
D. Plane wave

8. The equation under scaling invariant of Porous media is represents as
A. $u_t - \Delta(u^{\gamma}) = 0$
B. $u_{tt} - \Delta(u^{\gamma}) = 0$
C. $u_t - \mathrm{D}(u^{\gamma}) = 0$

D. $u_{tt} - \mathrm{Ds}(u^{\gamma}) = 0$

9. The Barenblatt's solution is of which kind of equation
A. Diffusion equation
B. Porous media equation
C. Wave equation
D. Laplace equation

10. The equation $u_t + u_{xxx} = 0$ is knows as
A. Airy's equation
B. Non characteristic surface equation
C. Water surface equation
D. None of these

11. The Schrodinger's equations is represent as
A. $iu_t + \Delta u = 0$
B. $u_t + iDu_x = 0$
C. $u_{tt} + iDu = 0$
D. $iu_{tt} + u_{xx} = 0$

AnswersforSelf Assessment

l.	A	2.	B	3.	B	4.	D	5.	A
6.	C	7.	B	8.	A	9.	B	10.	A
11.	A								

Review Questions

1. State and prove the Korteweg-de Vries (KDV) equation for solitons.
2. Derive the exponential solution for plane and travelling wave equation by considering heat equation and wave equation.
3. Derive the exponential solution for plane and travelling wave equation by considering Airy'sequation.
4. State and prove Barenblatt's solution to the porous medium equation by the method of similarity under scaling.

Further Readings

1. I.N. Sneddon(1957), Elements Of Partial Differential Equations,Mcgraw Hill Education.

2. Lawrence C. Evans (1998), Partial Differential Equations, Universities Press Pvt. Ltd

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Unit 14 : Heat Equations

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of heat equation of diffusion equation.
- understand about fundamental of solution of heat equation.
- know about theelementary solution of diffusion equation.
- determine green function and find uniqueness through energy functions.

Introduction

Next we study the heat equation

$$u_t - \Delta u = 0 \tag{14.0.1}$$

and the nonhomogeneous heat equation

$$u_t - \Delta u = f \tag{14.0.2}$$

subject to appropriate initial and boundary conditions. Here t > 0 and $x \in U$,where $U \subset R^n$is open. The unknown is u: $\bar{U} \times [0, \infty) \to R$, u = u(x, t), and the Laplacian Δ is taken with respect to the spatial variables

$x = (x_1, x_2, \dots \dots . x_n): \Delta u = \Delta u_x = \sum_{i=1}^{n} u_{x_i x_i}$.

In (14.0.2) the function f: $U \times [0, \infty) \to R$ is given.

A guiding principle is that any assertion about harmonic functions yields an analogous (but more complicated) statement about solutions of the heat equation. Accordingly our development will largely parallel the corresponding theory for Laplace's equation.

Physical interpretation. The heat equation, also known as the diffusion equation, describes in typical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. If $V \subset U$is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt}\int_V udx = -\int_{\partial V} F.vds$$

F being the flux density. Thus

$u_t = -div\ F$ (14.0.3)

as V was arbitrary. In many situations F is proportional to the gradient of u, but points in the opposite direction (since the flow is from regions of higher to lower concentration):

F=-aDu (a > 0).

Substituting into (14.0.3), we obtain the PDE

u_t = a div(Du) = aΔu,

which for a =1 is the heat equation.

The heat equation appears as well in the study of Brownian motion.

14.1 Fundamental Solution

Derivation of the fundamental solution:

An important first step in studying any PDE is often to come up with some specific solutions. We observe that the heat equation involves one derivative with respectto the time variable t, but two derivatives with respect to the space variablesx_i (i = 1,...,n). Consequently we see that if u solves (14.0.1), then so does $u(\lambda x, \lambda^2 t)\ for\ \lambda \in R$. This scaling indicates the ratio $\frac{r^2}{t}$, $(r = |x|)$ is important for the heat equation and suggests that we search for a solution of (14.0.1) having the form

$$u(x,t) = v\left(\frac{|x|^2}{t}\right), t > 0, x \in R^n,$$

for some function v as yet undetermined.

Although this approach eventually leads to what we want, it is quicker to seek a solution u having the special structure

$$u(x,t) = \frac{1}{t^\alpha}\left(\frac{x}{t^\beta}\right), \quad (x \in R^n, t > 0) \tag{14.1.1}$$

where the constants α, β and the function $v: R^n \to R,$ must be found. We come to (14.1.1) if we look for a solution u of the heat equation invariant under the dilation scaling

$$u(x,t) \to \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

That is, we ask

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for all $\lambda > 0, x \in R^n, t > 0$. Setting $\lambda = t^{-1}$, we obtain (13.3.2) for $v(y) = u(y, 1)$.

Let us insert (14.1.1) into (14.0.1), and thereafter compute

$$\alpha t^{-(\alpha+1)}v(y) + \beta t^{-(\alpha+1)}y.Dv(y) + t^{-(\alpha\gamma+2\beta)}\Delta v(y) = 0 \tag{14.1.2}$$

For$y = t^{-\beta}x$. In order to transform (14.1.2) into an expression involving the variable y alone, we take ß=1/2. Then the terms with t are identical, and so (14.1.2) reduces to

$$\alpha v + \frac{1}{2}y.Dv + \Delta v = 0 \tag{14.1.3}$$

We simplify further by guessing v to be radial; that is, $v(y) = w(|y|)$for some w: R→R. Thereupon (14.1.3) becomes

$$\alpha w + \frac{1}{2}rw' + w'' + \frac{n-1}{r}w' = 0 \tag{14.1.4}$$

For $r = |y|$, $\quad ' = \frac{d}{dr}$Now if we set $\alpha = \frac{1}{2}$, this simplifies to read

$$(r^{n-1}w')' + \beta(r^n w)' = 0$$

Thus

$$r^{n-1}w' + \beta r^n w = a$$

for some constant a.

Assuming $\lim_{r\to\infty} w, w' = 0,$ we conclude a = 0; whence

$$w' = -\frac{1}{2}rw.$$

But for some constant b

$$w = be^{-\frac{r^2}{4}} \tag{14.1.5}$$

Combining (14.1.1), (14.1.5) and our choices for, β, we conclude that

$$\frac{b}{t^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

solves the heat equation (14.0.1).

This computation motivates the following

DEFINITION. The functions

$$\phi(x,t) := \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}, & (x \in R^n, t > 0) \\ 0, & (x \in R^n, t > 0) \end{cases}$$

is called the fundamental solution of the heat equation.

Remark: 14.1.2: Notice that ϕ is singular at the point (0,0). We will sometimes write $\phi(x,t) = \phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variable x.

14.2 Elementary Solutions of the Diffusion Equation

In this section we shall consider elementary solutions of the one - dimensional diffusion equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \tag{14.2.1}$$

We begin by considering the expression

$$\theta = \frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4\kappa t}\right) \tag{14.2.2}$$

For this function it is readily seen that

$$\frac{\partial^2\theta}{\partial x^2} = \frac{x^2}{4\kappa^2 t^{5/2}}e^{-x^2/4\kappa t} - \frac{1}{2\kappa t^{3/2}}e^{-x^2/4\kappa t}$$

and

$$\frac{\partial\theta}{\partial t} = \frac{x^2}{4\kappa t^{5/2}}e^{-x^2/4\kappa t} - \frac{1}{2t^{3/2}}e^{-x^2/4\kappa t}$$

showing that the function (14.2.2) is a solution of the equation (14.2.1). It follows immediately that

$$\frac{1}{2\sqrt{\pi\kappa t}}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right) \tag{14.2.3}$$

Where ξ is an arbitrary real constant, is also a solution. Furthermore, if the function $\phi(x)$ is bounded for all real values of x, then it is possible that the integral

$$\frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty}\phi(\xi)\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d\xi \tag{14.2.4}$$

is also, in some sense, a solution of the equation (14.2.1). It may readily be proved that the integral (14.2.4) is convergent if t > 0 and that the integrals obtained from it by differentiating under the integral sign with respect to x and t are uniformly convergent in the neighborhood of the point (x,t). The function θ(x,t) and its derivatives of all orders therefore exist for t > 0, and since the integrand satisfies the one -dimensional diffusion equation, it follows that θ(x,t) itself satisfies that equation for t > 0.

Now

$$\left|\frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty}\phi(\xi)\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d\xi-\phi(x)\right|$$

$$=|I_1+I_2+I_3-I_4|$$

Where

$$I_1=\frac{1}{\sqrt{\pi}}\int_{-N}^{N}\{\phi(x+2u\sqrt{\kappa t})-\phi(x)\}e^{-u^2}\,du$$

$$I_2=\frac{1}{\sqrt{\pi}}\int_{N}^{\infty}\{\phi(x+2u\sqrt{\kappa t})\}e^{-u^2}\,du$$

$$I_3=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-N}\{\phi(x+2u\sqrt{\kappa t})\}e^{-u^2}\,du$$

$$I_4=\frac{2\phi(x)}{\sqrt{\pi}}\int_{N}^{\infty}e^{-u^2}\,du$$

If the function $\phi(x)$ is bounded, we can make each of the integrals I_2, I_3, I_4 as small as we please by taking N to be sufficiently large, and by the continuity of the function ϕwe can make the integralI_1 as small as we please by taking t sufficiently small. Thus as $t\to 0, \theta(x,t)\to\phi(x)$. Thus the Poisson integral

$$\theta(x,t)=\frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty}\phi(\xi)\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d\xi \qquad (14.2.5)$$

is the solution of the initial value problem

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\kappa}\frac{\partial\theta}{\partial t} \qquad -\infty<x<\infty$$

$$\theta(x,0)=\phi(x) \qquad (14.2.6)$$

It will be observed that by a simple change of variable we can express the solution (14.2.5) in the form

$$\theta(x,t)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\{\phi(x+2u\sqrt{\kappa t})\,\mathrm{e}^{-u^2}\,du \qquad (14.2.7)$$

We shall now show how this solution may be modified to obtain the solution of the boundary value problem

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\kappa}\frac{\partial\theta}{\partial t} \qquad 0\le x<\infty$$

$$\theta(x,0)=f(x) \quad x>0 \qquad (14.2.8)$$

$$\theta(0,t)=0, \qquad t>0$$

14.3 Energy Methods

a. Uniqueness.

Let us investigate again the initial/boundary-value problem

$$\begin{cases} u_t-\Delta u=f \text{ in } U_T \\ u=g \text{ on } \Gamma_T \end{cases} \qquad (14.3.1)$$

We earlier invoked the maximum principle to show uniqueness, and now-by analogy -provide an alternative argument based upon integration by parts. We assume as usual that UC R" is open, bounded and that ∂U is C[1]. The terminal time T > 0 is given.

Theorem14.3.1: (Uniqueness). There exists at most one solution $u \in C_1^2(\bar{U}_T)$of (14.3.1).

Proof. 1. If $\bar{u}$ is another solution, $w = u - \bar{u}$solves

$$\begin{cases} w_t - \Delta w = 0 \; in \;\; U_T \\ \quad w = 0 \; on \; \Gamma_T \end{cases} \tag{14.3.2}$$

2. Set

$$e(t) := \int_U w^2(x,t)dx, \quad (0 \le t < T).$$

Then

$$\dot{e}(t) := 2\int_U ww_t dx \quad \left(. = \frac{d}{dt}\right)$$

$$= 2\int_U w\Delta w dx$$

$$-2\int_U |Dw|^2 \, dx \le 0,$$

and so $e(t) \le e(0) = 0 \;\; (0 \le t \le T)$.

Consequently $w = u - \bar{u} \;\; in \; U_T$.

b. **Backwards uniqueness.**

A rather more subtle question concerns uniqueness backwards in time for the heat equation. For this, suppose u and ũ are both smooth solutions of the heat equation in U_T, with the same boundary conditions on ∂U:

$$\begin{cases} \;\; u_t - \Delta u = f \; in \;\; U_T \\ u = g \; on \; \partial U \;\; \times [0,T], \end{cases} \tag{14.3.3}$$

$$\begin{cases} \;\; u_t - \Delta u = f \; in \;\; U_T \\ u = g \; on \; \partial U \;\; \times [0,T], \end{cases} \tag{14.3.4}$$

for some function g. Note carefully that we are not supposing u = ũ at time t = 0.

Theorem 14.3.2 (Backwards uniqueness). Suppose u, ũ € $C^2(\bar{U}_T)$ solve (14.3.3), (14.3.4). If

u(x,T) = ũ (x,T) ($x \in U$),thenu =ũ within U_T.

In other words, if two temperature distributions on U agree at some time T > 0, and have had the same boundary values for times 0 <u>≤</u>t<u>≤</u>T, then these temperatures must have been identically equal within U at all earlier times. This is not at all obvious.

Proof. 1. Write $w = u - \bar{u}$and, as in the proof of Theorem (14.3.1), set

$$e(t) := \int_U w^2(x,t)dx, \quad (0 \le t < T).$$

As before

$$\dot{e}(t) := -2\int_U |Dw|^2 \, dx, \tag{14.3.5}$$

Further more

$$\ddot{e}(t) := -4\int_U Dw.Dw_t \, dx$$

$$:= 4\int_U \Delta w w_t \, dx \tag{14.3.6}$$

$$= 4\int_U (\Delta w)^2\, dx \quad by\ (14.3.1)$$

Now since w =0 on∂U,

$$\int_U |Dw|^2\, dx = -\int_U w\Delta w\, dx$$

$$\leq \left(\int_U w^2 dx\right)^{\frac{1}{2}} \left(4\int_U (\Delta w)^2 dx\right)^{1/2}$$

$$= e(t)\ddot{e}(t)$$

Thus (143.5) and (14.3.6) imply

$$\left(\dot{e}(t)\right)^2 := 4\left(\int_U |Dw|^2\, dx\right)^2$$

$$\leq \left(\int_U w^2 dx\right)\left(4\int_U (\Delta w)^2\, dx\right)$$

$$= e(t)\ddot{e}(t)$$

Hence

$$e(t)\ddot{e}(t) \geq \left(\dot{e}(t)\right)^2, 0 \leq t \leq T. \qquad (14.3.7)$$

2. Now if e(t) = 0 for all 0 ≤t≤ T, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$, with $e(t) > 0$ for $t_1 < t < t_2$, $e(t_2) = 0$. (14.3.8)

3. Now write

$$f(t) = \log e(t)(t_1 < t < t_2). \qquad (14.3.9)$$

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \quad by \quad (14.3.7)$$

and so f is convex on the interval (t_1, t_2). Consequently if $0 < \tau < 1$,

t_1<t< t_2, we have

Then

$$f\big((1-\tau)t_1 + \tau t\big) \leq (1-\tau)t_1 + \tau f(t)$$

Recalling (14.3.9), we deduce

$$e\big((1-\tau)t_1 + \tau t\big) \leq e(t_1)^{(1-\tau)} + e(t)^{\tau}$$

and so

$$0 \leq e\big((1-\tau)t_1 + \tau t\big) \leq e(t_1)^{(1-\tau)} + e(t)^{\tau} \ (0 < \tau < 1$$

But in view of (14.3.8) this inequality implies e(t) = 0 for all times $t_1 \leq t \leq t_2$, a contradiction.

14.4 Green's Function

We saw earlier how Green's functions may be employed with advantage in the determination of solutions of Laplace's equation. We proceed now to show how a similar function may be used conveniently in the mathematical theory of diffusion processes.

Suppose we are considering the solution $\theta(r, t)$ of the diffusion equation

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta \qquad (14.4.1)$$

in the volume V, which is bounded by the simple surface S, subject to the boundary condition

$$\theta(r, t) = \phi(r, t) \qquad if\ r \in S \qquad (14.4.2)$$

and the initial condition

$$\theta(r,0) = f(r) \qquad if\ r \in V \tag{14.4.3}$$

We then define the Green's function G(r, r', t - t') (t > t') of our problem as the function which satisfies the equation

$$\frac{\partial G}{\partial t} = \kappa\nabla^2 G \tag{14.4.4}$$

the boundary condition

$$G(r, r', t - t') = 0 \qquad it\ r' \in S \tag{14.4.5}$$

and the initial condition that $\lim_{t \to t'} G$ is zero at all points of V except at the point r where G takes the form

$$\frac{1}{8\left(\pi\kappa(t-t')\right)^{3/2}} \exp\left(-\frac{(r-r')^2}{4\kappa(t-t')}\right) d\xi \tag{14.2.6}$$

Because G depends on t only in that it is a function of t - t', it follows that equation (14.2.4) is equivalent to

$$\frac{\partial G}{\partial t'} + \kappa\nabla^2 G = 0 \tag{14.4.7}$$

The physical interpretation of the Green's function G is obvious from these equations : ' G(r, r', t -- t') is the temperature at r' at time t due to an instantaneous point source of unit strength generated at time t' at the point r, the solid being initially at zero temperature, and its surface being maintained at zero temperature.

Since the time t' lies within the interval of t for which equations (14.4.1) and (14.4.2) are valid, we may rewrite these equations in the form

$$\frac{\partial \theta}{\partial t'} = \kappa\nabla^2\theta \qquad (t' < t) \tag{14.4.8}$$

$$\theta(r', t') = \phi(r', t') \qquad if\ r' \in S \tag{14.4.9}$$

It follows immediately from equations (14.4.7) and (14.4.8) that

$$\frac{\partial(\theta G)}{\partial t'} = \theta\frac{\partial G}{\partial t'} + G\frac{\partial\theta}{\partial t'} = \kappa(G\nabla^2\theta - \theta\nabla^2 G)$$

so that if ϵ is an arbitrarily small positive constant,

$$\int_0^{t-\epsilon}\left\{\int_V \frac{\partial(\theta G)}{\partial t'} d\tau'\right\} dt' = \int_0^{t-\epsilon}\{\kappa(G\nabla^2\theta - \theta\nabla^2 G)d\tau'\}dt' \tag{14.4.10}$$

If we interchange the order in which we take the integrations on the left -hand side, we find that it takes the form

$$\int_V (\theta G)_{t'=t-\epsilon}\, d\tau' - \int_V (\theta G)_{t'=0} d\tau' = \theta(r,t)\int_V [G(r, r', t-t']_{t'=t-\epsilon} d\tau' - \int_V G(r, r', t') f(r') d\tau'$$

Now from the expression (14.4.6) for G(r, r', t - t') we can readily show that

$$\int_V [G(r, r', t-t']_{t'=t-0} d\tau' = 1$$

so that if we let $\epsilon \to 0$, the left -hand side of equation (14.4.10) becomes

$$\theta(r,t) - \int_V G(r, r', t') f(r') d\tau'$$

On the other hand, if we apply Green's theorem to the right -hand side of equation (14.4.10) and make use of equations (14.4.2) and (14.4.5), we find that it reduces to

$$-\kappa\int_0^t dt' \int_S \phi(r', t)\frac{\partial G}{\partial n}\, dS'$$

in the limit as $\epsilon \to 0$. It will be recalled that a an denotes differentiation along the outward -drawn normal to S. We therefore obtain finally

$$\theta(r,t) = \int_V f(r')G(r,r',t)d\tau' - \kappa \int_0^t dt' \int_S \phi(r',t)\frac{\partial G}{\partial n}\, dS' \qquad (14.4.11)$$

as the solution of the boundary value problem formulated in equations (14.4.1), (14.4.2), and (14.4.3).

Summary

- The heat (diffusion) equation with its physical interpretation isdiscussed.
- Theelementary and fundamental solutions are determined.
- The energy method to find the uniqueness of boundary value problem is derived
- The Green' function for diffusion equation is elaborated.

Keywords

- Diffusion equation
- Elementary solution
- Fundamental solution
- Energy method
- Green's function

Self Assessment

1. The function defined below for $x \in R^n, x \neq 0,$ is the fundamental solution of Heat Equation.

A. $\varphi(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ (x \in R^n, t \in R)$

B. $\varphi(x) = -\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ (x \in R^n, t \in R)$

C. $\varphi(x) = -\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}},\ (x \in R^n, t \in R)$

D. $\varphi(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}},\ (x \in R^n, t \in R)$

2. The solution of heat equation in invariant under

A. Dilation scaling
B. Rotational scaling
C. Magnification scaling
D. None of these

3. The energy function for the diffusion equation over the domain U is given by

A. $e(t) = -\int w^2(x,t)dx$
B. $e(t) = \int w^2(x,t)dx$
C. $e(t) = \int w_t^2(x,t)dx$
D. $e(t) = -\int w_t^2(x,t)dx$

4. The one dimensional diffusion equation is defined as

A. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial\theta}{\partial x}$

B. $\kappa \frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}$

C. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial^2\theta}{\partial x^2}$

D. $\kappa \frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial x}$

5. The function $\theta = \frac{1}{\sqrt{t}} exp\left(-\frac{x^2}{4\kappa t}\right)$ is the solution of the equation

A. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial\theta}{\partial x}$

B. $\kappa \frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}$

C. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial^2\theta}{\partial x^2}$

D. $\kappa \frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial x}$

E. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial\theta}{\partial x}$

6. The function $\phi(x)$ is bounded for all real value for x then the integral formula $\frac{1}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} \phi(\xi) exp\{-\frac{(x-\xi)^2}{4\kappa t}\} d\xi$ will satisfy the equation

A. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial\theta}{\partial x}$

B. $\kappa \frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}$

C. $\kappa \frac{\partial^2\theta}{\partial t^2} = \frac{\partial^2\theta}{\partial x^2}$

D. $\kappa \frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial x}$

7. The green function $G(r, r', t-t')$ $(t > t')$ for the diffusion problem is satisfy the equation

A. $\frac{\partial G}{\partial t'} = \kappa\nabla^2 G$

B. $\frac{\partial G}{\partial t'} = -\kappa\nabla^2 G$

C. $\frac{\partial^2 G}{\partial t^2} = \kappa\nabla^2 G$

D. $\frac{\partial^2 G}{\partial t^2} = -\kappa\nabla^2 G$

8. The initial/bounday value problem $\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$ must have

A. There exists at most one solution

B. There exists more than one solution

C. There exists no solution

D. There exists infinite many solution

9. If two temperature distributions on U agree at some time T > 0, and had the same boundary values for times $0 \leq t \leq T$, then
A. these temperature must be identically equal with in U at all earlier times.
B. these temperature must be identically equal with in U at all later times.
C. these temperature must not be identically equal with in U at all earlier times.
D. these temperature must not be identically equal with in U at all later times

10. Which kind of boundary value problems holds backward uniqueness?
A. Heat Equation
B. Wave equation
C. Laplace equation
D. None of these

11. Which method helps to prove the uniqueness of solution of boundary value problem of heat equation?
A. Green function method
B. Energy method
C. Elementary method
D. Fundamental method

12. Which kind of solution is not possible for heat equation?
A. Particular unique solution
B. Fundamental solution
C. Elementary solution
D. Radial vector solution

13. For the heat equation , the solution u having the special structure in term of function $v: R^n \to R$.where α, β are constants
A. $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) (x \in R^n, t > 0)$
B. $u(x,t) = \frac{\alpha}{t^\beta} v\left(\frac{x}{t^\beta}\right) (x \in R^n, t > 0)$
C. $u(x,t) = \frac{\alpha}{t^\alpha} v\left(\frac{x}{t^{\alpha\beta}}\right) (x \in R^n, t > 0)$
D. $u(x,t) = \frac{1}{t^{\alpha\beta}} v\left(\frac{x}{t^\beta}\right) (x \in R^n, t > 0)$

AnswersforSelf Assessment

l.	A	2.	A	3.	B	4.	B	5.	B
6.	B	7.	B	8.	A	9.	A	10.	A
11.	B	12.	D	13.	A				

Review Questions

1. Derive the fundamental solution using dilation scaling for heat equation.

2. Derive elementary solution for heat equation.
3. Discuss the uniqueness of solution using energy method.
4. State and prove the backward uniqueness.
5. Derive the Green's function for diffusion problem.

Further Readings

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LOVELY PROFESSIONAL UNIVERSITY
Jalandhar-Delhi G.T. Road (NH-1)
Phagwara, Punjab (India)-144411
For Enquiry: +91-1824-521360
Fax.: +91-1824-506111
Email: odl@lpu.co.in