Complex Analysis - I

DEMTH542

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Unit 01: Complex Function

CONTENTS

Objectives

The study of functions of a complex variable is an attempt to extend calculus to the complex space. We will now look at the functions of a complex variable and build a limit and differentiation theory for them. The section's major purpose is to introduce analytic functions, which are essential in complex analysis.

After this unit, you would be able to

- understand the concept of complex functions and their different forms and types
- learn the limit and continuity of a complex-valued function
- explain the differentiability and Analyticity of a complex function
- compute the domain where a complex-valued function is continuous, differentiable, and analytic

Introduction

We explore functions $f(x)$ of a real variable x in one-variable calculus. Similarly, in complex analysis, we investigate functions $f(z)$ of a complex variable $z \in C$. (or in some region of C).

A complex variable is a symbol, such as z, that may represent any of a set of complex integers. Assume that each value of a complex variable z corresponds to one or more values of a complex variable w. Then we claim that w is a function of z and write $w = f(z)$. The variable z is sometimes referred to as an independent variable, whereas the variable w is referred to as a dependent variable.

In this unit of complex variable functions, we shall show how the operations of taking a limit and using differentiation rules, finding the derivatives, which we are familiar with for functions of a real variable, extend in a natural way to the complex plane.

1.1 Complex Functions

Let the complex variable Z be defined by $Z = x + iy$ where x and y are real variables and i is, as usual, given by$i^2 = 1$.

Now let a second complex variable w be defined by $w = u + iv$ where u and v are real variables. If there is a relationship between w and z such that to each value of z in a given region of the z −plane there is assigned one, and only one, the value of w then w is said to be a function of z, defined on the given region.

In this case, we write $w = f(z)$.

Example: Consider $w = z^2 - z$,

which is defined for all values of z (that is, the right-hand side can be computed for every value of z).

Then, remembering that $z = x + iy$,

$$\begin{aligned} w &= u + iv \\ &= (x + iy)^2 - (x + iy) \\ &= x^2 + 2ixy - y^2 - x - iy. \end{aligned}$$

Hence, equating real and imaginary parts: $u = x^2 - x - y^2$ and $v = 2xy - y$.

Question: Compute the value of $w = z^2 - z$ for $z = 2 + 3i$?

Solution

$$\begin{aligned} w &= z^2 - z \\ &= x^2 + 2ixy - y^2 - x - iy \\ &= 2^2 + 2i * 2 * 3 - 3^2 - 2 - 3i \\ &= -7 + 9i \end{aligned}$$

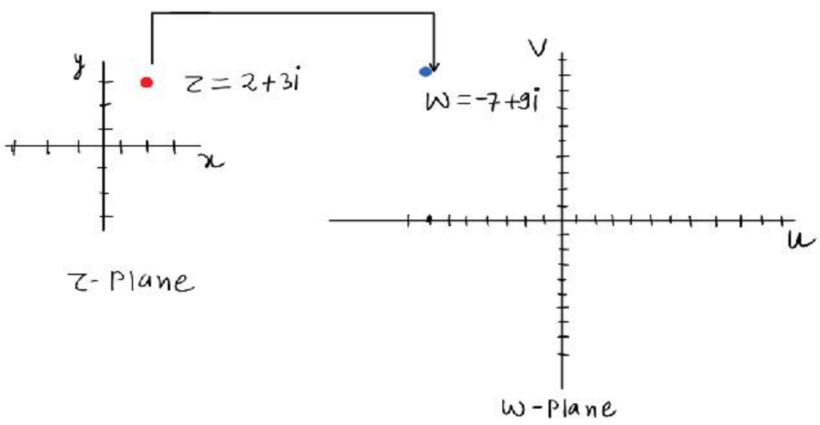

Figure 1: The mapping of z = 2+3i in the w plane by w = z²-z

Task: (i) For which values of z is $w = 1/z$ defined?

(ii) For these values obtain u and v and evaluate w when $z = 2 - i$.

Solution

(a) $w \text{ is defined for all } z \neq 0.$

(b) $Let\ w = u + iv$

$$u + iv = \frac{1}{z} = \frac{1}{x + iy}$$

$$= \frac{1}{x + iy} \cdot \frac{(x - iy)}{(x - iy)}$$

$$= \frac{x - iy}{x^2 + y^2}$$

$$Hence\ u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2}.$$

$$If\ z = 2 - i, \quad then\ x = 2, y = -1$$

$so\ that\ x^2 + y^2 = 5.\ Then\ u = 2/5, v = -1/5\ and\ w = 2/5 - 1/5\, i.$

1.2 The Exponential Function

Let z be the complex variable then using Euler's relation we are led to define

$e^z = e^{x+iy}$

$e^z = e^x . e^{iy}$

$e^z = e^x . (Cosy + iSiny)$

Task: Find the solutions for z of the equation $e^z = i$

$e^z = e^x . (Cosy + iSiny)$

$$e^x . (Cosy + iSiny) = 0 + i$$

$e^x . Cosy = 0 \Rightarrow y = (2k + 1)\pi/2, \forall k \in z$

$e^x . Siny = i \Rightarrow e^x . \frac{Sin(2k+1)\pi}{2} = 1 \Rightarrow x = 0.$

$z = 0 + \frac{i(2k+1)\pi}{2}.$

1.3 Trigonometric Functions

We denote the complex counterparts of the real trigonometric functions cos x and sin x by cos z and sin z and we define these functions by the relations:

$e^{iz} = Cosz + iSinz$, and

$e^{-iz} = Cosz - iSinz.$

$Cosz = (e^{iz} + e^{-iz})/2$, and

$Sinz = (e^{iz} - e^{-iz})/2i$

Example:

Prove that $Cos^2 z + Sin^2 z = 1$

Solution

$$Cos^2z + Sin^2z = (e^{i2z} + e^{-i2z} + 2)/4 - (e^{i2z} + e^{-i2z} - 2)/4$$

= 1

1.4 Logarithmic Function

Since the exponential function is one-to-one it possesses an inverse function, which we call ln z.

If w = u + iv is a complex number such that $e^w = z$ then the logarithm function is defined through the statement: w = ln z. To see what this means it will be convenient to express the complex number z in exponential form as $z = re^{i\theta}$ and so

$$w = u + iv$$
$$= ln\, ln(re^{i\theta})$$
$$= ln\, r + i\theta$$

Example

(a) $ln\, ln(1 + i) = ln\, ln\left(\sqrt{2}e^{\frac{i\pi}{4}}\right) = ln\, ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right) = \frac{lnln2}{2} + i\,(\pi/4 + 2k\pi)$

(b) $If\ ln\, lnz = 1 - i\pi\ then\ z = e^{1-i\pi} = e^1.e^{-i\pi} = -e.$

1.5 The Limit of a Complex Function

The limit of $w = f(z) as\ z \to z_0$ is a number l such that $|f(z) - l|$ can be made as small as we wish by making $|z - z_0|$ suﬃciently small. With the function *f (z)* we are allowed to approach the point $z = z_0$ along any path in the z-plane; we require merely that the distance $|z - z_0|$ decreases to zero.

In some cases, the limit is simply f (z_0).

Example

Let $w = z^2 - z$, the limit of this function as $z \to i$ is $f(i) = i^2 - i = -1 - i$.

Task:

(a) Find the $\frac{z^3 - iz^2 + z - i}{z - i}$

(b) Find the $\frac{z}{\underline{z}}$ where $\underline{z} = (x - iy)$ is the complex conjugate of z

Solution

(a) $\frac{z^3 - iz^2 + z - i}{z - i}$

$$= \frac{z^2(z - i) + z - i}{z - i}$$

$$= \frac{(z^2 + 1)(z^2 - i)}{z - i} = 0$$

(b) $\frac{z}{\underline{z}} = \frac{x + iy}{\underline{x - iy}} = \frac{x + iy}{x - iy} = \frac{x}{x} = 1, and$

$\frac{x + iy}{x - iy} = \frac{iy}{-iy} = -1.$ Hence limit does not exist.

1.6 Continuity

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ as well as at $z = z_0$. The function $f(z)$ is said to be continuous at $z = z_0$ if $f(z) = f(z_0)$.

Note that this implies three conditions that must be met so that $f(z)$ be continuous at $z = z_0$:

1. $f(z)$ must exist
2 $f(z_0)$ must exist, i.e., $f(z)$ is defined at z_0
3. $f(z) = f(z_0)$

Equivalently, if f (z) is continuous at z_0, we can write this in the suggestive form

Example

The function $f(z) = \frac{z}{z^2+4}$ is discontinuous at $z = \pm 2i$ as $f(2i)$ and $f(-2i)$ do not exist.

Task: Check the continuity of $f(z) = \{z^2 z \neq i0z = i\}$?

Solution

Here $f(i) = 0$ and $f(z) = i^2 = -1$. So $f(z)$ is discontinuous at $z = i$

Task: Check the continuity of $f(z) = \left\{\frac{(z^2+4)}{z-2i} z \neq 2i3 + 4iz = 2i\right\}$?

Solution

Here $f(2i) = 3 + 4i$ and $f(z) = \frac{(z-2i)(z+2i)}{z-2i} = 4i$. So $f(z)$ is discontinuous at $z = 2i$

Task: Check the continuity of $f(z) = \frac{z^2+1}{z^2-3z+2}$?

Solution

Here $f(z)$ is not defined as $z^2 - 3z + 2 = 0$ .So $f(z)$ is discontinuous at $z = 2$ and $z = 1$. We can also say $f(z)$ is continuous for all z outside of $|z| = 2$

1.7 Differentiability

The function $f(z)$ is said to be differentiable at $z = z_0$ if the limit $\left\{\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z}\right\}$ exist.

Here $\Delta z = \Delta x + i\Delta y$. The derivative of $f(z)$ at $z = z_0$ is denoted by $\left(\frac{df}{dz}\right)_{z=z_0}$ or by $f'(z_0)$

Singular Point

A point at which the derivative of function $f(z)$ does not exist is called a singular point of the function.

A. Analytic function

A function $f(z)$ is said to be analytic at a point z_0 if it is di erentiable throughout a neighborhood of z_0, however small.

Notes: (A neighborhood of z_0 is the region contained within some circle $|z - z_0| = r$.

Example: The function $f(z) = \frac{1}{Z^2+1}$ has the

$f'(z) = -\frac{2z}{(z^2+1)^2}$. It is clear that $f'(z)$ is not defined at $z = -i$, and $z = i$. So, the singular points of $f(z) = \frac{1}{Z^2+1}$ are $z = -i$, and $z = i$ and it is not analytic at $z = -i$, and $z = i$.

Task: Using differentiation rules, find the derivatives of $f(z) = e^z$

Solution

$$f(z) = e^z$$

$$f(z_0) = e^{z_0}$$

$$f(z_0 + \Delta z) = e^{(z_0+\Delta z)}$$

Now

$$f'(z_0) = \left\{\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{e^{(z_0+\Delta z)} - e^{z_0}}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{e^{z_0}e^{\Delta z} - e^{z_0}}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{e^{z_0}(e^{\Delta z} - 1)}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{e^{z_0}(1 + \frac{\Delta z}{1!} + \frac{(\Delta z)^2}{2!} + \cdots, -1)}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{e^{z_0}\ \Delta z(\frac{1}{1!} + \frac{(\Delta z)^1}{2!} + \frac{(\Delta z)^2}{3!} \ldots,)}{\Delta z}\right\}$$

$$f'(z_0) = e^{z_0}\left\{\left(1 + \frac{(\Delta z)^1}{2!} + \frac{(\Delta z)^2}{3!} \ldots,\right)\right\}$$

$$f'(z_0) = e^{z_0}$$

Task: Using differentiation rules, find the derivatives of $f(z) = z^2$

Solution

$$f(z) = z^2$$

$$f(z_0) = z_0^2$$

$$f(z_0 + \Delta z) = (z_0 + \Delta z)^2$$

Now

$$f'(z_0) = \left\{\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{z_0^2 + (\Delta z)^2 + 2z_0\Delta z - z_0^2}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{\Delta z(\Delta z + 2z_0)}{\Delta z}\right\}$$

$$f'(z_0) = \{\Delta z + 2z_0\}$$

$$f'(z_0) = 2z_0$$

Task: Show that the function $f(z)$ = $\underline{z}$ is not analytic anywhere in the complex plane.

Solution

$$f(z) = \underline{z}$$

$$f(z) = \underline{x + iy}$$

$$f(z) = x - iy$$

$$z_0 = x_0 + iy_0$$

$$\Delta z = \Delta x + i\Delta y$$

$$f\ (z_0) = x_0 - iy_0$$

$$f\ (\Delta z) = \Delta x - i\Delta y$$

$$f(z_0 + \Delta z_0) = x_0 + \Delta x - i(y_0 + \Delta y)$$

Now

$$f'(z_0) = \left\{\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z}\right\}$$

$$f'(z_0) = \left\{\frac{x_0 + \Delta x - i(y_0 + \Delta y) - (x_0 - iy_0)}{\Delta x - i\Delta y}\right\}$$

$$f'(z_0) = \left\{\frac{x_0 + \Delta x - iy_0 + i\Delta y - x_0 + iy_0}{\Delta x - i\Delta y}\right\}$$

$$f'(z_0) = \left\{\frac{\Delta x + i\Delta y}{\Delta x - i\Delta y}\right\}$$

$f'(z_0) = \left\{\lim_{\Delta y \to 0} \left(\frac{\Delta x + i\Delta y}{\Delta x - i\Delta y}\right)\right\} = \left(\frac{\Delta x}{\Delta x}\right) = 1$

And

$f'(z_0) = \left\{\lim_{\Delta x \to 0} \left(\frac{\Delta x + i\Delta y}{\Delta x - i\Delta y}\right)\right\} = \left(-\frac{i\Delta y}{i\Delta y}\right) = -1$

Here the left- and right-hand side limit does not exist so $f'(z_0) = \left\{\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z}\right\}$ does not exist.

Notes:

- If a function is analytic at any given point in the complex plane, then it's also differentiable at the given point.
- If a function is differentiable at any given point in the complex plane, then it's also continuous at the given point.
- If a function is continuous at any given point in the complex plane, then the limit also exists at the given point.

Summary

- The limit of $w = f(z) as\ z \to z_0$ is a number l such that $|f(z) - l|$ can be made as small as we wish by making $|z - z_0|$ su ciently small.
- Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ as well as at $z = z_0$. The function $f(z)$ is said to be continuous at $z = z_0$ if $f(z) = f(z_0)$.
- The function $f(z)$ is said to be differentiable at $z = z_0$ if the limit $\left\{\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z}\right\}$ exist.
- A point at which the derivative of function $f(z)$ does not exist is called a singular point of the function.
- A function $f(z)$ is said to be analytic at a point z_0 if it is di erentiable throughout a neighborhood of z_0, however small.
- If a function is analytic at any given point in the complex plane, then it's also differentiable at the given point.
- If a function is differentiable at any given point in the complex plane, then it's also continuous at the given point.
- If a function is continuous at any given point in the complex plane, then the limit also exists at the given point.

Keywords

Limit: The limit of $w = f(z) as\ z \to z_0$ is a number l such that $|f(z) - l|$ can be made as small as we wish by making $|z - z_0|$ su ciently small.

Continuity: $f(z)$ be continuous at $z = z_0$:

1. $f(z)$ must exist
2 $f(z_0)$ must exist, i.e., $f(z)$ is defined at z_0
3. $f(z) = f(z_0)$

Differentiability: The function $f(z)$ is said to be differentiable at $z = z_0$ if the limit $\left\{\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z}\right\}$ exist.

Analytic function: function $f(z)$ is said to be analytic at a point z_0 if it is di erentiable throughout a neighborhood of z_0.

Self Assessment

1. What is the value of $\frac{z^2-4}{z-2}$?

A. 2

B. 4

C. 1

D. 0

2. What is the value of $\frac{z^2-3z-4}{z-3}$?

A. 2

B. 0

C. 1

D. I

3. Which one of the following is true?

A. If the limit of any function exists at a point, then the function is also continuous at the same point
B. If the limit of any function does not exist at a point, then the function is continuous at the same point
C. If a function is continuous at a point then the limit of function exists at the same point
D. If a function is continuous at a point then the limit of function does not exist at the same point

4. The function $f(z) = e^z$ is continuous at

A. Everywhere in the complex plane
B. Nowhere in the complex plane
C. In only the positive quadrant of the complex plane
D. In only the negative quadrant of the complex plane

5. The function $f(z) = Sinz/z$ is continuous at z=0

A. True
B. False

6. Which one of the following is true for the real part(x) value to solve $e^z = 1$?

A. x=1
B. x=2
C. x=0
D. x=3

7. Which one of the following is true for the imaginary part(y) value to solve $e^z = 1$?

A. y=1
B. y=$n\pi, nisodd$
C. y=$n\pi, niseven$
D. y=3

8. The points where $f(z) = 1/(z-5)$ is not defined?

A. z=1
B. z=2
C. z=5
D. z=3

9. Which one of the following is true for the real part(x) value to solve $e^z = i$?

A. x=1
B. x=2
C. x=0
D. x=3

10. Which one of the following is true for the imaginary part(y) value to solve $e^z = i$?

A. y=1
B. y=$\frac{(n+1)\pi}{2}, n \text{ is odd}$
C. y=$\frac{(n+1)\pi}{2}, n \text{ is even}$
D. y=3

11. Which of the following is true about $f(z) = z^2$?

A. Continuous and differentiable
B. Continuous but not differentiable
C. Neither continuous nor differentiable
D. Differentiable but not continuous?

12. Which of the following is true about $f(z) = z + iz$?

A. Continuous and differentiable
B. Continuous but not differentiable
C. Neither continuous nor differentiable
D. Differentiable but not continuous

13. Which of the following is true about $f(z) = \frac{z+iz}{z^2}$?

A. Continuous and differentiable
B. Continuous but not differentiable
C. Neither continuous nor differentiable
D. Differentiable but not continuous

14. Which of the following is one of the singular points $f(z) = \frac{iz}{z^2-4}$?

A. 3

B. 4

C. 2

D. 1

15. Which of the following is one of the singular points $f(z) = \frac{i}{z^2+4}$?

A. 3

B. 4

C. 2i

D. 1

Answers for Self Assessment

1.	B	2.	B	3.	C	4.	A	5.	A
6.	C	7.	C	8.	C	9.	C	10.	C
11.	A	12.	A	13.	C	14.	C	15.	C

Review Questions

1. Evaluate the $z^2 - 5z + 10$
2. Evaluate the $\frac{z^3+8}{z^4+4z^2+16}$
3. Evaluate the $\frac{3z^4-2z^3+8z^2-2z+5}{z-i}$
4. Let $w = f(z) = z(2-z)$. Find the values of w corresponding to $z = \pi i$.
5. Let $w = f(z) = z/(2-z)$. Find the values of w corresponding to $z = 2i$.
6. Check the continuity of $f(z) = -Cosz$
7. Suppose $f(z)$ *and* $g(z)$ is continuous at $z = a$. Prove that $3f(z) - 4ig(z)$ is also continuous at $z = a$
8. Check the continuity of $f(z) = \frac{3z^2+4}{z^4-16}$
9. Check the continuity of $f(z) = \frac{z^2+1}{z^3+9}$
10. Using the definition, find the derivative of function $f(z) = 3z^2$ at the $z = 2$
11. Using the definition, find the derivative of function $f(z) = Sinz$ at the $z = \frac{\pi}{2}$
12. Using the definition, find the derivative of function $f(z) = 1/z$ at the $z = \frac{1}{2}$
13. Find the points where the $\frac{3z^4-2z^3+8z^2-2z+5}{z-i}$ is not analytic?
14. Find the points where the $\frac{2z+5}{z^2-1}$ is not analytic?
15. Check whether the function $f(z) = z(2-z)$ is analytic everywhere?

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

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3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 02: Cauchy-Riemann Equations

CONTENTS

Objectives

The study of functions of a complex variable is an attempt to extend calculus to the complex space. We will now look at the functions of a complex variable and build a limit and differentiation theory for them. The section's major purpose is to introduce analytic functions, which are essential in complex analysis.

After this unit, you would be able to

- understand the concept of the Cauchy-Riemann equation and their different forms
- learn the necessary conditions for a complex-valued function to be analytic at a point
- explain the sufficient conditions for a complex-valued function to be analytic at a point
- derive the Polar form of the Cauchy-Riemann equation

Introduction

When considering real-valued functions $f(x,y) : R^2 \to R$ of two variables, there is no notion of 'the' derivative of a function. For such functions, we instead only have partial derivatives $\frac{\partial f(x,y)}{\partial x}$ *and* $\frac{\partial f(x,y)}{\partial y}$ (and also directional derivatives) which depend on how we approach a point $(x,y) \in R^2$.

For a complex-valued function $f(z) = f(x,y): C \to R$, we now have a new concept of derivative, $f'(z)$, which by definition cannot depend on how we approach a point $(x,y) \in C$. It is logical, then, that there should be a relationship between the complex derivative $f'(z)$and the partial derivatives $\frac{\partial f(z)}{\partial x}$ *and* $\frac{\partial f(z)}{\partial y}$ (defined exactly as in the real-valued case). The relationship between the complex derivative and partial derivatives is very strong and is a powerful computational tool. It is described by the Cauchy–Riemann Equations, named after Augustin Louis Cauchy (1789–1857) and Georg Friedrich Bernhard Riemann (1826–1866). In this unit of complex variable functions, we shall understand the concept of the Cauchy-Riemann equation and their different forms and learn the necessary and sufficient conditions for a complex-valued function to be analytic at a point and then extend in a natural way to derive the Polar form of the Cauchy-Riemann equation.

2.1 The Cauchy-Riemann Equations

Let the $z = x + iy$ and $w = u(x, y) + iv(x, y)$. A function $w = f(z)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at this point, the first-order partial derivative of $u(x, y)$ and $v(x, y)$ exist and satisfy Cauchy-Riemann equations. These state that w= f(z) is differentiable at a point $z = z_0$if, and only if,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{and} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof:

If $f(z)$ is differential at z then according to the definition of differentiability

$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists and unique along every path along which $\Delta z \to 0$

$$f'(z) = \lim_{\Delta x \to 0 \Delta y \to 0} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y}$$

Along the path $\Delta x = 0$.

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y + \Delta y) + iv(x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{0 + i\Delta y}$$

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y + \Delta y) - u(x, y)\} + \{iv(x, y + \Delta y) - iv(x, y)\}}{i\Delta y}$$

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y + \Delta y) - u(x, y)\}}{i\Delta y} + \lim_{\Delta y \to 0} \frac{\{v(x, y + \Delta y) - v(x, y)\}}{\Delta y}$$

$$f'(z = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Along the path $\Delta y = 0$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x + \Delta x, y) + iv(x + \Delta x, y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i0}$$

$$f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x + \Delta x, y) - u(x, y)\} + \{iv(x + \Delta x, y) - iv(x, y)\}}{\Delta x}$$

$$f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x + \Delta x, y) - u(x, y)\}}{\Delta x} + \lim_{\Delta x \to 0} \frac{\{iv(x + \Delta x, y) - iv(x, y)\}}{\Delta x}$$

$$f'(z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$$

Since $f(z)$ is differential, the two limits in equation (2.7) and (2.11) are equal:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equate real and imaginary part

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ **which is the Cauchy-Riemann(C-R) equation.**

2.2 The Necessary Condition for Analyticity of Complex Functions

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic or differentiable in a region R is that, in R, u *and* v satisfy the Cauchy- Riemann equations i.e.: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

- The C-R equations are necessary conditions for a function to be differentiable or analytic at a point. Thus, a function not satisfying C-R equations at a point will neither be differentiable nor analytic at that point.
- These conditions are not sufficient. Thus, there exist functions that satisfy C-R equations at a point but are not differentiable at that point.

2.3 Sufficient Condition for Analyticity of Complex Function

The sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic at a point z is

1. $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, and\ \frac{\partial v}{\partial x}$ are a continuous function of x *and* y in a certain neighborhood of z.

2. The C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are satisfied in the neighborhood of z.

Example 1:

Show that $\boldsymbol{w} = \boldsymbol{e^z}$ is analytic in the entire complex plane

Solution

$$w = e^z$$
$$= e^{x+iy}$$

$= e^x.e^{iy} = e^x(cos\ cosy + isiny)$

$= e^x cos\ y + i\ e^x sin\ y$. Here

$$u = e^x \cos y$$
$$v = e^x \sin y$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x cos\ y$, and

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y$

Here The C-R equations are satisfied in the neighborhood of z.

Also $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, and\ \frac{\partial v}{\partial x}$ being polynomial is a continuous function of x *and hence* $w = e^z$ ***is analytic in the entire complex plane.***

Task: Show that the simple function $f(z) = z = x - iy$ is not analytic anywhere in the complex plane.

Solution

Here $f(z) = x - iy$ and

$u = x,\ v = -y$

$\frac{\partial u}{\partial x} = 1$, and $\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial x} = -1$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$

The C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$are not satisfied so function $f(z)$ is not analytic anywhere in the complex plane.

Task: Show that$w = z^2 - z$is analytic in the entire complex plane

Solution

$w = (x + iy)^2$- $(x + iy)$

$= x^2 - y^2 + 2xyi - x - iy$

$= x^2 - y^2 - x + (2xy - y)i$

Here $u = x^2 - y^2 - x$, and $v = (2xy - y)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$ **and** $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$

Here The C-R equations are satisfied in the neighborhood of z.

Also $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$, *and* $\frac{\partial v}{\partial x}$ being polynomial is a continuous function of *x and hence* $w = z^2 - z$ ***is analytic in the entire complex plane.***

Task: Check whether$w = 1/z$is analytic in the entire complex plane

Solution:

$w = \frac{1}{z} = \frac{1}{x + iy} \times \frac{x - iy}{x - iy}$

$$= \frac{x}{x^2 + y^2} - \frac{i\, y}{x^2 + y^2}$$

Here $u = \frac{x}{x^2 + y^2}$ and $v = -\frac{y}{x^2 + y^2}$

$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2) - x \times 2x}{(x^2 + y^2)^2}$

$\frac{\partial v}{\partial y} = \frac{-(x^2 + y^2) + y \times 2x}{(x^2 + y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{0 - x \times 2y}{(x^2 + y^2)^2}$

$\frac{\partial v}{\partial x} = \frac{0 + y \times 2x}{(x^2 + y^2)^2}$

$u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$, *and* $\frac{\partial v}{\partial x}$ are not defined at $z = 0$. So w is not analytic at $z = 0$.

2.4 Polar Form Cauchy- Riemann Equations.

The polar form of the Cauchy-Riemann equations is-

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Proof:

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$.

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$

The Cauchy- Riemann equations.

Remembering that $z = x + iy$ *and* $w = u + iv$ *a function* $w = f(z)$ *is analytic at a point.*

This is provided by the Cauchy –Riemann equations

These state that $w = f(z)$ *is differentiable at a point* $z = z_0$ *if, and only if,*

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at that point.

Polar Form of Cauchy- Riemann Equations.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

As $r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times 2x^2 = \left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{x}{r} = \cos\theta$

$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}.2y = \left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{y}{r} = \sin\theta$

$\frac{\partial \theta}{\partial x} = \frac{1}{\frac{x^2+y^2}{x^2}} \times \left[\frac{0-y}{x^2}\right] = \left(\frac{-y}{x^2+y^2}\right) = -\frac{1}{r}\sin\theta$

$\frac{\partial \theta}{\partial y} = \frac{1}{\frac{x^2+y^2}{x^2}} \times \frac{1}{x} = \left(\frac{x}{x^2+y^2}\right) = \frac{1}{r}\cos\theta$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r} \times \cos\theta + \frac{\partial u}{\partial \theta}\left(\frac{-1}{r}\sin\theta\right)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r} \times \sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{1}{r}\cos\theta\right)$

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r} \times \cos\theta + \frac{\partial v}{\partial \theta}\left(\frac{-1}{r}\sin\theta\right)$

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r} \times \sin\theta + \frac{\partial v}{\partial \theta}\left(\frac{1}{r}\cos\theta\right)$

As $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial r} \times \cos\theta + \frac{\partial u}{\partial \theta}\left(\frac{-1}{r}\sin\theta\right) = \frac{\partial v}{\partial r} \times \sin\theta + \frac{\partial v}{\partial \theta}\left(\frac{1}{r}\cos\theta\right)$$

$\Rightarrow \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos^2\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta.\cos\theta = 0$

As $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ so

$$\frac{\partial u}{\partial r} \times \sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{1}{r}\cos\theta\right) = -\left(\frac{\partial v}{\partial r} \times \cos\theta + \frac{\partial v}{\partial \theta}\left(\frac{-1}{r}\sin\theta\right)\right)$$

$\Rightarrow (\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta})\sin^2\theta + (\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta})\cos\theta.\sin\theta = 0$

After adding (2.26) and (2.27) we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

After adding (2.26) from (2.27) we get

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Task: Using the polar form of the C-R equations show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Proof

The polar form of the Cauchy-Riemann equations is-

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Now differentiate both sides of equation (2.30) partially concerning r and both sides of equation (2.31) partially concerning θ

$$\frac{\partial^2 u}{\partial r^2} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}$$

$$\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

Now multiply equation (2.33) by *1/r* and add (2.32)

$$\frac{\partial^2 u}{\partial r^2} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Using C-R equation (2.31)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Summary

- Let the $z = x + iy$ and $w = u(x,y) + iv(x,y)$. A function $w = f(z)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at this point, the first-order partial derivative of $u(x,y)$ and $v(x,y)$ exist and satisfy Cauchy-Riemann equations. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

- A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic or differentiable in a region R is that, in R, u and v satisfy the Cauchy- Riemann
- The C-R equations are necessary conditions for a function to be differentiable or analytic at a point. Thus, a function not satisfying C-R equations at a point will neither be differential nor analytic at that point.
- These conditions are not sufficient. Thus, there exist functions that satisfy C-R equations at a point but are not differential at that point.
- The sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic at a point z is
  a. $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, and\ \frac{\partial v}{\partial x}$ are a continuous function of x and y in a certain neighborhood

  of z.

  b. The C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are satisfied in the neighborhood of z.

- The polar form of the Cauchy-Riemann equations is-
  $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Keywords

Cauchy-Riemann equations: Let the $z = x + iy$ and $w = u(x,y) + iv(x,y)$. then

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The polar form of the Cauchy-Riemann equations -

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Self Assessment

1. Let $z = r.e^{i\theta}$ and $f(z) = u + iv = \sqrt{z}$ then $\frac{\partial u}{\partial r}$ =?

A. $-\frac{\sqrt{r}}{2} sin(\theta/2)$

B. $\frac{1}{2\sqrt{r}} Cos(\theta/2)$

C. $\frac{1}{2\sqrt{r}} sin(\theta/2)$

D. $\frac{\sqrt{r}}{2} Cos(\theta/2)$

2. Let $z = r.e^{i\theta}$ and $f(z) = u + iv = \sqrt{z}$ then $\frac{\partial u}{\partial \theta}$ =?

A. $-\frac{\sqrt{r}}{2} sin(\theta/2)$

B. $\frac{1}{2\sqrt{r}} Cos(\theta/2)$

C. $\frac{1}{2\sqrt{r}} sin(\theta/2)$

D. $\frac{\sqrt{r}}{2} Cos(\theta/2)$

3. Let $z = r.e^{i\theta}$ and $f(z) = u + iv = \sqrt{z}$then $\frac{\partial v}{\partial r}$ =?

A. $-\frac{\sqrt{r}}{2} sin(\theta/2)$
B. $\frac{1}{2\sqrt{r}} Cos(\theta/2)$
C. $\frac{1}{2\sqrt{r}} sin(\theta/2)$
D. $\frac{\sqrt{r}}{2} Cos(\theta/2)$

4. Let $z = r.e^{i\theta}$ and $f(z) = u + iv = \sqrt{z}$then $\frac{\partial v}{\partial \theta}$ =?

A. $-\frac{\sqrt{r}}{2} sin(\theta/2)$
B. $\frac{1}{2\sqrt{r}} Cos(\theta/2)$
C. $\frac{1}{2\sqrt{r}} sin(\theta/2)$
D. $\frac{\sqrt{r}}{2} Cos(\theta/2)$

5. Let $z = r.e^{i\theta}$ and $f(z) = u + iv$ then $\frac{\partial u}{\partial r}$ =?

A. $\frac{\partial v}{\partial \theta}$
B. $\frac{1}{2\sqrt{r}} \frac{\partial v}{\partial \theta}$
C. $\frac{1}{r} \frac{\partial v}{\partial \theta}$
D. $\frac{r}{2} \frac{\partial v}{\partial \theta}$

6. Let $z = x + iy$ and $f(z) = u + iv = e^z$then $\frac{\partial u}{\partial x}$ =?

A. $e^x.Cosy$
B. $e^x.Siny$
C. $-e^x.Cosy$
D. $-e^x.Siny$

7. Let $z = x + iy$ and $f(z) = u + iv = e^z$then $\frac{\partial v}{\partial x}$ =?

A. $e^x.Cosy$
B. $e^x.Siny$
C. $-e^x.Cosy$
D. $-e^x.Siny$

8. The function $f(x + iy) = x^2 + y + i(y^2 - x)$ is …

A. analytic everywhere in the complex plane
B. analytic on the real axis
C. only analytic on the line $y = x$
D. differentiable on the line $y = x$ and nowhere else. So, it is nowhere analytic.

9. Let $z = x + iy$ and $f(z) = u + iv = \underline{z}$ then $\frac{\partial u}{\partial x}$ =?

A. 1

B. 0

C. -1

D. 2

10. Let $z = x + iy$ and $f(z) = u + iv = e^z$ then $\frac{\partial v}{\partial x}$ =?

A. 1

B. 0

C. -1

D. -2

11. The function $f(x + iy) = x^2 - y^2 + i(2xy)$ is …

A. Analytic everywhere in the complex plane

B. Not analytic anywhere in the complex plane

C. Only analytic on the line $y = x$

D. Only analytic on the real axis

12. Let $f(x + iy) = x^2 + y^2$ then

A. The Cauchy Remain equations for $f(x + iy)$ are satisfied $at\ x = 0, and\ y = 0$.

B. The $f(x + iy)$ is analytic.

C. The $f(x + iy)$ is differentiable everywhere.

D. Only analytic on the real axis

13. Let $f(x + iy) = x - 2ay + i(bx - cy)$ is analytic then the value of c is…

A. -1

B. $2k$, k is any real number

C. k, k is any real number

D. $1 + 2i$

14. Let $f(x + iy) = x - 2ay + i(bx - cy)$ is analytic then the value of b is…

A. -1

B. $2k$, k is any real number

C. k, k is any real number

D. $1 + 2i$

15. Let $f(x + iy) = x - 2ay + i(bx - cy)$ is analytic then the value of a is…

A. -1

B. $2k$, k is any real number

C. k, k is any real number

D. $1 + 2i$

Answers for Self Assessment

1.	B	2.	A	3.	C	4.	D	5.	C
6.	A	7.	B	8.	D	9.	A	10.	B
11.	A	12.	A	13.	A	14.	B	15.	C

Review Questions

1. Check whether the Cauchy-Remain equation is satisfied for $f(z) = 5z + 10$
2. Check whether the Cauchy-Remain equation is satisfied for $f(z) = z^2$
3. Find the polar form $f(r, \theta)$ of $f(z) = z + 5$
4. Let $w = f(z) = z(2 - z)$. Find the values of z where $f(z)$ *is not analytic?*
5. Let $w = f(z) = z/(2 - z)$. Find the values of z where $f(z)$ *is not analytic?*
6. Does every function which satisfies the C-R equation is analytic?
7. Show that $f(z) = z^3$ is analytic in the entire complex plane?
8. Show that $f(z) = Sinz$ is analytic in the entire complex plane?
9. Show that $f(z) = z.e^z$ is analytic in the entire complex plane?
10. Check whether $f(z) = \underline{z}$ is analytic everywhere?
11. Check whether $f(z) = |z|^2$ is analytic everywhere?
12. Check whether $f(z) = 1/z$ is analytic at the $z = \frac{1}{2}$
13. Find the points where the $\frac{z^2}{z-1}$ is not analytic?
14. Verify the Cauchy-Remain equation for $f(z) = e^{-z}$?
15. Verify the Cauchy-Remain equation for $f(z) = z(2 - z)$?

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.
2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.
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Unit 03: Harmonic Function

CONTENTS

Objectives

Harmonic functions arise often and serve an important role in mathematics, physics, and engineering and they are governed by their singularities and boundary conditions (such as Dirichlet boundary conditions or Neumann boundary conditions). In areas with no boundary, adding the real or imaginary portion of any whole function generates a harmonic function with the same singularity. The purpose for this section is to have complete understanding of the role of harmonic function in complex domains.

After this unit, you would be able to

- describes the harmonic function of two variables in the provided domain.
- check whether a particular component of a complex function is harmonic in the given domain.
- compute the harmonic conjugate of a particular component of a complex function

Introduction

In this section, we will study the definition, several essential features, of a harmonic function and how they are related to complex analysis. We will learn the fundamental relationship for the analytic function,the real and imaginary portions are both harmonic. We shall demonstrate that this is a straightforward result of the Cauchy-Riemann equations and in the last we will learn how to compute the harmonic conjugate of one of the component of a complex function.

3.1 Harmonic Function

A real-valued function$F(x,y)$of two real variables x *and* y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$\frac{\partial^2 F(x,y)}{\partial x^2}+\frac{\partial^2 F(x,y)}{\partial y^2}=0$$

Example:

The temperatures $T\ (x,y)$ thin plates lying in the xy plane are often harmonic.

It is easy to verify that the function $T\ (x, y)\ =\ e^{-y}\ sin\,x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip(shown in Figure 1) $0\ <\ x\ <\ \pi, y\ >\ 0$.

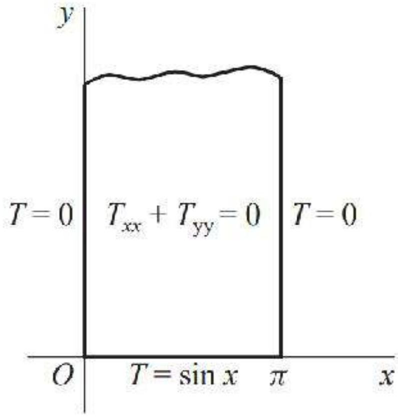

Figure 1: The geometry of plate for temperature distribution.

Here-

$$\frac{\partial T}{\partial x}\ = e^{-y}\ Cos\,x$$

$\frac{\partial^2 T}{\partial x^2}\ = -e^{-y}\ Sin\,x\ .$

$$\frac{\partial T}{\partial y}\ = -e^{-y}\ Sin\,x$$

$\frac{\partial^2 T}{\partial y^2}\ = e^{-y}\ Sin\,x\ .$

And $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

So we can say that T is harmonic and the following graph shows the temperature distribution in x and y direction.

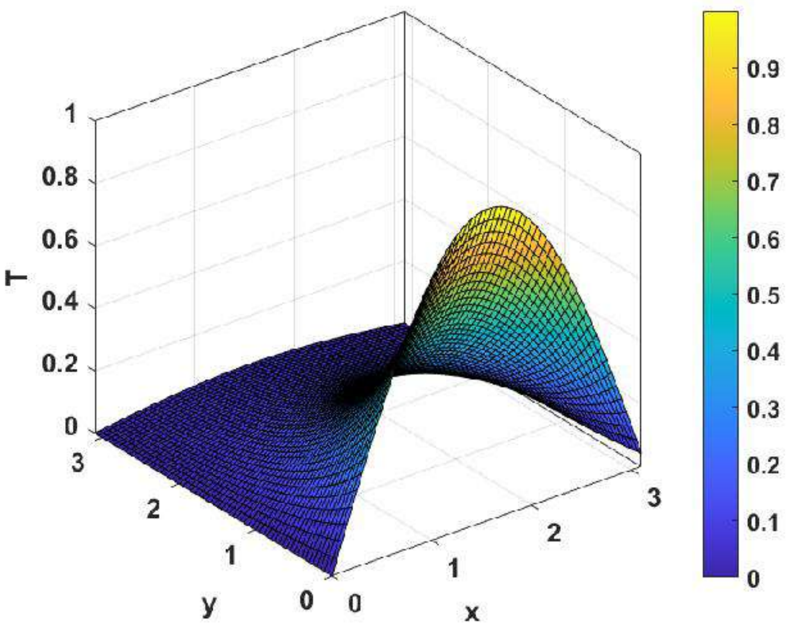

Figure 2: The temperature distribution in x and y direction.

Theorem 1. If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof:

To prove this, we need a result that is , if a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all orders at that point.

Let the $z = x + iy$ and $w = u(x,y) + iv(x,y)$. A function $w = f(z)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself.Assuming that $f(z)$ is analytic in D. Then at this point, the first-order partial derivative of $u(x,y)$ and $v(x,y)$ exist. and satisfy Cauchy-Riemann equations. These state that w= f(z) is differentiable at a point $z = z_0$if, and only if,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating both sides of equation (3.2) with respect to x andequation (3.3) with respect to y, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

As $v$is continuous so $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ .Now add the equations (3.4) and (3.5).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly

Differentiating both sides of equation (3.2) with respect to y andequation (3.3) with respect to x, we have

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

As $u$is continuous so $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ .Now subtract the equations (3.8) from (3.7).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Task: Show that the component of $w = z^2 - z$are harmonic in the entire complex plane

Solution

$w = (x + iy)^2$- $(x + iy)$

$= x^2 - y^2 + 2xyi - x - iy$

$= x^2 - y^2 - x + (2xy - y)i$

Here $u = x^2 - y^2 - x$, and $v = (2xy - y)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$

Here The C-R equations are satisfied in the neighborhood of z.

Also $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$, *and* $\frac{\partial v}{\partial x}$ being polynomial is a continuous function of x *and hence* $w = z^2 - z$ *is analytic in the entire complex plane.*

Now

$\frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial^2 u}{\partial y^2} = -2.$

So $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Proved)

Similarly

$\frac{\partial^2 v}{\partial x^2} = 0$ and $\frac{\partial^2 v}{\partial y^2} = 0.$

So $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ (Proved)

Task: Show that the component of $u = x^2 - y^2 - y$is harmonic.

Solution

Here $u = x^2 - y^2 - y$

$\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = -2y - 1$

Now

$\frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial^2 u}{\partial y^2} = -2.$

So $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Proved)

3.2 Harmonic Conjugate

If two given functions u *and* v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations ($\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) throughout D, then v is said to be a harmonic conjugate of u.

Notes: A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

3.3 Method to find the Harmonic Conjugate of a Function

To find the harmonic conjugate of a funcion$u(x,y)$, assuming that $f(z) = u(x,y) + iv(x,y)$is analytic in D. Then at this domain the first-order partial derivative of $u(x,y)$ and $v(x,y)$ exist. and satisfy Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Holding x fixed and integrating each side of equation (3.10)here with respect to y,

$$\int \left(\frac{\partial u}{\partial x}\right)\partial y + \varphi(x) = v(x,y)$$

Where $v(x,y)$ is the harmonic conjugate of $u(x,y)$ and $\varphi(x)$ is, at present, an arbitrary function of x.

Now partially differentiate each side of equation (3.12) with respect to x and partially differentiate the $u(x,y)$ with respect to y and then using the equation (3.11) find the $\varphi(x)$. Put the $\varphi(x)$ in the equation (3.12) and the $v(x,y)$.

Example:Show that $u(x,y) = y^3 - 3x^2y$is harmonic and then find the harmonic conjugate of $u(x,y)$?

Solution

Given that $u(x,y) = y^3 - 3x^2y$

Here $\frac{\partial^2 u}{\partial x^2} = -6y$, and

$\frac{\partial^2 u}{\partial y^2} = 6$y.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So $u(x,y) = y^3 - 3x^2y$is harmonic.

Now Harmonic Conjugates of $u(x,y)$

As we know that A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

Let $v(x,y)$ is the harmonic conjugate of $u(x,y)$ now using the above results we can say that $f(z)$ is analytic.

If $f(z)$ is analytic then $u(x,y), v(x,y)$ must satisfy the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

As $u(x,y) = y^3 - 3x^2y$.Then

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = -6xy = \frac{\partial u}{\partial x}$$

Holding x fixed and integrating each side of $\frac{\partial v}{\partial y} = -6xy$ with respect to y,

$$\int (-6xy)\partial y + \varphi(x) = v(x,y)$$

So $v(x,y) = -3xy^2 + \varphi(x)$

Now differentiate the obtained $v(x,y) = -3xy^2\varphi(x)$ with respect to x, we have

$$\frac{\partial v}{\partial x} = -3y^2 + \varphi'(x)$$

Now put the expression $\frac{\partial v}{\partial x} = -3y^2 + \varphi'(x)$ in the other form of C-R equation

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 = -\frac{\partial v}{\partial x} = -(-3y^2 + \varphi'(x))$$

Hence

$\varphi'(x) = 3x^2$, and

$$\varphi(x) = \int 3x^2 dx + c$$

$\varphi(x) = x^3 + c$, here c is the arbitrary constant.

Now put the value of $\varphi(x)$ in the $v(x,y) = -3xy^2 + \varphi(x)$

$$v(x,y) = -3xy^2 + x^3 + c$$

Let $c = 0$,then $v(x,y) = -3xy^2 + x^3 +$ is the harmonic conjugate of $u(x,y) = y^3 - 3x^2y$.

Task: Let $u(x,y) = 2x(1-y)$is harmonic and then find the harmonic conjugate $v(x,y)$ of $u(x,y)$ and show that $v(x,y)$ is harmonic?

Solution

Given that $u(x,y) = 2x - 2xy$

Now Harmonic Conjugates of $u(x,y)$

As we know that A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

Let $v(x,y)$ is the harmonic conjugate of $u(x,y)$ now using the above results we can say that $f(z)$ is analytic.

If $f(z)$ is analytic then $u(x,y), v(x,y)$ must satisfy the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

As $u(x,y) = 2x - 2xy$.Then

$$\frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = 2 - 2y = \frac{\partial u}{\partial x}$$

Holding x fixed and integrating each side of $\frac{\partial v}{\partial y} = 2 - 2y$ with respect to y,

$$\int (2-2y)\partial y + \varphi(x) = v(x,y)$$

So $v(x,y) = 2y - y^2 + \varphi(x)$

Now differentiate the obtained $v(x,y) = -3xy^2\varphi(x)$ with respect to x, we have

$$\frac{\partial v}{\partial x} = 0 + \varphi'(x)$$

Now put the expression $\frac{\partial v}{\partial x} = 0 + \varphi'(x)$ in the other form of C-R equation

$$\frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} = -(0 + \varphi'(x))$$

Hence

$\varphi'(x) = 2x$, and

$$\varphi(x) = \int 2xdx + c$$

$\varphi(x) = x^2 + c$, here c is the arbitry constant.

Now put the value of $\varphi(x)$ in the $v(x,y) = 2y - y^2 + \varphi(x)$

$$v(x,y) = 2y - y^2 + x^2 + c$$

Let $c = 0$,then $v(x,y) = 2y - y^2 + x^2$ is the harmonic conjugate of $u(x,y) = 2x(1-y)$.

Now

$\frac{\partial^2 v}{\partial x^2} = 2$, and

$\frac{\partial^2 v}{\partial y^2} = -2$.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

So$v(x,y) = 2y - y^2 + x^2$is harmonic.

Summary

- A real-valued function$F(x,y)$of two real variables x *and* y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$\frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2} = 0$$

- If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions u *and* v are harmonic in D.
- A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.
- If two given functions u *and* v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations ($\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) throughout D,
- then v is said to be a harmonic conjugate of u.

Keywords

Harmonic: A real-valued function$F(x,y)$of two real variables x *and* y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$\frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2} = 0$$

Harmonic conjugate: harmonic conjugate of u If two given functions u *and* v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations ($\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) throughout D, then v is said to be a harmonic conjugate of u.

Self Assessment

1. A real function $f(x,y)$ is called harmonic function in domain D if it satisfies…

A. $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

B. $\frac{\partial f}{\partial x}=0$

C. $\frac{\partial f}{\partial y}=0$

D. $\frac{\partial^2 f}{\partial y^2}=0$

2. If $f(x,y)=u(x,y)+iv(x,y)$ is analytic in some domain D, then …

A. Only $u(x,y)$ is harmonic in *D*

B. Only $v(x,y)$ is harmonic in *D*

C. Both $u(x,y)$ and $v(x,y)$ is harmonic in *D*

D. Neither $u(x,y)$ nor $v(x,y)$ is harmonic in *D*

3. If $f(x,y)=u(x,y)+iv(x,y)$ is analytic in some domain *D*, and $u(x,y)=ax^2-by^2$ then the relation between $a, and\ b$ is …

A. $a=b$

B. $a\neq b$

C. $a+b=0$

D. $ab=2$

4. If $f(x,y)=u(x,y)+iv(x,y)$ is analytic in some domain *D*, and $v(x,y)=x^4+y^4-kx^2y^2$ then what is the value of k such that $v(x,y)$is harmonic?

A. 5

B. 6

C. 7

D. 8

5. If $f(x,y)=u(x,y)+iv(x,y)$ is analytic in some domain *D*, and $u(x,y)=e^{kx}Cosy$ then what is the value of k such that $u(x,y)$is harmonic?

A. ±1

B. 0

C. 2

D. 3

6. If $f(x,y)=u(x,y)+iv(x,y)$ is analytic in some domain *D*, and $u(x,y)=e^{x}Sinky$ then what is the value of k such that $u(x,y)$is harmonic?

A. ±1

B. 0

C. 2

D. 3

7. If $z^2 - z = u(x,y) + iv(x,y)$ is analytic in some domain *D*, and $u(x,y) = x^2 - y^2 + x$ is harmonic?

A. True

B. False

8. If $z^2 - z = u(x,y) + iv(x,y)$ is analytic in some domain *D*, and $u(x,y) = 2xy + y$ is harmonic?

A. True

B. False

9. If $z^2 - z = u(x,y) + iv(x,y)$ is analytic in some domain *D*, and $v(x,y) = x^2 - y^2 + x$ is harmonic?

A. True

B. False

10. If $z^2 - z = u(x,y) + iv(x,y)$ is analytic in some domain *D*, and $v(x,y) = 2xy + y$ is harmonic?

A. True

B. False

11. If $U = x^2 - y^2 - y$ is harmonic and $f(z) = U + iV$ is analytic in entire complex plane then the harmonic conjugate of U is?

A. $x(2y + 1)$ +C

B. xy+C

C. $x(2y - 1)$ +C

D. $x - y$+C

12. If $V = xy$ is harmonic and $f(z) = U + iV$ is analytic in entire complex plane then the harmonic conjugate of U is?

A. $(x^2 - y^2) + C$

B. $x - y + C$

C. $\frac{x^2-y^2}{4} + C$

D. $\frac{x^2-y^2}{2} + C$

13. If $U = 5x + 2xy$ is harmonic and $f(z) = U + iV$ is analytic in entire complex plane then the harmonic conjugate of U is?

A. $(y^2 - x^2 + 5y) + C$
B. $x - y + C$
C. $\frac{x^2-y^2+5y}{4} + C$
D. $\frac{x^2-y^2}{2} + C$

14. If $U = 2x^2 - 2y^2 + 4xy$ is harmonic and $f(z) = U + iV$ is analytic in entire complex plane then the harmonic conjugate of U is?

A. $(y^2 - x^2y) + C$
B. $4xy - 2x^2 + 2y^2 + C$
C. $\frac{x^2-y^2+5y}{6} + C$
D. $\frac{x^2-y^2}{3} + C$

15. The value of p such that $2x - x^2 + py^2$ is harmonic?

A. 1
B. 2
C. 3
D. 4

Answers for Self Assessment

1.	A	2.	C	3.	A	4.	B	5.	A
6.	A	7.	A	8.	B	9.	B	10.	A
11.	A	12.	D	13.	A	14.	B	15.	A

Review Questions

1) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.
2) Suppose A is real or, more generally, suppose Im A is harmonic.
   Prove that $|curl\ grad\ A| = 0$
3) Determine whether the functions $u = x^2 - y^2 + 2xy - 2x + 3y$ is harmonic
4) Determine whether the functions $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic.
5) Determine whether the functions $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic.
6) Determine whether the functions $u = 3y^2x + 2xy - 2y^3$ is harmonic.
7) Let $u(x,y) = y^3 - 3x^2y$ is harmonic and then find the harmonic conjugate $v(x,y)$ of $u(x,y)$
8) Let $v(x,y) = x^3 - 3y^2x$ is harmonic and then find the harmonic conjugate $u(x,y)$ of $v(x,y)$
9) Let $u(x,y) = 2x - x^3 + 3y^2x$ is harmonic and then find the harmonic conjugate $v(x,y)$ of $u(x,y)$
10) Let $u(x,y) = \frac{y}{x^2+y^2}$ is harmonic and then find the harmonic conjugate $v(x,y)$ of $u(x,y)$

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

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Unit 04: Curves in the Complex Plane

CONTENTS

Objectives

Integrals of complex-valued functions of a complex variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

After this unit, you would be able to

- describe different types of curves in the complex plane.
- calculated the length of the curve in the given interval.
- evaluate the line integration of various complex functions in the given domain.

Introduction

To understand the different notions of the curves in the complex plane we need to know the parametric representation of curves. Suppose the continuous real-valued functions $x = \emptyset(t)$ and $y = \psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \leq t \leq t_2$. Then the parametric equation $z = x + iy$ is defined as

$z(t) = x(t) + iy(t), t_1 \leq t \leq t_2$.

$z(t)$define a continuous curve or arc C in the z plane joining points $a = z(t_1)$ and $b = z(t_2)$. See the following Figure 1:

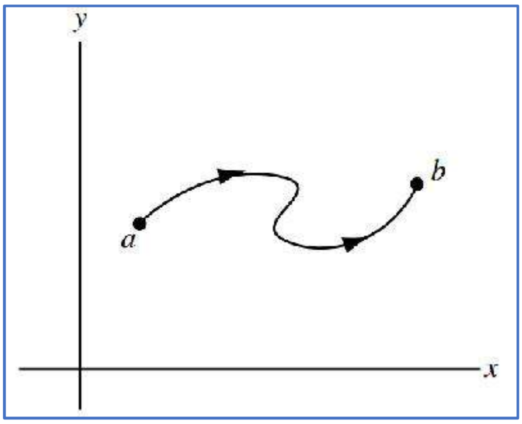


Figure 1: a continuous curve in the z plane

Here $z(t)$ be a complex-valued function of a real variable t called a *parametrization* of $\boldsymbol{C}$.

The point $z(a) = x(a) + iy(a)$ or $z_0 = (x(a), y(a))$ is called the initial point of $\boldsymbol{C}$ and $z(b) = x(b) + iy(b)$ or $z_1 = (x(b), y(b))$ is its terminal point.

The expression $z(t) = x(t) + iy(t)$ could also be interpreted as a two-dimensional vector function. Consequently, *z(a) and z(b)* can be interpreted as position vectors.

As t varies from *t=a to t=b* we can envision the curve $\boldsymbol{C}$ being traced out by the moving arrowhead of *z(t)*.

4.1 Smooth Curve

Let $z(t) = x(t) + iy(t), t_1 \leq t \leq t_2$ be the parametric representation of any curve ***C***.

Suppose the derivative of $z(t)$is $z'(t) = x'(t) + iy'(t)$. We say a curve $\boldsymbol{C}$ in the complex plane is smooth if *z'(t)* is continuous and never zero in the interval $a \leq t \leq b$.

As shown in the following Figure 2(a), since the vector *z'(t)* is not zero at any point *P on* C, the vector *z'(t)* is tangent to *C at P*. Thus, a smooth curve has a continuously turning tangent; or in other words, a smooth curve can have no sharp corners or cusps. Figure 2(b) is an example of a not smooth curve.

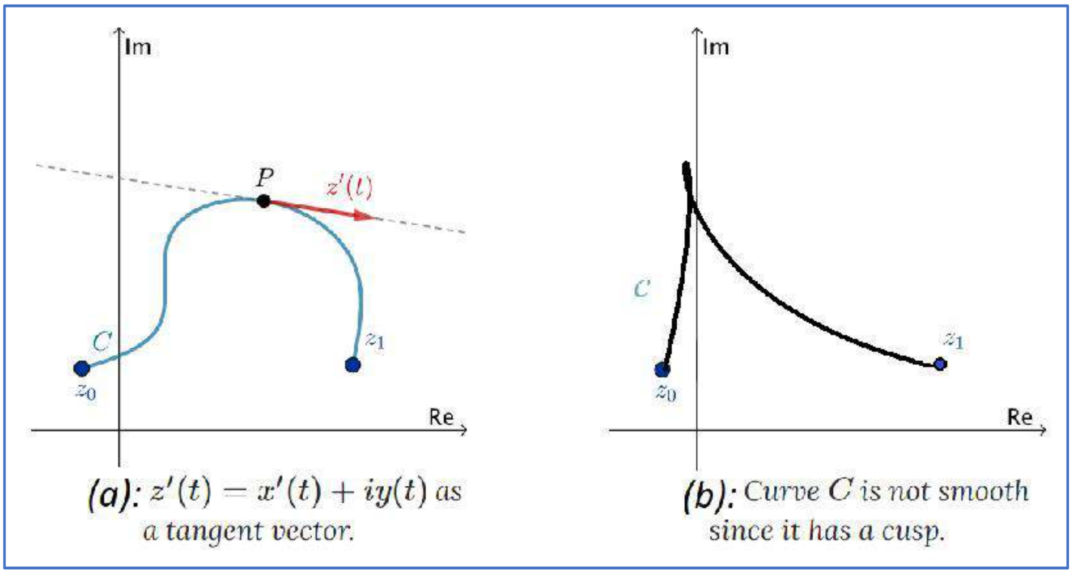


Figure 2: Example of a smooth and not a smooth curve

4.2 Piecewise-Smooth Curve

A *piecewise smooth curve* C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together.

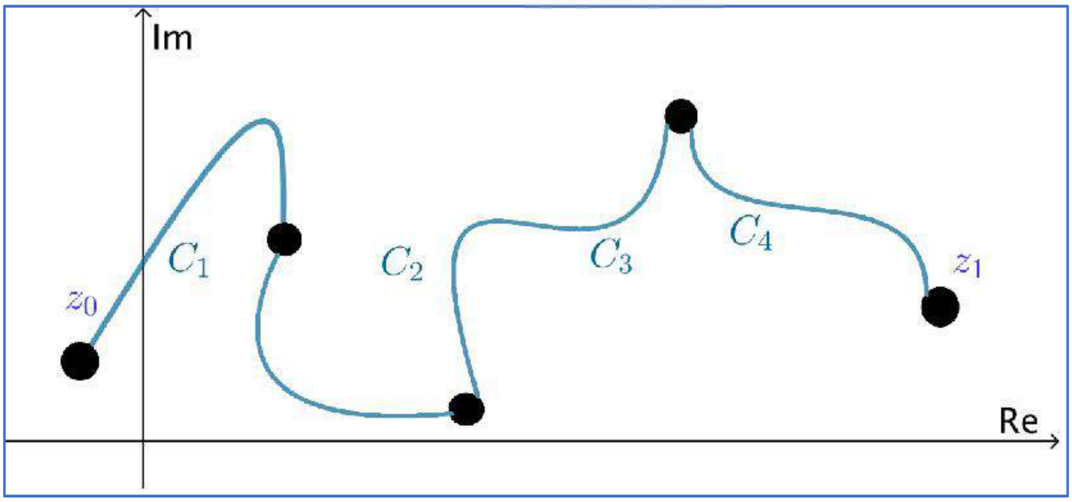


Figure 3: Piecewise-smooth curve

4.3 Simple Curve

A curve C in the complex plane is said to be simple if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$. Figure 4 depicts the simple curve in the complex plane and Figure 5 is an example of the non-simple curve.

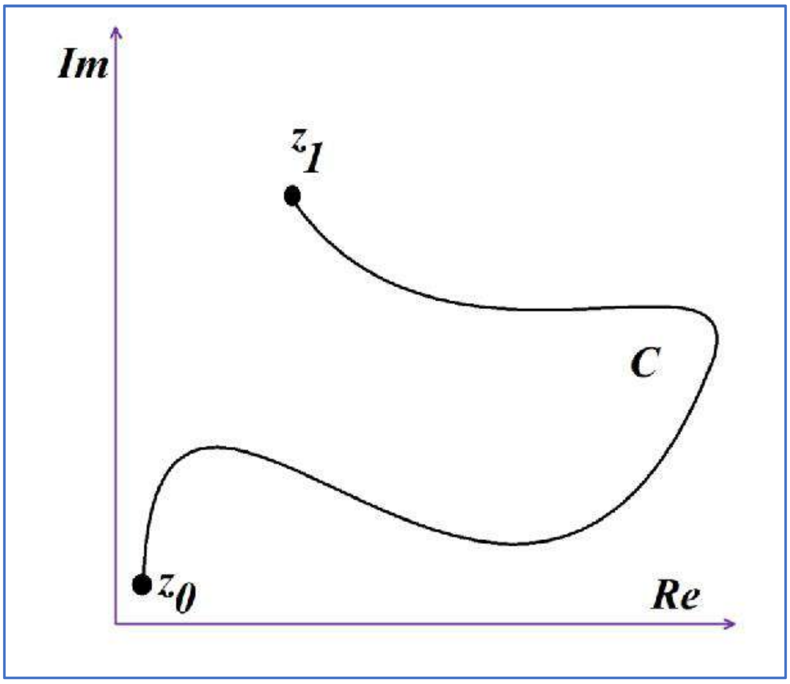


Figure 4: Simple curve

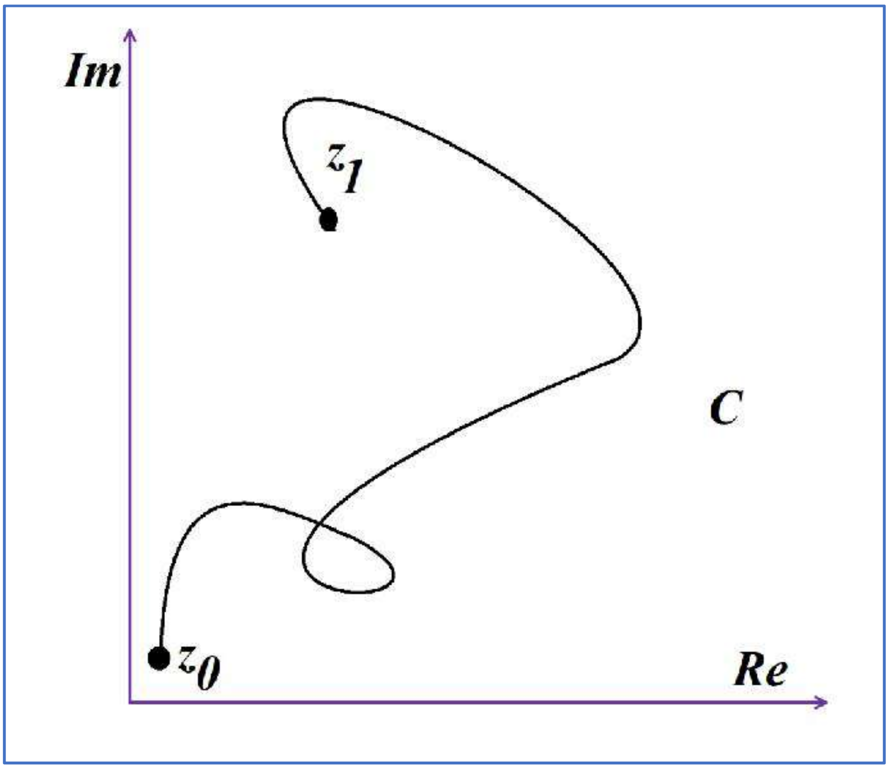


Figure 5: A non-simple curve

4.4 Simple Closed Curve

A curve C in the complex plane is said to be simpleclosed if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, *and* $z(a) = z(b)$. Here a and b are initial and ending points of the path.

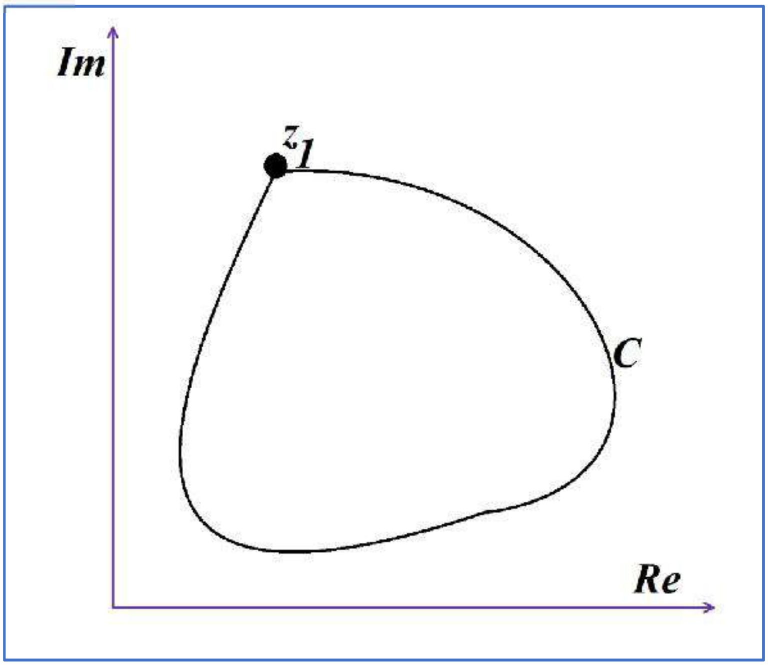


Figure 6: Simple closed curve

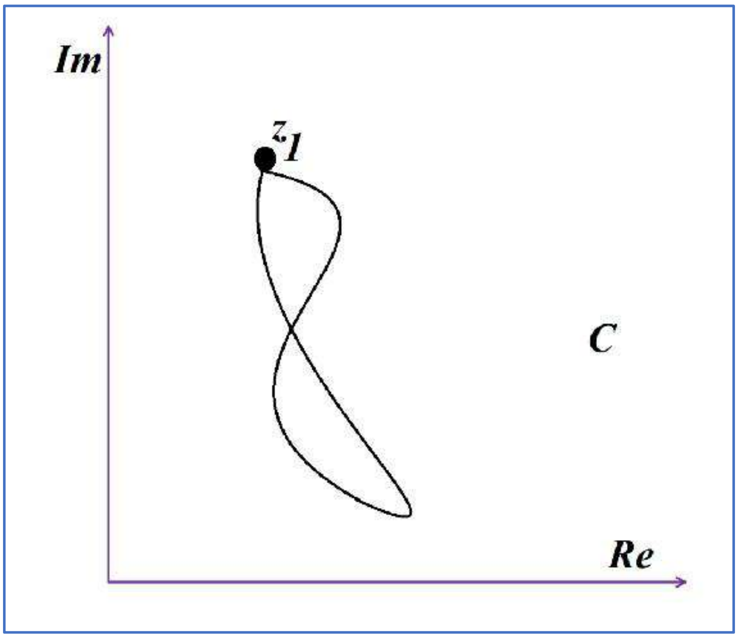


Figure 7: non-simple closed curve

4.5 Contour

In complex analysis, a piecewise smooth curve C is called a *contour* or *path*. We define the *positive direction* on a contour C to be the direction on the curve corresponding to increasing values of the parameter t.

It is also said that the curve C has *positive orientation*. In the case of a simple closed contour C, the *positive* direction corresponds to the *counterclockwise* direction.

For example, the circle $z(t) = e^{it}$, $0 \le t \le 2\pi$, has positive orientation. Figure 8 shows the positive orientation of the curve.

The *negative direction* on a contour C is the direction opposite the positive direction. If C has an orientation, the *opposite curve*, that is, a curve with opposite orientation, is denoted by $-C$.

On a simple closed curve, the negative direction corresponds to the clockwise direction.

For instance, the circle $z(t) = e^{-it}$, $0 \le t \le 2\pi$, has negative orientation. Figure 9 shows the negative orientation of the curve.

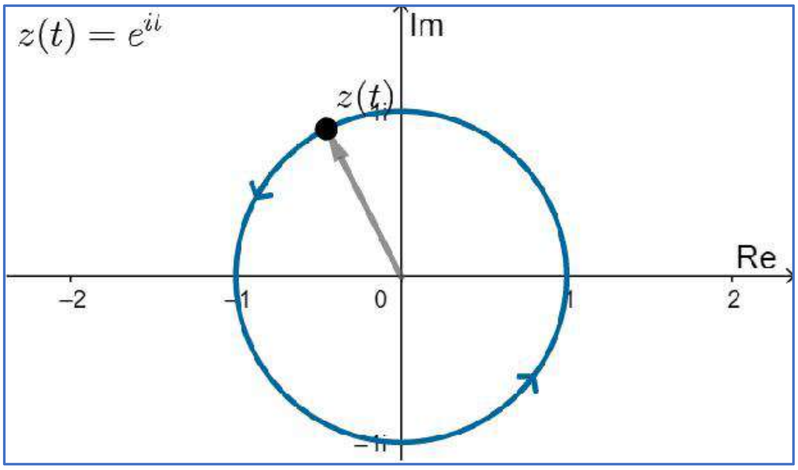


Figure 8: Positive orientation of the contour

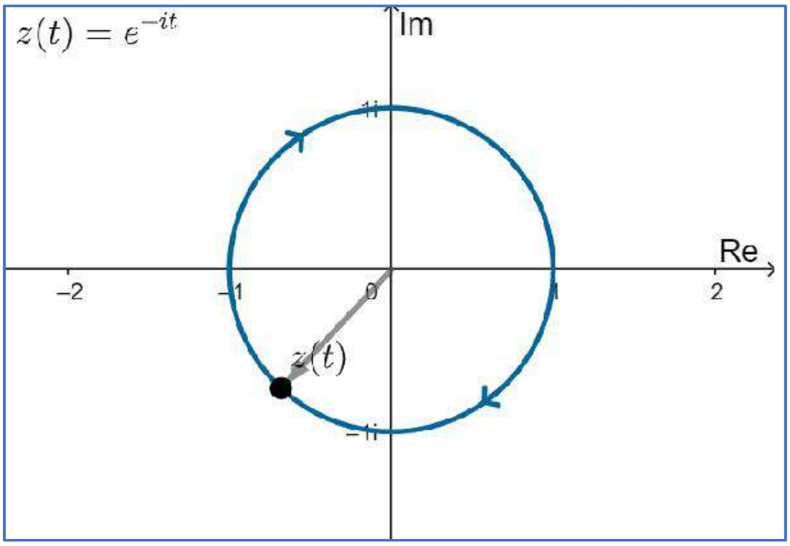


Figure 9: Negative orientation of the contour

4.6 Length of Arch/Curve

Let $z(t) = x(t) + iy(t), t_1 \leq t \leq t_2$ be the parametric representation of any curve ***C.***

Suppose the derivative of $z(t)$is $z'(t) = x'(t) + iy'(t)$ then the arc is called a differentiable arc, and the real-valued function$|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ is integrable over the interval $a \leq t \leq b$. In fact, according to the definition of arc length in calculus, the length of C is the number

$$L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

length of the curve ***C.***

Example

Consider the curve defined by $z: [0,2\pi] \to C$ where $z(t) = R(\cos t + i\sin t)$. Then the length of the curve is calculated as:

$$z(t) = R(\cos t + i\sin t)$$
$$z'(t) = R(-\sin t + i\cos t)$$
$$|z'(t)| = \sqrt{R^2(\sin^2 t + \cos^2 t)} = \sqrt{R^2}$$
$$L = \int_a^b |z'(t)|\, dt = \int_0^{2\pi} \sqrt{R^2}\, dt$$
$$L = R[t]_0^{2\pi} = 2\pi R$$

Task: Find the length of the curve ***C***whose parametric representation is given by $x(t) = t^3,\ y(t) = \frac{t^3}{3}$ as $0 \leq t \leq 5$

Solution:

$$z(t) = t^3 + \frac{t^3}{3} i$$

$$z'(t) = 3t^2 - 3\frac{t^2}{3}i$$

$$|z'(t)| = \sqrt{9t^4 + t^4} = \sqrt{10}\,t^2$$

$$L = \int_0^5 \sqrt{10}\,t^2\,dt$$

$$L = \sqrt{10}\left[\frac{t^3}{3}\right]_0^5 = \frac{125}{3}\sqrt{10}$$

4.7 Complex line Integral

Let $f(z)$ be continuous at all points of a curve $\boldsymbol{C}$, which we shall assume has a finite length shown in the Figure 10.

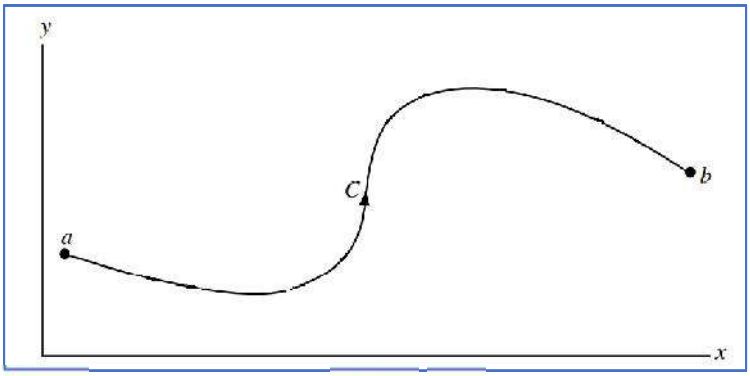


Figure 10: A continuous curve with finite length.

Subdivide C into n parts by means of points$z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0, b = z_n$.

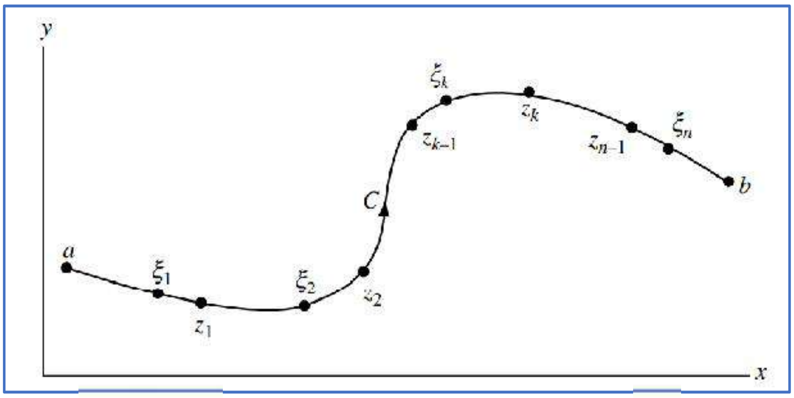


Figure 11:Curve C is divided into n small arc.

On each arc joining z_{k-1} *to* z_k [where k goes from 1 to n], choose a point γ_k. Form the sum

$$S_N = f_1(\gamma_1).(z_1 - a) + f_2(\gamma_2).(z_2 - z_1) + \cdots, + f_n(\gamma_n).(b - z_{n-1})$$

$$S_N = \sum_{k=1}^{n} f_k(\gamma_k).(z_k - z_{k-1}) = \sum_{k=1}^{n} f_k(\gamma_k).\Delta z_k$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths

$|\Delta z_k| \to 0$. Then, since $f(z)$ is continuous, the sum S_N approaches a limit that does not depend on the mode of subdivision, and we denote this limit by

$$n \to \infty, |\Delta z_k| \to 0, \sum_{k=1}^{n} f_k(\gamma_k).\Delta z_k = \int_a^b f(z)\,dz$$

called the complex line integral or simply line integral of $f(z)$ along curve C, or the definite integral of $f(z)$ from a to b along curve C. In such a case, $f(z)$ is said to be integrable along C. If $f(z)$ is analytic at all points of a region R and if C is a curve lying in R, then $f(z)$ is continuous and therefore integrable along C.

Example:

Suppose that $f(z) = u(x,y) + iv(x,y)$.Then

$$\int_c f(z)dz = \int_c \big(u(x,y) + iv(x,y)\big)(dx + idy)$$

$$\int_c f(z)dz = \int_c u(x,y)dx - v(x,y)dy + i\int_c v(x,y)dx + u(x,y)dy$$

Task: Obtain the complex integral: $\int_c zdz$ where C is the straight-line path from $z = 1 + i$ to $z = 3 + i$.

Solution

Consider the following diagram for the straight-line path from $z = 1 + i$ to $z = 3 + i$.

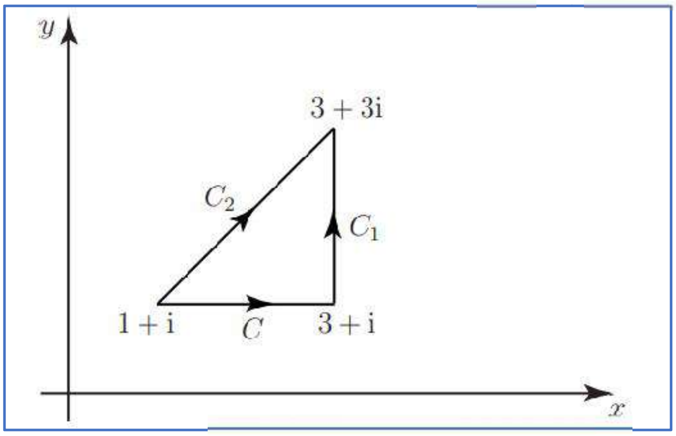


Figure 12: the straight-line path from z = 1 + i to z = 3 + i.

$$\int_c zdz = (x + iy)(dx + idy)$$

Here, since y is constant $(y = 1)$ along the given path then $z = x + i$, implying that $u = x$ and $v = 1$. Also, as y is constant, $dy = 0$.

$$\int_c zdz = \int_1^3 (x + 1i)dx$$

$$\int_c zdz = \left[\frac{x^2}{2} + xi\right]_1^3 = 4 + 2i$$

Task: Obtain the complex integral: $\int_{c_1} zdz$ where C_1 is the straight-line path from $z = 3 + i$ to $z = 3 + 3i$.

Solution

Consider the following diagram for the straight-line path from $z = 3 + i$ to $z = 3 + 3i$.

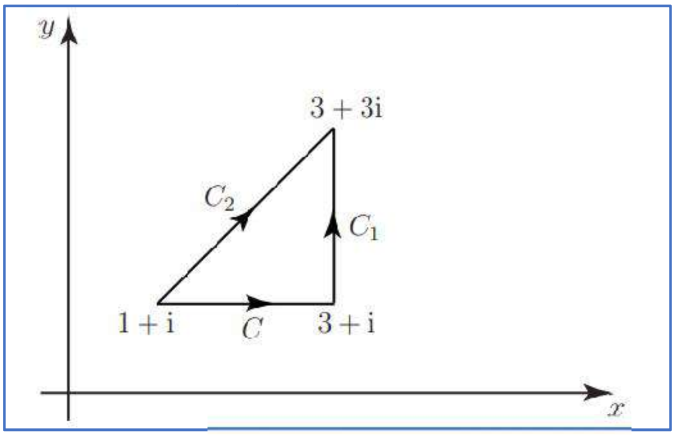


Figure 13: the straight-line path from z = 3 + i to z = 3 +3i.

$$\int_c zdz = (x+iy)(dx+idy)$$

Here, since x is constant $(x = 3)$ along the given path then $z = 3 + yi$, implying that $u = 3$ *and* $v = y$. Also, as x is constant, $dx = 0$.

$$\int_c zdz = \int_1^3 (3+yi)idy$$

$$\int_c zdz = \left[-\frac{y^2}{2} + 3yi\right]_1^3 = -4 + 6\text{i}.$$

Task: Obtain the complex integral: $\int_{C_1} |z|dz$ where C_1 is the straight-line path from $z = -i$ *to* $z = i$.

Solution

Consider the following diagram for the straight-line path from $z = -i$ *to* $z = i$.

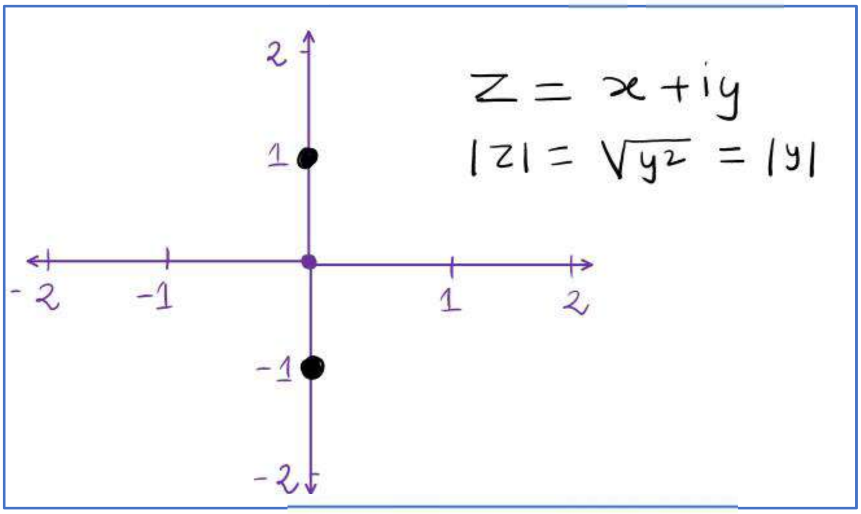


Figure 14: the straight-line path from z = -i to z = i.

$$\int_c |z|dz = |(x+iy)|(dx+idy)$$

Here, since x is constant $(x = 0)$ along the given path then $z = 0 + yi$, implying that $u = 0$ *and* $v = y$. Also, as x is constant, $dx = 0$.

$$\int_{C_1} |z|dz = \int_{-1}^1 (\sqrt{y^2})idy$$

$\int_{C_1} |z| dz = \left[\frac{y^2 i}{2}\right]_{-1}^{1}$ =0

Summary

This section deals with some basic definitions and operations. These are summarized below:

- Let $z(t) = x(t) + iy(t)$, $t_1 \leq t \leq t_2$ be the parametric representation of any curveSuppose the derivative of $z(t)$is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is smooth if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$.
- A curve C in the complex plane is said to be simple if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$.
- A curve C in the complex plane is said to be simple closed if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, $and\ z(a) = z(b)$. Here a and b are initial and ending points of the path.
- The length of C is the number $L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$.
- Let$f(z)$ is continuous, the sum S_N approaches a limit that does not depend on the mode of subdivision and we denote this limit by

$$n \to \infty, |\Delta z_k| \to 0\,, \sum_{k=1}^{n} f_k(\gamma_k).\, \Delta z_k = \int_a^b f(z)\, dz$$

called the complex line integral or simply line integral of $f(z)$ along curve C

Keywords

Length of arch/curve

Let $z(t) = x(t) + iy(t)$, $t_1 \leq t \leq t_2$ be the parametric representation of any curve ***C.***

Suppose the derivative of $z(t)$is $z'(t) = x'(t) + iy'(t)$ then the arc is called a differentiable arc, and the real-valued function$|z'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$ is integrable over the interval $a \leq t \leq b$. In fact, according to the definition of arc length in calculus, the length of C is the number

$L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$ is length of the curve ***C.***

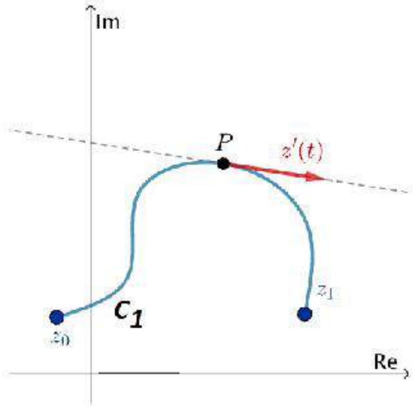
Line integral: Let $f(z)$ is continuous, the sum S_N approaches a limit that does not depend on the mode of subdivision and we denote this limit by

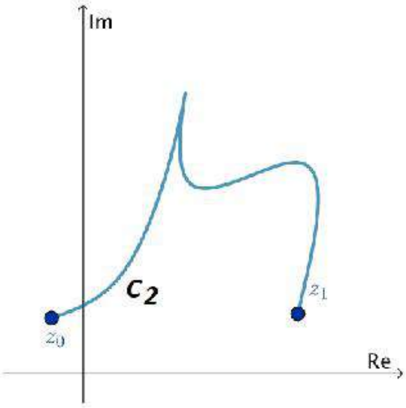
$$n \to \infty, |\Delta z_k| \to 0\,, \sum_{k=1}^{n} f_k(\gamma_k).\, \Delta z_k = \int_a^b f(z)\, dz$$

called the complex line integral or simply line integral of $f(z)$ along curve C.

Self Assessment

1. Consider the following figure for curves C_1 and C_2 in the complex plane then





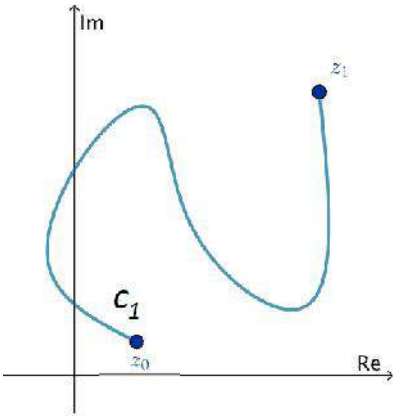
A. Only C_1 is smooth.

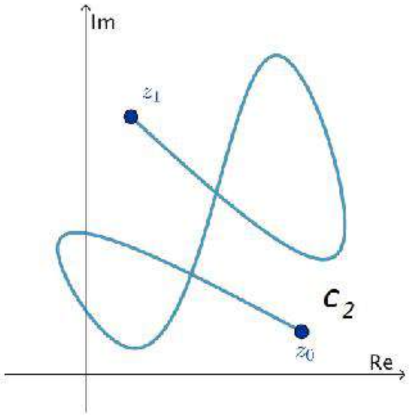
B. Only C_2 is smooth.

C. Both C_1 and C_2 are smooth

D. Neither C_1 nor C_2 is smooth

2. Consider the following figure for curve C_1 and C_2 then





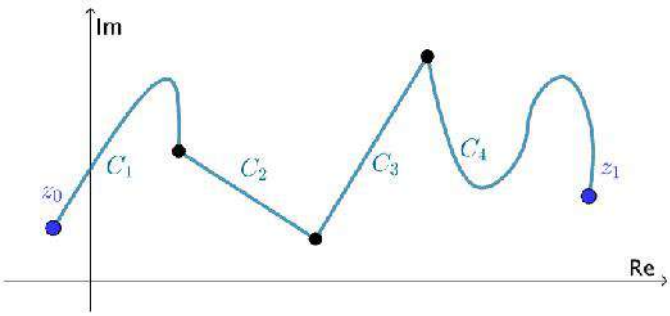
A. Only C_1 is not simple.

B. Only C_2 is not simple.

C. Both C_1 and C_2 are simple

D. Neither C_1 nor C_2 is simple

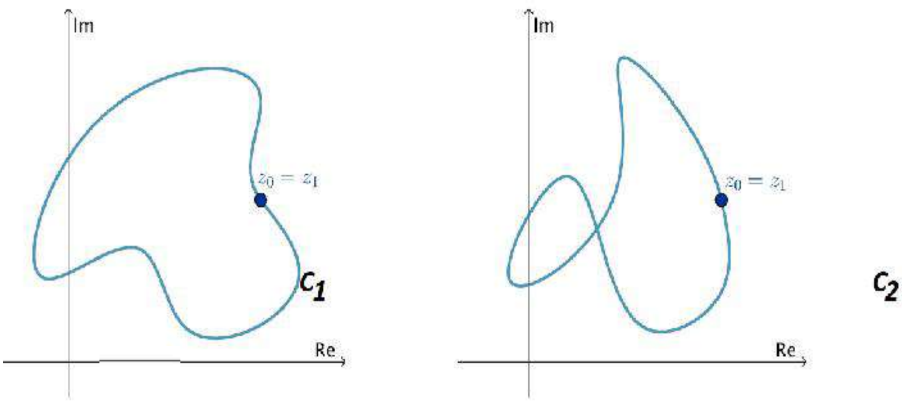
3. The statement "following figure3 represent the piecewise smooth curve" is…



A. True

B. False

4. Which one of the following in Figure 4 is the simple closed curve?



A. Only C_2

B. Both $C_1$and C_2

C. Neither C_1nor C_2

D. Only C_1

5. A piecewise continuous closed smooth curve is called a contour?

A. False

B. True

6. The length of the arc for $\boldsymbol{z}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{i}\boldsymbol{y}(\boldsymbol{t})$ where $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{t}, \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{2t}, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{1}]$

A. 5

B. 10

C. $\sqrt{5}$

D. $\sqrt{10}$

7. The length of the arc for $\boldsymbol{z}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{i}\boldsymbol{y}(\boldsymbol{t})$ where $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{sint}, \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{cost}, \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{1}]$

A. 5

B. 10

C. 1

D. 2

8. The length of the arc for $z(t) = x(t) + iy(t)$ where $x(t) = t^2, y(t) = 6t^2, t \in [0, 1]$

A. 5

B. 6

C. $\sqrt{37}$

D. $\sqrt{35}$

9. The length of the arc for $z(t) = x(t) + iy(t)$ where $x(t) = 2t, y(t) = 6t, t \in [0, 1]$

A. $\sqrt{20}$

B. $2\sqrt{10}$

C. $\sqrt{10}$

D. $2\sqrt{35}$

10. The length of the arc for $z(t) = x(t) + iy(t)$ where $x(t) = t, y(t) = 10, t \in [0, 10]$

A. 1

B. $2\sqrt{10}$

C. $\sqrt{10}$

D. 10

11. Evaluate $\int_0^{1+i}(x - y + ix^2)dz$ along the straight line from (0,0) to (1,1)

A. $\frac{i-1}{3}$

B. $\frac{i-1}{2}$

C. $\frac{i+1}{3}$

D. $\frac{i+1}{2}$

12. Evaluate $\int_0^{1+i}(x - y + ix^2)dz$ over the path along the lines $y = 0$ and $x = 1$.

A. $\frac{1}{3}$

B. $\frac{-1}{2} + \frac{5}{6}i$

C. $\frac{i}{3}$

D. $\frac{i+6}{2}$

13. Evaluate $\int_0^{1+i}(x - y + ix^2)dz$ over the path along the lines $x = 0$ and $y = 1$.

A. $\frac{-1}{2} - \frac{1}{6}i$

B. $\frac{-1}{2} + \frac{1}{6}i$

C. $\frac{i+6}{2}$

D. $\frac{1}{3}$

14. Evaluate $\int_0^{1+i}(x-y+ix^2)dz$ over the path $x=y^2$.

A. $\frac{-11}{30}+\frac{1}{6}i$

B. $\frac{-11}{30}-\frac{1}{6}i$

C. $\frac{-1}{3}+\frac{2}{6}i$

D. $\frac{-11}{30}+0i$

15. Evaluate $\int_0^{1+i}(x-y+ix^2)dz$ along the curve C; $x=t \; and \; y=t^2$.

A. $\frac{1}{3}$

B. $\frac{-1}{3}+\frac{1}{2}i$

C. $\frac{i}{3}$

D. $\frac{i+6}{2}$

Answers for Self Assessment

1.	A	2.	A	3.	A	4.	D	5.	B
6.	C	7.	C	8.	C	9.	C	10.	D
11.	A	12.	B	13.	A	14.	B	15.	B

Review Questions

1. Obtain the complex integral: $\int_c zdz$ where C is the straight-line path from $z = 1 + i \; to \; z = 3 + 3i$.?
2. Obtain the complex integral: $\int_c zdz$ where C is the straight-line path from $z = 2 + 2i \; to \; z = 5 + 2i$?
3. Obtain the complex integral: $\int_c zdz$ where C is the straight-line path from $z = 5 + 2i \; to \; z = 5 + 5i$?
4. Obtain the complex integral: $\int_c zdz$ where C is the straight-line path from $z = 2 + 2i \; to \; z = 5 + 5i$.?
5. Obtain the complex integral: $\int_c |z| dz$ where C is the path from left half of the unit circle from $z = -i \; to \; z = i$?
6. Obtain the complex integral: $\int_c 1/z \, dz$ where $C$is the unit circle.
7. Obtain the complex integral: $\int_c (z^2 + z)dz$ where C is the path from left half of the unit circle from $z = 1 \; to \; z = i$?

8. Obtain the complex integral: $\int_c z^2 dz$ where C is the straight-line path from

$z = 1 + i \; to \; z = 3 + 3i.$?

9. Obtain the complex integral: $\int_c z^2 dz$ where C is the straight-line path from

$z = 2 + 2i \; to \; z = 5 + 2i$?

10. Obtain the complex integral: $\int_c z^2 dz$ where C is the straight-line path from

$z = 5 + 2i \; to \; z = 5 + 5i$?

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.
2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.
3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.
4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 05: Cauchy-Goursat Theorem

CONTENTS

Objectives

The integration of a function of a complex variable along an open or close curve in the plane of the complex variables is known as complex integration. Cauchy's integral theorem is part of complex integration. The study of complex integration is very useful in engineering physics and mathematics as well as the concept of center of mass, the center of gravity, mass moment of inertia of vehicles, etc. It can be used in placing a satellite in its orbit to calculate the velocity and trajectory. After this section, you will be able to-

- understand the concept of a simple and multi-connected domain.
- Learn the Cauchy-Goursat theorem and apply it in to solve the complex integration problem.
- Solve the complex integral problem using the Cauchy integral formula.

Introduction

InthisSection,weintroduceCauchy'stheoremwhichallowsustosimplifythecalculationofcertain contourintegrals.Asecondresult,knownasCauchy'sintegralformula,allowsustoevaluatesome integrals of theform $\oint_c \left(\frac{f(z)}{z-z_0}\right) dz.$

Where z_0 lies inside the closed curve c.

5.1 Simply and Multiply Connected Regions

A region R is called simply connected if any simple closed curve, which lies in R, can be shrunk to a point without leaving R. A region R, which is not simply connected, is called multiply connected.

For example, suppose R is the region defined by $|z| < 2$,shown shaded in Figure 1: Simply connected domain. If G is any simple closed curve lying in R [i.e., whose points are in R], we see that it can be shrunk to a point that lies in R, and thus does not leave R, so that R is simply-connected.

On the other hand, if R is the region defined by $1 < |z| < 2$ shown shaded in Figure 2, then there is a simple closed curve G lying in R that cannot possibly be shrunk to a point without leaving R, so that R is multiply-connected. Intuitively, a simply connected region is one that does not have any "holes" in it, while a multiply connected region is one that does. The multiply connected regions of Figure 2 and Figure 3 have, respectively, one and three holes in them.

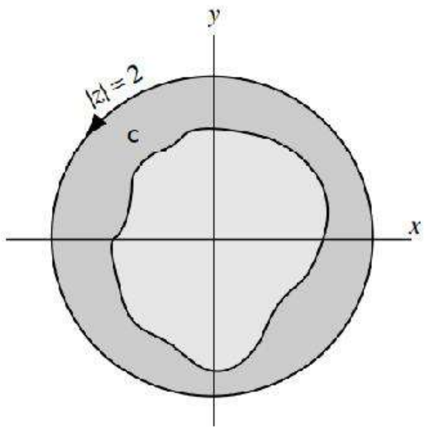


Figure 1: Simply connected domain.

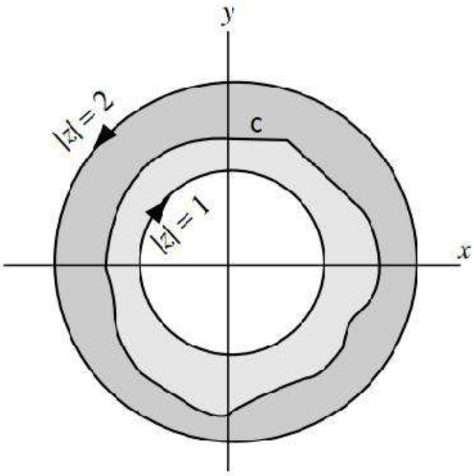


Figure 2: Multiply connected domain, $1 < |z| < 2$.

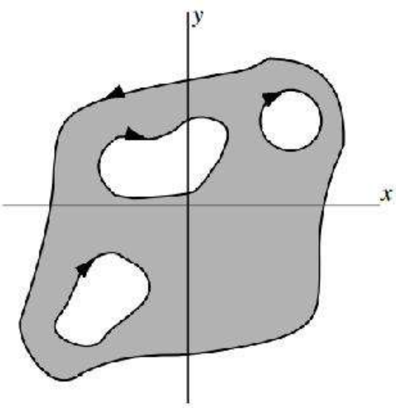


Figure 3: Multiply connected domain

5.2 Jordan Curve

Any continuous, closed curve that does not intersect itself and may or may not have a finite length, is called a Jordan curve.

5.3 Jordan Curve Theorem

A Jordan curve divides the plane into two regions having the curve as a common boundary. That region, which is bounded [i.e., is such that all points of it satisfy $|z| < M$ where M is some positive constant], is called the interior or inside of the curve, while the other region is called the exterior or outside of the curve. Using the Jordan curve theorem, it can be shown that the region inside a simple closed curve is a simply-connected region whose boundary is the simple closed curve.

5.4 Cauchy's Theorem. The Cauchy–Goursat Theorem

Let f(z) be analytic in a region R and on its boundary C. Then

$\oint_c f(z)dz = 0.$

This fundamental theorem, often called Cauchy's integral theorem or simply Cauchy's theorem, is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in R. However, Goursat gave a proof which removed this restriction. For this reason, the theorem is sometimes called the Cauchy – Goursat theorem when one desires to emphasize the removal of this restriction.

Wewillprovethetheoremunderanextrahypothesisthatf'isacontinuous function.

Green'sTheorem: *Let Cbe a simple closed curve with positive orientation.LetR bethedomainthatformstheinteriorofC. IfP andQ are continuousand have continuous partial derivatives* P_x, P_y, Q_x *and* Q_y *at all points on Cthen*

$$\oint_c (P(x,y)dx + Q(x,y)dy) = \oiint [Q_x(x,y) - P_y(x,y)]dxdy.$$

Proof. Let$f(z)= f(x+iy)=u(x,y)+iv(x,y)$.Then

$$\oint_c \big(u(x,y) + iv(x,y)\big)(dx + idy) = \oint_c (u(x,y)dx - v(x,y)dy) + i \oint_c (v(x,y)dx + u(x,y)dy)$$

$= \oiint -v_x(x,y) - u_y(x,y)]dxdy + i \oiint [u_x(x,y) - v_y(x,y)]dxdy$

=0.

Example:

Let $C: |z| \le 1$ then $\oint_c \left(\frac{z}{z-2}\right) dz = 0$. Clearly $f(z) = z$ is analytic inside the $C: |z| \le 1$ and $f(z) = z$ is not analytic at z =2, which does not lie inside the C. So then $\oint_c \left(\frac{z}{z-2}\right) dz = 0$.

Consider the contour shown in Figure 4 and assume *f (z)* is analytic everywhere on and inside the contour C.

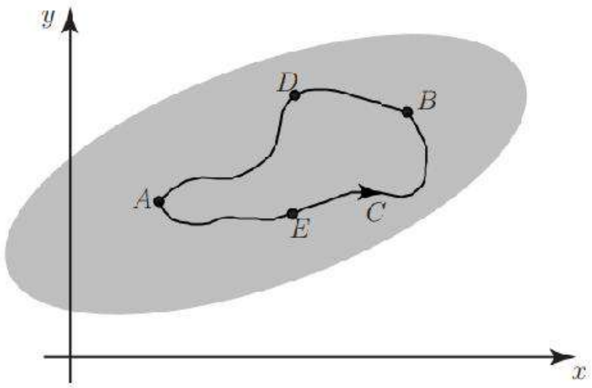


Figure 4: analytic function inside the contour

Then by analogy with real line integrals

$\int_{AEB} f(z)dz + \int_{BDA} f(z)dz = \oint_c f(z)dz = 0$

By Cauchy's theorem, since reversing the direction of integration reverses the sign of the integral.

This implies that we may choose any path between A and B and the integral will have the same value providing f (z) is analytic in the region concerned.

5.5 Cauchy's Integral Formula

If *f(z)* is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside C. Then

$$\oint_c \left(\frac{f(z)}{z - z_0}\right) dz = 2\pi i\, f(z).$$

Example: Evaluate the $\oint_c \left(\frac{z}{z^2+1}\right) dz$, where c is the path $|z - i| = \frac{1}{2}$.

Solution

It is clear that the center of the circle is i and the radius is ½.The following figure shows the path $|z - i| = \frac{1}{2}$.

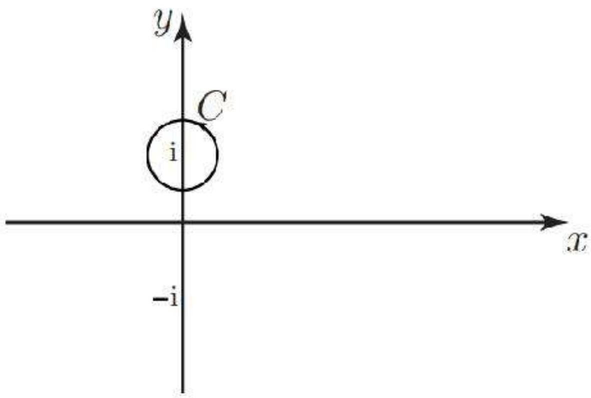


Figure 5: the path $|z - i| = \frac{1}{2}$.

$$\oint_c \left(\frac{z}{z^2 + 1}\right) dz = \oint_c \left(\frac{z}{(z + i)(z - i)}\right) dz$$

$f(z) = \frac{z}{(z+i)}$ is analytic inside and on the curve c.Using Cauchy integral formula

$$\oint_c \left(\frac{\frac{z}{(z+i)}}{z-i}\right) dz = 2\pi i\, f(z=i)$$

$$\oint_c \left(\frac{\frac{z}{(z+i)}}{z-i}\right) dz = 2\pi i * \frac{1}{2} = \pi i.$$

Task: Evaluate the $\oint_c \left(\frac{z}{z^2+1}\right) dz$, where c is the path $|z+i| = \frac{1}{2}$.

Solution

It is clear that the center of the circle is i and the radius is ½.The following figure shows the path $|z+i| = \frac{1}{2}$.

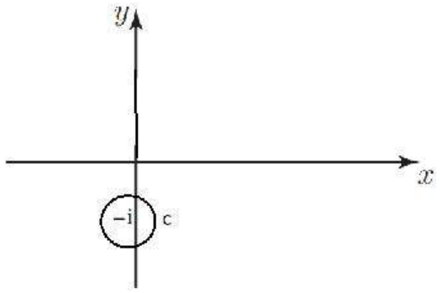


Figure 6: the path $|z+i| = \frac{1}{2}$.

$$\oint_c \left(\frac{z}{z^2+1}\right) dz = \oint_c \left(\frac{z}{(z+i)(z-i)}\right) dz$$

$f(z) = \frac{z}{(z-i)}$ is analytic inside and on the curve c.Using Cauchy integral formula

$$\oint_c \left(\frac{\frac{z}{(z-i)}}{z+i}\right) dz = 2\pi i\, f(z=-i)$$

$$\oint_c \left(\frac{\frac{z}{(z-i)}}{z+i}\right) dz = 2\pi i * \frac{-1}{-2} = \pi i.$$

Task: Evaluate the $\oint_c \left(\frac{z}{z^2+1}\right) dz$, where c is the path $|z| = 2$.

Solution

It is clear that the center of the circle is i and the radius is 2.The following figure shows the path $|z| = 2$.

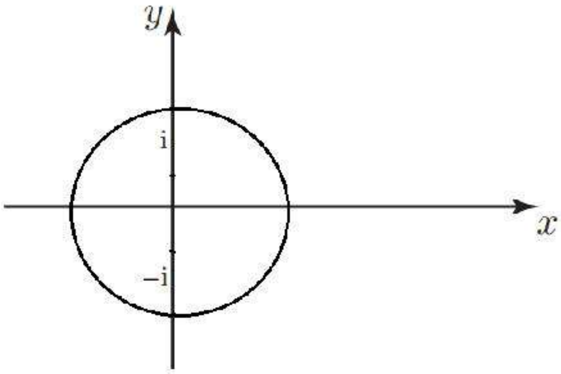


Figure 7: the path $|z| = 2$.

$$\oint_c \left(\frac{z}{z^2+1}\right) dz = \oint_c \left(\frac{z}{(z+i)(z-i)}\right) dz$$

$$\oint_c \left(\frac{z}{z^2+1}\right) dz = \oint_c \left(\frac{1}{2(z+i)} + \frac{1}{2(z-i)}\right) dz$$

$f(z) = \frac{1}{2}$ is analytic inside and on the curve c execpt the $z = i$ .Similarly $f(z) = \frac{1}{2}$ is analytic inside and on the curve c execpt the $z = -i$ .Using Cauchy integral formula

$$\oint_c \left(\frac{z}{z^2+1}\right) dz = 2\pi i[\, f(z=i) + f(z=-i)]$$

$$\oint_c \left(\frac{z}{z^2+1}\right) dz = 2\pi i[\frac{1}{2} + \frac{1}{2}]$$

$$\oint_c \left(\frac{z}{z^2+1}\right) dz = 2\pi i.$$

5.6 The Derivative of an Analytic Function

If f(z) is analytic in a simply-connected region then at any interior point of the region, z_0 inside C. Then say, the derivatives of f(z) of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$\oint_c \left(\frac{f(z)}{(z-z_0)^{n+1}}\right) dz = 2\pi i \frac{f^n(z)}{n!}.$

where C is any simple closed curve, in the region, which encloses z_0. Note the case $n = 1$:

$\oint_c \left(\frac{f(z)}{(z-z_0)^2}\right) dz = 2\pi i \frac{f'(z)}{1!}.$

Example

Evaluate the $\oint_c \left(\frac{z^3}{(z+1)^2}\right) dz$, where c is the path $|z| = 2$.

Solution

Let $g(z) = z^3$ and it is analytic within and on the circle C we use Cauchy's integral formula for derivatives to show that

$\oint_c \left(\frac{z^3}{(z+1)^2}\right) dz = 2\pi i \frac{g'(z=-1)}{1!}.$

$\oint_c \left(\frac{z^3}{(z+1)^2}\right) dz = 2\pi i \frac{[3z^2]_{z=-1}}{1!}.$

$\oint_c \left(\frac{z^3}{(z+1)^2}\right) dz = 2\pi i \frac{3}{1!}.$

$\oint_c \left(\frac{z^3}{(z+1)^2}\right) dz = 6\pi i.$

Summary

This section deals with some basic definitions and operations. These are summarized below:

- A region R is called simply-connected if any simple closed curve, which lies in R, can be shrunk to a point without leaving R. A region R, which is not simply-connected, is called multiply connected.
- Let f(z) be analytic in a region R and on its boundary C. Then
  $\oint_c f(z) dz = 0.$
- If *f(z)* is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside *C*. Then

$$\oint_c \left(\frac{f(z)}{z - z_0}\right) dz = 2\pi i\, f(z).$$

- If f(z) is analytic in a simply-connected region then at any interior point of the region, z_0 inside *C*. Then say, the derivatives of f(z) of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:
  $\oint_c \left(\frac{f(z)}{(z-z_0)^{n+1}}\right) dz = 2\pi i \frac{f^n(z)}{n!}.$

Keywords

Simply connected domain:

A region R is called simply-connected if any simple closed curve, which lies in R, can be shrunk to a point without leaving R.

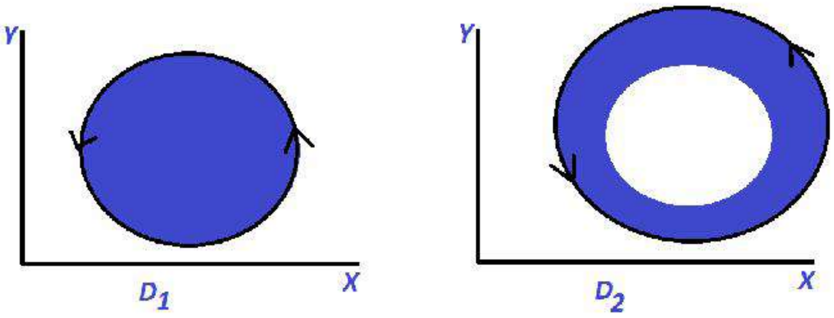
Cauchy's Integral Formula:

If f(z) is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside C. Then

$$\oint_c \left(\frac{f(z)}{z - z_0}\right) dz = 2\pi i\, f(z).$$

Self Assessment

1. Which one of the following region is simply connected?



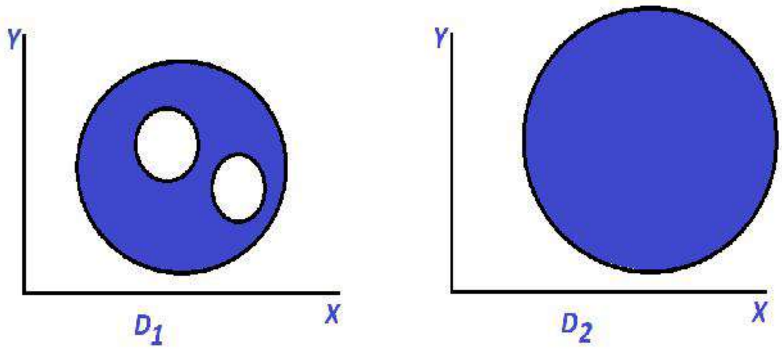
A. Only D_1

B. Only D_2

C. Both D_1 and D_2

D. Neither D_1 nor D_2

2. Which one of the following region is not simply connected?



A. Only D_1

B. Only D_2

C. Both D_1 and D_2

D. Neither D_1 nor D_2

3. If C is a simple closed contour, then the conclusion follows from the Cauchy-Goursat theorem is $\oint_c \boldsymbol{f(z)dz = 0}$

A. True

B. False

4. The value of $\int_c \frac{z^2}{z-2} dz$, where C is the closed curve: $|z| = 2$

A. $8\pi i$

B. πi

C. $2\pi i$

D. $10\pi i$

5. The value of $\int_c \frac{z}{z^2+1} dz$, where C is the closed curve:$|z - i| = \frac{1}{2}$

A. $8\pi i$

B. πi

C. $2\pi i$

D. $10\pi i$

6. The value of $\int_c \frac{z}{z^2+1} dz$, where C is the closed curve:$|z + i| = \frac{1}{2}$

A. $8\pi i$

B. πi

C. $2\pi i$

D. $10\pi i$

7. The value of $\int_c \frac{z}{z^2+1} dz$, where C is the closed curve :$|z| = 2$

A. $8\pi i$

B. πi

C. $2\pi i$

D. $10\pi i$

8. The value of $\int_c \frac{1}{z-1} dz$, where C is the closed curve :$|z| = 2$

A. $8\pi i$

B. πi

C. $2\pi i$

D. $10\pi i$

9. The value of $\int_c \frac{e^z}{z-1} dz$, where C is the closed curve:$|z - 1 - i| = 2$

A. $2\pi e i$

B. $e\pi i$

C. $2\pi i$

D. 0

10. The value of $\int_c \frac{e^z}{z+2} dz$, where C is the closed curve :$|z - 1 - i| = 2$

A. $8\pi i$

B. $e\pi i$

C. $2\pi i$

D. 0

11. The value of $\int_c \frac{6}{z(z-1)(z-2)} dz$, where C is the closed curve :$|z - 3| = 1$

A. $8\pi i$

B. $6\pi i$

C. $2\pi i$

D. $10\pi i$

12. The value of $\int_C \frac{6}{z(z-2)} dz$, where C is the closed curve :$|z-3| = 1$

A. $8\pi i$

B. $6\pi i$

C. $2\pi i$

D. $10\pi i$

13. The value of $\int_C \frac{1}{z-10} dz$, where C is the closed curve :$|z| = 2$

A. $8\pi i$

B. πi

C. 0

D. $10\pi i$

14. The value of $\int_C \frac{z^3}{(z-1)^2} dz$, where C is the closed curve :$|z| = 2$

A. $6\pi i$

B. $e\pi i$

C. $2\pi i$

D. 0

15. The value of $\int_C \frac{e^{5z}}{(z-i)^3} dz$, where C is the closed curve:$|z-1| = 4$

A. $25\pi i$

B. $25\pi i * e^{5i}$

C. $2\pi i$

D. 0

Answers for Self Assessment

1.	A	2.	A	3.	A	4.	A	5.	B
6.	B	7.	C	8.	C	9.	A	10.	D
11.	B	12.	B	13.	C	14.	A	15.	B

Review Questions

1. Obtain the complex integral: $\int_c z^2 dz$ where $C: |z| \leq 2$?
2. Obtain the complex integral: $\int_c z + 1dz$ where$C: |z| \leq 2$?
3. Obtain the complex integral: $\int_c \frac{1}{z} dz$ where$C: |z| \leq 1$?
4. Obtain the complex integral: $\int_c \frac{1}{z-2} dz$ where$C: |z| \leq 1$?

5. Obtain the complex integral: $\int_C \frac{z}{z+5}\, dz$ where $C: |z-1| \leq 1$?
6. Obtain the complex integral: $\int_C (z-1)/z^2\, dz$ where C is the unit circle.
7. Obtain the complex integral: $\int_C \frac{e^z}{z}\, dz$ where $C: |z| \leq 1$?
8. Obtain the complex integral: $\int_C \frac{e^z}{z-5}\, dz$ where $C: |z| \leq 2$?
9. Obtain the complex integral: $\int_C \frac{5z}{(z-10)^3}\, dz$ where $C: |z-9| \leq 5$?
10. Obtain the complex integral: $\int_C \frac{5z^2}{(z-10)^3}\, dz$ where $C: |z-9| \leq 5$?

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Rajesh Kumar Chandrawat,Lovely Professional University

Unit 06: Gauss Mean Value Theorem

CONTENTS

Objectives

The complex analysis revolves around complex analytic functions. These are functions that have a complex derivative. Unlike calculus using real variables, the mere existence of a complex derivative has strong implications for the properties of the function. There are a small number of far-reaching theorems that we will explore in this section and along the way, we will touch on some main theorems. After this unit, you will be able to-

- understand the concept of the Gauss mean value theorem.
- prove the Cauchy inequality using the Gauss mean value theorem.
- find the maximum value of a complex-valued function in the given domain.

Introduction

In this section first, the gauss means value theorem is discussed for an analytic function inside and on the domain. After that using the Cauchy integral formula, the Cauchy inequality would be proved and then using the Maximum modulus principal, the maximum value of $|f(z)|$ would be discussed.

6.1 Gauss' Mean Value Theorem

Suppose $f(z)$ is analytic inside and on a circle C with center at a and radius r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})d\theta$$

Proof

Let $f(z)$ is analytic inside and on a circle C with center at a and radius r shown in the following Figure 1.

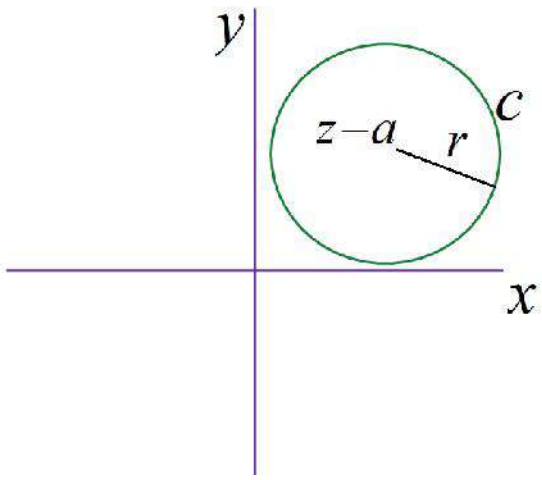


Figure 1: An analytic function inside the $|z-a| \leq r$.

Using the Cauchy integral formula

$$\oint_c \left(\frac{f(z)}{z-z_0}\right) dz = 2\pi i\, f(z_0).$$

Here $z_0 = a$ so

$$f(a) = \frac{1}{2\pi i} \oint_c \left(\frac{f(z)}{z-a}\right) dz$$

If the center of the circle is a and the radius is r.The equation of circle is $|z-a| = r$.

Let $z - a = re^{i\theta}$

$$z = a + re^{i\theta}$$

$$dz = re^{i\theta}.i.d\theta$$

Now put the $z = a + re^{i\theta}$, and $dz = re^{i\theta}.i.d\theta$ in the above equation.

$$f(a) = \frac{1}{2\pi i} \oint_c \left(\frac{f(a+re^{i\theta})}{a+re^{i\theta}-a}\right) re^{i\theta}.i.d\theta$$

$f(a) = \frac{1}{2\pi i} \oint_c \left(\frac{f(a+re^{i\theta})}{re^{i\theta}}\right) re^{i\theta} d\theta, 0 \leq \theta \leq 2\pi.$

$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta}) d\theta.$

Example

Evaluate the $\frac{1}{2\pi}\int_0^{2\pi} \sin^2\left(\frac{\pi}{6} + 2e^{i\theta}\right) d\theta.$

Suppose $f(z) = \sin^2(z)$ is analytic inside and on a circle C with center at a and radius r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta}) d\theta$$

Here $\left|z - \frac{\pi}{6}\right| = 2$, so $a = \frac{\pi}{6}$and radius $r = 2$.Now using Gauss mean value thorem,

$\frac{1}{2\pi}\int_0^{2\pi} \sin^2\left(\frac{\pi}{6} + 2e^{i\theta}\right) d\theta = f\left(\frac{\pi}{6}\right).$

$f\left(\frac{\pi}{6}\right) = \sin^2\left(\frac{\pi}{6}\right).$

$f\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$

Task: Find the mean value of $x^2 - y^2 + 2y$ over the circle $|z - 5 + 2i| = 3$.

Solution

Here the $f(z) = x^2 - y^2 + 2y + 0i$ is analtytic inside the domianthat is the center of the circle is $a = 5 - 2i$, and the radius is $r = 3$unit.The following Figure 2 depicts the circle $|z - 5 + 2i| = 3$.

Using the Gauss mean value thorem.

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})d\theta$$

$a = 5 - 2i$,

$$a = 5 - 2i,$$

$$x = 5, y = -2$$

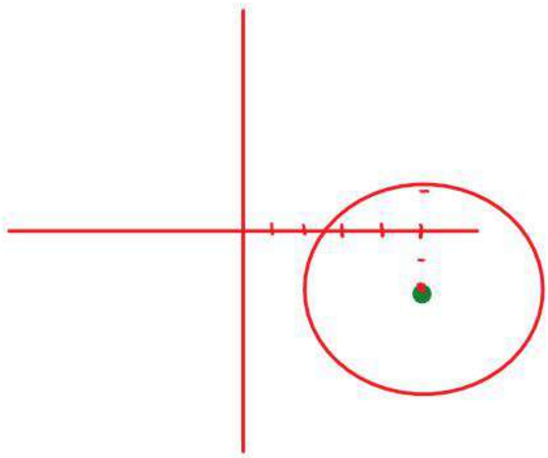


Figure 2: The domain with center 5-2i, and radius 3

$f(a)$ is the mean value so

$$f(a) = 5^2 - (-2)^2 + 2(-2)$$

$$f(a) = 25 - 4 - 4.$$

$$f(a) = 17.$$

6.2 Cauchy's Inequality

Suppose $f(z)$ is analytic inside and on a circle C of radius r and center at $z = a$. Then

$|f^n(a)| \leq \frac{M.n!}{r^n}, n = 0,1,2 \ldots$

where M is a constant such that$|f(z)| < M$ on C i.e., M is an upper bound of $|f(z)|$ on C.

Proof

We have by Cauchy's integral formulas

$\oint_c \left(\frac{f(z)}{(z-z_0)^{n+1}}\right) dz = 2\pi i \frac{f^n(z_0)}{n!}.$

$f^n(a) = \frac{n!}{2\pi i}\oint_c \left(\frac{f(z)}{(z-a)^{n+1}}\right) dz.$

It is also given that $|z - a| = r$ and $|f(z)| < M$.

$|f^n(a)| = \left|\frac{n!}{2\pi i}\oint_c \left(\frac{f(z)}{(z-a)^{n+1}}\right) dz\right| \ldots(*)$

Now

$$z - a = re^{i\theta}$$

$$z = a + re^{i\theta}$$

$$dz = re^{i\theta}.i.d\theta$$

$$|dz| = |re^{i\theta}.i.d\theta|$$

$$|dz| = |r|.|e^{i\theta}|.|i|.|d\theta|$$

Here $|r| = r$

$|e^{i\theta}| = |cos\theta + isin\theta| = 1,$

$|i| = 1,$

$$|d\theta| = d\theta.$$

Finally

$$|dz| = r.d\theta$$

Now using equation …(*)

$$|f^n(z_0 = a)| = \left|\frac{n!}{2\pi i}\right|\left|\oint_c\left(\frac{f(z)}{(z-a)^{n+1}}\right)\right||dz|$$

$$|f^n(z_0 = a)| = \frac{n!}{2\pi|i|}\left|\oint_c\left(\frac{f(z)}{(z-a)^{n+1}}\right)\right|r.d\theta$$

$$|f^n(z_0 = a)| = \frac{n!}{2\pi}\left|\oint_c\left(\frac{f(z)}{(z-a)^{n+1}}\right)\right|r.d\theta$$

$$|f^n(z_0 = a)| = \frac{n!}{2\pi}\oint_c\left(\frac{|f(z)|}{|(z-a)^{n+1}|}\right)r.d\theta$$

It is also given that $|z-a| = r$ and $|f(z)| < M$.

$$|f^n(a)| \le \frac{n!}{2\pi}\oint_c\left(\frac{M}{r^{n+1}}\right)r.d\theta$$

As $0 \le \theta \le 2\pi$.

$$|f^n(a)| \le \frac{n!}{2\pi}\int_0^{2\pi}\left(\frac{M.r}{r^n.r}\right).d\theta$$

$$|f^n(a)| \le \frac{n!}{2\pi}\frac{M}{r^n}\int_0^{2\pi}d\theta$$

$$|f^n(a)| \le \frac{n!}{2\pi}\frac{M}{r^n}[\theta]_0^{2\pi}$$

$$|f^n(a)| \le \frac{n!}{2\pi}\frac{M}{r^n}2\pi$$

$$|f^n(a)| \le \frac{n!}{r^n}M$$

6.3 Maximum Modulus Theorem

Suppose $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

Proof

Suppose $f(z)$ is analytic inside and on a circle C with center at a and radius r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})d\theta$$

$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(a + re^{i\theta})|d\theta$ ....(#)

Let us suppose that $|f(a)|$ is a maximum so that $|f(a + re^{i\theta})| \leq |f(a)|$.

If $|f(a + re^{i\theta})| < |f(a)|$ for one value of θ then, by continuity of f, it would hold for a finite arc, say $\theta_1 < \theta < \theta_2$.

But, in such case, the mean value of $|f(a + re^{i\theta})|$ is less than $|f(a)|$, which would contradict ....(#).

It follows, therefore, that in any δ −neighborhood of a, i.e., for $|z - a| < \delta, f(z)$ must be a constant.

If $f(z)$ is not a constant, the maximum value of $|f(z)|$ must occur on C.

Example

Let $f(z) = 2z + 5i$, then let us find

(a) the maximum value of $f(z)$ inside $|z| \leq 1$.

(b) the point where $f(z)$ attains it maximum inside $|z| \leq 1$.

Solution

(a) As it is clear that the $f(z) = 2z + 5i$ is analytic inside $|z| \leq 1$.

It also mentioned that the center of the domain is 0. The radius of the disc is 1.

$$z = r.e^{i\theta} = e^{i\theta}.$$

$$z = e^{i\theta} = cos\theta + isin\theta.$$

$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Now

$$f(z) = 2z + 5i.$$

$$f(z) = 2cos\theta + i2sin\theta + 5i.$$

$$f(z) = 2cos\theta + (2sin\theta + 5)i.$$

$$|f(z)| = |2cos\theta + (2sin\theta + 5)i|.$$

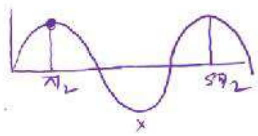
$$|f(z)| = \sqrt{4cos^2\theta + 4\sin^2\theta + 25 + 20sin\theta}$$

$$|f(z)| = \sqrt{4 + 25 + 20sin\theta}$$

$$|f(z)| = \sqrt{29 + 20sin\theta}.$$

As we need the maximum value so we have to consider the maximum value of $sin\theta$.

$sin\theta$ has the maximum value 1 at $\theta = (2n + 1)\pi/2$.



$$|f(z)| = \sqrt{49} = 7$$

So the maximum value of $f(z)$ is 7.

(b) Now we will find the point where $f(z)$ attained its maximum value.

$$z = e^{i\theta} = \cos\theta + i\sin\theta.$$

Since the maximum value 1 at $\theta = (2n+1)\pi/2$. So

$$z = \cos(2n+1)\pi/2 + i\sin(2n+1)\pi/2$$

$z = 0 + i$ is the point where $f(z)$ attained its maximum value.

Task: Let $f(z) = z + 5$, then find

(a) the maximum value of $f(z)$ inside $|z| \leq 2$.

(b) the point where $f(z)$ attains it maximum inside $|z| \leq 2$.

Solution

(a) As it is clear that the $f(z) = z + 5$ is analytic inside $|z| \leq 2$.

It also mentioned that the center of the domain is 0. The radius of the disc is 2.

$$z = r.e^{i\theta} = 2.e^{i\theta}.$$

$$z = 2.e^{i\theta} = 2.\cos\theta + i2.\sin\theta.$$

$$|z| = \sqrt{4.\cos^2\theta + 4.\sin^2\theta} = 2.$$

Now

$$f(z) = z + 5.$$

$$f(z) = 2\cos\theta + i2\sin\theta + 5.$$

$$f(z) = 2\cos\theta + 5 + (2\sin\theta)i.$$

$$|f(z)| = |2\cos\theta + 5 + (2\sin\theta)i|.$$

$$|f(z)| = \sqrt{4\cos^2\theta + 4\sin^2\theta + 25 + 20\cos\theta}$$

$$|f(z)| = \sqrt{4 + 25 + 20\cos\theta}$$

$$|f(z)| = \sqrt{29 + 20\cos\theta.}$$

As we need the maximum value so we have to consider the maximum value of $\cos\theta$.

$\cos\theta$ has the maximum value 1 at $\theta = 2n\pi$.

$$|f(z)| = \sqrt{49} = 7$$

So the maximum value of $f(z)$ is 7.

(b) Now we will find the point where $f(z)$ attained its maximum value.

$$z = 2.e^{i\theta} = 2.\cos\theta + i2.\sin\theta.$$

Since the maximum value 1 at $\theta = 2n\pi.$. So

$z = 2.\cos 2n\pi. + i2.\sin 2n\pi.$

$z = 2 + 0i$ is the point where $f(z)$ attained its maximum value.

Summary

This section deals with some basic definitions and operations. These are summarized below:

- Suppose $f(z)$ is analytic inside and on a circle C with center at a and radius r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})d\theta$$

- Suppose $f(z)$ is analytic inside and on a circle C of radius r and center at $z = a$. Then

$|f^n(a)| \leq \frac{M\,n!}{r^n}, n = 0,1,2 \ldots$

where M is a constant such that $|f(z)| < M$ on C i.e., M is an upper bound of $|f(z)|$ on C.

- Suppose $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

Keywords

Maximum value of $|f(z)|$

The value where $f(z)$ attains its maximum

Self Assessment

1. Suppose $\boldsymbol{f(z) = sin(z)}$ is analytic inside and on the curve C with center "$\frac{\pi}{2}$" and radius "2" then using mean value theorem what is the value of-

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(\frac{\pi}{2} + 2e^{i\theta}\right) d\theta ?$$

A. 3
B. 1
C. 21
D. 45

2. Suppose $\boldsymbol{f(z) = cos(z)}$ is analytic inside and on the curve C with center "$\frac{\pi}{2}$" and radius "2" then using mean value theorem what is the value of-

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\left(\frac{\pi}{2} + 2e^{i\theta}\right) d\theta ?$$

A. 3
B. 1
C. 0
D. 45

3. Suppose $f(z) = sin^2(z)$ is analytic inside and on the curve C with center "$\frac{\pi}{6}$" and radius "2" then using mean value theorem what is the value of-

$$\int_0^{2\pi} sin^2\left(\frac{\pi}{6} + 2e^{i\theta}\right) d\theta ?$$

A. π
B. $\frac{\pi}{2}$
C. 2π
D. $\frac{\pi}{4}$

4. Suppose $f(z) = cos^2(z)$ is analytic inside and on the curve C with center "0+0i" and radius "2" then using mean value theorem what is the value of-

$$\int_0^{2\pi} Cos^2(2e^{i\theta}) d\theta ?$$

A. π
B. 0
C. 2π
D. $\frac{\pi}{4}$

5. Suppose $f(z) = x^2 + y^2$ is analytic inside the curve C with center "-5+2i" and radius "3" then using mean value theorem what is the value of $f(z)$

A. 20
B. 29
C. 21
D. 0

6. Suppose $f(z) = x^2 - y^2 + 2y$ is analytic inside the curve C with center "5-2i" and radius "3" then using mean value theorem what is the value of $f(z)$

A. 20
B. 29
C. 21
D. 0

7. Suppose $f(z)$ is analytic inside and on the curve C with center "$\frac{\pi}{2}$" and radius "2" and M is the upper bound of $f(z)$ on C then using Cauchy inequality which one of the following is true-

A. $\left|f'\left(\frac{\pi}{2}\right)\right| \leq \frac{M}{2}$

B. $\left|f'\left(\frac{\pi}{2}\right)\right| \leq \frac{M}{3}$

C. $\left|f'\left(\frac{\pi}{2}\right)\right| > \frac{M}{2}$

D. $\left|f'\left(\frac{\pi}{2}\right)\right| \geq \frac{M}{2}$

8. Suppose $\boldsymbol{f(z)}$ is analytic inside and on the curve C with center *"0+i0"* and radius *"5"* and $\mathbf{10}$ is the upper bound of $\boldsymbol{f(z)}$ on C then using Cauchy inequality which one of the following is true-

A. $|f'(0)| \leq 2$

B. $|f'(0)| \leq \frac{10}{3}$

C. $|f'(0)| > \frac{10}{2}$

D. $|f'(0)| \geq \frac{10}{5}$

9. Suppose $\boldsymbol{f(z)}$ is analytic inside and on the curve C with center "$\frac{\boldsymbol{\pi}}{\mathbf{2}}$" and radius "2" and $\mathbf{20}$ is the upper bound of $\boldsymbol{f(z)}$ on C then using Cauchy inequality which one of the following is true-

A. $\left|f'\left(\frac{\pi}{2}\right)\right| \leq 10$

B. $\left|f'\left(\frac{\pi}{2}\right)\right| \leq 5$

C. $\left|f'\left(\frac{\pi}{2}\right)\right| > 0$

D. $\left|f'\left(\frac{\pi}{2}\right)\right| \geq 15$

10. Suppose $\boldsymbol{f(z)}$ is analytic inside and on the curve C with center *"10+i0"* and radius *"5"* and $\mathbf{15}$ is the upper bound of $\boldsymbol{f(z)}$ on C then using Cauchy inequality which one of the following is true-

A. $|f'(10)| \leq 5$

B. $|f'(0)| \leq \frac{15}{3}$

C. $|f'(10)| > 0$

D. $|f'(0)| \geq 5$

11. Suppose $\boldsymbol{f(z)} = \mathbf{2z + 5i}$ is analytic inside and on the curve C with center *"0+0i"* and radius *"1"* then the maximum value of $\boldsymbol{f(z)}$ is__?

A. 7

B. 10

C. 3

D. 4

12. Suppose $f(z) = 2z + 5i$ is analytic inside and on the curve C with center "*0+0i*" and radius "*1*" then $f(z)$attains its maximum value at__?

A. z=7
B. z=10
C. z=3
D. z=i

13. Suppose $f(z) = 2z$ is analytic inside and on the curve C with center "*0+0i*" and radius "*1*" then the maximum value of $f(z)$ is__?

A. 2
B. 10
C. 1
D. 4

14. Suppose $f(z) = z$ is analytic inside and on the curve C with center "*0+0i*" and radius "5" then the maximum value of $f(z)$ is__?

A. 2
B. 5
C. 1
D. 4

15. Suppose $f(z) = z + 5$ is analytic inside and on the curve C with center "*0+0i*" and radius "2" then the maximum value of $f(z)$ is__?

A. 5
B. 10
C. 3
D. 7

Answers for Self Assessment

1.	B	2.	C	3.	B	4.	B	5.	B
6.	C	7.	A	8.	A	9.	A	10.	A
11.	A	12.	D	13.	A	14.	B	15.	D

Review Questions

1. Let $f(z) = 2z + 3i$, then let us find the maximum value of $f(z)$ inside $|z| \leq 1$.
2. Let $f(z) = 2z + 3i$, then let us find the point where $f(z)$ attains it maximum inside $|z| \leq 1$.
3. Let $f(z) = z + i$, then let us find the maximum value of $f(z)$ inside $|z| \leq 1$.
4. Let $f(z) = z + i$, then let us find the point where $f(z)$ attains it maximum inside $|z| \leq 1$.
5. Suppose $f(z)$ is analytic inside and on a circle C of radius 2 and center at $z = 1$, such that $|f(z)| < 10$ on C i.e., 10 is an upper bound of $|f(z)|$ on C. Then using Cauchy inequality find the $|f^2(a)| =?$
6. Suppose $f(z)$ is analytic inside and on a circle C of radius 1 and center at $z = 0$, such that $|f(z)| < 5$ on C i.e., 5 is an upper bound of $|f(z)|$ on C. Then using Cauchy inequality find the $|f^2(a)| =?$
7. Evaluate the $\frac{1}{2\pi}\int_0^{2\pi} \text{Cos}^2\left(\frac{\pi}{6} + 2e^{i\theta}\right) d\theta$
8. Evaluate the $\frac{1}{2\pi}\int_0^{2\pi} sin\left(\frac{\pi}{4} + e^{i\theta}\right) d\theta$
9. Find the mean value of $x^2 - y^2$ over the circle $|z - 2 + i| = 1$.
10. Find the mean value of $x^2 + 2y$ over the circle $|z - 5 + 2i| = 5$.

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 07: Liouville's Theorem

CONTENTS

Objectives

The analytic functions are the core of complex analysis. Unlike real-valued calculus, the presence of a complex derivative has significant ramifications for the function's characteristics. We'll look at a few far-reaching theorems in this part, as well as several key theorems along the way. After this unit, you will be able to-

- understand the concept of entire function.
- prove the Liouville's Theorem using the Cauchy integral formula.
- prove the fundamental theorem of algebra using the Liouville's Theorem.
- prove Morera's theorem using the Cauchy's theorem.

Introduction

In this section first, the Liouville's Theorem is discussed for the entire function. After that using Liouville's Theorem, the fundamental theorem of algebra is discussed and then the Morera's theorem would be proved using the Cauchy's theorem.

7.1 Entire function

If $f(z)$ is analytic on the whole complex plane, then it is said to be entire.

Or

A function $f(z)$ is called entire if it has a representation of the form $f(z) = \sum_{k=0}^{\infty} a_k z^k$ valid for $|z| < \infty$. This class of functions is designated by E. E is a linear space.

Example

Some examples of entire functions are-

- $\frac{\text{Sin}(z)}{z}$
- 2^z
- $\int_0^z e^{t^2} dt$

Task: Check whether the $f(z) = \frac{e^z}{z}$ is entire?

Solution

$$f(z) = \frac{e^z}{z}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z} \right\}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{\frac{e^{(z_0+\Delta z)}}{z_0+\Delta z} - \frac{e^{z_0}}{z_0}}{\Delta z} \right\}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{\frac{\left(1+\frac{(z_0+\Delta z)}{1!}+\frac{(z_0+\Delta z)^2}{2!}+\cdots\right)}{(z_0+\Delta z)} - \frac{\left(1+\frac{z_0}{1!}+\frac{z_0^2}{2!}+\cdots\right)}{z_0}}{\Delta z} \right\}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{\left(1/(z_0+\Delta z) + \frac{1}{1!} + \frac{z_0+\Delta z}{2!} + \cdots\right) - 1/z_0 + \frac{1}{1!} + \frac{z_0}{2!} + \cdots}{\Delta z} \right\}$$

$f'(z_0)$does not exist at $z_0 = 0$.

Hence its not entire function.

7.2 Liouville's Theorem

In complex analysis, Liouville's Theorem states that a bounded holomorphic function on the entire complex plane must be constant. It is named after Joseph Liouville.

Statement:

Let $f(z): C \to C$ be an entire function. Suppose there exists some real number$M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.

Or

If a function $f(z)$ is entire and bounded in the complex plane then $f(z)$ is constant throughout the plane

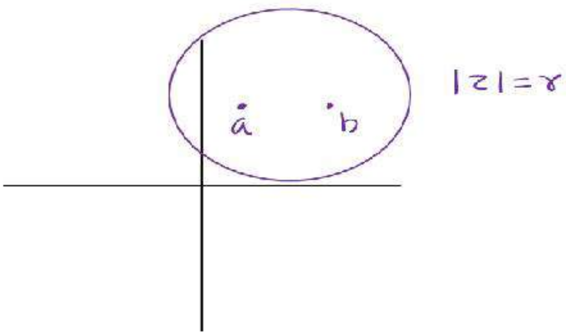
Proof:

It is given that

i. A function $f(z)$ is analytic in the entire complex plane

ii. A function $f(z)$is bounded, that $|f(z)| \leq M$.

Let us consider two points a and b inside a particular domain.



Then using Cauchy integral formula

$$\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-a}\right)dz = f(a)$$
$$\frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-b}\right)dz = f(b)$$

If $f(z)$ is constant throughout the domain then $f(a) = f(b)$.

Now let's prove $f(a) - f(b) = 0$.

$$f(a)-f(b) = \frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-a}\right)dz - \frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-b}\right)dz$$
$$f(a)-f(b) = \frac{1}{2\pi i}\oint_c\left(\frac{f(z)}{z-a} - \frac{f(z)}{z-b}\right)dz$$
$$f(a)-f(b) = \frac{1}{2\pi i}\oint_c f(z)\left(\frac{1}{z-a} - \frac{1}{z-b}\right)dz$$
$$f(a)-f(b) = \frac{1}{2\pi i}\oint_c f(z)\left(\frac{z-b-z+a}{(z-a)(z-b)}\right)dz$$
$$f(a)-f(b) = \frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz$$
$$|f(a)-f(b)| = \left|\frac{a-b}{2\pi i}\oint_c f(z)\left(\frac{1}{(z-a)(z-b)}\right)dz\right|$$
$$|f(a)-f(b)| \le \left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{|(z-a)(z-b)|}\right)|dz|$$
$$|f(a)-f(b)| \le \left|\frac{a-b}{2\pi i}\right|\oint_c |f(z)|\left(\frac{1}{(|z|-|a|)(|z|-|b|)}\right)|dz|$$

Let

$$z = re^{i\theta}$$
$$dz = re^{i\theta}.i.d\theta$$
$$|dz| = \left|re^{i\theta}.i.d\theta\right|$$
$$|dz| = |r|.\left|e^{i\theta}\right|.|i|.|d\theta|$$

Here $|r| = r$

$\left|e^{i\theta}\right| = |cos\theta + isin\theta| = 1,$

$|i| = 1,$

$$|dz| = r.|d\theta|$$
$$|f(a)-f(b)| \le \frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(r-a)(r-b)}\right) r.|d\theta|$$
$$|f(a)-f(b)| \le \frac{a-b}{2\pi}\oint_c M\left(\frac{1}{(1-a/r)(1-b/r)}\right).|d\theta|$$

If $f(z)$ is analytic in the entire complex plane then $|z| = r \to \infty$. So

$$|f(a)-f(b)| \le 0$$
$$f(a)-f(b) = 0$$

Hence, we can say that $f(a) = f(b)$. It means that $f(z)$ is a constant.

Liouville's Theorem proof using Cauchy integral formula for Derivatives.

If f(z) is analytic in a simply-connected region then at any interior point of the region, z_0 inside C. Then say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$\oint_c\left(\frac{f(z)}{(z-z_0)^{n+1}}\right)dz = 2\pi i\frac{f^n(z_0)}{n!}.$

where C is any simple closed curve, in the region, which encloses z_0. Note the case $n = 1$:

$\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{(z-z_0)^2}\right) dz = f'(z_0)$.

$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint_c \left(\frac{f(z)}{(z-z_0)^2}\right) dz\right|$.

$|f'(z_0)| \leq \left|\frac{1}{2\pi i}\right| \oint_c \left(\frac{|f(z)|}{|(z-z_0)^2|}\right) |dz|$.

Here $z = re^{i\theta}$

$dz = re^{i\theta}.i.d\theta$.

$|dz| = \left|re^{i\theta}.i.d\theta\right|$.

$|dz| = |r|.\left|e^{i\theta}\right|.|i|.|d\theta|$.

Here $|z - z_0| = r$

$\left|e^{i\theta}\right| = |cos\theta + isin\theta| = 1$,

$|i| = 1$,

$|dz| = r.|d\theta|$.

$|f'(z_0)| \leq \frac{1}{2\pi}\oint_c \left(\frac{M}{r^2}\right) r.|d\theta|$.

$|f'(z_0)| \leq \frac{1}{2\pi}\oint_c \left(\frac{M}{r}\right).d\theta$.

If $f(z)$ is analytic in the entire complex plane then $r \to \infty$. So

$$|f'(z_0)| \leq 0$$

$$f'(z_0) = 0$$

$$f(z) = constant.$$

7.3 Fundamental Theorem of Algebra

The fundamental theorem of algebra states that every nonconstant polynomial with complex coefficients has a complex root.

In fact, every known proof of this theorem involves some analysis, since the result depends on certain properties of the complex numbers that are most naturally described in topological terms.

It follows from the division algorithm that every complex polynomial of degree n has n complex roots, counting multiplicities. In other words, every polynomial over Csplits over Cor decomposes into linear factors.

Proof

We use Liouville's Boundedness Theorem of complex analysis, which says that every bounded entire function is constant.

Suppose that $P(z)$ is a complex polynomial of degree n with no complex roots; without loss of generality, suppose that $P(z)$ is monic. Then $1/P(z)$ is an entire function; we wish to show that it is bounded. It is clearly bounded when $n = 0$; we now consider the case when $n > 0$.

Let R be the sum of absolute values of the coefficients of $P(z)$, so that $R \geq 1$. Then

for $|z| \geq S$,

$$|P(z)| \geq |z^n| - (R-1)|z^{n-1}| = |z^{n-1}| \cdot \left[|z| - (R-1)\right] \geq R^{n-1}.$$

It follows that $1/P(z)$ is a bounded entire function for $|z| > R$. On the other hand, by the Heine-Borel Theorem, the set of z for $|z| \leq R$which is a compact set so its image under $1/P(z)$ is also compact; in particular, it is bounded.

Therefore, the function is bounded on the entire complex plane when $n > 0$.

Now we apply Liouville's theorem and see that $1/P(z)$ is constant, so $P(z)$ is a constant polynomial. The theorem then follows.

7.4 Morera's Theorem

Let $f(z)$ be continuous in a simply-connected region R and suppose that

$$\oint_c f(z)dz = 0$$

around every simple closed curve C in R. Then *f(z)* is analytic in *R*. This theorem, due to Morera, is often called the converse of Cauchy's theorem. It can be extended to multiply-connected regions.

Proof

For a proof, which assumes that $f'(z)$ is continuous in R.If $f(z)$ has a continuous derivative in$R$, then we can apply Green's theorem to obtain

IfP andQ are continuousand have continuous partial derivatives $P_x, P_y, Q_x$and $Q_y$at all points on Cthen

$$\oint_c (P(x,y)dx + Q(x,y)dy) = \iint [Q_x(x,y) - P_y(x,y)]dxdy.$$

Let*f(z)*= *f(x+iy)*=*u(x,y)*+*iv(x,y)*.Then

$$\oint_c (u(x,y) + iv(x,y))(dx + idy) = \oint_c (u(x,y)dx - v(x,y)dy) + i\oint_c (v(x,y)dx + u(x,y)dy)$$

$$= \iint [-v_x(x,y) - u_y(x,y)]dxdy + i\iint [u_x(x,y) - v_y(x,y)]dxdy$$

If $\oint_c f(z)dz = 0$ around every closed path C in R, we must have

$$\oint_c (u(x,y)dx - v(x,y)dy) = 0, \oint_c (v(x,y)dx + u(x,y)dy) = 0.$$

It means

$$\oint -v_x(x,y) - u_y(x,y)]dxdy = 0, \iint [u_x(x,y) - v_y(x,y)]dxdy = 0$$

C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

are satisfied and thus (since these partial derivatives are continuous) it follows that$f(z) = u + iv$ is analytic.

Summary

- Let $f(z): C \to C$ be an entire function. Suppose there exists some real number$M \geq 0$ such that $|f(z)| \leq M$ for all $z \in C$ Then $f(z)$ is a constant function.
- Let $f(z)$ be continuous in a simply-connected region R and suppose that

$\oint_c f(z)dz = 0$around every simple closed curve C in R. Then *f(z)* is analytic in *R*.

- The fundamental theorem of algebra states that every nonconstant polynomial with complex coefficients has a complex root.

Keywords

- **Entire function:** If $f(z)$ is analytic on the whole complex plane, then it is said to be entire.
- **Bounded function**: Suppose there exists some real number$M \geq 0$ such that $|f(z)| \leq M$, then $f(z)$ is bounded

Self Assessment

1. Which of the following is true in the reference of Liouville's Theorem?

A. If a function f is entire and bounded in the whole complex plane, then f is constant throughout the entire complex plane.

B. If a function f is entire and unbounded in the whole complex plane, then f is constant throughout the entire complex plane

C. If a function f is entire and bounded in the $|z| \leq 1$ complex plane, then f is constant throughout the entire complex plane

D. If a function f is entire and bounded in the $|z| \geq 1$ complex plane, then f is constant throughout the entire complex plane

2. If a function $\boldsymbol{f(z)}$ is entire and bounded in the whole complex plane, and $\boldsymbol{a}, \boldsymbol{b}$ are two points in the plane then which of the following is true in the reference of Liouville's Theorem?

A. $f(a) = f(b)$

B. $f(a) \neq f(b)$

C. $f(a) = 10\, f(b)$

D. $f(a) = -f(b)$

3. If a function $\boldsymbol{f(z)}$ is entire and bounded in the whole complex plane, then which of the following is true in the reference of Liouville's Theorem?

A. $f'(z) = 0$

B. $f'(z) \neq 0$

C. $f'(z) = 1$

D. $f'(z) = 10$

4. Let $\boldsymbol{f(z)}$ is the monic polynomial with degree 2 and real coefficients such that $\boldsymbol{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}$ is one of the roots of $\boldsymbol{f(z)}$ then $\boldsymbol{f(0)}$ =?

A. 1

B. 2

C. 0

D. 3

5. Let $\boldsymbol{f(z)}$ is the monic polynomial with degree 2 and real coefficients such that $\boldsymbol{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}$ is one of the roots of $\boldsymbol{f(z)}$ then $\boldsymbol{f(1)}$ =?

A. 1

B. 2

C. 0

D. 3

6. Let $f(z)$ is the monic polynomial with degree 2 and real coefficients such that $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is one of the roots of $f(z)$ then $f(2)$ =?

A. 1
B. 2
C. 0
D. 3

7. Let $f(z)$ is the monic polynomial with degree 2 and real coefficients such that $z = 1 + i$ is one of the roots of $f(z)$ then $f(0)$ =?

A. 1
B. 2
C. 0
D. 3

8. Let $f(z)$ is the monic polynomial with degree 2 and real coefficients such that $z = 1 + i$ is one of the roots of $f(z)$ then $f(1)$ =?

A. 1
B. 2
C. 0
D. 3

9. Let $f(z)$ is the monic polynomial with degree 2 and real coefficients such that $z = 1 + i$ is one of the roots of $f(z)$ then $f(2)$ =?

A. 1
B. 2
C. 0
D. 3

10. Let $f(z)$ is the monic polynomial with degree 3 and real coefficients such that $z = 1 + i, 1$ both are the roots of $f(z)$ then $f(0)$ =?

A. 1
B. -2
C. 0
D. 3

11. Let $f(z)$ is the monic polynomial with degree 3 and real coefficients such that $z = 1 + i, 1$ both are the roots of $f(z)$ then $f(1)$ =?

A. 0
B. 3
C. 1
D. 2

12. Let $f(z)$ is the monic polynomial with degree 3 and real coefficients such that $z = 1 + i, 1$ both are the roots of $f(z)$ then $f(2)$ =?

A. 1
B. 2
C. 0
D. 3

13. Let $f(z)$ is the monic polynomial with degree 3 and real coefficients such that $z = 1 + i, 1$ both are the roots of $f(z)$ then $f(3)$ =?

A. 1
B. 2
C. 10
D. 3

14. If $f(z)$ is continuous in a simply connected domain *D* and if $\int_c f(z)dz = 0$ for every simple closed curve *C in D* then ___?

A. $f(z)$ is analytic in *D*
B. $f(z)$ is not analytic in *D*
C. $f(z)$ is not continuous in *D*
D. $f(z)$ is not differentiable in *D*

15. The Moreira's theorem is converse of Cauchy Integral theorem.

A. True
B. False

Answers for Self Assessment

1.	A	2.	A	3.	A	4.	A	5.	A
6.	D	7.	B	8.	B	9.	B	10.	B
11.	A	12.	B	13.	C	14.	A	15.	A

Review Questions

1. A force field is given by $F = 3z + 5$. Find the work done in moving an object in this force field along the parabola $z = t^2 + it$ from $z = 0, to\ z = 4 + 2i$.
2. Suppose P(x, y) and Q(x, y) are conjugate harmonic functions and C is any simple closed curve. Prove that $\oint_c (P(x,y)dx + Q(x,y)dy) = 0$.
3. Prove that every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots, a_n z^n = 0$, where the degree n ≥ 1 and $a_n \neq 0$, has exactly n roots.
4. Determine all the roots of the equation $1 + z + z^2 = 0$
5. Determine all the roots of the equation $1 + 2z + 2z^2 = 0$
6. Determine all the roots of the equation $1 + 2z + 5z^2 = 0$
7. Determine all the roots of the equation $10 + 2z + 2z^2 = 0$

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 08: Zeroes and Singularities

CONTENTS

Objective

After this unit, you will be able to

- understand the concept of zeroes of complex function.
- understand the different type singularities of complex function

Introduction

In this unit first, we will discuss the zeroes of complex plane, then the singularity of a complex function will be classified. The singularity can be explored using the Laurent series expansion, but in this unit, we will classify the different types like removable, essential, pole without the Laurent series.

8.1 Zeroes of theAnalytic Function

Suppose that a function $f(z)$ is analytic at a point z_0. Then the zeroes of $f(z)$ are the points $z_0$where $f\ (z_0)\ =\ 0$.

We know that all of the derivatives $f^{(n)}(z)\ \ (n\ =\ 1,2,\ldots)$ exist at z_0. If $f\ (z_0)\ =\ 0$ and if there is a positive integer m such that $f^{(m)}(z_0)\ \neq\ 0$ and each derivative of lower-order vanishes at z_0 , means

$$f^{(')}(z_0) = 0,$$

$$f^{('')}(z_0) = 0,$$

$$.$$

$$.$$

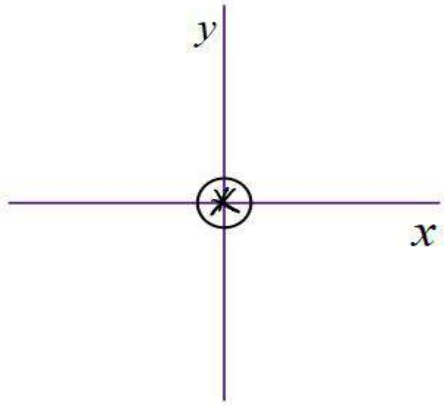
$$.$$

$$f^{(m)}(z_0) \neq 0,$$

then $f(z)$ is said to have a zero of order m at z_0.

Example:

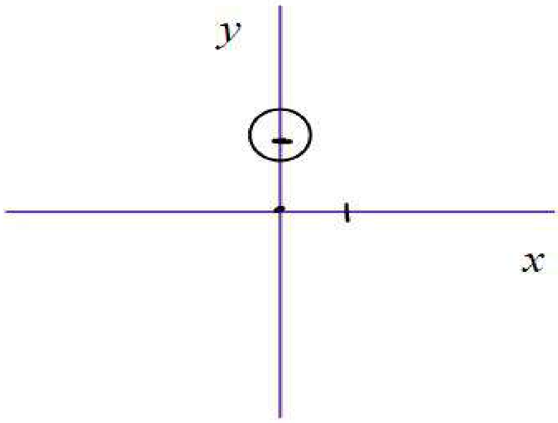
$f(z) = z$ has a simple zero at $z = 0$.



As $f(z) = z$ and $f'(z) = 1 \neq 0$. So $f(z) = z$ has a simple zero or zero of order one at $z = 0$.

Example:

$f(z) = (z - i)^2$ has a zero of order two at $z = i$.



As $f(z) = (z - i)^2$ and

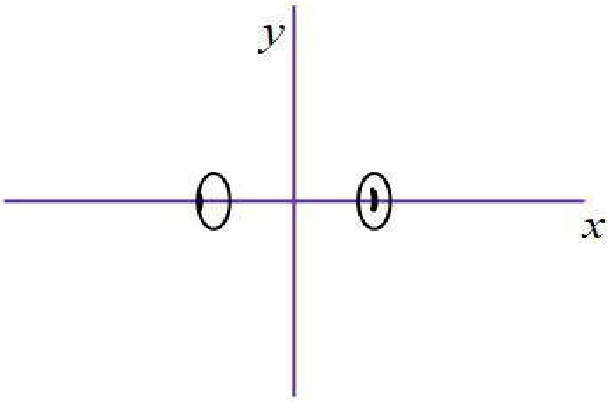
$f'(z) = 2(z - i) = 0$ at $z = i$.

$f''(z) = 2 \neq 0$ at $z = i$.

So $f(z) = (z - i)^2$ has a zero of order two at $z = i$.

Example:

$f(z) = z^2 - 1$ has two simple zeros at $z = \pm 1$.



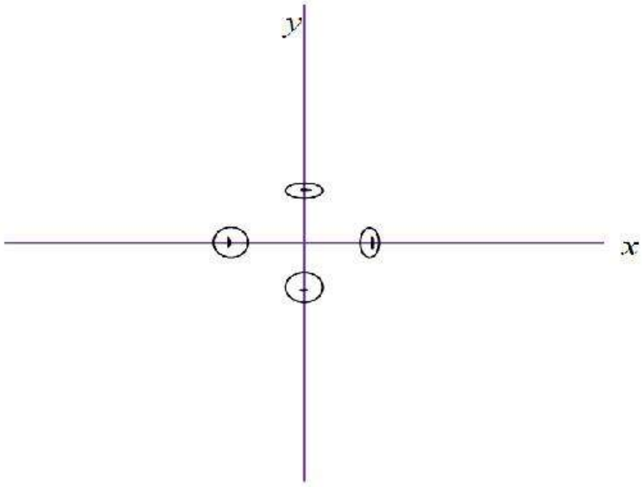
As $f(z) = z^2 - 1$ and

$f'(z) = 2z \neq 0$ at $z = 1$, and $z = -1$.

So $f(z) = z^2 - 1$ has a simple zero or order one at $z = 1$, and $z = -1$.

Example

$f(z) = z^4 - 1$ has a zero of order one at $z = -i, i, -1, and\ 1$.



As $f(z) = z^4 - 1$ and

$f(z) = (z - 1)(z + i)(z - i)$.

$f(z) = 0$ at $z = -i, i, -1, and\ 1$.

$f'(z) = 4z^3 \neq 0$ at $z = i$.

So $f(z) = z^4 - 1$ has a zero of order one at $z = -i, i, -1, and\ 1$.

Theorem 8.1. Let a function $f(z)$ be analytic at a point z_0. It has a zero of order $m\ at\ z_0$ if and only if there is a function $g(z)$, which is analytic and nonzero at z_0, such that $f(z) = (z - z_0)^m g(z)$.

Example

The polynomial $f(z) = z^3 - 8 = (z - 2)(z^2 + 2z + 4)$ has azero of order $m = 1$ at $z_0 = 2$ since

$f(z) = (z - 2)g(z).$

where $g(z) = z^2 + 2z + 4$, and because $f(z)$ and $g(z)$ are entire and $g(2) = 12 \neq 0$.

Note how the fact that $z_0 = 2$ is a zero of order $m = 1$ of $f(z)$ also follows from theobservations that $f(z)$ is entire and that

$f(2) = 0$ and $f'(2) = 12 \neq 0.$

Example

The entire function $f(z) = z(e^z - 1)$ has a zero of order$m = 2$ at the point $z_0 = 0$ since

$f(0) = f'(0) = 0$, and $f''(0) = 2 \neq 0.$

$f(z) = (z - 0)^2 g(z).$

where $g(z)$ is the entire function

$g(z) = (e^z - 1)/z$ $when$ $z = 0,$

$g(z) = 1$ $when$ $z = 0.$

Task: Find the zeroes of $f(z) = e^z + e^{-z}$.

Solution

$f(z) = 0.$

$e^z + e^{-z} = 0.$

$e^z + \frac{1}{e^z} = 0.$

$e^{2z} + 1 = 0.$

$e^{2z} = -1.$

$e^{2(x+iy)} = -1.$

$e^{2x}.e^{i2y} = -1.$

$e^{2x}.(cos2y + isin2y) = -1.$

$e^{2x}.cos2y = -1.$

$ie^{2x}.sin2y = 0.$

$x = 0.$

$cos2y = -1.$

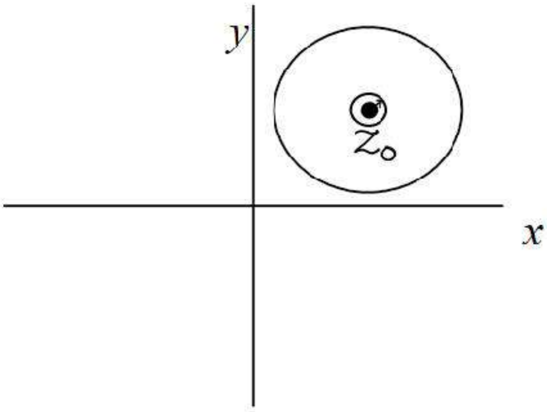
$y = (2n + 1)\pi/2.$

$z_0 = x + iy = 0 + i(2n + 1)\pi/2.$

Hence we can say that there are infinitely many zeroes.

8.2 Singularities

A point z_0 is called a singular point of a function $f(z)$ if $f(z)$ fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.



Example:

Behavior of following functions at $z = 0$.

$$f(z) = \frac{1}{z^9}$$

$$f(z) = \frac{Sinz}{z}$$

$$f(z) = \frac{e^z - 1}{z}$$

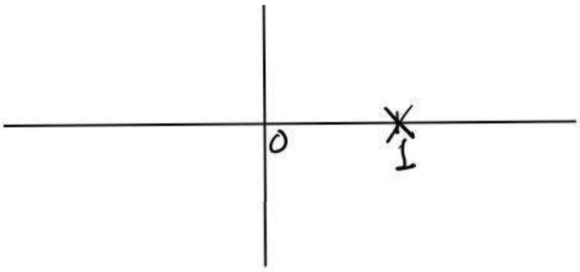
$$f(z) = \frac{1}{\sin(1/z)}$$

We observed that all the functions mentioned above are not analytic at $z = 0$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic.

Example

Behavior of following function at $z = 1$.

$$f(z) = \frac{z}{1 - z}$$



We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

Example

$f(z) = z^2$ is analytic everywhere so it has no singular point.

Example

Behavior of following function in the entire z plane

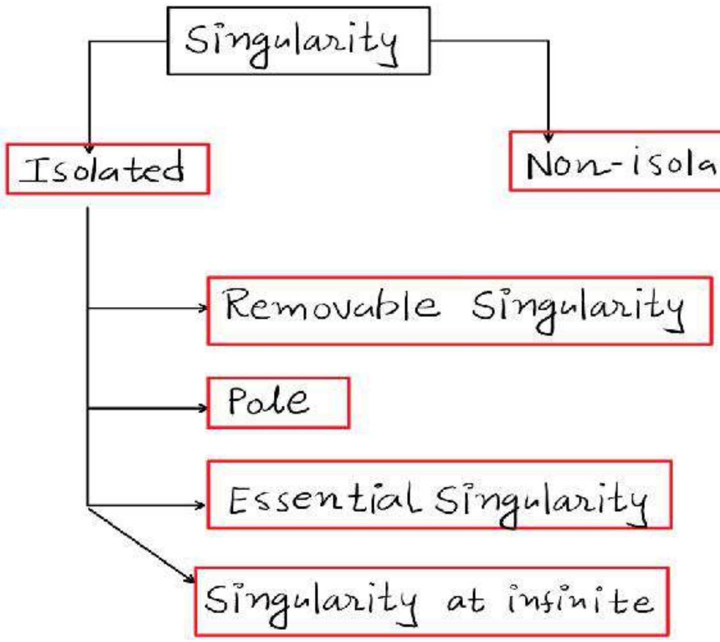
$$f(z) = |z|^2$$

We observed that the $f(z)$is not analytic at $z = 1$.However in every neighbourhood of $z = 0$, there is point at which $f(z)$ is analytic. So $z = 1$ is the singular point of $f(z)$.

8.3 Classification of Singularity

The singularity of a complex function can be classified into two groups, isolated and non-isolated. It can be done via Laurent series expansion, but we can also classify the singularity without the Laurent series expansion. In the forthcoming units we will consider the classification using the Laurent series.

The isolated singularity further can be classified into different type. The following diagram shows the different types of the singularities.



8.4 Isolated Singularity

A point a is called an isolated singularity for $f(z)$ if $f(z)$ is not analytic at $z = a$ and there exist $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$. The neighbourhood$|z - a| < r$ contains no singularity of $f(z)$ except a.

Example

$f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities $z = 0, i, -i$.

Example

$f(z) = \frac{1}{sinz}$ has three isolated singularities $z = 0, \pm\pi, \pm 2\pi, \ldots,$

8.5 Removable singularity

Let $f(z)$is analytic everywhere execpt the point $z_0$inside and on the domain then if the $\lim_{z \to z_0} f(z)$ exists then z_0 is the removable singularity.

Example

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{sinz}{z} = 1.$

$\lim_{z\to z_0} f(z)$exists then $z_0 = 0$ is the removable singularity.

Example

Let $f(z) = \frac{z-sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{z-sinz}{z^3}$

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{1-cosz}{3z^2}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{0+sinz}{6z^1}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{cosz}{6}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{1}{6}$

$\lim_{z\to z_0} f(z)$exists then $z_0 = 0$ is the removable singularity.

8.6 Pole

Let $f(z)$is analytic everywhere execpt the point $z_0$inside and on the domain then if the $\lim_{z\to z_0}(z -$ $z0kfz=\lambda$, where $\lambda\neq 0$, then $z0$ is the pole of order k.

If $k = 1$, then z_0 is the simple pole.

Example

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{e^z}{z}$

$\lim_{z\to z_0} (z - 0)^1 f(z) = \lim_{z\to 0} z\frac{e^z}{z}$

$\lim_{z\to z_0} (z - 0)^1 f(z) = e^0 = 1 \neq 0.$

So $z_0 = 0$ is the pole of order *1* or simple pole.

Example

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{cosz}{z^2}$

$\lim_{z\to z_0} (z - 0)^2 f(z) = \lim_{z\to 0} z^2 \frac{cosz}{z^2}.$

$\lim_{z\to z_0} (z - 0)^2 f(z) = \lim_{z\to 0} cosz = 1 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

Example

Consider $f(z) = \frac{1-e^{2z}}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{1-e^{2z}}{z^3}$

$\lim_{z\to z_0} (z-0)^3 f(z) = \lim_{z\to 0} z^3 \frac{(1-e^{2z})}{z^3} = 0.$

$\lim_{z\to z_0} (z-0)^2 f(z) = \lim_{z\to 0} z^2 \frac{(1-e^{2z})}{z^3}.$

$\lim_{z\to z_0} (z-0)^2 f(z) = \lim_{z\to 0} \frac{(1-e^{2z})}{z}.$

$\lim_{z\to z_0} (z-0)^2 f(z) = \lim_{z\to 0} \frac{-2e^{2z}}{1} = -2 \neq 0.$

So $z_0 = 0$ is the pole of order 2.

8.7 Essential Singularity

Let $f(z)$is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z\to z_0}(z-$ *z0nfz=∞* , then *z0* is essential singularity.

Example

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} e^{1/z}$

$\lim_{z\to z_0} (z-0)^n f(z) = \lim_{z\to 0} z^n e^{1/z}.$

$\lim_{z\to z_0} (z-0)^n f(z) = \lim_{z\to 0} z^n [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,].$

$\lim_{z\to z_0} (z-0)^n f(z) = \infty.$

So $z_0 = 0$ is an essential singularity.

Singularity at infinity

We classify the types of singularities at infinity by letting w = 1/z and analyzing the resulting function at w = 0.

Example

$f(z) = z^3.$

$f(z) = g(w) = 1/w^3.$

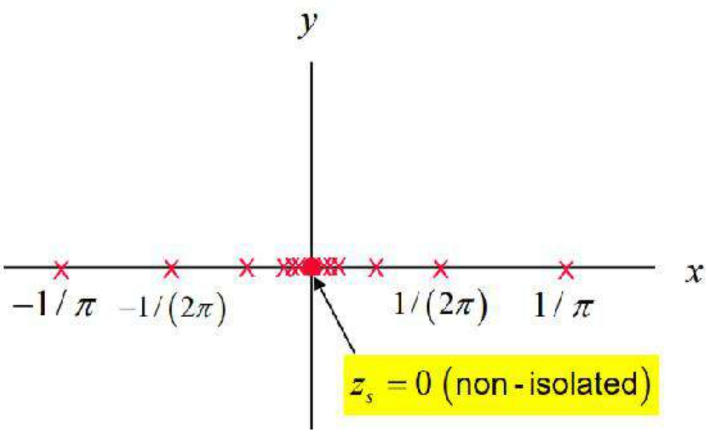
$g(w)$ has a pole of order 3 at w = 0 The function f(z) has a pole of order 3 at infinity.

Non-isolated singularity

A point a is called a non-isolated singularity for $f(z)$ if $f(z)$is not is not isolated at $z = a$.

Example

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$



The function is not analytic in any region 0 < |z| < δ.

Summary

- Suppose that a function $f(z)$ is analytic at a point z_0. Then the zeroes of $f(z)$ are the points $z_0$where $f\ (z_0)\ =\ 0$.
- A point z_0 is called a singular point of a function $f(z)$ if $f(z)$ fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.
- A point a is called an isolated singularity for $f(z)$ if $f(z)$ is not analytic at $z = a$ and there exist $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$. The neighbourhood $|z - a| < r$ contains no singularity of $f(z)$ except a.
- Let $f(z)$is analytic everywhere execpt the point $z_0$inside and on the domain then if the $\lim_{z \to z_0} f(z)$ exists then z_0 is the removable singularity.
- Let $f(z)$is analytic everywhere execpt the point $z_0$inside and on the domain then if the $\lim_{z \to z_0}(z - z_0)^k f(z) = \lambda$, where $\lambda \neq 0$, then z_0 is the pole of order k.
- A point a is called a non-isolated singularity for $f(z)$ if $f(z)$is not is not isolated at $z = a$.
- Let $f(z)$is analytic everywhere execpt the point z_0 inside and on the domain then if the $\lim_{z \to z_0}(z - z_0)^n f(z) = \infty$, then z_0 is essential singularity.

Keywords

- Zero: Suppose that a function $f(z)$ is analytic at a point z_0. Then the zeroes of $f(z)$ are the points $z_0$where $f\ (z_0)\ =\ 0$.
- Pole: Let $f(z)$is analytic everywhere execpt the point $z_0$inside and on the domain then if the $\lim_{z \to z_0}(z - z_0)^k f(z) = \lambda$, where $\lambda \neq 0$, then z_0 is the pole of order k.

Self Assessment

1. The polynomial $\boldsymbol{f(z) = z^3 - 8}$ has a____?

A. Zero of order one at $z_0 = 2$

B. Zero of order three at $z_0 = 2$

C. Zero of order two at $z_0 = 2$

D. Zero of order one at $z_0 = 8$

2. The polynomial $f(z) = e^z(z-1)$ has a____?

A. Zero of order one at $z_0 = 0$

B. Zero of order three at $z_0 = 0$

C. Zero of order two at $z_0 = 0$

D. Zero of order one at $z_0 = 1$

3. The polynomial $f(z) = z^4 - 1$ has a____?

A. Zero of order one at $z_0 = 1$

B. Zero of order four at $z_0 = 1$

C. Zero of order two at $z_0 = 0$

D. Zero of order one at $z_0 = 0$

4. Consider the $f(z) = \frac{z}{1-z}$then

A. $z_0 = 1$ is the singular point of $f(z)$

B. $z_0 = 0$ is the singular point of $f(z)$

C. $z_0 = 10$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

5. Consider the $f(z) = z^2$then

A. $z_0 = 1$ is the singular point of $f(z)$

B. $z_0 = 0$ is the singular point of $f(z)$

C. $z_0 = 10$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

6. Consider the $f(z) = \frac{z^2-9}{z^2(z-1)(z-1-2i)}$then

A. $z_0 = 1$ is one of the singular points of $f(z)$

B. $z_0 = 3$ is the singular point of $f(z)$

C. $z_0 = -3$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

7. Consider the $f(z) = \frac{z^2-9}{z^2(z-1)(z-1-2i)}$then

A. $z_0 = 1 + 2i$ is one of the singular points of $f(z)$

B. $z_0 = 3$ is the singular point of $f(z)$

C. $z_0 = -3$ is the singular point of $f(z)$

D. There is no singular point of $f(z)$

8. $f(z) = \frac{e^{-z}}{(z+2)^3}$ has ___

A. a pole of order three at $z = -2$

B. a simple pole at $z = -2$

C. a pole of order two at $z = -2$

D. a simple pole at $z = 0$

9. $f(z) = \frac{\tan z}{z}$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order two at $z = 0$

D. essential singularity at $z = 0$

10. $f(z) = e^{\frac{1}{z^3}}$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order two at $z = 0$

D. essential singularity at $z = 0$

11. $f(z) = \frac{1}{z(e^z - 1)}$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order two at $z = 0$

D. essential singularity at $z = 0$

12. $f(z) = \frac{1}{z^2(e^z)}$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order two at $z = 0$

D. essential singularity at $z = 0$

13. $f(z) = z^3$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order three at $z = \infty$

D. essential singularity at $z = 0$

14. $f(z) = \tan\left(\frac{1}{z}\right)$ has ___

A. Non-isolated singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order three at $z = 0$

D. essential singularity at $z = 0$

15. $\boldsymbol{f(z) = e^z}$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order three at $z = \infty$

D. essential singularity at $z = \infty$

Answers for Self Assessment

1.	A	2.	C	3.	A	4.	A	5.	D
6.	A	7.	A	8.	A	9.	A	10.	D
11.	C	12.	C	13.	C	14.	A	15.	D

Review Questions

1. Determine the zeros of the $z^2 sinz$
2. Determine the zeros of the $(z-1)^2/e^z$
3. Determine the zeros of the $(z-1)(z+2)/(z-5)(z-2)$
4. Determine the singularity of the z^2/e^z
5. Determine the singularity of the $z^2/(z-5)$
6. Determine the singularity of the $z/(z-6)(z-5)$
7. Determine the singularity of the $e^z/(z+sinz)$
8. Determine the singularity of the $2z+1/(z-2)(z-1)$
9. Determine the singularity of the $e^z/(z-sinz)$
10. Determine the singularity of the $sinz + z/sinz)$

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 09: Taylor and Laurent Series

CONTENTS

Objective

A power series with non-negative power terms is called a Taylor series. In complex variable theory, it is common to work with power series with both positive and negative power terms. This type of power series is called a Laurent series. The primary goal of this unit is to establish the relation between convergent power series and analytic functions. More precisely, we try to understand how the region of convergence of a Taylor series or a Laurent series is related to the domain of analyticity of an analytic function. The knowledge of Taylor and Laurent series expansion is linked with more advanced topics, like the classification of singularities of complex functions, residue calculus, analytic continuation, etc. After this unit, you will be able to

- find the Taylor series expansion of a complex function.
- find the Laurent series expansion of a complex function.

Introduction

We originally defined an analytic function as one where the derivative, defined as a limit of ratios, existed. We went on to prove Cauchy's theorem and Cauchy's integral formula. These revealed some deep properties of analytic functions, **e.g.,** the existence of derivatives of all orders. Our goal in this topic is to express analytic functions as infinite power series. This will lead us to Taylor series. When a complex function has an isolated singularity at a **point,** we will replace Taylor series by Laurent series.

9.1 Convergence of Power Series

When we include powers of the variable z in the series, we will call it a power series. In this section we'll state the main theorem we need about the convergence of power series.

Theorem: Consider the power series $(z) = \sum^{\infty} (z - z_0)^n$. There is a number ⍰ ≥ 0 such that:

1. If > 0 then the series converges absolutely to an analytic function for $|z - z_0| <$ ⍰.

2. The series diverges for $|z - z_0| >$, is called the radius of convergence. The disk $|z - z_0| <$ is called the disk of convergence.

3. The derivative is given by term-by-term differentiation $'(z) = \sum_{n=0}^{\infty}(z - z_0)^{n-1}$. The series for $'(z)$ also has radius of convergence R.

4. If γ is a bounded curve inside the disk of convergence then the integral is given by term-by-term integration

$$\int (z)\,dz = \sum_{n=0}^{\infty} \int_{\gamma} {}_n (z - z_0)^n$$

The theorem doesn't say what happens when $|z - z_0| =$.

If $= \infty$ the function (z) is entire.

If $= 0$ the series only converges at the point $z = z_0$. In this case, the series does not represent an analytic function on any disk around z_0. Often (not always) we can find using the ratio test.

9.2 Taylor's Series

Let (z) be analytic in a region containing z_0. Then (z) can be represented as a power series in $z - z_0$ given by

$$(z) = (z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \dots + \frac{f^n(z_0)}{n!}(z - z_0)^2 + \cdots$$

The expansion is valid in the largest open disc with center z_0 contained in D.

Proof

Let > 0 be such that the disc $|z - z_0| < {}_1 <$. Let ${}_1$ be the circle $|z - z_0| = {}_1$.

By Cauchy's integral formula, we have $(z) = \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z}\,d\zeta$

Also, by theorem on higher derivatives we have

$${}^n(z) = \frac{n!}{2\pi i}\int \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\zeta$$

Now $\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0 - z + z_0)}$

$$= \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)},$$

$$= \frac{1}{(\zeta - z_0)}\left(1 - \frac{z - z_0}{\zeta - z_0}\right)^{-1},$$

$$= \frac{1}{(\zeta - z_0)}\left(1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \left(\frac{z - z_0}{\zeta - z_0}\right)^3 + \cdots,\right),$$

Now multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C and using Cauchy integral theorem we get

$$\frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z_0}\,d\zeta + \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta - z_0)^2}\,\zeta(z - z_0) + \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta - z_0)^3}\,\zeta(z - z_0)^2 + \cdots, + \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta - z_0)^n}\,\zeta(z - z_0)^{n-1}.$$

Taking limit as n→∞ in (3) we get,

$$(z) = (z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \dots + \frac{f^n(z_0)}{n!}(z - z_0)^2 + \cdots$$

Example

The Taylor's series for $(z) = \frac{1}{z}$ about $z = 1$ is given by,

$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots$.

Now $f(z) = \frac{1}{z} \Rightarrow f(1) = 1$

$f'(z) = -\frac{1}{z^2} \Rightarrow f'(1) = -1.$

$$f''(z) = 2\frac{1}{z^3} \quad \Rightarrow \quad f''(1) = 2.$$

$$f'''(z) = -6\frac{1}{z^4} \quad \Rightarrow \quad f'''(1) = -6.$$

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$$\frac{1}{z} = f(1) + \frac{-1}{1!}(z-1) + \frac{2}{2!}(z-1)^2 + \frac{-6}{3!}(z-1)^3 + \cdots,$$

This expansion is valid in the disc $|z-1| < 1$.

Example

The Taylor's series for $f(z) = \sin z$ about $z = \frac{\pi}{4}$ is given by,

$$\sin z = f\left(\frac{\pi}{4}\right) + \frac{f'\left(\frac{\pi}{4}\right)}{1!}\left(z - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots.$$

Now $f(z) = \sin z \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$f'(z) = \cos z \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

$$f''(z) = -\sin z \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

$$f'''(z) = -\cos z \quad \Rightarrow \quad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

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$$\sin z = \frac{1}{\sqrt{2}} + \frac{\frac{1}{\sqrt{2}}}{1!}\left(z - \frac{\pi}{4}\right) - \frac{\frac{1}{\sqrt{2}}}{2!}\left(z - \frac{\pi}{4}\right)^2 - \frac{\frac{1}{\sqrt{2}}}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots.$$

This expansion is valid in the entire complex plane.

9.3 Laurent's Series

Any function which is analytic in a region containing the annulus $r_1 < |z - z_0| < r_2$ can be represented in a series of the form

$\sum_{-\infty}^{\infty} a_n (z - z_0)^n$.

Let C_1 and C_2 denote respectively the concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ with $r_1 < r_2$.

Let $f(z)$ be analytic in a region containing the circular annulus $r_1 < |z - z_0| < r_2$. Then $f(z)$ can be represented as a convergent series of positive and negative powers of $z - z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Where

$$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}}\, d\zeta$$

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

Proof:

Let z be any point in the circular annulus $r_1 < |z - z_0| < r_2$.

we have $f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z)} d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta$

As in the proof of Taylor's theorem, we have

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z_0} d\zeta + \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta (z - z_0) + \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta (z - z_0)^2 + \cdots + \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^n} d\zeta (z - z_0)^{n-1}$$

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, + a_{n-1}(z - z_0)^{n-1} + R_n(z).$$

Here

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$R_n(z) = \frac{(z - z_0)^n}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^n(\zeta - z)} d\zeta.$$

Now $\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0 - z + z_0)}$

$$= \frac{1}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)}$$

$$= \frac{1}{(z - z_0)}\left(1 - \frac{\zeta - z_0}{z - z_0}\right)^{-1},$$

$$= \frac{1}{(z - z_0)}\left(1 + \left(\frac{\zeta - z_0}{z - z_0}\right) + \left(\frac{\zeta - z_0}{z - z_0}\right)^2 + \left(\frac{\zeta - z_0}{z - z_0}\right)^3 + \cdots,\right),$$

Now multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C and using Cauchy integral theorem we get

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z - z_0} d\zeta + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(z - z_0)^2} d\zeta(\zeta - z_0) + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(z - z_0)^3} d\zeta(\zeta - z_0)^2 + \cdots, + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(z - z_0)^n} d\zeta(\zeta - z_0)^{n-1}.$$

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots, + \frac{b_{n-1}}{(z - z_0)^{n-1}} + S_n(z).$$

$$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}} d\zeta$$

$$S_n(z) = \frac{1}{2\pi i(z - z_0)^n}\int_{C_1} \frac{f(\zeta)(\zeta - z_0)^n}{(z - \zeta)} d\zeta$$

Now using the maximal modulus principle $R_n(z) \to 0$, $S_n(z) \to 0$, As $n \to \infty$.

Hence

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z)} d\zeta + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(z - \zeta)} d\zeta$$

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, + a_{n-1}(z - z_0)^{n-1} + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots, + \frac{b_{n-1}}{(z - z_0)^{n-1}}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Example

Find the Laurent's series expansion of $(z) = z^2 \square^{1/z}$ about $z = 0$.

Solution

$(z) = z^2 \square^{1/z} \square|$ $z = 0$

Clearly $f(z)$ is analytic at all point $z \neq 0$.

Now, $f(z) = z^2 \left[1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \dots,\right]$

$f(z) = \left[z^2 + \frac{z^2}{z} + z^2 \frac{\left(\frac{1}{z}\right)^2}{2!} + z^2 \frac{\left(\frac{1}{z}\right)^2}{3!} + \dots.\right]$

$f(z) = \left[\frac{1}{2!} + z + z^2 + \frac{1}{3!z} + \frac{1}{4!z^2} \dots,\right]$

This is the required Laurent's series expansion for $f(z)$ at $z = 0$.

Summary

- Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \dots + \frac{f^n(z_0)}{n!}(z - z_0)^n + \cdots$$

The expansion is valid in the largest open disc with center z_0 contained in D.

- Any function which is analytic in a region containing the annulus $R_1 < |z - z0| < R_2$ can be represented in a series of the form

$\sum_{-\infty}^{\infty} a_n(z - z_0)^n$.

Keywords

Consider the power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. There is a number $R \geq 0$ such that:

1. If $R > 0$ then the series converges absolutely to an analytic function for $|z - z_0| < R$.

2. The series diverges for $|z - z_0| > R$, R is called the radius of convergence. The disk $|z - z_0| < R$ is called the disk of convergence.

Self Assessment

1. Expand $f(z) = z^{e^z}$ in a Taylor series around z = 0 then what is the coefficient of $z^{}$?

A. 1

B. 2

C. 3

D. 4

2. Expand $f(z) = z^{e^z}$ in a Taylor series around z = 0 then what is the coefficient of $z^{}$?

A. 1

B. 2

C. 3

D. 4

3. Expand $f(z) = z^{e^z}$ in a Taylor series around z = 0 then what is the coefficient of $z^{}$?

A. 4.5

B. 2

C. 3

D. 4

4. Expand $(z) = z^{|}$ ^{z} in a Taylor series around z = 0 then what is the coefficient of $z^{|}$? A. 1

B. 2

C. 4.5

D. 4

5. Expand $(z) = l[$ $(+ z)$ in a Taylor series around z = 0 then what is the coefficient of $z^{|}$? A. 1

B. 2

C. 3

D. 4

6. Expand $(z) = l[$ $(+ z)$ in a Taylor series around z = 0 then what is the coefficient of $z^{|}$? A. 1

B. -1/2

C. 1/3

D. -1/4

7. Expand $(z) = l[$ $(+ z)$ in a Taylor series around z = 0 then what is the coefficient of $z^{|}$? A. 1

B. -1/2

C. 1/3

D. -1/4

8. Expand $(z) = l[$ $(+ z)$ in a Taylor series around z = 0 then what is the coefficient of $z^{|}$? A. 1

B. -1/2

C. 1/3

D. -1/4

9. Expand $(z) = \frac{z - si[\ z}{z^{|5}}$ in a Laurent series then what is the coefficient of $z^{|}$?

A. 1/6

B. 2

C. 3

D. 4

10. Expand $(z) = \frac{z - \sin z}{z^{\text{[illegible]}}}$ in a Laurent series then what is the coefficient of $z^{\text{[illegible]}}$?

A. 0

B. 2

C. 3

D. 4

11. Expand $(z) = \frac{z - \sin z}{z^{\text{[illegible]}}}$ in a Laurent series then what is the coefficient of $z^{\text{[illegible]}}$?

A. 1/6

B. $-\frac{1}{5!}$

C. $\frac{1}{7!}$

D. 0

12. Expand $(z) = \frac{z - \sin z}{z^{\text{[illegible]}}}$ in a Laurent series then what is the coefficient of $z^{\text{[illegible]}}$?

A. 0

B. 2

C. 3

D. 4

13. Expand $(z) = \frac{z - \sin z}{z^{\text{[illegible]}}}$ in a Laurent series then what is the coefficient of $z^{\text{[illegible]}}$?

A. 1/6

B. $-\frac{1}{5!}$

C. $\frac{1}{7!}$

D. 0

14. Expand $(z) = \frac{z - \sin z}{z^{\text{[illegible]}}}$ in a Laurent series then what is the coefficient of $z^{\text{[illegible]}}$?

A. 0

B. 2

C. 3

D. 4

15. Expand $(z) = \frac{z - \sin z}{z^{\text{[illegible]}}}$ in a Laurent series then what is the coefficient of $z^{\text{[illegible]}}$?

A. 0

B. 2

C. 3

D. 4

Answers for Self Assessment

1. A	2. C	3. A	4. C	5. A
6. B	7. C	8. D	9. A	10. A
11. C	12. A	13. C	14. A	15. A

Review Questions

1. Expand $z\square^{2z}$ in Taylor's series about z=-1
2. Expand 1/ ^{2} in Taylor's series when |z+1|<1
3. Expand 1/ ^{2} in Taylor's series when |z-2|<2
4. Expand cos z into Taylor's series about the point $z = \pi/2$ and determine the region of convergence.
5. Expand $-\frac{1}{(z-1)(z-2)}$, as a power series in z in the region |z|<1.
6. Expand $-\frac{1}{(z-1)(z-2)}$, as a power series in z in the region 1<|z|<2.
7. Expand $-\frac{1}{(z-1)(z-2)}$, as a power series in z in the region |z|≥1.
8. Find Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.
9. If $(z) = \frac{z+4}{(z+3)(z-1)^2}$ .Find Laurent's series expansion in 0 < |z-1| < 4.
10. If $(z) = \frac{z+4}{(z+3)(z-1)^2}$ .Find Laurent's series expansion in |z-1| > 4.

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Rajesh Kumar Chandrawat,Lovely Professional University

Unit 10: Singularity by Laurent Series, Residue, Cauchy Residue Theorem

CONTENTS

Objective

After this unit, you will be able to

- classify the singularity through the Laurent series expansion.
- find the residue at different type of singularity
- evaluate the complex integration using residue theorem.

Introduction

In this unit first, we will discuss the singularity of a complex function using Laurent series expansion. Then the residue of a complex function will be explored and then we will understand the complex integration using residue theorem.

10.1 Classification of Singularity by Laurent Series Expansion

It is also possible to classify the singularity using the Laurent series expansion.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z-a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n$$

Were

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - a)^{-n+1}}\, d\zeta$$

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta$$

The series consisting of the negative powers of $z-a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the principal part or singular part $of\ f(z)\ at\ z = a$.

The singular part of $f(z)$ at $z = a$ determines the character of the singularity.

There are three types of singularities. They are

(i) Removable singularities

(ii) Poles

(iii) Essential singularities.

Removable singularity

Let $\boldsymbol{a}$ be an isolated singularity for $\boldsymbol{f(z)}$. Then $\boldsymbol{a}$ is called a removable singularity if the principal part of $\boldsymbol{f(z)}$ $\boldsymbol{at\ z = a}$ has no terms.

If $\boldsymbol{a}$ is a removable singularity for $\boldsymbol{f(z)}$ then the Laurent's series expansion of $\boldsymbol{f(z)}$ $\boldsymbol{about\ z = a}$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Hence as $\boldsymbol{z \to a}$, $\boldsymbol{f(z) = a_0}$ Hence by defining $\boldsymbol{f(a) = a_0}$ the function $\boldsymbol{f(z)}$ becomes analytic at $\boldsymbol{a}$.

Example

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

Now $f(z) = \frac{sinz}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.)$

$$f(z) = \frac{sinz}{z} = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots.)$$

Here the principal part of $f(z)$ at $z = 0$ has no terms. Hence z = 0 is a removable singularity.

$\lim_{z \to z_0} f(z)$also exists then $z_0 = 0$ is the removable singularity.

Example

Let $f(z) = \frac{z - sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots,)}{z^3}$$

$$f(z) = \frac{\frac{z^3}{3!} - \frac{z^5}{5!} - \ldots,)}{z^3}$$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} - \ldots,$$

$z = 0$ is a removable singularity. By defining $f(0) = 1/6$ the function becomes analytic at $z =$ 0.Also $\lim_{z \to z_0} f(z)$exists then $z_0 = 0$ is the removable singularity.

Pole

Let a be an isolated singularity of $f(z)$. The point a is called a pole if the principal part of $f(z)$ at $z = a$ has a finite number of terms.

If the principal part of $f(z)at$ $z = a$ is given by

$\frac{b1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_r}{(z-a)^r}$, . where $b_r \neq 0$.

We say that a is a pole of order r for $f(z)$. Note: A pole of order 1 is called a simple pole and a pole of order 2 is called double pole.

Example

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$f(z) = \frac{e^z}{z} = \frac{1}{z}(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots,)$

$f(z) = \frac{e^z}{z} = (1/z + 1 + \frac{z}{2} + \frac{z^2}{6} + \ldots,)$

Here the principal part of $f(z)$ $at\ z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$. Also

$\lim_{z \to z_0}(z - 0)^1 f(z) = e^0 = 1 \neq 0$. So $z_0 = 0$ is the pole of order *1* or simple pole.

Example:

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{cosz}{z^2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,}{z^2}$$

The principal part of $f(z)$ $at\ z = 0$ contains the term $1/z^2$. Hence z=0 is a double pole of f(z).

Also $\lim_{z \to z_0}(z - 0)^2 f(z) = \lim_{z \to 0} cosz = 1 \neq 0$.

So $z_0 = 0$ is the pole of order 2.

Essential singularity

Let a be an isolated singularity of $f(z)$. The point a is called an essential singularity of $f(z)$ at $z = a$ if the principal part of $f(z)$ $at\ z = a$ has an infinite number of terms.

Example

Consider $f(z) = e^{1/z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = e^{1/z}$$

$f(z) = [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$.

The principal part of $f(z)$ has infinite number of terms. Hence $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

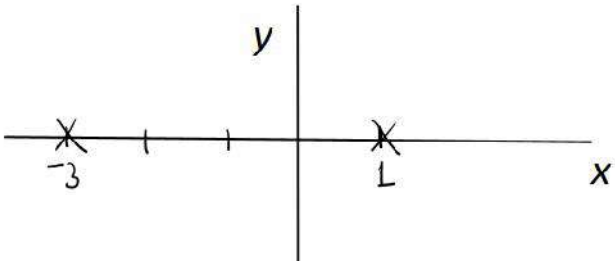
Also $\lim_{z \to z_0}(z - 0)^n f(z) = \infty$.

So $z_0 = 0$ is an essential singularity.

10.2 Meromorphic Functions

A function is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example: $f(z) = \frac{z}{(z-1)(z+3)^2}$



$f(z)$ is analytic everywhere in the complex plane except $z = 1$ *and* $z = -3$.Here $z = 1$ is a simple pile and $z = -3$ is the pole of order 3. We can say that the $f(z)$ has finite number of poles and it's a meromorphic function.

10.3 Residue at a Singularity

The following lemmas provide methods for calculation of residues.

Lemma 1

If $z = a$ is a removable singularity for f(z) then $Res(f(z);\ a) = 0$.

The principal part of $f(z)$ *at* $z = a$ has no terms. $\lim_{z\to a} f(z)$also exists then $z_0 = a$ is the removable singularity.

Example

Let $f(z) = \frac{sinz}{z}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

Now $f(z) = \frac{sinz}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ ....)$

$$f(z) = \frac{sinz}{z} = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ ....)$$

Here the principal part of $f(z)$ *at* $z = 0$ has no terms. Hence z = 0 is a removable singularity.

$\lim_{z\to z_0} f(z)$also exists then $z_0 = 0$ is the removable singularity.

So $Res\left(\frac{sinz}{z};\ 0\right) = 0$.

Example

Let $f(z) = \frac{z-sinz}{z^3}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ ...,)}{z^3}$$

$$f(z) = \frac{\frac{z^3}{3!} - \frac{z^5}{5!} - \ ...,)}{z^3}$$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} - \ ...,$$

$z = 0$ is a removable singularity. By defining $f(0) = 1/6$ the function becomes analytic *at* $z = 0$.

Also

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{z-sinz}{z^3}$

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{1-cosz}{3z^2}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{0+sinz}{6z^1}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{cosz}{6}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0} \frac{1}{6}$

$\lim_{z\to z_0} f(z)$exists then $z_0 = 0$ is the removable singularity.

So $Res\left(\frac{z-sinz}{z^3}, ; 0\right) = 0$.

Lemma 2

If $z = a$ is a simple pole for f(z) then $Res(f(z);\ a) = \lim_{z\to a}(z-a)f(z)$.

Lemma 3

If a is a simple pole for $f(z) and\ f(z) = \frac{g(z)}{z-a}$

where $g(z)$ is analytic at a and $g(a) \neq 0$ then $Res(f(z);\ a) = g(a)$.

Lemma 4

Let a be a pole of order $m > 1$ for $f(z)$ and let $f(z) = \frac{g(z)}{(z-a)^m}$

where $g(z)$ is analytic at a and $g(a) \neq 0$. Then $Res\ (f(z);\ a) = \frac{g^{m-1}(a)}{(n-1)!}$

Example

Consider $f(z) = \frac{e^z}{z}$,clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$f(z) = \frac{e^z}{z} = \frac{1}{z}(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots,)$

$f(z) = \frac{e^z}{z} = (1/z + 1 + \frac{z}{2} + \frac{z^2}{6} + \ldots,)$

Here the principal part of $f(z)\ at\ z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$. Also

$\lim_{z\to z_0}(z-0)^1 f(z) = e^0 = 1 \neq 0$. So $z_0 = 0$ is the pole of order *1* or simple pole.

Hence

$$Res\left(\frac{e^z}{z};\ 0\right) = g(0) = 1$$

Example

Consider $f(z) = \frac{cosz}{z^2}$, clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{cosz}{z^2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,}{z^2}$$

The principal part of $f(z)\ at\ z = 0$ contains the term $1/z^2$. Hence z=0 is a double pole of f(z).

Also $\lim_{z\to z_0}(z-0)^2 f(z) = \lim_{z\to 0} cosz = 1 \neq 0$.

So $z_0 = 0$ is the pole of order 2. Here $g(z) = cosz$

Hence

$$Res\left(\frac{cosz}{z^2};\ 0\right) = g'(0) = -1$$

Lemma 4

Let $f(z)$is analytic everywhere except the point a inside and on the domain then if the $\lim_{z\to z_0}(z-$ $anfz=\infty$, then $z0$ is essential singularity. Then the coefficient of $z{-}a{-}1$ in the Laurent series expansion is the residue of $f(z)$ at $z=a$

Example

Consider $f(z) = ze^{1/z}$, clearly $z = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to a} f(z) = \lim_{z\to 0} ze^{1/z}$

$\lim_{z\to a}(z-0)^1 f(z) = \lim_{z\to 0} z^1 e^{1/z}$.

$\lim_{z\to a}(z-0)^1 f(z) = \lim_{z\to 0} z^1[1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^2+\cdots,]$.

$\lim_{z\to a}(z-0)^1 f(z) = \infty$.

So $z = 0$ is an essential singularity. Now the coefficient of $(z-0)^{-1}$ is $\frac{1}{2!}$

So

$$Res\left(ze^{1/z};\ 0\right) = 1/2$$

10.4 Cauchy Residue Theorem

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C

except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside C. Then

$$\int_c f(z)dz \ = \ 2\pi i\sum_{j=1}^{n} Res\{f(z);\ z_j\}$$

Proof

Let $C_1, C_2, \ldots, C_n$be circles with centres $z_1, z_2, \ldots, z_n$ respectively such that all circles are interior to C and are disjoint with each other. By Cauchy's theorem for multiply connected regions we have,

$$\int_c f(z)dz \ = \ \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \cdots, \int_{c_n} f(z)dz$$

$$\int_c f(z)dz \ = \ 2\pi i\ Res\{f(z);\ z_1\} + 2\pi i\ Res\{f(z);\ z_2\} + \cdots, 2\pi i\ Res\{f(z);\ z_n\}$$

$\int_c f(z)dz \ = \ 2\pi i\sum_{j=1}^{n} Res\{f(z);\ z_j\}$

Example

$\int_c \frac{sinz}{z}dz$,where $C\ is\ |z| \le 2$

Clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

Now $f(z) = \frac{sinz}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.)$

$$f(z) = \frac{sinz}{z} = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots.)$$

Here the principal part of $f(z)\ at\ z\ =\ 0$ has no terms. Hence z = 0 is a removable singularity.

$\lim_{z\to z_0} f(z)$also exists then $z_0 = 0$ is the removable singularity.

So $Res\left(\frac{sinz}{z};\ 0\right)\ =\ 0$.

$$\int_c \frac{sinz}{z}dz = 2\pi i Res\left(\frac{sinz}{z};\ 0\right) = 2\pi i * 0 = 0.$$

Example

$\int_c \frac{z-sinz}{z^3} dz$,where C is $|z| \leq 2$,

clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{z-(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ ...,)}{z^3}$$

$$f(z) = \frac{\frac{z^3}{3!}-\frac{z^5}{5!}-\ ...,)}{z^3}$$

$$f(z) = \frac{1}{3!}-\frac{z^2}{5!}-\ ...,$$

$z = 0$ is a removable singularity. By defining $f(0) = 1/6$ the function becomes analytic $at\ z = 0$.

Also

$\lim_{z\to z_0} f(z) = \lim_{z\to 0}\frac{z-sinz}{z^3}$

$\lim_{z\to z_0} f(z) = \lim_{z\to 0}\frac{1-cosz}{3z^2}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0}\frac{0+sinz}{6z^1}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0}\frac{cosz}{6}$ [L-Hosptital rule]

$\lim_{z\to z_0} f(z) = \lim_{z\to 0}\frac{1}{6}$

$\lim_{z\to z_0} f(z)$exists then $z_0 = 0$ is the removable singularity.

So $Res\left(\frac{z-sinz}{z^3},\ ;\ 0\right) = 0$.

$$\int_c \frac{sinz}{z} dz = 2\pi i Res\left(\frac{z-sinz}{z^3};\ 0\right) = 2\pi i * 0 = 0.$$

Example

Consider $\int_c \frac{e^z}{z} dz$,where C is $|z| \leq 2$,

Clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$f(z) = \frac{e^z}{z} = \frac{1}{z}(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\ldots,)$

$f(z) = \frac{e^z}{z} = (1/z+1+\frac{z}{2}+\frac{z^2}{6}+\ldots,)$

Here the principal part of $f(z)$ $at\ z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$. Also

$\lim_{z\to z_0}(z-0)^1 f(z) = e^0 = 1 \neq 0$. So $z_0 = 0$ is the pole of order *1* or simple pole.

Hence

$$Res\left(\frac{e^z}{z};\ 0\right) = g(0) = 1$$

$$\int_c \frac{e^z}{z} dz = 2\pi i Res\left(\frac{e^z}{z};\ 0\right) = 2\pi i * 1 = 2\pi i.$$

Example

Consider $\int_c \frac{cosz}{z^2}dz$,where $C\ is\ |z| \le 2,$

Clearly $z_0 = 0$ is an isolated singular point for $f(z)$.

$$f(z) = \frac{cosz}{z^2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,}{z^2}$$

The principal part of $f(z)\ at\ z = 0$ contains the term $1/z^2$. Hence z=0 is a double pole of f(z).

Also $\lim_{z\to z_0}(z-0)^2 f(z) = \lim_{z\to 0} cosz = 1 \ne 0.$

So $z_0 = 0$ is the pole of order 2. Here $g(z) = cosz$

Hence

$$Res\left(\frac{cosz}{z^2};\ 0\right) = g'(0) = -1$$

$$\int_c \frac{e^z}{z}dz = 2\pi i Res\left(\frac{cosz}{z^2};\ 0\right) = 2\pi i * -1 = -2\pi i.$$

Example

Consider $\int_c ze^{1/z}dz$, where $C\ is\ |z| \le 2,$

Clearly $z = 0$ is an isolated singular point for $f(z)$.

$\lim_{z\to a} f(z) = \lim_{z\to 0} ze^{1/z}$

$\lim_{z\to a}(z-0)^1 f(z) = \lim_{z\to 0} z^1 e^{1/z}.$

$\lim_{z\to a}(z-0)^1 f(z) = \lim_{z\to 0} z^1 [1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots,]$

$\lim_{z\to a}(z-0)^1 f(z) = \infty.$

So $z = 0$ is an essential singularity. Now the coefficient of $(z-0)^{-1}$ is $\frac{1}{2!}$

So

$Res\left(ze^{1/z};\ 0\right) = 1/2.$

$$\int_c ze^{1/z}dz = 2\pi i Res\left(ze^{1/z};\ 0\right) = 2\pi i * \frac{1}{2} = \pi i.$$

<u>Summary</u>

- Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, \dots, z_n$ inside C.
  Then$\int_c f(z)dz = 2\pi i \sum_{j=1}^{n} Res\{f(z);\ z_j\}$

- If $z = a$ is a removable singularity for f(z) then $Res(f(z);\ a) = 0.$
- If $z = a$ is a simple pole for f(z) then $Res(f(z);\ a) = \lim_{z\to a}(z-a)f(z).$
- If a is a simple pole for $f(z) and\ f(z) = \frac{g(z)}{z-a}$

where $g(z)$ is analytic at a and $g(a) \ne 0$ then $Res(f(z);\ a) = g(a).$

- Let $f(z)$is analytic everywhere except the point a inside and on the domain then if the $\lim_{z\to z_0}(z-a)^n f(z) = \infty$, then z_0 is essential singularity. Then the coefficient of $(z-a)^{-1}$ in the Laurent series expansion is the residue of $f(z)$ at $z = a$

- Let a be a pole of order $m > 1$ for $f(z)$ and let $f(z) = \frac{g(z)}{(z-a)^m}$

where $g(z)$ is analytic at a and $g(a) \neq 0$. Then $Res\ (f(z);\ a) = \frac{g^{m-1}(a)}{(n-1)!}$

Keywords

- If $z = a$ is a removable singularity for f(z) then $Res(f(z);\ a) = 0.$
- If $z = a$ is a simple pole for f(z) then $Res(f(z);\ a) = \lim_{z\to a}(z-a)f(z)$

Self Assessment

1. $\boldsymbol{f(z)} = \frac{e^{2z}}{(z-1)^3}$ has ___

A. a pole of order 3 at $z = -1$

B. a simple pole at $z = -1$

C. a pole of order 2 at $z = -1$

D. a simple pole at $z = 0$

2. $\boldsymbol{f(z)} = \frac{Sin\,z}{z}$ has ___

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order 2 at $z = 0$

D. essential singularity at $z = 0$

3. In the Laurent series expansion of $\boldsymbol{f(z)}$ about z = 0, the principal part is 0 then $\boldsymbol{f(z)}$ has

A. removable singularity at $z = 0$

B. a simple pole at $z = 0$

C. a pole of order 2 at $z = 0$

D. essential singularity at $z = 0$

4. In the Laurent series expansion of $\boldsymbol{f(z)}$ about z = a, the principal part has 5 terms then $\boldsymbol{f(z)}$ has

A. removable singularity at $z = a$

B. a simple pole at $z = a$

C. a pole of order 5 at $z = a$

D. essential singularity at $z = a$

5. $\boldsymbol{f(z)} = \frac{z}{(z-1)}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?

A. 5
B. 1
C. 2
D. 3

6. $f(z) = \frac{z}{(z-1)(z-2)^3}$ is meromorphic because it has finite number (say K) of pole in the entire complex plane then K =?

A. 5
B. 1
C. 2
D. 3

7. Which one of the following functions is meromorphic?

$f(z) = \frac{e^z}{z}$, and $g(z) = \frac{sinz}{(z-1)^2}$.

A. Only $f(z)$
B. Only $g(z)$
C. Both $f(z)$ and $g(z)$
D. Neither $f(z)$ nor $g(z)$

8. In the Laurent series expansion of $f(z)$ about z = a, the principal part has infinite many terms then $f(z)$ has

A. removable singularity at $z = a$
B. a simple pole at $z = a$
C. a pole of order a at $z = a$
D. essential singularity at $z = a$

9. $f(z) = e^{\frac{1}{z^2}}$ has ___

A. removable singularity at $z = 0$
B. a simple pole at $z = 0$
C. a pole of order 2 at $z = 0$
D. essential singularity at $z = 0$

10. Residues of $f(z) = \frac{z}{(z-1)}$ at z =1

A. 1
B. 2

C. 3

D. 4

11. Residues of $f(z) = \frac{1}{z(z+2)^3}$ at z =0

A. 5

B. 1

C. 1/8

D. -1/8

12. Residues of $f(z) = \frac{1}{z(z+2)^3}$ at$z = -2$

A. 5

B. 1

C. 1/8

D. -1/8

13. Residues of $f(z) = \frac{Sinz}{z}$ at$z = 0$

A. 0

B. 1

C. 1/8

D. -1/8

14. Residues of $f(z) = 1/(z+1)^3$ $at\ z = -1$

A. 1

B. 2

C. 0

D. 4

15. Residues of $f(z) = \frac{tanz}{z}$ at$z = 0$

A. 0

B. 1

C. 1/8

D. -1/8

16. Evaluate the$\int_c \left(\frac{z^2}{(z-1)^2(z+2)}\right) dz$*where* C:$|z| = 3$

A. $2\pi i$

B. πi

C. 0

D. π

17. Evaluate the $\int_C z.e^{\frac{1}{z}}\,dz$ *where* $C: |z| = 1$

A. $2\pi i$

B. πi

C. 0

D. π

18. Evaluate the $\int_C \frac{e^z}{z-1}\,dz$ *where* $C: |z| = 2$

A. $2\pi i\, e$

B. πi

C. 0

D. π

19. Evaluate the $\int_C \frac{e^z}{(z+1)^2}\,dz$ *where* $C: |z-3| = 3$

A. $2\pi i\, e$

B. πi

C. 0

D. π

Answers for Self Assessment

1.	A	2.	A	3.	A	4.	C	5.	A
6.	C	7.	C	8.	D	9.	D	10.	A
11.	C	12.	D	13.	A	14.	C	15.	A
16.	A	17.	B	18.	A	19	C		

Review Questions

1. Find the residue of $\frac{z+1}{z^2+2z+4}$ inside the circle $|z+1+i| \leq 2$

2. Evaluate $\int_C \frac{z+1}{z^2+2z+4}\,dz$ by using residue theorem, where C is the circle $|z+1+i| = 2$.

3. Find the residue of $\frac{1}{2z+3}$ inside the circle $|z| \leq 2$

4. Evaluate $\int_C \frac{1}{2z+3} dz$ by using residue theorem, where C is the circle $|z| \leq 2$.

5. Find the poles of $f(z) = \frac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

6. Find the poles of $f(z) = \frac{e^z}{2z^2+2z}$ and determine the residues at the poles.

7. Find the poles of $f(z) = \frac{z^2}{z^3+2z^2+2z}$ and determine the residues at the poles.

8. Find the poles of $f(z) = \frac{e^z}{z^2+2}$ and determine the residues at the poles.

9. Find the residue of $\frac{z}{z^2+2z}$ inside the circle $|z| \leq 5$

10. Evaluate $\int_C \frac{1}{z^2+2z}$ dz by using residue theorem, where C is the circle $|z| \leq 5$.

Further Readings

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2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

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Unit 11: Argument Principle, Rouche's Theorem

CONTENTS

Objective

After studying this unit, you will be able to:

- discuss the concept of the argument principle.
- describe Rouhe's theorem.

Introduction

In the last unit, you have studied the Taylor series, singularities of complex-valued functions and use the Laurent series to classify these singularities. This unit will explain the concept related to argument principle and Rouche's theorem.

11.1 Argument Principle

Let C be a simple closed curve, and suppose $f(z)$ is analytic on C. Suppose moreover that the only singularities of $f(z)$inside C are poles.

If $f(z) \neq 0$ for all $z \in C$, then d $= (C)$ is a closed curve which does not pass through the origin. Let N be the number of zeros and P the number of poles then

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N - P$$

Proof

If $f(z) \neq 0$ for all $z \in C$, then d $= (C)$ is a closed curve which does not pass through the origin.

If ¥$(t), a \leq t \leq b$ is a complex description of C, then ¶$(t) = f(\gamma(t)), \alpha \leq t \leq \beta$ is a complex description of d.

Now,

$$\int_c \frac{f'(z)}{f(z)}dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)dt$$

But notice that $\xi'(t) = f'(\gamma(t))\gamma'(t)$.

Hence,

$$\int_c \frac{f'(z)}{f(z)} dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\xi(t))} \gamma'(t) dt = \int_\alpha^\beta \frac{\xi'(t)}{\xi(t)} dt$$

Next, we shall use the Residue Theorem to evaluate the integral $\int_c \frac{f'(z)}{f(z)} dz$. The singularities of the integrand $\frac{f'(z)}{f(z)}$ are the poles of $f(z)$together with the zeros of $f(z)$.

Let's find the residues at these points.

First, let $Z = \{z_1, z_2, \ldots, z_K\}$ be set of all zeros of $f(z)$. Suppose the order of the zero z_j is n_j.

Then $f(z) = (z - z_j)^{n_j} h(z)$ and $h(z_j) \neq 0$. Thus

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} - \frac{m_0}{(z - p_J)^{m_j}}$$

$$(z - p_J)^{m_j} \frac{f'(z)}{f(z)} = \frac{(z - p_J)^{m_j} h'(z) - m_0}{h(z)(z - p_J)^{m_j}}$$

Since $h'(z)/h(z)$ is analytic at p_J, it has a Taylor series representation about that point; and so above equation tells us that $\frac{f'(z)}{f(z)}$ has a pole of order n_j at p_J, with residue m_0.

Using the Cauchy residue theorem $\int_c \frac{f'(z)}{f(z)} dz = 2\pi i[\text{Sum of all the residue}]$ Hence

$\frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)} dz = N - P.$

where n is the winding number, or the number of times d winds around the origin-n > 0 means Γwinds in the positive sense, and n negative means it winds in the negative sense. Finally, we

$n = N - P$, where $N = n_1 + n_2 + \ldots + n_K$ is the number of zeros inside *C*, counting multiplicity, or the order of the zeros, and $P = m_1 + m_2 + \ldots + m_J$ is the number of poles, counting the order. This result is the celebrated argument principle

Example

Let $f(z) = \sin\pi z$ then evaluate $\int_c \frac{f'(z)}{f(z)} dz$ Where $c: |z| \leq \pi$.

Here $f(z) = \sin\pi z$

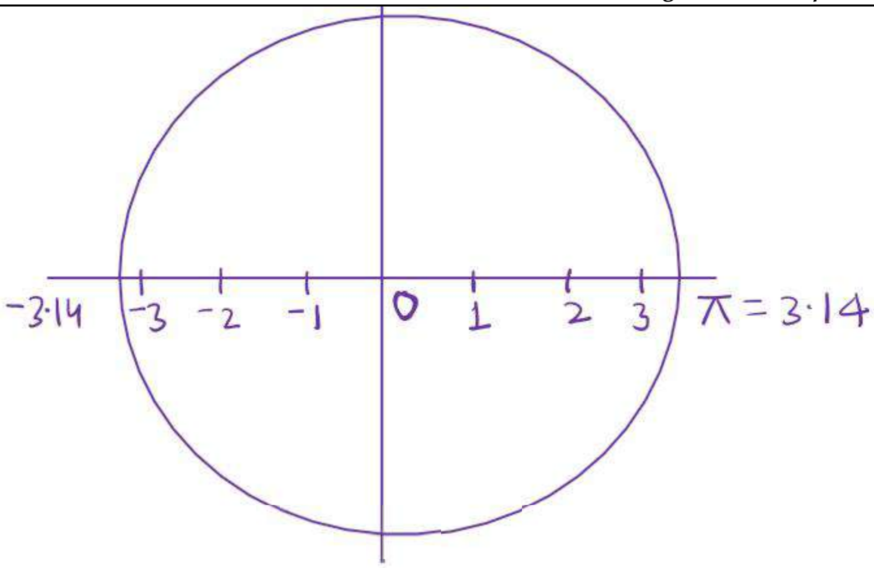
$$f'(z) = \pi\cos\pi z$$

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{\pi\cos\pi z}{\sin\pi z} dz.$

Using argument principle

$$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{\pi\cos\pi z}{\sin\pi z} dz = 2\pi i[N - P]$$

Now the zeroes of $f(z) = \sin\pi z$ inside the $|z| \leq \pi$ are $\{-3, -2, -1, 0, 1, 2, 3\}$, N=7



Now there is no poles of $f(z) = \sin\pi z$ inside the $|z| \leq \pi$, P=0.

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{\pi \cos\pi z}{\sin\pi z} dz = 2\pi i[7 - 0]$.

$\int_c \frac{\pi \cos\pi z}{\sin\pi z} dz = 14\pi i.$

$\int_c \cot z dz = 7i.$

Example

Let $f(z) = z^2 - z$ then evaluate $\int_c \frac{f'(z)}{f(z)} dz$ Where c: $|z| \leq \pi$.

Here $f(z) = z^2 - z$

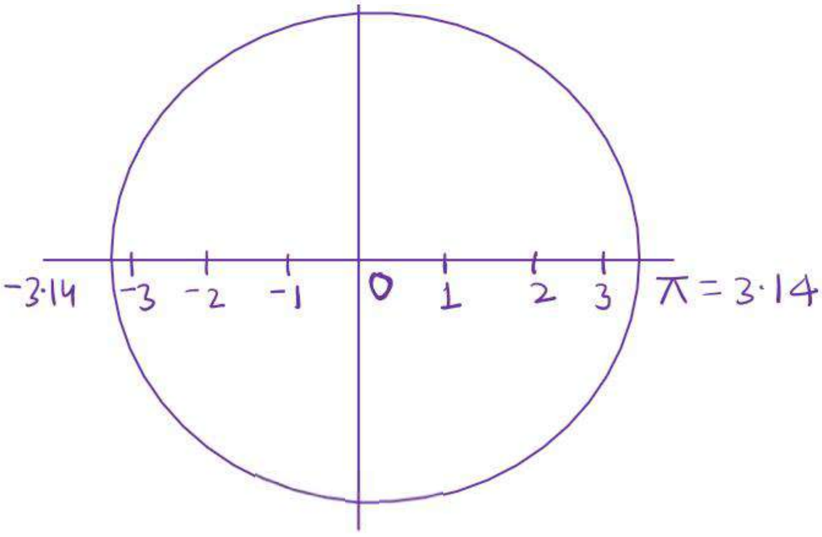
$$f'(z) = \pi \cos\pi z$$

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z-1}{z^2-z} dz.$

Using argument principle

$$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z-1}{z^2-z} dz = 2\pi i[N - P]$$

Now the zeroes of $f(z) = z^2 - z$ inside the $|z| \leq \pi$ are $\{0, 1\}$, N=2.



Now there is no poles of $f(z) = z^2 - z$ inside the $|z| \leq \pi$, P=0.

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z-1}{z^2-z} dz = 2\pi i[N - P].$

$\int_c \frac{2z-1}{z^2-z} dz = 2\pi i[2 - 0].$

$\int_c \frac{2z-1}{z^2-z} dz = 4\pi i.$

Example

Let $f(z) = z^2 - 5z + 6$ then evaluate $\int_c \frac{f'(z)}{f(z)} dz$ Where c: $|z| \leq \pi$.

Here $f(z) = z^2 - 5z + 6$

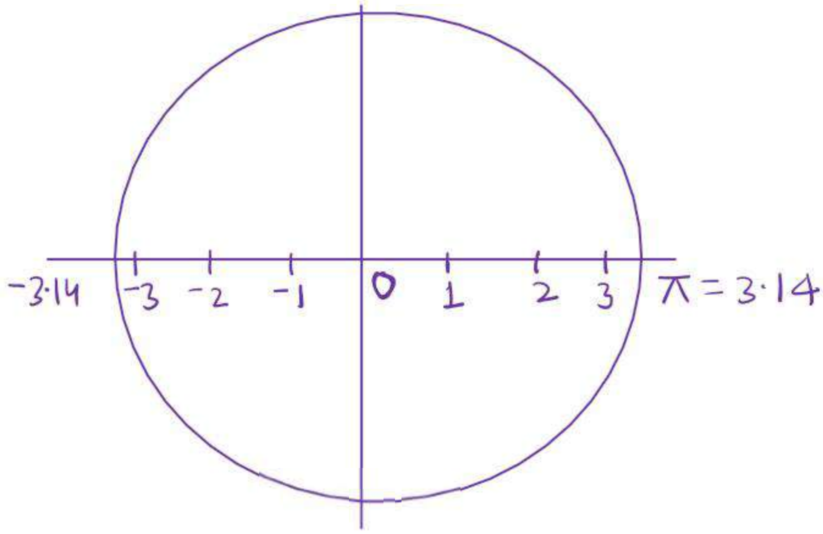
$$f'(z) = 2z - 5$$

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z-5}{z^2-5z+6} dz.$

Using argument principle

$$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z-5}{z^2-5z+6} dz = 2\pi i[N - P]$$

Now the zeroes of $f(z) = z^2 - 5z + 6$ inside the $|z| \leq \pi$ are $\{2, 3\}$, N=2.



Now there is no poles of $f(z) = z^2 - 5z + 6$ inside the $|z| \leq \pi$, P=0.

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z-1}{z^2-5z+6} dz = 2\pi i[N - P].$

$\int_c \frac{2z-1}{z^2-5z+6} dz = 2\pi i[2 - 0].$

$\int_c \frac{2z-1}{z^2-5z+6} dz = 4\pi i.$

We can also evaluate $\int_c \frac{2z-1}{z^2-5z+6} dz$inside the $|z| \leq \pi$ using the Cauchy Residue theorem.

$$\int_c \frac{2z-1}{z^2-5z+6} dz = \int_c \frac{2z-1}{(z-2)(z-3)} dz$$

$z = 2, 3$ are inside the $|z| \leq \pi$.

$Res\left(\frac{2z-1}{z^2-5z+6}; 2\right) = \frac{3}{-1} = -3.$

$Res\left(\frac{2z-1}{z^2-5z+6}; 3\right) = \frac{5}{1} = 5.$

$\int_c \frac{2z-1}{z^2-5z+6} dz = \int_c \frac{2z-1}{(z-2)(z-3)} dz = 2\pi i\left[Res\left(\frac{2z-1}{z^2-5z+6}; 2\right) + Res\left(\frac{2z-1}{z^2-5z+6}; 3\right)\right].$

$\int_c \frac{2z-1}{z^2-5z+6} dz = \int_c \frac{2z-1}{(z-2)(z-3)} dz = 2\pi i[-3 + 5].$

$\int_c \frac{2z-1}{z^2-5z+6} dz = 4\pi i.$

Example

Let $f(z) = z^2 - 5$ then evaluate $\int_c \frac{f'(z)}{f(z)} dz$ Where c: $|z| \leq \pi$.

Here $f(z) = z^2 - 5$

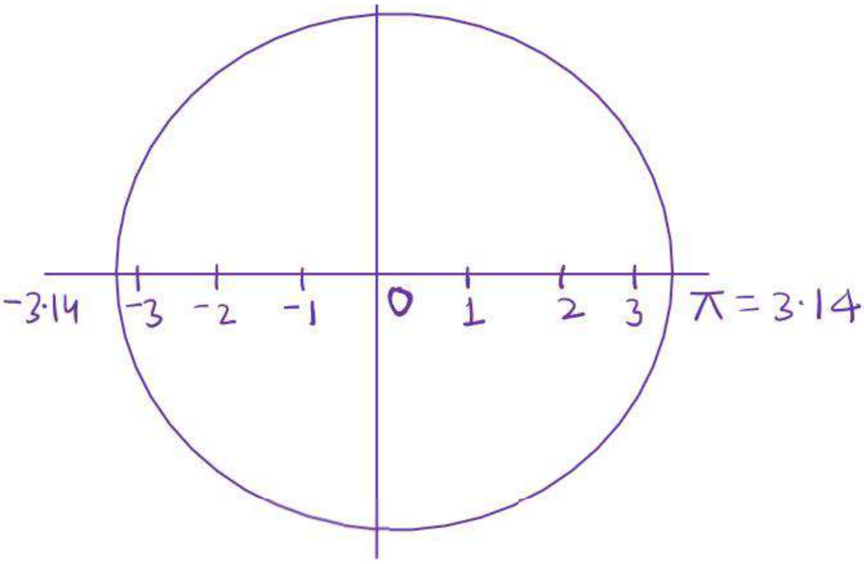
$$f'(z) = 2z$$

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z}{z^2-5} dz.$

Using argument principle

$$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z}{z^2-5} dz = 2\pi i[N - P]$$

Now the zeroes of $f(z) = z^2 - 5$ inside the $|z| \leq \pi$ are $\{\pm\sqrt{5}\}$, N=2.



Now there is no poles of $f(z) = z^2 - 5$ inside the $|z| \leq \pi$, P=0.

$\int_c \frac{f'(z)}{f(z)} dz = \int_c \frac{2z}{z^2-5} dz = 2\pi i[N - P].$

$\int_c \frac{2z}{z^2-5} dz = 2\pi i[2 - 0].$

$\int_c \frac{2z}{z^2-5} dz = 4\pi i.$

We can also evaluate $\int_c \frac{2z}{z^2-5} dz$inside the $|z| \leq \pi$ using the Cauchy Residue theorem.

$$\int_c \frac{2z}{z^2-5} dz = \int_c \frac{2z}{(z-\sqrt{5})(z+\sqrt{5})} dz$$

$z = -\sqrt{5}, \sqrt{5}$ are inside the $|z| \leq \pi$.

$Res\left(\frac{2z}{z^2-5}; \sqrt{5}\right) = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.$

$Res\left(\frac{2z}{z^2-5}; -\sqrt{5}\right) = \frac{-2\sqrt{5}}{-2\sqrt{5}} = 1.$

$\int_c \frac{2z}{z^2-5} dz = \int_c \frac{2z}{(z-\sqrt{5})(z+\sqrt{5})} dz = 2\pi i\left[Res\left(\frac{2z}{z^2-5}; \sqrt{5}\right) + Res\left(\frac{2z}{z^2-5}; -\sqrt{5}\right)\right].$

$\int_c \frac{2z}{z^2-5} dz = \int_c \frac{2z}{(z-\sqrt{5})(z+\sqrt{5})} dz = 2\pi i[1 + 1].$

$\int_c \frac{2z}{z^2-5} dz = \int_c \frac{2z}{(z-\sqrt{5})(z+\sqrt{5})} dz = 4\pi i.$

11.2 Rouche's Theorem

The Rouche's theorem is a consequence of the argument principle. It can be useful in locating regions of the complex plane in which a given analytic function has zeros.

Theorem. Let C denote a simple closed contour, and suppose that

(a) two functions $f\ (z)\ and\ g(z)$ are analytic inside and on C;

(b) $|f\ (z)|\ >\ |g(z)|$ at each point on C.

Then $f\ (z)$ and $f\ (z)\ +\ g(z)$ have the same number of zeros, counting multiplicities, inside C.

Proof

The orientation of C in the statement of the theorem is evidently immaterial.

Thus, in the proof here, we may assume that the orientation is positive. We begin with the observation that neither the function $f\ (z)$ nor the sum $f\ (z)\ +\ g(z)$ has a zero on C, since

$$|f\ (z)|\ >\ |g(z)|\ \geq\ 0\ and\ |f\ (z)\ +\ g(z)| \geq ||f\ (z)| - |g(z)|\ |\ >\ 0$$

when $z\ is\ on\ C$.

If $Z_f\ and\ Z_{f+g}$ denote the number of zeros, counting multiplicities, of $f\ (z)\ and\ f\ (z)\ +\ g(z)$,

respectively, inside C, we know that

$$Z_f\ = \frac{1}{2\pi}\Delta_c arg\ f(z) and\ Z_{f+g} = \frac{1}{2\pi}\Delta_c arg\ arg[f\ (z)\ +\ g(z)].$$

Consequently, since

$Z_f\ = \frac{1}{2\pi}\Delta_c \text{arg}\ f(z) and\ Z_{f+g} = \frac{1}{2\pi}\Delta_c \text{arg}\ [f\ (z)(1\ + \frac{g(z)}{f(z)}]$

$Z_f\ = \frac{1}{2\pi}\Delta_c \text{arg}\ f(z) and\ Z_{f+g} = \frac{1}{2\pi}\Delta_c \text{arg}\ [f\ (z)] + \frac{1}{2\pi}\Delta_c \text{arg}\ (\frac{g(z)}{f(z)})].$

it is clear that,

$$Z_{f+g} =\ Z_f\ + \frac{1}{2\pi}\Delta_c\ \text{arg}\, F\ (z),$$

$$F\ (z) =\ 1\ + \frac{g(z)}{f(z)}$$

But $|F\ (z) - \ 1| = \frac{|g(z)|}{|f\ (z)|} <\ 1;$

and this means that under the transformation $w\ =\ F\ (z)$, the image of C lies in the open disk $|w\ -\ 1|\ <\ 1$. That image does not, then, enclose the origin $w\ =\ 0$. Hence $\Delta_c \text{arg} F\ (z)\ =\ 0$ and,

$Z_{f+g}\ =\ Z_f$ Rouche's theorem is proved.

Example

In order to determine the number of roots of the equation $z^7\ -\ 4z^3\ +\ z\ -\ 1\ =\ 0$ inside the circle $|z| = 1$,

write $f\ (z)\ =\ -4z^3\ and\ g(z)\ =\ z^7\ +\ z\ -\ 1$.

Then observe that $|f\ (z)|\ =\ 4|z|^3\ =\ 4$

and $|g(z)| \leq |z|^7\ +\ |z|\ +\ 1\ =\ 3$ when $|z| = 1$.

The conditions in Rouche's theorem are thus satisfied. Consequently, since $f\ (z)$ has three zeros, counting multiplicities, inside the circle $|z|\ =\ 1$, so does $f\ (z)\ +\ g(z)$. That is, equation has three roots there.

Summary

Let C denote a simple closed contour, and suppose that

(a) two functions $f\ (z)$ *and* $g(z)$ are analytic inside and on C;

(b) $|f\ (z)|\ >\ |g(z)|$ at each point on C.

Then $f\ (z)$ and $f\ (z)\ +\ g(z)$ have the same number of zeros, counting multiplicities, inside C.

Keywords

Let C be a simple closed curve, and suppose $f(z)$ is analytic on C. Suppose moreover that the only singularities of $f(z)$inside C are poles.

If $f(z) \neq 0$ for all $z \in C$, then d $=$ (C) is a closed curve which does not pass through the origin. Let N be the number of zeros and P the number of poles then

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} dz = N - P$$

Self Assessment

1. Let $f(z) = \frac{2z+1}{z^2+5z+6}$ and *N* be the number of zero and *P* be the number of pole then *N=?*

A. 1
B. 2
C. 3
D. 4

2. Let $f(z) = \frac{2z+1}{z^2+5z+6}$ and *N* be the number of zero and *P* be the number of pole then *P=?*

A. 1
B. 2
C. 3
D. 4

3. Let $f(z) = sin\pi z$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = \pi$ then *N=?*

A. 7
B. 2
C. 3
D. 4

4. Let $f(z) = sin\pi z$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = \pi$ then *P=?*

A. 0
B. 2
C. 3
D. 4

5. Let $f(z) = z - 2$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = 5$ then *N=?*

A. 1
B. 2
C. 3

D. 4

6. Let $f(z) = z - 2$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = 5$ then *P=?*

A. 0
B. 2
C. 3
D. 4

7. Let $f(z) = z^2 - z$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = 2$ then *N=?*

A. 7
B. 2
C. 3
D. 4

8. Let $f(z) = z^2 - z$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = 2$ then *P=?*

A. 0
B. 2
C. 3
D. 4

9. Let $f(z) = \sin\pi z$ and *N* be the number of zero, *P* be the number of pole inside C: $|z| = \pi$ then $\int_c \frac{f'(z)}{f(z)} dz$=?

A. 14i
B. 2i
C. 7
D. 4

10. Let $f(z)$ is analytic inside C: $|z| = 2$,and N=10 be the number of zero, P=5 be the number of pole then $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} dz$=?

A. 5
B. 2
C. 3
D. 1

11. Let $f(z) = z - 2$ and N be the number of zero, P be the number of pole inside C: $|z| = 5$ then then $\int_c \frac{f'(z)}{f(z)} dz$=?

A. $2\pi i$
B. πi
C. $2i$
D. 2π

12. Let $\boldsymbol{f(z) = z - 5}$ and N be the number of zero, P be the number of pole inside C: $|\boldsymbol{z}| = \boldsymbol{5}$ then

then $\int_c \frac{f'(z)}{f(z)} \boldsymbol{dz}$=?

A. $2\pi i$
B. πi
C. $2i$
D. 2π

13. Let $\boldsymbol{f(z) = z^2 - z}$ and N be the number of zero, P be the number of pole inside C: $|\boldsymbol{z}| =$ $\boldsymbol{2}$ then $\int_c \frac{f'(z)}{f(z)} \boldsymbol{dz}$=?

A. $4\pi i$

B. πi

C. $2i$

D. 2π

14. Let $\boldsymbol{f(z)}$ is analytic inside C: $|\boldsymbol{z}| = \boldsymbol{2}$,and N=100 be the number of zero, P=50 be the number of poles, then $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} \boldsymbol{dz}$=?

A. 50
B. 20
C. 30
D. 40

15. Let $\boldsymbol{f(z)}$ is analytic inside C: $|\boldsymbol{z}| = \boldsymbol{2}$,and N=1 be the number of zero, P=0 be the number of pole then $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} \boldsymbol{dz}$=?

A. 1
B. 2
C. 3
D. 4

Answers for Self Assessment

l.	A	2.	B	3.	A	4.	A	5.	A
6.	A	7.	B	8.	A	9.	A	10.	A
11.	A	12.	A	13.	A	14.	A	15.	A

Review Questions

1. Let $\boldsymbol{f(z) = z - 10}$ and N be the number of zero, P be the number of poles inside C: $|\boldsymbol{z}| =$ $\boldsymbol{5}$ then then $\int_c \frac{f'(z)}{f(z)} \boldsymbol{dz}$=?

2. Let $f(z) = z/10$ and N be the number of zero, P be the number of poles inside C: $|z| = 5$ then then $\int_c \frac{f'(z)}{f(z)} dz$=?

3. Let $f(z) = (z-10)(z-5)$ and N be the number of zero, P be the number of poles inside C: $|z| = 10$then then $\int_c \frac{f'(z)}{f(z)} dz$=?

4. Let $f(z) = 1/10z$ and N be the number of zero, P be the number of poles inside C: $|z| = 10$then then $\int_c \frac{f'(z)}{f(z)} dz$=?

5. Let $f(z) = \frac{2z+1}{z^2-6z+8}$ and *N* be the number of zero and *P* be the number of pole then *P*=*?*

6. Let $f(z) = \frac{1+z^2}{z^2-6z+8}$ and *N* be the number of zero and *P* be the number of pole then N=*?*

7. Let $f(z) = \frac{2z}{z^2-10z+40}$ and *N* be the number of zero and *P* be the number of pole then *P*=*?*

8. Let $f(z) = \frac{1+z^4}{z^2-12z+32}$ and *N* be the number of zero and *P* be the number of pole then N=*?*

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

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Unit 12: Integrals Involving Sines and Cosines Functions

CONTENTS

Objective

We turn now to some important applications of the theory of residues, which was developed in Unit 10. Cauchy residue theorem is used to evaluate certain types of definite and improper integrals occurring in real analysis and applied mathematics. These integrals are first transformed to associate counter integral. The counter integrals are then evaluated using Cauchy residue theorem. After studying this unit, you will be able to:

- evaluate the definite integral of type $\int_0^{2\pi} f(cos\theta, sin\theta)d\theta$.
- evaluate the improper integral over semi-infinite and infinite interval

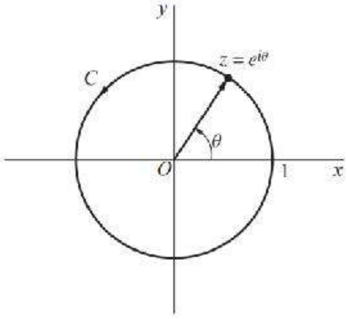
Introduction

In this unit first the definite integral of type $\int_0^{2\pi} f(cos\theta, sin\theta)d\theta$ would be explained and then we will evaluate the improper integral over semi-infinite$(0, \infty)$, $(-\infty, 0)$ and infinite $(-\infty, \infty)$interval.

12.1 Integrals Involving Sines and Cosines Functions

The integral around the unit circle of the type $\int_0^{2\pi} f(cos\theta, sin\theta)d\theta$, where f is a rational function of $cos\theta, and\ sin\theta$ can be obtained by setting $z = e^{i\theta}$.

The fact that θ varies from 0 $to\ 2\pi$ leads us to consider θ as an argument of a point z on a positively oriented circle C centered at the origin. Taking the radius to be unity, we use the parametric representation $z = e^{i\theta}, (0 \le \theta \le 2\pi)$.



to describe C (in above figure). We then refer to the differentiation formula to write $\frac{dz}{d\theta} = ie^{i\theta} = iz.$

We know that $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$and $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$. These relations suggest that we make the substitutions $\sin\theta = \frac{z - z^{-1}}{2i} \cos\theta = \frac{z + z^{-1}}{2}, d\theta = \frac{dz}{iz}$which transform integral into the contour integral of a function of z around the unit radius circle C.

$$\int_c \left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}$$

The original integral is, of course, simplya parametric form of integral and we can evaluate that integral by means of Cauchy's residue theorem once the zeros in the denominator have been located and provided that none lie on *C*.

Example

Evaluate $\int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta}$ around the unit radius circle *C*.

Solution

Let

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta}$$

Put $\sin\theta = \frac{z - z^{-1}}{2i} \cos\theta = \frac{z + z^{-1}}{2}, d\theta = \frac{dz}{iz}.$

Then

$$I = \int_0^{2\pi} \frac{dz}{iz\left(3 - 2.\frac{z + z^{-1}}{2} + \frac{z - z^{-1}}{2i}\right)}$$

$$I = \int_0^{2\pi} \frac{2dz}{(1 - 2i)z^2 + 6iz - 1 - 2i}$$

Now obtain the poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$

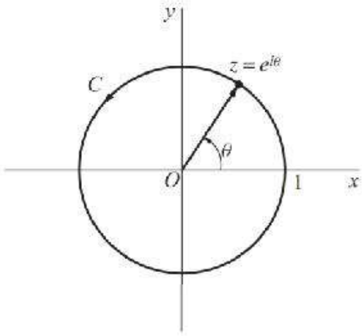
$$(1 - 2i)z^2 + 6iz - 1 - 2i = 0$$

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1 - 2i)(-1 - 2i)}}{2(1 - 2i)}$$

$$z = \frac{-6i \pm 4i}{2(1 - 2i)}$$

$$z = 2 - i, (2 - i)/5$$

Only $\frac{2-i}{5}$ lies inside C.



$$Res\left(\frac{2}{(1-2i)z^2+6iz-1-2i};(2-i)/5\right) = \lim_{z\to(2-i)/5}\{z-(2-i)/5\}\frac{2}{(1-2i)z^2+6iz-1-2i}.$$

$$Res\left(\frac{2}{(1-2i)z^2+6iz-1-2i};(2-i)/5\right) = 1/2i.$$

$$I = \int_0^{2\pi}\frac{d\theta}{3-2cos\theta+sin\theta} = \int_0^{2\pi}\frac{dz}{iz\left(3-2.\frac{z+z^{-1}}{2}+\frac{z-z^{-1}}{2i}\right)}$$

$$= 2\pi i[Res\left(\frac{2}{(1-2i)z^2+6iz-1-2i};(2-i)/5\right)].$$

$$I = \int_0^{2\pi}\frac{d\theta}{3-2cos\theta+sin\theta} = 2\pi i[1/2i]$$

$$I = \int_0^{2\pi}\frac{d\theta}{3-2cos\theta+sin\theta} = \pi$$

Example

Evaluate $\int_0^{2\pi}\frac{d\theta}{5+4sin\theta}$ around the unit radius circle *C*.

Solution

Let

$$I = \int_0^{2\pi}\frac{d\theta}{5+4sin\theta}$$

Put $\sin\theta = \frac{z-z^{-1}}{2i}\cos\theta = \frac{z+z^{-1}}{2}d\theta = \frac{dz}{iz}$.

Then

$$I = \int_0^{2\pi}\frac{dz}{iz\left(5+4.\frac{z-z^{-1}}{2i}\right)}$$

$$I = \int_0^{2\pi}\frac{dz}{iz\left(5+2.\frac{z^2-1}{zi}\right)}$$

$$I = \int_0^{2\pi}\frac{iz.dz}{iz(5zi+2z^2-2)}$$

$$I = \int_0^{2\pi}\frac{dz}{2z^2+5iz-2}$$

Now obtain the poles of $\frac{1}{2z^2+5iz-2}$

$$2z^2+5iz-2=0$$

$$z = \frac{-5i \pm \sqrt{(5i)^2 + 16}}{4}$$

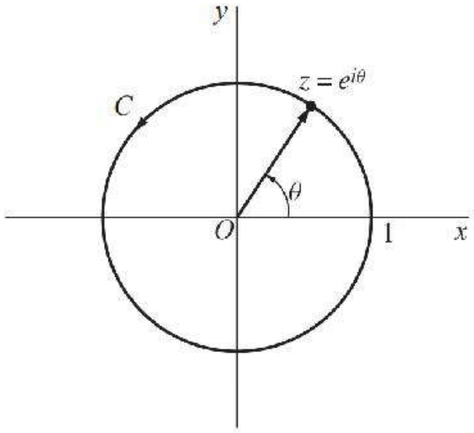
$$z = \frac{-5i \pm \sqrt{-25 + 16}}{4}$$

$$z = \frac{-5i \pm \sqrt{-9}}{4}$$

$$z = \frac{-5i \pm 3i}{4}$$

$$z = -\frac{i}{2}, z = -2i$$

Only $z = -\frac{i}{2}$ lies inside C.



$Res\left(\frac{1}{2z^2+5iz-2}; -\frac{i}{2}\right) = \lim_{z\to -\frac{i}{2}}\{z - (-\frac{i}{2})\}\frac{1}{2z^2+5iz-2}.$

$Res\left(\frac{1}{2z^2+5iz-2}; -\frac{i}{2}\right) = \lim_{z\to -\frac{i}{2}}\frac{1}{4z+5i}.$

$Res\left(\frac{1}{2z^2+5iz-2}; -\frac{i}{2}\right) = \frac{1}{-2i+5i}.$

$Res\left(\frac{1}{2z^2+5iz-2}; -\frac{i}{2}\right) = \frac{1}{3i}.$

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta} = 2\pi i Res\left(\frac{1}{2z^2 + 5iz - 2}; -\frac{i}{2}\right).$$

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta} = 2\pi i[1/3i]$$

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2cos\theta + sin\theta} = 2\pi/3$$

12.2 Improper Integrals, Integration Along Indented Contours

In calculus, the improper integral of a continuous function $f(x)$ over the semi-infinite interval $0 \le x < \infty$ is defined by means of the equation $\int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$

When the limit on the right exists, the improper integral is said to converge to that limit. If $f(x)$ is continuous for all x, its improper integral over the infinite interval $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R\to\infty}\int_{0}^{R_2} f(x)\,dx.$

We now describe a method involving sums of residues, to be illustrated in the next section, that is often used to evaluate improper integrals of rational functions.

$\int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = g(x)/h(x)$ and $g(x), h(x)$ are polynomials in x and the degree of $h(x)$ exceeds that of $g(x)$ by atleast two.

To evaluate this type of integral we *take* $f(z) = g(z)\,h(z)$.

The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0$.

Case (i) No pole of $f(z)$ lies on the real axis.

We choose the curve C consisting of the interval $[-r, r]$ on the real axis and the semi-circle $|z| = r$ lying in the upper half of the plane.

Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C.

Then we have $\int_c f(z)dz = \int_{-r}^{r} f(x)\,dx + \int_{c_1} f(z)dz.$

Where c_1 is the semi circle.

Since deg $h(x)$ – $deg\ f(x) \geq 2$ it follows that $\int_{c_1} f(z)dz \to 0$ as $r \to \infty$ and hence

$$\int_{c_1} f(z)dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$\int_{-\infty}^{\infty} f(x)\,dx$can be evaluated by using Cauchy's residue theorem.

Case (ii) $f(z)$ has poles lying on the real axis.

Suppose a is a pole lying on the real axis.

In this case we indent the real axis by a semi circleC_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small.

Such an indenting must be done for every pole of $f(z)$ lying on the real axis.

It can be proved that $\int_{C_2} f(z)dz = -\pi i\, Res\,\{f(z);\ a\}$. By taking limit as $r\to\infty$ *and* $\varepsilon\to 0$ we obtain the value of $\int_{-\infty}^{\infty} f(x)\,dx$.

Example

Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{1+x^4}dx$

Solution

Let $f(z) = 1/(1 + z^4)$

The poles of $f(z)$ are given by the roots of the equation $z^4 + 1 = 0$ which are the

four fourth roots of -1.

$$z^4 = -1$$

$$z = (-1)^{\frac{1}{4}}$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}$$

$$= \cos\frac{(2n+1)\pi}{4} + \sin\frac{(2n+1)\pi}{4}, n = 0,1,2,3.$$

$= e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, and\ e^{\frac{i7\pi}{4}}$ which are all simple poles.

We choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi-circle $|z| = r$ which we denote by C_r .

$\therefore \int_c f(z)dz = \int_{-r}^{r} f(x)\,dx + \int_{c_1} f(z)dz.$

The poles of $f(z)$ lying inside the contour C are obviously $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$ only.

We find the residues of $f(z)$ at these points.

$$Res\left\{f(z);\ e^{\frac{i\pi}{4}}\right\} = h(e^{i\pi 4})/k'(e^{i\pi 4})$$

$$where\ h(z) = 1\ and\ k(z) = z^4 + 1$$

$$k'(z) = 4z^3$$

$\Rightarrow$

$$k'\left(e^{\frac{i\pi}{4}}\right) = 4\left(e^{\frac{i\pi}{4}}\right)^3$$

$$k'\left(e^{\frac{i\pi}{4}}\right) = 4\left(e^{\frac{i3\pi}{4}}\right)$$

$$Res\left\{f(z);\ e^{\frac{i\pi}{4}}\right\} = \frac{1}{4\left(e^{\frac{i3\pi}{4}}\right)}$$

Now

$$Res\left\{f(z);\ e^{\frac{i3\pi}{4}}\right\} = h(e^{i3\pi 4})/k'(e^{i3\pi 4})$$

$$where\ h(z) = 1\ and\ k(z) = z^4 + 1$$

$$k'(z) = 4z^3$$

$\Rightarrow$

$$k'\left(e^{\frac{i3\pi}{4}}\right) = 4\left(e^{\frac{i3\pi}{4}}\right)^3$$

$$k'\left(e^{\frac{i3\pi}{4}}\right) = 4\left(e^{\frac{i9\pi}{4}}\right)$$

$$Res\left\{f(z);\ e^{\frac{i3\pi}{4}}\right\} = \frac{1}{4\left(e^{\frac{i9\pi}{4}}\right)}$$

By residue theorem,

$$\int_c f(z)dz = 2\pi i\left(Res\left\{f(z);\ e^{\frac{i\pi}{4}}\right\} + Res\left\{f(z);\ e^{\frac{i3\pi}{4}}\right\}\right)$$

$$\int_c f(z)dz = 2\pi i\left(\frac{1}{4\left(e^{\frac{i3\pi}{4}}\right)} + \frac{1}{4\left(e^{\frac{i9\pi}{4}}\right)}\right)$$

$$\int_c f(z)dz = 2\pi i\left(\frac{1}{4(\cos 3\pi/4 + i\sin 3\pi/4)} + \frac{1}{4(\cos 9\pi/4 + i\sin 9\pi/4)}\right)$$

$$\int_c f(z)dz = \left(\frac{2\pi i}{4(1/\sqrt{2} + i\,1/\sqrt{2})} + \frac{2\pi i}{4(1/\sqrt{2} + i1/\sqrt{2})}\right)$$

$$\int_c f(z)dz = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_c f(z)dz = \int_{-r}^{r} f(x)\,dx + \int_{c_1} f(z)dz$$

$$\int_{-r}^{r} f(x)\,dx + \int_{c_1} f(z)dz = \frac{\pi}{\sqrt{2}}$$

$$\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{c_1} f(z)dz = \frac{\pi}{\sqrt{2}}$$

Since deg $h(x)$ – $deg\ f(x) \geq 2$ it follows that $\int_{c_1} f(z)dz \to 0$ as $r \to \infty$ and hence

$$\int_{c_1} f(z)dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{c_1} f(z)dz = \frac{\pi}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Summary

- The integral around the unit circle of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$, where f is a rational function of $\cos\theta, and \sin\theta$ can be obtained by setting $z = e^{i\theta}$.
- The improper integral of a continuous function $f(x)$ over the semi-infinite interval $0 \leq x < \infty$ is defined by means of the equation $\int_0^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx$.
- When the limit on the right exists, the improper integral is said to converge to that limit. If $f(x)$ is continuous for all x, its improper integral over the infinite interval $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R\to\infty} \int_0^{R_2} f(x)\,dx.$$

Keywords

Improper integral: The improper integral of a continuous function $f(x)$ over the semi-infinite interval $0 \leq x < \infty$ is defined by means of the equation $\int_0^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx$.

Self Assessment

1. If $\int_0^{\pi} \frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{17-8cos\theta}$ is

A. $\frac{\pi}{15}$
B. $\frac{\pi}{10}$
C. $\frac{\pi}{5}$
D. $\frac{\pi}{100}$

2. If $\int_0^{\pi} \frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{10-8cos\theta}$ is

A. $\frac{\pi}{30}$
B. $\frac{\pi}{10}$
C. $\frac{\pi}{3}$
D. $\frac{\pi}{100}$

3. If $\int_0^{\pi} \frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{2+1cos\theta}$ is

A. $\frac{2\pi}{\sqrt{3}}$
B. $\frac{\pi}{\sqrt{3}}$
C. $\frac{2\pi}{6}$
D. $\frac{2\pi}{10}$

4. If $\int_0^{\pi} \frac{d\theta}{a+bSin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{17-8sin\theta}$ is

A. $\frac{1}{15}$
B. $\frac{\pi}{10}$
C. $\frac{\pi}{15}$
D. $\frac{\pi}{100}$

5. If $\int_0^{\pi} \frac{d\theta}{a+bSin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{10-8sin\theta}$ is

A. $\frac{\pi}{3}$
B. $\frac{\pi}{10}$
C. $\frac{\pi}{5}$
D. $\frac{\pi}{100}$

6. If $\int_0^{\pi} \frac{d\theta}{a+bSin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{2+1Sin\theta}$ is

A. $\frac{\pi}{\sqrt{3}}$

B. $\frac{2\pi}{\sqrt{3}}$

C. $\frac{2\pi}{6}$

D. $\frac{2\pi}{10}$

7. If $\int_0^{\pi} \frac{d\theta}{a+bSin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{10+8sin\theta}$ is

A. $\frac{\pi}{3}$

B. $\frac{\pi}{10}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

8. If $\int_0^{\pi} \frac{d\theta}{a+bSin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ then the value of $\int_0^{\pi} \frac{d\theta}{2-1Sin\theta}$ is

A. $\frac{\pi}{\sqrt{3}}$

B. $\frac{2\pi}{\sqrt{3}}$

C. $\frac{2\pi}{6}$

D. $\frac{2\pi}{10}$

9. The value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2+25)^2}$

A. $\frac{\pi}{16}$

B. $\frac{\pi}{250}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

10. The value of $\int_0^{\infty} \frac{dx}{(x^2+25)^2}$

A. $\frac{\pi}{500}$

B. $\frac{\pi}{10}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

11. The value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2+100)^2}$

A. $\frac{\pi}{160}$

B. $\frac{\pi}{2000}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{1000}$

12. The value of $\int_{0}^{\infty} \frac{dx}{(x^2+4)^2}$

A. $\frac{\pi}{4000}$

B. $\frac{\pi}{10}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{1000}$

13. The value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$

A. $\frac{\pi}{16}$

B. $\frac{\pi}{10}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

14. The value of $\int_{0}^{\infty} \frac{dx}{(x^2+4)^2}$

A. $\frac{\pi}{32}$

B. $\frac{\pi}{10}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

15. The value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2}$

A. $\frac{\pi}{16}$

B. $\frac{\pi}{54}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

16. The value of $\int_{0}^{\infty} \frac{dx}{(x^2+4)^2}$

A. $\frac{\pi}{108}$

B. $\frac{\pi}{10}$

C. $\frac{\pi}{5}$

D. $\frac{\pi}{100}$

Answers for Self Assessment

l.	A	2.	C	3.	A	4.	C	5.	A
6.	B	7.	A	8.	B	9.	B	10.	A
11.	B	12.	A	13.	A	14.	A	15.	B
16.	A								

Review Questions

1. Evaluate $\int_0^{\pi} \frac{d\theta}{10-5cos\theta}$
2. Evaluate $\int_0^{\pi} \frac{d\theta}{15-5cos\theta}$
3. Evaluate $\int_0^{\pi} \frac{d\theta}{3.5-1.5cos\theta}$
4. Evaluate $\int_0^{\pi} \frac{d\theta}{10-5sin\theta}$
5. Evaluate $\int_0^{\pi} \frac{d\theta}{15-5sin\theta}$
6. Evaluate $\int_0^{\pi} \frac{d\theta}{3.5-1.5sin\theta}$
7. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{1+x^2} dx$
8. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{1+x^3} dx$
9. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{1+x^6} dx$
10. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{1+x} dx$
11. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{(1+x^2)^2} dx$
12. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{(100+x^2)^2} dx$
13. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{(81+x^2)^2} dx$
14. Use contour integration method to evaluate $\int_0^{\infty} \frac{1}{(49+x^2)^2} dx$
15. Use contour integration method to evaluate $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$
16. Use contour integration method to evaluate $\int_{-\infty}^{\infty} \frac{1}{(100+x^2)^2} dx$

17. Use contour integration method to evaluate $\int_{-\infty}^{\infty} \frac{1}{(81+x^2)^2} dx$

18. Use contour integration method to evaluate $\int_{-\infty}^{\infty} \frac{1}{(49+x^2)^2} dx$

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 13: Conformal Mapping

CONTENTS

Objectives

A large number of problems arising in fluid mechanics, electrostatics, heat conduction, and many other physical situations can be mathematically formulated in terms of the Laplace's equation. i.e, all these physical problems reduce to solving the equation $\Phi_{xx} + \Phi_{yy} = 0$, in a certain region D of the z −plane.

The function $\Phi(x, y)$, in addition to satisfying this equation also satisfies certain boundary conditions on the boundary C of the region D. From the theory of analytic functions, we know that the real and the imaginary parts of an analytic function satisfy Laplace's equation. It follows that solving the above problem reduces to finding a function that is analytic in D and that satisfies certain boundary conditions on C. Using the conformal mapping it turns out that the solution of this problem can be greatly simplified if the region D is either the upper half of the z plane or the unit disk.

If in the z −plane we are given a potential $\Phi(x, y)$, and apply to it a conformal transformation, in the w-plane we obtain a potential $\varphi(x, y)$ that is a solution of the Laplace equation $\varphi_{uu} + \varphi_{vv} = 0$. After this unit you will be able to

- understand the principle of conformal mapping and their different types.
- learn the necessary and sufficient conditions for conformal mappings.
- find the image of the curve C in the z −plane into w −plane under the transformation $w = f(z)$.

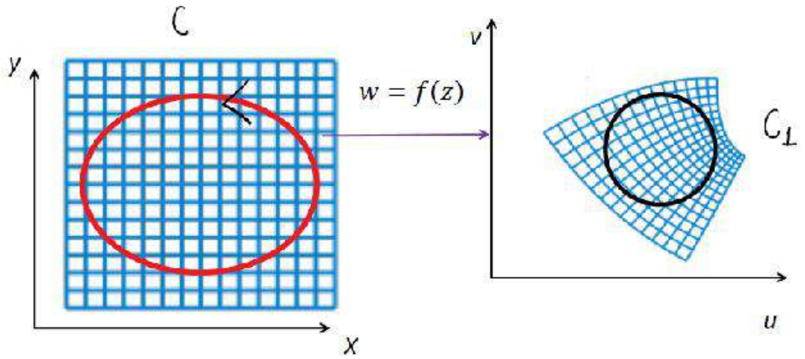
Introduction

A complex number $z = x + iy$ can be represented by a point P whose coordinates are (x, y). The axis of x is called real axis and the axis of y is called imaginary axis. The plane is called as a z −plane or a complex plane or Argand plane.

A number of points (x, y) are plotted on z −plane by taking different values of z (i.e., different values of x and y).

The curve C is drawn by joining the plotted points in the z −plane. Now let $w = u + iv = f(z) = f(x + iy)$. To draw a curve of w, we take u-axis and v-axis.

By plotting different (u, v) on w-plane and joining them, we get a curve C_1 on w −plane. For every point (x, y) in the z −plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w −plane shown as in the following figure.



We call this as" transformation" or mapping of z −plane into w −plane. If a point z_0 maps into the point w_0, w_0 is known as the image of z_0. As the point $P(x, y)$ traces a curve C in z −plane the transformed point P_0 (u, v) will trace a curve C_1 in w-plane.

We say that a curve C in the z −plane is mapped in to the corresponding curve C_1 in w −plane by the relation $w = f(z)$.

Example:

Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$.

Solution:

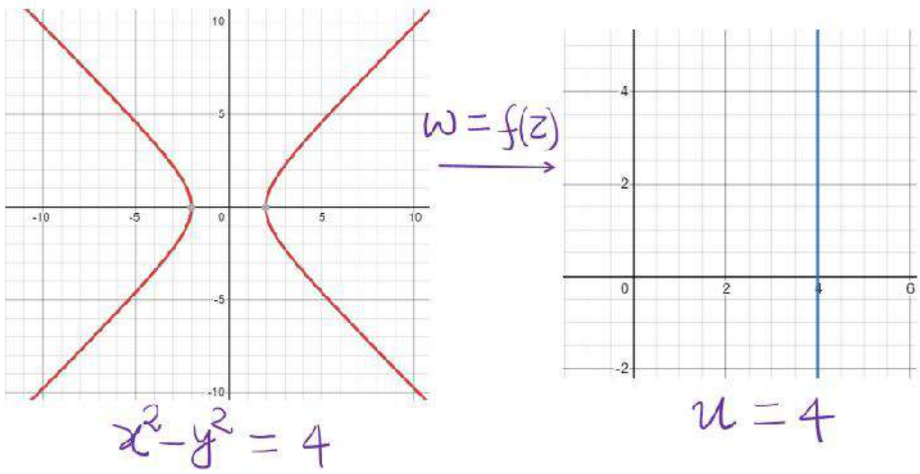
$$w = z^2$$

$$\Rightarrow u + iv = (x + iy)^2$$

$$= (x^2 - y^2 + i(2xy).$$

This gives $u = x^2 - y^2$ and $v = 2xy$

Image of the curve $x^2 - y^2$ is a straight line, $u = 4$ parallel to the v-axis in w-plane.



13.1 Conformal Transformation

Let two curves C_1, C_2 in the z −plane intersect at the point P and the corresponding curve C_{01} , C_{02} in the z −plane intersect at P_0 .

If the angle of intersection of the curves at P in z −plane is the same as the angle of intersection of the curves of w −plane at P_0 in magnitude and sense, then the transformation is called conformal transformation at P.

13.2 Necessary and Sufficient Conditions for Conformal Mappings

The Necessary and sufficient condition for the toning formation $w = f(z)$ to be conformal in that $f(z)$ is analytic.

Theorem

Let $f(z)$ be an analytic function of z in a region D of the z −plane and $f'(z) \neq 0$ in D. Then the mapping $w = f(z)$ is conformal at all points of D.

13.3 Classification of Conformal Transformation

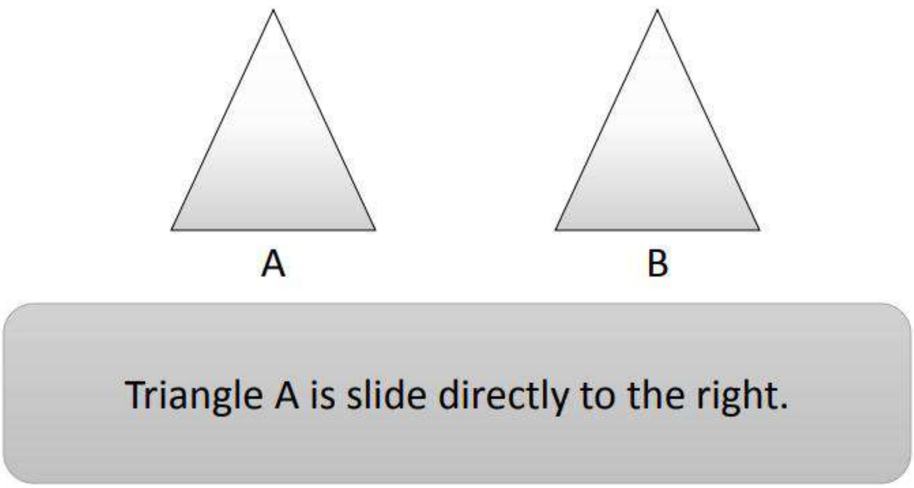
1. **Translation:**

A translation is a transformation that slides a figure across a plane or through space. With translation all points of a figure move the same distance and the same direction.

Basically, translation means that a figure has moved. An easy way to remember what translation means is to remembera translation is a change in location. A translation is usually specified by a direction and a distance.

The mapping is $w = z + c$, where c is a complex constant.
Let $z = x + iy, w = u(x,y) + iv(x,y)$ and $c = c_1 + ic_2$. Then $w = z + c$ will imply $u + iv = (x + iy) + (c_1 + ic_2)$
$= (x + c_1) + i(y + c_2)$.

By comparing real and imaginary parts, we get, $u = x + c_1$, and $v = y + c_2$ Thus, the transformation of a point $P(x,y)$ in the $z - plane$ onto a point $P_0(x + c_1, y + c_2)$. Hence, the transformation is a translation of the axes and preserves the shape and size.



2. Rotation and Magnification:

A rotation is a transformation that turns a figure about (around) a point or a line. The point a figure turns around is called the center of rotation. Basically, rotation means to spin a shape. The center of rotation can be on or outside the shape.

Dilation changes the size of the shape without changing the shape. When you go to the eye doctor, they dilate you eyes. Let's try it by turning off the lights.

When you enlarge a photograph or use a copy machine to reduce a map, you are making dilations. Enlarge means to make a shape bigger.

Reduce means to make a shape smaller. The scale factor tells you how much something is enlarged or reduced.

This mapping is $w = cz$, where c is a complex constant.

a. Cartesian form: Let $w = u(x,y) + iv(x,y), z = x + iy$ and $c = c_1 + ic_2$. Then $w = cz$ will imply that

$$u + iv = (c_1 + ic_1)(x + iy)$$

$$= (c_1x - c_2y) + i(c_1y + c_2x)$$

By comparing real and imaginary parts, $u(x,y) = c_1x - c_2y$ and

$$v(x,y) = c_1y - c_2x.$$

Thus, the transformations of a point $P(x,y)$ in the z −plane into a point

$P_0\ (c_1x - c_2y, c_1y + c_2x)$ in w-plane.

b. Polar form: Let $w = Re^{i\varphi}, z = re^{i\theta}$ and $c = \rho e^{i\alpha}$.

Then transformation $w = cz$ becomes $Re^{i\varphi} = \rho e^{i\alpha} \cdot re^{i\theta} = \rho re^{i(\alpha+\theta)}$.

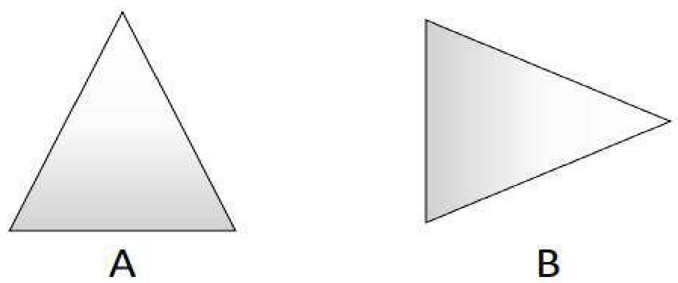
By comparing, we have $R = \rho r and \varphi = \theta + \alpha$

Thus, the transformation maps a point $P(r,\theta)$ in the z −plane into a point $P_0(\rho r, \theta + \alpha)$ in the w −plane.

Hence, the transformations consist of magnification of the radius vector of P by $\rho = |c|$ and its rotation through the angle α.

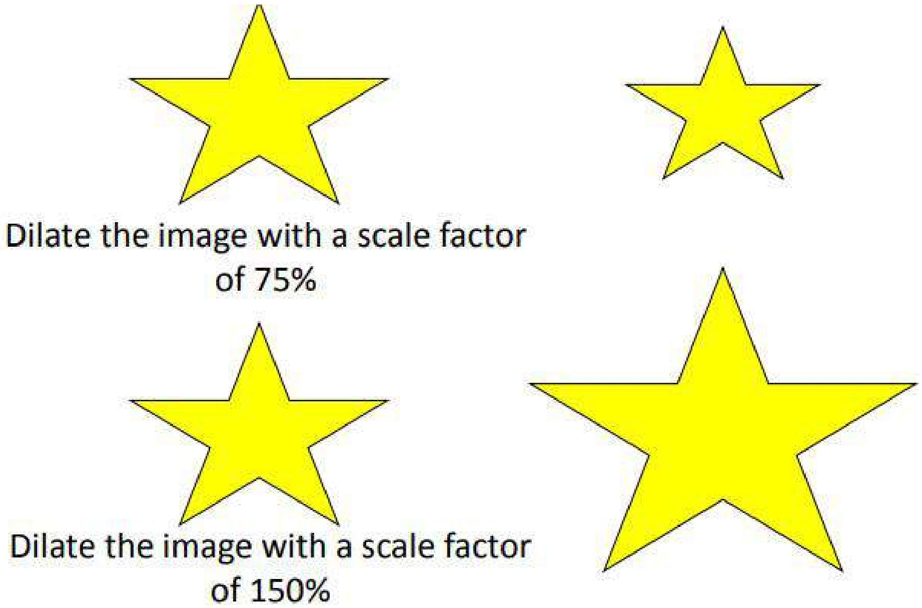
Example

Describe how the triangle A was transformed to make triangle B



Triangle A was **rotated** right 90°

Notice each time the shape transforms the shape stays the same and only the size changes.



3. **Inversion and Reflection:** The transformation $w = \frac{1}{z}$ is known as inversion and reflection.
   a) Cartesian Form: Let $w = u + iv, z = x + iy$, then

$$w = \frac{1}{z}$$
$$\Rightarrow u + iv = \frac{1}{x + iy}$$
$$u + iv = \frac{x - iy}{(x + iy)(x - iy)}$$
$$u + iv = \frac{x - iy}{(x^2 + y^2)}$$

By comparing the real and the imaginary parts, we get

$$u = \frac{x}{(x^2 + y^2)}$$
$$v = \frac{-y}{(x^2 + y^2)}$$

Thus, the transformation maps a point $P(x, y)$ in the z-plane into a point $P_0\left(\frac{x}{(x^2+y^2)}, \frac{-y}{(x^2+y^2)}\right)$in the w −plane.

(b) Polar Form:

Let $w = Re^{i\varphi}$ and $z = re^{i\theta}$. Then the transformation becomes $Re^{i\varphi} = \frac{1}{r}e^{-i\theta}$.

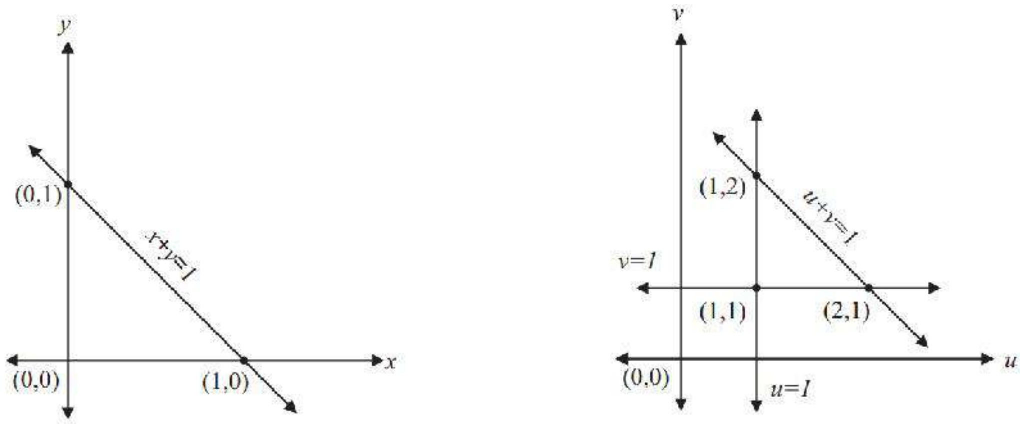
so that $R = \frac{1}{r}$ *and* $\varphi = -\theta$. Thus under the transformation w = 1/z , any point $P(r, \theta)$ in z-plane is mapped into the point $P_0(\frac{1}{r}, -\theta)$. Note that, the origin z = 0 is mapped to the point w = ∞, called the point at infinity.

Example

Consider the transformation $w = z + (1 + i)$ and determine the region in the w-plane corresponding to the triangular region bounded by the lines $x = 0, y = 0$ *and* $x + y = 1$ in the z-plane.

Solution:

The given triangular region bounded by the lines $x = 0$, $y = 0$ and $x+y = 1$ is shown in the following figure. Then the vertices of the triangular region are (0, 0),(1, 0) and (0, 1). Now, the given transformation is



$$w = z + (1 + i)$$

$$\Rightarrow u + iv = (x + iy) + (1 + i)$$

$$\Rightarrow u + iv = (x + 1) + i(y + 1)$$

$$\Rightarrow u = x + 1 \text{ and } v = y + 1.$$

$$x = 0 \Rightarrow u = 1;$$

$$y = 0 \Rightarrow v = 1;$$

$$x + y = 1 \Rightarrow u - 1 + v - 1 = 1 \Rightarrow u + v = 3$$

The line $x = 0$ maps into $u = 1$, which is also the vertical line in w-plane.

Also, the line $y = 0$ maps into $v = 1$, which is the horizontal line in w-plane. And, the line $x + y = 1$ maps into the line $u + v = 3$. Hence, the region becomes triangle bounded by the lines $u = 1, v = 1, u + v = 3$; which is shown in above figure.

Example

Find the image of the circle $|z| = 2$ under the transformation $w = iz + 1$.

Solution:

Let $w = u + iv$ and $z = x + iy$.

Then $w = iz + 1$

$$\Rightarrow u + iv = i(x + iy) + 1 = ix - y + 1$$

$$= (1 - y) + ix$$

$$\Rightarrow u = 1 - y \text{ and } v = x$$

$$\Rightarrow y = 1 - u \text{ and } x = v \text{ Now, } |z| = 2$$

$$\Rightarrow |z|^2 = 4$$

$$\Rightarrow |x + iy|^2 = 4$$

$$\Rightarrow x^2 + y^2 = 4$$

$$\Rightarrow v^2 + (1 - u)^2 = 4$$

$$\Rightarrow (u - 1)^2 + v^2 = 4$$

which is equation of the circle centered at (1, 0) and radius is 2. Thus the transformation rotates the circle by $\pi/2$ and translate its by unity to the right.

Example

Find the image of the line $y - x + 1 = 0$ under the mapping $w = \frac{1}{z}$. Show it graphically

Solution:

Let $w = u + iv$,

$z = x + iy$. Then the transformation $w = \frac{1}{z}$

$$\Rightarrow u + iv = \frac{1}{x + iy}$$

$$\Rightarrow x + iy = \frac{1}{u + iv}$$

$$\Rightarrow x + iy = \frac{1}{u + iv} \times \frac{u - iv}{u - iv}$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

comparing real and imaginary parts

$$x = \frac{u}{u^2 + v^2}$$

$$y = -\frac{v}{u^2 + v^2}$$

$$Now\ y - x + 1 = 0$$

$$\Rightarrow x - y = 1$$

$$\Rightarrow \frac{u}{u^2 + v^2} + \frac{v}{u^2 + v^2} = 1$$

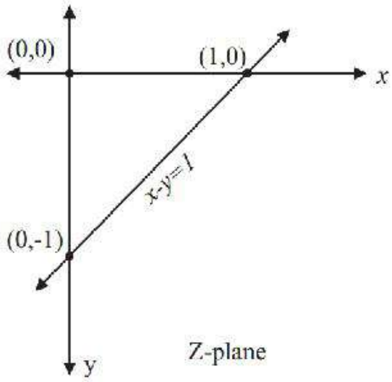
$$\Rightarrow u + v = u^2 + v^2$$

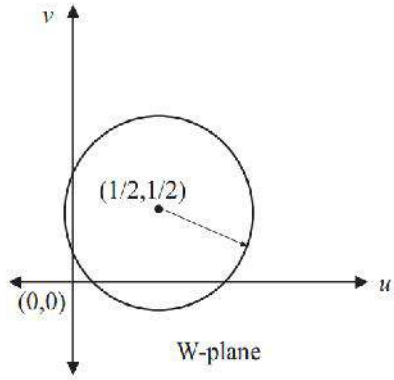
$$\Rightarrow u^2 - u + v^2 - v = 0$$

$$\Rightarrow u^2 - u + \frac{1}{4} + v^2 - v + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2}$$

which is the equation of the circle centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ with radius $\left(\frac{1}{2}\right)^{\frac{1}{2}}$





Example

Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscate $R^2 = cos\, 2\varphi$, where $w = Re^{i\varphi}$

Solution:

Let $w = u + iv$,

$z = x + iy$. Then the transformation $w = \frac{1}{z}$

$$\Rightarrow u + iv = \frac{1}{x + iy}$$

$$\Rightarrow x + iy = \frac{1}{u + iv}$$

$$\Rightarrow x + iy = \frac{1}{u + iv} \times \frac{u - iv}{u - iv}$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

comparing real and imaginary parts

$$x = \frac{u}{u^2 + v^2}$$

$$y = -\frac{v}{u^2 + v^2}$$

$$x^2 - y^2 = 1$$

$$\left(\frac{u}{u^2 + v^2}\right)^2 - \left(\frac{v}{u^2 + v^2}\right)^2 = 1$$

$$\Rightarrow u^2 - v^2 = (u^2 + v^2)^2$$

Putting $u = R\cos\varphi$ and $v = R\sin\varphi$ ($\because w = Re^{i\varphi}$),

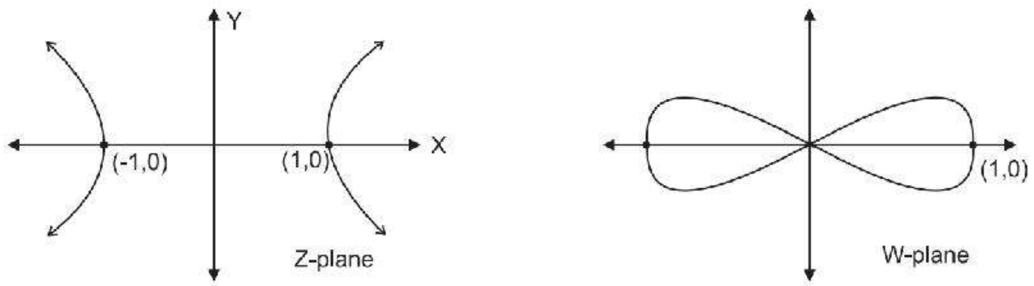
we get,

$$(R\cos\varphi)^2 - (R\sin\varphi)^2 = ((R\cos\varphi)^2 + (R\sin\varphi)^2)^2$$

$\Rightarrow R^2 = \cos^2\varphi - \sin^2\varphi$

$R^2 = cos\, 2\varphi$.

Hence, the image of hyperbola in z-plane is $R^2 = cos\, 2\varphi$, which is lemniscate shown in the following figure.



Summary

- A complex number $z = x + iy$ can be represented by a point P whose coordinates are (x, y). The axis of x is called real axis and the axis of y is called imaginary axis. The plane is called as a z −plane or a complex plane or Argand plane.
- A curve C in the z −plane is mapped in to the corresponding curve C_1 in w −plane by the relation $w = f(z)$.
- If the angle of intersection of the curves at P in z −plane is the same as the angle of intersection of the curves of w −plane at P_0 in magnitude and sense, then the transformation is called conformal transformation at P.
- The Necessary and sufficient condition for the toning formation $w = f(z)$ to be conformal in that $f(z)$ is analytic.
- Let $f(z)$ be an analytic function of z in a region D of the z −plane and $f'(z) \neq 0$ in D. Then the mapping $w = f(z)$ is conformal at all points of D.

Keywords

- **Translation:** The mapping is $\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{c}$, where $\boldsymbol{c}$ is a complex constant.
- **Rotation and Magnification:** This mapping is $w = cz$, where c is a complex constant.
- **Inversion and Reflection:** The transformation $w = \frac{1}{z}$ is known as inversion and reflection.

Self Assessment

1. What is the region's shape formed by the set of complex numbers z satisfying |z-ω|≤ α?

A. circle of radius ω

B. circle with center ω

C. disk of radius α

D. disk with center α

2. Find the area of the region given by 11≤|z| ≤ 19.

A. 120π sq. units

B. 180π sq. units

C. 240π sq. units

D. 320π sq. units

3. Describe the region given by $|z - i|z|| - |z + i|z|| = 0$.

A. real axis

B. imaginary axis

C. circle centered at origin

D. quadrant 2

4. The mapping $w = e^z$ is conformal in whole complex plane

A. True

B. False

5. The image of $|z - 1| = 2$ under the $w = 2z$

A. $|w - 2| = 4$

B. $|w - 1| = 4$

C. $|w| = 4$

D. $|w - 2| = 2$

6. The image of $|z| = 2$ under the $w = 2z$

A. $|w| = 4$

B. $|w - 1| = 4$

C. $|w| = 4$

D. $|w - 2| = 2$

7. The image of $|z - 1| = 2$ under the $w = 3z$

A. $|w - 3| = 6$

B. $|w - 1| = 6$

C. $|w| = 6$

D. $|w - 2| = 4$

8. The image of $|z| = 2$ under the $w = 3z$

A. $|w| = 6$

B. $|w - 1| = 6$

C. $|w| = 4$

D. $|w - 2| = 2$

9. The image of $z = i$ under the $w = 3z + 4 - 2i$

A. $w = 4 + i$

B. $w = 6 + i$

C. $w = 4$

D. $w = 2$

10. The image of $\boldsymbol{z = 1 + i}$ under the $\boldsymbol{w = 3z + 4 - 2i}$

A. $w = 4 + i$

B. $w = 7 + i$

C. $w = 4$

D. $w = 2$

11. The image of $\boldsymbol{z = 1 - i}$ under the $\boldsymbol{w = 3z + 4 - 2i}$

A. $w = 4 + i$

B. $w = 7 - 5i$

C. $w = 4$

D. $w = 2$

12. The image of $|\boldsymbol{z - 1}| \geq \boldsymbol{2}$ under the $\boldsymbol{w = 2z}$

A. $|w - 2| \geq 4$

B. $|w - 1| \geq 4$

C. $|w| \geq 4$

D. $|w - 2| = 2$

13. The image of $|\boldsymbol{z}| > 2$ under the $\boldsymbol{w = 2z}$

A. $|w| > 4$

B. $|w - 1| = 4$

C. $|w| = 4$

D. $|w - 2| = 2$

14. The image of $|\boldsymbol{z - 1}| > 2$ under the $\boldsymbol{w = 3z}$

A. $|w - 3| > 6$

B. $|w - 1| < 6$

C. $|w| = 6$

D. $|w - 2| = 4$

15. The image of $|\boldsymbol{z}| = \boldsymbol{1}$ under the $\boldsymbol{w = 3z}$

A. $|w| = 3$

B. $|w - 1| = 3$

C. $|w| = 4$

D. $|w - 2| = 2$

Answers for Self Assessment

l.	D	2.	B	3.	A	4.	A	5.	A

6.	A	7.	A	8.	A	9.	A	10.	B
11.	B	12.	A	13.	A	14.	A	15.	A

Review Questions

1. Find the image of $|z - 10| = 2$ under the $w = 2z + 5$

2. Find the image of $|z - 10| = 2$ under the $w = 5z$

3. Find the image of $|z - 10| = 2$ under the $w = \frac{z}{3}$

4. Find the image of $z^2 = 2$ under the $w = 2z + 5$

5. Find the image of $z^2 - 5z = 2$under the $w = 5z$

6. Find the image of $z^2 - 1 = 2$under the $w = \frac{z}{3}$

7. Find the image of $|z - 10| = 2$ under the $w = 1/z$

8. Find the image of $|z - 10| = 2$ under the $w = 5/z$

9. Find the image of $|z - 10| = 2$ under the $w = \frac{1}{z}$

10. Find the image of $z^2 = 2$ under the $w = 1/z$

Further Readings

1. Complex Variables and Applications by Churchill, R. V., And Brown, J. W., McGraw Hill Education.

2. Foundations Of Complex Analysis by S. Ponnusamy, Narosa Publishing House.

3. Complex Analysis by Lars V. Ahlfors, Mcgraw Hill Education.

4. Complex Variables Theory and Applications by H. S. Kasana, Prentice-Hall.

Unit 14: Mobius Transformation

CONTENTS

Objectives

Mobius transformation are among those fundamental mapping in the geometry with application from brain mapping to relativity theory. A Mobius transformation sense each point in a plane to a corresponding point.In this section we investigate the Mobius transformation which provides very convenient methods of finding a one-to-one mapping of one domain into another.

After this unit you will be able to

- understand the Mobius transformation and its property
- learn theCross Ratio of cross-ratio of four pointsand its invariance property
- solve the problem of fixed points and Mobius Maps.

Introduction

In the previous unit we have studied a linear transformation $w = \varphi(z) := Az + B$, where A and B are fixed complex numbers, $A \neq 0$. We write $w = \varphi(z)$ as $|A|e^{iA(a)} z + B$.

As we see this transformation is a composition of a rotation about the origin through the angle $Arg\ (a)$.

$w_1 := e^{iArg(a)}\ z$, a magnification.

$w_2 = |A|w_1$ and a translation $w = w_3 = w_2 + B$.

Each of these transformations are one-to-one mappings of the complex plane onto itself and gap geometric objects onto congruent objects. In this unit first, we will understand the Mobius transformation and its property and then we will consider the fixed point, and last, we will deduce the learn the Cross Ratio of cross-ratio of four points and its invariance property.

14.1 Mobius Transformation

A transformation of the form $w = f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0, a, b, c, d$ are complex constants is called a bilinear transformation or mobius transformation. Bilinear transformation is conformal since $\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0.$

The inverse mapping of the above transformation is $f^{-1}(w) = z = \frac{-dw+b}{cw-a}$

which is also a bilinear transformation.

We can extend f and f^{-1} to mappings in the extended complex plane. The value $f(\infty)$ should be chosen, so that $f(z)$ has a limit ∞.

Therefore, we define $f(\infty) = \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = a/c$

and the inverse is $f^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, the value $f^{-1}(\infty)$ is obtained by

$$f^{-1}(\infty) = \lim_{w\to\infty} f(w) = \lim_{w\to\infty} \frac{-d+\frac{b}{w}}{c-\frac{a}{w}} = -d/c$$

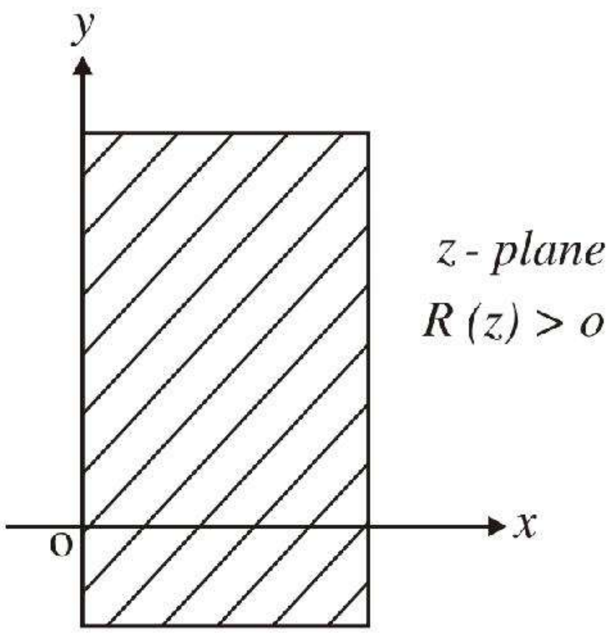
and the inverse is $f^{-1}\left(\frac{-d}{c}\right) = \infty$.

With these extensions we conclude that the transformation $w = f(z)$ is a one-to-one mapping of the extended complex z −plane into the extended complex w-plane.

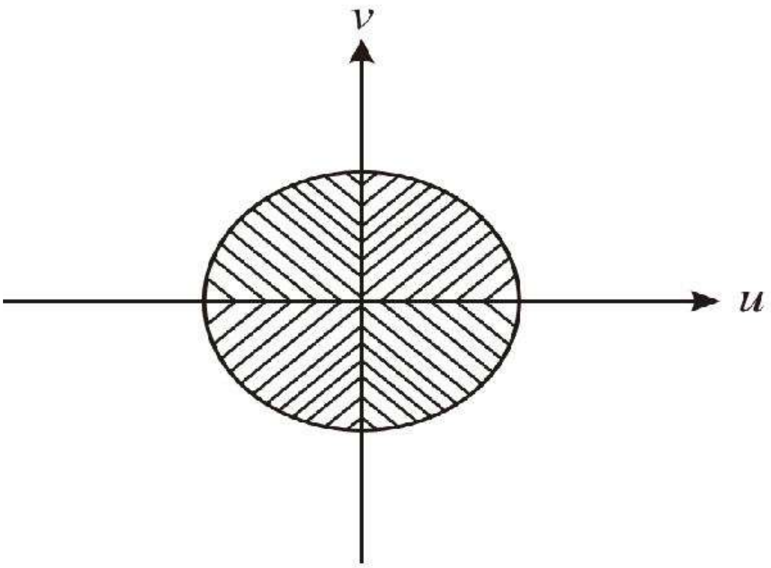
Example

The Bilinear Transformation which transforms R(z)≥ 0 into the unit circle $|w| \leq 1$

Here the region for R(z)≥ 0 is shown below.



And the following figure represents the region $|w| \leq 1$in the w -plane.



Suppose the bilinear transformation $w = \frac{az+b}{cz+d}$ .... (1)

The transforms the half plane R (z) ≥ 0 into the circle$|w| \leq 1$.

w is expressible as $w = \frac{a}{c}\frac{z+b/a}{z+d/c}$ … … … … … (2)

This Ĝ $c \neq 0$, otherwise the points at infinity in the two planes would correspond.

We have seen that the transformation (2) transforms a straight line of z-plane into a circle of w-plane and points symmetrical about the line transform into the inverse points w.r.t.$|w| = 1$.

Here the points $z\, and\, -\bar{z}$ symmetrical about the imaginary axis $x = R(z) = 0$ will correspond to $w = 0$ and $w = \infty$, the inverse points of the circle $|w| = 1$.

Thus we may write, $\frac{b}{a} = -\alpha$and $\frac{d}{c} = \bar{\alpha}$ on putting above in eqn (2) ,$we\ get\ w = 0, w = \infty$.

Then (2) takes the form $w = \frac{a}{c} * \frac{z-\alpha}{z+\bar{\alpha}}+$ .... (3)

The point z = 0 on the boundary of the half plane R (z) ≥ 0 must correspond to a point on the boundary of $|w| = 1$so that

$$|w| = \left|\frac{a}{c}\right| * \left|\frac{z-\alpha}{z+\bar{\alpha}}\right|$$

$$1 = \left|\frac{a}{c}\right| * \left|\frac{0-\alpha}{0+\bar{\alpha}}\right|$$

$$\left|\frac{a}{c}\right| = 1$$

$$\frac{a}{c} = e^{i\lambda}$$

Where λ is real. Hence

$$w = e^{i\lambda} * \frac{z-\alpha}{z+\bar{\alpha}}$$

Evidently z =α gives w = 0. But w = 0 is an interior point of circle $|w| = 1$.

Hence z =α must be a point of the right half plane i.e., R (α) > 0. With this condition

$$w = e^{i\lambda} * \frac{z - \alpha}{z + \overline{\alpha}}$$

is the required transformation.

Theorem

The composition of two Möbius transformations is again a Möbius transformation.

Proof

Just as translations and rotations of the plane can be constructed from reflections across lines, the general Möbius transformation can be constructed from inversions about clines.

14.2 Property of Mobius Transformation

Every bilinear transformation $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ is the combination of basic transformations translations, rotations and magnification and inversion.

Möbius transformations = products of inversions (or sometimes orientation-preserving products) Forms group of geometric transformations Contains all circle-preserving transformations in higher dimensions (but not 2d) contains all conformal transformations.

14.3 Fixed Point

A point z_0 in complex plane is called a fixed point for the function f if $f(z_0) = z_0$.

To visualize Möbius transformations it is helpful to focus on fixed points and, in the case of two fixed points, on two families of clines with respect to these points.

Given two points p and q in complex plane in the following figure, a ***type I cline of*** p ***and*** q is a cline that goes through p and q, and a ***type II cline of*** p ***and*** q is a cline with respect to which p and q are symmetric.

Type II clines are also called circles of Apollonius shows some type I and type II clines of p and qq. The type II clines of p and q are dashed.

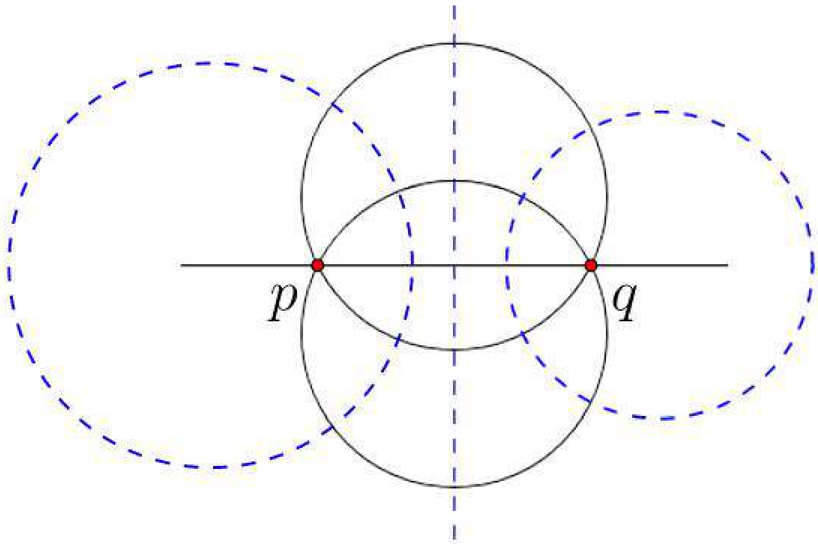


Figure 1. Type I clines (solid) and Type II clines (dashed) of pp and q.

Example

Find the fixed point/s of $f(z) = \frac{z-1}{z}$

Solution

$$f(z) = \frac{z-1}{z} = z$$

$$1 = \frac{z-1}{z^2}$$

$$z^2 - z + 1 = 0$$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \ast 1 \ast 1}}{2}$$

$$z = \frac{1 \pm \sqrt{3}i}{2}$$

So

$z_0 = \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$ are the fixed points

Example

Find the fixed point/s of $f(z) = \frac{z-5}{z}$

Solution

$$f(z) = \frac{z-5}{z} = z$$

$$1 = \frac{z-5}{z^2}$$

$$z^2 - z + 5 = 0$$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \ast 1 \ast 5}}{2}$$

$$z = \frac{1 \pm \sqrt{19}i}{2}$$

So

$z_0 = \frac{1+19i}{2}, \frac{1-\sqrt{19}i}{2}$ are the fixed points

14.4 Cross Ratio

We have already seen that Möbius transformations map circles to circles. In this section we want to find a specific Möbius transformation that takes a specific circleto another specific circle. Recall from Euclidean geometry that three points uniquely determine a circle. Let us denote one circle by C_1 and one by C_2. We choose points z_1, z_2 *and* z_3 on C_1 *and* w_1, w_2 *and* w_3 *on* C_2.

Then if we find a Möbius transformation h that takes

$$h(z_1) = w_1,$$

$$h(z_2) = w_2,$$

$$h(z_3) = w_3.$$

then h must map C_1 *to* C_2.

If the points $z_i \neq \infty$, we define a Möbius transformation f by $f(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

which clearly takes

$$f(z_1) = 0,$$

$$f(z_2) = 1,$$

$$f(z_3) = \infty.$$

If one of the three points $z_i = \infty$ (which means that C_1 is a line) we have

$$f(z) = \frac{z_2 - z_3}{z - z_3}\ as(z_1 = \infty),$$

$f(z) = \frac{z - z_1}{z - z_3}\ as(z_2 = \infty),$

$f(z) = \frac{z - z_1}{z_2 - z_1}\ as(z_1 = \infty),$

which satisfy $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty.$

Now let g be another Möbius transformation which takes

$$g(w_1) = 0, g(w_2) = 1, g(w_3) = \infty.$$

Then we notice that the equation $h(z) = w$ can be written as

$$g^{-1}(f(z)) = w \iff g(w) = f(z)$$

which means that $\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$

These fractions are called cross ratios.

Example

Find a Möbius transformation that takes $0\ to\ i, 1\ to\ 2\ and\ -1\ to\ 4$ ratio?

Solution

We calculate the appropriate cross

$$(z, 0, 1, -1) = \frac{(z - 0)(1 - (-1))}{(z - (-1))(1 - 0)}$$

$$= \frac{2z}{z + 1}$$

$$(w, i, 2, 4) = \frac{(w - i)(2 - 4)}{(w - 4)(2 - i)}$$

$$= \frac{-2(w - i)}{(w - 4)(2 - i)}$$

Now

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$$

$$\frac{2z}{z + 1} = \frac{-2(w - i)}{(w - 4)(2 - i)}$$

which gives $w = h(z) = \frac{(16 - 6i)z + 2i}{(6 - 2i)z + 2}$

which is the desired Möbius transformation.

Example

Find a Möbius transformation that takes the region $D_1 = \{z \in C \mid |z| > 1\}$ to the region $D_2 = \{z \in C \mid < 0\}$.

We choose both D_1 *and* D_2 to be left regions.

That is accomplished by choosing

$$z_1 = 1,$$

$$z_2 = -i,$$

$$z_3 = -1, and$$

$$w_1 = 0,$$

$$w_2 = i,$$

$$w_3 = \infty.$$

Since Möbius transformations take left regions to left regions, a solution to the problems is a Möbius transformation that takes $1\ to\ 0, -i\ to\ i\ and\ -1\ to\ \infty$.

As in the previous example we find such a Möbius transformation by equating the two cross ratios, i.e.,$(w, 0, i, \infty) = (z, 1, -i, -1)$

which is the same as $\frac{w-0}{i-0} = (z-1)(-i+1)/(z+1)(-i-1)$,

where we have used the first formula in Equation 6 to calculate the cross ratio for w. This gives the desired Möbius transformation $w = h(z) = \frac{1-z}{1+z}$

There exists a unique bilinear transformation that maps four distinct points z_1, z_2, z_3 *and* z_4 on to four distinct points w_1, w_2, w_3 *and* w_4 respectively.

An implicit formula for the mapping is given by, $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)}$

The above expression is known as cross-ratio of four points.

Summary

- Every bilinear transformation $w = (az+b)/(cz+d)$,$ad - bc \neq 0$ is the combination of basic transformations translations, rotations and magnification and inversion.
- The cross-ratio of four points is bilinear transformation that maps four distinct points z_1, z_2, z_3 *and* z_4 on to four distinct points w_1, w_2, w_3 *and* w_4 respectively is given by, $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)}$.

Keywords

Fixed point: A point $\boldsymbol{z_0}$ in complex plane is called a fixed point for the function $\boldsymbol{f}$

if $f(z_0) = z_0$.

Self Assessment

1. What is/are the fixed points of $\frac{2z-4}{z-1}$?

A. $\frac{3\pm\sqrt{7}i}{2}$

B. $\frac{3\pm\sqrt{5}i}{2}$

C. $\frac{4\pm\sqrt{7}i}{2}$

D. $\frac{3\pm\sqrt{7}i}{4}$

2. What is/are the fixed points of $\frac{2z}{z-1}$?

A. 3

B. 2

C. $\frac{5}{2}$

D. 1

3. What is/are the fixed points of $\frac{2z-4}{z}$?

A. $\frac{1\pm\sqrt{3}i}{2}$

B. $\frac{1\pm\sqrt{3}i}{5}$

C. $\frac{4\pm\sqrt{7}i}{2}$

D. $1 \pm \sqrt{3}i$

4. What is/are the fixed points of $\frac{1}{z}$?

A. 3

B. 2

C. $\frac{5}{2}$

D. 1

5. What is/are the fixed points of $\frac{z-5}{z}$?

A. $\frac{1\pm\sqrt{19}i}{2}$

B. $\frac{3\pm\sqrt{3}i}{2}$

C. $\frac{4\pm\sqrt{8}i}{2}$

D. $\frac{2\pm\sqrt{7}i}{4}$

6. What is/are the fixed points of $\frac{z-3}{z}$?

A. $\frac{1\pm\sqrt{19}i}{2}$

B. $\frac{3\pm\sqrt{3}i}{2}$

C. $\frac{4\pm\sqrt{8}i}{2}$

D. $\frac{1\pm\sqrt{11}i}{4}$

7. What is/are the fixed points of $\frac{z-3}{z-2}$?

A. $\frac{1\pm\sqrt{19}i}{2}$

B. $\frac{3\pm\sqrt{3}i}{2}$

C. $\frac{4\pm\sqrt{8}i}{2}$

D. $\frac{3\pm\sqrt{10}i}{4}$

8. What is/are the fixed points of $\frac{z}{z-2}$?

A. 0

B. 2

C. $\frac{5}{2}$

D. 1

9. A transformation of the form $w = f(z) = \frac{z+b}{z+5}$ is called a bilinear transformation or mobius transformation if____?

A. $b \neq \frac{10}{3}$

B. $b \neq 5$

C. $b \neq 7$

D. $b \neq \frac{7}{3}$

10. A transformation of the form $w = f(z) = \frac{z+1}{cz+5}$ is called a bilinear transformation or mobius transformation if____?

A. $c \neq \frac{7}{3}$

B. $c \neq 5$

C. $c \neq 7$

D. $c \neq \frac{1}{3}$

11. A transformation of the form $w = f(z) = \frac{2z+b}{3z+5}$ is called a bilinear transformation or mobius transformation if____?

A. $b \neq \frac{10}{3}$

B. $b \neq \frac{10}{3}$

C. $b \neq \frac{7}{3}$

D. $b \neq \frac{7}{3}$

12. A transformation of the form $w = f(z) = \frac{3z+b}{5z+5}$ is called a bilinear transformation or mobius transformation if____?

A. $b \neq 5$

B. $b \neq 6$

C. $b \neq 7$

D. $b \neq 3$

13. A transformation of the form $w = f(z) = \frac{3z+5}{cz+5}$ is called a bilinear transformation or mobius transformation if____?

A. $c \neq \frac{7}{3}$

B. $c \neq 3$

C. $c \neq 7$

D. $c \neq \frac{1}{3}$

14. A transformation of the form $w = f(z) = \frac{4z+b}{5z+5}$ is called a bilinear transformation or mobius transformation if____?

A. $b \neq 4$

B. $b \neq 6$

C. $b \neq 7$

D. $b \neq 3$

15. A transformation of the form $w = f(z) = \frac{4z+5}{cz+5}$ is called a bilinear transformation or mobius transformation if____?

A. $c \neq \frac{1}{4}$

B. $c \neq 4$

C. $c \neq 7$

D. $c \neq \frac{1}{7}$

Answers for Self Assessment

l.	A	2.	A	3.	D	4.	D	5.	A
6.	D	7.	B	8.	A	9.	B	10.	B
11.	B	12.	D	13.	B	14.	A	15.	B

Review Questions

1. Find the fixed point of $\frac{z}{2z-1}$
2. Find the fixed point of $\frac{-z}{(1+i)z-i}$
3. Find the fixed point of $\frac{3iz-5}{z-i}$
4. Find the fixed point of $\frac{z-1}{2z-1}$
5. Find the fixed point of $\frac{z-2}{z-1}$
6. Analyze the Möbius transformation of $\frac{z}{2z-1}$
7. Analyze the Möbius transformation of $\frac{-z}{(1+i)z-i}$
8. Analyze the Möbius transformation of $\frac{3iz-5}{z-i}$
9. Analyze the Möbius transformation of $\frac{z-1}{2z-1}$
10. Analyze the Möbius transformation of $\frac{z-2}{z-1}$

Further Readings

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