Real Analysis II

DEMTH528

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CONTENTS

Unit 01: Lebesgue Outer Measure

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Objectives

After studying this unit, students will be able to:

- understand extended real numbers
- define outer measure
- identify properties of outer measure
- define F_σ −set and G_δ −set
- explain Cantor set and its measure

Introduction

The goal of this unit is to study a set function, on a collection of setstaking values in the non-negative extended real numbers, that generalizes the notion of length of an interval. All the sets, considered in this unit are subsets of the set of real numbers, unless stated otherwise. This unit provides the basis for the forthcoming study of Lebesgue measurable sets, Lebesgue measurable functions, and the Lebesgue integral.

1.1 Extended Real Numbers

Let $\mathbb{R}^* = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, where $\mathbb{R} = (-\infty, \infty)$.

That is, we can write $\mathbb{R}^* = [-\infty, \infty]$.

Here $+\infty$ and $-\infty$ are two symbols.

Order relation on $\mathbb{R}^*$

For every $x \in \mathbb{R}, -\infty < x < +\infty$

Here $-\infty$ is the smallest element in $\mathbb{R}^*$ and $+\infty$ is the largest element in $\mathbb{R}^*$.

Algebraic operations on $\mathbb{R}^*$

Addition: For every $x \in \mathbb{R}$

1. $(-\infty) + x = -\infty$
2. $(+\infty) + x = +\infty$
3. $(+\infty) + (+\infty) = +\infty$
4. $(-\infty) + (-\infty) = -\infty$

Notes: $(+\infty) + (-\infty)$ is not defined.

Multiplication:If $x > 0$, then

$$x(+\infty) = (+\infty)(x) = +\infty$$
$$x(-\infty) = (-\infty)(x) = -\infty$$

and if $x < 0$, then

$$(+\infty)x = (+\infty)(x) = -\infty$$
$$(-\infty)x = (-\infty)(x) = +\infty.$$

Further, we have,

$$(+\infty)(0) = (-\infty)(0) = 0$$
$$(\pm\infty)(+\infty) = \pm\infty$$
$$(\pm\infty)(-\infty) = \mp\infty$$

The set $\mathbb{R}^*$ also denoted as $[-\infty, \infty]$ with the above properties is called the set of **extended real numbers**.

Supremum and infimum in $\mathbb{R}^*$**:** Let $A \subseteq \mathbb{R}^*$ be any non-empty set.

$Sup\ (A) = +\infty$ if $A \cap \mathbb{R}$ is not bounded above.

$Inf\ (A) = -\infty$ if $A \cap \mathbb{R}$ is not bounded below.

Notes: Sup(A) and Inf(A) always exists for every non-empty subset A of $\mathbb{R}^*$.

1.2 Set Function

Let $\mathcal{C}$ be the class of subset of$X$. A function $\mu: \mathcal{C} \to [0, \infty]$ is called a set function. It is a function whose domain is a collection ofsets. Therefore, it is called a set function.

- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be**monotone** if for all A, $B \in \mathcal{C}, \mu(A) \leq \mu(B)$ whenever $\subseteq B$.
- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be **finitely additive** if

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

whenever $A_1, A_2, \dots, A_n \in \mathcal{C}$ and$\bigcup_{i=1}^{n} A_i \in \mathcal{C},\ A_i \cap A_j = \phi_j$ for $i \neq j$.

- A set function$\mu: \mathcal{C} \to [0, \infty]$ is said to be **countably additive** if

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

whenever $A_1, A_2, \ldots \in \mathcal{C}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$, $A_i \cap A_j = \phi$ for $i \neq j$.

- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be **countably subadditive** if

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

whenever $A_1, A_2, \ldots \in \mathcal{C}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

1.3 Outer Measure

Length of an interval: The length of an interval is defined as the difference of endpoints of the interval.

If I is any one of the intervals [a,b]/ (a, b)/ [a, b)/ (a, b], then the length of each interval will be $b - a$ and is denoted as $l(I)$ or | I |.

If I is of the form $(a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$ or $(-\infty, \infty)$ then $l(I) = \infty$.

Length of an open set: If O is an open subset of $\mathbb{R}$ then O can be written as a countable union of pairwise disjoint open intervals, say $\{I_n\}$ i.e.,

$$O = \bigcup_n I_n.$$

Then the length of an open set O is defined as

$$l(O) = \sum_n I_n.$$

Length of a closed set: Let F be a closed subset of $\mathbb{R}$ contained in some interval (a, b), then the length of the closed set F is defined as

$$l(F) = b - a - l(F^c) \text{ where } F^c = (a, b) - F.$$

Lebesgue outer measure of a set: The Lebesgue outer measure or simply outer measure of a subset A of $\mathbb{R}$ is denoted by $m^*(A)$ and is defined as

$$m^*(A) = inf\left\{\sum_n l(I_n) : \{I_n\} \text{is a countable collection of open intervals such that } A \subseteq \bigcup_n I_n\right\}$$

Here infimum is taken over all possible countable coverings of A.

Notes: is taken ov). $m^*(A) \geq$ C

Notes: If {$_{n}$} is any countable collection of open intervals such that $A \subseteq \bigcup_n I_n$ then

$$m^*(A) \leq \sum_n l(I_n).$$

Theorem1.1: (i)$m^*(\phi) = 0$, where ϕ is an empty set.

(ii) If A and B are subsets of $\mathbb{R}$ such that $A \subseteq B$ then $m^*(A) \leq m^*(B)$.

(iii) $m^*(\{x\}) = 0$ for any $x \in \mathbb{R}$.

(iv) If E is a countable subset of $\mathbb{R}$ then $m^*(E) = 0$.

Proof: (i)Let $\epsilon > 0$ be given, then

$$\phi \subseteq \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right), x \in \mathbb{R}$$

$$\Rightarrow m^*(\phi) \le l\left(\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)\right)$$

$$\Rightarrow m^*(\phi) \le \epsilon.$$

Since ϵ is arbitrary, therefore we have,

$$m^*(\phi) = 0.$$

(ii) Let $\{I_n\}$ be a countable collection of open intervals such that

$$\bigcup_n I_n \supseteq B$$

then

$$\bigcup_n I_n \supseteq A.$$

$$\Rightarrow m^*(A) \le \sum_n l(I_n).$$

But $\{I_n\}$ is an arbitrary collection of open intervals such that

$$\bigcup_n I_n \supseteq B.$$

Therefore,

$$m^*(A) \le \inf\left\{\sum_n l(I_n) : B \subseteq \bigcup_n I_n\right\}$$
$$= m^*(B).$$

Thus, we get $m^*(A) \le m^*(B)$ whenever $A \subseteq B$.

(iii)Since $\{x\} \subseteq \left(x-\frac{1}{n},x+\frac{1}{n}\right); n \in \mathbb{N}$

$$\therefore m^*(\{x\}) \le l\left(\left(x-\frac{1}{n},x+\frac{1}{n}\right)\right); n \in \mathbb{N}$$

$$= \frac{2}{n} \to 0 \ as n \to \infty$$

$$\Longrightarrow m^*(\{x\}) = 0.$$

(iv) Let E be a countable subset of $\mathbb{R}$ then we may express E as

$$E = \{x_1, x_2, \dots x_n, \dots\}$$

Let $\epsilon > 0$ be given

Then the collection $\{I_n\}$, where

$$I_n = \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\right)$$

is a sequence of open intervals such that

$$\bigcup_{n=1}^{\infty} I_n \supseteq E.$$

Therefore,

$$m^*(E) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$= \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}$$

$$= \epsilon \sum_{n=}^{\infty} \frac{1}{2^n}$$

$$= \epsilon \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)$$

$$= \epsilon \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \epsilon$$

$$\Rightarrow m^*(E) \leq \epsilon.$$

Since $\epsilon > 0$, is arbitrary.

Therefore, we get $m^*(E) = 0$.

This completes the proof.

Example:If proof. $E = \{x \in \mathbb{Q}$: $0 \leq$ $x \leq 2$}, $\mathbb{Q}$ the set of rationalsthen find $m^*(E)$.

Solution: We have

$$E = \{x \in \mathbb{Q}: 0 \leq x \leq 2\}$$

$$\Rightarrow E \subseteq \mathbb{Q}$$

$$\Rightarrow m^*(E) \leq m^*(Q)$$

Since $\mathbb{Q}$ is countable.

Therefore, $m^*(Q) = 0$

$$\Rightarrow m^*(E) = 0.$$

$\boldsymbol{F_\sigma}$ **− set**: A set which is a countable union of closed sets is called F_σ −set.

Example:

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right].$$

$\boldsymbol{G_\delta}$ **− set:** A set which is a countable intersection of open sets is known as $G_\delta$set.

Example:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}).$$

Theorem 1.2: The outer measure of an interval is its length.

Proof:Let I be an interval.

We have three cases.

Case (i):I=[a,b]

For every $\varepsilon > 0$, we have

$$I \subseteq (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$$

$$m^*(I) \leq l\left(\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)\right)$$

$$= b + \frac{\epsilon}{2} - a + \frac{\epsilon}{2}$$

$$= b - a + \epsilon$$

$$i.e., \; m^*(I) \leq \; b - a + \epsilon, \varepsilon > 0$$

$$\Rightarrow m^*(I) \leq \; b - a \qquad \ldots (1)$$

Also, we have

$$m^*(I) = inf\left\{\sum_n l(I_n) : \text{I} \subseteq \bigcup_n I_n\right\}$$

$\therefore$ for given $\varepsilon > 0$,there exists a countable collection $\{I_n\}$ of open intervals such that

$$\bigcup_n I_n \supseteq I$$

and

$$\sum_n l(I_n) < m^*(I) + \varepsilon \qquad \ldots (2)$$

Now $\{I_n\}$ is an open cover of I and I, being closed and bounded subset of $\mathbb{R}$,is a compact set.

$\therefore \exists$ a finite subcover I , say $\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$ such that

$$a_1 < a < b_1 \text{ and } b_1 < b$$

$$a_2 < b_1 < b_2 \text{ and } b_2 < b$$

$$\ldots \quad \ldots \quad \ldots$$

$a_{k-1} < b_{k-2} < b_{k-1}$and $b_{k-1} < b$

$a_k < b_{k-1} < b_k$and $b_k > b$

Then from (2),we have

$$m^*(I) + \varepsilon > \sum_{n=1}^{\infty} l(I_n)$$

$$\geq \sum_{n=1}^{k} l\big((a_n, b_n)\big)$$

$$= l\big((a_1, b_1)\big) + l\big((a_2, b_2)\big) + \cdots + l\big((a_k, b_k)\big)$$

$$= (b_1 - a_1) + (b_2 - a_2) + \cdots (b_k - a_k)$$

$$= b_k + [(b_{k-1} - a_k)] + \cdots + [(b_1 - a_2)] - a_1$$

$$\geq b_k - a_1$$

$$> b - a \qquad \because b_k > b > a > a_1$$

$$i.e., \qquad m^*(I) + \varepsilon > b - a \quad \forall \varepsilon > 0$$

$$\Rightarrow m^*(I) \geq b - a \qquad \ldots (3)$$

From (1) and (3), we get

$$m^*(I) = b - a = l(I).$$

Case (ii):I is oneof form (a,b), (a,b] or [a, b).

For any $\varepsilon > 0$, we can find a closed interval J such that $J \subseteq I$ and $l(J) > l(I) - \varepsilon$

$$\Rightarrow l(I) - \varepsilon < l(J)$$

$$= m^*(J)\{by\ case\ (i)\}$$

$$\leq m^*(I) \quad \because J \subseteq I$$

$$\leq m^*(\bar{I}) \quad \because I \subseteq \bar{I}$$

$=l(\bar{I})$

$$= l(I)$$

$$\Rightarrow l(I) - \varepsilon < m^*(I) \leq l(I)\ \forall \epsilon > 0$$

$$\Rightarrow l(I) \leq m^*(I) \leq L(I)$$

$$\Rightarrow l(I) = m^*(I).$$

Case (iii):If I is one of the forms [a,∞), (a, ∞), $(-\infty, b)$ or$(-\infty, b]$ or $(-\infty, \infty)$, then $l(I) = \infty$.

Here we will show that $m^*(I) = \infty = l(I)$

Given any real number $k > 0$,∃ closed bounded interval $J \subset I$ such that $l(J) = k$

Since

$$J \subset I$$

$$\Rightarrow m^*(J) \leq m^*(I)$$

$$m^*(I) \geq m^*(J)$$

$$= l(J) \qquad \{by\ case\ (i)\}$$

$$= k$$

$m^*(I) \geq k$,for any positive real number$k$.

∴ for given $k > 0$, however large $m^*(I) \geq k$

Hence, $m^*(I) = \infty = l(I)$.

This completes the proof.

Theorem 1.3 (Countable Subadditivity Property):

If $\{A_n\}$ is a countable collection of sets of reals, then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n)$$

Proof: If$m^*(A_n) = \infty$ for somenthen the result holds trivially.

Therefore, let $m^*(A_n) < \infty\ \forall n$

Now,

$$m^*(A_n) = inf\left\{\sum_i l(I_{n,i}) : \{I_{n,i}\} \text{ a countable collection of open intervals such that} A_n \subseteq \bigcup_i I_{n,i}\right\}.$$

∴ for given $\varepsilon > 0$ and for each A_n,there exists a collection $\{I_{n,i}\}_i$ of open intervals such that

$$A_n \subseteq \bigcup_i I_{n,i}$$

and

$$\sum_i l(I_{n,i}) < m^*(A_n) + \frac{\varepsilon}{2^n} \qquad \dots (1)$$

Now,

$$A_n \subseteq \bigcup_i I_{n,i}$$

$$\Rightarrow \bigcup_n A_n \subseteq \bigcup_n \bigcup_i I_{n,i}$$

$$= \bigcup_{n,i} I_{n,i}$$

i.e. the collection $\{I_{n,i}\}_{n,i}$ forms a countable collection of open intervals which covers $\cup_n A_n$.

Therefore,

$$m^*\left(\bigcup_n A_n\right) \leq \sum_{n,i} l(I_{n,i})$$

$$= \sum_n \left(\sum_i l(I_{n,i})\right)$$

$$< \sum_n [m^*(A_n) + \frac{\varepsilon}{2^n}] \qquad \{by(1)\}$$

$$= \sum_n m^*(A_n) + \sum_n \frac{\varepsilon}{2^n}$$

$$= \sum_n m^*(A_n) + \varepsilon \sum_n \frac{1}{2^n}$$

$$= \sum_n m^*(A_n) + \varepsilon \left(\frac{\frac{1}{2}}{1 - \frac{1}{2}}\right)$$

$$= \sum_n m^*(A_n) + \varepsilon$$

$$i.e.\ m^*\left(\bigcup_n A_n\right) < \sum_n m^*(A_n) + \varepsilon$$

$$\Rightarrow \ m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

This completes the proof.

Theorem 1.4: If $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$

Solution: We have

$m^*(A \cup B) \leq m^*(A) + m^*(B)$

$$\therefore m^*(A \cup B) \leq 0 + m^*(B)$$

$$\Rightarrow m^*(A \cup B) \leq m^*(B) \qquad \dots (1)$$

Now,

$$B \subseteq A \cup B$$

$$\Rightarrow m^*(B) \leq m^*(A \cup B) \qquad \dots (2)$$

From (1) and (2), we get

$m^*(B) = m^*(A \cup B)$.

This completes the proof.

Theorem 1.5: Prove that m^* is translation invariant.

Or

If $E \subseteq \mathbb{R}$, then $m^*(E+y) = m^*(E), y \in \mathbb{R},\ E+u = \{x+y\ : x \in E\}$.

Proof:We know "If I be any interval with endpoints a and b then I+y is also an interval with endpoints $a+y, b+y$ and is defined as

$$I+y = \{x+y : x \in I\} \text{ and } l(I+y) = l(I)"$$

Let $\varepsilon > 0$ be given. We have

$$m^*(E) = inf\left\{\sum_n l(I_n) : \{I_n\} \text{is a countable collection of open intervals such that } E \subseteq \bigcup_n I_n\right\}$$

$\therefore\ \exists$ acollection $\{I_n\}$ of open intervals such that

$$E \subseteq \bigcup_n I_n$$

and

$$\sum_n l(I_n) < m^*(E) + \varepsilon \quad \dots (1)$$

Now,

$$E \subseteq \bigcup_n I_n$$

$$\Rightarrow E + y \subseteq \bigcup_n (I_n + y)$$

$$\Rightarrow m^*(E+y) \leq m^*\left(\bigcup_n (I_n + y)\right)$$

$$\leq \sum_n m^*(I_n + y)$$

$$= \sum_n l(I_n + y)$$

$$= \sum_n l(I_n)$$

$$\therefore m^*(E+y) \leq \sum_n l(I_n)$$

$$< m^*(E) + \varepsilon \qquad \{\text{by}(1)\}$$

Thus, we get
$m^*(E+y) < m^*(E) + \varepsilon\ , \varepsilon > 0$

$$\Rightarrow m^*(E+y) \leq m^*(E) \quad \dots .(2)$$

Now, we have

$$E = (E+y) - y$$

$$\therefore m^*\big((E+y) + (-y)\big) \leq m^*(E+Y) \qquad (by\ 2)$$

$$\Rightarrow m^*(E) \leq m^*(E+y) \quad \dots (3)$$

From (2) and (3), we get

$$m^*(E) = m^*(E+y).$$

This completes the proof.

Theorem1.6: Let A be any set of reals.

Then,

i. for given $\varepsilon > 0$, there exists an open set $O \geq A$ such that $m^*(O) < m^*(A) + \varepsilon$and the inequality is strict in case we have $m^*(A) < \infty$.

ii. there exists aG_δ −set$G$ ⊇A such that $m^*(A) = m^*(G)$.

Proof:(i) Let $m^*(A) < \infty$. We have,

$$m^*(A) = \inf\left\{\sum_n l(I_n) : \{I_n\} \text{is a countable collection of open intervals such that } A \subseteq \bigcup_n I_n\right\}$$

Therefore, for given $\varepsilon > 0$, there exists a countable collection $\{I_n\}$ of open intervals such that

$$A \subseteq \bigcup_n I_n$$

and

$$\sum_n l(I_n) < m^*(A) + \varepsilon \quad \ldots (1)$$

Let

$$O = \bigcup_n I_n.$$

Therefore, $O$is an open set and $O \supseteq A$.

Also,

$$m^*(O) = m^*\left(\bigcup_n I_n\right)$$

$$\leq \sum_n m^*(I_n)$$

$$= \sum_n l(I_n)$$

$< m^*(A) + \varepsilon\{by(1)\}$

$$i.e.\ m^*(O) < m^*(A) + \varepsilon \quad \ldots (2)$$

Let $m^*(A) = \infty$.

Since $O \supseteq A$, therefore

$$m^*(O) \geq m^*(A) = \infty$$

$$\Rightarrow m^*(O) = \infty.$$

$$\Rightarrow m^*(O) = m^*(A) + \varepsilon = \infty \quad \ldots (3)$$

From (2) and from (3), we get

$$m^*(O) \leq m^*(O) + \varepsilon.$$

(ii) Taking $\varepsilon = \frac{1}{n}, n \in \mathbb{N}$ in (i), there exists an open set $O_n \supseteq A$ such that

$$m^*(O_n) \leq m^*(A) + \frac{1}{n} \quad \ldots (4)$$

Let

$$G = \bigcap_{n=1}^{\infty} O_n.$$

$\Rightarrow G$ is a G_δ − set and $G \supseteq A$.

Now,

$$G \subseteq O_n \quad \forall n$$

$$\Rightarrow m^*(G) \le m^*(O_n)$$

$$\le m^*(A) + \frac{1}{n}, n \in \mathbb{N} \quad (by\ 4)$$

$$\Rightarrow m^*(G) \le m^*(A) \qquad \ldots (5)$$

Also,

$$A \subseteq G$$

$$\Rightarrow m^*(A) \le m^*(G) \qquad \ldots (6)$$

From (5) and (6) we get

$m^*(A) = m^*(G)$.

This completes the proof.

Cantor set: Let $\boldsymbol{C_0} = [\mathbf{0}, \mathbf{1}]$.

Step-1: We divide $[0,1]$ into 3 subintervals of the same length and remove open middle subinterval, to get

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Step-2: We divide each of the 2 resulting intervals above into 3 subintervals again and remove the open middle subintervals, to get

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Similarly, we get

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right].$$

Continuing this way, we get a sequence $\{C_n\}$ of sets such that

i. $C_1 \supseteq C_2 \supseteq C_3 \ldots$
ii. C_n is union of 2^n disjoint closed intervals each of length $\frac{1}{3^n}$.

Then the set $C = \bigcap_{n=1}^{\infty} C_n$ is defined as a Cantor set.

Notes: If the outer measure of a set is zero, then the set may not be countable.

e.g., Cantor set C where $= \bigcap_{n=1}^{\infty} C_n$, C_n is union of 2^n disjoint closed intervals each of length $\frac{1}{3^n}$.

Now,

$$C \subseteq C_n \quad \forall n$$

$$\Rightarrow m^*(C) \le m^*(C_n)\ \forall n$$

$$\le \frac{1}{3^n} + \frac{1}{3^n} + \cdots (2^n times)$$

$$= \left(\frac{2}{3}\right)^n \to 0 \text{ as } n \to \infty$$

$$\Rightarrow m^*(C) = 0.$$

Thus, we get outer measure of an uncountable set (Cantor set) is zero.

Summary

- We denote the set of extended real numbers by $\mathbb{R}^*$ which is defined as $\mathbb{R}^* = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, where $\mathbb{R} = (-\infty, \infty)$. That is, we can write $\mathbb{R}^* = [-\infty, \infty]$. Here $+\infty$ and $-\infty$ are two symbols.
- For every $\boldsymbol{x} \in \mathbb{R}, -\infty < \boldsymbol{x} < +\infty$. Here $-\infty$ is the smallest element in $\mathbb{R}^*$ and $+\infty$ is the largest element in $\mathbb{R}^*$.
- For every $\boldsymbol{x} \in \mathbb{R}$, $(-\infty) + \boldsymbol{x} = -\infty, (+\infty) + \boldsymbol{x} = +\infty, (+\infty) + (+\infty) = +\infty, (-\infty) + (-\infty) = -\infty$.
- If $x > 0$, then $x(+\infty) = (+\infty)(x) = +\infty$, $x(-\infty) = (-\infty)(x) = -\infty$ and if $x < 0$, then $(+\infty)x = (+\infty)(x) = -\infty$, $(-\infty)x = (-\infty)(x) = +\infty$.
- Let $A \subseteq \mathbb{R}^*$ be any non-empty set. $Sup\ (A) = +\infty$ if $A \cap \mathbb{R}$ is not bounded above. $Inf\ (A) = -\infty$ if $A \cap \mathbb{R}$ is not bounded below.
- Let $\mathcal{C}$ be the class of subset of X. A function $\mu: \mathcal{C} \to [0, \infty]$ is called a set function. It is a function whose domain is a collection of sets. Therefore, it is called a set function.
- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be monotone if for all $A, B \in \mathcal{C}, \mu(A) \leq \mu(B)$ whenever $\subseteq B$.
- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be finitely additive if $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$

  whenever $A_1, A_2, \dots, A_n \in \mathcal{C}$ and $\bigcup_{i=1}^n A_i \in \mathcal{C}$, $A_i \cap A_j = \phi_j$ for $i \neq j$.
- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be countably additive if $\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

  $$\text{whenever } A_1, A_2, \dots \in \mathcal{C} \text{ and } \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}, A_i \cap A_j = \phi \text{ for } i \neq j.$$
- A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be countably subadditive if $\mu(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$

  $$\text{whenever } A_1, A_2, \dots \in \mathcal{C} \text{ and } \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}.$$
- The length of an interval is defined as the difference of endpoints of the interval.
- If O is an open subset of $\mathbb{R}$ then O can be written as a countable union of pairwise disjoint open intervals, say $\{I_n\}$ i.e.,

  $O = \bigcup_n I_n$. Then the length of an open set O is defined as $l(O) = \sum_n I_n$.
- Let F be a closed subset of $\mathbb{R}$ contained in some interval (a, b), then the length of the closed set F is defined as $l(F) = b - a - l(F^c)$ where $F^c = (a, b) - F$.
- The Lebesgue outer measure or simply outer measure of a subset A of $\mathbb{R}$ is denoted by $m^*(A)$ and is defined as

  $$m^*(A) = inf\left\{\sum_n l(I_n) : \{I_n\} \text{is a countable collection of open intervals such that } A \subseteq \bigcup_n I_n\right\}.$$
- A set which is a countable union of closed sets is called F_σ –set.
- A set which is a countable intersection of open sets is known as G_δ set.
- If $\{A_n\}$ is a countable collection of sets of reals, then $m^*(\bigcup_n A_n) \leq \sum_n m^*(A_n)$.

- If $\{I_n\}$ is any countable collection of open intervals such that $A \subseteq \cup_n I_n$ then $m^*(A) \leq \sum_n l(I_n)$.
- $m^*(\phi) = 0$, where ϕ is an empty set.
- If A and B are subsets of $\mathbb{R}$ such that $A \subseteq B$ then $m^*(A) \leq m^*(B)$.
- $m^*(\{x\}) = 0$ for any $x \in \mathbb{R}$.
- If E is a countable subset of $\mathbb{R}$ then $m^*(E) = 0$.
- The outer measure of an interval is its length.
- If $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$
- $m^*$is translation invariant i.e., if $E \subseteq \mathbb{R}$, then $m^*(E + y) = m^*(E), y \in \mathbb{R},\ E + u = \{x + y\ : x \in E\}$.
- Let A be any set of reals.Then, for given $\varepsilon > 0$, there exists an open set $O \geq A$ such that $m^*(O) < m^*(A) + \varepsilon$and the inequality is strict in case we have $m^*(A) < \infty$ *and* there exists a G_δ –set $G \supseteq$A such that $m^*(A) = m^*(G)$.
- If the outer measure of a set is zero, then the set may not be countable.

Keywords

Extended Real Numbers: We denote the set of extended real numbers by$\mathbb{R}^*$which isdefined as

$$\mathbb{R}^* = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}, \text{where } \mathbb{R} = (-\infty, \infty).$$

That is, we can write $\mathbb{R}^* = [-\infty, \infty]$. Here $+\infty$ and $-\infty$ are two symbols.

Order relation on $\mathbb{R}^*$: For every$\boldsymbol{x} \in \mathbb{R}, -\infty < \boldsymbol{x} < +\infty$

Here $-\infty$ is the smallest element in $\mathbb{R}^*$ and $+\infty$ is the largest element in $\mathbb{R}^*$.

Additionon $\mathbb{R}^*$**:** For every $\boldsymbol{x} \in \mathbb{R}$

- $(-\infty) + x = -\infty$
- $(+\infty) + x = +\infty$
- $(+\infty) + (+\infty) = +\infty$
- $(-\infty) + (-\infty) = -\infty$

Multiplicationon $\mathbb{R}^*$: If $x > 0$, then

- $x(+\infty) = (+\infty)(x) = +\infty,\ x(-\infty) = (-\infty)(x) = -\infty$

and if $x < 0$, then

- $(+\infty)x = (+\infty)(x) = -\infty,\ (-\infty)x = (-\infty)(x) = +\infty.$

Supremum and infimum in $\mathbb{R}^*$**:** Let $A \subseteq \mathbb{R}^*$ be any non-empty set.

- $Sup\ (A) = +\infty$ if $A \cap \mathbb{R}$ is not bounded above.
- $Inf\ (A) = -\infty$ if $A \cap \mathbb{R}$ is not bounded below.

Set function: Let $\mathcal{C}$ be the class of subset of$X$. A function $\mu: \mathcal{C} \to [0, \infty]$ is called a set function. It is a function whose domain is a collection ofsets. Therefore, it is called a set function.

Length of an interval: The length of an interval is defined as the difference of endpoints of the interval.

Length of an open set: If O is an open subset of $\mathbb{R}$ then O can be written as a countable union of pairwise disjoint open intervals, say $\{I_n\}$ i.e.,

$$O = \bigcup_n I_n.$$

Then the length of an open set O is defined as

$$l(O) = \sum_n I_n.$$

Length of a closed set: Let F be a closed subset of $\mathbb{R}$ contained in some interval (a, b), then the length of the closed set F is defined as

$$l(F) = b - a - l(F^c) \text{where} F^c = (a, b) - F.$$

Lebesgue outer measure of a set: The Lebesgue outer measure or simply outer measure of a subset A of $\mathbb{R}$ is denoted by $m^*(A)$ and is defined as

$$m^*(A) = inf\left\{\sum_n l(I_n) : \{I_n\} \text{is a countable collection of open intervals such that } A \subseteq \bigcup_n I_n\right\}.$$

F_σ – set: A set which is a countable union of closed sets is called F_σ –set.

G_δ – set: A set which is a countable intersection of open sets is known as G_δ set.

Countable Subadditivity Property: If $\{A_n\}$ is a countable collection of sets of reals, then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

Self Assessment

1) Select the incorrect option:

$A.$ $(+\infty) + (\infty) = +\infty$

$B.$ $(-\infty) + (-\infty) = -\infty$

$C.$ $(+\infty) + (-\infty)$ is not defined

D. none of these

2) Consider the following statements:

(I) $(s)(+\infty) = +\infty$ and $(s)(-\infty) = -\infty, 0 < s \in \mathbb{R}$

(II) $(s)(+\infty) = +\infty$ and $(s)(-\infty) = -\infty, s \in \mathbb{R}$ and $s < 0$

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

3)Let $M \subseteq \mathbb{R}^*$ be a non-empty set. Consider the following statements:

(I) $sup\ M = +\infty$ if M is bounded above in $\mathbb{R}$.

(II) $inf\ M = -\infty$ if M is not bounded below in $\mathbb{R}$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

4)Range of a set function is a subset of the set of non-negative real numbers.

A. True

B. False

5)A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be countably additive if $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_1, \ldots, A_i, \ldots \in \mathcal{C}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

A. True

B. False

6) Consider the following statements:

(I)A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be finitely additive if $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ whenever $A_1, \ldots, A_n \in \mathcal{C}$ such that $\bigcup_{i=1}^{n} A_i \in \mathcal{C}$ and A_i's are pairwise disjoint.

(II)A set function $\mu: \mathcal{C} \to [0, \infty]$ is said to be countably subadditive if $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_1, \ldots, A_i, \ldots \in \mathcal{C}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

7) Consider the following statements:

(I) Length of the interval $(-\infty, b] = \infty$

(II)Length of the interval $(-\infty, \infty) = \infty$

(III)Length of the interval $(a, b) = b - a$

(IV)Length of the interval $(a, \infty) = \infty$

A. only (I) is correct

B. only (III) is correct

C. all are correct

D. all are incorrect

8) If A is an open set such that $A = \bigcup_n I_n$ where I_n is an open interval for each n then the length of the set A is equal to $\sum_n l(I_n)$.

A. True

B. False

9) Outer measure of any set is non-negative.

A. True

B. False

10) If $A \subseteq \bigcup_n I_n$ then $m^*(A) \leq \sum_n l(I_n)$.

A. True

B. False

11) Consider the following statements:

(I) Outer measure of an empty set can be any positive number.

(II) Outer measure of a singleton set can be any positive number.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

12) Consider the following statements:

(I) If $A \subseteq B$ then $m^*(A) = m^*(B)$

(II) Outer measure of a countable set is zero.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

13) If $A \subset \mathbb{Q}$ then $m^*(A) = 0$

A. True

B. False

14) Intersection of an arbitrary collection of closed sets in $\mathbb{R}$ need not be closed.

A. True

B. False

15) Intersection of an arbitrary collection of open sets in $\mathbb{R}$ need not be open.

A. True

B. False

16) Union of a finite collection of closed sets in $\mathbb{R}$ is closed.

A. True

B. False

17) Union of an arbitrary collection of closed sets in $\mathbb{R}$ need not be closed.

A. True

B. False

18) Union of an arbitrary collection of open sets in $\mathbb{R}$ need not be open.

A. True

B. False

19) Consider the following statements:

(I) A set which is a countable union of closed sets is called F_σ-set.

(II) A set which is a countable intersection of open sets is called G_δ-set

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

20) Consider the following statements:

(I) If $\{B_n\}$ is a countable collection of sets of reals then $m^*(\cup B_n) = \sum m^*(B_n)$.

(II) If set A is compact then every open cover of A has a finite subcover. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

21) Consider the following statements:

(I) $l(I) > m^*(I), I = (-\infty, b]$

(II)$l(I) < m^*(I), I = (-\infty, \infty)$

(III)$l(I) = m^*(I), I = (a, b)$

A. (I) and (II)are correct

B. only (III) is correct

C. (I) and (III)are correct

D. all are incorrect

22) Consider the following statements:

(I) Cantor set is countable.

(II) Outer measure of the Cantor set is zero. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

23) Consider the following statements:

(I)Outer measure of a countable set is always zero.

(II) If the outer measure of a set is zero then the set need not be countable. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

24) Consider the following statements:

(I)Outer measure need not be translation invariant.

(II) Let A be any set of reals. Then there exists a G_δ – set $G \supseteq A$ such that $m^*(A) = m^*(G)$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	D	2.	A	3.	B	4.	A	5.	B
6.	C	7.	C	8.	A	9.	A	10.	A
11.	D	12.	B	13.	A	14.	B	15.	A
16.	A	17.	A	18.	B	19.	C	20.	B
21.	B	22.	B	23.	C	24.	B		

Review Questions

Find $m^*(E)$if:

1)$E = \{x \in \mathbb{R}: 1 \leq x \leq 10\}$.

2)$E = \{x \in \mathbb{N}: 1 \leq x \leq 10\}$.

3)$E = \{x \in \mathbb{R}: 1/2 \leq x \leq 5\} \cap \mathbb{Q}$.

4)$E = \{x \in \mathbb{N}: 1/2 \leq x \leq 5\} \cap \mathbb{R}$.

5)$E = \{x \in \mathbb{R}: -\infty < x \leq 2\}$.

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Graduate Texts in Mathematics, Measure Integration and Real Analysis, Sheldon Axler, Springer.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

https://www.youtube.com/watch?v=YIrx8W5nyq8&t=29s

Unit 02: Measurable Sets

CONTENTS

Objectives

After studying this unit, students will be able to:

- determine the conditions to prove that a set is measurable or not
- understand the algebra of measurable sets
- define algebra and sigma algebra
- describe Borel sigma algebra
- define Lebesgue measure
- demonstrate the concept of sum modulo 1

Introduction

Outer measure has four properties: (i) outer measure is defined for all sets of real numbers, (ii) outer measure of an interval is its length, (iii) outer measure is countably sub additive, (iv) outer measure is translation invariant. But Outer measure fails to be countably additive.

In order to have the property of countable additivity satisfied, we have to restrict the domain of definition for the function m^* to some suitable subset, (say) $\mathcal{M}$, of the power set $P(\mathbb{R})$. The members of $\mathcal{M}$ are called measurable sets. There are various ways to define a measurable set but we follow an approach due to Constantine Caratheodory.

2.1 Measurable Set

A subset E of $\mathbb{R}$ is said to be **Lebesgue measurable** or, briefly **measurable** if for each set $A \subseteq \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

Notes: Since $A = A \cap \mathbb{R}$

$$= A \cap (E \cup E^c)$$

$$= (A \cap E) \cup (A \cap E^c)$$

$$\therefore\ m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c)$$

Thus, in order to improve E to be a measurable set, we only need to show that

$$m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c).$$

Theorem2.1: If $m^*(E) = 0$, then E is a measurable set.

Proof: Let A be any subset of $\mathbb{R}$ then

$$A \cap E \subseteq E$$
$$\Rightarrow m^*(A \cap E) \leq m^*(E) = 0$$
$$\Rightarrow m^*(A \cap E) \leq 0$$
$$\Rightarrow m^*(A \cap E) = 0 \because m^*(A \cap E) \geq 0$$
$$\therefore m^*(A \cap E) + m^*(A \cap E^c) = 0 + m^*(A \cap E^c) \leq m^*(A)$$
$$\because A \cap E^c \subseteq A \Rightarrow m^*(A \cap E^c) \leq m^*(A)$$
$$i.e. m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

Therefore, E is a measurable set.

This completes the proof.

Cor1: Every countable set is measurable.

Proof: Let E be a countable set.

Then $m^*(E) = 0$, therefore by above theorem, E is measurable.

Hence every countable set is measurable.

Cor 2: Cantor set is measurable.

Proof: Let $'C'$ be the cantor set then $m^*(C) = 0$.

Therefore, by above theorem, C is a measurable set.

Hence Cantor-set is measurable.

Theorem 2.2: Prove the following.

(i) If E is a measurable set, then E^c is also measurable.

(ii) If E_1 and E_2 are measurable then $E_1 \cup E_2$.

Proof: (i) Since E is a measurable set,

$$\therefore m^*(A) = m^*(A \cap E) + m^*(A \cap E^c), A \subseteq \mathbb{R}$$
$$\Rightarrow m^*(A) = m^*(A \cap E^c) + m^*(A \cap (E^c)^c), A \subseteq \mathbb{R}$$
$$\Rightarrow E^c \text{ is measurable.}$$

(ii) Since E_1 is a measurable set,

$$\therefore m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c), A \subseteq \mathbb{R} \quad \ldots(1)$$

Again as E_2 is a measurable set,

$$\therefore m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c), A \subseteq \mathbb{R} \quad \ldots(2)$$

Using (2) in (1), we get

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$
$$\geq m^*[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + m^*(A \cap E_1^c \cap E_2^c)$$
$$= m^*[(A \cap E_1) \cup \{A \cap (E_2 - E_1)\}] + m^*[A \cap (E_1 \cup E_2)^c]$$
$$= m^*[A \cap \{E_1 \cup (E_2 - E_1)\}] + m^*[A \cap (E_1 \cup E_2)^c]$$
$$= m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$
$$i.e. m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$
$$\Rightarrow E_1 \cup E_2 \text{ is measurable}$$

This completes the proof.

Cor 1: If E_1 and E_2 are measurable then $E_1 \cap E_2$ is also measurable.

Proof: Since E_1 and E_2 are measurable therefore E_1^c and E_2^c are also measurable

$$\Rightarrow E_1^c \cup E_2^c \text{ is measurable}$$

$\Rightarrow \ (E_1 \cap E_2)^c$ is measurable

$\Rightarrow \ [(E_1 \cap E_2)^c]^c$ is measurable

$\Rightarrow \ E_1 \cap E_2$ is measurable

Cor 2: If E_1 and E_2 are measurable then $E_1 - E_2$ is also measurable.

Proof: Since E_1 and E_2 are measurable therefore E_1 and E_2^c are also measurable

$\Rightarrow \ E_1 \cap E_2^c$ is measurable $i.e.$, $E_1 - E_2$ is also measurable.

Algebra:

A non-empty collection $\mathcal{A}$ of subsets of a set X is called an algebra if

i) Empty set $\Phi \in \mathcal{A}$.

ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

Notes: If $\mathcal{A}$ is an algebra then

$$A, B \in \mathcal{A} \Rightarrow A^c, B^c \in \mathcal{A} \Rightarrow A^c \cup B^c \in \mathcal{A} \Rightarrow (A \cap B)^c \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}.$$

σ –algebra:

A non-empty collection $\mathcal{A}$ of subsets of a set X is called σ –algebra if

(i) $\Phi \in \mathcal{A}$

$(ii) A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

$(iii) A_n \in \mathcal{A}, n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Notes: If $\mathcal{A}$ is a σ –algebra then

$$A_n \in \mathcal{A}, n \in \mathbb{N} \Rightarrow A_n^c \in \mathcal{A}, n \in \mathbb{N}$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{A} \Rightarrow \left(\bigcap_{n=1}^{\infty} A_n\right)^c \in \mathcal{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Notes: Every –algebra is algebra.

Theorem 2.3: If $E_1, E_2, \ldots, E_n$ are disjoint measurable sets, then

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i), A \subseteq \mathbb{R}$$

Proof: We prove the result by induction on n.

For $n = 1$, we have

$$m^*(A \cap E_1) = m^*(A \cap E_1)$$

which is true.

$\therefore$ result is true for $n = 1$.

Let the result be true for $(n-1)$ sets $i.e.$

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right] = \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

Now since E_n is measurable, we have

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n\right] + m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n^c\right]$$

$$= m^*(A \cap E_n) + m^*\left[A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right]$$

$\because$ $E_1, E_2, \ldots, E_n$ are disjoint

$$\therefore\ A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n = A \cap E_n$$

and

$$A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n^c = A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)$$

Therefore

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = m^*(A \cap E_n) + m^*\left[A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right]$$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i) \text{ [by induction hypothesis]}$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i).$$

This completes the proof.

Cor: If $E_1, E_2, \ldots, E_n$ are disjoint measurable sets, then

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$$

Proof: Taking $A = \mathbb{R}$ in above theorem, we get

$$m^*\left[\mathbb{R} \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(\mathbb{R} \cap E_i)$$

$$\Rightarrow\ m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i).$$

Theorem 2.4: If $\{E_n\}$ is a sequence of disjoint measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.

Proof: Let $E = \bigcup_{n=1}^{\infty} E_n$ and $F_n = \bigcup_{i=1}^{n} E_i$ then F_n being finite union of measurable sets, is measurable.

$$\therefore\ m^*[A] = m^*[A \cap F_n] + m^*[A \cap F_n^c], A \subseteq \mathbb{R} \quad \ldots (1)$$

Now for $n \in \mathbb{N}$, we have

$F_n \subseteq E,$

$$\Rightarrow\ F_n^c \supseteq E^c$$

$$\Rightarrow A \cap F_n^c \supseteq A \cap E^c$$

$$\Rightarrow m^*(A \cap F_n^c) \geq m^*(A \cap E^c)$$

$$\Rightarrow m^*(A \cap F_n) + m^*(A \cap F_n^c) \geq m^*(A \cap F_n) + m^*(A \cap E^c) \quad \ldots (2)$$

From (1) and (2), we get

$$m^*(A) \geq m^*(A \cap F_n) + m^*(A \cap E^c)$$

$$= m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] + m^*(A \cap E^c)$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$\because m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i), A \subseteq \mathbb{R}, E_1, E_2, \dots, E_n \text{ are disjoint measurable sets.}$$

$$\therefore\ m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c), n \in \mathbb{N}$$

$$\Rightarrow\ m^*(A) \geq \lim_{n\to\infty} \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$\Rightarrow\ m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$\geq m^*\left[\bigcup_{i=1}^{\infty}(A \cap E_i)\right] + m^*(A \cap E^c)$$

$$= m^*\left[A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right] + m^*(A \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap E^c)$$

$$\therefore\ m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c), A \subseteq \mathbb{R}$$

$$\Rightarrow E = \bigcup_{n=1}^{\infty} E_n \text{ is measurable.}$$

This completes the proof.

Theorem 2.5: If $\{E_n\}$ is a sequence of measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.

Proof: Let

$$F_1 = E_1$$

$$F_2 = E_2 - E_1$$

$$F_3 = E_3 - (E_1 \cup E_2)$$

$$\dots$$

$$F_n = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right)$$

$$\dots$$

Since $E_1, E_2, \dots, E_{n-1}$ are measurable sets, $\bigcup_{i=1}^{n-1} E_i$ is also measurable.

$$\therefore\ F_n = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right) \text{ is measurable and } F_1 = E_1$$

$$\Rightarrow\ \{F_n\} \text{ is a sequence of disjoint measurable sets.}$$

$$\Rightarrow \bigcup_{n=1}^{\infty} F_n \text{ is measurable.} \quad \dots (1)$$

Claim

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$$

We have

$$F_n = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right), F_1 = E_1$$

$$\therefore\ F_n \subseteq E_n, \forall n$$

$$\Rightarrow \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n \qquad \dots (2)$$

Now, $x \in \bigcup_{n=1}^{\infty} E_n$

Then $x \in E_k$ for some $k \in \mathbb{N}$. Let m be the least positive integer such that $x \in E_m$

$$i.e.\, x \in E_m \text{ and } x \notin E_i \text{ for } i = 1,2,\dots,m-1$$

$$\Rightarrow x \in E_m - \bigcup_{i=1}^{m-1} E_i$$

$$\Rightarrow x \in F_m$$

$$\Rightarrow x \in \bigcup_{n=1}^{\infty} F_n .$$

$$\therefore \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} F_n \qquad \dots (3)$$

From (2) and (3), we get

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$$

$\therefore$ by(1), $\bigcup_{n=1}^{\infty} E_n$ is a measurable set.

This completes the proof.

Cor: If $\{E_n\}$ is a sequence of measurable sets, then $\bigcap_{n=1}^{\infty} E_n$ is also measurable.

Proof: Since E_n is measurable, $\forall n \in \mathbb{N}$ therefore E_n^c is also measurable $\forall n \in \mathbb{N}$

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n^c \text{ is measurable.}$$

$$\Rightarrow \left(\bigcap_{n=1}^{\infty} E_n\right)^c \text{ is measurable}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} E_n \text{ is measurable}$$

Theorem 2.6: If E_1 and E_2 are measurable sets such that $E_2 \subseteq E_1$ then

$$m^*(E_1 - E_2) = m^*(E_1) - m^*(E_2)$$

Proof: Since E_1 and E_2 are measurable sets, therefore E_2 and $E_1 - E_2$ are also measurable.

E_2 and $E_1 - E_2$ are disjoint sets.

$$\therefore m^*[E_2 \cup (E_1 - E_2)] = m^*(E_2) + m^*(E_1 - E_2)$$

$$\Rightarrow m^*(E_1) = m^*(E_2) + m^*(E_1 - E_2) \because E_2 \subseteq E_1$$

$$\Rightarrow m^*(E_1 - E_2) = m^*(E_1) - m^*(E_2).$$

This completes the proof.

Theorem 2.7: If E_1 and E_2 are measurable sets, then show that

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2)$$

Proof: Since E_1 is measurable, therefore

$$m^*(E_1 \cup E_2) = m^*[(E_1 \cup E_2) \cap E_1] + m^*[(E_1 \cup E_2) \cap E_1^c]$$

$$= m^*(E_1) + m^*[(E_1 \cap E_1^c) \cup (E_2 \cap E_1^c)]$$

$$= m^*(E_1) + m^*[\Phi \cup (E_2 - E_1)]$$
$$= m^*(E_1) + m^*(E_2 - E_1)$$

Adding $m^*(E_1 \cap E_2)$ on both sides, we get

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2 - E_1) + m^*(E_1 \cap E_2)$$
$$= m^*(E_1) + m^*[(E_2 - E_1) \cup (E_1 \cap E_2)]$$

$\because$ $E_2 - E_1$ and $E_1 \cap E_2$ are also disjoint

$$= m^*(E_1) + m^*(E_2).$$

This completes the proof.

Theorem 2.8: The collection $\mathcal{M}$ of measurable subsets of $\mathbb{R}$ is a $\sigma - algebra$.

Proof: We know "If a non-empty collection $\mathcal{A}$ of subsets of a set X is given, then $\mathcal{A}$ is called $\sigma - algebra$ if

1) $\Phi \in \mathcal{A}$
2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3) $A_n \in \mathcal{A}, \forall\, n \in \mathbb{N} \Rightarrow \cup_n A_n \in \mathcal{A}$

i) We know Φ is a measurable set.

ii) Let $E \in \mathcal{M}$

$\Rightarrow$ E is a measurable set.

$\Rightarrow$ E^c is a measurable set.

$\Rightarrow$ $E^c \in \mathcal{M}$

iii) Let $E_n \in \mathcal{M}, \forall$ n $\in \mathbb{N}$

$\Rightarrow E_n$ is a measurable set $, \forall$ n $\in \mathbb{N}$

$\Rightarrow \bigcup_{n=1}^{\infty} E_n$ is a measurable set.

$\Rightarrow \bigcup_n E_n \in \mathcal{M}$

Hence $\mathcal{M}$ is a $\sigma - algebra$.

This completes the proof.

Theorem 2.9: Every interval is a measurable set.

Proof: Let $I = (a, \infty)$ be any interval and A be any subset of $\mathbb{R}$.

Let $A_1 = A \cap (a, \infty)$ and $A_2 = A \cap \left((a, \infty)\right)^c = A \cap (-\infty, a]$.

Therefore, to show 'I' is measurable, we will show that

$$m^*(A) \geq m^*(A_1) + m^*(A_2)$$

If $m^*(A) = \infty$, then result holds trivially.

If $m^*(A) < \infty$, then for given $\epsilon > 0, \exists$ a countable collection $\{I_n\}$ of open intervals such that

$$A \subseteq \bigcup_n I_n \text{ and } \sum_n l(I_n) < m^*(A) + \epsilon \qquad \dots (1)$$

Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$

Then

$$I_n' \cup I_n'' = [I_n \cap (a, \infty)] \cup [I_n \cap (-\infty, a]]$$
$$= I_n \cap [(a, \infty) \cup (-\infty, a]]$$
$$= I_n \cap \mathbb{R} = I_n$$

Also

$$I_n' \cap I_n'' = [I_n \cap (a,\infty)] \cap \; I_n \cap (-\infty, a]]$$
$$= I_n \cap [(a,\infty) \cap (-\infty, a]]$$
$$= I_n \cap \Phi = \Phi$$

Thus, we get

$$I_n = I_n' \cup I_n'' \text{and } I_n' \cap I_n'' = \Phi$$
$$\Rightarrow \; l(I_n) = l(I_n') + l(I_n'') \qquad \dots (2)$$

Now

$$\bigcup_{n=1}^{\infty} I_n' = \bigcup_{n=1}^{\infty} [I_n \cap (a,\infty)]$$
$$= \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (a,\infty) \supseteq A \cap (a,\infty) = A_1 \left(\because A \subseteq \bigcup_{n=1}^{\infty} I_n\right)$$
$$i.e., \quad \bigcup_{n=1}^{\infty} I_n' \supseteq A_1$$
$$\Rightarrow m^*(A_1) \le \sum_n l(I_n')$$

Now,

$$\bigcup_{n=1}^{\infty} I_n'' = \bigcup_{n=1}^{\infty} [I_n \cap (-\infty, a]]$$
$$= \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (-\infty, a] \supseteq A \cap (-\infty, a] = A_2$$
$$i.e. \bigcup_{n=1}^{\infty} I_n'' \supseteq A_2$$
$$\Rightarrow \; m^*(A_2) \le \sum_n l(I_n'')$$
$$\therefore \; m^*(A_1) + m^*(A_2) \le \sum_n l(I_n') + \sum_n l(I_n'')$$
$$= \sum_n (l(I_n') + l(I_n'')) = \sum_n l(I_n) \; (by\ (2))$$
$$< m^*(A) + \epsilon \qquad (by\ (1))$$
$$i.e., \qquad m^*(A_1) + m^*(A_2) < m^*(A) + \epsilon, \epsilon > 0$$
$$\Rightarrow m^*(A_1) + m^*(A_2) \le m^*(A)$$
$$i.e., \qquad m^*(A) \ge m^*(A_1) + m^*(A_2)$$

Hence $I = (a,\infty)$ is a measurable set.

Now, since the class $\mathcal{M}$ of measurable sets is a $\sigma -$ algebra. Therefore, it gives the result for other types of intervals also.

This completes the proof.

Cor 1: Every open set is measurable.

Proof: Let O be an open set. Then O can be expressed as a countable union of disjoint collection of open intervals. Since open intervals are measurable sets and countable union of measurable sets is measurable, therefore O is a measurable set.

Cor 2: Every closed set is measurable.

Proof: Let F be a closed set. Then F^c is an open set. ⇒ F^c is measurable.

⇒ $(F^c)^c = F$ is also measurable.

Cor 3: F_σ −set and G_δ −sets are measurable sets.

Proof: Since F_σ −set is countable union of closed sets and closed sets are measurable. Also, countable union of measurable sets is measurable. Therefore F_σ −set is measurable

Now G_δ −set is countable intersection of open sets and open sets are measurable, therefore G_δ −sets are measurable, since countable intersection of measurable sets is measurable.

Borel σ − algebra

The intersection of all the σ −algebras of subsets of $\mathbb{R}$ that contain the open sets is a σ −algebra called the Borel σ −algebra.Members of this collection are called Borel sets.

Notes: The Borel σ −algebra is contained in every σ −algebra that contains all open sets. Therefore, since the measurable sets are a σ −algebra containing all open sets, every Borel set is measurable.

Lebesgue Measure: The restriction of the set function outer measure to the class of measurable sets is called Lebesgue measure. It is denoted by m, so that if E is a measurable set, its Lebesgue measure, $m(E)$, is defined by

$$m(E) = m^*(E)$$

Theorem 2.10: For a given set E, the following statements are equivalent:

i) E is measurable.

ii) For given $\epsilon > 0, \exists$ an open set $O \supseteq E$ such that $m^*(O - E) < \epsilon$.

iii) There exists a G_δ −set $G \supseteq E$ such that $m^*(G - E) = 0$.

iv) For given $\epsilon > 0$, there exists a closed set $F \subseteq E$ such that $m^*(E - F) < \epsilon$.

v) There exists a F_σ − set, $F \subseteq E$ such that $m^*(E - F) = 0$.

Proof: $(i) \Rightarrow (ii)$

We have following two cases:

Case I: $m^*(E) < \infty$. By the definition of outer measure, for given $\epsilon > 0, \exists$ a countable collection $\{I_n\}$ of open intervals such that $E \subseteq \bigcup_n I_n$ and

$$\sum_n l(I_n) < m^*(E) + \epsilon \qquad \ldots (1)$$

Let $O = \bigcup_n I_n$.

Then O is an open set such that $O \supseteq E$.

Also

$$m^*(O) = m^*\left(\bigcup_n I_n\right)$$

$$\leq \sum_n m^*(I_n)$$

$$= \sum_n l(I_n)$$

$$< m^*(E) + \epsilon \qquad (\text{by}(1))$$

$$\Rightarrow m^*(O) - m^*(E) < \epsilon \qquad \ldots (2)$$

Now, O being open set, is measurable and E is also measurable.

$$\therefore O \sim E \text{ is a measurable set.}$$

Now

$$m^*(O) = m^*[E \cup (O - E)][\because O \supseteq E]$$

$$= m^*(E) + m^*(O - E) \; [\because E \text{ and } O - E \text{ are disjoint}$$

$$i.e. m^*(O) - m^*(E) = m^*(O - E) \Rightarrow m^*(O - E) < \epsilon \quad (by\ (2))$$

Case II) $m(E) = \infty$

We can find a sequence $\{I_n\}$ of open disjoint intervals such that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n, l(I_n) < \infty$$

$$\therefore E = \mathbb{R} \cap E$$
$$= \left(\bigcup_{n=1}^{\infty} I_n\right) \cap E$$
$$= \bigcup_{n=1}^{\infty} (I_n \cap E)$$
$$= \bigcup_{n=1}^{\infty} E_n, E_n = I_n \cap E, n \in \mathbb{N}$$

Now,

$$m^*(E_n) = m^*(I_n \cap E)$$
$$\leq m^*(I_n)$$
$$= l(I_n) < \infty$$

and each E_n is measurable as I_n and E are measurable.

$\therefore$ By case I, $\exists$ an open set $O_n \supseteq E_n$ such that

$$m^*(O_n - E_n) < \frac{\epsilon}{2^n} \quad \dots (3)$$

Let $O = \bigcup_{n=1}^{\infty} O_n$. Then O is an open set such that $O \supseteq E$.

And

$$O - E = \bigcup_n O_n - \bigcup_n E_n \subseteq \bigcup_n (O_n - E_n),$$

$$\therefore m^*(O - E) = m^*\left(\bigcup_n O_n - \bigcup_n E_n\right)$$
$$\leq m^*\left(\bigcup_n (O_n - E_n)\right)$$
$$\leq \sum_n m^*(O_n - E_n)$$
$$< \sum_n \frac{\epsilon}{2^n} \text{ (by(3))}$$
$$= \epsilon \sum_n \frac{1}{2^n}$$
$$= \epsilon \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \epsilon$$

$$\Rightarrow m^*(O - E) < \epsilon$$

$(ii) \Rightarrow (iii)$

By (ii), for given $\epsilon = \frac{1}{n} > 0$ $(n \in \mathbb{N})$, $\exists$ an open set $O_n \supseteq E$ such that

$$m^*(O_n - E) < \frac{1}{n} \quad \dots (4)$$

Let $G = \bigcap_n O_n$, so that G is a G_δ –set and $G \supseteq E$. Also

$$m^*(G - E) = m^*\left(\bigcap_n O_n - E\right)$$

$$= m^*\left(\left(\bigcap_n O_n\right) \cap E^c\right)$$

$$= m^*\left(\bigcap_n (O_n \cap E^c)\right)$$

$$= m^*\left(\bigcap_n (O_n - E)\right), n \in \mathbb{N}$$

$$< \frac{1}{n} \quad \forall n \in \mathbb{N} \qquad (\text{by}(4))$$

$$i.e., \quad m^*(G - E) < \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad m^*(G - E) = 0$$

$(iii) \Rightarrow (i)$

By (iii), there exists a G_δ −set and $G \supseteq E$ such that $m^*(G - E) = 0$

$\Rightarrow$ $G - E$ is measurable.

Also, G, being G_δ −set is measurable.

$\therefore G - (G - E) = E$ is measurable.

$(i) \Rightarrow (iv)$

Since E is measurable therefore E^c is also measurable.

By (ii), for given $\epsilon > 0, \exists$ an open set $O \supseteq E^c$ such that

$$m^*(O - E^c) < \epsilon \qquad \ldots (5)$$

Let $F = O^c$. Then F is a closed set and $F \subseteq E$.

$\because O \supseteq E^c$

$\Rightarrow F = O^c \subseteq E$

Also

$$m^*(E - F) = m^*(E \cap F^c)$$
$$= m^*(E \cap O)$$
$$= m^*(O \cap (E^c)^c)$$
$$= m^*(O - E^c) < \epsilon \quad (\text{by}(5))$$

$i.e., \quad m^*(E - F) < \epsilon$

$(iv) \Rightarrow (v)$

By (iv), for given $\epsilon = \frac{1}{n} > 0, n \in \mathbb{N}$ there exists closed set F_n such that $F_n \subseteq E$ and

$$m^*(E - F_n) < \frac{1}{n} \qquad \ldots (6)$$

Let $F = \bigcup_n F_n$, so that F is F_σ − setand $F \subseteq E$.

Also

$$m^*(E - F) = m^*\left(E - \bigcup_n F_n\right)$$

$$= m^*\left[E \cap \left(\bigcup_n F_n\right)^c\right]$$

$$= m^*\left[E \cap \left(\bigcap_n F_n^c\right)\right]$$

$$= m^*\left[\bigcap_n (E \cap F_n^c)\right]$$

$$= m^*\left[\bigcap_n (E - F_n)\right]$$

$$\leq m^*(E - F_n), \forall n \in \mathbb{N}$$

$$< \frac{1}{n} \quad \forall n \in \mathbb{N} \qquad (\text{by}(6))$$

$$i.e.\, m^*(E - F) < \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad m^*(E - F) = 0$$

$(v) \Rightarrow (i)$

By (v), $\exists\, a\, F_\sigma$ –set such that $F \subseteq E$ and $m^*(E - F) = 0$

$\Rightarrow E - F$ is a measurable set.

Also, F, being a F_σ –set, is measurable.

Therefore $(E - F) \cup F = E$ is measurable.

This completes the proof.

Theorem 2.11: If $\{E_i\}$ is a sequence of disjoint measurable sets then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

Proof: Since $\{E_i\}$ is a sequence of measurable sets, so $\bigcup_{i=1}^{\infty} E_i$ is also measurable.

Therefore, by countable subadditivity property of outer measure, we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i) \qquad \ldots(1)$$

Let $F_n = \bigcup_{i=1}^{n} E_i$

then

$$m(F_n) = m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i) \qquad \ldots(2)$$

Now

$$F_n = \bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i, \quad \forall n$$

$$\therefore\ m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m(F_n) = \sum_{i=1}^{n} m(E_i), \forall n \quad (by(2))$$

$$\rightarrow\ m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i)$$

$$\Rightarrow\ m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m(E_i) \qquad \ldots(3)$$

From (1) and (3), we get

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

This completes the proof.

Theorem 2.12: Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets $(i.e. E_{n+1} \subseteq E_n \ \ \forall n)$ such that $m(E_1) < \infty$. Then

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$$

Proof: We have $m(E_1) < \infty$ and $E_n \subseteq E_1 \ \ \forall n \in \mathbb{N}$.

$\therefore \ m(E_n) \leq m(E_1) < \infty \ \Rightarrow \ m(E_n) < \infty, n \in \mathbb{N}$.

Let $F_i = E_i - E_{i+1}$, so that $F_i's$ are pairwise disjoint.

Let$E = \bigcap_{i=1}^{\infty} E_i$.

Then

$$m(E_1 - E) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i)$$

$$i.e., \qquad m(E_1 - E) = \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \qquad \text{... (1)}$$

Now

$$E_1 = E \cup (E_1 - E) \text{and } E, E_1 - E \text{ are disjoint.}$$

$$\therefore m(E_1) = m(E) + m(E_1 - E)$$

$$\Rightarrow m(E_1 - E) = m(E_1) - m(E) \qquad \text{... (2)}$$

Also

$$E_i = E_{i+1} \cup (E_i - E_{i+1}) \text{and} E_{i+1}, E_i - E_{i+1} \text{are disjoint.}$$

$$\therefore m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1})$$

$$\Rightarrow m(E_i - E_{i+1}) = m(E_i) - m(E_{i+1}) \qquad \text{... (3)}$$

From (1), (2) and (3), we get

$$m(E_1) - m(E) = \sum_{i=1}^{\infty} [m(E_i) - m(E_{i+1})]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [m(E_i) - m(E_{i+1})]$$

$$= \lim_{n\to\infty} [m(E_1) - m(E_{n+1})]$$

$$= m(E_1) - \lim_{n\to\infty} m(E_n)$$

$$\Rightarrow m(E) = \lim_{n\to\infty} m(E_n)$$

$$i.e., \qquad m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

This completes the proof.

Notes: The condition $m(E_1) < \infty$ in above theorem cannot be dropped.

e.g., Let $E_n = (n, \infty)$. Then $\{E_n\}$ is an infinite decreasing sequence of measurable sets, such that $m(E_n) = \infty$ for each $n \in \mathbb{N}$ and

$$\bigcap_{n=1}^{\infty} E_n = \Phi$$

Now since

$$\bigcap_{n=1}^{\infty} E_n = \Phi$$

$$\Rightarrow m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$$

But

$$m(E_n) = \infty, n \in \mathbb{N},$$

$$\therefore \lim_{n\to\infty} m(E_n) = \infty$$

$$\Rightarrow m\left(\bigcap_{n=1}^{\infty} E_n\right) \neq \lim_{n\to\infty} m(E_n).$$

Theorem2.13:Let$\{E_n\}$ be an infinite increasing sequence of measurable sets ($i.e. E_{n+1} \supseteq E_n$for each n). Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$$

Proof: **Case I**: $m(E_k) = \infty$ for some $k \in \mathbb{N}$ we have,

$$\bigcup_{i=1}^{\infty} E_i \supseteq E_k.$$

$$\therefore m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m(E_k) = \infty$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \infty \qquad \ldots(1)$$

Now as $\{E_n\}$is increasing sequence, therefore

$$E_n \supseteq E_k, \forall n \geq k$$

$$\Rightarrow m(E_n) \geq m(E_k) = \infty \;\; \forall n \geq k$$

$$\Rightarrow \lim_{n\to\infty} m(E_n) = \infty \qquad \ldots(2)$$

From (1) and (2), we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

Case II)$m(E_i) < \infty$ for all $i \in \mathbb{N}$.

Let $F_1 = E_1$ and $F_{i+1} = E_{i+1} - E_i$ for each $i \in \mathbb{N}$. Then

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

and $F_i's$ are disjoint.

$$\therefore m\left(\bigcup_{i=1}^{\infty} E_i\right) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} m(F_i)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \lim_{n\to\infty} m(E_n).$$

This completes the proof.

Theorem 2.14: If A is a measurable set and $m^*(A\Delta B) = 0$, then show that B is also measurable and

$$m(B) = m(A)$$

Proof: We have

$$A\Delta B = (A - B) \cup (B - A)$$

$$\therefore\ A - B \subseteq A\Delta B \text{ and } B - A \subseteq A\Delta B$$

$$\Rightarrow m^*(A - B) \leq m^*(A\Delta B) = 0 \text{ and} m^*(B - A) \leq m^*(A\Delta B) = 0$$

$$\therefore\ m^*(A - B) = 0 \text{ and} m^*(B - A) = 0$$

$$\Rightarrow A - B \text{ and } B - A \text{ are measurable sets.}$$

Now A and $A - B$ are measurable, therefore $A - (A - B) = A \cap B$ is also measurable.

$$\therefore B - A \text{ and } A \cap B \text{ are measurable.}$$

$$\Rightarrow (B - A) \cup (A \cap B) = \text{B is also measurable.}$$

Further,

$$m(B) = m[(B - A) \cup (A \cap B)]$$

$$= m(B - A) + m\,(A \cap B)$$

$$= 0 + m\,(A \cap B)$$

$$i.e,\quad m(B) = m\,(A \cap B) \qquad \ldots (1)$$

Similarly, we can obtain,

$$m(A) = m\,(A \cap B) \qquad (2)$$

Therefore from (1) and (2), we get

$$m(A) = m\,(B)$$

Sum Modulo 1

If $x, y \in [0,1)$, then we define sum modulo 1 as

$$x\dot{+}y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$

$e.g.,$ If $x = 0.6, y = 0.3$ then $x\dot{+}y = x + y$ andIf $x = 0.6, y = 0.9$ then $x\dot{+}y = x + y - 1$.

Notes: For 0.6, y ,1) and en x),1), we define the translation of then E mod ulo one by y − 1. y as

$$E\dot{+}y = \{x\dot{+}y : x \in E\}$$

Theorem 2.15: Let $E \subseteq [0,1)$ be a measurable set and $y \in [0,1)$. Then the set

$$E\dot{+}y = \{x\dot{+}y : x \in E\}$$

is measurable and $m(E\dot{+}y) = m(E)$.

Proof: Let

$$E_1 = E \cap [0,1 - y) \text{ and } E_2 = E \cap [1 - y, 1)$$

Then E_1 and $E_2$are measurable sets and $E_1 \cup E_2 = E, E_1 \cap E_2 = \Phi$

$$\square\ m(E) = m(E_1 \cup E_2)$$

$$= m(E_1) + m(E_2) \qquad \ldots (1)$$

Also, if

$$y_1 \in E_1 = E \cap [0,1 - y)$$

$$\square\ y_1 \in E \text{ and} y_1 \in [0,1 - y)$$

$$\square\ 0 \leq y_1 < 1 - y$$

$$\Rightarrow y + y_1 < 1$$

$$\square\ E_1 \dot{+} y = E_1 + y \qquad \ldots (2)$$

If $y_2 \in E_2 = E \cap [1 - y, 1)$

$$\Rightarrow y_2 \in [1 - y, 1)$$

$$\Rightarrow 1 - y \leq y_2 < 1 \Rightarrow y + y_2 \geq 1$$

$$\Rightarrow y \dot{+} y_2 = y_2 + (y - 1)$$

$$\square\ E_2 \dot{+} y = E_2 + (y - 1) \qquad \ldots (3)$$

Since RHS of (2) and (3) are measurable. Therefore, LHS of (2) and (3) are also measurable

$$\Rightarrow E_1 \dot{+} y \text{ and } E_2 \dot{+} y \text{ are measurable.}$$

and

$$m(E_1 \dot{+} y) = m(E_1 + y) = m(E_1) \qquad \ldots (4)$$

$$m(E_2 \dot{+} y) = m(E_2 + y) = m(E_2) \qquad \ldots (5)$$

Now let $z \in (E_1 \dot{+} y) \cap (E_2 \dot{+} y)$

$$\Rightarrow z = e_1 + y = e_2 + y - 1, \text{ for some } e_1, e_2 \in E \subseteq [0,1)$$

$$\Rightarrow e_1 = e_2 - 1 < 0, \text{ which is a contradiction.}$$

$\therefore E \dot{+} y = (E_1 \dot{+} y) \cup (E_2 \dot{+} y)$ where $(E_1 \dot{+} y) \cap (E_2 \dot{+} y) = \Phi$.

Hence $E \dot{+} y$ being the union of two measurable sets is also measurable and

$m(E \dot{+} y) = m(E_1 \dot{+} y) + m(E_2 \dot{+} y) = m(E_1) + m(E_2) = m(E). \qquad \{by\ (1), (4), (5)\}$

This completes the proof.

Theorem 2.16: There exists a non-measurable subset of [0,1).

Proof: Two elements $x, y \in [0,1)$ are said to be related and we write $x \sim y$ if and only if $x - y$ is a rational number.

$$x - x = 0 \in \mathbb{Q} \quad \forall x \in [0,1) \therefore x \sim x\ \forall x \in [0,1) \Rightarrow \sim \text{ is reflexive.}$$

Let

$$x \sim y, x, y \in [0,1) \ \square\ x - y \in \mathbb{Q} \ \square\ y - x \in \mathbb{Q} \ \square\ y \sim x$$

Therefore ~ is symmetric.

Now let $x \sim y$ and $y \sim z$, $x, y, z \in [0,1)$ then

$$x - y, y - z \in \mathbb{Q}$$

$$\square\ (x - y) + (y - z) \in \mathbb{Q}$$

$$\square\ x - z \in \mathbb{Q}$$

$$\square\ x \sim z$$

$$\square\ \sim \text{ is transitive.}$$

Thus ~ is an equivalence relation in [0,1) and partitions [0,1) into mutually disjoint equivalence classes.

Axiom of choice says, "If $\{E_\alpha | \alpha \in A\}$ is a non-empty collection of non-empty disjoint subsets of a set $\mathcal{X}$, then □ a set $V \subseteq \mathcal{X}$ containing just one element from each set E_α."

We can construct a set $P \subseteq [0,1)$ by taking exactly one element from each of these equivalence classes.

Let $[0,1) \cap \mathbb{Q} = \{r_0, r_1, r_2, \ldots,\}, r_0 = 0$.

Define $P_i = P \dot{+} r_i \quad \forall i$.

Claim:

i) $P_i \cap P_j = \Phi$ for $i \neq j$

ii) $\bigcup_i P_i = [0,1)$

For (i) let $x \in P_i \cap P_j, i \neq j$. {Here $i \neq j \Rightarrow r_i \neq r_j$ }

□ $x \in P_i$ and $x \in P_j$

□ $x = p \dot{+} r_i$ and $x = q \dot{+} r_j$, for some $p, q \in P$

$\Rightarrow p \dot{+} r_i = q \dot{+} r_j$

Following possibilities arises.

$a)\ p + r_i = q + r_j$

$b)\ p + r_i - 1 = q + r_j$

$c)\ p + r_i = q + r_j - 1$

$d)\ p + r_i - 1 = q + r_j - 1$

$p - q \in \mathbb{Q}$ in all cases, therefore $p \sim q$.

But P contains one and only one element from each equivalence class. If $p = q$ then

$$r_i = r_j \text{ or } r_i = r_j - 1 \text{ or } r_j = r_i - 1$$

which is impossible because r_i and r_j are different and $r_i, r_j \in [0,1)$ implies $r_i = r_j - 1 < 0, r_j = r_i - 1 < 0$.

Hence $P_i \cap P_j = \Phi$ for $i \neq j$.

For (ii) let $x \in [0,1)$ then x belongs to some equivalence class. Then □p □ P such that $x \sim p$.

$$\Rightarrow x - p \text{ must be a rational number.}$$

Now if $x > p$. Then

$$x = p + (x - p) \text{ □ } P \dot{+} r_i, r_i = x - p \in [0,1)$$

If $x < p$ then

$$x = p + (x - p + 1) - 1 \text{ □ } P \dot{+} r_i, r_i = x - p + 1 \text{ □ } [0,1)$$

□ $x \in P \dot{+} r_i$ for some i

$$\text{□ } x \in \bigcup_i P \dot{+} r_i$$

$$\Rightarrow x \in \bigcup_i P_i$$

□ $[0,1)$ □ $\bigcup_i P_i$ but $\bigcup_i P_i \subseteq [0,1)$ is obviously true.

Therefore $[0,1) = \bigcup_i P_i$.

Thus $\{P_i\}$ is a sequence of pairwise disjoint sets and

$$\bigcup_i P_i = [0,1)$$

Now, if possible, let P be a measurable set. Then $P_i = P \dot{+} r_i$ is measurable □ i and

$$m(P_i) = m(P)$$

$$\text{□ } 1 = m([0,1))$$

$$= m\left(\bigcup_i P_i\right)$$

$$= \sum_i m(P_i)$$

$$= \sum_i m(P)$$

$$= \begin{cases} 0 \text{ if } m(P) = 0 \\ \infty \text{ if } m(P) > 0 \end{cases}$$

which is impossible.

Hence P is a non-measurable subset of [0,1).

Summary

- If $m^*(E) = 0$, then E is a measurable set.
- If E is a measurable set, then E^c is also measurable.
- If E_1andE_2 are measurable then $E_1 \cup E_2$.
- If E_1 and E_2 are measurable then $E_1 \cap E_2$ is also measurable.
- If E_1 and E_2 are measurable then $E_1 - E_2$ is also measurable.
- If $E_1, E_2, \ldots, E_n$ are disjoint measurable sets, then $m^*[A \cap (\bigcup_{i=1}^{n} E_i)] = \sum_{i=1}^{n} m^*(A \cap E_i), A \subseteq \mathbb{R}$
- If $E_1, E_2, \ldots, E_n$ are disjoint measurable sets, then$m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(E_i)$.
- If $\{E_n\}$ is a sequence of disjoint measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.
- If $\{E_n\}$ is a sequence of measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.
- If $\{E_n\}$ is a sequence of measurable sets, then $\bigcap_{n=1}^{\infty} E_n$ is also measurable.
- If E_1andE_2 are measurable sets such that $E_2 \subseteq E_1$ then$m^*(E_1 - E_2) = m^*(E_1) - m^*(E_2)$.
- If E_1andE_2 are measurable sets, then$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2)$.
- The collection $\mathcal{M}$ of measurable subsets of $\mathbb{R}$ is a $\sigma -$ algebra.
- Every interval is a measurable set.
- Every open set is measurable.
- Every closed set is measurable.
- F_σ −set and G_δ −sets are measurable sets.
- If $\{E_i\}$ is a sequence of disjoint measurable sets then$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.
- Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets $(i.e. E_{n+1} \subseteq E_n \ \ \forall n)$ such that $m(E_1) < \infty$. Then $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.
- Let $\{E_n\}$ be an infinite increasing sequence of measurable sets $(i.e. E_{n+1} \supseteq E_n$for each $n)$. Then$m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.
- If A is a measurable set and $m^*(A \Delta B) = 0$, then show that B is also measurable and $m(B) = m(A)$.
- Let $E \subseteq [0,1)$ be a measurable set and $y \in [0,1)$. Then the set$E\dot{+}y = \{x\dot{+}y : x \in E\}$ is measurable and $m(E\dot{+}y) = m(E)$.
- There exists a non-measurable subset of [0,1).

Keywords

Algebra: A non-empty collection $\mathcal{A}$ of subsets of a set X is called an algebra if

(i) Empty set $\Phi \in \mathcal{A}$.

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \sqcup A \cup B \in \mathcal{A}$.

σ −algebra: A non-empty collection $\mathcal{A}$ of subsets of a set X is called σ −algebra if

(i) $\Phi \in \mathcal{A}$

$(ii) A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

$$(iii) A_n \in \mathcal{A}, n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Borel σ − algebra

The intersection of all the σ −algebras of subsets of $\mathbb{R}$ that contain the open sets is a σ −algebra called the Borel σ −algebra. Members of this collection are called Borel sets.

Lebesgue Measure: The restriction of the set function outer measure to the class of measurable sets is called Lebesgue measure. It is denoted by m, so that if E is a measurable set, its Lebesgue measure, $m(E)$, is defined by

$$m(E) = m^*(E)$$

Sum Modulo 1

If $x, y \in [0,1)$, then we define sum modulo 1 as

$$x \dot{+} y = \begin{cases} x + y & \text{if } x + y < 1, \\ x + y - 1 & \text{if } x + y \geq 1. \end{cases}$$

Self Assessment

1)Consider the following statements:

(I) $m^*(E) = 0$ □ E is measurable.

(II) E is measurable □ $m^*(E) = 0$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

2) Consider the following statements:

(I) Every measurable set is countable.

(II) Every countable set is measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

3) Consider the following statements:

(I) Cantor set is uncountable.

(II) Cantor set is measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

4) Consider the following statements:

(I) $(E_1 \cap E_2)^c$ is measurable $\Rightarrow$ $E_1 \cap E_2$ is measurable.

(II) E_1, E_2 are measurable □ $E_1 \cap E_2^c$ is measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

5) If □ is algebra then empty set $\phi \in$ □.

A. True
B. False

6) If □ is sigma algebra, $A_k \in \mathcal{A}$, $k \in \mathbb{N}$ then $\bigcup_{k=1}^{n} A_k \in \mathcal{A}$ but $\bigcup_{k=1}^{\infty} A_k \notin \mathcal{A}$.

A. True
B. False

7) Every sigma algebra is algebra.

A. True
B. False

8) Every algebra is sigma algebra.

A. True
B. False

9) If $E_1, E_2, \dots, E_n$ are disjoint measurable sets then

A. $\sum_{i=1}^{n} m^*(E_i) < m^*(\bigcup_{i=1}^{n} E_i)$
B. $\sum_{i=1}^{n} m^*(E_i) > m^*(\bigcup_{i=1}^{n} E_i)$
C. $\sum_{i=1}^{n} m^*(E_i) = m^*(\bigcup_{i=1}^{n} E_i)$
D. cannot say anything

10) Consider the following statements:

(I) If $\{E_n\}$ is a sequence of disjoint measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.

(II) If $\{E_n\}$ is a sequence of measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ need not be measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

11) Consider the following statements:

(I) If $\{E_n\}$ is a sequence of measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.

(II) If $\{E_n\}$ is a sequence of measurable sets, then $\bigcap_{n=1}^{\infty} E_n$ need not be measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

12) Choose the correct statement:

A. Every open set is measurable but every closed set need not be measurable.

B. Every closed set is measurable but every open set need not be measurable.
C. Every open set is measurable and every closed set is measurable.
D. Every open set need not be measurable and every closed set need not be measurable.

13) G_δ sets are measurable.

A. True
B. False

14) Not all F_σ sets are measurable.

A. True
B. False

15) Consider the following statements:

(I)$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) - m^*(E_2)$

(II)The collection of measurable subsets of $\mathbb{R}$ is sigma algebra.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

16) Consider the following statements:

(I) E is measurable □ for given □> 0, there exists an open set $O \supseteq E$ such that $m^*(O - E) < \epsilon$.

(II) E is measurable □ for given □> 0, there exists a closed set $F \subseteq E$ such that $m^*(E - F) < \epsilon$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

17) Consider the following statements:

(I) E is measurable □ there exists a G_δ set $G \supseteq E$ such that $m^*(G - E) = 0$.

(II) E is measurable □ there exists a F_σ set $F \subseteq E$ such that $m^*(E - F) = 0$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

18) Consider the following statements:

(I) If $\{E_n\}$ be an infinite decreasing sequence of measurable sets such that $m(E_1) < \infty$ then $m(\bigcap_{n=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.

(II) If $\{E_n\}$ be an infinite increasing sequence of measurable sets then $m(\bigcap_{n=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.

A. only (I) is correct
B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

19) Consider the following statements:

(I) If $\{E_n\}$ be an infinite increasing sequence of measurable sets such that $m(E_1) < \infty$ then $m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.

(II) If $\{E_n\}$ be an infinite increasing sequence of measurable sets then $m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

20) Consider the following statements:

(I) If A is a measurable set such that $m^*(A\Delta B) = 0$ then B is measurable.

(II) If A is a measurable set such that $m^*(A\Delta B) = 0$ then $m(B) < m(A)$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

21) There exists a non-measurable subset of [0, 1).

A. True

B. False

22) Let $E \subseteq [0,1)$ be a measurable set and $y \in [0,1)$. Consider the following statements:

(I)$m(E+y) > m(E)$

(II)The set $E+y = \{x+y: x \in E\}$ is measurable.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

Answersfor Self Assessment

1.	A	2.	B	3.	C	4.	C	5.	A
6.	B	7.	A	8.	B	9.	C	10.	A
11.	A	12.	C	13.	A	14.	B	15.	B
16.	C	17.	C	18.	A	`9.	C	20.	A

21. A 22. B

Review Questions

1) Show that union of countable collection of measurable sets is measurable.

2) Prove that if sigma algebra of subsets of real numbers contains intervals of the form (a, ∞), then it contains all intervals.

3)Show that intersection of countable collection of measurable sets is measurable.

4)Prove or disprove: Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets $(i.e. E_{n+1} \subseteq E_n \ \forall n)$, then $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.

5) Prove that any set of outer measure zero is measurable.

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

Unit 03: Measurable Functions

CONTENTS

Objectives

After studying this unit, students will be able to:

- define measurable functions
- understand the algebra of measurable functions
- demonstrate the theorems related to measurable functions
- describeBorel measurable functions and its related theorems
- explain characteristic function and simple function

Introduction

In this unit we study the measurable functions to lay the foundation for studying Lebesgue integral, which we discuss in the next unit. We discuss the algebra of measurable functions and some other related concepts.

3.1 Measurable Functions

Let f be an extended real valued function defined on a measurable set E. Then f is Lebeseque measurable function or briefly, a measurable function, if, for each $\alpha \in \mathbb{R}$, the set $\{x: f(x) > \alpha\}$ is measurable.

Theorem 3.1: Let f be an extended real valued function with measurable domain $E \subseteq \mathbb{R}$. Then the following statements are equivalent.

i) $\{x: f(x) > \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.

ii) $\{x: f(x) \geq \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.

iii) $\{x: f(x) < \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.

iv) $\{x: f(x) \leq \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.

Proof: $(i) \Rightarrow (ii)$

Firstly, we show that for any $\alpha \in \mathbb{R}$,

$$\{x: f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\}$$

Let

$$y \in \{x: f(x) \geq \alpha\}$$

$$\Rightarrow f(y) \geq \alpha$$

$$\Rightarrow f(y) > \alpha - \frac{1}{n}, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \forall n \in \mathbb{N}$$

$$\therefore \{x: f(x) \geq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \qquad \dots (1)$$

Again let

$$y \in \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\}$$

$$\Rightarrow y \in \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \forall n \in \mathbb{N}$$

$$\Rightarrow f(y) > \alpha - \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow f(y) \geq \alpha \Rightarrow y \in \{x: f(x) \geq \alpha\}$$

$$\therefore \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \subseteq \{x: f(x) \geq \alpha\} \qquad \dots (2)$$

From (1) and (2), we get

$$\{x: f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\}$$

Now by (i), $\left\{x: f(x) > \alpha - \frac{1}{n}\right\}$ is measurable for each $'n'$ and countable intersection of measurable sets is measurable.

$$\therefore \bigcap_{n=1}^{\infty} \left\{x: f(x) > \alpha - \frac{1}{n}\right\} \text{is measurable.}$$

$$\Rightarrow \{x: f(x) \geq \alpha\} \text{is measurable.}$$

$$(ii) \Rightarrow (iii)$$

By (ii), $\{x: f(x) \geq \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$.

$\Rightarrow \{x: f(x) \geq \alpha\}^c$ is also measurable for each $\alpha \in \mathbb{R}$.

$\Rightarrow \{x: f(x) < \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$.

$(iii) \Rightarrow (iv)$

Firstly, we show that for any $\alpha \in \mathbb{R}$,

$$\{x: f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x: f(x) < \alpha + \frac{1}{n}\right\}$$

Let

$$y \in \{x: f(x) \leq \alpha\}$$

$$\Rightarrow f(y) \leq \alpha$$

$$\Rightarrow f(y) < \alpha + \frac{1}{n}, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \left\{x: f(x) < \alpha + \frac{1}{n}\right\} \forall n \in \mathbb{N}$$

$$\therefore \{x: f(x) \leq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \left\{x: f(x) < \alpha + \frac{1}{n}\right\} \qquad \dots (3)$$

Again let

$$y \in \bigcap_{n=1}^{\infty}\left\{x: f(x) < \alpha + \frac{1}{n}\right\}$$

$$\Rightarrow y \in \left\{x: f(x) < \alpha + \frac{1}{n}\right\} \ \forall n \in \mathbb{N}$$

$$\Rightarrow f(y) < \alpha + \frac{1}{n}, \ \ \forall n \in \mathbb{N}$$

$$\Rightarrow f(y) \le \alpha$$

$$\Rightarrow y \in \{x: f(x) \le \alpha\}$$

$$\therefore \bigcap_{n=1}^{\infty}\left\{x: f(x) < \alpha + \frac{1}{n}\right\} \subseteq \{x: f(x) \le \alpha\} \qquad \ldots (4)$$

From (3) and (4), we get

$$\{x: f(x) \le \alpha\} = \bigcap_{n=1}^{\infty}\left\{x: f(x) < \alpha + \frac{1}{n}\right\}$$

Now by (iii), $\left\{x: f(x) < \alpha + \frac{1}{n}\right\}$ is measurable for each n and countable intersection of measurable sets is measurable.

$$\therefore \bigcap_{n=1}^{\infty}\left\{x: f(x) < \alpha + \frac{1}{n}\right\} \text{is measurable.}$$

$$\Rightarrow \{x: f(x) \le \alpha\} \text{is measurable.}$$

$(iv) \Rightarrow (i)$

By (iv), $\{x: f(x) \le \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$.

$\Rightarrow \{x: f(x) \le \alpha\}^c$ is also measurable for each $\alpha \in \mathbb{R}$.

$\Rightarrow \{x: f(x) > \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$.

Theorem 3.2: Show that if f is measurable then $\{x: f(x) = \alpha\}$ is measurable for each extended real number α.

Proof: Let $\alpha \in \mathbb{R}$ then

$$\{x: f(x) = \alpha\} = \{x: f(x) \ge \alpha\} \cap \{x: f(x) \le \alpha\}$$

Now f is a measurable function. Therefore $\{x: f(x) \ge \alpha\}$ and $\{x: f(x) \le \alpha\}$ are measurable.

$$\Rightarrow \{x: f(x) \ge \alpha\} \cap \{x: f(x) \le \alpha\} \text{is measurable.}$$

$$\Rightarrow \{x: f(x) = \alpha\} \text{is a measurable set.}$$

If $\alpha = \infty$ then

$$\{x: f(x) = \infty\} = \bigcap_{n=1}^{\infty}\{x: f(x) > n\}$$

Since f is measurable function, therefore $\{x: f(x) > n\}$ is measurable set for each $n \in \mathbb{N}$.

$\Rightarrow \bigcap_{n=1}^{\infty}\{x: f(x) > n\}$ is a measurable set.

$\Rightarrow \{x: f(x) = \alpha\}$ is a measurable set.

If $\alpha = -\infty$ then

$$\{x: f(x) = -\infty\} = \bigcap_{n=1}^{\infty}\{x: f(x) < -n\}$$

Since f is measurable function, therefore $\{x: f(x) < -n\}$ is measurable set for each $n \in \mathbb{N}$.

$\Rightarrow \bigcap_{n=1}^{\infty}\{x: f(x) < -n\}$ is a measurable set.

$\Rightarrow \{x: f(x) = -\infty\}$ is a measurable set.

Hence $\{x: f(x) = \alpha\}$ is a measurable set for each extended real number α.

Theorem 3.3: Show that every function defined on a set of measure zero is measurable.

Proof: Let $f: E \to \mathbb{R}$ be a function with $m^*(E) = 0$.

Let α be any real number then

$$\{x \in E : f(x) > \alpha\} \subseteq E$$

$$\Rightarrow m^*\{x \in E : f(x) > \alpha\} \leq m^*(E) = 0$$

$$\Rightarrow m^*\{x \in E : f(x) > \alpha\} = 0$$

$\Rightarrow \{x \in E : f(x) > \alpha\}$ is a measurable set.

$\Rightarrow f$ is a measurable function.

Example: Constant function with measurable domain is a measurable function.

Solution: Let the function f defined by $f(x) = c$ with measurable domain E. Let α be any real number. Then

$$\{x \in E : f(x) > \alpha\} = \begin{cases} E: \alpha < c \\ \phi: \alpha \geq c \end{cases}$$

Since sets on RHS are measurable therefore set on LHS is measurable,

$\Rightarrow f$ is a measurable function.

Theorem 3.4: If f is a measurable function on a set E and $E_1 \subseteq E$ is a measurable set, then f is a measurable function on E_1.

Proof: Let α be any real number. Then,

$$\{x \in E_1 : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \cap E_1$$

Now, $\{x \in E : f(x) > \alpha\}$ is a measurable set as f is a measurable function on E.

Also E_1 is a measurable set.

$\Rightarrow \{x \in E : f(x) > \alpha\} \cap E_1$ is a measurable set.

$\Rightarrow \{x \in E_1 : f(x) > \alpha\}$ is a measurable set.

$\Rightarrow f$ is a measurable function on E_1.

Theorem 3.5: If f is a measurable function on each of the sets in a countable collection $\{E_n\}$ of measurable set, then f is measurable on $\cup_n E_n$.

Proof: Let $E = \cup_n E_n$.

Since each E_n is a measurable set and countable union of measurable sets is measurable, therefore E is a measurable set. Let α be any real number. Then

$$\{x \in E: f(x) > \alpha\} = \left\{x \in \bigcup_n E_n : f(x) > \alpha\right\}$$

$$= \bigcup_n \{x \in E_n: f(x) > \alpha\}$$

Now, f is a measurable function on E_n for each n.

$\Rightarrow \{x \in E_n : f(x) > \alpha\}$ is a measurable set for each n.

$\Rightarrow \bigcup_n \{x \in E_n: f(x) > \alpha\}$ is a measurable set.

$\Rightarrow \{x \in E : f(x) > \alpha\}$ is a measurable set.

$\Rightarrow f$ is a measurable function on $E = \bigcup_n E_n$.

Example: Every continuous function with measurable domain is measurable.

Solution: Let f be a continuous function defined on measurable domain E.

Now, (α, ∞) is an open set in $\mathbb{R}$ and f is continuous on E.

Therefore $f^{-1}\big((\alpha, \infty)\big)$ is an open set in E.

□ $\{x \in E : f(x) > \alpha\}$is open set in E.

□ $\{x \in E : f(x) > \alpha\}$is a measurable set for each α □ ℝ.

□ f is a measurable function on E.

Notes: A measurable function may not be continuous.

e.g.,define$f: [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 2 \text{ if } x \in (1,2] \end{cases}$$

Let α be any real number.

$$\{x \in [0,2] \,□\, f(x) > \alpha\} = \begin{cases} [0,2] \text{ if } \alpha < 1 \\ (1,2] \text{ if } 1 \le \alpha < 2 \\ \phi \text{ if } \alpha \ge 2 \end{cases}$$

Since the sets on RHS are measurable. Therefore, set on LHS is also measurable $i.e. f$ is a measurable function on $[0,2]$. But f is discontinuous at $x = 1$. Hence f is measurable function on $[0,2]$ which is discontinuous on $[0,2]$.

Theorem 3.6: Let f be a measurable function on E. Then $\{x \in E : a \le f(x) \le b\} = f^{-1}([a,b])$ is a measurable set.

Proof: We have

$$\{x \in E : a \le f(x) \le b\} = \{x \in E : f(x) \ge a\} \cap \{x \in E : f(x) \le b\}$$

Now, the sets on RHS are measurable. Therefore

$$\{x \in E : f(x) \ge a\} \cap \{x \in E : f(x) \le b\} \text{is measurable.}$$

$$\Rightarrow \{x \in E : a \le f(x) \le b\} \text{is a measurable set.}$$

$$i.e., \quad f^{-1}([a,b]) \text{is a measurable set.}$$

Similarly, we can show that

$$\{x \in E : a < f(x) \le b\} = f^{-1}((a,b])$$

$f^{-1}\big([a,b)\big)$ and $f^{-1}\big((a,b)\big)$ are measurable sets.

Positive and negative parts of a function: Let f be a function. Then positive part of f written as f^+ and negative part of f written as f^-, defined to be the non-negative functions given by

$$f^+ = \max\{f, 0\} \text{ and } f^- = \max\{-f, 0\}.$$

$f^+ =$ ma $f, 0\}$ and $f_- =$ r

Notes: $|f| = f^+ + f^-$ and $f = f^+ - f^-$

Theorem 3.7: Let $c \in \mathbb{R}$ and f, g be measurable functions with same measurable domain E. Then each of the following functions are measurable on E.

(1) $f + c$ (2) cf (3) $f + g$ (4) $f - g$ (5)f^2(6) fg (7) $|f|$

Proof: (1) For each $\alpha \in \mathbb{R}$, we have $\{x \in E : f(x) + c > \alpha\} = \{x \in E : f(x) > \alpha - c\}$.

Since f is measurable, therefore $\{x \in E : f(x) > \alpha - c\}$ is a measurable set.

□ $\{x \in E : f(x) + c > \alpha\}$is a measurable set.

□ $f + c$ is a measurable function.

(2)Case(i)If $c = 0$ then $cf = 0$ (constant) and constant function is measurable. Therefore cf is measurable.

Case(ii)If $c > 0$ then$cf(x) > \alpha$ □ $f(x) > \frac{\alpha}{c}$

 $\{x \in E : cf(x) > \alpha\} = \left\{x \in E : f(x) > \frac{\alpha}{c}\right\}$

Since f is measurable, therefore $\left\{x \in E : f(x) > \frac{\alpha}{c}\right\}$ is a measurable set.

 $\{x \in E : cf(x) > \alpha\}$is a measurable set.

 cf is a measurable function.

Case(iii)If $c < 0$ then$cf(x) > \alpha \Rightarrow f(x) < \frac{\alpha}{c}$

 $\{x \in E : cf(x) > \alpha\} = \left\{x \in E : f(x) < \frac{\alpha}{c}\right\}$

Since f is measurable, therefore $\left\{x \in E : f(x) < \frac{\alpha}{c}\right\}$ is a measurable set.

 $\{x \in E : cf(x) > \alpha\}$is a measurable set.

 cf is a measurable function.

(3) Since $f(x) + g(x) < \alpha \Rightarrow f(x) > \alpha - g(x)$ therefore the set

$$\{x \in E : f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} [\{x \in E: f(x) > r\} \cap \{x \in E: r > \alpha - g(x)\}]$$

$$= \bigcup_{r \in \mathbb{Q}} [\{x \in E: f(x) > r\} \cap \{x \in E: g(x) > \alpha - r\}] \quad ..(1)$$

Now LHS of (1) is measurable as RHS of (1) is measurable.

 $f + g$ is a measurable function.

(4)We have $f - g = f + (-g)$.

Since g is measurable $-g$ is measurable.

Now $f, -g$ are measurable functions

 $f + (-g)$is a measurable function.

⇒ $f - g$ is a measurable function.

(5)If $\alpha \geq 0$, then

$\{x \in E : f^2(x) > \alpha\} = \{x \in E : f(x) > \sqrt{\alpha}\}$ $\{x \in E : f(x) < -\sqrt{\alpha}\}$ ... (2)

Since f is a measurable function therefore each of sets on RHS of (2) is measurable and hence their union is also measurable.

 LHS of (2) is a measurable set

 $f^2$is a measurable function.

If $\alpha < 0$, then the set $\{x \in E : f^2(x) > \alpha\} = E$

Since E is a measurable set

 $\{x \in E : f^2(x) > \alpha\}$is a measurble set

 $f^2$is a measurable function.

(6)Since f and g are measurable functions on E.

 $f + g$ and $f - g$ are measurable functions on E

 $(f + g)^2$and$(f - g)^2$ are measurable functions on E

 $(f + g)^2 - (f - g)^2$ are measurable functions on E

 $4fg$ is measurable function.

 $\frac{1}{4}(4fg) = fg$ is a measurable function.

$$(7)\{x \in E: |f| > \alpha\} = \begin{cases} E & \text{if } \alpha < 0 \\ \{x \in E: f(x) > \alpha \text{ } \{x \in E: f(x) < -\alpha & \text{otherwise} \end{cases} \quad ... (3)$$

Now both the sets on RHS of (3)are measurable.

 $|f|$is measurable function.

Theorem 3.8: For a finite family $\{f_k\}_{k=1}^{n}$ of measurable functions with common domain E, the functions $\max\{f_1, f_2, \dots, f_n\}$ and $\min\{f_1, f_2, \dots, f_n\}$ are also measurable.

Proof: For any $\alpha \in \mathbb{R}$, we have

$$\{x \in E : \max\{f_1, f_2, \dots, f_n\}(x) > \alpha\} = \bigcup_{k=1}^{n} \{x \in E : f_k(x) > \alpha\} \qquad \dots (1)$$

Now the set on RHS of (1) is measurable since it is the finite union of measurable sets. Therefore, the set on LHS of (1) is measurable. Thus, the function $\max\{f_1, f_2, \dots, f_n\}$ is measurable.

Now $\min\{f_1, f_2, \dots, f_n\} = -\max\{-f_1, -f_2, \dots, -f_n\}$.

Now, $f_1, f_2, \dots, f_n$ are measurable function.

□ $-f_1, -f_2, \dots, -f_n$ are also measurable.

□ $\max\{-f_1, -f_2, \dots, -f_n\}$ is a measurable function.

□ $-\max\{-f_1, -f_2, \dots, -f_n\}$ is a measurable function.

□ $\min\{f_1, f_2, \dots, f_n\}$ is a measurable function.

Cor: Since $|f|(x) = \max\{f(x), -f(x)\}$, $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$.

Therefore, if f is measurable on E, so are the functions $|f|, f^+, f^-$.

Theorem 3.9: f is measurable if and only if f^+ and f^- are measurable.

Proof: f is measurable

□ f^+, f^- are measurable.

Conversely, let f^+ and f^- are measurable.

□ $f^+ - f^-$ is measurable

□ f is measurable function

Theorem 3.10: Let f be a function defined on a measurable set E. Then f is a measurable function if and only if $f^{-1}(G)$ is a measurable set for every open set G in $\mathbb{R}$.

Proof: Let $'f'$ be a measurable function on measurable set E and $'G'$ be any open set in $\mathbb{R}$. Then G can be written as countable union of disjoint open interval. Suppose

$$G = \bigcup_n (a_n, b_n)$$

Then

$$f^{-1}(G) = f^{-1}\left(\bigcup_n (a_n, b_n)\right)$$

$$= \bigcup_n f^{-1}[(a_n, b_n)]$$

Now $f^{-1}[(a_n, b_n)]$ is a measurable set for each n as f is a measurable function.

□ $\bigcup_n f^{-1}[(a_n, b_n)]$ is a measurable set.

⇒ $f^{-1}(G)$ is a measurable set.

Conversely, suppose inverse image of each open set is measurable. Now,

$\{x \in E : f(x) > \alpha\} = f^{-1}[(\alpha, \infty)]$ is a measurable set.

□ f is a measurable function.

Theorem 3.11: Let f be a measurable real valued function defined on a measurable set E. If g is a continuous function defined on $\mathbb{R}$, then show that the composition $g \circ f$ is a measurable function on E.

Proof: Let α be any real number. Now,

$$\{x \in E : (g \circ f)(x) > \alpha\} = \{x \in E : g(f(x)) > \alpha\}$$
$$= \{x \in E : g(f(x) \in (\alpha, \infty)\}$$

$$= \{x \in E: f(x) \in g^{-1}[(\alpha,\infty)]\} = f^{-1}(G), \text{where } G = g^{-1}[(\alpha,\infty)]$$

Now g is continuous and (α,∞) is an open set therefore $g^{-1}[(\alpha,\infty)]$ is an open set. $i.e. G$ is an open set.

Also f is given measurable

□ $f^{-1}(G)$is a measurable set.

□ $\{x \in E: (g \circ f)(x) > \alpha\}$is a measurabe set.

□ $g \circ f$ is a measurable funcion on E

Theorem 3.12: Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable domain E. Then

$i)$ $\sup\limits_n f_n$ is measurable on E.

$ii)$ $\inf\limits_n f_n$ is measurable on E.

$iii)$ $\lim\sup f_n$ is measurable on E.

$iv)$ $\lim i \inf f_n$ is measurable on E.

Proof: Let α be any real number. Let $g = \sup\limits_n f_n$. Then

$$\{x \in E: g(x) > \alpha\} = \bigcup_n \{x \in E: f_n(x) > \alpha\}$$

Now f_n is a measurable function for each n.

□ $\{x \in E: f_n(x) > \alpha\}$is a measurable set for each n.

□ $\bigcup\limits_n \{x \in E: f_n(x) > \alpha\}$ is a measurable set

□ $\{x \in E: g(x) > \alpha\}$is a measurable set.

$\Rightarrow g = \sup\limits_n f_n$ is a measurable function.

$ii)$ $\inf\limits_n f_n = -\sup\limits_n(-f_n)$

Since f_n is measurable for each $'n'$,

□ $-f_n$is measurable for each $'n'$

□ $\sup\limits_n(-f_n)$ is measurable.

□ $-\sup\limits_n(-f_n)$ is measurable.

$\Rightarrow$ $\inf\limits_n(f_n)$ is measurable.

$iii)$ We know

$\lim\sup f_n = \inf\limits_n \sup\limits_{m\geq n} f_m = \inf\limits_n F_n$where $F_n = \sup\limits_{m\geq n} f_m$is a measurable function for each $'n'$by using(i). Therefore by (ii), $\inf\limits_n F_n = \lim\sup f_n$ is measurable function on E.

(iv) Since $\lim\inf f_n = -\lim\sup(-f_n)$ and so is measurable.

Cor: If $\{f_n\}$ is a sequence of measurable functions defined on E, then $\lim\ f_n$ is measurable on E, if it exists.

Proof: If $\lim\ f_n$ exists, then

$$\lim\sup f_n = \lim\inf f_n = \lim f_n$$

Now by above theorem, both $\lim\sup f_n$ and $\lim\inf f_n$ are measurable on E. Therefore $\lim\ f_n$ is also measurable on E.

Definition: If a property holds except on a set of measure zero, we say that it holds almost everywhere, usually abbreviated to $a.e.$

Notes: Two functions f and g defined on the same domain E are said to be equal almost everywhere on E if

$f(x) = g(x), \forall x \in E - F$ and $f(x) \neq g(x), \forall x \in F$ with $m(F) = 0$.

Theorem 3.13: If f is a measurable function on measurable domain E and $f = g a.e$ on E then g is measurable on E.

Proof: We define $A = \{x \in E: f(x) \neq g(x)\}$.

Since $f = g a.e$ on E

$\Rightarrow m(A) = 0$.

Now

$$\{x \in E: g(x) > \alpha\} = \{x \in A: g(x) > \alpha\} \cup [\{x \in E: f(x) > \alpha\} \cap (E - A)] \quad \ldots (1)$$

Since $m(A) = 0$, therefore, $\{x \in A: g(x) > \alpha\}$ is a measurable set, since it is a subset of a set of measure zero, $\{x \in E: f(x) > \alpha\}$ is measurable, since f is given measurable on E.

Also $E - A$ is measurable, since E and A are measurable sets.

$\Rightarrow$ RHS of (1) is measurable.

$\Rightarrow$ LHS of (1) is measurable.

$\Rightarrow$ g is measurable function on E

Essential Supremum: Let f be a measurable function then $\inf\{\alpha: f \leq \alpha \ a.e.\}$ is called the essential supremum of f denoted by ess sup f.

Essential infimum: Let f be a measurable function then $\sup\{\alpha: f \geq \alpha \ a.e.\}$ is called the essential infimum of f denoted by ess inf f.

Essential Bounded: If f is a measurable function and ess sup$|f| < \infty$, then f is said to be essentially bounded.

Notes:

- $f \leq$ ess sup f a. e.
- ess sup(f + g) $\leq$ ess sup f + ess sup g
- ess sup f $\leq$ sup f
- ess sup f $\leq$ $-$ess inf $(-f)$
- Let $f = g \ a.e.$ where f is a continuous function then
- ess sup f = ess sup g = sup f.

Borel Measurable Function: A function f is said to be Borel measurable provided its domain E is a Borel set and for each $\alpha \in \mathbb{R}$, $\{x \in E: f(x) > \alpha\}$ is a Borel set.

Notes: Every Borel measurable function is Lebesgue measurable but converse need not be true.

Notes: If f is a Borel function and B is a Borel set then $f^{-1}(B)$ is a Borel set.

Theorem 3.14: Let f be an extended real valued function defined on a Borel set E. Then the following statements are equivalent:

i) $\{x \in E: f(x) > \alpha\}$ is a Borel set for each $\alpha \in \mathbb{R}$

ii) $\{x \in E: f(x) \geq \alpha\}$ is a Borel set for each $\alpha \in \mathbb{R}$

iii) $\{x \in E: f(x) < \alpha\}$ is a Borel set for each $\alpha \in \mathbb{R}$

iv) $\{x \in E: f(x) \leq \alpha\}$ is a Borel set for each $\alpha \in \mathbb{R}$

Theorem 3.15: Let 'c' be any real number and let f and g be real-valued Borel measurable functions defined on same Borel set E.

Then $f + c, cf, f + g, f - g, fg$ are also Borel measurable functions on E.

Theorem 3.16: Let $< f_n >$ be a sequence of Borel measurable functions defined on same Borel set E. Then each of the following functions are Borel measurable:

$i)$ $\sup_{1\leq i\leq n} f_i$

$ii)$ $\inf_{1\leq i\leq n} f_i$

$iii)$ $\sup_n f_n$

$iv)$ $\inf_n f_n$

$v)$ $\lim \sup f_n$

$vi)$ $\lim \inf f_n$

Theorem 3.17: If f and g are Borel measurable functions then the composition $f \circ g$ is also Borel measurable.

Proof: Let α be any real number then

$$\{x: (f \circ g)(x) > \alpha\} = \{x: f\big(g(x)\big) > \alpha\}$$

$$= \{x: g(x) \in A\} = g^{-1}(A), \text{ where } A = \{y: f(y) > \alpha\}.$$

Now f is Borel function therefore A is a Borel set.

Since g is a Borel function and A is a Borel set.

□ $g^{-1}(A)$is a Borel set.

□ $\{x: (f \circ g)(x) > \alpha\}$is a Borel set.

□ $f \circ g$ is a Borel measurable function.

Characteristic Function: If A is any subset of $\mathbb{R}$, then the characteristic function of A, written as χ_A, is the function on $\mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \neq A \end{cases}$$

Theorem 3.18: Let E be a measurable set and $A \subseteq E$. Then A is measurable if and only if χ_A is a measurable function.

Proof: Let A be a measurable set and let α be any real number.

$$\{x \in E: \chi_A(x) > \alpha\} = \begin{cases} \Phi & \text{if} \quad \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ E & \text{if} \quad \alpha < 0 \end{cases}$$

Since the sets on RHS are measurable therefore the set on LHS is measurable.

□ $\chi_A$is a measurable function.

Conversely, suppose χ_A is measurable.

Now for each $x \in A$, $\chi_A(x) = 1 > 0$

□ $A = \{x: \chi_A(x) > 0\}$

Since χ_A is measurable therefore $\{x: \chi_A(x) > 0\}$ is a measurable set.

□ A is a measurable set.

Properties of characteristic functions: Let A and B be subsets of E. Then

1) $\chi_\Phi = 0$ and $\chi_E = 1$
2) $A \subsetneq B \Rightarrow \chi_A \leq \chi_B$
3) $\chi_{A\cap B} - \chi_A \cdot \chi_B$
4) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$
5) $\chi_{A^c} = 1 - \chi_A$
6) $\chi_{A-B} = \chi_A - \chi_{A\cap B}$
7) If $A = \bigcup_{n=1}^{\infty} A_n$ and $A_i \cap A_j = \Phi, i \neq j$ then $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$.

Simple Function: Let s be a real valued function defined on X. If the range of s is finite, we say that s is simple function.

Suppose range of s consists distinct numbers $c_1, c_2, \ldots, c_n$.

Let $E_i = \{x: s(x) = c_i\}$ $(i = 1,2, \ldots, n)$then $s = \sum_{i=1}^{\infty} c_i\, \chi_{E_i}$,that is every simple function is finite linear combination of characteristic functions. This expression of s is called the canonical representation of the simple function s.

s is measurable if and only if the sets $E_1, E_2, \ldots, E_n$ are measurable.

Step function: A function $\Phi: [a, b] \to \mathbb{R}$ is said to be a step function if there exists a partition

$$\{a = x_0, x_1, x_2, \ldots, x_n = b\}$$

of the interval $[a, b]$ such that in every subinterval (x_{i-1}, x_i) the function Φ is constant.

$$i.e., \quad \Phi(x) = c_i, \forall x \in (x_{i-1}, x_i) (i = 1,2, \ldots, n).$$

Summary

- Let f be an extended real valued function with measurable domain $E \subseteq \mathbb{R}$. Then the following statements are equivalent.
    1. $\{x: f(x) > \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.
    2. $\{x: f(x) \geq \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.
    3. $\{x: f(x) < \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.
    4. $\{x: f(x) \leq \alpha\}$ is a measurable set for each $\alpha \in \mathbb{R}$.
- If f is measurable then $\{x: f(x) = \alpha\}$ is measurable for each extended real number α.
- Every function defined on a set of measure zero is measurable.
- Constant function with measurable domain is a measurable function.
- If f is a measurable function on a set E and $E_1 \subseteq E$ is a measurable set, then f is a measurable function on E_1.
- If f is a measurable function on each of the sets in a countable collection $\{E_n\}$ of measurable set, then f is measurable on $\cup_n E_n$.
- Every continuous function with measurable domain is measurable.
- A measurable function may not be continuous.
- Let f be a measurable function on E. Then $\{x \in E : a \leq f(x) \leq b\} = f^{-1}([a, b])$ isa measurable set.
- Let $c \in \mathbb{R}$ and f, g be measurable functions with same measurable domain E. Then each of the following functions are measurable on E.

  (1) $f + c$ (2) cf (3) $f + g$ (4) $f - g$ (5)f^2(6) fg (7) $|f|$
- For a finite family $\{f_k\}_{k=1}^{n}$ of measurable functions with common domain E, the functions $\max\{f_1, f_2, \ldots, f_n\}$ and $\min\{f_1, f_2, \ldots, f_n\}$ are also measurable.
- f is measurable if and only if f^+ and f^- are measurable.
- Let f be a function defined on a measurable set E. Then f is a measurable function if and only if $f^{-1}(G)$ is a measurable set for every open set G in $\mathbb{R}$.
- Let f be a measurable real valued function defined on a measurable set E. If g is a continuous function defined on $\mathbb{R}$, then show that the composition $g \circ f$ is a measurable function on E.
- Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable domain E. Then

  $i)$ $\sup_n f_n$ is measurable on E.

  $ii)$ $\inf_n f_n$ is measurable on E.

  $iii)$ $\lim\sup f_n$ is measurable on E.

  $iv)$ $\lim\inf f_n$ is measurable on E.

- If f is a measurable function on measurable domain E and $f = g a.e$ on E then g is measurable on E.
- Let E be a measurable set and $A \subseteq E$. Then A is measurable if and only if χ_A is a measurable function.

Keywords

Measurable functions: Let f be an extended real valued function defined on a measurable set E. Then f is Lebeseque measurable function or briefly, a measurable function, if, for each α □ $\mathbb{R}$, the set $\{x: f(x) > \alpha\}$ is measurable.

Positive and negative parts of a function: Let f be a function. Then positive part of f written as f^+ and negative part of f written as f^-, defined to be the non-negative functions given by

$$f^+ = \max\{f, 0\} \text{ and } f^- = \max\{-f, 0\}.$$

Essential supremum: Let f be a measurable function then $\inf\{\alpha: f \leq \alpha \ a.e.\}$ is called the essential supremum of f denoted by $\operatorname{ess\,sup} f$.

Essential infimum: Let f be a measurable function then $\sup\{\alpha: f \geq \alpha \ a.e.\}$ is called the essential infimum of f denoted by $\operatorname{ess\,inf} f$.

Essentially bounded: If f is a measurable function and $\operatorname{ess\,sup}|f| < \infty$, then f is said to be essentially bounded.

Borelmeasurable function: A function f is said to be Borel measurable provided its domain E is a Borel set and for each α □ $\mathbb{R}$, $\{x \in E: f(x) > \alpha\}$ is a Borel set.

Characteristic function: If A is any subset of $\mathbb{R}$, then the characteristic function of A, written as χ_A, is the function on $\mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \neq A. \end{cases}$$

Simple function: Let s be a real valued function defined on X. If the range of s is finite, we say that s is simple function.

Self Assessment

1) Consider the following statements:

(I)If f is measurable then $\{x: f(x) = \alpha\}$ is measurable for each extended real number α.

(II)If f is measurable then $\{x: f(x) = \alpha\}$ is measurable only for α □ $\mathbb{R}$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

2) Consider the following statements:

(I)$\{x: f(x) > \alpha\}$ is measurable □$\{x: f(x) < \alpha\}$ is measurable.

(II)$\{x: f(x) < \alpha\}$ is measurable □$\{x: f(x) > \alpha\}$ is measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

3) Let the function f be defined on a measurable set E such that $m(E) = 0$, then f must be measurable.

A. True
B. False

4) Let the function f be defined on a measurable set E such that $f(x) = 2$, then f need not be measurable.

A. True
B. False

5) Consider the following statements:

(I) Every measurable function with measurable domain is continuous.

(II)Every continuous function with measurable domain is measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

6) $f^+ = max\{f, 0\}$

A. True
B. False

7) $f^- = -max\{f, 0\}$

A. True
B. False

8) Consider the following statements:

(I) $|f| = f^+ - f^-$

(II)$f = f^+ + f^-$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

9) Let f be a measurable function. Consider the following statements:
(I) $f^+ = max(f, 0)$ is measurable.

(II)$f^- = -min(f, 0)$ is not measurable. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

10) Consider the following statements:

(I)If f is measurable then $f^2$is also measurable.

(II)If f^2 is measurable then f is also measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

11) Let f be a function defined on a measurable set E. Consider the following statements:

(I)f is measurable function □ $f^{-1}(O)$ is a measurable set for every open set O in $\mathbb{R}$.

(II)If $f^{-1}(O)$ is a measurable set for every open set O in $\mathbb{R}$ □ f is measurable function.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

12) $inf\{\alpha: f \leq \alpha\ a.e.\}$ is known as *ess* $sup f$.

A. True
B. False

13) $sup\{\alpha: f \geq \alpha\ a.e.\}$ is known as *ess* $sup f$.

A. True
B. False

14)*ess* $sup f = -$*ess* $inf(-f)$

A. True
B. False

15) Consider the following statements:

(I)$sup\ f \leq ess\ sup\ f$

(II)$f \leq ess\ sup\ f\ a.e.$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

16) Consider the following statements:

(I)$\chi_{A\cap B} = \chi_A \cdot \chi_B$

(II)$\chi_{A^c} = 1 - \chi_A$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

17) Consider the following statements:

(I) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

(II) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect.

Answers for Self Assessment

1.	A	2.	C	3.	A	4.	B	5.	B
6.	A	7.	B	8.	D	9.	A	10.	A
11.	C	12.	A	13.	B	14.	A	15.	B
16.	C	17.	C						

Review Questions

1) Every measurable function is continuous. Prove or disprove.

2) If $f(x) = 3$, then show that f is measurable.

3) Show that if f is measurable then $\{x: f(x) = -\infty\}$ is measurable.

4) Define $f: [0,3] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 3 \text{if } x \in (1,3]. \end{cases}$$

Check the measurability of f.

5) Show that the real valued function f defined on $[-1, 1]$ by $f(x) = x$ is measurable.

Further Readings

Measure theory and integration by G DE BARRA, NewAgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

Unit 04: The Lebesgue Integral of Bounded Functions Over a Set of Finite Measure

CONTENTS

Objectives

After studying this unit, students will be able to:

- determine the integral of simple functions
- define upper and lower Lebesgue integral
- demonstrate Lebesgue integrable function
- understand the relation between Riemann integral and Lebesgue integral
- explain theorems related to Lebesgue integral

Introduction

In this unit, we first define the integral of simple functions. The Lebesgue integral of a bounded function over a set of finite measure is defined with the help of the integral of simple functions. We also compare the Lebesgue integral and Riemann integral.

4.1 Lebesgue Integral of Simple Functions:

A measurable real-valued function ϕ defined on a set E is said to be simple provided it takes only a finite number of real values. If ϕ takes the distinct values$a_1, a_2, \ldots, a_n$ on E, then the canonical representation of Φ on E is defined as:

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i}, E_i = \{x \in E : \phi(x) = a_i\} \ (i = 1,2, \ldots, n)$$

The canonical representation is characterized by the $E_i's$ being disjoint and the $a_i's$ being distinct.

Definition: For a simple function ϕ defined on a set of finite measure E, we define the integral of ϕ over E by

$$\int_E \phi = \sum_{i=1}^{n} a_i\, m(E_i)$$

where ϕ has the canonical representation given as

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

Lemma 4.1: Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number.

If

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i} \text{ on } E$$

Then

$$\int_E \phi = \sum_{i=1}^{n} a_i \, m(E_i).$$

Proof: The collection $\{E_i\}_{i=1}^{n}$ is disjoint but the above may not be the canonical representation since $a_i{}'s$ may not be distinct.

Let $'a'$ be an element in the range of ϕ. Then the canonical representation of ϕ is given by

$$\phi = \sum_{a} a \, \chi_{B_a},$$

where $'a'$ varies over the range of ϕ and the set B_a is given by

$$B_a = \{x \in E : \phi(x) = a\} = \bigcup_{a_i = a} E_i$$

$$\therefore \int_E \phi = \sum_{a} a \, m(B_a)$$

$$= \sum_{a} a \, m\left(\bigcup_{a_i = a} E_i\right)$$

$$= \sum_{a} a \left(\sum_{a_i = a} m(E_i)\right)$$

$$= \sum_{i=1}^{n} a_i m(E_i).$$

This completes the proof.

Theorem 4.2: (Linearity and Monotonicity of Integration)

Let ϕ and ψ be simple functions defined on a set of finite measure E. Then for any α and β,

$$1) \int_E (\alpha\phi + \beta\psi) = \alpha \int_E \phi + \beta \int_E \psi$$

$$2) \text{ If } \phi \le \psi \text{ on } E \text{ then } \int_E \phi \le \int_E \psi$$

Proof: 1) Since both ϕ and ψ takes only a finite number of values on E, we may choose a finite disjoint collection on $\{E_i\}_{i=1}^{n}$ of measurable subsets of E, the union of which is E, such that ϕ and ψ are constant on each E_i.

For each $i, 1 \le i \le n$, let $a_i$andb_i, respectively be the values taken by ϕ andψ on E_i.

By the preceding lemma,

$$\int_E \phi = \sum_{i=1}^{n} a_i \, m(E_i)$$

$$\int_E \psi = \sum_{i=1}^{n} b_i \, m(E_i)$$

The simple function $\alpha\phi + \beta\psi$ takes the constant value $\alpha a_i + \beta b_i$ on E_i. Thus

$$\int_E (\alpha\phi + \beta\psi) = \sum_{i=1}^{n} (\alpha a_i + \beta b_i)\, m(E_i)$$

$$= \alpha \sum_{i=1}^{n} a_i\, m(E_i) + \beta \sum_{i=1}^{n} b_i\, m(E_i)$$

$$= \alpha \int_e \phi + \beta \int_E \psi.$$

2) Assume $\eta = \psi - \phi$ on E.

Since $\phi \leq \psi$ on E, therefore, $\eta \geq 0$.

Consider

$$\int_E \psi - \int_E \phi = \int_E (\psi - \phi) = \int_E \eta \geq 0$$

Since the non-negative simple function has a non-negative integral.

$$\Rightarrow \int_E \psi \geq \int_E \phi$$

$$i.e. \int_E \phi \leq \int_E \psi, \phi \leq \psi \text{on } E.$$

This completes the proof.

4.2 Lebesgue Integrable Function

Lower and Upper Lebesgue Integral:

Let f be a bounded function defined on a measurable set E with $m(E) < \infty$.

Let $\alpha \leq f(x) \leq \beta\ \forall x \in E, \alpha, \beta \in \mathbb{R}$.

Let ϕ and ψ be the simple functions such that $\phi \leq f \leq \beta$. Now $\phi \leq f \leq \beta$ and $\alpha \leq f \leq \Psi$

$$\Rightarrow \int_E \phi \leq \int_E \beta = \beta\, m(E)$$

and

$$\Rightarrow \int_E \psi \geq \int_E \alpha = \alpha m(E)$$

$\therefore \left\{\int_E \phi : \phi \text{is simple and} \phi \leq f\right\}$ is a non-empty subset of $\mathbb{R}$ which is bounded above and $\left\{\int_E \psi : \psi \text{is simple and} \psi \geq f\right\}$ is a non-empty subset of $\mathbb{R}$ which is bounded below.

$\sup_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$ and $\inf_{\substack{\psi \geq f \\ \psi - simple}} \int_E \psi$ exist.

We define $\sup_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$ as lower Lebesgue integral of f over E and denote it by $\mathcal{L} \int_{\underline{E}} f$ and $\inf_{\substack{\psi \geq f \\ \psi - simple}} \int_E \psi$ as upper Lebesgue Integral of f over E and denote it by $\mathcal{L} \int_E^{-} f$.

A bounded measurable function f defined on a measurable set E of finite measure is said to be Lebesgue Integrable if

$$\mathcal{L} \int_{\underline{E}} f = \mathcal{L} \int_E^{-} f$$

and common value is denoted by $\mathcal{L} \int_E f$ or $\int_E f$.

Theorem 4.3: Let f be a bounded function defined on the closed bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.

Proof: Let f be R −integrable over $[a, b]$. Then

$$\mathcal{R}\int_a^b f\,dx = \mathcal{R}\int_{\underline{a}}^{b} f\,dx$$

$$= \sup_{\substack{\phi\le f\\ \phi-step}} \int_a^b \phi\,dx$$

$$\le \sup_{\substack{\varphi\le f\\ \phi-simple}} \int_a^b \phi\,dx$$

$$= \mathcal{L}\int_{\underline{a}}^{b} f\,dx$$

$$\le \mathcal{L}\int_a^{\overline{b}} f\,dx$$

$$= \inf_{\substack{\psi\ge f\\ \psi-simple}} \int_a^b \psi\,dx$$

$$\le \inf_{\substack{\psi\ge f\\ \psi-step}} \int_a^b \psi\,dx$$

$$= \mathcal{R}\int_a^{\overline{b}} f\,dx$$

$$= \mathcal{R}\int_a^b f\,dx$$

□ $\mathcal{R}\int_a^b f\,dx \le \mathcal{L}\int_{\underline{a}}^{b} f\,dx \le \mathcal{L}\int_a^{\overline{b}} f\,dx \le \mathcal{R}\int_a^b f\,dx$

□ $\mathcal{L}\int_{\underline{a}}^{b} f\,dx = \mathcal{L}\int_a^{\overline{b}} f\,dx = \mathcal{R}\int_a^b f\,dx$

□ $\mathcal{L}\int_a^b f\,dx = \mathcal{R}\int_a^b f\,dx.$

This completes the proof.

Notes: Lebesgue Integrable functions may not be Riemann Integrable.

Example 4.4: Let $f:[0,1] \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is Lebesgue integrable but not Riemann integrable.

Solution: Let $P = \{0 = x_0, x_1, \dots, x_n = 1\}$ be any partition of [0,1].

Let

$$m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\} = 0$$

$$M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\} = 1$$

Therefore,

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0$$

$$U(P,f) = \sum_{i=1}^{n} M_i\,(x_i - x_{i-1}) = 1$$

Now

$$\mathcal{R}\underline{\int_0^1} f\,dx = \sup_P L(P,f) = 0$$

$$\mathcal{R}\overline{\int_0^1} f\,dx = \inf_P U(P,f) = 1$$

 f is not $\mathcal{R}$ integrable over [0,1]

Now if $A = \mathbb{Q} \cap [0,1]$ then $f = \chi_A$

$$\mathcal{L}\int_0^1 f\,dx = \int_0^1 1 \cdot \chi_A = m(A) = 0.$$

This completes the proof.

Theorem 4.5: Let f and g be bounded measurable functions defined on a set E of finite measure. Then

1) $\int_E cf = c\int_E f, c \in \mathbb{R}$

2) $\int_E (f+g) = \int_E f + \int_E g$

3) If $f = g$ $a.e.$ on E then $\int_E f = \int_E g$

4) If $f \le g$ $a.e.$ on E then $\int_E f \le \int_E g$

5) $\left|\int_E f\right| \le \int_E |f|$

6) If $\alpha \le f \le \beta$ on E then $\alpha\, m(E) \le \int_E f \le \beta m(E)$

7) If A and B are disjoint measurable subsets of E then $\int_{A\cup B} f = \int_A f + \int_B f$

Proof: If ϕ and ψ are simple on E such that $m(E) < \infty$ and $\phi \ge \psi$ $a.e.$ then

$$\int_E \phi \ge \int_E \psi$$

And if E_1 and E_2 are disjoint then

$$\int_{E_1\cup E_2} \phi = \int_{E_1} \phi + \int_{E_2} \phi$$

1) Since f is a bounded measurable function on E.

 cf is also bounded measurable function on E.

 cf is Lebesgue integrable on E.

Now, if $c = 0$, the result holds trivially.

For $c \ne 0$,

Case I: $c > 0$

Here

$$\int_E cf = \inf_{\substack{c\psi \geq cf \\ c\psi - simple}} \int_E c\psi$$

$$= c \inf_{\substack{\psi \geq f \\ \psi - simple}} \int_E \psi$$

$$= c\int_E f$$

Case II) If $c < 0$

$$\int_E cf = \inf_{\substack{c\psi \geq cf \\ c\psi - simple}} \int_E c\psi$$

$$= c \sup_{\substack{\psi \leq f \\ \psi - simple}} \int_E \psi$$

$$= c\int_E f.$$

2) Since f and g are bounded measurable functions on E. Therefore $f + g$ is integrable over E.

Let ϕ_1 and ϕ_2 be any two simple functions such that $\phi_1 \leq f$ and $\phi_2 \leq g$ then $\phi_1 + \phi_2$ is simple such that

$$\phi_1 + \phi_2 \leq f + g$$

$$\Rightarrow \int_E (f+g) \geq \int_E (\phi_1 + \phi_2) = \int_E \phi_1 + \int_E \phi_2$$

$$\Rightarrow \int_E (f+g) \geq \sup_{\substack{\phi_1 \leq f \\ \phi_1 - simple}} \int_E \phi_1 + \sup_{\substack{\phi_2 \leq g \\ \phi_2 - simple}} \int_E \phi_2$$

$$\Rightarrow \int_E (f+g) \geq \int_E f + \int_E g \qquad \ldots (1)$$

Let ψ_1 and ψ_2 be any two simple functions such that $\psi_1 \geq f$ and $\psi_2 \geq g$ then $\psi_1 + \psi_2$ is simple such that

$$\psi_1 + \psi_2 \geq f + g$$

$$\Rightarrow \int_E (f+g) \leq \int_E (\psi_1 + \psi_2)$$

$$= \int_E \psi_1 + \int_E \psi_2$$

$$\Rightarrow \int_E (f+g) \leq \inf_{\substack{\psi_1 \geq f \\ \psi_1 - simple}} \int_E \psi_1 + \inf_{\substack{\psi_2 \geq g \\ \psi_2 - simple}} \int_E \psi_2$$

$$\Rightarrow \int_E (f+g) \leq \int_E f + \int_E g \qquad \ldots (2)$$

$\therefore$ From (1) and (2),

$$\int_E (f+g) = \int_E f + \int_E g$$

3) Since $f = g$ $a.e.$ on E, therefore, $f - g = 0$ $a.e.$ on E.

Let ϕ and ψ be simple functions such that

$$\phi \leq f - g \leq \psi \Rightarrow \phi \leq 0 \ a.e. \text{ on } E, \psi \geq 0 \ a.e. \text{ on } E$$

$$\Rightarrow \int_E \phi \leq 0 \text{ and } \int_E \psi \geq 0$$

$$□ \sup_{\substack{\phi \le f-g \\ \phi-simple}} \int_E \phi \le 0 \text{ and } \inf_{\substack{\psi \ge f-g \\ \psi-simple}} \int_E \psi \ge 0$$

$$□ \int_E (f-g) \le 0 \text{ and } \int_E (f-g) \ge 0$$

$$□ \int_E (f-g) = 0$$

$$□ \int_E f - \int_E g = 0$$

$$□ \int_E f = \int_E g \,.$$

The converse of this part is not true.

e.g., If $f: [-1,1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 2 \text{ if } x \le 0 \\ 0 \text{ if } x > 0 \end{cases}$$

and $g: [-1,1] \to \mathbb{R}$, such that $g(x) = 1, \forall x \in [-1,1]$, then

$$\int_{-1}^{1} f\, dx = \int_{-1}^{1} g\, dx = 2 \text{ but } f \ne g \; a.e.$$

4) Since $f \le g \; a.e.$ on E

□ $g - f \ge 0 \; a.e.$ on E.

Let ψ be a simple function such that $\psi \ge g - f$ then

$$\psi \ge 0 \; a.e. \text{ on } E$$

$$□ \int_E \psi \ge 0$$

$$□ \inf_{\substack{\psi \ge g-f \\ \psi-simple}} \int_E \psi \ge 0$$

$$□ \int_E (g-f) \ge 0 \;□ \int_E g - \int_E f \ge 0 \;□ \int_E g \ge \int_E f.$$

5) The function $|f|$ is measurable and bounded on E.

□ $|f|$ is integrable on E.

Now

$$-|f| \le f \le |f| \text{ on } E$$

$$□ -\int_E |f| \le \int_E f \le \int_E |f| \text{ on } E$$

$$\Rightarrow \left| \int_E f \right| \le \int_E |f|$$

6) Let ϕ and ψ be simple functions such that $\phi \le f \le \psi$ then $\phi \le \beta$ and $\alpha \le \psi$

$$\Rightarrow \int_E \phi \le \int_E \beta \text{ and } \int_E \alpha \le \int_E \psi$$

$$\sup_{\substack{\phi \le f \\ \phi-simple}} \int_E \phi \le \int_E \beta \text{ and } \int_E \alpha \le \inf_{\substack{\psi \ge f \\ \psi-simple}} \int_E \psi$$

$$□ \int_E f \le \beta\, m(E) \text{ and } \alpha\, m(E) \le \int_E f$$

$$□ \alpha\, m(E) \le \int_E f \le \beta\, m(E)$$

7) Both $f \cdot \chi_A$ and $f \cdot \chi_B$ are bounded measurable function on E. Since A and B are disjoint,

$$\therefore\ f \cdot \chi_{A\cup B} = f \cdot \chi_A + f \cdot \chi_B$$

Now, for any measurable subset E_1 of E,

$$\int_{E_1} f = \int_E f\ \chi_{E_1}$$

Therefore, bythe linearity of integration,

$$\int_{A\cup B} f = \int_E f\ \chi_{A\cup B} = \int_E f\ \chi_A + \int_E f\ \chi_B = \int_A f + \int_B f.$$

This completes the proof.

Lemma 4.6: Let $\{\phi_n\}$ and $\{\psi_n\}$ be the sequences of measurable functions, each of which is integrable over E such that $\{\phi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property that $\phi_n \leq f \leq \psi_n$ on E for all n.

If

$$\lim_{n\to\infty} \int_E [\psi_n - \phi_n] = 0$$

Then

1) $\{\phi_n\} \to f$ pointwise $a.e.$ on E.

2) $\{\psi_n\} \to f$ pointwise $a.e.$ on E.

3) f is intergrable over E.

4) $\lim_{n\to\infty} \int_E \phi_n = \int_E f$

5) $\lim_{n\to\infty} \int_E \psi_n = \int_E f$

Proof: For x in E, define

$$\phi^*(x) = \lim_{n\to\infty} \phi_n(x) \text{ and } \psi^*(x) = \lim_{n\to\infty} \psi_n(x)$$

$$\phi_n \leq \phi^* \leq f \leq \psi^* \leq \psi_n \text{ on } E\ \forall n \qquad \dots (1)$$

$$0 \leq \int_E (\psi^* - \phi^*) \leq \int_E (\psi_n - \phi_n)\ \ \exists n$$

$$\square\ 0 \leq \int_E (\psi^* - \phi^*) \leq \lim_{n\to\infty} \int_E (\psi_n - \phi_n) = 0$$

Since $\psi^* - \phi^*$ is anon-negativemeasurable function and

$$\int_E (\psi^* - \phi^*) = 0 \therefore\ \psi^* = \phi^*\ a.e. \text{ in } E \text{ but } \phi^* \leq f \leq \psi^* \text{ on } E$$

$$\Rightarrow \phi_n \to f \text{ and } \psi_n \to f \text{ pointwise } a.e. \text{ on } E$$

$$\therefore f \text{ is measurable}$$

$0 \leq f - \phi_1 \leq \psi_1 - \phi_1$ on E and ψ_1 and ϕ_1 are integrable over E, f is integrable over E.

From (1), we have

$$0 \leq \int_E \psi_n - \int_E f = \int_E (\psi_n - f) \leq \int_E (\psi_n - \phi_n)$$

And

$$0 \leq \int_E f - \int_E \phi_n = \int_E (f - \phi_n) \leq \int_E (\psi_n - \phi_n)$$

$$\lim_{n\to\infty} \int_E \phi_n = \int_E f = \lim_{n\to\infty} \int_E \psi_n.$$

This completes the proof.

Theorem 4.7: Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if it is measurable.

Proof: Simple Approximation Lemma says "Let f be a measurable function on X that is bounded on X, $i.e.$ there exists $M \geq 0$ for which $|f| \leq M$ on X. Then for each $\epsilon > 0$, there are simple functions ϕ_ϵ and ψ_ϵ defined on X

1) $\phi_\epsilon \leq f \leq \psi_\epsilon$
2) $0 \leq \psi_\epsilon - \phi_\epsilon < \epsilon$ on X"

Let n be a natural number. By the Simple Approximation Lemma, with $\epsilon = \frac{1}{n}$, there are two simple functions ϕ_n and ψ_n defined on E for which $\phi_n \leq f \leq \psi_n$ on E and $0 \leq \psi_n - \phi_n < \frac{1}{n}$ on E.

$$0 \leq \int_E \psi_n - \int_E \phi_n = \int_E [\psi_n - \phi_n] \leq \frac{1}{n} m(E)$$

$$0 \leq \inf_{\substack{\psi \geq f \\ \psi - simple}} \int_E \psi - \sup_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi \leq \int_E \psi_n - \int_E \phi_n \leq \frac{1}{n} m(E)$$

$$\Rightarrow \inf_{\substack{\psi \geq f \\ \psi - simple}} \int_E \psi = \sup_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$$

$$\Rightarrow \mathcal{L}\overline{\int_E} f = \mathcal{L}\underline{\int_E} f$$

$\Rightarrow f$ is integrable over E.

Conversely, suppose f is integrable. There are sequences of simple function $\{\phi_n\}$ and $\{\psi_n\}$ for which $\phi_n \leq f \leq \psi_n$ on E for all n and

$$\lim_{n \to \infty} \int_E (\psi_n - \phi_n) = 0$$

Using monotonicity of integration and by possibly replacing ϕ_n by $\max_{1 \leq i \leq n} \phi_i$ and ψ_n by $\min_{1 \leq i \leq n} \psi_i$, we may suppose that $\{\phi_n\}$ is increasing and $\{\psi_n\}$ is decreasing. By preceding lemma, $\phi_n \to f$ pointwise $a.e.$ on E. Therefore f is measurable since it is the pointwise limit $a.e.$ of a sequence of measurable functions.

This completes the proof.

Theorem 4.8 (Bounded Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure. Let $|f_n(x)| \leq M, \forall x \in E$ and $\forall n \in \mathbb{N}$ for some $0 < M \in \mathbb{R}$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$ then

$$\int_E f = \lim_{n\to\infty} f_n$$

$$i.e. \int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_E f_n$$

Proof: Since f_n is a measurable function on E for all $n \in \mathbb{N}$, therefore $f = \lim_{n\to\infty} f_n$ is also a measurable function on E.

Since $|f_n(x)| \leq M \ \forall n \in \mathbb{N}, \forall x \in E \ \therefore \ |f(x)| \leq M, \forall x \in E$.

Thus f is a bounded measurable function on E.

$\Rightarrow f$ is Lebesgue integrable over E.

Therefore, for given $\frac{\epsilon}{2[m(E)+1]} > 0$ and $\delta = \frac{\epsilon}{4M} > 0, \exists$ a measurable set $A \subseteq E$ with $m(A) < \delta = \frac{\epsilon}{4M}$ and a positive integer N such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2[m(E)+1]} \quad \forall n \geq N, \forall x \in E - A \qquad \dots (1)$$

$$\Rightarrow \left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right|$$

$$= \left| \int_E (f_n - f) + \int_{E-A} (f_n - f) \right|$$

$$\leq \left| \int_E (f_n - f) \right| + \left| \int_{E-A} (f_n - f) \right|$$

$$\leq \int_E |f_n - f| + \int_{E-A} |f_n - f|$$

$$\leq \int_E |f_n - f| + \int_{E\sim A} \frac{\epsilon}{2[m(E)+1]}$$

$$\leq \int_A [|f_n| + |f|] + \frac{\epsilon}{2[m(E)+1]} m(E-A) \quad \forall n \geq N$$

$$\leq \int_A (M+M) + \frac{\epsilon}{2[m(E)+1]} m(E) \because E - A \subseteq E$$

$$\leq 2M\, m(A) + \frac{\epsilon}{2}$$

$$< 2M \frac{\epsilon}{4m} + \frac{\epsilon}{2} = \epsilon$$

$$i.e. \left| \int_E f_n - \int_E f \right| < \epsilon \; \forall n \geq N$$

$$\square \lim_{n\to\infty} \int_E f_n = \int_E f.$$

This completes the proof.

Summary

- For a simple function ϕ defined on a set of finite measure E, we define the integral of ϕ over E by $\int_E \phi = \sum_{i=1}^{n} a_i\, m(E_i)$ where ϕ has the canonical representation given as $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$.
- Let ϕ and ψ be simple functions defined on a set of finite measure E. Then for any α and β, (1) $\int_E (\alpha\phi + \beta\psi) = \alpha \int_E \phi + \beta \int_E \psi$ (2) If $\phi \leq \psi$ on E then $\int_E \phi \leq \int_E \psi$.
- A bounded measurable function f defined on a measurable set E of finite measure is said to be Lebesgue Integrable if $\mathcal{L} \underline{\int_E} f = \mathcal{L} \overline{\int_E} f$ and common value is denoted by $\mathcal{L} \int_E f$ or $\int_E f$.
- Let f be a bounded function defined on the closed bounded interval $[a,b]$. If f is Riemann integrable over $[a,b]$, then it is Lebesgue integrable over $[a,b]$ and the two integrals are equal.
- Lebesgue Integrable functions may not be Riemann Integrable.
- Let f and g be bounded measurable functions defined on a set E of finite measure. Then

  1) $\int_E cf = c \int_E f, c \in \mathbb{R}$

  2) $\int_E (f+g) = \int_E f + \int_E g$

  3) If $f = g$ $a.e.$ on E then $\int_E f = \int_E g$

  4) If $f \leq g$ $a.e.$ on E then $\int_E f \leq \int_E g$

  5) $\left| \int_E f \right| \leq \int_E |f|$

  6) If $\alpha \leq f \leq \beta$ on E then $\alpha\, m(E) \leq \int_E f \leq \beta m(E)$

  7) If A and B are disjoint measurable subsets of E then $\int_{A\cup B} f = \int_A f + \int_B f$

- Let $\{\phi_n\}$ and $\{\psi_n\}$ be the sequences of measurable functions, each of which is integrable over E such that $\{\phi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property that $\phi_n \le f \le \psi_n$ on E for all n. If $\lim_{n\to\infty}\int_E[\psi_n - \phi_n] = 0$. Then

  1) $\{\phi_n\} \to f$ pointwise $a.e.$ on E.
  2) $\{\psi_n\} \to f$ pointwise $a.e.$ on E.
  3) f is intergrable over E.
  4) $\lim_{n\to\infty}\int_E \phi_n = \int_E f$
  5) $\lim_{n\to\infty}\int_E \psi_n = \int_E f$

- Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if it is measurable.

Keywords

Lebesgue Integral of Simple Functions: For a simple function ϕ defined on a set of finite measure E, we define the integral of ϕ over E by

$$\int_E \phi = \sum_{i=1}^{n} a_i\, m(E_i)$$

Whereϕ has the canonical representation given as $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$

Lower and Upper Lebesgue Integral: We define $\sup_{\substack{\phi\le f\\ \phi-simple}} \int_E \phi$ as lower Lebesgue integral of f over E and denote it by $\mathcal{L}\underline{\int_E} f$ and $\inf_{\substack{\psi\ge f\\ \psi-simple}} \int_E \psi$ as upper Lebesgue Integral of f over E and denote it by $\mathcal{L}\overline{\int_E} f$.

Lebesgue Integrable Function: A bounded measurable function f defined on a measurable set E of finite measure is said to be Lebesgue Integrable if $\mathcal{L}\underline{\int_E} f = \mathcal{L}\overline{\int_E} f$ and common value is denoted by $\mathcal{L}\int_E f$ or $\int_E f$.

Bounded Convergence Theorem: Let $\{f_n\}$be a sequence of measurable functions defined on a set E of finite measure. Let $|f_n(x)| \le M, \forall x \in E$ and $\forall n \in \mathbb{N}$ for some $0 < M \in \mathbb{R}$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$ then $\int_E f = \lim_{n\to\infty} f_n$ $i.e.,$ $\int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty}\int_E f_n$.

Self Assessment

1)If $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, where $E_i's$ are disjoint measurable sets, then

A. $\int \phi = \sum_{i=1}^{n} m(E_i)$

B. $\int \phi = a_1 \sum_{i=1}^{n} m(E_i)$

C. $\int \phi = \sum_{i=1}^{n} a_i m(E_i)$

D. none of these

2) If ϕ is a simple function which vanishes outside a set of finite measureand E_1 and E_2 are disjoint measurable sets then $\int_{E_1\cup E_2} \phi > \int_{E_1} \phi + \int_{E_2} \phi$.

A. True

B. False

3) If φ is zero almost everywhere then $\int \varphi$ need not be zero.

A. True

B. False

4)If φ and ψ are simple functions which vanish outside a set of finite measuresuch that

$\phi \geq \psi \ a.e.$, then $\int \phi \geq \int \psi$.

A. True

B. False

5) The lower Lebesgue integral of bounded measurable function f over E is defined as

$A. \ \sup_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$

$B. \ \sup_{\substack{\phi \leq f \\ \phi - step}} \int_E \phi$

$C. \ \inf_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$

D. None of these

6) The upper Lebesgue integral bounded measurable function f over E is defined as

$A. \ \sup_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$

$B. \ \sup_{\substack{\phi \leq f \\ \phi - step}} \int_E \phi$

$C. \ \inf_{\substack{\phi \leq f \\ \phi - simple}} \int_E \phi$

D. None of these

7) Every Lebesgue integrable function is Riemann integrable.

A. True

B. False

8)Every Riemann integrable function is Lebesgue integrable.

A. True

B. False

9) Let f be an integrable function, then

$A. \ \left|\int f\right| = \int |f|$

B. $|\int f| \geq \int |f|$

C. $\left|\int f\right| \leq \int |f|$

D. None of these

10) $\int_E 1 =$

A. 0

B. 1

C. m(E)

D. None of these

11) $\int_E cf < c\int_E f$, where $c \in \mathbb{R}^-$ and f is bounded measurable functions defined on a set E of finite measure.

A. True

B. False

12)Consider the following statements:

(I)Let f and g be bounded measurable functions defined on a set E of finite measure.

$$\text{If } f = g \ a.e. \text{ on } E \text{ then } \int_E f = \int_E g.$$

(II)Let f and g be bounded measurable functions defined on a set E of finite measure.

$$\text{If } \int_E f = \int_E g \text{ then } f = g \ a.e. \ on \ E.$$

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. d)both (I) and (II) are incorrect

13)Let f be bounded measurable function defined on measurable set E. Then for any measurable subset E_1 of E, $\int_{E_1} f = \int_E f\chi_{E_1}$

A. True

B. False

14)Let $E = E_1 \cup E_2$ where E_1 and E_2 are measurable and disjoint and f is bounded measurable function on E. Then $\int_E f = \int_{E_1} f + \int_{E_2} f$.

A. True

B. False

15)Let $m(E) < \infty$ and f_n defined on E is measurable for each n such that for $0 < M \in \mathbb{R}$, $|f_n(x)| \leq M$ for all x and all n. If $f_n \to f$ as $n \to \infty$ then

A. $\int_E f < \lim_{n\to\infty} \int_E f_n$

B. $\int_E f = \lim_{n\to\infty} \int_E f_n$

$C. \int_E f > \lim_{n \to \infty} \int_E f_n$

D. None of these

Answers for Self Assessment

1.	C	2.	B	3.	B	4.	A	5.	A
6.	D	7.	B	8.	A	9.	C	10.	C
11.	B	12.	A	13.	A	14.	A	15.	B

Review Questions

1) Every Lebesgue integrable function is Riemann integrable. Prove or disprove this statement.

2) Prove the linearity property of integration for bounded measurable function on a set of finite measure.

3) Prove the monotonicity property of integration for bounded measurable function on a set of finite measure.

4)Let $\{f_n\}$be a sequence of measurable functions defined on a set E of finite measure. Let $|f_n(x)| \leq M, \forall x \in E$ and $\forall n \in \mathbb{N}$ for some $0 < M \in \mathbb{R}$. If $f_n \to f$ $a.e.$ on E, then $\int_E f = \lim_{n \to \infty} f_n$.

5) Bounded convergence theorem may not hold in case of Riemann integration. Give an example in support of this statement.

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

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Unit 05: The Lebesgue Integral of Non-negative Measurable Functions

CONTENTS

Objectives

After studying this unit, students will be able to:

- determine the Lebesgue integral of non-negative functions
- define monotonicity property of Lebesgue integral of non-negative functions
- explain linearity property of Lebesgue integral of non-negative functions
- demonstrate Fatou's lemma
- explain monotone convergence theorem

Introduction

In the previous unit, we have discussed the Lebesgue integral of bounded functions over a set of finite measure. Here, we will discuss Lebesgue integral of non-negative functions in terms of bounded measurable functions.

5.1 The Lebesgue integral of non-negative functions

Definition: For f a non-negative measurable function on E, we define the integral of f over E by

$$\int_E f = \sup\{\int_E h : 0 \le h \le f, h \text{ is bounded measurable function such that } h$$

$$\text{vanishes outside a set of finite measure i.e. } m\{x: h(x) \neq 0\}\}.$$

Theorem 5.1: Let f and g be non-negative measurable functions defined on a measurable set E. Then

(i) $\int_E cf = c\int_E f, c > 0$

(ii) $\int_E (f+g) = \int_E f + \int_E g$

(iii) If $f \le g$ on E then $\int_E f \le \int_E g$

(iv) If A and B are measurable subsets of E such that $A \supseteq B$ then $\int_A f \ge \int_B g$.

Proof: (i)

$$\int_E cf = \sup_{0 \le ch \le cf} \int_E cf,\ c > 0$$

where ch is a bounded measurable function which vanishes outside a set of finite measure.

$$\therefore \int_E cf = \sup_{0 \le h \le f} c \int_E h = c \sup_{0 \le h \le f} \int_E h = c \int_E f$$

$ii)$ Let h, k be bounded measurable functions such that $0 \le h \le f, 0 \le k \le g$ and both vanishes outside a set of finite measure.

Then $h + k$ is a bounded measurable function such that $0 \le h + k \le f + g$ and $h + k$ vanishes outside a set of finite measure.

$$\therefore \int_E (h + k) \le \int_E (f + g)$$

$$\Rightarrow \int_E h + \int_E k \le \int_E (f + g)$$

$$\Rightarrow \sup_{0 \le h \le f} \int_E h + \sup_{0 \le k \le g} \int_E k \le \int_E (f + g)$$

$$\Rightarrow \int_E f + \int_E g \le \int_E (f + g) \qquad \dots (1)$$

Let l be a bounded measurable function which vanishes outside a set of finite measure and $0 \le l \le f + g$.

Let $h(x) = \min\{f(x), l(x)\}, k(x) = l(x) - h(x)$. Then

$a)$ $h(x) \le f(x)$

$b)$ $k(x) = l(x) - h(x)$

$$= \begin{cases} l(x) - f(x) \text{ if } h(x) = f(x) \\ l(x) - l(x) \text{ if } h(x) = l(x) \end{cases}$$

$$= \begin{cases} l(x) - f(x) \text{ if } h(x) = f(x) \\ 0 \qquad\qquad \text{ if } h(x) = l(x) \end{cases}$$

$\le g(x) [\because g \ge 0, l \le f + g]$.

$c)$ $|h(x)| = \begin{cases} |f(x)| & if\ h(x) \le l(x) \\ |l(x)| & if\ l(x) \le f(x) \end{cases} \le |l(x)|$

$d)$ $|k(x)| = |l(x) - h(x)|$

$$= \begin{cases} |l(x) - f(x)|\ ;\ f(x) \le l(x) \\ |l(x) - l(x)|\ ;\ l(x) \le f(x) \end{cases}$$

$$= \begin{cases} |l(x) - f(x)|\ ;\ f(x) \le l(x) \\ 0 \qquad\qquad\quad ;\ l(x) \le f(x) \end{cases}$$

$$= \begin{cases} l(x) - f(x)\ ;\ f(x) \le l(x) \\ 0 \qquad\qquad ;\ l(x) \le f(x) \end{cases} \le |l(x)|$$

From (c) and (d), we get $h(x)$ and $k(x)$ are bounded measurable function.

$$\square \int_E l = \int_E (h + k) = \int_E h + \int_E k \le \int_E f + \int_E g$$

$$\square \sup_{0 \le l \le f + g} \int_E l \le \int_E f + \int_E g$$

$$\Rightarrow \int_E (f + g) \le \int_E f + \int_E g \qquad \dots (2)$$

$iii)$ Let h be a bounded measurable function such that $0 \le h \le f$ and h vanishes outside a set of finite measure.

Now $h \le g$ as $f \le g$

$$\square \int_E h \le \int_E g$$

□ $\sup_{0\le h\le f} \int_E h \le \int_E g$

$\Rightarrow \int_E f \le \int_E g$.

$iv)$ Since $A \supseteq B$, therefore, $\chi_A f \ge \chi_B f$

$$\int_A f = \int_E \chi_A f \ge \int_E \chi_B f = \int_B f$$

□ $\int_A f \ge \int_B f$.

This completes the proof.

Theorem 5.2: Show that if f is a non-negative measurable function on E then $f = 0\ a.e.$ on E if and only if $\int_E f = 0$.

Proof: Suppose $f = 0\ a.e.$ on E. Let ϕ be a simple function and h a bounded measurable function which $0 \le \phi \le h < f$ on E. Then $\phi = 0\ a.e.$ on E

$$\square \int_E \phi = 0$$

$$\square \sup_{0\le\phi\le h} \int_E \phi = 0$$

$$\square \int_E h = 0$$

$$\square \sup_{0\le h\le f} \int_E h = 0$$

$$\square \int_E f = 0.$$

Conversely, suppose $\int_E f = 0$.

If $E_n = \left\{x \in E: f(x) > \frac{1}{n}\right\}$ then $0 = \int_E f \ge \int_E \frac{1}{n}\chi_{E_n} = \frac{1}{n}\ m(E_n) \Rightarrow m(E_n) = 0$

but $\{x: f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n \therefore m\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$

$i.e. m\{x: f(x) > 0\} = 0$ □ $f = 0\ a.e.$ on E.

This completes the proof.

Theorem 5.3 (Fatou's Lemma)

Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. If $f_n \to f\ a.e.$ on E then

$$\int_E f \le \liminf \int_E f_n$$

Proof: Since the integral over the set of measure zero is zero. Therefore, we may assume that $f_n \to f$ everywhere on E. Let h be a bounded measurable function such that $0 \le h \le f$ and h vanishes outside a set of finite measure. Let

$$E_1 = \{x \in E: h(x) \ne 0\} \Rightarrow m(E_1) < \infty \text{ and } h(x) = 0, \square\ x \in E - E_1$$

For every $n \in \mathbb{N}$, define $h_n(x) = \min\big(h(x), f_n(x)\big)$, then h_n is bounded measurable function bounded by the bounds of h and vanishes outside E_1.

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min\{h(x), f_n(x)\} = h(x)$$

Therefore by bounded convergence theorem,

$$\lim_{n\to\infty} \int_{E_1} h_n(x) = \int_{E_1} h$$

$$\square \int_E h = \int_{E_1} h$$

$$= \lim_{n\to\infty} \int_{E_1} h_n$$

$$= \lim_{n\to\infty} \int_{E} h_n$$

$$= \liminf \int_E h_n$$

$$\leq \liminf \int_E f_n$$

$$\square \sup_{0\leq h\leq f} \int_E h \leq \liminf \int_E f_n$$

$$\square \int_E f \leq \liminf \int_E f_n .$$

This completes the proof.

Notes: The inequality in Fatou’s lemma may be strict.

Example 5.4: $\{f_n\}$ is non-negative measurable functions on $\mathbb{R}$, such that

$$f_n = \chi_{E_n}, E_n = [n, n+1].$$

Show that strict inequality holds in Fatou’s lemma.

Solution: $f_n = \chi_{E_n}, E_n = [n, n+1]$

Then

$$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

$$\square \int_{\mathbb{R}} f = 0$$

and

$$\int_{\mathbb{R}} f_n = \int_{E_n} f_n + \int_{\mathbb{R}-E_n} f_n$$

$$= \int_{E_n} 1 + \int_{\mathbb{R}-E_n} 0$$

$$= 1 \cdot m(E_n) = 1$$

Thus

$$\int_{\mathbb{R}} f < \liminf \int_{\mathbb{R}} f_n.$$

Theorem 5.5 (Monotone Convergence Theorem)

Let $\{f_n\}$ be a monotonically increasing sequence of non-negative measurable functions on E. If $f_n \to f$ $a.e.$ on E then

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof: By Fatou’s Lemma, we have

$$\int_E f \leq \liminf \int_E f_n \qquad \ldots (1)$$

Since $\{f_n\}$is increasing and $\lim f_n = f$ $a.e.$

$$\square\ f_n \leq f \ \forall n \ a.e. \text{ on } E$$

$$\square \int_E f_n \leq \int_E f$$

$$\square \limsup \int_E f_n \leq \int_E f \qquad \ldots (2)$$

From (1) and (2), we get

$$\int_E f \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E f$$

$$\square \liminf \int_E f_n = \limsup \int_E f_n = \int_E f$$

$$\square \lim \int_E f_n = \int_E f.$$

This completes the proof.

Cor1: Let $\{u_n\}$ be a sequence of non-negative measurable functions on E. If

$$f = \sum_{n=1}^{\infty} u_n \ a.e. \text{ on } E$$

Then

$$\int_E f = \sum_{n=1}^{\infty} \int_E u_n$$

$$i.e. \int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_E u_n.$$

Proof: Let $f_n = \sum_{i=1}^{n} u_i$. Then $\{f_n\}$ is increasing sequence of non-negative measurable functions such that $f_n \to f \ a.e.$ Therefore by monotone convergence theorem, we have

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

$$= \lim_{n\to\infty} \int_E \sum_{i=1}^{n} u_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i$$

$$= \sum_{i=1}^{\infty} \int_E u_i$$

$$= \sum_{n=1}^{\infty} \int_E u_n.$$

This completes the proof.

Cor 2: Let f be a non-negative measurable function on E and

$$E = \bigcup_{n=1}^{\infty} E_n,$$

where $E_n's$ are pairwise disjoint measurable sets. Then

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

Proof: Let $u_n = f \cdot \chi_{E_n}$ then $\{u_n\}$ is a sequence of non-negative measurable functions on E. Also

$$f \cdot \chi_E = f \cdot \chi_{\bigcup_{n=1}^{\infty} E_n}$$

$$= f \cdot (\chi_{E_1} + \chi_{E_2} + \cdots)$$

$$= \sum_{n=1}^{\infty} \int_{E_n} f \cdot \chi_{E_n}$$

$$= \sum_{n=1}^{\infty} u_n$$

$$\Rightarrow \int_E f = \int f \cdot \chi_E$$

$$= \int \sum_{n=1}^{\infty} u_n$$

$$= \sum_{n=1}^{\infty} \int u_n$$

$$= \sum_{n=1}^{\infty} \int f \cdot \chi_{E_n}$$

$$= \sum_{n=1}^{\infty} \int_{E_n} f.$$

This completes the proof.

Definition: A non-negative measurable function f defined on a measurable set E is said to be integrable over E provided $\int_E f < \infty$.

Theorem 5.6: Let f and g be two non-negative measurable functions on E. If f is integrable over E and $g \leq f$ on E then g is also integrable over E and

$$\int_E (f - g) = \int_E f - \int_E g$$

Proof: Since $g \leq f$ on E

$$\Rightarrow \int_E g \leq \int_E f < \infty \Rightarrow \int_E g < \infty$$

$\Rightarrow$ g is integrable over E

Now $f - g$ and g are non-negative measurable functions.

$$\Rightarrow \int_E f = \int_E [(f - g) + g]$$

$$\Rightarrow \int_E f = \int_E (f - g) + \int_E g$$

$$\Rightarrow \int_E (f - g) = \int_E f - \int_E g.$$

This completes the proof.

Summary

- Let f and g be non-negative measurable functions defined on a measurable set E. Then
  (i) $\int_E cf = c\int_E f, c > 0$
  (ii) $\int_E (f + g) = \int_E f + \int_E g$
  (iii) If $f \leq g$ on E then $\int_E f \leq \int_E g$
  (iv) If A and B are measurable subsets of E such that $A \supseteq B$ then $\int_A f \geq \int_B g$.
- If f is a non-negative measurable function on E then $f = 0$ $a.e.$ on E if and only if $\int_E f = 0$.
- Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. If $f_n \to f$ $a.e.$ on E then

$$\int_E f \leq \liminf \int_E f_n.$$

- The inequality in Fatou's lemma may be strict.

- Let $\{f_n\}$ be a monotonically increasing sequence of non-negative measurable functions on E. If $f_n \to f$ $a.e.$ on E then

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

- Let $\{u_n\}$ be a sequence of non-negative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n$ $a.e.$ on E, then

$$\int_E f = \sum_{n=1}^{\infty}\int_E u_n \ i.e. \int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\int_E u_n.$$

- Let f be a non-negative measurable function on E and $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n's$ are pairwise disjoint measurable sets. Then

$$\int_E f = \sum_{n=1}^{\infty}\int_{E_n} f.$$

- A non-negative measurable function f defined on a measurable set E is said to be integrable over E provided $\int_E f < \infty$.
- Let f and g be two non-negative measurable functions on E. If f is integrable over E and $g \le f$ on E then g is also integrable over E and $\int_E (f - g) = \int_E f - \int_E g$.

Keywords

Lebesgue integral of non-negative function: For f a non-negative measurable function on E, we define the integral of f over E by

$$\int_E f = \sup\{\int_E h : 0 \le h \le f, h \text{ is bounded measurable function such that } h \text{ vanishes outside a set of finite measure i.e. } m\{x: h(x) \neq 0\}\}.$$

Fatou's Lemma: Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. If $f_n \to f$ $a.e.$ on E then

$$\int_E f \le \liminf \int_E f_n.$$

Monotone Convergence Theorem: Let $\{f_n\}$ be a monotonically increasing sequence of non-negative measurable functions on E. If $f_n \to f$ $a.e.$ on E then

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

Self Assessment

1) Let f be a non-negative measurable function defined on a set E and h is a bounded non-negative measurable function such that $m\{x: h(x) \neq 0\} < \infty$. Then $\int_E f =$

A. $\sup_{h \le f} \int_E h$

B. $\inf_{h \le f} \int_E h$

C. $\int_E h$

D. None of these

2) Consider the following statements:

(I) Let f be non-negative measurable functions defined on a set E. If $f = 0$ $a.e.$ on E then $\int_E f = 0$

(II) Let f be non-negative measurable functions defined on a set E. If $\int_E f = 0$ then $f = 0$ $a.e.$ on E.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

3) Let f be non-negative function defined on a measurable set E and A, B are susets of E such that A⊆B then $\int_E \chi_A f = \int_E \chi_B f$.

A. True

B. False

4) Let f be a non-negative measurable function defined on measurable set E. Then $\int_E cf = c\int_E f$, $c > 0$.

A. True

B. False

5) Let $\{f_n\}$ be a sequence of non-negative measurable functions such that $\{f_n\} - f\, a.e.$ on E. Then

A. $\int_E f = \lim\inf \int_E f_n$

B. $\int_E f \geq \lim\inf \int_E f_n$

C. $\int_E f \leq \lim\inf \int_E f_n$

D. None of these

6) Let $\{f_n\}$ be a monotonically increasing sequence of non-negative measurable functions such that $\{f_n\} - f$ on E. Then

A. $\int_E f = \lim \int_E f_n$

B. $\int_E f \geq \lim\inf \int_E f_n$

C. $\int_E f \leq \lim\inf \int_E f_n$

D. None of these

7) A non-negative measurable function f defined on measurable set E is said to be integrable over E if $\int_E f < \infty$.

A. True

B. False

8) Let $\{u_n\}$ be a sequence of non-negative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n\, a.e.$ on E then

A. $\int_E f < \sum_{n=1}^{\infty} \int_E u_n$

B. $\int_E f > \sum_{n=1}^{\infty} \int_E u_n$

C. $\int_E f = \sum_{n=1}^{\infty} \int_E u_n$

D. Cannot say anything

9) Let f be non-negative measurable function on E and $E = \cup_{n=1}^{\infty} E_n$, where sets E_n are pairwise disjoint measurable sets. Then

A. $\int_E f = \int_{E_n} f$

B. $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$

C. $\int_E f < \sum_{n=1}^{\infty} \int_{E_n} f$

D. Cannot say anything

10)Let f be a measurable function over E. If there is an integrable function g such that $|f| \leq g$, then f$f$is integrable over E.

A. True

B. False

11) Strict inequality is not possible in Fatou Lemma.

A. True

B. False

12) Monotone convergence theorem also holds for every decreasing sequence.

A. True

B. False

13)Fatou Lemma may not hold if functions are not non-negative.

A. True

B. False

14)Let h be non-negative bounded measurable function such that it vanishes outside the set of finite measure. Then $m\{x: h(x) \neq 0\} < \infty$.

A. True

B. False

15)Let $E = \cup_{n=1}^{\infty} E_n$, where sets E_n are pairwise disjoint measurable sets. Then $f\chi_E = \sum_{n=1}^{\infty} f\chi_{E_n}$

A. True

B. False

Answersfor Self Assessment

1.	A	2.	C	3.	B	4.	A	5.	C
6.	A	7.	A	8.	C	9.	B	10.	A

11. B 12. B 13. A 14. A 15. A

Review Questions

1) Show with the help of an example that Fatou's lemma may not hold if f_n's are not non-negative.

2) Monotone convergence theorem may not hold for monotonically decreasing sequence of functions. Give an example in support of this statement.

3)Prove the linearity property of integration for non-negative measurable function on a set of finite measure.

4)Prove the monotonicity property of integration for non-negative measurable function on a set of finite measure.

5) Let f be a non-negative measurable function on E. If $f = 0$ $a.e.$ on$E$ then $\int_E f = 0$. Is the converse true?

Further Readings

Measure theory and integration by G DE BARRA, NewAgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

Unit 06: The General Lebesgue Integral

CONTENTS

Objectives

After studying this unit, students will be able to:

- determine the Lebesgue integral of measurable functions
- define monotonicity property of Lebesgue integral of measurable functions
- explain linearity property of Lebesgue integral of measurable functions
- demonstrate Lebesgue dominated convergence theorem
- explain generalized Lebesgue dominated convergence theorem

Introduction

In the previous unit, we have discussed the Lebesgue integral of non-negative functions. Here we discuss the general Lebesgue integral with the help of positive and negative parts of a function. Recall that

- $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.
- f^+ and f^- both are non-negative functions.
- $f = f^+ - f^-$ and $|f| = f^+ + f^-$.
- f is measurable if and only if both f^+ and f^- are measurable.

6.1 The General Lebesgue Integral

Definition: Let f be a measurable function on E. Then f is said to be Lebesgue integrable over E if

$$\int_E f^+ < \infty \text{ and } \int_E f^- < \infty \text{ and } \int_E f = \int_E f^+ - \int_E f^-$$

Definition: If E is a measurable set, f is a measurable function and $\chi_E f$ is integrable, we say that f is integrable over E and its integral is given by

$$\int_E f = \int f \cdot \chi_E \,.$$

Notes: Let f be integrable over E and C a measurable subset of E. Then

$$\int_C f = \int_E f \cdot \chi_C$$

Theorem 6.1: A measurable function f is integrable over E if and only if $|f|$ is integrable over E.

Proof: Let f be integrable over E. Then f^+ and f^- are integrable over E.

$$\Rightarrow \int_E f^+ < \infty \text{ and} \int_E f^- < \infty$$

$$\therefore \int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^- < \infty$$

$\Rightarrow |f|$ is integrable over E.

Conversely, let $|f|$ be integrable over E.

$$\Rightarrow \int_E |f| < \infty$$

$$\Rightarrow \int_E (f^+ + f^-) < \infty$$

$$\Rightarrow \int_E f^+ + \int_E f^- < \infty$$

$$\Rightarrow \int_E f^+ < \infty \text{ and} \int_E f^- < \infty$$

$\Rightarrow$ f^+ and f^- are integrable over E.

$\Rightarrow$ f is integrable over E.

This completes the proof.

Theorem 6.2: Let f and g be two integrable functions over E. Then

1) cf is integrable over E and $\int_E cf = c\int_E f$, $c \in \mathbb{R}$.
2) $f + g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$
3) If $f \le g$ $a.e.$ then $\int_E f \le \int_E g$.
4) If E_1 and E_2 are disjoint measurable subsets of E, then

$$\int_{E_1 \cup E_2} f = \int_{E_2} f + \int_{E_2} f$$

Proof: 1) We have

$$(cf)^+ = \begin{cases} cf^+; c \ge 0 \\ -cf^-; \quad c < 0 \end{cases}$$

and

$$(cf)^- = \begin{cases} cf^-; c \ge 0 \\ -cf^+; c < 0 \end{cases}$$

$$\therefore \int_E (cf)^+ = \begin{cases} \int_E cf^+; c \ge 0 \\ \int_E -cf^-; \quad c < 0 \end{cases} = \begin{cases} c\int_E f^+; c \ge 0 \\ -c\int_E f^-; \ c < 0 \end{cases} < \infty$$

and

$$\int_E (cf)^- = \begin{cases} \int_E cf^-; c \ge 0 \\ \int_E -cf^+; \quad c < 0 \end{cases} = \begin{cases} c\int_E f^-; c \ge 0 \\ -c\int_E f^+; \ c < 0 \end{cases} < \infty$$

$\therefore$ cf is integrable over E.

Now

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^-$$

$$= \begin{cases} c\int_E f^+ - c\int_E f^-; \quad c \ge 0 \\ -c\int_E f^- - (-c)\int_E f^+; \ c < 0 \end{cases}$$

$$= c\int_E f$$

2) We have $(f+g)^+ \le f^+ + g^+$ and $(f+g)^- \le f^- + g^-$

Since f and g are integrable.

□ f^+, f^-, g^+, g^- are integrable

□ $(f+g)^+$ and $(f+g)^-$ are integrable.

□ $f+g$ is integrable.

Also

$$(f+g)^+ - (f+g)^- = f + g = f^+ + g^+ - f^- - g^-$$

$$\Rightarrow (f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$□ \int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+$$

$$□ \int_E (f+g)^+ - \int_E (f+g)^- = \left(\int_E f^+ - \int_E f^-\right) + \left(\int_E g^+ - \int_E g^-\right)$$

$$□ \int_E (f+g) = \int_E f + \int_E g$$

3) Since $f \le g$ $a.e.$

□ $g - f \ge 0$ $a.e.$

□ $\int_e g - f \ge 0$ $a.e.$

□ $\int_E g = \int_E [(g-f)+f]$

$= \int_E (g-f) + \int_E f$

□ $\int_E g \ge \int_E f \Rightarrow \int_E f \le \int_E g.$

4) $\int_{E_1 \cup E_2} f = \int f \cdot \chi_{E_1 \cup E_2}$

$= \int f \cdot (\chi_{E_1} + \chi_{E_2})$

$= \int_E f\,\chi_{E_1} + \int_E f\chi_{E_2}$

$= \int_{E_1} f + \int_{E_2} f.$

This completes the proof.

Theorem 6.3: Let f be a measurable function over E. If there is an integrable function g such that $|f| \le g$, then f is integrable over E.

Proof:We have

$$f^+ \le |f| \le g \Rightarrow f^+ \le g \Rightarrow \int_E f^+ \le \int_E g < \infty \quad □ \int_E f^+ < \infty$$

$$f^- \le |f| \le g \Rightarrow f^- \le g \Rightarrow \int_E f^- \le \int_E g < \infty \quad □ \int_E f^- < \infty$$

□ f^+ and f^- are integrable over E

⇒ f is integrable over E.

This completes the proof.

Example 6.4: Let f be a measurable function and g be an integrable function such that

$$\alpha \le f \le \beta \ a.e., \alpha, \beta \in \mathbb{R}$$

Show that there exists $\nu \in \mathbb{R}$ such that

$$\alpha \le \nu \le \beta \ and \int f\,|g| = \nu \int |g|.$$

Solution: Since

$$\alpha \leq f \leq \beta \; a.e. \therefore |fg| \leq (|\alpha| + |\beta|)|g| \; a.e.$$

Since g is integrable.

 fg is integrable.

Now, $\alpha \leq f \leq \beta \; a.e.$

$\Rightarrow \alpha|g| \leq f|g| \leq \beta|g| \; a.e.$

$\Rightarrow \alpha \int |g| \leq \int f\, |g| \leq \beta \int |g|$ ... (1)

If $\int |g| = 0$ then $g = 0 \; a.e.$ and the result is trivial.

If $\int |g| \neq 0$, take $v = \frac{\int f|g|}{\int |g|}$

Then from (1), we have, $\alpha \leq v \leq \beta$ and also,

$$\int f|g| = v \int |g|$$

Example 6.5: If f is an integrable function over E, then show that f is finite valued $a.e.$ on E.

Solution: Since f is integrable over $E, \therefore |f|$ is also integrable over E.

$$\Rightarrow \int |f| < \infty \qquad \text{... (1)}$$

If possible, let $|f| = \infty$ on a set $A \subseteq E$ and $m(A) > 0$ then $|f| > n$ on A.

$$\int_E |f| \geq \int_A |f| > n\, m(A) \text{ for all } n.$$

$$\Rightarrow \int_E |f| = \infty, \text{ which contradicts (1).}$$

Hence f is finite valued $a.e.$ on E.

Example 6.6: Let f be an integrable function. Show that

$$\left|\int f\right| \leq \int |f|.$$

When does the equality occur?

Solution: Since $f \leq |f|$ and $-f \leq |f|$

$\int f \leq \int |f|$ and $\int -f \leq \int |f|$

$\int f \leq \int |f|$ and $-\int f \leq \int |f|$

$\Rightarrow \left|\int f\right| \leq \int |f|$.

Case I: $\int f \geq 0$

If $\left|\int f\right| = \int |f|$ then $\int f = \int |f|$

$\int (|f| - f) = 0$

$|f| - f = 0 \; a.e.$

$|f| = f \; a.e$

$f \geq 0 \; a.e.$

Case II: $\int f \leq 0$

If $\left|\int f\right| = \int |f|$ then $-\int f = \int |f|$

$\int (|f| + f) = 0$

$|f| + f = 0 \; a.e.$

$\Rightarrow |f| = -f \; a.e$

$\Rightarrow f \leq 0 \; a.e.$

Hence equality occurs when either $f \geq 0 \; a.e.$ or $f \leq 0 \; a.e.$

Theorem 6.7: Let f be a Lebesgue integrable function over E. Then for given $\epsilon > 0, \exists \delta > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta$, we have

$$\left|\int_A f\right| < \epsilon$$

Proof: Case I:When f is a non-negative function.

Subcase I: When f is bounded on E. Here $|f(x)| \leq M, \forall x \in E$ and for some $0 < M \in \mathbb{R}$.

For given $\epsilon > 0$, let $A \subseteq E$ with $m(A) < \delta = \frac{\epsilon}{M}$ then

$$\int_A f \leq \int_A M = M\, m(A) < M \cdot \frac{\epsilon}{M} = \epsilon \; i.e. \int_A f \; < \epsilon$$

Subcase II: When f is unbounded on E. Let $\{f_n\}$ be a sequence of functions defined by

$$f_n(x) = \begin{cases} f(x); f(x) \leq n \\ n\,; f(x) > n \end{cases}$$

Then $\{f_n\}$ is an increasing sequence of bounded functions such that $f_n \to f \; a.e.$

Therefore, by monotone convergence theorem, we have

$$\int_E f = \lim \int_E f_n$$

Therefore, for given $\epsilon > 0, \exists$ a positive integer N such that

$$\int_E f - \int_E f_N < \frac{\epsilon}{2} \Rightarrow \int_E (f - f_N) < \frac{\epsilon}{2} \qquad \dots (1)$$

Now f_N is bounded non-negativemeasurable function on E. Therefore by subcase I, for given

$\epsilon > 0, \exists \delta > 0$Such that for every measurable set $A \subseteq E$with $m(A) < \delta$, we have

$$\int_A f_N < \frac{\epsilon}{2} \qquad \dots (2)$$

Then

$$\int_A f = \int_A [(f - f_N) + f_N]$$

$$= \int_A (f - f_N) + \int_A f_N$$

$$\leq \int_E (f - f_N) + \int_A f_N$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$i.e. \int_A f < \epsilon$$

Case II) When f is any Lebesgue Integrable function.

We have, $f = f^+ - f^-, f^+ \geq 0, f^- \geq 0$

Therefore, by case I, for given $\epsilon > 0, \exists \delta_1 > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta_1$, we have

$$\int_A f^+ < \frac{\epsilon}{2} \qquad \dots (3)$$

Similarly, $\exists \delta_2 > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta_2$, have

$$\int_A f^- < \frac{\epsilon}{2} \qquad \dots (4)$$

Choose $\delta = \min(\delta_1, \delta_2)$, so that (3) and (4) hold for every set $A \subseteq E$ with $m(A) < \delta$.

Thus if $m(A) < \delta$, then we have

$$\left|\int_A f\right| \le \int_A |f|$$
$$= \int_A (f^+ + f^-)$$
$$= \int_A f^+ + \int_A f^-$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
$$i.e.\ \left|\int_A f\right| < \epsilon.$$

This completes the proof.

Example 6.8: Let f be an integrable function over E. Show that for given $\epsilon > 0$, ∃ a simple function s such that

$$\int_E |f - s| < \epsilon$$

Solution: Since f is integrable over E. Therefore f^+ and f^- are integrable over E.

$f^+ \ge 0$ and $f^- \ge 0$, ∃ increasing sequences $\{s_n\}$ and $\{t_n\}$ of non-negative simple functions such that

$$\lim_{n\to\infty} s_n = f^+, \lim_{n\to\infty} t_n = f^- \text{ for every } x \in E.$$

Therefore, by the monotone convergence theorem, we have

$$\int_E f^+ = \lim \int_E s_n, \int_E f^- = \lim \int_E t_n$$

Therefore ∃ positive integers n_1 and n_2 such that

$$\int_E (f^+ - s_{n_1}) < \frac{\epsilon}{2} \text{ and } \int_E (f^- - t_{n_2}) < \frac{\epsilon}{2}$$

Taking $s = s_{n_1} - t_{n_2}$, we have

$$\int_E |f - s| \le \int_E |f^+ - s_{n_1}| + \int_E |f^- - t_{n_2}|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

∴ $\int_E |f - s| < \epsilon.$

Theorem 6.9 (Lebesgue Dominated Convergence Theorem)

Let g be an integrable function over E and $\{f_n\}$ be a sequence of measurable functions such that

$|f_n| \le g$ on E for all n and $f_n \to f$ $a.e.$ on E. Then f is integrable over E and

$$\int_E f = \lim \int_E f_n.$$

Proof: Since $|f_n| \le g$ on E for all n and g is integrable over E, therefore, every f_n is integrable over E. Also $\lim_{n\to\infty} f_n = f$ $a.e.$ on E and $|f_n| \le g$ on E for all n.

∴ $|f| \le g$ $a.e.$ on E

∴ f is integrable over E.

Now, $|f_n| \le g$ on E for all n

⇒ $-g \le f_n \le g$ on E for all n

∴ $g + f_n \ge 0$ and $g - f_n \ge 0$ for all n.

⇒ $\{g + f_n\}$ and $\{g - f_n\}$ are sequences of non-negative measurable function on E such that

$g + f_n \to g + f$ $a.e.$ and $g - f_n \to g - f$

Therefore, by Fatou's Lemma, we have

$$\int_E (g+f) \le \liminf \int_E (g+f_n) \text{ and } \int_E (g-f) \le \liminf \int_E (g-f_n)$$

$$□ \int_E g + \int_E f \le \int_E g + \liminf \int_E f_n \text{ and } \int_E g - \int_E f \le \int_E g - \limsup \int_E f_n$$

$$□ \int_E f \le \liminf \int_E f_n \text{ and } \int_E f \ge \limsup \int_E f_n$$

$$□ \int_E f \le \liminf \int_E f_n \le \limsup \int_E f_n \le \int_E f$$

$$\Rightarrow \liminf \int_E f_n = \limsup \int_E f_n = \int_E f$$

$$□ \int_E f = \lim \int_E f_n.$$

This completes the proof.

Cor 1: Let $\{u_n\}$ be a sequence of integrable functions over E such that $\sum_{n=1}^{\infty} u_n$ converges $a.e.$ on E. Let g be an integrable function over E such that

$$\left|\sum_{i=1}^{n} u_i\right| \le g, n \in \mathbb{N}$$

Then $\sum_{n=1}^{\infty} u_n$ is integrable over E and

$$\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_E u_n.$$

Proof: Let

$$f_n = \sum_{i=1}^{n} u_i \text{ and } f = \sum_{i=1}^{\infty} u_i$$

Then $\{f_n\}$ is a sequence of integrable functions over E such that $|f_n| \le g$ on E for all n and $f_n \to f$ $a.e.$ on E. Therefore, by Lebesgue Dominated convergence theorem, we have

$$\int_E f = \lim \int_E f_n$$

$$□ \int_E \sum_{n=1}^{\infty} u_n = \lim_{n\to\infty} \int_E \sum_{i=1}^{n} u_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i$$

$$= \sum_{i=1}^{\infty} \int_E u_i$$

$$= \sum_{n=1}^{\infty} \int_E u_n.$$

This completes the proof.

Cor 2: If f is integrable over E and $\{E_i\}$ is a sequence of disjoint measurable sets such that $E = \bigcup_{i=1}^{\infty} E_i$ then

$$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f$$

Proof: Let $u_n = f \cdot \chi_{E_n}$ then

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} f \cdot \chi_{E_n} = f \sum_{n=1}^{\infty} \chi_{E_n} = f \cdot \chi_{\cup E_n} = f \cdot \chi_E$$

$$□ \sum_{n=1}^{\infty} u_n = f \cdot \chi_E$$

Now, $|u_n| = |f \cdot \chi_{E_n}| \leq |f|$ and $|f|$ is integrable over E, therefore u_n is integrable over E for all n also

$$\left|\sum_{i=1}^{n} u_i\right| \leq |f| \text{ for all } n$$

Therefore, by cor 1, we have

$$\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_E u_n$$

$$\Rightarrow \int f \cdot \chi_E = \sum_{n=1}^{\infty} \int_E f \cdot \chi_{E_n}$$

$$\Rightarrow \int_E f = \sum_{n=1}^{\infty} \int_{E_n} f = \sum_{i=1}^{\infty} \int_{E_i} f.$$

This completes the proof.

Theorem 6.10: (General Lebesgue Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to f. Suppose there is a $\{g_n\}$ of non-negative measurable functions on E that converges $a.e.$ on E to g and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g_n$ on E for all n.

If $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$ then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof: Since $|f_n| \leq g_n \ \forall n$

$\Rightarrow |f_n| \leq g \ a.e. \ \forall n$

$\Rightarrow$ each f_n is integrable.

Also $f_n \to f \ a.e.$ and $|f_n| \leq g \ \forall n$

$\Rightarrow |f| \leq g \ a.e.$

$\Rightarrow f$ is integrable.

Now $|f_n| \leq g_n \forall n$

$\Rightarrow -g_n \leq f_n \leq g_n \ \forall n$

$\Rightarrow g_n + f_n \geq 0$ and $g_n - f_n \geq 0 \ \forall n$

$\{g_n + f_n\}$ and $\{g_n - f_n\}$ are sequences of non-negative measurable functions such that

$\{g_n + f_n\} \to g + f \ a.e.$ and $\{g_n - f_n\} \to g - f \ a.e.$

Therefore, by Fatou's Lemma, we have

$$\int_E (g+f) \leq \liminf \int_E (g_n + f_n) \text{ and } \int_E (g-f) \leq \liminf \int_E (g_n - f_n)$$

$$\Rightarrow \int_E g + \int_E f \leq \liminf \int_E (g_n + f_n) \text{ and } \int_E g - \int_E f \leq \liminf \int_E (g_n - f_n)$$

$$\Rightarrow \int_E f \leq \liminf \int_E f_n \text{ and } \int_E f \geq \limsup \int_E f_n$$

$$\Rightarrow \lim \int_E f_n = \int_E f .$$

This completes the proof.

Summary

- If E is a measurable set, f is a measurable function and $\chi_E f$ is integrable, we say that f is integrable over E and its integral is given by

$$\int_E f = \int f \cdot \chi_E .$$

- Let f be integrable over E and C a measurable subset of E. Then

$$\int_C f = \int_E f \cdot \chi_C .$$

- A measurable function f is integrable over E if and only if $|f|$ is integrable over E.
- Let f and g be two integrable functions over E. Then
  1) cf is integrable over E and $\int_E cf = c\int_E f$, $c \in \mathbb{R}$.
  2) $f+g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$
  3) If $f \le g$ $a.e.$ then $\int_E f \le \int_E g$.
  4) If E_1 and E_2 are disjoint measurable subsets of E, then

$$\int_{E_1 \cup E_2} f = \int_{E_2} f + \int_{E_2} f$$

- Let f be a measurable function over E. If there is an integrable function g such that $|f| \le g$, then f is integrable over E.
- If f is an integrable function over E, then show that f is finite valued $a.e.$ on E.
- Let f be a measurable function and g be an integrable function such that $\alpha \le f \le \beta$ $a.e.$, $\alpha, \beta \in \mathbb{R}$ then there exists $\nu \in \mathbb{R}$ such that $\alpha \le \nu \le \beta$ and $\int f\,|g| = \nu \int |g|$.
- Let f be an integrable function, then $|\int f| \le \int |f|$. Equality occurs when either $f \ge 0$ $a.e.$ or $f \le 0$ $a.e.$
- Let f be a Lebesgue integrable function over E. Then for given $\epsilon > 0$, □$\delta > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta$, we have $|\int_A f| < \epsilon$
- Let f be an integrable function over E. Then for given $\epsilon > 0$, □ a simple function s such that $\int_E |f - s| < \epsilon$
- Let $\{u_n\}$ be a sequence of integrable functions over E such that $\sum_{n=1}^{\infty} u_n$ converges $a.e.$ on E. Let g be an integrable function over E such that $|\sum_{i=1}^{n} u_i| \le g, n \in \mathbb{N}$.
- If f is integrable over E and $\{E_i\}$ is a sequence of disjoint measurable sets such that $E = \cup_{i=1}^{\infty} E_i$ then

$$\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f.$$

Keywords

General Lebesgue Integral: Let f be a measurable function on E. Then f is said to be Lebesgue integrable over E if

$$\int_E f^+ < \infty \text{ and } \int_E f^- < \infty \text{ and } \int_E f = \int_E f^+ - \int_E f^- .$$

Lebesgue Dominated Convergence Theorem: Let g be an integrable function over E and $\{f_n\}$ be a sequence of measurable functions such that

$|f_n| \le g$ on E for all n and $f_n \to f$ $a.e.$ on E. Then f is integrable over E and

$$\int_E f = \lim \int_E f_n.$$

General Lebesgue Dominated Convergence Theorem: Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to f. Suppose there is a $\{g_n\}$ of non-negative measurable functions on E that converges $a.e.$ on E to g and dominates $\{f_n\}$ on E in the sense that $|f_n| \le g_n$ on E for all n. If $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$ then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Self Assessment

1) f is integrable if both non-negative functions f^+ and f^- are integrable.

A. True
B. False

2) $\int f = \int f^+ + \int f^-$

A. True
B. False

3) A measurable function f is integrable over E if, and only if $|f|$ is integrable over E.

A. True
B. False

4) $\int |f| = \int f^+ - \int f^-$

A. True
B. False

5) $\int (cf)^+ = c\int f^+, c \geq 0$

A. True
B. False

6) $\int (cf)^- = c\int f^-, c \geq 0$

A. True
B. False

7) $\int (cf)^+ = -c\int f^-, c < 0$

A. True
B. False

8) $\int (cf)^- = -c\int f^-, c < 0$

a) True

b) False

9) $\int (cf)^- = -c\int f^+, c < 0$

A. True
B. False

10) $\int (cf)^- = c\int f^+, c < 0$

A. True
B. False

11) If f is integrable function over E, then f is finite valued $a.e.$ on E.

A. True
B. False

12) Consider the following statements:

(I) Let f be an integrable function and $f \geq 0$ $a.e.$ then $\left|\int f\right| = \int |f|$.

(II) Let f be an integrable function and $f \leq 0$ $a.e.$ then $\left|\int f\right| \leq \int |f|$. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

13) Let $\{u_n\}$ be a sequence of integrable functions over E such that $\sum_{n=1}^{\infty} u_n$ converges a.e. on E. Let g be an integrable function over E such that $|\sum_{i=1}^{n} u_i| \leq g$ a.e. on E then

A. $\int_E \sum_{n=1}^{\infty} u_n \leq \sum_{n=1}^{\infty} \int_E u_n$
B. $\int_E \sum_{n=1}^{\infty} u_n \geq \sum_{n=1}^{\infty} \int_E u_n$
C. $\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_E u_n$
D. None of these

14) If f is integrable function over E and $\{E_i\}$ is a sequence of disjoint measurable sets such that $E = \bigcup_{i=1}^{\infty} E_i$, then $\int_E f = \sum_{i=1}^{\infty} f.m(E_i)$.

A. True
B. False

15) If f is integrable function over E and $\{E_i\}$ is a sequence of disjoint measurable sets such that $E = \bigcup_{i=1}^{\infty} E_i$, then $\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f$

A. True
B. False

Answers for Self Assessment

1.	A	2.	B	3.	A	4.	B	5.	A
6.	A	7.	A	8.	B	9.	A	10.	B
11.	A	12.	A	13.	C	14.	B	15.	A

Review Questions

1) Let $\{f_n\}$ be a sequence of integrable functions such that $\{f_n\} - f$ a.e. with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

2) For $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 2n, & x \in \left(\frac{1}{2n}, \frac{1}{n}\right) \\ 0, & x \in \left(0, \frac{1}{2n}\right) \cup \left(\frac{1}{n}, 1\right). \end{cases}$

Show that Fatou's lemma holds but Lebesgue Dominated Convergence theorem does not.

3) Define positive and negative parts of a function. When a measurable function f is said to be integrable over E?

4) Prove the linearity property of integration for measurable function.

5)Prove the monotonicity property of integration for measurable function.

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

Unit 07: Functions of Bounded Variation

CONTENTS

Objectives

After studying this unit, students will be able to:

- Define the functions of bounded variation
- Investigate the properties of the functions of bounded variation
- Explain Jordan decomposition theorem
- understand variation function
- solve problems on functions of bounded variation

Introduction

In this unit, we explore those functions which do not behave too erratically over an interval. The members of this special class of functions on closed and bounded intervals can be expressed as the difference of two increasing functions.

7.1 Functions of Bounded Variation

Definition: Let f be a real-valued function defined on the closed and bounded interval $[a, b]$ and let

$$P = \{a = x_0, x_1, \dots, x_n = b\}$$

be a partition of $[a, b]$.

We define

$$p = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+,$$

$$n = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^-,$$

$$t = p + n$$

$$= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

Notes: $t, p, n \geq 0$.

Further, we write $P = P_f[a,b] = \sup p$,

$$N = N_f[a,b] = \sup n,$$

$$T = T_f[a,b] = \sup t.$$

Here the suprema are taken over all possible partitions of $[a,b]$ are defined as positive variation, negative variation, and total variation of f on $[a,b]$ respectively.

If $T_f[a,b] < \infty$ then f is called a function of bounded variation on $[a,b]$.

We denote the class of functions with this property by $BV[a,b]$.

A function is said to belong to $BV(-\infty,\infty)$ if it belongs to $BV[a,b]$ for all finite a and b.

Theorem 7.1: A monotonic function f defined on $[a,b]$ is a function of bounded variation and

$T_f[a,b] = |f(b) - f(a)|$.

Proof: Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a,b]$.

Then

$$t = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= \begin{cases} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] ; f \text{ is increasing} \\ \sum_{i=1}^{n} [-f(x_i) + f(x_{i-1})] ; f \text{ is decreasing} \end{cases}$$

$$= |f(b) - f(a)|$$

$$\therefore T_f[a,b] = \sup_P t$$

$$= |f(b) - f(a)| < \infty$$

$\Rightarrow f$ is a function of bounded variation.

This completes the proof.

Theorem 7.2: If f is a function of bounded variation on $[a,b]$ then

$T_f[a,b] = P_f[a,b] + N_f[a,b]$

and

$f(b) - f(a) = P_f[a,b] - N_f[a,b]$.

Proof: Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of $[a,b]$ then

$$p - n = \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]^+ - \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]^-$$

$$= \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$$

$$\Rightarrow p - n = f(b) - f(a) \quad ...(1)$$

$$\Rightarrow p = n + f(b) - f(a)$$

$$\leq N_f[a,b] + f(b) - f(a)$$

$$\Rightarrow \sup_p p \leq N_f[a,b] + f(b) - f(a)$$

$$\Rightarrow P_f[a,b] \leq N_f[a,b] + f(b) - f(a)$$

$$\Rightarrow P_f[a,b] - N_f[a,b] \leq f(b) - f(a) \quad ...(2)$$

Again,

$$n = p + f(a) - f(b)$$

$$\leq P_f[a,b] + f(a) - f(b)$$

$$\Rightarrow \sup_p n \leq P_f[a,b] + f(a) - f(b)$$

$$\Rightarrow f(b) - f(a) \leq P_f[a,b] - N_f[a,b] \quad ...(3)$$

From (2) and (3)

$$f(b) - f(a) = P_f[a,b] - N_f[a,b] \quad ...(4)$$

Since

$$t = n + p$$

$$\Rightarrow T_f[a,b] \geq n + p$$

$$\Rightarrow T_f[a,b] \geq p - [f(b) - f(a)] + p$$

$$= 2p - \left[P_f[a,b] - N_f[a,b]\right]$$

$$\Rightarrow T_f[a,b] \geq 2P_f[a,b] - P_f[a,b] + N_f[a,b]$$

$$\Rightarrow T_f[a,b] \geq P_f[a,b] + N_f[a,b] \quad ...(5)$$

Also

$$T_f[a,b] = \sup_p(t)$$

$$= \sup_p(p+n)$$

$$\leq \sup_p p + \sup_p n$$

$$= P_f[a,b] + N_f[a,b]$$

$$\therefore T_f[a,b] \leq P_f[a,b] + N_f[a,b] \quad ...(6)$$

From (5) and (6),

$T_f[a,b] = P_f[a,b] + N_f[a,b]$.

This completes the proof.

Theorem 7.3: For any function f and g defined on $[a,b]$,

$i)\ T_{f+g}[a,b] \le T_f[a,b] + T_g[a,b]$

$ii)\ T_{cf}[a,b] = |c|\, T_f[a,b],\ c \in \mathbb{R}$.

Proof: Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$.

$$i)\ \sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})| = \sum_{i=1}^{n} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})|$$

$$\le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$$\le \sup_P \sum_{i=1}^{n} | f(x_i) - f(x_{i-1})| + \sup_P \sum_{i=1}^{n} | g(x_i) - g(x_{i-1})|$$

$$= T_f[a,b] + T_g[a,b]$$

$$\Rightarrow \sup_P \sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})| \le T_f[a,b] + T_g[a,b]$$

$$\Rightarrow T_{f+g}[a,b] \le T_f[a,b] + T_g[a,b].$$

$$ii)\ \sum_{i=1}^{n} |(cf)(x_i) - (cf)(x_{i-1})| = \sum_{i=1}^{n} |c||f(x_i) - f(x_{i-1})|$$

$$= |c| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$\le |c|\, T_f[a,b]$$

$$\Rightarrow T_{cf}[a,b] \le |c| T_f[a,b] \quad \ldots(1)$$

Also

$$|c| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |(cf)(x_i) - (cf)(x_{i-1})|$$

$$\le T_{cf}[a,b]$$

$$\Rightarrow |c| T_f[a,b] \le T_{cf}[a,b] \quad \ldots(2)$$

From (1) and (2),

$T_{cf}[a,b] = |c| T_f[a,b]$.

This completes the proof.

Cor 1: If $f, g \in BV[a,b]$ then so is $f+g$.

Proof: Since $f, g \in BV[a,b]$ therefore $T_f[a,b] < \infty$ and $T_g[a,b] < \infty$.

Now,

$$T_{f+g}[a,b] \le T_f[a,b] + T_g[a,b]$$

$$\Rightarrow T_{f+g}[a,b] < \infty$$

$$\Rightarrow f + g \in BV[a,b]$$

Cor 2: If $f \in BV[a,b]$ then so is $cf, c \in \mathbb{R}$.

Proof: Since $f \in BV[a,b]$ $\therefore$ $T_f[a,b] < \infty$. Now,

$$T_{cf}[a,b] \le |c| T_f[a,b] < \infty$$

$$\Rightarrow cf \in BV[a,b]$$

Theorem 7.4: A function of bounded variation is bounded.

Proof: Let $f:[a,b] \to \mathbb{R}$ be a function of bounded variation, $T_f[a,b] < \infty$.

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$

Be a partition of $[a,b]$. Now for any $x \in [a,b]$ we have

$$|f(x)| - |f(a)| \le |f(x) - f(a)|$$

$$\Rightarrow |f(x)| \le |f(a)| + |f(x) - f(a)|$$

$$\le |f(a)| + \sup_p \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= |f(a)| + T_f[a,b]$$

$\Rightarrow f$ is bounded.

This completes the proof.

Notes: A bounded function may not be a function of bounded variation.

Notes: A continuous function need not be a function of bounded variation.

Notes: A function of bounded variation need not be continuous.

Example: Let $f:[0,1] \to \mathbb{R}$, such that

$$f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right); x \ne 0 \\ 0; x = 0 \end{cases}.$$

Show that f is not of bounded variation.

Solution: Let $P = \left\{0, \frac{2}{2n+1}, \frac{2}{2n-1}, \frac{2}{2n-3}, \dots, \frac{2}{5}, \frac{2}{3}, 1\right\}$ be the partition of $[0,1], n \in \mathbb{N}$.

$$\left|f(1)-f\left(\frac{2}{3}\right)\right|+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+\cdots+\left|f\left(\frac{2}{2n-1}\right)-f\left(\frac{2}{2n+1}\right)\right|+\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|$$

$$=\frac{2}{3}+\frac{2}{3}+\frac{2}{5}+\cdots+\frac{2}{2n+1}$$

$$=4\left(\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right)$$

Since the series $\sum\frac{1}{2n+1}$ is a divergent series, therefore its n^{th} partial sum

$\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}$ is not bounded above.

Hence $T_f[0,1]\to\infty\Rightarrow f\notin BV[0,1]$.

Notes: Let $f:[0,2]\to\mathbb{R}$ defined by $f(x)=[x]$ then f, being increasing is a function of bounded variation.

Theorem 7.5: The product of two functions of bounded variation is also of bounded variation.

Proof: Let $f,g\in BV[a,b]$ then f and g are bounded on $[a,b]$. $\therefore\ \exists\ 0<k_1\in\mathbb{R}$ and $0<k_2\in\mathbb{R}$ such that

$|f(x)|\le k_1$ for all $x\in[a,b]$ and $|g(x)|\le k_2$ for all $x\in[a,b]$.

Let $k=\max\{k_1,k_2\}\ \therefore\ |f(x)|\le k$ and $|g(x)|\le k$ for all $x\in[a,b]$.

Let $P=\{a=x_0,x_1,\ldots,x_n=b\}$

then

$$\sum_{i=1}^{n}|(fg)(x_i)-(fg)(x_{i-1})|=\sum_{i=1}^{n}|f(x_i)g(x_i)-f(x_{i-1})\,g(x_{i-1})|$$

$$=\sum_{i=1}^{n}|f(x_i)g(x_i)-f(x_i)\,g(x_{i-1})+f(x_i)\,g(x_{i-1})-f(x_{i-1})\,g(x_{i-1})|$$

$$\le\sum_{i=1}^{n}|f(x_i)||g(x_i)-g(x_{i-1})|+\sum_{i=1}^{n}|g(x_{i-1})||f(x_i)-f(x_{i-1})|$$

$$\le k\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})|+k\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

$$\le k\left[T_g[a,b]+T_f[a,b]\right]$$

$$\Rightarrow T_{fg}[a,b]\le k\left[T_g[a,b]+T_f[a,b]\right]$$

Since

$f,g\in BV[a,b]\ \therefore\ T_f[a,b]<\infty$ and $T_g[a,b]<\infty$

$\Rightarrow T_{fg}[a,b]<\infty\ \Rightarrow fg\in BV[a,b]$.

This completes the proof.

Theorem 7.6: Let $f\in BV[a,b]$ and $|f(x)|\ge k$ for all $x\in[a,b]$ for some $0<k\in\mathbb{R}$ then

$\frac{1}{f} \in BV[a,b]$.

Proof: Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be the partition of [a, b].

Then

$$\sum_{i=1}^{n}\left|\left(\frac{1}{f}\right)(x_i) - \left(\frac{1}{f}\right)(x_{i-1})\right| = \sum_{i=1}^{n}\left|\frac{1}{f(x_i)} - \frac{1}{f(x_{i-1})}\right|$$

$$= \sum_{i=1}^{n}\left|\frac{f(x_{i-1}) - f(x_i)}{f(x_i)f(x_{i-1})}\right|$$

$$\leq \frac{1}{k^2}\sum_{i=1}^{n}|f(x_i) - f(x_{i-1})|$$

$$\leq \frac{1}{k^2} T_f[a,b]$$

$$\Rightarrow T_{\frac{1}{f}}[a,b] \leq \frac{1}{k^2} T_f[a,b] < \infty$$

$$\Rightarrow \frac{1}{f} \in BV[a,b].$$

This completes the proof.

Cor: If $f, g \in BV[a,b]$ and $|g(x)| \geq k$ for all $x \in [a,b]$ for some $0 < k \in \mathbb{R}$ then

$\frac{f}{g} \in BV[a,b]$.

Proof: By preceding theorem,

$\frac{1}{g} \in BV[a,b]$

Also,

$f \in BV[a,b] \therefore f\left(\frac{1}{g}\right) = \frac{f}{g} \in BV[a,b]$.

Theorem 7.7: If $f \in BV[a,b]$ and $a < c < b$ then $f \in BV[a,c]$, $f \in BV[c,b]$ and

$T_f[a,b] = T_f[a,c] + T_f[c,b]$.

Proof: Let $P_1 = \{a = x_0, x_1, \dots, x_k = c\}$ and $P_2 = \{c = x_k, x_{k+1}, \dots, x_n = b\}$ be partitions of $[a,c]$ and $[c,b]$ respectively so that $P = P_1 \cup P_2 = \{a = x_0, x_1, \dots, x_n = b\}$ is a partition of $[a,b]$.

Then

$$\sum_{i=1}^{k}|f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})|$$

$$\leq T_f[a,b]$$

$$\Rightarrow \sup_{P_1}\sum_{i=1}^{k}|f(x_i) - f(x_{i-1})| \leq T_f[a,b]$$

$\Rightarrow T_f[a,c] \leq T_f[a,b] < \infty$

$\Rightarrow f \in BV[a,c]$

Similarly, we can show $f \in BV[c,b]$.

Now

$$\sum_{i=1}^{k}|f(x_i)-f(x_{i-1})| + \sum_{i=k+1}^{n}|f(x_i)-f(x_{i-1})| = \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| \leq T_f[a,b]$$

$$\Rightarrow \sup_{P_1}\sum_{i=1}^{k}|f(x_i)-f(x_{i-1})| + \sum_{i=k+1}^{n}|f(x_i)-f(x_{i-1})| \leq T_f[a,b]$$

$$\Rightarrow T_f[a,c] + \sup_{P_2}\sum_{i=k+1}^{n}|f(x_i)-f(x_{i-1})| \leq T_f[a,b]$$

$$\Rightarrow T_f[a,c] + T_f[c,b] \leq T_f[a,b] \qquad \ldots(1)$$

Let $P_1^* = \{a = y_0, y_1, \ldots, y_{i-1}, c\}$ and $P_2^* = \{c, y_j, y_{j+1}, \ldots, y_n = b\}$ be partitions of $[a,c]$ and $[c,b]$ respectively.

Let $P^* = \{a =, y_0, y_1, \ldots, y_j, \ldots, y_n = b\}$.

Now,

$$\sum_{i=1}^{n}|f(y_i)-f(y_{i-1})| \leq \left[\sum_{i=1}^{j-1}|f(y_i)-f(y_{i-1})| + |f(c)-f(y_{j-1})|\right]$$

$$+\left[|f(y_j)-f(c)| + \sum_{i=j+1}^{n}|f(y_i)-f(y_{i-1})|\right]$$

$$\leq T_f[a,c] + T_f[c,b]$$

$$\Rightarrow T_f[a,b] \leq T_f[a,c] + T_f[c,b] \qquad \ldots(2)$$

From (1) and (2),

$T_f[a,b] = T_f[a,c] + T_f[c,b]$.

This completes the proof.

Variation Function: Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. The function $V_f:[a,b]\to\mathbb{R}$ defined by

$V_f(x) = T_f[a,x]$

is called variation function.

V_f is monotonically increasing because if $y > x$ then

$T_f[a,y] = T_f[a,x] + T_f[x,y] \geq T_f[a,x]$

$i.e.\ y > x \Rightarrow V_f(y) \geq V_f(y)$.

Theorem 7.8 (Jordan's Theorem):

A function f is of bounded variation on $[a,b]$ if and only if it is the difference of two monotonically increasing real-valued functions on $[a,b]$.

Proof: Suppose $f \in BV[a,b]$.

Let $g = \frac{1}{2}(V_f + f), h = \frac{1}{2}(V_f - f)$, then $f = g - h$.

Let $x_1, x_2 \in [a,b]$ and $x_1 < x_2$ then

$$g(x_2) - g(x_1) = \frac{1}{2}[V_f(x_2) + f(x_2)] - \frac{1}{2}[V_f(x_1) + f(x_1)]$$

$$= \frac{1}{2}[V_f(x_2) - V_f(x_1)] + \frac{1}{2}[f(x_2) - f(x_1)] \quad \dots(1)$$

and

$$h(x_2) - h(x_1) = \frac{1}{2}[V_f(x_2) - f(x_2)] - \frac{1}{2}[V_f(x_1) - f(x_1)]$$

$$= \frac{1}{2}[V_f(x_2) - V_f(x_1)] - \frac{1}{2}[f(x_2) - f(x_1)] \quad \dots(2)$$

Now

$f \in BV[a,b]$

$\Rightarrow f \in BV[x_1, x_2]$

$$\therefore |f(x_2) - f(x_1)| \leq T_f[x_1, x_2]$$

$$= \left[T_f[a, x_1] + T_f[x_1, x_2]\right] - T_f[a, x_1]$$

$$= T_f[a, x_2] - T_f[a, x_1]$$

$$= V_f(x_2) - V_f(x_1)$$

$$\Rightarrow |f(x_2) - f(x_1)| \leq V_f(x_2) - V_f(x_1)$$

$$\Rightarrow f(x_2) - f(x_1) \leq V_f(x_2) - V_f(x_1) \text{ and } -[f(x_2) - f(x_1)] \leq V_f(x_2) - V_f(x_1)$$

$$\Rightarrow [V_f(x_2) - V_f(x_1)] - [f(x_2) - f(x_1)] \geq 0 \text{ and} [V_f(x_2) - V_f(x_1)] + [f(x_2) - f(x_1)] \geq 0$$

$$\Rightarrow g(x_2) - g(x_1) \geq 0 \text{ and } h(x_2) - h(x_1) \geq 0$$

$\Rightarrow g$ and h are monotonically increasing functions $f = g - h$.

Converse: Let $f = g - h$ where g and h are monotonically increasing real-valued functions on $[a,b]$.

Then for any $P = \{a = x_0, x_1, \dots, x_n = b\}$ of $[a,b]$, we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |(g-h)(x_i) - (g-h)(x_{i-1})|$$

$$\leq \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |h(x_i) - h(x_{i-1})|$$

$$= [g(b) - g(a)] + [h(b) - h(a)]$$

$\Rightarrow T_f[a,b] < \infty$

$\Rightarrow f \in BV[a,b]$.

This completes the proof.

Theorem 7.9: Show that if f' exists and is bounded on $[a,b]$ then $f \in BV[a,b]$.

Proof: Since f' is bounded on $[a,b]$, $\therefore |f'(x)| \le M$, for all $x \in [a,b]$

and for some $0 < M \in \mathbb{R}$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ then

$$\frac{|f(x_i) - f(x_{i-1})|}{|x_i - x_{i-1}|} \le M, 1 \le i \le n$$

$$\Rightarrow |f(x_i) - f(x_{i-1})| \le M(x_i - x_{i-1})$$

$$\Rightarrow \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le M \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= M(b-a)$$

$$\Rightarrow \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le M(b-a)$$

$$\Rightarrow T_f[a,b] \le M(b-a) < \infty$$

$$\Rightarrow f \in BV[a,b].$$

This completes the proof.

Example: Show that the function $f:[0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \cos\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is of bounded variation on $[0,1]$.

Solution: f is continuous on $[0,1]$.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \cos\frac{1}{h}}{h} = 0$$

and for $x \ne 0$,

$$f'(x) = x^2\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x\cos\frac{1}{x}$$

$$= \sin\frac{1}{x} + 2x\cos\frac{1}{x}$$

$$\Rightarrow f'(x) \text{ exists in } [0,1]$$

$$|f'(x)| = \left|\sin\frac{1}{x} + 2x\cos\frac{1}{x}\right|$$

$$\leq \left|\sin\frac{1}{x}\right| + 2|x|\left|\cos\frac{1}{x}\right|$$

$$\leq 1 + 2 = 3$$

Therefore $f \in BV[0,1]$

Summary

- A monotonic function f defined on $[a,b]$ is a function of bounded variation and $T_f[a,b] = |f(b) - f(a)|$.
- If f is a function of bounded variation on $[a,b]$ then $T_f[a,b] = P_f[a,b] + N_f[a,b]$ and $f(b) - f(a) = P_f[a,b] - N_f[a,b]$.
- For any function f and g defined on $[a,b]$, we have
  $i)\ T_{f+g}[a,b] \leq T_f[a,b] + T_g[a,b]$
  $ii)\ T_{cf}[a,b] = |c|\ T_f[a,b],\ c \in \mathbb{R}$
- If $f, g \in BV[a,b]$ then so is $f + g$.
- If $f \in BV[a,b]$ then so is $cf, c \in \mathbb{R}$
- A function of bounded variation is bounded.
- A bounded function may not be a function of bounded variation.
- A continuous function need not be a function of bounded variation.
- A function of bounded variation need not be continuous.
- The product of two functions of bounded variation is also of bounded variation.
- Let $f \in BV[a,b]$ and $|f(x)| \geq k$ for all $x \in [a,b]$ for some $0 < k \in \mathbb{R}$ then

$$\frac{1}{f} \in BV[a,b]$$

- If $f, g \in BV[a,b]$ and $|g(x)| \geq k$ for all $x \in [a,b]$ for some $0 < k \in \mathbb{R}$ then

$$\frac{f}{g} \in BV[a,b]$$

- If $f \in BV[a,b]$ and $a < c < b$ then $f \in BV[a,c]$, $f \in BV[c,b]$ and

$$T_f[a,b] = T_f[a,c] + T_f[c,b]$$

- If f' exists and is bounded on $[a,b]$ then $f \in BV[a,b]$.

Keywords

Functions of bounded variation: If $T_f[a,b] < \infty$ then f is called a function of bounded variation on $[a,b]$. We denote the class of functions with this property by $BV[a,b]$.

Variation Function: Let $f: [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. The function $V_f: [a,b] \to \mathbb{R}$ defined by $V_f(x) = T_f[a,x]$ is called variation function.

Jordan's Theorem: A function f is of bounded variation on $[a, b]$ if and only if it is the difference of two monotonically increasing real-valued functions on $[a, b]$.

Self Assessment

1) Consider the following statements:

(I) A monotonically increasing function f defined on $[a, b]$ is a function of bounded variation.

(II) A monotonically decreasing function f defined on $[a, b]$ is a function of bounded variation.Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

2) Consider the following statements:

(I) If f is a function of bounded variation then $T_f[a, b] < \infty$.

(II) $T_f[a,b] = P_f[a,b] + N_f[a,b]$ whenever $f \in BV[a,b]$.Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

3) $P_f[a,b] - N_f[a,b] = f(b) + f(a)$ whenever $f \in BV[a,b]$.

A. True
B. False

4) Total variation, positive variation, and negative variation need not be non-negative quantities.

A. True
B. False

5) Consider the following statements:

(I) $T_f[a,b] + T_g[a,b] \leq T_{(f+g)}[a,b]$.

(II) $T_{cf}[a,b] = cT_f[a,b]$, $c \in \mathbb{R}$. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

6) Consider the following statements:

(I) A function of bounded variation must be bounded.

(II) A bounded function must be a function of bounded variation. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

7) Consider the following statements:

(I) A function of bounded variation must be continuous.

(II) A continuous function must be a function of bounded variation. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

8) Consider the following statements:

(I) The product of two functions of bounded variation is also of bounded variation.

(II) $f \in BV[a,b]$ and $|f(x)| \leq c$, for all $x \in [a,b]$ and $0 < c \in \mathbb{R}$ implies $\frac{1}{f} \in BV[a,b]$. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

9) Consider the following statements:

(I) $f \in BV[a,b] \Rightarrow cf \in BV[a,b]$ only when c is a non-negative real number.

(II) $f \in BV[a,b] \Rightarrow cf \in BV[a,b]$ for every real number c. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

10) $f, g \in BV[a,b]$. Consider the following statements:

(I) $f + g \in BV[a,b]$

(II) $f - g \in BV[a,b]$. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

11) Let $f \in BV[a,b]$ and $a < c < b$. Consider the following statements:

(I) $T_f[a,b] - T_f[c,b] = T_f[a,c]$.

(II) $f \in BV[a,c]$. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

12) Variation function is a monotonically increasing function.

A. True
B. False

13) Consider the following statements:

(I) If $f \in BV[a,b]$ then it can be written as the difference of two monotonically increasing functions.

(II) If f is the difference of two monotonically increasing functions then f is a function of bounded

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

14) Choose the correct option.

A. If f is bounded then it is a function of bounded variation.
B. If f' exists then it is a function of bounded variation.
C. If f' exists and is bounded then it is a function of bounded variation.
D. none of these

15) Suppose f is a real-valued function on [0, 1] defined by

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) ; x \neq 0 \\ 0 ; x = 0 \end{cases},$$

then

A. f' exists.
B. f is a function of bounded variation.
C. f' is bounded.
D. all are correct.

Answers for Self Assessment

1.	C	2.	C	3.	B	4.	B	5.	D
6.	A	7.	D	8.	A	9.	B	10.	C
11.	C	12.	A	13.	C	14.	C	15.	D

Review Questions

1) A bounded function may not be a function of bounded variation. Give an example in support of this statement.

2) Let $f:[0,1] \to \mathbb{R}$, such that

$$f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right); x \neq 0 \\ 0; x = 0 \end{cases}.$$

Show that f is not of bounded variation.

3) Show with the help of an example that a continuous function need not be of bounded variation.

4) A function of bounded variation need not be continuous. Give an example in support of this statement.

5) Let $f:[-1,1] \to \mathbb{R}$, such that

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right); x \neq 0 \\ 0; x = 0 \end{cases}.$$

Show that f is of bounded variation on [-1, 1].

Further Readings

Measure theory and integration by G DE BARRA, New Age International.

Real Analysis by H L Royden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

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Unit 08: The Four Derivatives and Differentiation and Integration

CONTENTS

Objectives

After studying this unit, students will be able to:

- determine the four derivatives
- define the relationship between differentiation and integration
- explain Lebesgue point of a function
- demonstrate Lebesgue set of a function
- explain the Vitali covering lemma
- understand Lebesgue theorem

Introduction

Differentiation and integration are closely connected. The fundamental theorem of calculus tells us that differentiation and integration are inverse processes. In this unit, we will examine these concepts in detail.

8.1 The Four Derivatives

If f is an extended real-valued function, finite at x and defined in an open interval containing x, then the following four quantities, not necessarily finite, are called as the four derivatives or Dini Derivative,

1) $D^+ f(x) = \lim_{h\to 0^+} \sup \frac{f(x+h)-f(x)}{h}$ (Upper Right Derivative)

2) $D_+ f(x) = \lim_{h\to 0^+} \inf \frac{f(x+h)-f(x)}{h}$ (Lower Right Derivative)

3) $D^- f(x) = \lim_{h\to 0^-} \sup \frac{f(x+h)-f(x)}{h}$ (Upper Left Derivative)

4) $D_- f(x) = \lim_{h\to 0^-} \inf \frac{f(x+h)-f(x)}{h}$ (Lower Left Derivative)

Notes: 1) The function f is differentiable at x if and only if, the four derivatives have a finite common value.

2) $D^+ f(x) \geq D_+ f(x)$

3) $D^- f(x) \geq D_- f(x)$

4) $D^+(-f) = -D_+ (f)$

5) $D^-(-f) = -D_- (f)$

6) $D_+ f(x) = -D^+(-f(x))$

7) $D_- f(x) = -D^-(-f(x))$.

Example: If $f(x) = |x|$, find $D^+ f(0), D_+ f(0), D^- f(0), D_- f(0)$.

Solution:

$$D^+ f(0) = \lim_{h \to 0^+} \sup \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \sup \frac{f(h)}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h}$$

$$= 1.$$

$$D_+ f(0) = \lim_{h \to 0^+} \inf \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \inf \frac{h}{h} = 1$$

$$D^- f(0) = \lim_{h \to 0^-} \sup \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \sup \frac{-h}{h} = -1$$

$$D_- f(0) = \lim_{h \to 0^-} \inf \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \inf \frac{-h}{h} = -1.$$

Example: If $f(x) = \begin{cases} x \sin\frac{1}{x}; x \neq 0 \\ 0; x = 0 \end{cases}$, find $D^+ f(0), D_+ f(0), D^- f(0), D_- f(0)$.

$$D^+ f(0) = \lim_{h \to 0^+} \sup \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \sup \frac{h \sin\frac{1}{h}}{h}$$

$$= \lim_{h \to 0^+} \sup \sin\frac{1}{h}$$

$$= \inf_{\delta} \sup_{0<h<\delta} \sin\frac{1}{h} = 1$$

$$D_+ f(0) = \lim_{h \to 0^+} \inf \sin\frac{1}{h} = -1$$

$$D^- f(0) = \lim_{h \to 0^-} \sup \sin\frac{1}{h} = 1$$

$$D_- f(0) = \lim_{h \to 0^-} \inf \sin\frac{1}{h} = -1.$$

8: The Fou iva i s and I ifferenti tion an Integ

Notes: If $f(x) = \{1; x \notin \mathbb{Q}$, $f:[0,1] \to \mathbb{R}$, find $D_+ f(0)$, $D^+ f(0), D^- f(0), D_- f(0)$.

Solution: When $x \in \mathbb{Q}$,

$$D^+ f(x) = \lim_{h \to 0^+} \sup \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0^+} \sup \frac{f(x+h)}{h} = \lim_{h \to 0^+} \sup \left(0 \text{ or } \frac{1}{h}\right) = \infty$$

$$D_+ f(x) = \lim_{h \to 0^+} \inf \frac{f(x+h)}{h} = \lim_{h \to 0^+} \inf \left(0 \text{ or } \frac{1}{h}\right) = 0$$

$$D^- f(x) = \lim_{h \to 0^-} \sup \frac{f(x+h)}{h} = \lim_{h \to 0^-} \sup \left(0 \text{ or } \frac{1}{h}\right) = 0$$

$$D_- f(x) = \lim_{h \to 0^-} \inf \frac{f(x+h)}{h} = \lim_{h \to 0^-} \inf \left(0 \text{ or } \frac{1}{h}\right) = -\infty$$

If $x \notin \mathbb{Q}$,

$$D^+ f(x) = \lim_{h \to 0^+} \sup \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \sup \frac{f(x+h) - 1}{h}$$

$$= \lim_{h \to 0^+} \sup \frac{(0 \text{ or } 1) - 1}{h} = \lim_{h \to 0^+} \sup \left(\frac{-1}{h} \text{ or } 0\right) = 0$$

$$D_+ f(x) = \lim_{h \to 0^+} \inf \left(\frac{-1}{h} \text{ or } 0\right) = -\infty$$

$$D^- f(x) = \lim_{h \to 0^-} \sup \left(\frac{-1}{h} \text{ or } 0\right) = \infty$$

$$D_- f(x) = \lim_{h \to 0^-} \inf \left(\frac{-1}{h} \text{ or } 0\right) = 0$$

Example: If

$$f(x) = \begin{cases} ax \sin^2 \frac{1}{x} + bx \cos^2 \frac{1}{x}, & x > 0 \\ px \sin^2 \frac{1}{x} + qx \cos^2 \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases},$$

$a < b, p < q$, find $D_+ f(0), D^+ f(0), D^- f(0), D_- f(0)$.

Solution: $D^+ f(0) = \lim_{h \to 0^+} \sup \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0^+} \sup \frac{f(h)}{h}$$

$$= \lim_{h \to 0^+} \sup \frac{ah \sin^2 \frac{1}{h} + bh \cos^2 \frac{1}{h}}{h}$$

$$= \lim_{h \to 0^+} \sup \left[a \sin^2 \frac{1}{h} + b \cos^2 \frac{1}{h}\right]$$

$$= \lim_{h \to 0^+} \sup \left[\frac{a}{2}\left(1 - \cos \frac{2}{h}\right) + \frac{b}{2}\left(1 + \cos \frac{2}{h}\right)\right]$$

$$= \frac{1}{2} \lim_{h \to 0^+} \sup \left[(a+b) + (b-a) \cos \frac{2}{h}\right]$$

$$= \frac{1}{2}(a + b + b - a) = b$$

$$D_+ f(0) = \lim_{h \to 0^+} \inf \frac{f(h)}{h}$$

$$= \frac{1}{2} \lim_{h \to 0^+} \inf \left[(a+b) + (b-a) \cos \frac{2}{h}\right] = \frac{1}{2}(a + b - b + a) = a$$

$$D^- f(0) = \lim_{h \to 0^-} \sup \frac{f(h)}{h}$$

$$= \lim_{h \to 0^-} \sup \left[p \sin^2 \frac{1}{h} + q \cos^2 \frac{1}{h}\right]$$

$$= \frac{1}{2} \lim_{h \to 0^-} \sup \left[(p+q) + (q-p) \cos \frac{2}{h} \right]$$

$$= \frac{1}{2} [p + q + q - p] = q$$

$$D_- f(0) = \lim_{h \to 0^-} \inf \frac{f(h)}{h}$$

$$= \frac{1}{2} \lim_{h \to 0^-} \inf \left[(p+q) + (q-p) \cos \frac{2}{h} \right]$$

$$= \frac{1}{2} [p + q - q + p] = p$$

Indefinite Integral: If f is an integrable function on $[a,b]$ then the function F defined by

$$F(x) = \int_a^x f(t)dt + F(a)$$

is called the indefinite integral of f.

Theorem 8.1: Let f be an integrable function on $[a,b]$. Then the indefinite integral of f is a continuous function of bounded variation on $[a,b]$.

Proof: Let F be an indefinite integral of f defined on $[a,b]$

$$\Rightarrow F(x) = \int_a^x f(t)dt + F(a)$$

Let c be any point of $[a,b]$. Then

$$|F(x) - F(c)| = \left| \int_a^x f(t)dt - \int_a^c f(t)dt \right| = \left| \int_a^x f(t)dt - \int_c^a f(t)dt \right|$$

$$= \left| \int_c^x f(t)dt \right|$$

$$\leq \int_c^x |f(t)|dt \qquad \ldots (1)$$

Now f is integrable over $[a,b]$.

$\Rightarrow |f|$ is integrable over $[a,b]$.

Therefore, for given $\epsilon > 0$, $\exists \delta > 0$ such that for every measurable set $A \subseteq [a,b]$ with $m(A) < \delta$, we have

$$\int_A |f| < \epsilon$$

$$\Rightarrow \int_c^x |f(t)|dt < \epsilon \text{ for } |x - c| < \delta \qquad \ldots (2)$$

$$\Rightarrow |F(x) - F(c)| < \epsilon \text{ for } |x - c| < \delta$$

$\Rightarrow F$ is continuous at c and hence on $[a,b]$.

Let $P = \{a = x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$.

Then

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f(t)dt \right|$$

$$\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t)|dt$$

$$= \int_a^b |f(t)|dt$$

$$\Rightarrow \sup_P \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \int_a^b |f(t)|dt$$

$$\Rightarrow T_f[a,b] \leq \int_a^b |f(t)|dt < \infty.$$

$$\Rightarrow F \in BV[a,b].$$

This completes the proof.

Theorem 8.2: If f is an integrable function on $[a,b]$and

$$\int_a^x f(t)dt = 0$$

for all $x \in [a,b]$ then $f = 0\ a.e.$ in$[a,b]$.

Proof: If possible, let $f \neq 0\ a.e.$ in$[a,b]$. Let $f(t) > 0$ on set $E \subseteq [a,b]$ with $m(E) > 0$. Then $\exists$ a closed set$F \subseteq E$such that $m(F) > 0$.

Let $O = F^c = [a,b] - F$ so that O is an open set. Let $O = \cup_n (a_n, b_n)$ where $\{(a_n, b_n)\}$ is a sequence of pairwise disjoint open intervals.

Now

$$\int_a^x f(t)dt = 0 \text{ for all } x \in [a,b]$$

$$\Rightarrow \int_a^b f(t)dt = 0$$

$$\Rightarrow \int_{OUF} f(t)\,dt = 0$$

$$\Rightarrow \int_O f(t)\,dt + \int_F f(t)\,dt = 0$$

$$\Rightarrow \int_O f(t)\,dt = -\int_F f(t)\,dt$$

Since $f(t) > 0$ on F and $m(F) > 0$.

$$\Rightarrow \int_F f(t)dt \neq 0$$

$$\Rightarrow \int_O f(t)\,dt \neq 0$$

$$\Rightarrow \int_{\cup_n (a_n, b_n)} f(t)\,dt \neq 0$$

$$\Rightarrow \sum_n \int_{a_n}^{b_n} f(t)dt \neq 0$$

$$\Rightarrow \int_{a_N}^{b_N} f(t)dt \neq 0 \text{ for some positive integer } N.$$

$$\Rightarrow \int_a^{b_N} f(t)dt - \int_a^{a_N} f(t)dt \neq 0$$

$$\Rightarrow \int_a^{b_N} f(t)dt \neq 0 \text{ or} \int_a^{a_N} f(t)dt \neq 0$$

which is contrary to the given. Hence $f = 0$ $a.e.$ on $[a, b]$.

This completes the proof.

Lebesgue Point of a Function: Let f be an integrable function on $[a, b]$. A point x in $[a, b]$ is said to be a Lebesgue point of f if

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} |f(t) - f(x)|dt = 0$$

Lebesgue set of a function:

The set of all Lebesgue points of a function f in $[a, b]$ is called the Lebesgue set of a function f.

Theorem 8.3: Every point of continuity of an integrable function f is a Lebesgue point of f.

Proof: Let f be continuous at x_0.

 for given $\epsilon > 0$, $\delta > 0$ such that

$$|f(t) - f(x_0)| < \epsilon \text{ whenever} |t - x_0| < \delta$$

Now for $0 < |h| < \delta$, we have

$$\frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|dt < \epsilon \Rightarrow \lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|dt = 0.$$

This completes the proof.

8.2 Differentiation and Integration

Theorem 8.4: Let f be bounded measurable function defined on $[a, b]$. If

$$F(x) = \int_a^x f(t)dt + F(a)$$

Then

$$F'(x) = f(x) \text{ } a.e. \text{ on } [a, b]$$

Proof: Since f is bounded and measurable on $[a, b]$.

 f is integrable on $[a, b]$.

 F is a continuous function of bounded variation on $[a, b]$.

 F' exists a.e. in [a, b].

Now, f is bounded on [a, b].

Therefore a positive real number k such that $|f(x)| \leq k$ for all $x \in [a, b]$.

For each positive integer n, define

$$f_n(x) = \frac{F(x+h) - F(x)}{h}, h = \frac{1}{n}$$

Then

$$f_n(x) = \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right]$$

$$= \frac{1}{h}\int_x^{x+h} f(t)dt$$

$$\square\ |f_n(x)| = \frac{1}{h}\left|\int_x^{x+h} f(t)dt\right|$$

$$\leq \int_x^{x+h} |f(t)|dt$$

$$\leq \frac{k}{h}\int_x^{x+h} dt = k$$

$$\square\ |f_n(x)| \leq k.$$

Also

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{F\left(x+\frac{1}{n}\right) - F(x)}{\frac{1}{n}}$$

$$= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= F'(x)\ a.e.$$

Therefore, by bounded convergence theorem, for every $c \in [a, b]$,

$$\int_a^c F'(x)dx = \lim_{n\to\infty} \int_a^c f_n(x)dx$$

$$= \lim_{h\to 0} \int_a^c \frac{F(x+h) - F(x)}{h} dx$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^c F(x+h)dx - \int_a^c F(x)dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{a+h}^{c+h} F(x)dx - \int_a^c F(x)dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{a+h}^{c} F(x)dx + \int_c^{c+h} F(x)dx - \int_a^c F(x)dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{a+h}^{a} F(x)dx + \int_c^{c+h} F(x)dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_c^{c+h} F(x)dx - \int_a^{a+h} F(x)dx\right] \quad \ldots (1)$$

F is continuous therefore $\lim_{n\to\infty} \frac{1}{h}\int_c^{c+h} F(x)dx = F(c)$

$$\square\ (1)\ \square\ \int_a^c F'(x)dx = F(c) - F(a) = \int_a^c f(x)dx$$

$$\square\ \int_a^c [F'(x) - f(x)]\, dx = 0\ \forall c \in [a, b]$$

$$\Rightarrow F'(x) - f(x) = 0\ a.e. \text{ in}[a, b]$$

$$\square\ F'(x) = f(x)\ a.e. \text{ in}[a, b].$$

This completes the proof.

Theorem 8.5: Let f be an integrable function on $[a, b]$ and

$$F(x) = \int_a^x f(t)dt + F(a)$$

Then $F'(x) = f(x)\ a.e.\ \text{in}[a,b]$.

Proof: We may assume that $f \geq 0$. Define a sequence $\{f_n\}$ of function

$$f_n(x) = \begin{cases} f(x)\ ; & f(x) \leq n \\ n\ ; & f(x) > n \end{cases}$$

Each f_n is bounded and measurable on $[a,b]$.

 $\frac{d}{dx}\int_a^x f_n(t)dt = f_n(x)\ a.e.\ \text{in}[a,b]$ ... (1)

Let

$$G_n(x) = \int_a^x (f - f_n)\,, n \in \mathbb{N}$$

Then $G_n(x)$ is monotonically increasing function of x.

 G_n is differentiable $a.e.$ in $[a,b]$ by using the result "If f is monotonically increasing real-valued function on $[a,b]$ then f is differentiable a.e. and f' is measurable,

$$\int_a^b f'(x)dx \leq f(b) - f(a)."$$

$$ F(x) = \int_a^x f(t)dt + F(a)$$

$$= \int_a^x (f - f_n)\,dt + \int_a^x f_n\,dt + F(a)$$

$$= G_n(x) + \int_a^x f_n(t)\,dt + F(a)$$

 $F'(x) = G_n'(x) + f_n(x)\ a.e.$

 $F'(x) \geq f_n(x)\ a.e.\ \forall n$

 $F'(x) \geq f(x)\ a.e.$

$$ \int_a^b F'(x)dx \geq \int_a^b f(x)dx$$

$$= F(b) - F(a)$$

$$ \int_a^b F'(x)dx \geq F(b) - F(a) \qquad \text{... (2)}$$

Also, we have

$$\int_a^b F'(x)dx \leq F(b) - F(a) \qquad \text{... (3)}$$

$$ \int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$$

$$ \int_a^b F'(x)dx = \int_a^b f(x)dx$$

$$\Rightarrow \int_a^b [F'(x) - f(x)]dx$$

$$\Rightarrow F'(x) - f(x) = 0 \; a.e.$$

$$\Rightarrow F'(x) = f(x) \; a.e. \text{ in} [a,b].$$

This completes the proof.

Vitali Cover: Let $E \subseteq \mathbb{R}$, then a collection V of closed intervals of positive lengths is said to be Vitali Cover of E if for given $\epsilon > 0$ and any $x \in E, \exists$ an interval $I \in V$ such that $x \in I$ and $l(I) < \epsilon$.

Vitali's Covering Lemma: Let E be a set of finite outer measure and V be a Vitali cover of E.

Given $\epsilon > 0$, $\exists$ a finite disjoint collection $\{I_1, I_2, \ldots, I_N\}$ of intervals in V such that

$$m^*\left(E - \bigcup_{n=1}^{N} I_n\right) < \epsilon$$

Proof: Since $m^*(E) < \infty$, therefore we can find an open set $O \supseteq E$ such that $m^*(O) < \infty$

Since V is a Vitali cover of E, we may assume that each interval in V is contained in O.

Now we choose a sequence $\{I_n\}$ of disjoint intervals of V by induction as follows:

Let I_1 be any arbitrary interval of V and let k_1 be the supremum of the lengths of the intervals in V which do not have any point in common with I_1.

Then $k_1 < \infty$ as $k_1 \leq m(O) < \infty$.

Now, we choose an interval I_2 from V, disjoint from I_1, such that $l(I_2) > \frac{1}{2}k_1$.

Let k_2 be the least upper bound of the lengths of the intervals in V which do not have any point common with I_1 or I_2 and $k_2 < \infty$.

Choose I_3 from V which is disjoint from $I_1 \cup I_2$ such that $l(I_3) > \frac{1}{2}k_2$.

In general, having already chosen n disjoint intervals, $I_1, I_2, \ldots, I_n$, we denote $k_n < \infty$, the $l.u.b$ of the lengths of all intervals in V which do not have any point common with $\bigcup_{i=1}^{n} I_i$ and choose an interval I_{n+1} from V such that it is disjoint from the preceding intervals and

$$l(I_{n+1}) > \frac{1}{2}k_n.$$

Now, if for some n, the set $\bigcup_{i=1}^{n} I_i$ contains almost every point of the set E, then the lemma is proved; otherwise, we get an infinite sequence $\{I_n\}$ of intervals from V such that

$$I_i \cap I_j = \phi, \forall i \neq j$$

and

$$l(I_{n+1}) > \frac{k_n}{2}, k_n < \infty, n = 1,2,3, \ldots$$

Here, we note that the sequence $\{k_n\}$ is a monotonically decreasing sequence of non-negative real numbers.

Since

$$\bigcup_{n=1}^{\infty} I_n \subseteq O$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \leq m(O) < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \text{ converges.}$$

$\Rightarrow$ for given $\epsilon > 0$, $\exists$ an integer N such that $\displaystyle\sum_{n=N+1}^{\infty} l(I_n) < \frac{\epsilon}{5}$ ... (1)

Let $J = E - \bigcup_{n=1}^{N} I_n$.

Claim: $m^*(J) < \epsilon$.

Let $x \in J$ then $x \notin \bigcup_{n=1}^{N} I_n$.

$\bigcup_{n=1}^{N} I_n$ is closed set not containing x, we can find an interval I in V such that $x \in I$ and $l(I)$ so small that I does not meet any of the intervals $I_1, I_2, \dots, I_N$

$$i.e. I \cap I_i = \phi, i = 1,2,\dots,N.$$

Then we may have

$$l(I) \le k_N < 2\, l(I_{N+1})$$

Since $\lim_{n\to\infty} l(I_n) = 0$.

Therefore, interval I must meet at least one of the intervals in the sequence $\{I_n\}$.

Let n_0 be the smallest integer such that I meets I_{n_0}.

Then $n_0 > N$ and $l(I) \le k_{n_0-1} < 2l(I_{n_0})$ ... (2)

Since $x \in I$ and I has a common point with I_{n_0}.

Therefore, the distance of x from the midpoint of I_{n_0} is at most

$$l(I) + \frac{1}{2} l(I_{n_0}) < 2\, l(I_{n_0}) + \frac{1}{2} l(I_{n_0}) = \frac{5}{2}\, l(I_{n_0})$$

Therefore, if J_{n_0} is an interval concentric with I_{n_0} such that

$$l(J_{n_0}) = 5\, l(I_{n_0}), \text{we finu that } x \in J_{n_0}\ i.e. \forall x \in J, \exists\, n \ge N+1$$

such that $x \in J_n$ and

$$l(J_n) = 5\, l(I_n)$$

$$\Rightarrow J \subseteq \bigcup_{n=N+1}^{\infty} J_n$$

$$\Rightarrow m^*(J) \le \sum_{n=N+1}^{\infty} l(J_n)$$

$$= 5 \sum_{n=N+1}^{\infty} l(I_n) < \epsilon. \quad \{by(1)\}$$

Thus, we get

$$m^*(J) < \epsilon.$$

This completes the proof.

Theorem 8.6 (Lebesgue Theorem):

Let f be monotonically increasingthe real-valued function on $[a,b]$. Then f is differentiable $a.e.$ and $f'$is measurable.

Further,

$$\int_a^b f'(x)\,dx \le f(b) - f(a).$$

Proof: We first show that the set of points of (a,b) where any of two Dini derivatives are unequal is of measure zero.

Let

$$E = \{x \in (a,b): D^+ f > D_- f\}$$

$$F = \{x \in (a,b): D^- f > D_+ f\}$$

$$G = \{x \in (a,b): D^- f > D_- f\}$$
$$H = \{x \in (a,b): D^+ f > D_+ f\}$$

Since

$$D^+ f(x) \geq D_+ f(x), \qquad D^- f(x) \geq D_- f(x)$$

Therefore, it is sufficient to show that the measure of each of the sets E, F, G, H is zero.

We shall show that $m^*(E) = 0$ and the results in respect of the sets F, G and H can be proved similarly.

For any rationals r and s, with $r > s$, we define the set

$$E_{rs} = \{x: D^+ f(x) > r > s > D_- f(x)\}$$

Then

$$E = \bigcup_{rs \in \mathbb{Q}} E_{rs} \qquad \dots (1)$$

Let $m^*(E_{r,s}) = \alpha$.

Then for given $\epsilon > 0$, $\exists$ an open set O containing E_{rs}, such that $m(O) < \alpha + \epsilon$ ... (2)

Since $D_- f(x) < s \ \forall x \in E_{rs}$, therefore, there exists an arbitrary small interval $[x-h, x] \subseteq O$ such that

$$\frac{f(x) - f(x-h)}{h} < s$$

$$\Rightarrow f(x) - f(x-h) < sh \qquad \dots (3)$$

Thus, the collection $V = \{I_x: x \in E_{rs}\}$, where $I_x = [x-h, x]$ forms a Vitali cover of E_{rs}.

Hence by Vitali's covering lemma, for given $\epsilon > 0$, there exists finite disjoint collection $\{I_{x_1}, I_{x_2}, \dots, I_{x_N}\}$ of intervals of V such that

$$m^*\left(E_{rs} - \bigcup_{i=1}^{N} I_{x_i}\right) < \epsilon$$

Let $I_{x_i} = [x_i - h_i, x_i]$, $i = 1,2, \dots, N$.

Then,

$$\sum_{i=1}^{N} [f(x_i) - f(x_i - h_i)] < s \sum_{i=1}^{N} h_i \leq s\, m(O) < s(\alpha + \epsilon).$$

Let,

$$A = E_{r,s} \bigcap \left[\bigcup_{i=1}^{N} I_{x_i}^{O}\right], \text{and } y \in A$$

Since $D^+ f(y) > r$, therefore, there exist arbitrary small interval $[y, y+k]$ contained in some I_{x_i} such that $f(y+k) - f(y) > rk$.

Again, by using Vitali's covering lemma, $\exists$ a finite disjoint collection $\{J_{y_1}, J_{y_2}, \dots, J_{y_M}\}$ of intervals, where $J_{y_j} = [y_j, y_j + k_j]$, $j = 1,2, \dots, M$ such that

$$m^*\left(A - \bigcup_{j=1}^{M} J_{y_j}\right) < \epsilon,$$

$$\text{Now, } m^*(A) > \alpha - \epsilon$$

$$\Rightarrow m^*\left[A \cap \left(\bigcup_{j=1}^{M} J_{y_j}\right)\right] > \alpha - 2\epsilon,$$

and

$$\sum_{j=1}^{M}[f(y_j+k_j)-f(y_j)] > r\sum_{j=1}^{M}k_j > r(\alpha-2\epsilon)$$

But each J_{y_j} is contained in some I_{x_i}, therefore, summing over those j for which $J_{y_j} \subseteq I_{x_i}$, we get

$$\sum{}'[f(y_j+k_j)-f(y_j)] \le \sum(f(x_i)-f(x_i-h_i))$$

□ $\sum_{i=1}^{N}[f(x_i)-f(x_i-h_i)] \ge \sum_{j=1}^{M}[f(y_j+k_j)-f(y_j)]$

□ $s(\alpha+\epsilon) > r(\alpha-2\epsilon)$

Since $\epsilon > 0$, is arbitrary. Therefore, we have

$$s\alpha \ge r\alpha$$

□ $\alpha = 0 \because s < r$

□ $m^*(E_{r,s}) = 0$

□ $m^*(E) = 0$

This shows that

$$g(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is defined *a.e.* in $[a,b]$ and f is differentiable whenever g is finite.

Now we write

$$g_n(x) = \frac{f\left(x+\frac{1}{n}\right)-f(x)}{\frac{1}{n}}, f(x)=f(b), \forall x \ge b$$

Since f is increasing, therefore, $\{g_n\}$ is a sequence of nonnegative measurable functions.

Also,

$$\lim_{n\to\infty} g_n(x) = g(x)\ a.e.\ \text{in}[a,b]$$

□ g is measurable.

So, by Fatou's Lemma, we have

$$\int_a^b g \le \liminf \int_a^b g_n$$

$$= \liminf n\int_a^b \left[f\left(x+\frac{1}{n}\right)-f(x)\right]$$

$$= \liminf\left[n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f - n\int_a^b f\right]$$

$$= \liminf n\left[\int_{a+\frac{1}{n}}^{b} f + \int_{b}^{b+\frac{1}{n}} f - \int_a^{a+\frac{1}{n}} f - \int_{a+\frac{1}{n}}^{b} f\right]$$

$$= \liminf\left[n\int_b^{b+\frac{1}{n}} f - n\int_a^{a+\frac{1}{n}} f\right]$$

$$= \liminf\left[n\frac{f(b)}{n} - n\int_a^{a+\frac{1}{n}} f\right]$$

$$= f(b) - \limsup n \int_{a}^{a+\frac{1}{n}} f$$

$$\leq f(b) - f(a) < \infty$$

□ g is integrable and hence finite $a.e.$ in $[a, b]$.

Hence f is differentiable $a.e.$ in $[a, b]$ and $g = f'$ $a.e.$ in $[a, b]$.

This completes the proof.

Summary

- Upper Right Derivative:

$$D^{+} f(x) = \lim_{h \to 0^{+}} \sup \frac{f(x+h) - f(x)}{h}.$$

- Lower Right Derivative:

$$D_{+} f(x) = \lim_{h \to 0^{+}} \inf \frac{f(x+h) - f(x)}{h}.$$

- Upper Left Derivative:

$$D^{-} f(x) = \lim_{h \to 0^{-}} \sup \frac{f(x+h) - f(x)}{h}.$$

- Lower Left Derivative:

$$D_{-} f(x) = \lim_{h \to 0^{-}} \inf \frac{f(x+h) - f(x)}{h}.$$

- If f is an integrable function on $[a, b]$ then the function F defined by

$$F(x) = \int_{a}^{x} f(t)dt + F(a)$$

  is called the indefinite integral of f.
- Let f be an integrable function on $[a, b]$. A point x in $[a, b]$ is said to be a Lebesgue point of f if

$$\lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)| dt = 0$$

- The set of all Lebesgue points of a function f in $[a, b]$ is called the Lebesgue set of a function f.
- Let $E \subseteq \mathbb{R}$, $V$is the collection of closed intervals of positive lengths is said to be Vitali Cover of E if for given $\epsilon > 0$ and any $x \in E, \exists$ an interval $I \in V$ such that $x \in I$ and $l(I) < \epsilon$.
- Let E be a set of finite outer measure and V be a Vitali cover of E. Given $\epsilon > 0$, □ a finite disjoint collection $\{I_1, I_2, \dots, I_N\}$ of intervals in V such that

$$m^{*}\left(E - \bigcup_{n=1}^{N} I_n\right) < \epsilon.$$

- Let f be increasing the real-valued function on $[a, b]$. Then f is differentiable $a.e.$, $f'$is measurable and

$$\int_{a}^{b} f'(x)\, dx \leq f(b) - f(a).$$

- Let f be an integrable function on $[a, b]$. Then the indefinite integral of f is a continuous function of bounded variation on $[a, b]$.
- If f is an integrable function on $[a, b]$and $\int_{a}^{x} f(t)dt = 0$ for all $x \in [a, b]$ then $f = 0$ $a.e.$ in$[a, b]$.
- Every point of continuity of an integrable function f is a Lebesgue point of f.

- Let f be bounded measurable function defined on $[a,b]$. If $F(x)=\int_a^x f(t)dt+F(a)$ then $F'(x)=f(x)\ a.e.$ on$[a,b]$.
- Let f be an integrable function on $[a,b]$ and $F(x)=\int_a^x f(t)dt+F(a)$, then $F'(x)=f(x)\ a.e.$ in$[a,b]$.

Keywords

Upper Right Derivative: $D^+ f(x)=\lim_{h\to 0^+}\sup\frac{f(x+h)-f(x)}{h}$.

LowerRight Derivative: $D_+ f(x)=\lim_{h\to 0^+}\inf\frac{f(x+h)-f(x)}{h}$.

UpperLeft Derivative: $D^- f(x)=\lim_{h\to 0^-}\sup\frac{f(x+h)-f(x)}{h}$.

LowerLeft Derivative: $D_- f(x)=\lim_{h\to 0^-}\inf\frac{f(x+h)-f(x)}{h}$.

Indefinite Integral: If f is an integrable function on $[a,b]$ then the function F defined by

$$F(x)=\int_a^x f(t)dt+F(a)$$

is called the indefinite integral of f.

Lebesgue Point of a Function: Let f be an integrable function on $[a,b]$. A point x in $[a,b]$ is said to be a Lebesgue point of f if

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}|f(t)-f(x)|dt=0$$

Lebesgue set of a function: The set of all Lebesgue points of a function f in $[a,b]$ is called the Lebesgue set of a function f.

Vitali Cover: Let $E\subseteq\mathbb{R}$, $V$is the collection of closed intervals of positive lengths is said to be Vitali Cover of E if for given $\epsilon>0$ and any $x\in E,\exists$ an interval $I\in V$ such that $x\in I\ and\ l(I)<\epsilon$.

Vitali's Covering Lemma: Let E be a set of finite outer measure and V be a Vitali cover of E.

Given $\epsilon>0$, $\exists$ a finite disjoint collection $\{I_1,I_2,\dots,I_N\}$ of intervals in V such that

$$m^*\left(E-\bigcup_{n=1}^{N} I_n\right)<\epsilon.$$

Lebesgue Theorem: Let f be increasing the real-valued function on $[a,b]$. Then f is differentiable $a.e.$, $f'$is measurable and

$$\int_a^b f'(x)\,dx\le f(b)-f(a).$$

Self Assessment

1) Consider the following statements:

(I) $D^+f(x)=\lim_{h\to 0^+}\inf\frac{f(x+h)-f(x)}{h}$

(II) $D^-f(x)=\lim_{h\to 0^+}\sup\frac{f(x+h)-f(x)}{h}$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

2) Consider the following statements:

(I) $D_{-}f(x) = \lim_{h\to 0^{+}} \inf \frac{f(x+h)-f(x)}{h}$

(II) $D_{+}f(x) = \lim_{h\to 0^{+}} \inf \frac{f(x+h)-f(x)}{h}$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

3) Consider the following statements:

(I) $D^{+}f(x) \geq D_{+}f(x)$

(II) $D^{-}f(x) \geq D_{-}f(x)$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

4) Consider the following statements:

(I) $D^{+}(-f) = -D^{+}(f)$

(II) $D^{-}(-f) = D_{-}f(x)$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

5) Consider the following statements:

(I) $-D^{+}(-f) = D_{+}(f)$

(II) $-D^{-}(-f) = D_{-}(f)$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

6) Consider the following statements:

(I) Let f be an integrable function on [a, b]. Then the indefinite integral of f is a continuous function on [a, b].

(II) Let f be an integrable function on [a, b]. Then the indefinite integral of f is a function of bounded variation on [a, b].

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

7) If f is an integrable function on [a, b] and $\int_a^x f(t)dt = 0, \forall x \in [a,b]$ then $f = 0\ a.e.$ on [a, b].

A. True
B. False

8) Let x be a Lebesgue point of integrable function f then

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt < 0.$$

A. True
B. False

9) Every point of continuity of an integrable function f need not be Lebesgue point of f.

A. True
B. False

10) Let f be an integrable function on [a, b] and $F(x) = \int_a^x f(t)dt + F(a)$, then $F'(x) = f(x) a.e. on\ [a,b]$.

A. True
B. False

11) Let $f_n(x) = \begin{cases} f(x); & f(x) \le n \\ n; & f(x) > n \end{cases}$ then each f_n need not be bounded.

A. True
B. False

12) Let $f - g \ge 0$ and $\int_a^b (f - g) = 0$ then $f = g\ a.e.$

A. True
B. False

13) A family $\mathcal{H}$ of closed intervals of positive lengths is said to be Vitali cover of set E if for given □> 0 and any $x \in E$, there exists an interval $I \in \mathcal{H}$ such that $x \in I$ and $l(I) > \epsilon$.

A. True
B. False

14) Let $\mathcal{H}$ be Vitali cover of $E, m^*(E) < \infty$, then for any $\in > 0$, there exists a finite disjoint collection $\{I_n\}, n = 1,2, \dots, N$ of intervals in $\mathcal{H}$ such that

$$m^*(E - \bigcup_{n=1}^{N} I_n) < \epsilon.$$

A. True
B. False

15) Let I_1, I_2 be two intervals such that I_1 has a point in common with I_2. Let $x \in I_1$, then the distance of x from the midpoint of I_2 is at most $l(I_1) + \frac{1}{3} l(I_2)$.

A. True
B. False

16) Let f be monotonically increasing real-valued function defined on [a, b]. Then f is differentiable almost everywhere.

A. True
B. False

17) Let f be monotonically increasing real-valued function defined on [a, b]. Then f' is measurable.

A. True
B. False

18) Let f be monotonically increasing real-valued function defined on [a, b]. Then

A. $\int_a^b f'(x) \leq f(b) - f(a)$

B. $\int_a^b f'(x) \leq f(a) - f(b)$

C. $\int_a^b f'(x) \geq f(b) + f(a)$

D. $\int_a^b f'(x) \leq f(b) + f(a)$

19) Let $h_n(x) = f\left(x + \frac{1}{n}\right) - f(x)$, f is monotonically increasing function then $\{h_n\}$ is a sequence of nonnegative measurable functions.

A. True
B. False

Answers for Self Assessment

1.	D	2.	B	3.	C	4.	D	5.	C
6.	C	7.	A	8.	B	9.	B	10.	A
11.	B	12.	A	13.	B	14.	A	15.	B
16.	A	17.	A	18.	A	19.	A		

Review Questions

1) If $f(x) = \begin{cases} x\cos\frac{1}{x}; x \neq 0 \\ 0; x = 0 \end{cases}$, find $D_+ f(0), D^+ f(0), D^- f(0), D_- f(0)$.

2) If $f(x) = 2|x|$, find $D_+ f(0), D^+ f(0), D^- f(0), D_- f(0)$.

3) If $f(x) = \begin{cases} 0; x \in \mathbb{Q} \\ 5; x \notin \mathbb{Q} \end{cases}$, find $D_+ f(0), D^+ f(0), D^- f(0), D_- f(0)$.

4) If

$$f(x) = \begin{cases} 2x\sin^2\frac{1}{x} + 3x\cos^2\frac{1}{x}, & x > 0 \\ 3x\sin^2\frac{1}{x} + 5x\cos^2\frac{1}{x}, & x > 0' \\ 0 \quad , & x = 0 \end{cases}$$

$a < b, p < q$, find $D_+ f(0), D^+ f(0), D^- f(0), D_- f(0)$.

5) Let f be bounded measurable function defined on $[a, b]$. If F be the indefinite integral of f F. Then $F'(x) = f(x)$ $a.e.$ on$[a, b]$.

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Graduate Texts in Mathematics, Measure Integration and Real Analysis, Sheldon Axler, Springer.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

https://www.youtube.com/watch?v=YIrx8W5nyq8&t=29s

Unit 09: Abstract Measure Spaces

CONTENTS

Objectives

After studying this unit, students will be able to:

- define ring and σ −ring
- explain algebra and σ − algebra
- definehereditaryσ −ring
- demonstrate measure on ring $\mathcal{R}$
- understand measurable space and measure space

Introduction

In this unit, the definition of measurable sets and functions are provided for abstract measure spaces. We present in this general setting, the main results of measurable sets, measurable functions, and Lebesgue integral on the real line provided in the previous units.

9.1 Abstract Measure Spaces

Definition: A class of sets $\mathcal{R}$, of some fixed space X, is called a ringif whenever $E \in \mathcal{R}$and $F \in \mathcal{R}$, then $E \cup F$ and $E - F$ belong to $\mathcal{R}$.

$e.g.$ The class of finite unions of intervals of the form $[a, b)$ forms a ring.

Notes: Since the class is closed under the union of two sets, it is closed under the union of the finite number of sets.

Notes: $A\Delta B = (A - B) \cup (B - A) \in \mathcal{R}$
$A \cap B = (A \cup B) - (A\Delta B) \in \mathcal{R}$.

Notes: The intersection of a finite collection of rings on X is again a ring.

Notes: If E is a class of subsets of X, then the intersection of all rings containing E is a ring which is called the ring generated by E and is denoted by $\mathcal{R}(E)$. $i.e.$ $\exists$ the smallest ring containing a given class of subsets of space X called a generated ring.

Definition: If the ring $\mathcal{R}$ has the property that $A \in \mathcal{R} \Rightarrow A^c \in \mathcal{R}$ for every $A \in \mathcal{R}$ then $\mathcal{R}$ is called algebra.

Notes: Every algebra is a ring but the converse is not true.

e.g. In an infinite set, if we take the collection of all finite subsets then it is a ring but not an algebra.

Definition: A ring is called a $\sigma -$ ring if it is closed under the formation of countable unions.

Notes: $\sigma -$ ring is closed under a countable intersection.

Notes: A σ – ring is a ring but the converse is not true.

Notes: The intersection of a family of σ – rings is again a σ – ring.

Notes: We define $\mathcal{S}(E)$ to be the smallest σ – ring containing a class of sets which is called the

σ – ring generated by E.

Definition: A $\sigma -$ ring is called a $\sigma -$ algebra if it is closed under complementation.

Definition: A class of sets $\mathcal{H}$ with the property, namely that every subset of one of its members belongs to the class, is said to be hereditary *i.e.* if $A \in \mathcal{H}$ and $B \subset A \Rightarrow B \in \mathcal{H}$ then $\mathcal{H}$ is said to be hereditary.

Definition: A class of sets $\mathcal{H}$, is called hereditary $\sigma -$ ring if it is $\sigma -$ ring and is hereditary.

Notes: $\mathcal{S}(\mathcal{R})$ denotes the σ – ring generated by the ring $\mathcal{R}$.

Notes: $\mathcal{H}(\mathcal{R})$ denotes the class consisting of $\mathcal{S}(\mathcal{R})$ together with all subsets of the sets of $\mathcal{S}(\mathcal{R})$.

Notes: $\mathcal{H}(\mathcal{R})$ is a σ – ring and is the smallest hereditary σ – ring containing $\mathcal{R}$.

Notes: The intersection of hereditary σ – ring is again a hereditary σ – ring.

Notes: $\mathcal{H}(\mathcal{R}) = \mathcal{H}(\mathcal{S}(\mathcal{R})) = \mathcal{H}(\mathcal{H}(\mathcal{R}))$.

Definition: A set function μ defined on a ring $\mathcal{R}$ is a measure if

1) μ is non-negative
2) $\mu(\phi) = 0$
3) for any sequence $\{A_n\}$ of disjoint sets of $\mathcal{R}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition: If $\mathcal{R}$ is a ring, a set function μ^* defined on the class $\mathcal{H}(\mathcal{R})$ is anouter measure if:

1) μ^* is non-negative.
2) If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$
3) $\mu^*(\phi) = 0$
4) for any sequence $< A_n >$ of sets of $\mathcal{H}(\mathcal{R})$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Notes: Lebesgue outer measure m^* is an outer measure in the sense of the above definition.

Example: Show that if $A, B \in \mathcal{R}$ and $A \subseteq B$ then $\mu(A) \leq \mu(B)$.

Solution: Since $B = A \cup (B - A)$

$$\Rightarrow \mu(B) = \mu(A) + \mu(B - A)$$

$$\Rightarrow \mu(A) \leq \mu(B)$$

Theorem 9.1: Let $\{A_i\}$ be a sequence in a ring $\mathcal{R}$, then there is a sequence $\{B_i\}$ of disjoint sets of $\mathcal{R}$ such that $B_i \subseteq A_i$ for each i and

$$\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$$

for each N, so that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i.$$

Proof: Define $\{B_i\}$as follows.

$B_1 = A_1,$

$$B_n = A_n - \bigcup_{i=1}^{n-1} B_i, n > 1 \qquad \ldots(1)$$

$B_i \in R$ and $B_i \subseteq A_i$ for each i.

Also, as B_n and $\bigcup_{i=1}^{n-1} B_i$ are disjoint, we have $B_n \cap B_m = \phi$ for $n > m$.

$B_1 = A_1$ and if

$$\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i \qquad \ldots(2)$$

then

$$B_{k+1} \cup \left(\bigcup_{i=1}^{k} B_i\right) = \left(A_{k+1} - \bigcup_{i=1}^{k} B_i\right) \cup \left(\bigcup_{i=1}^{k} B_i\right)$$

$$= (A_{k+1}) \cup \left(\bigcup_{i=1}^{k} B_i\right)$$

$$= (A_{k+1}) \cup \left(\bigcup_{i=1}^{k} A_i\right) \{\text{by } (2)\}$$

$$i.e. B_{k+1} \cup \left(\bigcup_{i=1}^{k} B_i\right) = A_{k+1} \cup \left(\bigcup_{i=1}^{k} A_i\right)$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i.$$

This completes the proof.

Theorem 9.2: If μ is a measure on ring$\mathcal{R}$and if the set function μ^* is defined on $\mathcal{H}(\mathcal{R})$ by

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in R, n = 1,2, \ldots, \quad E \subseteq \bigcup_{n=1}^{\infty} E_n\right\}, \qquad \ldots(1)$$

then

i) for $E \in \mathcal{R}, \mu^*(E) = \mu(E)$

ii) μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$.

Proof: (i) If $E \in \mathcal{R}$ then (1) gives $\mu^*(E) \leq \mu(E)$ ... (2)

Now, suppose $E \in \mathcal{R}$ and

$$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

where $E_n \in \mathcal{R}$ then we may replace the sequence $\{E_i \cap E\}$ by a sequence $\{F_i\}$ of disjoint sets of $\mathcal{R}$ such that $F_i \subseteq E_i \cap E$ and

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (E_i \cap E) = E$$

$$i.e. \bigcup_{i=1}^{\infty} F_i = E \qquad \text{... (3)}$$

Since

$F_i \subseteq E_i \cap E \Rightarrow F_i \subseteq E_i$

$\Rightarrow \mu(F_i) \leq \mu(E_i)$ for each i ... (4)

Now,

$$\mu(E) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} \mu(F_i)$$

$$\leq \sum_{i=1}^{\infty} \mu(E_i)$$

$$i.e.\ \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

$\Rightarrow \mu(E) \leq \mu^*(E)$ ... (5)

From (2) and (5), we get

$$\mu(E) = \mu^*(E).$$

(ii) Now we will show that μ^* is an outer measure on $H(\mathcal{R})$. By part (i) we get

1) μ^* is non-negative
2) If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$
3) $\mu^*(\phi) = 0$

Next, we will show that for any sequence $< E_i >$ of sets of $\mathcal{H}(\mathcal{R})$,

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

Suppose $\{Ei\}$ is a sequence of sets in $\mathcal{H}(\mathcal{R})$ then

$$\mu^*(E_i) = \inf\left\{\sum_{j=1}^{\infty} \mu(E_{i,j}) : \{E_{i,j}\} \text{is a sequence of sets of} \mathcal{R} \text{for each i such that} E_i \subseteq \bigcup_{j=1}^{\infty} E_{i,j}\right\}$$

Therefore, for given $\epsilon > 0$ and for each E_i, $\exists$ a collection $\{E_{i,j}\}$ of sets of $\mathcal{R}$ such

$$E_i = \bigcup_{j=1}^{\infty} E_{i,j}$$

and

$$\sum_{j=1}^{\infty} \mu(E_{i,j}) \le \mu^*(E_i) + \frac{\epsilon}{2^i} \qquad \ldots (6)$$

The sets $E_{i,j}$ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$, so

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu(E_{i,j})$$

$$\le \sum_{i=1}^{\infty} [\mu(E_i) + \frac{\epsilon}{2^i}]$$

$$= \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i}$$

$$= \sum_{j=1}^{\infty} \mu^*(E_i) + \epsilon$$

$$\therefore \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu^*(E_i)$$

Theorem 9.3: Show that

$$\mathcal{H}(\mathcal{R}) = \left\{E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\right\}$$

Proof: Here RHS defines a class of sets that is hereditary, contains $\mathcal{R}$, and is a σ −ring.

So, it contains $\mathcal{H}(\mathcal{R})$. $i.e.$

$$\mathcal{H}(\mathcal{R}) \subseteq \left\{E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\right\} \qquad \ldots (1)$$

Now, if $E_n \in \mathcal{R}$ for each n,

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}(\mathcal{R})$$

and so, each subset belongs to $\mathcal{H}(R)$. $i.e.$

$$\left\{E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\right\} \subseteq \mathcal{H}(\mathcal{R}) \qquad \ldots (2)$$

From (1) and (2), we get

$$\mathcal{H}(\mathcal{R}) = \left\{E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\right\}.$$

This completes the proof.

Definition: Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ then $E \in \mathcal{H}(\mathcal{R})$ is μ^* −measurable if for each

$$A \in \mathcal{H}(\mathcal{R}),$$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Theorem 9.4: Let μ^* be the outer measure on $\mathcal{H}(\mathcal{R})$ defined by μ on $\mathcal{R}$, then $\mathcal{S}^*$contains $\mathcal{S}(\mathcal{R})$, the σ −ring generated by $\mathcal{R}$. Here $\mathcal{S}^*$ denotes the class of μ^* −measurable sets which is a σ −ring.

Proof: Since $\mathcal{S}^*$ is a σ −ring, therefore it is sufficient to show that $\mathcal{R} \subseteq \mathcal{S}^*$.

If $E \in \mathcal{R}, A \in H(\mathcal{R}), \epsilon > 0$, ∃ a sequence $\{E_n\}$ of sets of $\mathcal{R}$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} E_n \qquad \ldots (1)$$

and

$$\mu^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \mu(E_n)$$

$$= \sum_{n=1}^{\infty} \mu[(E_n \cap E) \cup (E_n \cap E^c)]$$

$$= \sum_{n=1}^{\infty} \mu(E_n \cap E) + \sum_{n=1}^{\infty} \mu(E_n \cap E^c) \qquad \ldots (2)$$

Also, we have, $\mu^*(A \cap E) \leq \sum_{n=1}^{\infty} \mu(E_n \cap E)$, $\mu^*(A \cap E^c) \leq \sum_{n=1}^{\infty} \mu(E_n \cap E^c)$

$$(2) \Rightarrow \mu^*(A) + \epsilon \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\Rightarrow \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\therefore \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\Rightarrow E \text{ is } \mu^* - \text{measurable}$$

$$\Rightarrow E \in \mathcal{S}^* \Rightarrow R \subseteq \mathcal{S}^*.$$

This completes the proof.

Definition: A measure μ on $\mathcal{R}$ is complete if whenever $E \in \mathcal{R}, F \subseteq E$, and $\mu(E) = 0$ then $F \in \mathcal{R}$.

Definition: A measure μ on $\mathcal{R}$ is σ −finite if, for every set $E \in \mathcal{R}$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for each n.

Theorem 9.5: Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ and let $\mathcal{S}^*$ denote the class of μ^* −measurable sets which forms a σ −ring then μ^* restricted to $\mathcal{S}^*$ is a complete measure.

Proof: Suppose $\{E_i\}$ is a sequence of disjoint sets in $\mathcal{S}^*$, we get

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i)$$

So μ^* is a measure on the σ −ring $\mathcal{S}^*$. Also, every set $E \in \mathcal{H}(\mathcal{R})$ such that $\mu^*(E) = 0$ is μ^* −measurable because if $A \in \mathcal{H}(\mathcal{R})$ then

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\leq \mu^*(E) + \mu^*(A) = \mu^*(A)$$

and E is μ^*-measurable. In particular, if $E \in \mathcal{S}^*$ and $\mu^*(E) = 0$ and $F \subseteq E$ then it follows that $F \in \mathcal{S}^*$

$\Rightarrow$ So μ^* is a complete measure on $\mathcal{S}^*$.

This completes the proof.

Notes: If μ^* be the outer measure on $\mathcal{H}(\mathcal{R})$ defined by μ on $\mathcal{R}$ *i.e.* if

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{R}, n = 1,2,\ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n\right\},$$

we will denote the measure obtained by restricting μ^* to $\mathcal{S}^*$, by $\bar{\mu}$. Then $\bar{\mu}$ is an extension of μ.

Theorem 9.6: Show that if μ is a σ −finite measure on $\mathcal{R}$ then the extension $\bar{\mu}$ of μ to $\mathcal{S}^*$ is also σ −finite.

Proof: Let $E \in \mathcal{S}^*$. Then by definition of $\bar{\mu}$ there is a sequence $\{E_n\}$ of sets of $\mathcal{R}$ such that

$$\bar{\mu}(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

But each E_n is the union of a sequence $\{E_{n,i}, i = 1,2,\ldots\}$ of sets of $\mathcal{R}$ such that $\mu(E_n, i) < \infty$ for each n and i. Therefore,

$$\bar{\mu}(E) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu(E_n, i)$$

and so E is the union of a countable collection of sets of finite $\bar{\mu}$ −measure.

Definition: (X, S) where S is a σ −algebra of subsets of a space X,which is called a measurable space. The sets of S are called measurable sets.

Definition: (X, S, μ) is called a measure space if (X, S) is a measurable space and μ is a measure on S.

Definition: Let f be an extended real-valued function defined on X. Then f is said to be measurable if $\forall \alpha$,

$$\{x: f(x) > \alpha\} \in S$$

Integration with respect to a measure:

1) Let ϕ be a simple function defined as

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

then

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \mu_{A_i}$$

2) Let f be non-negative measurable function defined on X then

$$\int f \, d\mu = \sup\left\{\int \phi \, d\mu : \phi \leq f, \phi \text{ is a simple function}\right\}$$

3) Let f be non-negative measurable function defined on $E \in S$

$$\int_E f d\mu = \int_E f \, \chi_E \, d\mu$$

4) If f is measurable and both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite, then f is said to be integrable and

$$\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$$

5) f is integrable ifand only if$|f|$ is integrable.
6) $f \in L(X, \mu)$
7) $\int f \, d\mu$ means $\int f \, \chi_E \, d\mu, f \in L(X, \mu), E \in S$
8) $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$, provided at least one of the integrals on RHS is finite.

Theorem 9.12: Let f, g be integrable functions and a and b be constant, then $af + bg$ is integrable and

$$\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu$$

If $f = g \; a.e.$ then

$$\int f \, d\mu = \int g \, d\mu$$

Theorem 9.13: Let f be integrable then

$$\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$$

With equality if, and only if,

$$f \geq 0 \; a.e. \text{or } f \leq 0 \; a.e.$$

We also have Fatou's lemma, monotone convergence theorem, Lebesgue dominated convergence theorem in the case of fixed measure space (X, S, μ).

The proofs of the above theorems work on the same line as we have done in Lebesgue measure for real line with only changes in notation i.e., dx by $d\mu$.

Summary

- A class of sets $\mathcal{R}$, of some fixed space X, is called a ring if whenever $E \in \mathcal{R}$and $F \in \mathcal{R}$, then $E \cup F$ and $E - F$ belong to $\mathcal{R}$.
- If the ring $\mathcal{R}$ has the property that $A \in \mathcal{R}$ □ $A^c \in \mathcal{R}$ for every $A \in \mathcal{R}$ then $\mathcal{R}$ is called an Algebra.
- A ring is called a $\sigma -$ ring if it is closed under the formation of countable unions.
- A $\sigma -$ ring is called a $\sigma -$ algebra if it is closed under complementation.
- A class of sets $\mathcal{H}$, is called hereditary $\sigma -$ ring if it is $\sigma -$ ring and is hereditary.
- A set function μ defined on a ring $\mathcal{R}$ is a measure if

1) μ is non-negative
2) $\mu(\phi) = 0$
3) for any sequence $\{A_n\}$ of disjoint sets of $\mathcal{R}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

- If $\mathcal{R}$ is a ring, a set function μ^* defined on the class $\mathcal{H}(\mathcal{R})$ is an outer measure if:

1) μ^* is non-negative.
2) If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$
3) $\mu^*(\phi) = 0$
4) for any sequence $< A_n >$ of sets of $\mathcal{H}(\mathcal{R})$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

- A measure μ on $\mathcal{R}$ is complete if whenever $E \in \mathcal{R}, F \subseteq E$, and $\mu(E) = 0$ then $F \in \mathcal{R}$.
- A measure μ on $\mathcal{R}$ is $\sigma -$finite if, for every set $E \in \mathcal{R}$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for each n.
- $(X, \mathcal{S})$ where $\mathcal{S}$ is a $\sigma -$algebra of subsets of a space X, is called a measurable space. The sets of $\mathcal{S}$ are called measurable sets.
- $X, \mathcal{S}, \mu)$ is called a measure space if $(X, \mathcal{S})$ is a measurable space and μ is a measure on $\mathcal{S}$.
- The ring $\mathcal{R}$ is closed under the union of a finite number of sets.
- $A \Delta B = (A - B) \cup (B - A) \in \mathcal{R}$
- $A \cap B = (A \cup B) - (A \Delta B) \in \mathcal{R}$.
- The intersection of a finite collection of rings on X is again a ring.
- If E is a class of subsets of X, then the intersection of all rings containing E is a ring called the ring generated by E and is denoted by $\mathcal{R}(E)$. $i.e. \exists$ the smallest ring containing a given class of subsets of space X called a generated ring.
- Every algebra is a ring but the converse is not true.
- $\sigma -$ ring is closed under a countable intersection.
- A $\sigma -$ ring is a ring but the converse is not true.
- The intersection of a family of $\sigma -$ rings is again a $\sigma -$ ring.
- We define $S(E)$ to be the smallest $\sigma -$ ring containing a class of sets which is called the $\sigma -$ ring generated by E.
- A class of sets $\mathcal{H}$ with the property, namely that every subset of one of its members belongs to the class, is said to be hereditary $i.e.$ if $A \in \mathcal{H}$ and $B \subset A \Rightarrow B \in \mathcal{H}$ then $\mathcal{H}$ is said to be hereditary.

- $\mathcal{S}(\mathcal{R})$ denotes the $\sigma -$ ring$\mathcal{S}$ generated by the ring $\mathcal{R}$.
- $\mathcal{H}(\mathcal{R})$ denotes the class consisting of $\mathcal{S}(\mathcal{R})$ together with all subsets of the sets of $\mathcal{S}(\mathcal{R})$.
- $\mathcal{H}(\mathcal{R})$ is a $\sigma -$ ring and is the smallest hereditary $\sigma -$ ring containing $\mathcal{R}$.
- The intersection of hereditary $\sigma -$ ring is again a hereditary $\sigma -$ ring.
- $\mathcal{H}(\mathcal{R}) = \mathcal{H}(\mathcal{S}(\mathcal{R})) = \mathcal{H}(\mathcal{H}(\mathcal{R}))$.
- Let $\{A_i\}$ be a sequence in a ring $\mathcal{R}$, then there is a sequence$\{B_i\}$ of disjoint sets of $\mathcal{R}$ such that $B_i \subseteq A_i$ for each i and $\cup_{i=1}^{N} A_i = \cup_{i=1}^{N} B_i$for each N, so that $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$.
- If μ is a measure on ring$R$ and if the set function μ^* is defined on $\mathcal{H}(\mathcal{R})$ by

  $$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in R, n = 1,2,\dots,\ \ E \subseteq \bigcup_{n=1}^{\infty} E_n\right\},$$

  then for $E \in \mathcal{R}, \mu^*(E) = \mu(E)$ and μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$.
- $\mathcal{H}(\mathcal{R}) = \{E : E \subseteq \cup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\}$
- Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$then $E \in \mathcal{H}(\mathcal{R})$ is μ^* −measurable if for each $A \in \mathcal{H}(\mathcal{R})$,

  $$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$
- Let μ^* be the outer measure on $\mathcal{H}(\mathcal{R})$ defined by μ on $\mathcal{R}$, then $\mathcal{S}^*$contains $\mathcal{S}(\mathcal{R})$, the σ −ring generated by $\mathcal{R}$. Here $\mathcal{S}^*$ denotes the class of μ^* −measurable sets which is a σ −ring.
- Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ and let $\mathcal{S}^*$ denote the class of μ^* −measurable sets which forms a σ −ring then μ^* restricted to $\mathcal{S}^*$ is a complete measure.
- If μ^* be the outer measure on $\mathcal{H}(\mathcal{R})$ defined by μ on $\mathcal{R}$ $i.e.$ if$\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{R}, n = 1,2,\dots, E \subseteq \cup_{n=1}^{\infty} E_n\}$,we will denote the measureobtained by restricting μ^* to $\mathcal{S}^*$, by $\bar{\mu}$. Then $\bar{\mu}$ is an extension of μ.
- if μ is a σ −finite measure on $\mathcal{R}$ then the extension $\bar{\mu}$ of μ to $\mathcal{S}^*$ is also σ −finite.

Keywords

Ring: A class of sets $\mathcal{R}$, of some fixed space X, is called a ring if whenever $E \in \mathcal{R}$and $F \in \mathcal{R}$, then $E \cup F$ and $E - F$ belong to $\mathcal{R}$.

Algebra: If the ring $\mathcal{R}$ has the property that $A \in \mathcal{R} \Rightarrow A^c \in \mathcal{R}$ for every $A \in \mathcal{R}$ then $\mathcal{R}$ is called an Algebra.

σ − ring: A ring is called a $\sigma -$ ring if it is closed under the formation of countable unions.

σ −algebra: A $\sigma -$ ring is called a $\sigma -$ algebra if it is closed under complementation.

Hereditary σ − ring: A class of sets $\mathcal{H}$, is called hereditary $\sigma -$ ring if it is $\sigma -$ ring and is hereditary.

Measure on Ring $\mathcal{R}$: A set function μ defined on a ring $\mathcal{R}$ is a measure if

4) μ is non-negative
5) $\mu(\phi) = 0$
6) for any sequence $\{A_n\}$ of disjoint sets of $\mathcal{R}$ such that $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Outer measure on $\mathcal{H}(\mathcal{R})$: If $\mathcal{R}$ is a ring, a set function μ^* defined on the class $\mathcal{H}(\mathcal{R})$ is an outer measure if:

5) μ^* is non-negative.
6) If $A \subseteq B$ then $\mu^*(A) \le \mu^*(B)$
7) $\mu^*(\phi) = 0$
8) for any sequence $< A_n >$ of sets of $\mathcal{H}(\mathcal{R})$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Complete measure: A measure μ on $\mathcal{R}$ is complete if whenever $E \in \mathcal{R}, F \subseteq E$, and $\mu(E) = 0$ then $F \in \mathcal{R}$.

σ −finítemeasure: A measure μ on $\mathcal{R}$ is σ −finite if, for every set $E \in \mathcal{R}$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for each n.

Measurable space:$(X, \mathcal{S})$ where $\mathcal{S}$ is a σ −algebra of subsets of a space X, is called a measurable space. The sets of $\mathcal{S}$ are called measurable sets.

Measure space: $(X, \mathcal{S}, \mu)$ is called a measure space if $(X, \mathcal{S})$ is a measurable space and μ is a measure on $\mathcal{S}$.

Self Assessment

1) Consider the following statements:

(I) Every ring is an algebra.

(II) Every algebra is a ring.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

2) Consider the following statements:

(I) σ −ring is closed under the countable union.

(II) σ −ring is closed under the countable intersection.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

3) Consider the following statements:

(I) Every σ −ring is a σ −algebra.

(II) Every σ −algebra is a σ −ring.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

4) Let μ be a measure on a ring $\mathcal{R}$ and $\varkappa(\mathcal{R}) betheHereditary \sigma - ring$. If the set function μ^* defined on $\varkappa(\mathcal{R})$ is given by $\mu^*(E) = Inf\{\sum_{n=}^{\infty} \mu(E_n) : E_n \in \mathcal{R}, n = 1,2, \dots, E \subseteq \bigcup_{n=1}^{\infty} E_n\}$ then

A. $\mu^*(E) = \mu(E)$
B. $\mu^*(E) > \mu(E)$
C. $\mu^*(E) < \mu(E)$
D. none of these

5) Let μ be a measure on a ring $\mathcal{R}$ and $\varkappa(\mathcal{R})$be the Hereditary$\sigma - ring$. If the set function μ^* defined on $\varkappa(\mathcal{R})$ is given by

$\mu^*(E) = Inf\{\sum_{n=}^{\infty} \mu(En) : E_n \in \mathcal{R}, n = 1,2,\dots, E \subseteq \bigcup_{n=1}^{\infty} E_n\}$ then $\mu^*(E)$ is an outer measure on $\varkappa(\mathcal{R})$.

A. True

B. False

6) Let $\varkappa(\mathcal{R})$is the Hereditary$\sigma - ring$ then

$\varkappa(\mathcal{R})$=$\{E: E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}\}$.

A. True

B. False

7) Let μ^* be an outer measure on $\varkappa(\mathcal{R})$. Then $E \in \varkappa(\mathcal{R})$ is μ^* − measurable if for each $A \in \varkappa(\mathcal{R})$, we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

A. True

B. False

8) A measure μ on $\mathcal{R}$ is complete if whenever $E \in \mathcal{R}, F \subseteq E$, and $\mu(E) > 0$ then $F \in \mathcal{R}$.

A. True

B. False

9) A measure μ on $\mathcal{R}$ is σ −finite if, for every set $E \in \mathcal{R}$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) = \infty$ for each n.

A. True

B. False

10) Let μ^* be an outer measure on $\varkappa(\mathcal{R})$ and let S^* denote the class of μ^* − measurable sets which forms a σ −ring then μ^* restricted to S^* need not be a complete measure.

A. True

B. False

11) If μ is a σ −finite measure on $\mathcal{R}$ then the extension $\bar{\mu}$ of μ to S^* is also σ −finite.

A. True

B. False

12) If f is measurable and both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite then f is necessarily integrable.

A. True

B. False

13) Consider the following statements:

(I) If $\{E_n\}$ be an infinite decreasing sequence of measurable sets such that $\mu(E_1) < \infty$ then $\mu(\bigcap_{n=1}^{\infty} E_i) = \lim \mu(E_n)$.

(II) If $\{E_n\}$ be an infinite increasing sequence of measurable sets then $\mu(\cup_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

14) Consider the following statements:

(I)A measurable function f is integrable if and only if $| f |$ is integrable.

(II)$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$, provided at least one of the integrals on RHS is finite.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

15) Consider the following statements:

(I)Let f be integrable then $\int |f| d\mu \geq \left|\int f d\mu\right|$.

(II)If $f = g\ a.e.$, then $\int f d\mu < \int g d\mu$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

16) Let ϕ be a simple function whose canonical representation is given as

$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ then $\int \phi d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$.

A. True
B. False

17) Consider the following statements:

(I)$\int_E f d\mu \neq \int f \chi_E d\mu$

(II) If f is continuous and g is measurable then $f \circ g$ need not be measurable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	B	2.	C	3.	B	4.	A	5.	A
6.	A	7.	A	8.	B	9.	B	10.	B
11.	A	12.	A	13.	C	14.	C	15.	A
16.	A	17.	D						

Review Questions

1) Let $\{E_i\}$ be a sequence of measurable sets. We have

i) If $E_1 \subseteq E_2 \subseteq E_3 \ldots$, then $\mu(\cup_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

ii) If $E_1 \supseteq E_2 \supseteq E_3 \ldots$, and $\mu(E_1) < \infty$ then $\mu(\cap_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

2)If f_i is measurable, i=1, 2, …, then $\sup_{1 \le i \le n} f_i$, $\inf_{1 \le i \le n} f_i$, $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ are also measurable.

3) The measurability of f is equivalent to

i) $\forall \alpha, \{x: f(x) \geq \alpha\} \in S$

ii) $\forall \alpha, \{x: f(x) < \alpha\}$ □ S

iii) $\forall \alpha, \{x: f(x) \leq \alpha\}$ □ S

4) If f is measurable then $\{x: f(x) = \alpha\}$ is measurable for each extended real number α.

5) Constant functions are measurable.

6) χ_A is measurable if and only if $A \in S$

7) Let f be a continuous function and g is a measurable function then $f \circ g$ is measurable.

8) If $c \in \mathbb{R}$ and f, g are measurable functions then $cf, f+g, g-f, fg$ are measurable.

Further Readings

Measure theory and integration by G DE BARRA, NewAgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Graduate Texts in Mathematics, Measure Integration and Real Analysis, Sheldon Axler, Springer.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

https://www.youtube.com/watch?v=YIrx8W5nyq8&t=29s

Unit 10: The L^p-Spaces

CONTENTS

Objectives

After studying this unit, students will be able to:

- definenormed linear space
- understandL^p spaces
- explain L^p −norm
- determineL^∞ −norm
- explain theorems related to L^p −spaces

Introduction

In this unit, we discuss an important construction of function spaces which is useful in many branches of analysis. We will study about L^p spaces; their construction and properties. These spaces form an important class of Banach spaces in functional analysis.

10.1 The L^p Spaces

Normed Linear Space: Let X be a linear space over a field of real or complex. A norm on X is a real-valued function, (denoted as$\|\cdot\|$) on X which has the following properties:

i) $\|x\| \geq 0, x \in X.$

ii) $\|x\| = 0 \iff x = 0\ \forall x \in X$

iii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$

iv) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and $\alpha \in \mathbb{R}$(or $\mathbb{C}$)

A linear space $X,$ equipped with the norm $\|\cdot\|$ on it is called a normed linear space or simply a normed space. It is denoted by $(X, \|\cdot\|)$ or simply by $X.$

Metric Space: Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}$ is said to be metric ifand only if d satisfies the following properties:

i) $d(x, y) \geq 0\ \forall x, y \in X$

ii) $d(x, y) = 0 \iff x = y,\ \forall x, y \in X$

iii) $d(x, y) = d(y, x), \forall x, y \in X$

iv) $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$

If d is a metric on $X,$ then the (X, d) is called metric space.

A norm $\|\cdot\|$ on linear space X defines a metric d on X given by $d(x,y) = \|x-y\|$.

i) Let $x, y \in X \Rightarrow x - y \in X$

$\therefore \ \|x-y\| \geq 0 \ \Rightarrow d(x,y) \geq 0$

ii) $d(x,y) = 0$

$\Leftrightarrow \|x-y\| = 0$

$\Leftrightarrow x - y = 0$

$\Leftrightarrow x = y$

iii) $d(x,y) = \|x-y\|$

$= \|-(y-x)\|$

$= \|y-x\|$

$= d(y,x)$

iv) $d(x,z) = \|x-z\|$

$= \|x-y+y-z\|$

$= \|(x-y)+(y-z)\|$

$\leq \|x-y\| + \|y-z\|$

$= d(x,y) + d(y,z)$

Thus (X,d) is a metric space.

Here the metric d is known as the metric induced by the norm.

Definition: A sequence $\{x_n\}$ in a normed linear space X is said to converge to an element $x \in X$ if given an $\epsilon > 0$, there is a positive integer N such that

$$\|x_n - x\| < \epsilon, \forall n \geq N.$$

Definition: A sequence $\{x_n\}$ in a normed linear space X is a Cauchy sequence if given an $\epsilon > 0$, there is a positive integer N such that

$$\|x_n - x_m\| < \epsilon, \forall n, m \geq N.$$

Notes: In a normed linear space, every convergent sequence is a Cauchy sequence but converse may not be true.

Definition: A normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition: A complete normed linear space is called a Banach Space.

Definition: A series $\sum_{n=1}^{\infty} u_n$ in normed linear space X is said to be convergent to u if $u \in X$ and

$\lim_{n\to\infty} s_n = u$, where

$$s_n = u_1 + u_2 + \cdots + u_n.$$

In this case, we write

$$u = \sum_{n=1}^{\infty} u_n.$$

Notes: The series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if

$$\sum_{n=1}^{\infty} \|u_n\| < \infty.$$

Notes: A normed linear space X is complete ifand only ifevery absolutely convergent series is convergent.

Definition: If (X,S,μ) is a measure space and $p > 0$, we define $L^p(X,\mu)$or$L^p(\mu)$, to be the class of measurable functions $\{f: \int |f|^p \, d\mu < \infty\}$ with the convention that any two functions equal $a.e.$ specify the same element of $L^p(\mu)$.

Notes: The elements of the space $L^p(\mu)$ are not functions but classes of functions such that in each class any two functions are equal almost everywhere. Any two functions equal $a.e.$ have the same integrals over each set of S.

Notes: We will write $f \in L^p(\mu)$, then f is measurable and

$$\int |f|^p \, d\mu < \infty.$$

Notes: On the real line, if $X = (a,b)$ and μ is Lebesgue measure, we will write $L^p(a,b)$ for the corresponding space.

Definition: Let $f \in L^p(a,b)$, then the L^p −norm of f, denoted as $\|f\|_p$ is given by

$$\left[\int |f|^p \, d\mu\right]^{\frac{1}{p}}$$

Theorem 10.1: Let $f, g \in L^p(\mu)$ and let a, b be constants; then $af + bg \in L^p(\mu)$.

Proof: Since $f, g \in L^p(\mu)$ therefore

$$\int |f|^p \, d\mu < \infty$$

and

$$\int |g|^p \, d\mu < \infty$$

Consider $|f+g|^p \le [|f| + |g|]^p$

$$\le [2\max\{|f|, |g|\}]^p$$

$$= 2^p \max\{|f|^p, |g|^p\}$$

$$\le 2^p[|f|^p + |g|^p]$$

$$\therefore \int |f+g|^p \le \int 2^p[|f|^p + |g|^p]$$

$$= 2^p \int |f|^p + 2^p \int |g|^p < \infty$$

$\Rightarrow f + g \in L^p(\mu)$ ... (1)

Now, $\int |af|^p = \int |a|^p |f|^p$

$$= |a|^p \int |f|^p < \infty$$

$\Rightarrow af \in L^p(\mu)$ ... (2)

From (1) and (2), we get

$$af + bg \in L^p(\mu).$$

This completes the proof.

Definition: If (X, S, μ) is a measure space, we define $L^\infty(\mu)$ to be the class of measurable functions

$$\{f : ess\sup|f| < \infty\}.$$

L^∞ −norm of $f \in L^\infty(\mu)$ is denoted by $\|f\|_\infty$ and is defined as $\|f\|_\infty = ess\sup|f|$.

Theorem 10.2: Show that $L^\infty(\mu)$ is a vector space over the real numbers.

Proof: Let $f, g \in L^\infty(\mu)$ and $a, b \in \mathbb{R}$

$\Rightarrow ess\sup|f| < \infty$ ... (1)

and

$\Rightarrow ess\sup|g| < \infty$ ... (2)

Now,

$$ess\sup|af + bg| \le ess\sup|af| + ess\sup|bg|$$

$$= |a| ess \sup|f| + |b|\, ess \sup|g| < \infty$$

$$\Rightarrow af + bg \in L^{\infty}(\mu)$$

$\Rightarrow$ $L^{\infty}(\mu)$ is a vector space over the reals.

This completes the proof.

Theorem 10.3: Show that if $\mu(x) < \infty$ and $0 < p < q \leq \infty$ then

$$L^q(\mu) \subseteq L^p(\mu)$$

Proof: **Case I**: For $q < \infty$ let $f \in L^q(\mu)$

$$\Rightarrow \int |f|^q < \infty \qquad \ldots (1)$$

and

$|f|^p \leq 1 + |f|^q$ for $0 < p < q < \infty$ ... (2)

$$\Rightarrow \int |f|^p < \infty$$

$$\Rightarrow f \in L^p(\mu)$$

$$\Rightarrow L^q(\mu) \subseteq L^p(\mu)$$

Case II: For $q = \infty$. Let $f \in L^q(\mu)$ $i.e. f \in L^{\infty}(\mu)$

$$\Rightarrow ess \sup|f| < \infty.$$

Now

$$|f| \leq ess \sup|f| \;\; a.e.$$

$$\Rightarrow |f|^q \leq [ess \sup|f|]^p \; a.e.$$

$$\Rightarrow \int |f|^p < \infty$$

$$\Rightarrow f \in L^p(\mu)$$

Thus $L^q(\mu) \subseteq L^p(\mu)$ whenever $0 < p < q \leq \infty$.

This completes the proof.

Theorem10.4:Let $p > 0$ and $f \in L^p(\mu)$, $f \geq 0$ and let $f_n = \min(f, n)$. Showthat $f_n \in L^p(\mu)$ and $\lim\|f - f_n\|_p = 0$.

Proof:To prove this theorem we will use Lebesgue dominated convergence theorem which states that let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g, g$ is an integrable function and $\lim f_n = f \; a.e.$ Then f is integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu."$$

Since $0 \leq f_n \leq f$

$$\Rightarrow 0 \leq f_n^p \leq f^p,$$

Now, $f \in L^p(\mu) \Rightarrow \int f^p \, d\mu < \infty$

$$\Rightarrow \int f_n^p \, d\mu < \infty \Rightarrow f_n \in L^p(\mu)$$

Also, $0 \leq f - f_n \leq f$

$\Rightarrow 0 \leq (f - f_n)^p \leq f^p$, is an integrable function

and $\lim_{n\to\infty} f_n = \lim_{n\to\infty} \min(f, n) = f$

$$\Rightarrow \lim_{n\to\infty} \int (f - f_n)^p \, d\mu = 0$$

$$\Rightarrow \lim\|f - f_n\|_p = 0.$$

This completes the proof.

Theorem10.5: Let $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(\mu)$ that converges $a.e.$ to $f \in L^p(\mu)$ then $f_n \to f$ in $L^p(\mu)$ ifand only if

$$\lim_{n\to\infty} \int |f_n|^p \, d\mu = \int |f|^p \, d\mu$$

Proof: For each n, we have

$$\big| \|f_n\| - \|f\|_p \big| \leq \|f_n - f\|_p$$

Hence if $f_n \to f$ in$L^p(\mu)$, then

$$\lim_{n\to\infty} \int |f_n|^p \, d\mu = \int |f|^p \, d\mu.$$

Assume

$$\lim_{n\to\infty} \int |f_n|^p \, d\mu = \int |f|^p \, d\mu$$

Define $\psi(t) = |t|^p$ for all t then ψ is convex and thus

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{\psi(a) + \psi(b)}{2}, \text{for all } a, b$$

Hence

$$0 \leq \frac{|a|^p + |b|^p}{2} - \left|\frac{a-b}{2}\right|^p \text{ for all } a, b.$$

Therefore, for each n, a non-negative measurable function h_n is defined by

$$h_n(x) = \frac{|f_n(x)|^p + |f(x)|^p}{2} - \frac{|f_n(x) - f(x)|^p}{2^p} \text{for all } x \in E$$

Since $h_n \to |f|^p$, we inter from Fatou's Lemma that

$$\int |f|^p \, d\mu \leq \liminf \left[\int h_n \, d\mu\right]$$

$$= \liminf \left[\left\{\int \frac{|f_n(x)|^p + |f(x)|^p}{2} - \frac{|f_n(x) - f(x)|^p}{2^p}\right\} d\mu\right]$$

$$= \int |f|^p \, d\mu - \limsup \left[\int \left|\frac{f_n(x) - f(x)}{2}\right|^p d\mu\right]$$

Thus

$$\int |f|^p \, d\mu \leq \int |f|^p \, d\mu - \limsup \left[\int \left|\frac{f_n(x) - f(x)}{2}\right|^p d\mu\right]$$

$$\Rightarrow \limsup \left[\int \left|\frac{f_n(x) - f(x)}{2}\right|^p d\mu\right] \leq 0$$

$$\Rightarrow \lim \int |f_n - f|^p \, d\mu = 0$$

$$i.e. f_n \to f \text{ in} L^p(\mu)$$

This completes the proof.

Summary

- Let X be a linear space over a field of real or complex. A norm on X is a real-valued function, (denoted as $\|\cdot\|$) on X which has the following properties:
  i) $\|x\| \geq 0, x \in X.$
  ii) $\|x\| = 0 \Leftrightarrow x = 0 \; \forall x \in X$
  iii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$
  iv) $\|\alpha x\| = |\alpha| \|x\|, \forall x \in X$ and $\alpha \in \mathbb{R}$(or $\mathbb{C}$)
- A linear space X, equipped with the norm $\|\cdot\|$ on it is called a normed linear space or simply a normed space. It is denoted by $(X, \|\cdot\|)$ or simply by X.

- Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}$ is said to be metric if and only if d satisfies the following properties:

i) $d(x,y) \geq 0 \; \forall x, y \in X$

ii) $d(x,y) = 0 \Leftrightarrow x = y, \; \forall x, y \in X$

iii) $d(x,y) = d(y,x), \forall x, y \in X$

iv) $d(x,z) \leq d(x,y) + d(y,z), \forall x, y, z \in X$

- If d is a metric on X, then the (X,d) is called metric space.
- A sequence $\{x_n\}$ in a normed linear space X is a Cauchy sequence if given an $\epsilon > 0$, there is a positive integer N such that

$$\|x_n - x_m\| < \epsilon, \forall n, m \geq N.$$

- A normed linear space is said to be complete if every Cauchy sequence in it is convergent.
- A complete normed linear space is called a Banach Space.
- If (X, S, μ) is a measure space and $p > 0$, we define $L^p(X,\mu)$ or $L^p(\mu)$, to be the class of measurable functions $\{f: \int |f|^p \, d\mu < \infty\}$ with the convention that any two functions equal $a.e.$ specify the same element of $L^p(\mu)$.
- Let $f \in L^p(a,b)$, then the L^p −norm of f, denoted as $\|f\|_p$ is given by

$$\left[\int |f|^p \, d\mu\right]^{\frac{1}{p}}.$$

- If (X, S, μ) is a measure space, we define $L^\infty(\mu)$ to be the class of measurable functions

$$\{f: ess\sup|f| < \infty\}.$$

- L^∞ −norm of $f \in L^\infty(\mu)$ is denoted by $\|f\|_\infty$ and is defined as $\|f\|_\infty = ess\sup|f|$.
- A sequence $\{x_n\}$ in a normed linear space X is said to converge to an element $x \in X$ if given an $\epsilon > 0$, there is a positive integer N such that

$$\|x_n - x\| < \epsilon, \forall n \geq N.$$

- A sequence $\{x_n\}$ in a normed linear space X is a Cauchy sequence if given an $\epsilon > 0$, there is a positive integer N such that

$$\|x_n - x_m\| < \epsilon, \forall n, m \geq N.$$

- In a normed linear space, every convergent sequence is a Cauchy sequence but converse may not be true.
- A series $\sum_{n=1}^{\infty} u_n$ in normed linear space X is said to be convergent to u if $u \in X$ and $\lim_{n\to\infty} s_n = u$, where $s_n = u_1 + u_2 + \cdots + u_n$. In this case, we write $u = \sum_{n=1}^{\infty} u_n$.
- The series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|u_n\| < \infty$.
- A normed linear space X is complete if and only if every absolutely convergent series is convergent.
- The elements of the space $L^p(\mu)$ are not functions but classes of functions such that in each class any two functions are equal almost everywhere. Any two functions equal $a.e.$ have the same integrals over each set of S.
- We will write $f \in L^p(\mu)$, then f is measurable and $\int |f|^p \, d\mu < \infty$.
- On the real line, if $X = (a,b)$ and μ is Lebesgue measure, we will write $L^p(a,b)$ for the corresponding space.
- Let $f, g \in L^p(\mu)$ and let a, b be constants; then $af + bg \in L^p(\mu)$.
- $L^\infty(\mu)$ is a vector space over the real numbers.
- If $\mu(x) < \infty$ and $0 < p < q \leq \infty$ then $L^q(\mu) \subseteq L^p(\mu)$.
- Let $p > 0$ and $f \in L^p(\mu)$, $f \geq 0$ and let $f_n = \min(f, n)$ then $f_n \in L^p(\mu)$ and $\lim \|f - f_n\|_p = 0$.

- Let $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(\mu)$ that converges $a.e.$ to $f \in L^p(\mu)$ then $f_n \to f$ in $L^p(\mu)$ if and only if

$$\lim_{n\to\infty} \int |f_n|^p \, d\mu = \int |f|^p \, d\mu.$$

Keywords

Normed Linear Space: Let X be a linear space over a field of real or complex. A norm on X is a real-valued function, (denoted as $\|\cdot\|$) on X which has the following properties:

i) $\|x\| \geq 0, x \in X$.

ii) $\|x\| = 0 \Leftrightarrow x = 0 \, \forall x \in X$

iii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$

iv) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and $\alpha \subset \mathbb{R}$(or $\mathbb{C}$)

A linear space X, equipped with the norm $\|\cdot\|$ on it is called a normed linear space or simply a normed space. It is denoted by $(X, \|\cdot\|)$ or simply by X.

Metric Space: Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}$ is said to be metric if and only if d satisfies the following properties:

i) $d(x,y) \geq 0 \, \forall x, y \in X$

ii) $d(x,y) = 0 \Leftrightarrow x = y, \; \forall x, y \in X$

iii) $d(x,y) = d(y,x), \forall x, y \in X$

iv) $d(x,z) \leq d(x,y) + d(y,z), \forall x, y, z \in X$

If d is a metric on$X$, then the (X,d) is called metric space.

Cauchy sequence: A sequence $\{x_n\}$ in a normed linear space X is a Cauchy sequence if given an $\epsilon > 0$, there is a positive integer N such that

$$\|x_n - x_m\| < \epsilon, \forall n, m \geq N.$$

Complete normed linear space: A normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Banach space: A complete normed linear space is called a Banach Space.

L^p space: If (X, S, μ) is a measure space and $p > 0$, we define $L^p(X, \mu)$or$L^p(\mu)$, to be the class of measurable functions $\{f: \int |f|^p \, d\mu < \infty\}$ with the convention that any two functions equal $a.e.$ specify the same element of $L^p(\mu)$.

L^p − norm: Let $f \in L^p(a,b)$, then the L^p −norm of f, denoted as$\|f\|_p$ is given by

$$\left[\int |f|^p \, d\mu\right]^{\frac{1}{p}}.$$

L^∞ − space: If (X, S, μ) is a measure space, we define $L^\infty(\mu)$ to be the class of measurable functions

$$\{f: ess \sup|f| < \infty\}.$$

L^∞ −norm: L^∞ −norm of $f \in L^\infty(\mu)$ is denoted by $\|f\|_\infty$ and is defined as $\|f\|_\infty = ess \sup |f|$.

Self Assessment

1) In a normed linear space, every convergent sequence is a Cauchy sequence.

A. True

B. False

2) In a normed linear space, every Cauchy sequence is a convergent sequence.

A. True

B. False

3) A normed linear space is complete if every Cauchy sequence in it is convergent.

A. True
B. False

4) Consider the following statements:

(I) A complete normed linear space is known as Banach space.

(II) In a complete normed linear space every Cauchy sequence is convergent.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

5) Consider the following statements:

(I) A normed linear space is complete if every absolutely convergent series is convergent.

(II) In a complete normed linear space every absolutely convergent series is convergent.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

6) $\|f\|_p =$

A. $\int |f|^p d\mu$
B. $\int |f|^p d\mu]^p$
C. $\int |f|^p d\mu]^{1/p}$
D. $[\int |f|^{2p} d\mu]^p$

7) $L^p(\mu)$ is a vector space over the reals.

A. True
B. False

8)$L^\infty(\mu)$ need not be a vector space over the reals.

A. True
B. False

9) $\|f\|_\infty =$

A. $ess\ \sup|f|$
B. $ess\ \inf|f|$
C. $\sup|f|$
D. $\inf|f|$

10) Consider the following statements:

(I) If $\mu(X) < \infty$ and $0 < p < q < \infty$ then $L^q(\mu) \subseteq L^p(\mu)$.

(II) If $\mu(X) < \infty$ and $0 < p < q = \infty$ then $L^q(\mu) \subseteq L^p(\mu)$.

(III) If $\mu(X) < \infty$ and $0 < p < q \leq \infty$ then $L^p(\mu) \subseteq L^q(\mu)$.

A. only (I) is correct
B. only (III) is correct
C. both (I) and (II) are correct
D. both (I) and (III) are incorrect

11) If $f \in L^p(\mu)$ then $\int |f|^p d\mu < \infty$.

A. True
B. False

12) If $f \in L^\infty(\mu)$ then $esssup|f| = \infty$

A. True
B. False

13) Let $p > 0$ and $f \in L^p(\mu)$ where $f \geq 0$ and let $f_n = min(f, n)$ then $f_n \in L^p(\mu)$.

A. True
B. False

14) Let $p > 0$ and $f \in L^p(\mu)$ where $f \geq 0$ and let $f_n = min(f, n)$then $\lim \|f - f_n\|_p = 0$.

A. True
B. False

15) Let $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ that converges a.e. to $f \in L^p(\mu)$.Consider the following statements:

(I) $f_n \to f$ in $L^p(\mu)$ if $lim\int |f_n|^p d\mu < \int |f|^p d\mu$.

(II)$im\int |f_n|^p d\mu = \int |f|^p d\mu$) if $f_n \to f$ in $L^p(\mu)$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	A	2.	B	3.	A	4.	C	5.	C
6.	C	7.	A	8.	B	9.	A	10.	C
11.	A	12.	B	13.	A	14.	A	15.	B

Review Questions

1) Show that $L^p(\mu)$ is a vector space over the real numbers.

2) Show that in a normed linear space, every convergent sequence is a Cauchy sequence.

3) Let N be a normed linear space and let $x, y \in N$. Then prove that $|\|x\| - \|y\|| \leq \|x - y\|$.

4) If $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, then prove that $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

5) Show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/105/111105037/

https://www.youtube.com/watch?v=RJhX7JereNI

Unit 11: Convex Functions, Jensen's Inequality

CONTENTS

Objectives

After studying this unit, students will be able to:

- define convex function
- understandproperties of convex functions
- provide examplesof convex functions
- demonstrate theorems related to convex functions
- explain Jensen's inequality

Introduction

This unit covers the basic theory of convex functions.These functions appear in many problems in pure and applied mathematics. The theory of convex functions is part of the general subject of convexity since a convex function is one whose epigraph is a convex set. Here, we study the basic properties, proofs of important theorems, and some examples of convex functions.

11.1 Convex Functions

Definition: A function ψ defined on (a, b) is convex if for any non-negative numbers λ, μ such that $\lambda + \mu = 1$ and x, y such that $a < x < y < b$, we have

$$\psi(\lambda x + \mu y) \leq \lambda\, \psi(x) + \mu\, \psi(y).$$

Notes: The endpoints a, b can take values $-\infty, +\infty$ respectively.

s $-\infty, +\infty$ respectively.

Notes:The segment joining the points $X = (x, \psi(x))$ and $Y = (y, \psi(y))$ never lies below the graph of ψ.

Notes: If, for all positive number's λ, μ such that $\lambda + \mu = 1$,

$$\psi(\lambda x + \mu y) < \lambda\, \psi(x) + \mu\, \psi(y),$$

then ψ is said to be strictly convex.

Notes: A function f is said to be midpoint convex on (a,b) if for $x, y \in (a,b)$,

$$f\left(\frac{\lambda+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

Theorem 10.1: Let ψ be convex on (a,b) and $a < s < t < u < b$ then

$$\psi(s,t) \leq \psi(s,u) \leq \psi(t,u),$$

where $\psi(a,b)$ is defined as

$$\psi(a,b) = \frac{\psi(b)-\psi(a)}{b-a}.$$

Further, show that if ψ is strictly convex, equality will not occur.

Proof: First of all, we will show that $\psi(s,t) \leq \psi(s,u)$.

Since $\quad a < s < t < u < b$

$$\Rightarrow s < t < u$$

$$\Rightarrow 0 < t-s < u-s$$

$$\Rightarrow 0 < \frac{t-s}{u-s} < 1$$

So, let $\lambda = \frac{t-s}{u-s}$ ... (1)

$\therefore \mu = 1 - \lambda$

$$= 1 - \frac{t-s}{u-s}$$

$$= \frac{u-s}{u-s} \quad \text{... (2)}$$

$\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$represents the line segment joining the points x and y.

Therefore, for any point t such that $s < t < u$, we have

$$t = \lambda s + (1-\lambda)u$$

$$= \lambda s + \mu u \qquad \because \lambda + \mu = 1$$

$$\Rightarrow t = \left(\frac{t-s}{u-s}\right)s + \left(\frac{u-t}{u-s}\right)u \qquad \text{... (3)\{by(1)and(2)\}}$$

which is not true.

Again, consider

$$t = \lambda u + \mu s$$

$$= \left(\frac{t-s}{u-s}\right)u + \left(\frac{u-t}{u-s}\right)s \qquad \text{... (4)}$$

which is true.

Now

$$\Rightarrow \psi(t) \leq \left(\frac{t-s}{u-s}\right)\psi(u) + \left(\frac{u-t}{u-s}\right)\psi(s) \qquad \text{... (5)}$$

$$\Rightarrow (u-s)\psi(t) \leq (t-s)\psi(u) + u\,\psi(s) - t\,\psi(s) + [s\,\psi(s) - s\,\psi(s)]$$

$$\Rightarrow (u-s)\psi(t) \leq (t-s)\psi(u) + (u-s)\psi(s) - (t-s)\psi(s)$$

$$\Rightarrow (u-s)\big(\psi(t)-\psi(s)\big) \leq (t-s)[\psi(u)-\psi(s)]$$

$$\Rightarrow \frac{\psi(t)-\psi(s)}{t-s} \leq \frac{\psi(u)-\psi(s)}{u-s}$$

$$\Rightarrow \psi(s,t) \leq \psi(s,u) \qquad \text{... (6)}$$

If ψ is strictly convex then equality will not occur in (5) and so not in (6).

Next, we will show that $\psi(s,u) \leq \psi(t,u)$.

Since

$$s < t < u$$
$$\Rightarrow\ -s > -t > -u$$
$$\Rightarrow u - s > u - t > 0$$
$$\Rightarrow \frac{u-t}{u-s} < 1 \qquad \text{... (7)}$$
$$\therefore\ \mu = 1 - \lambda$$
$$= 1 - \frac{u-t}{u-s}$$
$$= \frac{t-s}{u-s} \qquad \text{... (8)}$$

Now

$$t = \lambda s + \mu u$$
$$= \left(\frac{u-t}{u-s}\right)s + \left(\frac{t-s}{u-s}\right)u$$
$$\Rightarrow\ \psi(t) \le \left(\frac{u-t}{u-s}\right)\psi(s) + \left(\frac{t-s}{u-s}\right)\psi(u)$$
$$\Rightarrow (u-s)\psi(t) \le (u-t)\psi(s) + t\,\psi(u) - s\,\psi(u) + [u\,\psi(u) - u\,\psi(u)]$$
$$\Rightarrow (u-s)\big(\psi(t) - \psi(u)\big) \le (u-t)[\psi(s) - \psi(u)]$$
$$\Rightarrow \frac{\psi(t)-\psi(u)}{u-t} \le \frac{\psi(s)-\psi(u)}{u-s}$$
$$\Rightarrow \frac{\psi(u)-\psi(s)}{u-s} \le \frac{\psi(u)-\psi(t)}{u-t}$$
$$\Rightarrow \psi(s,u) \le \psi(u,t).$$

This completes the proof.

Theorem 11.2: A differentiable function ψ is convex on (a,b) if and only if ψ' is a monotone increasing function. If ψ'' exists on (a,b) then ψ is convex if and only if $\psi'' \ge 0$ on (a,b).

Proof: Suppose ψ is differentiable and convex and let

$$a < s < t < u < v < b,$$

then for $s < t < u$, we have $\psi(s,t) \le \psi(s,u) \le \psi(t,u)$ ... (1)

and for $t < u < v$, we have $\psi(t,u) \le \psi(t,v) \le \psi(u,v)$ ... (2)

From (1) and (2),

$$\psi(s,t) \le \psi(u,v)$$
$$\Rightarrow \frac{\psi(t)-\psi(s)}{t-s} \le \frac{\psi(v)-\psi(u)}{v-u}$$
$$\Rightarrow \frac{\psi(t)-\psi(s)}{t-s} \le \frac{\psi(u)-\psi(v)}{u-v} \qquad \text{... (3)}$$

Let $t \to s$ and $u \to v$ then (3) $\Rightarrow$

$$\psi'(s) \le \psi'(v)\ \ \forall s < v$$
$$\Rightarrow\ \psi' \text{is monotone increasing.}$$

If ψ'' exists, it is never negative.

$\Rightarrow \psi'' \ge 0.$

Converse: Here we will show that if ψ' is monotone increasing then ψ is convex. Also, if $\psi'' \ge 0$ then ψ is convex. "

Let $x < y$ in (a,b) and $0 < c < 1$ and

$$z = cx + (1-c)y, \qquad x < z < y \qquad \text{... (4)}$$
$$\Rightarrow \psi(z) = \psi(cx + (1-c)y)$$

We will show that

$$\psi(z) \le c\,\psi(x) + (1-c)\psi(y)$$

$$i.e. \qquad \psi(z) \le c\,\psi(x) + (1-c)\psi(y) + [c\,\psi(z) - c\psi(z)]$$

$$i.e. \qquad c[\psi(x) - \psi(z)] + (1-c)[\psi(y) - \psi(z)] \ge 0 \qquad \ldots (5)$$

Now, by the mean value theorem, $\exists\ \theta_1 \in (x, z)$ such that

$$\psi'(\theta_1) = \frac{\psi(z) - \psi(x)}{z - x} \qquad \ldots (6)$$

And $\exists\ \theta_2 \in (z, y)$ such that

$$\psi'(\theta_2) = \frac{\psi(y) - \psi(z)}{y - z} \qquad \ldots (7)$$

$$(6) \Rightarrow \psi(x) - \psi(z) = (x - z)\psi'(\theta_1) \qquad \ldots (8)$$

$$(7) \Rightarrow \psi(y) - \psi(z) = (y - z)\psi'(\theta_2) \qquad \ldots (9)$$

Using (8) and (9) in (5), we get

$$c[(x - z)\psi'(\theta_1)] + (1-c)[(y - z)\psi'(\theta_2)]$$

$$= c\,\psi'(\theta_1)[x - cx - (1-c)y] + (1-c)\psi'(\theta_2)[y - cx - (1-c)y]$$

$$= c\,\psi'(\theta_1)[(1-c)(x-y)] + (1-c)\psi'(\theta_2)c\,(y - x)$$

$$= c\,(1-c)(y-x)[\psi'(\theta_2) - \psi'(\theta_1)] \ge 0$$

$$\because 0 < c < 1, x < y, \psi' \text{is monotone increasing}, \theta_1 < \theta_2$$

Thus ψ is convex if ψ' is monotone increasing. Now if $\psi'' \ge 0$ then ψ' is monotone increasing and hence ψ is convex.

This completes the proof.

Example: e^x is strictly convex on ℝ.

Example: x^α is convex on $(0, \infty)$ for $\alpha \ge 1$.

Example: $-x^\alpha$ is strictly convex on $(0, \infty)$ for $0 < \alpha < 1$.

Example: $x \log x$ is strictly convex on (0,1).

Theorem 11.3: Every function convex on an open interval is continuous.

Proof: Let f is convex on (a, b) and let $[c, d] \subset (a, b)$.

Choose c_1 and d_1 such that $a < c_1 < c < d < d_1 < b$

If $x, y \in [c, d]$ with $x < y$, then, for $x < y < d < d_1$, we have

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(d) - f(y)}{d - y} \le \frac{f(d_1) - f(d)}{d_1 - d} \qquad \ldots (1)$$

For $c_1 < c < x < y$, we have

$$\frac{f(c) - f(c_1)}{c - c_1} \le \frac{f(x) - f(c)}{x - c} \le \frac{f(y) - f(x)}{y - x} \qquad \ldots (2)$$

$$\Rightarrow \frac{f(c) - f(c_1)}{c - c_1} \le \frac{f(y) - f(x)}{y - x} \le \frac{f(d_1) - f(d)}{d_1 - d}$$

Thus, the set

$$\left\{\frac{f(y) - f(x)}{y - x} : c \le x < y \le d\right\}$$

is bounded.

Therefore, for all $x, y \in [c,d], x < y, \exists$ a positive real number M, such that

$$\left|\frac{f(y)-f(x)}{y-x}\right| \leq M$$

$$\Rightarrow |f(y)-f(x)| \leq M|y-x| < \epsilon \text{ whenever} |y-x| < \frac{\epsilon}{M} = \delta$$

$\Rightarrow f$ is uniformly continuous on $[c,d]$

$\Rightarrow f$ is continuous on [c, d]

$\therefore f$ is continuous on (a,b).

This completes the proof.

Notes: If in the definition of a convex function, the open interval is not specified then the above theorem would not hold.

$$e.g. \psi = 0 \text{ on}[0,1], \qquad \psi(1) = 1.$$

Here ψ is convex but not continuous.

11.2 Jensen's Inequality

Theorem 11.4 (Jensen's Inequality)

Let (X,S,μ) be a measure space with $\mu(X) = 1$. If ψ is convex on (a,b),

$-\infty < a < b < \infty$ and f is a measurable function, $a < f(x) < b$ for all x, then

$$\psi\left(\int f\, d\mu\right) \leq \int \psi \circ f\, d\mu$$

Proof: Since f is measurable, $a < f(x) < b, \mu(x) = 1$

$\Rightarrow f$ is integrable.

Put $t = \int f\, d\mu$.

Since

$$a < f(x) < b$$

$$\Rightarrow \int a\, d\mu < \int f\, d\mu < \int b\, d\mu$$

$$\Rightarrow a < t < b.$$

Now, $a < s < t < u < b \Rightarrow \psi(s,t) \leq \psi(s,u) \leq \psi(t,u)$

$$\text{where } \psi(a,b) = \frac{\psi(b)-\psi(a)}{b-a}.$$

Let

$$\beta = \sup_{x \in (a,t)} \frac{\psi(t)-\psi(x)}{t-x}.$$

Therefore,

$$\frac{\psi(t)-\psi(s)}{t-s} \leq \beta, s \in (a,t)$$

$$\Rightarrow \psi(s) \geq \psi(t) + \beta(s-t), s \in (a,t) \qquad \ldots (1)$$

Also,

$$\beta \leq \frac{\psi(u)-\psi(t)}{u-t}, u \in (t,b)$$

$$\Rightarrow \psi(u) \geq \psi(t) + \beta(u-t), \qquad u \in [t,b) \qquad \ldots (2)$$

For $\alpha \in (a,b)$,

$$\psi(\alpha) \geq \psi(t) + \beta(\alpha - t).$$

Put $\alpha = f(x) \in (a, b)$to get, for each x,

$$\psi(f(x)) \geq \psi(t) + \beta(f(x) - t) \qquad \dots (3)$$

Since f is measurable and ψ is continuous

$\Rightarrow \psi \circ f$ is measurable.

Now, since RHS of expression (3) is integrable.

$\therefore \int \psi \circ f \, d\mu$ exists.

{$\because$ if g is measurable and $g \geq h \in L(X, \mu)$ then $\int g d\mu$ exists.}

Now integrating both sides of relation (3), using the fact that $\mu(X) = 1$ and $t = \int f \, d\mu$, we get

$$\int \psi(f(x)) \, d\mu \geq \psi\left(\int f \, d\mu\right)$$

$$i.e. \psi\left(\int f \, d\mu\right) \leq \int \psi \circ f \, d\mu.$$

This completes the proof.

Summary

- A function ψ defined on (a, b) is convex if for any non-negative numbers λ, μ such that $\lambda + \mu = 1$ and x, y such that $a < x < y < b$, we have
$$\psi(\lambda x + \mu y) \leq \lambda \, \psi(x) + \mu \, \psi(y).$$
- The endpoints a, b can take values $-\infty, +\infty$ respectively.
- The segment joining the points $X = (x, \psi(x))$and$Y = (y, \psi(y))$ never lies below the graph of ψ.
- If, for all positive numbers λ, μ such that
$$\lambda + \mu = 1, \psi(\lambda x + \mu y) < \lambda \, \psi(x) + \mu \, \psi(y),$$
then ψ is said to be strictly convex.
- A function f is said to be midpoint convex on (a, b) if for $x, y \in (a, b)$,
$$f\left(\frac{x + y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y).$$
- Let ψ be convex on (a, b) and $a < s < t < u < b$ then $\psi(s, t) \leq \psi(s, u) \leq \psi(t, u)$, where $\psi(a, b)$ is defined as
$$\psi(a, b) = \frac{\psi(b) - \psi(a)}{b - a}.$$
Further, if ψ is strictly convex, equality will not occur.
- A differentiable function ψ is convex on (a, b) if and only if ψ' is a monotone increasing function. If ψ'' exists on (a, b) then ψ is convex if and only if $\psi'' \geq 0$ on (a, b).
- e^x is strictly convex on $\mathbb{R}$.
- $x^\alpha$is convex on $(0, \infty)$for $\alpha \geq 1$.
- $-x^\alpha$is strictly convex on $(0, \infty)$for $0 < \alpha < 1$.
- $x \log x$ is strictly convex on $(0,1)$.
- Every function convex on an open interval is continuous.
- Jensen's inequality: Let (X, S, μ) be a measure space with $\mu(X) = 1$. If ψ is convex on $(a, b), -\infty < a < b < \infty$ and f is a measurable function, $a < f(x) < b$ for all x, then
$$\psi\left(\int f \, d\mu\right) \leq \int \psi \circ f \, d\mu.$$

Keywords

Convex function: A function ψ defined on (a, b) is convex if for any non-negative numbers λ, μ such that $\lambda + \mu = 1$ and x, y such that $a < x < y < b$, we have

$$\psi(\lambda x + \mu y) \leq \lambda\, \psi(x) + \mu\, \psi(y).$$

Strictly convex functions: If, for all positive number's λ, μ such that $\lambda + \mu = 1$,

$$\psi(\lambda x + \mu y) < \lambda\, \psi(x) + \mu\, \psi(y),$$

then ψ is said to be strictly convex.

Midpoint convex function: A function f is said to be midpoint convex on (a, b) if for $x, y \in (a, b)$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y).$$

Jensen's Inequality 11.4: Let (X, S, μ) be a measure space with $\mu(X) = 1$. If ψ is convex on (a, b), $-\infty < a < b < \infty$ and f is a measurable function, $a < f(x) < b$ for all x, then

$$\psi\left(\int f\, d\mu\right) \leq \int \psi \circ f\, d\mu.$$

Self Assessment

1) Let a function ψ be defined on an open interval (a, b) and if for any non-negative numbers λ, μ where $\lambda + \mu = 1$ and x, y such that a<x<y<b, we have $\psi(\lambda x + \mu y) \leq \lambda\psi(x) + \mu\psi(y)$then ψ is a

A. simple function

B. step function

C. convex function

D. none of these

2) Let ψ be a convex function then the line segment joining the points $X = (x, \psi(x))$ and $Y = (y, \psi(y))$

A. never lies above the graph of ψ

B. lies below the graph of ψ

C. always lies below the graph of ψ

D. none of these

3) Let ψ be convex on (a, b) and a<s<t<u<b then

A. $\psi(s,t) \leq \psi(s,u) \leq \psi(t,u)$

B. $\psi(s,t) \geq \psi(s,u) \geq \psi(t,u)$

C. $\psi(s,t) = \psi(s,u) = \psi(t,u)$

D. none of these

4)The function f is said to be midpoint convex if

A. $f\left(\frac{x+y}{2}\right) = \frac{1}{2} f(x) + \frac{1}{2} f(xy)$

B. $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y)$

C. $f\left(\frac{x+y}{2}\right) \geq \frac{1}{2} f(x) + \frac{1}{2} f(xy)$

D. none of these

5) The function ψ is said to be strictly convex if, for all positive numbers λ, μ such that $\lambda + \mu = 1$, we have

A. $\psi(\lambda x + \mu y) = \lambda\psi(x) + \mu\psi(y)$

B. $\psi(\lambda x + \mu y) > \lambda\psi(x) + \mu\psi(y)$

C. $\psi(\lambda x + \mu y) < \lambda\psi(x) + \mu\psi(y)$

D. none of these

6)$\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ represents

A. circle

B. parabola

C. line segment

D. an ellipse

7) Consider the following statements

(I) $xlogx$ is strictly concave on (0, 1).

(II) $xlogx$ is not strictly convex on (0, 1). Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

8) Consider the following statements

(I) $-x^{\alpha}$ is strictly concave on $(0, \infty)$ for $0 < \alpha < 1$.

(II) $-x^{\alpha}$ is not strictly convex on $(0, \infty)$ for $0 < \alpha < 1$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

9) Consider the following statements

(I)x^{α} is not concave on $(0, \infty)$ for $\alpha \geq 1$.

(II)x^{α} is convex on $(0, \infty)$ for $\alpha \geq 1$.Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

10) e^x is strictly convex on R.

A. True

B. False

11) A differentiable function ψ is convex on (a, b) if and only if

A. ψ' is monotonically decreasing function

B. ψ' is step function

C. ψ' is simple function

D. ψ' is monotonically increasing function

12) Let ψ'' exists on (a, b). Consider the following statements

(I) ψ is convex on (a, b) if $\psi'' \geq 0$ on (a, b).

(II) $\psi'' \geq 0$ on (a, b) if ψ is convex on (a, b). Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

13) Consider the following statements:

(I) Every function convex on an open interval is continuous.

(II) Every function convex on a half − open interval is continuous. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

14) Let the function ψ be defined by $\psi = 1$ on $[0, 2)$ and $\psi(2) = 2$. Consider the following statements:

(I) ψ is convex.

(II) ψ is continuous.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

15) Let (X, S, μ) be a measure space with $\mu(X) = 1$. If ψ is convex on an open and bounded interval (a, b) and f is a measurable function such that $a < f(x) < b \ \forall x$ then $\int \psi \circ f \, d\mu \leq \psi(\int f d\mu)$.

A. True

B. False

16) If $\frac{1}{2}f(x)+\frac{1}{2}\geq f\left(\frac{x+y}{2}\right)$ then f is said to midpoint convex.

A. True
B. False

17) Constant functions are convex.

A. True
B. False

Answers for Self Assessment

1.	C	2.	B	3.	A	4.	B	5.	C
6.	C	7.	D	8.	D	9.	C	10.	A
11.	D	12.	C	13.	A	14.	A	15.	B
16.	A	17.	A						

Review Questions

Determine whether the following functions are convex or not on ℝ.

1) $e^{5x}+5$

2) $-8x^2$

3) $sinx$

4) $\cos x$

5) $ax+b$

6) x^2

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Graduate Texts in Mathematics, Measure Integration and Real Analysis, Sheldon Axler, Springer.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

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Unit 12: The Inequalities of Holder and Minkowski and Completeness of L^p Spaces

CONTENTS

Objectives

After studying this unit, students will be able to:

- define conjugate exponents
- demonstrate the inequality of Holder
- explainthe inequality of Minkowski
- demonstratecompleteness of L^p space
- understand completeness of L^∞ space

Introduction

The Minkowski's Inequality $\|f+g\|_p \leq \| f\|_p + \| g\|_p$ also called triangle inequality holds for $1 \leq p < \infty$. Thus for $1 \leq p < \infty$, the L^p −spaces are examples of normed linear spaces which are also complete under the metric $\|f-g\|_p$, the metric induced by the norm. Such normed linear spaces are called Banach spaces. Holder's inequality is used to prove Minkowski's inequality.

In this unit, we prove the inequalities of Holder and Minkowski and the completeness of L^p spaces.

12.1 The Inequality of Holder

Definition: Let p and q be two real numbers, $p, q > 1$. The conjugate of a number p is the number $q = \frac{p}{p-1}$, which is the unique number for which

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Here, p and q are known as conjugate exponents.

.

Notes: The conjugate of 1 is ∞.

Notes: The conjugate of ∞ is 1.

Theorem 12.1: Let $a > 0, b > 0, \frac{1}{p} + \frac{1}{q} = 1, p > 1, q > 1$.

Show that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$$

Proof: Since e^x is strictly convex.

$$\therefore e^{\frac{1}{p}\log a + \frac{1}{q}\log b} \leq \frac{1}{p} e^{\log a} + \frac{1}{q} e^{\log b}$$

$$= \frac{a}{p} + \frac{b}{q}$$

$$\Rightarrow e^{\frac{1}{p}\log a + \frac{1}{q}\log b} \leq \frac{a}{p} + \frac{b}{q}$$

$$\Rightarrow e^{\log(a^{\frac{1}{p}} b^{\frac{1}{q}})} \leq \frac{a}{p} + \frac{b}{q}$$

$$\Rightarrow a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

This completes the proof.

Theorem 12.2: (Holder's Inequality)

Let $1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\mu)$and $g \in L^q(\mu)$.

Then $fg \in L^1(\mu)$ and

$$\int |fg|\, d\mu \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int |g|^q\, d\mu\right)^{\frac{1}{q}}$$

$$i.e.\ \|fg\|_1 \leq \|f\|_p \|g\|_q \qquad \ldots (1)$$

Proof: If $a > 0, b > 0$, then

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \qquad \ldots (2)$$

If $\| f\|_p = 0$ or$\| g\|_q = 0$ then $fg = 0\ a.e.$

Therefore (1) is trivial.

If $\| f\|_p > 0$ and$\| g\|_q > 0$then in expression (2) we write

$$a = \frac{|f|^p}{(\|f\|_p)^p}$$

and

$$b = \frac{|g|^q}{(\|g\|_q)^q}$$

to get

$$\left[\frac{|f|^p}{(\|f\|_p)^p}\right]^{\frac{1}{p}} \left[\frac{|g|^q}{(\|g\|_q)^q}\right]^{\frac{1}{q}} \leq \frac{1}{p}\frac{|f|^p}{(\|f\|_p)^p} + \frac{1}{q}\frac{|g|^p}{(\|g\|_q)^q} \qquad \ldots (3)$$

Right hand side of expression (3) is integrable as $f \in L^p(\mu)$and $g \in L^q(\mu)$.

So $fg \in L^1(\mu)$.

Now on integrating both sides of relation (3), we get

$$\frac{\int |fg|\, d\mu}{\| f\|_p \| g\|_q} \leq \frac{1}{p}\frac{\int |f|^p\, d\mu}{(\| f\|_p)^p} + \frac{1}{q}\frac{\int |g|^q\, d\mu}{(\| g\|_q)^q}$$

$$\Rightarrow \frac{\int |fg|\, d\mu}{\| f\|_p \| g\|_q} \le \frac{1}{p}\frac{(\| f\|_p)^p}{(\| f\|_p)^p} + \frac{1}{q}\frac{(\| g\|_q)^q}{(\| g\|_q)^q}$$

$$\Rightarrow \frac{\int |fg|\, d\mu}{\| f\|_p \| g\|_q} \le \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \int |fg|\, d\mu \le \| f\|_p \| g\|_q \Rightarrow \|fg\|_1 \le \| f\|_p \| g\|_q$$

This completes the proof.

Notes: If $p = q = 2$ Holder's inequality is known as Cauchy Schwartz inequality.

12.2 The Inequality of Minkowski

Theorem 12.3 (The Inequality of Minkowski)

Let $1 \le p < \infty$ and $f, g \in L^p(\mu)$; then

$$\left(\int |f+g|^p\, d\mu\right)^{\frac{1}{p}} \le \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p\, d\mu\right)^{\frac{1}{p}}$$

$$i.e.\ \|f+g\|_p \le \| f\|_p + \| g\|_p$$

Proof: Case I for $p = 1$.

We have

$$|f+g| \le |f| + |g|$$

$$\Rightarrow \int |f+g|\, d\mu \le \int |f|\, d\mu + \int |f|\, d\mu$$

$$\Rightarrow \|f+g\|_1 \le \| f\|_1 + \| g\|_1 .$$

Case II If $1 < p < \infty$

Consider

$$\int |f+g|^p\, d\mu = \int |f+g|^{p-1}|f+g| d\mu$$

$$\le \int [|f+g|^{p-1}][|f| + |g|]\, d\mu$$

$$\Rightarrow \int |f+g|^p\, d\mu \le \int |f||f+g|^{p-1}\, d\mu + \int |g||f+g|^{p-1}\, d\mu \qquad \text{... (1)}$$

Let $1 < q < \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow p + q = pq$$

$$\Rightarrow pq - q = p$$

$$\Rightarrow q(p-1) = p \qquad \text{... (2)}$$

$$\therefore \int |f+g|^{(p-1)q}\, d\mu = \int |f+g|^p\, d\mu$$

$$< \infty \quad \because f, g \in L^p(\mu) \Rightarrow f+g \in L^p(\mu)$$

$$\Rightarrow \int |f+g|^{(p-1)q}\, d\mu < \infty$$

$$\Rightarrow |f+g|^{p-1} \in L^q(\mu).$$

Now, since $f, g \in L^p(\mu)$ and $|f+g|^{p-1} \in L^q(\mu)$ therefore by Holder's inequality, we have

$$\int |f||f+g|^{p-1}\, d\mu \le \|f\|_p \|(f+g)^{p-1}\|_q$$

and

$$\int |g||f+g|^{p-1}\,d\mu \le \|g\|_p\|(f+g)^{p-1}\|_q$$

$$□ \int |f||f+g|^{p-1}\,d\mu \le \|f\|_p\left[\int |f+g|^{(p-1)q}\,d\mu\right]^{\frac{1}{q}}$$

$$= \|f\|_p(\|f+g\|_p)^{\frac{p}{q}}$$

Similarly, we have

$$\int |g||f+g|^{p-1}\,d\mu \le \|g\|_p(\|f+g\|_p)^{\frac{p}{q}}$$

$$\therefore (1) \Rightarrow$$

$$\int |f+g|^p\,d\mu \le \|f\|_p(\|f+g\|_p)^{\frac{p}{q}} + \|g\|_p(\|f+g\|_p)^{\frac{p}{q}}$$

$$□ \ \|f+g\|_p]^p \le [\|f\|_p + \|g\|_p][\|f+g\|_p]^{\frac{p}{q}}$$

$$□ \ (\|f+g\|_p)^{\left(p-\frac{p}{q}\right)} \le \|f\|_p + \|g\|_p$$

$$□ \ \|f+g\|_p \le \|f\|_p + \|g\|_p$$

$$□ \ q(p-1) = p$$

$$□ \ p-1 = \frac{p}{q}$$

$$□ \ p - \frac{p}{q} = 1$$

This completes the proof.

Notes: For $0 < p < 1$, Minkowski's inequality does not hold.

12.3 Completeness of L^p Spaces

Theorem 12.4: (Completeness of L^p Spaces)

Show that L^p spaces are complete, $1 \le p < \infty$

Or

Every Cauchy sequence in L^p – spaces converges to some element in L^p – spaces, $1 \le p < \infty$.

Proof: Let $< f_n >$ be a Cauchy sequence in $L^p(\mu)$.

Therefore, for given $\epsilon > 0$, □ a positive integer N such that

$$\|f_m - f_n\|_p < \epsilon, \forall m,n \ge N$$

$$□ \left[\int |f_m - f_n|^p\right]^{\frac{1}{p}} < \epsilon \qquad \ldots (1)$$

Taking $\epsilon = \frac{1}{2}$, we can find a positive integer n_1 such that

$$\|f_m - f_n\|_p < \frac{1}{2}, □ m,n \ge n_1$$

Taking $\epsilon = \frac{1}{2^2}, n_2 > n_1$ such that

$$\|f_m - f_n\|_p < \frac{1}{2^2}, □ m,n \ge n_2.$$

For $\epsilon = \frac{1}{2^k}, n_k > n_{k-1}$ such that

$$\|f_m - f_n\|_p < \frac{1}{2^k}, □ m,n \ge n_k$$

$$□ \ \left\|f_{n_{k+1}} - f_{n_k}\right\|_p < \frac{1}{2^k} \qquad \ldots (2)$$

Let

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|,$$

$$g = \lim_{k\to\infty} g_k$$

$$= \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}| \qquad \dots (3)$$

Now

$$\| g_k \|_p \le \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p$$

$$< \sum_{i=1}^{k} \frac{1}{2^i}$$

$$\le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

$\Rightarrow \| g_k \|_p < 1 \qquad \dots (4)$

Now we apply Fatou's Lemmato the sequence of non-negative functions $\{g_k^p\}$, to get

$$(\| g \|_p)^p = \int \lim g_k^p \, d\mu$$

$$\le \liminf \int g_k^p \, d\mu$$

$$\le 1$$

$\Rightarrow g$ is finite $a.e.$

Thus

$$f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i})$$

is absolutely convergent $a.e.$

Let

$$f = \begin{cases} f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i}), & \text{where it converges otherwise} \\ 0, & \text{otherwise} \end{cases}$$

Since

$$f_{n_1} + \sum_{i=1}^{k-1} (f_{n_{i+1}} - f_{n_i}) = f_{n_k}$$

$$\Rightarrow \lim_{k\to\infty} f_{n_k} = f \ a.e. \qquad \dots (5)$$

From (1), we have $\| f_m - f_n \|_p < \epsilon \ \ \forall n, m \ge N$

So, by Fatou's Lemma, for each $m > N$,

$$\int |f - f_m|^p \, d\mu \le \liminf_{i\to\infty} \int |f_{n_i} - f_m|^p d\mu$$

$$\le \epsilon^p$$

$$\Rightarrow \| f - f_m \|_p \le \epsilon \quad \forall m > N \qquad \dots (6)$$

$$\text{Also } \| f \|_p = \| f - f_m + f_m \|_p$$

$$\le \| f - f_m \|_p + \| f_m \|_p < \infty \quad \{by\ (6)\}$$

$$\square\ f \in L^p(\mu)$$

This completes the proof.

Theorem12.5: Show that L^∞ space is a complete space.

Or

Let $< f_n >$ be a sequence in $L^\infty(\mu)$ such that

$$\|f_n - f_m\|_\infty \to 0 \text{ as } m, n \to \infty,$$

then there exists a function f such that

$$\lim_{n\to\infty} f_n = f \ a.e.,$$

$$f \in L^\infty(\mu)$$

$$\text{and} \lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Proof: $L^\infty(\mu)$ is a class of measurable functions $\{f: ess\ \sup|f| < \infty\}$ and $\|f\|_\infty = ess\ \sup|f|$.

Here we use the fact that "A function is greater than its essential supremum only on a set of measure zero.

Define

$$A_{n,m} = \{x: |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$$

$$B_n = \{x: |f_n(x)| > \|f_n\|_\infty\}$$

If $E = \left(\bigcup_{n\neq m} A_{n,m}\right) \cup \left(\bigcup_{k=1}^{\infty} B_k\right),$

then $\mu(E) = 0$

On E^c:

$$|f_n - f_m| \le \|f_n - f_m\|_\infty \to 0 \text{ as } m, n \to \infty$$

Therefore on E^c, $\{f_n(x)\}$ is a Cauchy sequence for each x, with limit $f(x)$ (say) and we define f arbitrarily on E.

So

$$\lim_{n\to\infty} f_n = f \ a.e.$$

Given $\epsilon > 0, \exists\ N$ such that

$$\|f_n - f_m\|_\infty < \epsilon \text{ for } n, m > N$$

So for $x \in E^c$

$$|f_n(x) - f_m(x)| < \|f_n - f_m\|_\infty < \propto,$$

Letting $n \to \infty$, we get

$$|f(x) - f_m(x)| \le \epsilon$$

$$\square\ |f| \le |f_m| + \epsilon\ a.e.$$

and hence $f \in L^\infty(u)$.

Also $\|f - f_m\|_\infty \le \epsilon$.

This completes the proof.

Summary

- Let p and q be two real numbers, $p, q > 1$. The conjugate of a number p is the number $q = \frac{p}{p-1}$, which is the unique number for which $\frac{1}{p} + \frac{1}{q} = 1$.
- The conjugate of 1 is ∞.
- The conjugate of ∞ is 1.
- Let $a > 0, b > 0, \frac{1}{p} + \frac{1}{q} = 1, p > 1, q > 1$ then

$$a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

- e^x is a strictly convex function.
- Let $1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\mu)$ and $g \in L^q(\mu)$.

  Then $fg \in L^1(\mu)$ and

$$\int |fg|\, d\mu \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int |g|^q\, d\mu\right)^{\frac{1}{q}}$$

$$i.e.\ \|fg\|_1 \leq \|f\|_p \|g\|_q \ .$$

- If $p = q = 2$ Holder's inequality is known as Cauchy Schwartz inequality.
- Let $1 \leq p < \infty$ and $f, g \in L^p(\mu)$; then

$$\left(\int |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p\, d\mu\right)^{\frac{1}{p}}$$

$$i.e.\ \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

- For $0 < p < 1$, Minkowski's inequality does not hold.
- L^p spaces are complete, $1 \leq p < \infty$.
- Every Cauchy sequence in $L^p -$ spaces converges to some element in $L^p -$ spaces, $1 \leq p < \infty$.
- L^∞ space is a complete space.
- Let $< f_n >$ be a sequence in $L^\infty(\mu)$ such that

$$\|f_n - f_m\|_\infty \to 0 \text{ as } m, n \to \infty,$$

then there exists a function f such that

$$\lim_{n\to\infty} f_n = f\ a.e., f \in L^\infty(\mu)$$

$$\text{and } \lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

- A function is greater than its essential supremum only on a set of measure zero.

Keywords

Conjugate exponents: Let p and q be two real numbers, $p, q > 1$. The conjugate of a number p is the number $q = \frac{p}{p-1}$, which is the unique number for which $\frac{1}{p} + \frac{1}{q} = 1$.

Here, p and q are known as conjugate exponents.

Holder's Inequality: Let $1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\mu)$ and $g \in L^q(\mu)$.

Then $fg \in L^1(\mu)$ and

$$\int |fg|\, d\mu \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int |g|^q\, d\mu\right)^{\frac{1}{q}}$$

$$i.e.\ \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Cauchy Schwarz inequality: If $p = q = 2$ Holder's inequality is known as Cauchy Schwartz inequality.

The Inequality of Minkowski: Let $1 \leq p < \infty$ and $f, g \in L^p(\mu)$; then

$$\left(\int |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p\, d\mu\right)^{\frac{1}{p}}$$

$$i.e.\ \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Completeness of L^p Spaces: L^p spaces are complete, $1 \leq p < \infty$ i.e., every Cauchy sequence in $L^p -$ spaces converges to some element in $L^p -$ spaces, $1 \leq p < \infty$.

Completeness of L^∞ Space: L^∞ space is a complete space i.e., let $< f_n >$ be a sequence in $L^\infty(\mu)$ such that

$$\|f_n - f_m\|_\infty \to 0 \text{ as } m, n \to \infty,$$

then there exists a function f such that

$$\lim_{n\to\infty} f_n = f \text{ a.e.},$$

$$f \in L^\infty(\mu)$$

$$\text{and } \lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Self Assessment

1) Let $p > 1$ be a real number. The conjugate of a number p is the number given by

A. $p/2$

B. $p/p - 1$

C. $p/p - 2$

D. none of these

2) If p and q are conjugate exponents then

A. $p + q = 1$

B. $p + q > 1$

C. $\frac{1}{p} + \frac{1}{q} = 1$

D. none of these

3)The conjugate of 1 is 1.

A. True

B. False

4) The conjugate of ∞ is ∞.

A. True

B. False

5) If p and q are conjugate exponents and $a > 0, b > 0$ then

$A.\ a^{1/p} b^{1/q} \le ap + bq$

$B.\ a^{1/p} b^{1/q} \le \frac{a^p}{p} + \frac{b^q}{q}$

$C.\ a^{1/p} b^{1/q} \le \frac{a^{1/p}}{p} + \frac{b^{1/q}}{q}$

$D.\ a^{1/p} b^{1/q} \le \frac{a}{p} + \frac{b}{q}$

6) $\exp\left(\frac{1}{p} \log a + \frac{1}{q} \log b\right) \le \frac{a}{p} + \frac{b}{q}$.

A. True
B. False

7) If p and q are conjugate exponents then $\|fg\|_1 \le \|f\|_p \|g\|_q$, this is known as

A. Minkowski's Inequality
B. Holder's Inequality
C. Jensen's Inequality
D. none of these

8) $\|fg\|_1 \le \|f\|_2 \|g\|_2$, this is known as

A. Minkowski's Inequality
B. Jensen's Inequality
C. Cauchy-Schwarz Inequality
D. none of these

9) Consider the following statements

(I) $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$.

(II) $\|f+g\|_q \le \|f\|_q + \|g\|_q, q > 1$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

10) $\|f+g\|_p \le \|f\|_p + \|g\|_p, p \ge 1$ this is known as

A. Jensen's Inequality
B. Holder's Inequality
C. Cauchy-Schwarz Inequality
D. Minkowski's Inequality

11) Consider the following statements

(I) For $1 \le p < \infty$, the L^p −spaces are normed linear spaces.

(II) For $0 < p < 1$, the L^p − spaces are normed linear spaces. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

12) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

A. True
B. False

13) $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$

A. True
B. False

14) Consider the following statements

(I) L^p spaces are complete, $1 \leq p < \infty$.

(II)L^∞ space is complete. Then

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

15) Every Cauchy sequence in L^p space may or may not converge to some element in L^p space, $1 \leq p < \infty$.

A. True
B. False

16) A function is greater than its essential supremum only on a set of measure zero.

A. True
B. False

17) $L^\infty(\mu)$ is a class of measurable functions $\{f: ess\ sup\ |f| < \infty\}$.

A. True
B. False

Answers for SelfAssessment

1.	B	2.	C	3.	B	4.	B	5.	D
6.	A	7.	B	8.	C	9.	C	10.	D
11.	A	12.	A	13.	A	14.	C	15.	B
16.	A	17.	A						

Review Questions

1) Prove that L^p −spaces are normed linear spaces for $1 \leq p < \infty$.

2)Let $1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^2(\mu)$and $g \in L^2(\mu)$.

Then $fg \in L^1(\mu)$ and

$$\int |fg|\, d\mu \leq \sqrt{\int |f|^2\, d\mu} \sqrt{\int |g|^2\, d\mu}$$

3) If $p + q = pq, f \in L^p, g \in L^q$. Show that if $0 < p < 1$ then

$$\int |fg|\, d\mu \geq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int |g|^q\, d\mu\right)^{\frac{1}{q}}$$

4)For nonnegative real numbers a, b and $0 < t < 1$, prove that

$$a^t b^{1-t} \leq ta + (1-t)b.$$

5) For nonnegative real numbers a, b and $0 < t < 1$, prove that

$$\left(\frac{a+b}{2}\right)^{1/t} \leq \frac{1}{2}\left(a^{1/t} + b^{1/t}\right).$$

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/105/111105037/

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Unit 13: Convergence in Measure and Almost Uniform Convergence

CONTENTS

Objectives

After studying this unit, students will be able to:

- defineconvergence in measure
- demonstrate algebra of functions in convergence in measure
- explain almost uniform convergence
- define Cauchy sequence in measure
- understand the relationship among convergence a.e., convergence in measure and almost uniform convergence

Introduction

We have already met uniform convergence, convergence a.e. and convergence in L^p −spaces. In this unit, we investigate other forms of convergence of measurable functions viz. convergence in measure, almost uniform convergence. We also discuss relation among convergence a.e., convergence in measure and almost uniform convergence. The notion of convergence in measure is of particular relevance to the theory of probability where it is often referred to as convergence in probability.

13.1 Convergence in Measure

Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite $a.e.$ on E. The $\{f_n\}$is said to converge in measure on E to f provided for each $\epsilon > 0$,

$$\lim_{n\to\infty} m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\} = 0$$

We write $f_n \xrightarrow{m} f$ on E.

Theorem 13.1: Let $f_n \xrightarrow{m} f$ and$g_n \xrightarrow{m} g$ on E then

i) $f_n + g_n \xrightarrow{m} f + g$

ii) $cf_n \xrightarrow{m} cf, c \in \mathbb{R}$

iii) $f_n^+ \xrightarrow{m} f^+$

iv) $f_n^- \xrightarrow{m} f^-$

v) $|f_n| \xrightarrow{m} |f|$

Proof:

i) Since $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$, therefore for given $\epsilon > 0$,

$$\lim_{n\to\infty} m\left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} = 0 \qquad \ldots (1)$$

and

$$\lim_{n\to\infty} m\left\{x \in E: |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\} = 0 \qquad \ldots (2)$$

Now,

$\{x \in E: |(f_n + g_n)(x) - (f + g)(x)| \geq \epsilon\}$

$$\subseteq \left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \cup \left\{x \in E: |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}$$

$\Rightarrow m\{x \in E: |(f_n + g_n)(x) - (f + g)(x)| \geq \epsilon\}$

$$\leq m\left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} + m\left\{x \in E: |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}$$

$$\Rightarrow \lim_{n\to\infty} m\{x \in E: |(f_n + g_n)(x) - (f + g)(x)| \geq \epsilon\} = 0 \qquad (\text{by}(1), (2))$$

$$\Rightarrow f_n + g_n \xrightarrow{m} f + g.$$

ii) If $c = 0$, the result is obvious.

Let $c \neq 0$ since $f_n \xrightarrow{m} f$, therefore for given $\epsilon > 0$,

$$\lim_{n\to\infty} m\left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{|c|}\right\} = 0 \qquad \ldots (3)$$

Now,

$$\{x \in E: |(cf_n)(x) - (cf)(x)| \geq \epsilon\} = \left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{|c|}\right\}$$

$$\Rightarrow m\{x \in E: |(cf_n)(x) - (cf)(x)| \geq \epsilon\} = m\left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{|c|}\right\}$$

$$\Rightarrow \lim_{n\to\infty} m\{x \in E: |(cf_n)(x) - (cf)(x)| \geq \epsilon\} = 0 \qquad (\text{by}(3))$$

$$\Rightarrow cf_n \xrightarrow{m} cf.$$

iii) Since $f_n \xrightarrow{m} f$ therefore for given $\epsilon > 0$,

$$\lim_{n\to\infty} m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\} = 0 \qquad \ldots (4)$$

Now,

$|f_n^+ - f^+| \leq |f_n - f|$

$$\therefore \{x \in E: |f_n^+(x) - f^+(x)| \geq \epsilon\} \subseteq \{x \in E: |f_n - f| \geq \epsilon\}$$

$$\Rightarrow m\{x \in E: |f_n^+(x) - f^+(x)| \geq \epsilon\} \leq m\{x \in E: |f_n - f| \geq \epsilon\}$$

$$\Rightarrow \lim_{n\to\infty} m\{x \in E: |f_n^+(x) - f^+(x)| \geq \epsilon\} = 0 \qquad (\text{by}(4))$$

$$\Rightarrow f_n^+ \xrightarrow{m} f^+$$

iv) Since,

$|f_n^- - f^-| \leq |f_n - f|$

$$\therefore \{x \in E: |f_n^-(x) - f^-(x)| \geq \epsilon\} \subseteq \{x \in E: |f_n - f| \geq \epsilon\}$$

$$\Rightarrow m\{x \in E: |f_n^-(x) - f^-(x)| \geq \epsilon\} \leq m\{x \in E: |f_n - f| \geq \epsilon\}$$

$\Rightarrow \lim_{n\to\infty} m\{x \in E: |f_n^-(x) - f^-(x)| \geq \epsilon\} = 0$

$$\Rightarrow f_n^- \xrightarrow{m} f^-$$

v) Since,

$$\left||f_n| - |f|\right| \leq |f_n - f|$$

$$\therefore \{x \in E: \left||f_n| - |f|\right| \geq \epsilon\} \subseteq \{x \in E: |f_n - f| \geq \epsilon\}$$

$$\Rightarrow m\{x \in E: \left||f_n| - |f|\right| \geq \epsilon\} \leq m\{x \in E: |f_n - f| \geq \epsilon\}$$

□ $\lim_{n\to\infty} m\{x \in E: |f_n^-(x) - f^-(x)| \geq \epsilon\} = 0$

$$\Rightarrow |f_n| \xrightarrow{m} |f|.$$

This completes the proof.

Theorem 13.2: If $m(E) < \infty$ and $f_n \xrightarrow{m} f, g_n \xrightarrow{m} g$

then

i) $f_n^2 \xrightarrow{m} f^2$

ii) $f_n g_n \xrightarrow{m} fg$

Proof: *i)* Since $f_n \xrightarrow{m} f$, so limit function f is finite valued $a.e.$

Therefore, for given $\epsilon > 0$, □ a set A and $k > 0$ such that

$$m(A) < \frac{\epsilon}{2} \text{ and } |f| < k \text{ on } A^c \qquad \dots (1)$$

Now, consider

$$E_\delta = \{x: |f_n(x) - f(x)| \geq \delta\}$$

□ $E_\delta^c = \{x: |f_n(x) - f(x)| < \delta\}$ ... (2)

On $(A \cup E_\delta)^c = A^c \cap E_\delta^c$,

$$\begin{aligned} |f_n^2 - f^2| &= |(f_n + f)(f_n - f)| \\ &= |f_n + 2f - f||f_n - f| \\ &\leq [|f_n - f| + 2|f|][|f_n - f|] \\ &< [\delta + 2k][\delta] < \epsilon, \end{aligned}$$

for an appropriate $\delta > 0$.

Also $m(E_\delta) < \frac{\epsilon}{2}$ for all large values of n.

$$\therefore m(A \cup E_\delta) \leq m(A) + m(E_\delta) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, for large values of n,

$$m\{x \in E: |f_n^2 - f^2| \geq \epsilon\} < \epsilon$$

□ $f_n^2 \xrightarrow{m} f^2$.

ii) Since $f_n \xrightarrow{m} f, g_n \xrightarrow{m} g$

□ $f_n + g_n \xrightarrow{m} f + g$ and $f_n - g_n \xrightarrow{m} f - g$

□ $(f_n + g_n)^2 \xrightarrow{m} (f + g)^2$ and $(f_n - g_n)^2 \xrightarrow{m} (f - g)^2$

□ $(f_n + g_n)^2 - (f_n - g_n)^2 \xrightarrow{m} (f + g)^2 - (f - g)^2$

□ $4 f_n g_n \xrightarrow{m} 4fg$

□ $f_n g_n \xrightarrow{m} fg$.

This completes the proof.

Notes: The condition $m(E) < \infty$ in the above theorem is necessary.

Example: Let $f_n(x) = x$ and $g_n(x) = c_n$, $\{c_n\}$ is a sequence of positive reals such that lim [illegible] 0 and [illegible], ∞) then show that [illegible]oes not converge to [illegible]sure.

Solution: Here,

$$m\{x \in E: |f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon\}$$

$$= m\{x \in E: |(c_n)(x)| \geq \epsilon\} = \infty \ \forall n.$$

Hence $f_n g_n$ does not converge to fg in measure.

Theorem 13.3: Let $f_n \xrightarrow{m} f$ on E and $g = f$ $a.e.$ on E. Then $f_n \xrightarrow{m} g$ on E.

Proof: Since $g = f$ $a.e.$

$$\Rightarrow m\{x \in E: f(x) \neq g(x)\} = 0 \qquad \dots (1)$$

Also, $f_n \xrightarrow{m} f$ therefore for given $\epsilon > 0$,

$$\lim_{n\to\infty} m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\} = 0 \qquad \dots (2)$$

Now

$\{x \in E: |f_n(x) - g(x)| \geq \epsilon\}$

$$\subseteq \{x \in E: f(x) \neq g(x)\} \cup \{x \in E: |f_n(x) - f(x)| \geq \epsilon\}$$

$\Rightarrow m\{x \in E: |f_n(x) - g(x)| \geq \epsilon\}$

$$\leq m\{x \in E: f(x) \neq g(x)\} + m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\}$$

$$\Rightarrow \lim_{n\to\infty} m\{x \in E: |f_n(x) - g(x)| \geq \epsilon\} = 0 \Rightarrow f_n \xrightarrow{m} g.$$

This completes the proof.

Theorem 13.4: If $f_n \xrightarrow{m} f$ on E, then limit function f is unique $a.e.$ on E.

Proof: If possible, let g be another function such that $f_n \xrightarrow{m} g$ then for given $\epsilon > 0$,

$$\lim_{n\to\infty} m\left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} = 0 \qquad \dots (1)$$

and

$$\lim_{n\to\infty} m\left\{x \in E: |f_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\} = 0 \qquad \dots (2)$$

$|f - g| = |(f - f_n) + (f_n - g)|$

$$\leq |f_n - f| + |f_n - g|$$

$\Rightarrow \{x \in E: |f(x) - g(x)| \geq \epsilon\}$

$$\subseteq \left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \cup \left\{x \in E: |f_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}$$

$\Rightarrow m\{x \in E: |f(x) - g(x)| \geq \epsilon\}$

$$\leq m\left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} + m\left\{x \in E: |f_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}$$

$\Rightarrow \lim_{n\to\infty} m\{x \in E: |f(x) - g(x)| \geq \epsilon\} = 0$

$\Rightarrow f = g$ $a.e.$ on E.

This completes the proof.

Theorem13.5: Let $\{f_n\}$ be a sequence of measurable functions which converges to f $a.e.$ on E with $m(E) < \infty$ then $f_n \xrightarrow{m} f$ on E.

Proof: For every $n \in \mathbb{N}$ and $\epsilon > 0$, consider the sets

$$S_n(\epsilon) = \{x \in E: |f_n(x) - f(x)| \geq \epsilon\}$$

Since sequence $\{f_n\}$is a sequence of measurable functions which converges to f $a.e.$ on E.

Therefore, for given $\delta > 0$, $\exists$ a measurable set $A \subseteq E$ with $m(A) < \delta$ and a positive integer N such that

$$|f_n(x) - f(x)| < \epsilon, \forall n \geq N, \text{and } \forall x \in E - A$$

□ $S_n(\epsilon)$ □ $A, \forall n \geq N$

$$\Rightarrow m(S_n(\epsilon)) \leq m(A)$$

$$< \delta, \forall n \geq N$$

□ $m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\} < \delta, \forall n \geq N$

□ $f_n \xrightarrow{m} f$ on E.

This completes the proof.

Cauchy Sequence in Measure: A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure or fundamental in measure if for any $\epsilon > 0$,

$$\lim_{n,n\to\infty} m\{x: |f_n(x) - f_m(x)| \geq \epsilon\} = 0$$

Or

A sequence $\{f_n\}$ of measurable function on E is said to be Cauchy in measure if for given $\eta > 0$ and $\epsilon > 0$, there is an index N such that for all $n, p \geq N$,

$$m\{x: |f_n(x) - f_p(x)| \geq \eta\} < \epsilon.$$

Theorem 13.6: If a sequence $\{f_n\} \xrightarrow{m} f$, then it is a Cauchy sequence in measure.

Proof: Since $\{f_n\} \xrightarrow{m} f$, therefore for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} m\{x: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\} = 0 \qquad \dots (1)$$

Now

$$|f_n(x) - f_p(x)| = |[f_n(x) - f(x)] + [f(x) - f_p(x)]|$$
$$\leq |f_n(x) - f(x)| + |f_p(x) - f(x)|$$

Therefore,

$$\{x: |f_n(x) - f_p(x)| \geq \epsilon\}$$
$$\subseteq \left\{x: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \cup \left\{x: |f_p(x) - f(x)| \geq \frac{\epsilon}{2}\right\}$$

□ $m\{x: |f_n(x) - f_p(x)| \geq \epsilon\} \leq m\left\{x: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} + m\left\{x: |f_p(x) - f(x)| \geq \frac{\epsilon}{2}\right\}$

□ $\lim_{n,p\to\infty} m\{x: |f_n(x) - f_p(x)| \geq \epsilon\} = 0$

□ $\{f_n\}$ is Cauchy sequence in measure.

This completes the proof.

13.2 Almost Uniform Convergence

Let $\{f_n\}$ be a sequence of measurable functions and let f be a measurable function then we say that $f_n \to f$ almost uniformly if for any $\epsilon > 0$, □ a set E with $m(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

We write

$$f_n \to f \ a.u.$$

Task: Uniform convergence a.e. implies almost uniform convergence.

Theorem 13.7: If $f_n \to f \ a.u.$ then $f_n \xrightarrow{m} f$.

Proof: Suppose $f_n \xrightarrow{m} f$ is not true, then ∃ positive numbers ϵ and δ such that

$$m\{x: |f_n(x) - f(x)| \geq \epsilon\} > \delta \qquad \dots (1)$$

for infinitely many n.

But since $f_n \to f \ a.u.$ therefore □ a set E with $m(E) < \delta$ such that $f_n \to f$ uniformly on E^c, we get a contradiction.

This completes the proof.

Theorem 13.8: If $\{f_n\}$ is a Cauchy sequence in measure, then there exists a measurable function f such that $f_n \overset{m}{\rightarrow} f$.

Proof: Since $\{f_n\}$ is a Cauchy sequence in measure, therefore $\exists n_1 \in \mathbb{N}$ such that

$$m\left\{x: |f_n(x) - f_p(x)| \geq \frac{1}{2}\right\} < \frac{1}{2} \quad \forall n, p \geq n_1$$

Similarly, □$n_2 > n_1$ such

$$m\left\{x: |f_n(x) - f_p(x)| \geq \frac{1}{2^2}\right\} < \frac{1}{2^2} \quad \forall n, p \geq n_2.$$

Continuing like this, we get a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$m\left\{x: |f_n(x) - f_p(x)| \geq \frac{1}{2^k}\right\} < \frac{1}{2^k} \quad □ n, p \geq n_k > n_{k-1}.$$

Define

$$E_k = \left\{x: |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\right\}$$

Let

$$x \in \left(\bigcup_{i=k}^{\infty} E_i\right)^c$$

□ $x \notin \bigcup_{i=k}^{\infty} E_i$

□ $x \notin E_i$ for any $i \geq k$

□ $|f_{n_i}(x) - f_{n_{i+1}}(x)| < \frac{1}{2^i}$, □$i \geq k$.

Then for $r > s \geq k$, we have

$$|f_{n_r}(x) - f_{n_s}(x)| = \left|\sum_{i=s+1}^{r} \left(f_{n_i}(x) - f_{n_{i-1}}(x)\right)\right|$$

$$\leq \sum_{i=s+1}^{r} |f_{n_i}(x) - f_{n_{i-1}}(x)|$$

$$\leq \sum_{i=s+1}^{\infty} |f_{n_i}(x) - f_{n_{i-1}}(x)|$$

$$< \sum_{i=s+1}^{\infty} \left(\frac{1}{2^i}\right)$$

$$= \frac{\frac{1}{2^{s+1}}}{1 - \frac{1}{2}}$$

$$= \frac{1}{2^s}.$$

□ $|f_{n_r} - f_{n_s}| < \frac{1}{2^s}$ for $r > s \geq k$ and □$x \in \left(\bigcup_{i=k}^{\infty} E_i\right)^c$.

□ The sequence $< f_{n_k} >$ is a uniformly Cauchy sequence in $\left(\bigcup_{i=k}^{\infty} E_i\right)^c$

$$m\left(\bigcup_{i=k}^{\infty} E_i\right) \leq \sum_{i=k}^{\infty} m(E_i)$$

$$< \sum_{i=k}^{\infty} \frac{1}{2^i}$$

$$= \frac{\frac{1}{2^k}}{1-\frac{1}{2}}$$

$$= \frac{1}{2^{k-1}}$$

$$□\ m\left(\bigcup_{i=k}^{\infty} E_i\right) < \frac{1}{2^{k-1}}.$$

Thus, the sequence $< f_{n_k} >$ is an almost uniformly Cauchy sequence converging to some measurable function f.

$$□\ m\left\{x: |f_{n_k}(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \to 0 \text{ as } k \to \infty$$

Now

$$|f_n - f| = |[f_n - f_{nk}] + [f_{n_k} - f\,]|$$

$$\leq |f_n - f_{nk}| + |f_{n_k} - f\,|$$

Therefore, for each $\epsilon > 0$, we have

$\{x: |f_n(x) - f(x)| \geq \epsilon\}$

$$□\ \left\{x: |f_n(x) - f_{n_k}(x)| \geq \frac{\epsilon}{2}\right\} \cup \left\{x: |f_{n_k}(x) - f(x)| \geq \frac{\epsilon}{2}\right\}$$

$$□\ m\{x: |f_n(x) - f(x)| \geq \epsilon\}$$

$$\leq m\left\{x: |f_n(x) - f_{n_k}(x)| \geq \frac{\epsilon}{2}\right\} + m\left\{x: |f_{n_k}(x) - f(x)| \geq \frac{\epsilon}{2}\right\}$$

Since $< f_n >$ is a Cauchy sequence in measure

$$□\ m\left\{x: |f_n(x) - f_{n_k}(x)| \geq \frac{\epsilon}{2}\right\}$$

is arbitrary small if n and n_k are sufficiently large.

$$□\ m\{x: |f_n(x) - f(x)| \geq \epsilon\} \to 0 \text{ as } n \to \infty$$

$$□\ f_n \xrightarrow{m} f.$$

This completes the proof.

Summary

- Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite $a.e.$ on E. The $\{f_n\}$ is said to converge in measure on E to f provided for each $\epsilon > 0$,

  $$\lim_{n\to\infty} m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\} = 0$$

  We write $f_n \xrightarrow{m} f$ on E.

- Let $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$ on E then
  a. $f_n + g_n \xrightarrow{m} f + g$
  b. $cf_n \xrightarrow{m} cf, c \in \mathbb{R}$
  c. $f_n^+ \xrightarrow{m} f^+$
  d. $f_n^- \xrightarrow{m} f^-$
  e. $|f_n| \xrightarrow{m} |f|$.

- If $m(E) < \infty$ and $f_n \xrightarrow{m} f, g_n \xrightarrow{m} g$ then

1. $f_n^2 \xrightarrow{m} f^2$
2. $f_n g_n \xrightarrow{m} fg$

   Here, the condition $m(E) < \infty$ is necessary.

- Let $f_n(x) = x$ and $g_n(x) = c_n, \{c_n\}$ is a sequence of positive reals such that $\lim_{n\to\infty} c_n = 0$ and $E = (0, \infty)$ then $f_n g_n$ does not converge to fg in measure.
- Let $f_n \xrightarrow{m} f$ on E and $g = f$ $a.e.$ on E. Then $f_n \xrightarrow{m} g$ on E.
- If $f_n \xrightarrow{m} f$ on E, then limit function f is unique $a.e.$ on E.
- Let $\{f_n\}$ be a sequence of measurable functions which converges to f $a.e.$ on E with $m(E) < \infty$ then $f_n \xrightarrow{m} f$ on E.
- A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure or fundamental in measure if for any $\epsilon > 0$,

$$\lim_{n,n\to\infty} m\{x: |f_n(x) - f_m(x)| \geq \epsilon\} = 0$$

  Or a sequence $\{f_n\}$ of measurable function on E is said to be Cauchy in measure if for given $\eta > 0$ and $\epsilon > 0$, there is an index N such that for all $n, p \geq N$,

$$m\{x: |f_n(x) - f_p(x)| \geq \eta\} < \epsilon.$$

- If a sequence $\{f_n\} \xrightarrow{m} f$, then it is a Cauchy sequence in measure.
- Let $\{f_n\}$ be a sequence of measurable functions and let f be a measurable function then we say that $f_n \to f$ almost uniformly if for any $\epsilon > 0$, ∃ a set E with $m(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. We write $f_n \to f$ $a.u.$
- Uniform convergence $a.e.$ implies almost uniform convergence.
- If $f_n \to f$ $a.u.$ then $f_n \xrightarrow{m} f$.
- If $f_n\}$ is a Cauchy sequence in measure, then there exists a measurable function f such that $f_n \xrightarrow{m} f$.

Keywords

Convergence in Measure:

Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite $a.e.$ on E. The $\{f_n\}$ is said to converge in measure on E to f provided for each $\epsilon > 0$,

$$\lim_{n\to\infty} m\{x \in E: |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

We write $f_n \xrightarrow{m} f$ on E.

Cauchy Sequence in Measure: A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure or fundamental in measure if for any $\epsilon > 0$,

$$\lim_{n,n\to\infty} m\{x: |f_n(x) - f_m(x)| \geq \epsilon\} = 0$$

Or

A sequence $\{f_n\}$ of measurable function on E is said to be Cauchy in measure if for given $\eta > 0$ and $\epsilon > 0$, there is an index N such that for all $n, p \geq N$,

$$m\{x: |f_n(x) - f_p(x)| \geq \eta\} < \epsilon.$$

Almost Uniform Convergence:

Let $\{f_n\}$ be a sequence of measurable functions and let f be a measurable function then we say that $f_n \to f$ almost uniformly if for any $\epsilon > 0$, ∃ a set E with $m(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

We write

$$f_n \to f \ a.u.$$

Self Assessment

1) A sequence $\{f_n\}$ of measurable functions is said to converge in measure to a measurable function f on a set E if for given $\epsilon > 0$, $\lim_{n\to\infty} m\{x \in E: |f_n(x) - f(x)| > \epsilon\} \neq 0$

A. True

B. False

2) Consider the following statements

(I)$|f_n^- - f^-| \leq |f_n - f|$.

(II)$\big||f_n| - |f|\big| \geq |f_n - f|$.Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

3) Consider the following statements

(I)Let $f_n \overset{m}{\to} f$ on E then $cf_n \overset{m}{\to} cf$ only for $c > 0$.

(II)Let $f_n \overset{m}{\to} f$on E then $f_n^+ \overset{m}{\to} f^+butf_n^- \overset{m}{\nrightarrow} f^-$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

4) Consider the following statements

(I) Let $f_n \overset{m}{\to} f$on E then $|f_n| \overset{m}{\to} |f|$.

(II)Let $f_n \overset{m}{\to} f$on E then $f_n^- \overset{m}{\to} f^-butf_n^+ \overset{m}{\nrightarrow} f^+$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

5) Consider the following statements

(I)Let$f_n \overset{m}{\to} f$ on E then $cf_n \overset{m}{\to} cf$ for $c \in \mathbb{R}$.

(II)Let $f_n \overset{m}{\to} f$on E then $f_n^+ \overset{m}{\to} f^+$and $f_n^- \overset{m}{\to} f^-$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

6)Consider the following statements:

(I) Let $f_n \xrightarrow{m} f$ on E then $f_n^2 \xrightarrow{m} f^2$ on E only if m(E) < ∞.

(II) m(E) < ∞ is the necessary condition in the following result. Let $f_n \xrightarrow{m} f$ on E then $f_n^2 \xrightarrow{m} f^2$ on E. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

7) Let $f_n(x)$ and $g_n(x)$ be defined on $E = (0,\infty)$ such that $f_n(x) = x$ and $g_n(x) = b_n, b_n \in \mathbb{N}$ for each n where $\lim_{n\to\infty} b_n = 0$ then $f_n g_n \xrightarrow{m} fg$ on E.

A. True

B. False

8) Consider the following statements:

(I) Let $f_n \xrightarrow{m} f$ on E and $g = f$ $a.e.$ on E. Then $f_n \xrightarrow{m} g$ on E.

(I) If f_n converges to f in measure on E then limit function f need not be unique. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

9) Consider the following statements:

(I) A sequence f_n of measurable functions is said to be a Cauchy sequence in measure if for given $\eta > 0$ and □> 0, there is an index N such that for all

$$n, p \geq N, m\{x: |f_n(x) - f_p(x)| \geq \eta\} < □.$$

(II) A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure if for any $\epsilon > 0$, $\lim_{m,n\to\infty} m\{x: |f_n(x) - f_m(x)| \geq \epsilon\} = 0$. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

10) Uniform convergence a.e. implies almost uniform convergence.

A. True

B. False

11) If $f_n \to f$ $a.u.$, then $f_n \to f$ in measure.

A. True

B. False

12) If $f_n \to f$ $a.u.$, then it is not necessary that $f_n \to f$ a.e.

A. True

B. False

13) Consider the following statements:

(I) If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f such that $f_n \to f$ in measure.

(II) If $f_n \to f\ a.u.$, then $f_n \to f$ a.e. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

14) If a sequence $\{f_n\}$ converges in measure to f, then it is a Cauchy sequence in measure.

A. True

B. False

15) Let $\{f_n\}$ be a sequence of measurable functions. If for any $\epsilon > 0$, there exists a set E with $m(E) < \epsilon$ and $f_n \to f$ uniformly on E^c then $f_n \to f$ almost uniformly.

A. True

B. False

16) A sequence$\{ f_n \}$ of measurable functions is said to be a Cauchy sequence in measure if for any $\epsilon > 0$, $\lim_{m,n\to\infty} m\{x: |f_n(x) - f_m(x)| \geq \epsilon\} = 0$.

A. True

B. False

17)If $f_n \to f\ a.u.$, then f_n need not to converge to f a.e.

A. True

B. False

18) Let $\{f_n\}$ be a sequence of measurable functions. If for any $\epsilon > 0$, there exists a set E with $m(E) > \epsilon$ and $f_n \to f$ uniformly on E^c then $f_n \to f$ almost uniformly.

A. True

B. False

Answers for Self Assessment

1.	B	2.	A	3.	D	4.	A	5.	C
6.	C	7.	B	8.	A	9.	C	10.	A
11.	A	12.	B	13.	C	14.	A	15.	B
16.	A	17.	B	18.	B				

Review Questions

1)Show that the condition $m(E) < \infty$ is necessary for the result $f_n \overset{m}{\rightarrow} f \Rightarrow f_n^2 \overset{m}{\rightarrow} f^2$.

2) Show that the condition $m(E) < \infty$ is necessary for the result $f_n \overset{m}{\rightarrow} f,\ g_n \overset{m}{\rightarrow} g \Rightarrow f_n g_n \overset{m}{\rightarrow} fg$.

3) If $f_n \rightarrow f_1$ in measure and also $f_n \rightarrow f_2$ in measure then show that $f_1 = f_2\ a.e.$

4) Show that uniform convergence $a.e.$ implies almost uniform convergence.

5)Let $f_n \overset{m}{\rightarrow} f$ where f and each f_n are measurable functions. Then there exists a subsequence $\{n_i\}$ such that $f_{n_i} \rightarrow f\ a.e.$

Further Readings

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Graduate Texts in Mathematics, Measure Integration and Real Analysis, Sheldon Axler, Springer.

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Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

https://www.youtube.com/watch?v=YIrx8W5nyq8&t=29s

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Unit 14: Egoroff's Theorem, Lusin's Theorem, F. Riesz's Theorem

CONTENTS

Objectives

After studying this unit, students will be able to:

- understand Egoroff's theorem
- explain the proof of Egoroff's theorem
- demonstrate Lusin's theorem
- understand proof of Lusin's theorem
- explain F. Riesz's theorem and its proof

Introduction

The main aim of this unit is to analyze the convergence of sequences of measurable functions. In this context, we will discuss three important theorems viz. Egoroff's theorem, Lusin's theorem, and F. Riesz's theorem.

14.1 Egoroff's Theorem

Lemma 14.1: Let E be a measurable set of finite measure and $\{f_n(x)\}$ be a sequence of measurable functions defined on E such that $f_n \to f$ on E. Then for given $\epsilon > 0$ and $\delta > 0, \exists$ a measurable set $A \subseteq E$ with $m(A) < \delta$ and a positive integer N such that

$$|f_n(x) - f(x)| < \epsilon \; \forall n \geq N, \forall x \in E - A.$$

Proof: Let

$$G_n = \{x: |f_n(x) - f(x)| \geq \epsilon\}, \forall n \in \mathbb{N}$$

Since $\{f_n(x)\}$ is a sequence of measurable functions, therefore $\lim_{n\to\infty} f_n(x) = f(x)$ is also a measurable function.

$\Rightarrow G_n$ is measurable $\forall n \in \mathbb{N}$.

Let

$$E_k = \bigcup_{n=k}^{\infty} G_n$$

Then $\{E_k\}$ is a sequence of measurable sets such that

$$E_{k+1} \subseteq E_k \text{ and } m(E_k) < \infty, \forall k \in \mathbb{N}$$

$$\bigcap_{k=1}^{\infty} E_k = \phi$$

If possible, let

$$x \in \bigcap_{k=1}^{\infty} E_k$$

$$\Rightarrow x \in E_k, \forall k \in \mathbb{N}$$

$$\Rightarrow x \in \bigcup_{n=k}^{\infty} G_n, \forall k \in \mathbb{N}$$

$$\Rightarrow \forall k \in \mathbb{N}, \exists n \geq k \text{ such that } x \in G_n$$

$$\Rightarrow \forall k \in \mathbb{N}, \exists n \geq k \text{ such that} |f_n(x) - f(x)| \geq \epsilon$$

$$\Rightarrow f_n \nrightarrow f, \text{which is a contradiction.}$$

Therefore

$$\bigcap_{k=1}^{\infty} E_k = \phi$$

Therefore

$$\lim_{k\to\infty} m(E_k) = m\left(\bigcap_{k=1}^{\infty} E_k\right)$$

$$= m(\phi)$$

$$= 0$$

$$i.e. \lim_{k\to\infty} m(E_k) = 0$$

$\Rightarrow \forall \delta > 0, \exists$ a positive N such that $m(E_n) < \delta, \forall n \geq N$.

Let$E_N = A$

$\Rightarrow A$ is a measurable subset of E with $m(A) < \delta$

and

$$x \in E - A$$

$$\Rightarrow x \notin A$$

$$\Rightarrow x \notin E_N$$

$$\Rightarrow x \notin \bigcup_{n=N}^{\infty} G_n$$

$$\Rightarrow x \notin G_n \text{ for any } n \geq N$$

$$\Rightarrow |f_n(x) - f(x)| < \epsilon, \forall n \geq N$$

Hence for given $\epsilon > 0$ and $\delta > 0, \exists$ a measurable set $A \subseteq E$ with $m(A) < \delta$ and a positive integer N such that

$$|f_n(x) - f(x)| < \epsilon \; \forall n \geq N, \forall x \in E - A.$$

This completes the proof.

Notes: Let E be a measurable ... h fir ... easure and ... a ... quer measurable functions such that $f_n \to f$ a.e. on E. Then for given ϵ $\{f_n\}$ be ... > 0 and sec... $\delta > 0, \exists$ ...nce of $A \subseteq E$ such that $m(A) < \delta$ and a positive integer N for which

$$|f_n - f| < \epsilon, \forall n \geq N \text{ and } \forall x \in E - A.$$

Theorem 14.2: (Egoroff's Theorem)

Assume E has a finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to be the function f. Then for each $\epsilon > 0, \exists$ a measurable set $A \subseteq E$ for which $f_n \to f$ uniformly on A and $m(E - A) < \epsilon$.

Proof:Here we use the theorem "Assume E has a finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to f. Then for each $\eta > 0$ and $\delta > 0, \exists$ a measurable set $A \subseteq E$ and an index N for which

$$|f_n - f| < \eta, \forall n \geq N \text{ on } A \text{ and } m(E - A) < \delta\text{"}$$

For each natural number n, Let A_n be a measurable subset of E, and $N(n)$ and index which satisfy the conclusion of the above mention result with above mention result with

$$\delta = \frac{\epsilon}{2^n} \text{ and } \eta = \frac{1}{n}$$

$$i.e.\ m(E - A_n) < \frac{\epsilon}{2^n} \qquad \ldots (1)$$

and

$$|f_k - f\ | < \frac{1}{n} \text{ on} A_n \text{for all } k \geq N(n) \qquad \ldots (2)$$

Define

$$A = \bigcap_{n=1}^{\infty} A_n.$$

By De Morgan's identities, the countable subadditivity of measure and (1), we get

$$m(E - A) = m\left(\bigcup_{n=1}^{\infty} (E - A_n)\right)$$

$$\leq \sum_{n=1}^{\infty} m(E - A_n)$$

$$< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

We claim that $f_n \to f$ uniformly on A.

Let $\epsilon > 0$, choose an index n_0 such that $\frac{1}{n_0} < \epsilon$ then by (2)

$$|f_k - f| < \frac{1}{n_0} \text{ on} A_{n_0} \text{for } k \geq N(n_0).$$

However, $A \subseteq A_{n_0}$ and $\frac{1}{n_0} < \epsilon$.

$\therefore |f_k - f| < \epsilon$ on A for $k \geq N(n_0)$.

Thus $f_n \to f$ uniformly on A and $m(E - A) < \epsilon$.

This completes the proof.

14.2 Lusin's Theorem

Theorem 14.3: (Lusin's Theorem)

Let f be a measurable function defined on E. Then for given $\varepsilon > 0$, there is closed set $F \subseteq E$ and a continuous function g on $\mathbb{R}$ such that

$$m(E - F) < \epsilon \text{ and } f = g \text{ on } F.$$

Notes: Tietze Extension Theorem:

Let F be a closed subset of $\mathbb{R}$. If $f: F \to \mathbb{R}$ is continuous on F then there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F.

Proof: Case I: f is a simple function.

Let

$$f = \sum_{i=1}^{n} a_i \chi_{E_i},$$

Where

$E_i = \{x \in E: f(x) = a_i\}$

are pairwise disjoint measurable sets.

Let

$$E_{n+1} = E - \bigcup_{i=1}^{n} E_i$$

so that E_{n+1} is measurable and

$$E = \bigcup_{i=1}^{n+1} E_i$$

Since E_i is measurable, $1 \leq i \leq n+1$.

Therefore, for given $\epsilon > 0, \exists$ closed set $F_i \subseteq E_i, 1 \leq i \leq n+1$

such that

$$m(E_i - F_i) < \frac{\epsilon}{n+1} \qquad \dots (1)$$

Let

$$F = \bigcup_{i=1}^{n+1} F_i.$$

Then F is closed, $F \subseteq E$ and

$$m(E - F) = m\left[\bigcup_{i=1}^{n+1} E_i - \bigcup_{i=1}^{n+1} F_i\right]$$

$$= m\left[\bigcup_{i=1}^{n+1} (E_i - F_i)\right]$$

$$= \sum_{i=1}^{n+1} (E_i - F_i)$$

$$< \sum_{i=1}^{n+1} \frac{\epsilon}{n+1}$$

$$- \frac{\epsilon}{n+1}(n+1)$$

$$= \epsilon$$

$$i.e.\ m(E - F) < \epsilon$$

F_i's are disjoint closed sets and f takes constant values a_i on each F_i.

Therefore f is continuous on F.

Case II: f is any measurable function on E.

Here we use the theorem "If f is a measurable function defined on E then $\exists$ a sequence $\{s_n\}$ of simple functions such that

$$s_n \to f \text{ on } E"$$

We can find a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ on E.

By case I, for each n and given $\epsilon > 0, \exists$ a closed set $A_n \subseteq E$ such that

$$m(E - A_n) < \frac{\epsilon}{2^{n+1}}$$

and f_n is continuous on A_n.

Let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Then each f_n is continuous on A and

$$m(E - A) = m\left[E - \bigcap_{n=1}^{\infty} A_n\right]$$

$$= m\left[\bigcup_{n=1}^{\infty}(E - A_n)\right]$$

$$\leq \sum_{n=1}^{\infty} m(E - A_n)$$

$$< \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}$$

$$i.e. m(E - A) < \frac{\epsilon}{2} \qquad \ldots (2)$$

Corresponding to each positive integer k, define

$$B_k = A \cap \{x: k - 1 \leq |x| \leq k\} \;\; \forall k \in \mathbb{N}$$

Each B_k is measurable and

$$A = \bigcup_{k=1}^{\infty} B_k$$

Now, as $f_n \to f$ on each B_k, so by Egoroff's theorem, for given $\epsilon > 0, \exists$ a measurable set $C_k \subseteq B_k$ such that

$$m(B_k - C_k) < \frac{\epsilon}{2^{k+2}}$$

and $f_n \to f$ uniformly on C_k.

Also corresponding to each measurable set C_k, there exists a closed set F_k such that

$$m(C_k - F_k) < \frac{\epsilon}{2^{k+2}}.$$

Therefore,$\exists$ a closed set $F_k \subseteq B_k$ such that

$$m(B_k - F_k) = m[(B_k - C_k) \cup (C_k - F_k)]$$

$$\{\because F_k \subseteq C_k \subseteq B_k\}$$

$$= m(B_k - C_k) + m(C_k - F_k)$$

$$< \frac{\epsilon}{2^{k+2}} + \frac{\epsilon}{2^{k+2}}$$

$$= \frac{\epsilon}{2^{k+1}}$$

$$i.e. \; m(B_k - F_k) < \frac{\epsilon}{2^{k+1}} \qquad \ldots (3)$$

and $f_n \to f$ uniformly on $F_k, \forall k \in \mathbb{N}$.

Hence f is continuous on $F_k, \forall k \in \mathbb{N}$

$$F = \bigcup_{k=1}^{\infty} F_k$$

As

$$F_k \subseteq B_k \subseteq \{x: k-1 \leq |x| \leq k\}, \forall k \in \mathbb{N}$$

Therefore

$$F = \bigcup_{k=1}^{\infty} F_k$$

is closed and f is continuous on F.

Also

$$m(E-F) = m(E-A) + m(A-F)$$

$$\because A_n \subseteq E \ \forall n$$

$$\Rightarrow \bigcap A_n \subseteq E$$

and

$$F_k \subseteq B_k$$

$$\Rightarrow \bigcup F_k \subseteq \bigcup B_k$$

$$\Rightarrow F \subseteq A$$

$\therefore F \subseteq A \subseteq E.$

Now,

$$m(E-F) < \frac{\epsilon}{2} + m(A-F) \quad \{by\ (2)\}$$

$$= \frac{\epsilon}{2} + m\left[\bigcup_{k=1}^{\infty} B_k - \bigcup_{k=1}^{\infty} F_k\right]$$

$$= \frac{\epsilon}{2} + m\left[\bigcup_{k=1}^{\infty} (B_k - F_k)\right]$$

$$< \frac{\epsilon}{2} + \sum_{k=1}^{\infty} m(B_k - F_k)$$

$$< \frac{\epsilon}{2} + \sum_{k=1}^{\infty} \frac{\epsilon}{2^{k+1}} \{by\ (3)\}$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$i.e.\ m(E-F) < \epsilon$$

f is continuous on closed set $F \subseteq E$ and $m(E-F) < \epsilon$.

This completes the proof.

14.3 F. Riesz's Theorem

Theorem 14.4: (F. Riesz Theorem)

Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \xrightarrow{m} f$ on E. Then $\exists$ a subsequence $\{f_{n_k}\}$ that converges to f $a.e.$ on E.

Proof: Consider two sequences $\{\epsilon_n\}$ and $\{\delta_n\}$ of positive real numbers such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \delta_n < \infty$$

Since $f_n \xrightarrow{m} f$ on E.

Therefore, for given $\epsilon > 0$ and $\delta > 0, \exists n_0 \in \mathbb{N}$ such that

$$m[\{x \in E: |f_n(x) - f(x)| \geq \epsilon\}] < \delta, \forall n \geq n_0 \qquad \text{... (1)}$$

By using (1), we form a strictly increasing sequence $< n_k >$ of positive integers as follows:

For $\epsilon_1 > 0$ and $\delta_1 > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$m[\{x \in E: |f_{n_1}(x) - f(x)| \geq \epsilon_1\}] < \delta_1, \forall n_1 > n_0$$

Also, there exists $n_2 \in \mathbb{N}$ such that

$$m[\{x \in E: |f_{n_2}(x) - f(x)| \geq \epsilon_2\}] < \delta_2, \forall n_2 > n_1$$

and so on.

In general, we get the number $n_k \in \mathbb{N}$ such that

$$m[\{x \in E: |f_{n_k}(x) - f(x)| \geq \epsilon_k\}] < \delta_k \ \ \forall n_k > n_{k-1}$$

Now we will prove that subsequences $< f_{n_k} >$ converges to f $a.e.$ on E.

Let

$$A_k = \bigcup_{i=k}^{\infty} \{x \in E: |f_{n_i}(x) - f(x)| \geq \epsilon_i\}, k \in \mathbb{N} \qquad \text{... (2)}$$

and

$$A = \bigcap_{k=1}^{\infty} A_k$$

Then $< A_k >$ is monotonically decreasing sequence of measurable sets and $m(A_1) < \infty$.

We have the result "If $\{E_n\}$ be monotonically decreasing sequence of measurable sets $i.e. E_{n+1} \subseteq E_n, \forall n$ such that $m(E_1) < \infty$ then

$$m\left(\bigcap E_n\right) = \lim_{n \to \infty} m(E_n)\text{"}$$

Therefore

$$m(A) = m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} m(A_k) \qquad \text{... (3)}$$

But

$$m(A_k) \leq \sum_{i=k}^{\infty} m\{x \in E: |f_{n_i}(x) - f(x)| \geq \epsilon_i\}$$

$$< \sum_{i=k}^{\infty} \delta_i \to 0 \; as \; k \to \infty$$

Therefore $m(A) = 0$.

Now we claim that $< f_{n_k} >$ converges to f on $E - A$.

For this let

$$x_0 \in E - A$$

$$\Rightarrow x_0 \notin A$$

$$\Rightarrow x_0 \notin \bigcap_{k=1}^{\infty} A_k$$

$$\Rightarrow x_0 \notin \text{at least one} A_k$$

$$\Rightarrow x_0 \notin A_{k_0} \text{for some positive integer} k_0$$

$$\Rightarrow x_0 \notin \bigcup_{i=k_0}^{\infty} \{x \in E: |f_{n_i}(x) - f(x)| \geq \epsilon_i\}$$

$$□ \; x_0 \notin \{x \in E: |f_{n_i}(x) - f(x)| \geq \epsilon_i\}, \forall i \geq k_0$$

$$□ \; |f_{n_i}(x_0) - f(x_0)| < \epsilon_i, □ i \geq k_0$$

But $\epsilon_i \to 0$ as $i \to \infty$.

Therefore,

$$\lim_{i \to \infty} f_{n_i}(x) = f(x_0)$$

$$□ \; f_{n_k} \to f \text{ on } E - A$$

$$□ \; f_{n_k} \to f a.e. \text{ on } E.$$

This completes the proof.

Summary

- Let E be a measurable set of finite measure and $\{f_n(x)\}$ be a sequence of measurable functions defined on E such that $f_n \to f$ on E. Then for given $\epsilon > 0$ and $\delta > 0$, □ a measurable set $A \subseteq E$ with $m(A) < \delta$ and a positive integer N such that
$$|f_n(x) - f(x)| < \epsilon \; \forall n \geq N, \forall x \in E - A.$$
- Let E be a measurable set with finite measure and $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ $a.e.$ on E. Then for given $\epsilon > 0$ and $\delta > 0$, □$A \subseteq E$ such that $m(A) < \delta$ and a positive integer N for which
$$|f_n - f| < \epsilon, \forall n \geq N \text{ and } □x \in E - A.$$
- Egoroff's Theorem:Assume E has a finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to be the function f. Then for each $\epsilon > 0$, □ a measurable set $A \subseteq E$ for which $f_n \to f$ uniformly on A and $m(E - A) < \epsilon$.
- Lusin's Theorem: Let f be a measurable function defined on E. Then for given $\epsilon > 0$, there is closed set $F \subseteq E$ and a continuous function g on $\mathbb{R}$ such that
$$m(E - F) < \epsilon \text{ and } f = g \text{ on } F.$$
- Tietze Extension Theorem:Let F be a closed subset of $\mathbb{R}$. If $f: F \to \mathbb{R}$ is continuous on F then there is a continuous function $g: \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F.
- If f is a measurable function defined on E then $\exists$ a sequence $\{s_n\}$ of simple functions such that
$$s_n \to f \text{ on } E$$
- F. Riesz Theorem: Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \overset{m}{\to} f$ on E. Then $\exists$ a subsequence $\{f_{n_k}\}$ that converges to f $a.e.$ on E.
- If $\{E_n\}$ be monotonically decreasing sequence of measurable sets $i.e. E_{n+1} \subseteq E_n, \forall n$ such that $m(E_1) < \infty$ then
$$m\left(\bigcap E_n\right) = \lim_{n \to \infty} m(E_n).$$

Keywords

Egoroff's Theorem: Assume E has a finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to be the function f. Then for each $\epsilon > 0$, □ a measurable set $A \subseteq E$ for which $f_n \to f$ uniformly on A and $m(E - A) < \epsilon$.

Lusin's Theorem: Let f be a measurable function defined on E. Then for given $\epsilon > 0$, there is closed set $F \subseteq E$ and a continuous function g on $\mathbb{R}$ such that

$$m(E - F) < \epsilon \text{ and } f = g \text{ on } F.$$

Tietze Extension Theorem:

Let F be a closed subset of $\mathbb{R}$. If $f: F \to \mathbb{R}$ is continuous on F then there is a continuous function $g: \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F.

F. Riesz Theorem: Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \overset{m}{\to} f$ on E. Then a subsequence $\{f_{n_k}\}$ that converges to f $a.e.$ on E

Self Assessment

1) Let $A = \{x \in E: \lim_{n\to\infty} f_n(x) \neq f(x)\}$ and $\lim_{n\to\infty} f_n(x) = f(x)\ a.e. on E$ then

A. $m(A) > 0$

B. $m(A) = 0$

C. $m(A) = \infty$

D. none of these

2) Let $m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E that converges a.e. on E to the real-valued function f. Then for each $\epsilon > 0$, there exists a measurable set $A \subseteq E$ for which $f_n \to f$ uniformly on A and $m(E - A) < \epsilon$. This is known as

A. Lusin's theorem

B. Egoroff's Theorem

C. Riesz's Theorem

D. Tietze Extension Theorem

3) Consider the following statements:

(I) Let $m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E such that $f_n \to f$ on E. Then for given $\epsilon > 0$ and $\delta > 0$, there exists a measurable set $A \subseteq E$ with $m(A) < \delta$ and a positive integer N such that $|f_n(x) - f(x)| < \epsilon\ \forall n \geq N, \forall x \in E - A$.

(II) Let $m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E such that $f_n \to f$ a.e. on E. Then for given $\epsilon > 0$ and $\delta > 0$, there exists a measurable set $A \subseteq E$ with $m(A) < \delta$ and a positive integer N such that $|f_n(x) - f(x)| < \epsilon\ \forall n \geq N, \forall x \in E - A$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

4) $m(\bigcup_n A_n) \leq \sum_n m(A_n)$.

A. True

B. False

5) $\sum_n \frac{1}{2^n} = \frac{1}{2}$.

A. True

B. False

6) Let f be a measurable function defined on a set E. Then for given $\epsilon > 0$, there is a closed set $F \subseteq E$ and a continuous function g on $\mathbb{R}$ such that $m(E - F) < \epsilon$ and $f = g$ on F. This is known as

A. Lusin's theorem

B. Egoroff's Theorem
C. Riesz's Theorem
D. Tietze Extension Theorem

7) Let F be a closed subset of $\mathbb{R}$. If $f: F \to \mathbb{R}$ is continuous on F then there is a continuous function $g: \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F. This is known as

A. Lusin's theorem
B. Egoroff's Theorem
C. Riesz's Theorem
D. Tietze Extension Theorem

8) If f is a measurable function defined on E, then there exists a sequence $\{s_n\}$ of simple functions such that $s_n \to f$ on E.

A. True
B. False

9) Let f be a simple function. Suppose $a_1, a_2, \dots, a_n$ are distinct and $E_i = \{x \in E: f(x) = a_i\}$ are pairwise disjoint measurable sets. Then $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ is the canonical representation of f.

A. True
B. False

10) Let $F_i, 1 \le i \le n$, be closed, then $F = \bigcup_{i=1}^{n} F_i$ is also closed.

A. True
B. False

11) Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \xrightarrow{m} f$ on E. Then there exists a subsequence $\{f_{n_k}\}$ that converges to f a.e. on E. This is known as

A. Lusin's theorem
B. Egoroff's Theorem
C. Riesz's Theorem
D. Tietze Extension Theorem

12) Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \xrightarrow{m} f$ on E. Then there exists a NO subsequence $\{f_{n_k}\}$ that converges to f a.e. on E.

A. True
B. False

13) If $\{E_n\}$ be a monotonically decreasing sequence of measurable sets such that $m(E_1) < \infty$ then $m(\bigcap E_n) = \lim_{n\to\infty} m(E_n)$.

A. True
B. False

14) Consider the following statements:

(I) Let $x \notin \bigcap_{k=1}^{\infty} A_k \Rightarrow x$ is not contained in at least one A_k.

(II) Let $x \in \bigcap_{k=1}^{\infty} A_k \Rightarrow x$ is contained in at most one A_k

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

15) Let $f_n \to f$ on $E - A$ and $m(A) = 0$. Then $f_n \to f$ a.e on E.

A. True
B. False

Answers for Self Assessment

1.	B	2.	B	3.	C	4.	A	5.	B
6.	A	7.	D	8.	A	9.	A	10.	A
11.	C	12.	B	13.	A	14.	A	15.	A

Review Questions

1) Let E be a measurable set with finite measure and $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ $a.e.$ on E. Then for given $\epsilon > 0$ and $\delta > 0, \exists A \subseteq E$ such that $m(A) < \delta$ and a positive integer N for which $|f_n - f| < \epsilon, \forall n \geq N$ and $\forall x \in E - A$.

2)If f is a measurable function defined on E then □ a sequence $\{s_n\}$ of simple functions such that

$s_n \to f$ on E.

3) If $\{E_n\}$ be monotonically decreasing sequence of measurable sets $i.e. E_{n+1} \subseteq E_n, \forall n$ such that $m(E_1) < \infty$ then $m(\cap E_n) = \lim_{n \to \infty} m(E_n)$.

4) Let f be a simple function defined on E. Then for given $\epsilon > 0$, there is closed set $F \subseteq E$ and a continuous function g on $\mathbb{R}$ such that

$$m(E - F) < \epsilon \text{ and } f = g \text{ on } F.$$

5)Assume E has a finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges $a.e.$ on E to f. Then for each $\eta > 0$ and $\delta > 0, \exists$ a measurable set $A \subseteq E$ and an index N for which

$$|f_n - f| < \eta, \forall n \geq N \text{ on } A \text{ and } m(E - A) < \delta.$$

Further Readings

Measure theory and integration by G DE BARRA, New AgeInternational.

Real Analysis by H LRoyden and P M Fitzpatrick, Pearson.

Lebesgue Measure and Integration by P K Jain, V P Gupta and Pankaj Jain, New Age International.

Graduate Texts in Mathematics, Measure Integration and Real Analysis, Sheldon Axler, Springer.

An Introduction to Measure and Integration, I K Rana.

Web Links

https://nptel.ac.in/courses/111/101/111101100/

https://www.youtube.com/watch?v=xUMRSOtM654&list=PL_a1TI5CC9RGKYvo8XNFTK9zkjMbYTEwS

https://www.youtube.com/watch?v=YIrx8W5nyq8&t=29s

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