Theory of Differential Equations

DEMTH517

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CONTENT

Unit 01: Introduction and initial value problem

Contents

Expected Learning Outcomes

After studying this unit, you will be able to

- understand about the different types of differential equations.
- analyze in the form of an explicit form, preferably in the form of elementary functions.
- find the qualitative property of the differential equation.
- understand the need of an initial value problem.

Introduction

Most dynamical systems-physical, social, biological, and engineering-are are often conveniently expressed in differential equations. Such equations can provide insight into a system's behaviour if they represent the various important factors governing the system. For instance, when a system is known to perform efficiently over a certain range of input, the solution of the differential equation governing the system over the interval is an important consideration in understanding its behavior.

This unit introduces the basic concept to define all kinds of differential equations, which can further help study the more behavior or different type of differential equations.

1.1 Notation and definition

In our discussion, the independent variable is always treated as real and is denoted by t. Further, the dependent variables, u for scalar equations and x for the vector-valued equations, as also all the functions are assumed to be real. However, the theory developed in this chapter can, with minor modifications, be extended to the complex case.

Let R be the set of all real numbers, and I be an open interval on the real line R, that is, $I = \{t: t\epsilon R, r_1 < t < r_2\}$, where $r_1$and r_2 are any two fixed points in R. Also, let R^n denote the real n dimensional Euclidean space with elements $\mathrm{x} = (\mathrm{t}, \mathrm{x}_1, \mathrm{x}_2, \dots \dots \dots, \mathrm{x}_\mathrm{n})$ or (t, x).

We shall often use R instead of R^1. Let B be a domain, i.e. an open-connected set in R^{n+1}, and $C[B, R^n]$ be a class of functions defined and continuous on B, taking values in R^n.

When f is a member of this class, we shall write $f\epsilon C[B, R^n]$.

Definition 1.1.1 An ordinary differential equation of the n-th order and of form $\mathrm{F}(\mathrm{t}, \mathrm{u}, \mathrm{u}', \mathrm{u}'', \dots \dots \dots, \mathrm{u}^{(\mathrm{n})}) = 0$ (1.1.1) where $u^{(n)}$ is the nth derivative of the unknown functions u with respect to t and F is defined in some subset of R^{n+2}, expressed relation between the $(n+2)$- variables $t, u, u', u'', \ldots\ldots u^{(n)}$. Because of its implicit nature, (1.1.1) may represent a collection of differential equations.

Example 1.1.2: The implicit equation $u'^3 - 3t^2u'^2 + 3uu' = 0$ leads to three equations, namely, $u' = 0, u' = (3t^2 + (9t^2 - 12u)^{\frac{1}{2}})/2,\ u' = (3t^2 - (9t^2 - 12u)^{\frac{1}{2}})/2$.

To avoid ambiguity the implicit equation (1.1.1) may exhibit, we shall assume that this equation is solvable for u^n; then, it can be written in the form $u^n = g(t, u, u', u'', \ldots\ldots\ldots, u^{(n-1)})$ (1.1.2)

where g is a given function defined on B.

If g is linear in $u, u', u'', \ldots\ldots\ldots, u^{n-1}$, then the differential equation (1.1.1) is called linear; otherwise, it is referred to as nonlinear.

Definition 1.1.3 A function $\varphi(t)$ is called a solution of (1.1.2) on $r_1 < t < r_2$ if $\varphi(t)$ is defined and n-times differentiable on $r_1 < t < r_2$ and satisfies

$$\varphi^n(t) = g(t, \varphi(t), \varphi'(t), \varphi''(t), \ldots\ldots\ldots, \varphi^{n-1}(t)), \quad t\epsilon(r_1, r_2).$$

Example 1.1.4: The functions $u_1(t) = t^2$ and $u_2(t) = \frac{1}{t}$ are the solution of the differential equation $u'' = 2u/t^2, t > 0$. Similarly, the functions $u_1(t) = 1, u_2(t) = \cos t$ and $u_3(t) = \sin t$ are the solutions of $u''' + u' = 0$ for all t.

The aim of the study of an ordinary differential equation is to find an explicit form, preferably in the form of elementary functions. In the absence of an explicit form, one needs to study the behavior of solutions by available analytical methods.

Before looking for a solution or any qualitative properties, we want to know the class or group in which the equations belong to.

Classification based on the dependent variable: linear or nonlinear

Classification based on condition: Initial value problem or boundary value problem

Initial value problem

We begin with the first-order scalar differential equations of the form

$$u' = g(t, u), \tag{1.1.3}$$

Where g is a real-valued continuous function on $I \times R$. Equation (1.1.3) is termed linear if the function $g(t, u)$ is linear in u; otherwise, it is called nonlinear.

We shall deal first with the elementary properties of the solutions of (1.1.3).

Definition 1.1.5: (General solution) A solution of a differential equation is said to be a general solution if it includes all the solutions of the differential equation.

In order to gain familiarity with differential equations and their solution, we start with linear equations of the form

$$u' = a(t)u + b(t), \tag{1.1.4}$$

Where $a(t)$ and $b(t)$ are continuous on I.

Example 1.1.6: As a special case of (1.1.4), consider the equation

$$u' = tu + 2t. \tag{1.1.5}$$

It can be easily verified that the function $u(t) = ce^{t^2/2} - 2$, (1.1.6)

Where c is an arbitrary constant, satisfy (1.1.5) for al t in R. Since all the solutions of (1.1.5) can be obtained from (1.1.6) by assigning suitable values to c, (1.1.6) is the general solution of (1.1.5). Further, for each fixed value of c, (1.1.6) represents a curve in the (t,u)-plane and, if c is arbitrary, it represents infinitely many curves. The totality of these curves is known as a one-parameter family of curves and c is the parameter of this family. Such curves are also called the integral curves of (1.1.5).

For most problems is science and engineering, we are interested not in a general solution but only in a particular solution satisfying a given initial condition. Determining a particular solution is equivalent to picking out a specific integral curve from the one-parameter family. This can be achieved by prescribing an initial condition, namely, at some $t = t_0$ the solution $u(t)$ must have a pre-assigned value u_0, that is

$$u(t_0) = u_0. \tag{1.1.7}$$

A differential equation equipped with an initial condition is said to form an initial value problem. For example, (1.1.3) with the initial condition (1.1.7) is an initial value problem.

In the next, we shall confine our study to initial value problems.

Definition 1.1.7: A real-valued function $u(t)$define on I is said to be a solution of the initial value problem (1.1.3), (1.1.7) on I if

(i) $u'(t)$exists for $t \in I$;

(ii) $u(t_0) = u_0, t_0 \in I$;

(iii) The points $(t, u(t))I \times R, t \in I$; and

(iv) $u'(t) = g(t, u), t \in I$.

It should be noted that nonlinear differential equations differ linear from ones. For instance, there are several methods are available to solve the linear differential equations, but no such methods are available for nonlinear differential equations., in particular, determining their explicit solution is usually very difficult, if not impossible. Consequently, the methods that yield approximate solution or qualitative information about the solution of nonlinear equations are very useful. Further, the concept of general solution for linear equations differs from that for nonlinear equations. More precisely, a (first order) linear equations has only one general solution where as a nonlinear have a general solution as well as singular solutions.

1.2 System of differential equations

We shall consider a system of first-order differential equations of the form

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \dots \dots x_n) \\ x_2' &= f_2(t, x_1, x_2, \dots \dots, x_n) \\ &. \\ &. \\ x_n' &= f_n(t, x_1, x_2, \dots \dots, x_n) \end{aligned} \tag{1.2.1}$$

Where are $f_1, f_2, \dots \dots, f_n$ given functions in some domain B of $(n+1)-$ dimensional Euclidean space R^{n+1} and $x_1, x_2, \dots \dots x_n$ are n- unknown functions.

Definition 1.2.1 A set of n-functions $\varphi_1, \varphi_2, \dots \dots, \varphi_n$ defined on I is said to be a solution of (1.2.1) on I if, for $t \in I$,

$\varphi_1'(t), \varphi_2'(t), \dots \dots \dots \varphi_n'(t)$ exists;

the point $(t, \varphi_1(t), \varphi_2(t), \dots \dots, \varphi_n(t))$ remains in B; and

$$\varphi_i'(t) = f_i(t, \varphi_1(t), \varphi_2(t), \dots \dots, \varphi_n(t)), \quad i = 1,2, \dots, n$$

Geometrically, this amount to saying that a solution of (1.1.2) is a curve in the (n+1)-dimensional region B with each point p on the curve and has the coordinates$(t, \varphi_1(t), \varphi_2(t), \dots \dots, \varphi_n(t))$, where $\varphi_i'(t)$is the $i\,th$ component of the tangent vector to the curve in the direction x_i. When $n = 1$, this interpretation is clear, and thus the curve in B defined by any solution of (1.2.1) is again a solution curve.

An $n\,th$ order differential equation of the form (1.1.2) may also be treated as a system of the type (1.2.1). To see this, let

$u = u_1, u' = u_2, \ldots\ldots.., u^{(n-1)} = u\_n$.

Then, (1.1.2) is equivalent to

$u_i' = u_{i+1},\ i = 1,2, \ldots \ldots, n-1,$

$u_n' = g(t, u_1, u_2, \ldots.., u_n).$

This set of equations is indeed of the form (1.2.1).

Example 1.2.2.: In particular, consider the second-order differential equation

$$u'' + u'^2 = g(t, u), \qquad (1.2.2)$$

where g is a given function. Setting $u = u_1, u' = u_2,$ we have the system

$$u_1' = u_2, u_2' = -u_2^2 + g(t, u_1). \qquad (1.2.3)$$

This is a special case of (1.2.3) with n=2, $f_1(t, u_1, u_2) = u_2$, and $f_2(t, u_1, u_2) = -u_2^2 + g(t, u_1)$.

It can be easily verified that (1.2.2) and (1.2.3) are equivalent. For this, let φ be a solution of (1.2.2) on I. Then, $u_1 = \varphi(t), u_2 = \varphi'(t)$ is a solution of (1.2.3) on I since

$u_1' = \varphi' = u_2,$

$u_2' = \varphi'' = -\varphi'^2 + g(t, \varphi) = -u_2^2 + g(t, u_1).$

Conversely, let (φ_1, φ_2) be a solution of (1.2.3) on I. Then $u_1 = \varphi_1(t)$, that is the first component, is a solution on (1.2.2) on I since

$u'' = \varphi_1'' = (\varphi_1')' = \varphi_2' = -\varphi_2^2 + g(t, \varphi_1) = -u_2^2 + g(t, u_1).$

Vector-matrix notation

A system of equations of the form (1.2.1) can always be written as a single vector-valued equation by introducing the n-dimensional column vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = col(x_1, x_2, \ldots..x_n).$$

Let $x(t)$be the vector-valued function defined by

$$x(t) = col\big(x_1(t), x_2(t), \ldots.., x_n(t)\big).$$

Similarly, let f be the vector-valued function given by

$$f(t, x) = \begin{bmatrix} f_1(t, x_1, x_2, \ldots.., x_n) \\ f_2(t, x_1, x_2, \ldots\ldots.x_n) \\ \vdots \\ f_n(t, x_1, x_2, \ldots\ldots..x_n) \end{bmatrix} = col\big(f_1(t, x), f_2(t, x), \ldots\ldots\ldots.., f_n(t, x)\big).$$

Then, (1.2.1) can be expressed as

$$x' = f(t, x). \qquad (1.2.4)$$

By a solution of (1.2.4) on I we mean a vector valued function φ with components $\varphi_1, \varphi_2, \ldots \ldots, \varphi_n$ which satisfies

$$\big(t, \varphi(t)\big) = \big(t, \varphi_1(t), \varphi_2(t), \ldots\ldots\ldots\ldots, \varphi_n(t)\big) \in B, \qquad t \in I$$

$$\varphi'(t) = f\big(t, \varphi(t)\big), \qquad t \in I.$$

Equation (1.2.4) is usually referred to as a *nonautonomous* differential system. A differential system of the form

$$x' = f(x), \qquad (1.2.5)$$

In which the right-hand side does not involve the independent variable t, is said to be *autonomous*. An important feature of (1.2.5) is that if $\varphi(t)$ is a solution of (1.2.5) on $r_1 < t < r_2$, then $\varphi(t - t_0)$ is a solution on $t_0 + r_1 < t < t_0 + r_2$. Further, it is sometimes convenient to represent the solutions of (1.2.5) in the $(t, x) -$ space as curves in the x-space with t as a curve parameter. Such curve are called trajectories and the space that contains these is known as the phase space of (1.2.5).

Linear case

Consider a system of first-order linear differential equations of the form $x_1' = a_{11}(t)x_1 + \ldots\ldots + a_{1n}(t)x_n + b_1(t)$

$x_2' = a_{21}(t)x_1 + \ldots\ldots + a_{2n}(t)x_n + b_2(t)$

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$x_n' = a_{n1}(t)x_1 + \ldots\ldots + a_{nn}(t)x_n + b_n(t)$

or $x_i' = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t), \qquad i = 1,2,\ldots\ldots,n,$ (1.2.6)

where $a_{ij}(t),\ i,j = 1,2,3,\ldots\ldots,n, b_i(t), i = 1,2,\ldots,n,$ are real-valued functions defined on I, and $x(t) = (x_1(t),\ldots\ldots\ldots,x_n(t))$ is an unknown i-dimensional vector-valued function. Let $A(t) = \left(a_{ij}(t)\right)$ be $n \times n$ matrix and B(t) be an n-vector $(b_1(t), b_2(t), \ldots\ldots\ldots, b_n(t))$. Then, (1.2.6) can be written as

$x' = A(t)x + B(t).$ (1.2.7)

This is a particular case of (1.2.6) with $f(t,x) = A(t)x + B(t),\ A(t)x$ being the usual matrix-vector product. Equation (1.2.7) is referred to as a non-homogeneous linear differential system, but when $B(t) = 0$, it is called a homogeneous linear system.

An important special case of (1.2.7) is the n-th order linear differential equation

$u^{(n)} + a_1(t)u^{(n-1)} + \cdots + a_n(t)u = b(t).$ (1.2.8)

This is of the type (1.2.6). To see this, let

$$u = u_1, u' = u_2, \ldots, u^{(n-1)} = u_n.$$

Then, (1.2.8) is equivalent to

$$u_i' = u_{i+1}, \quad i = 1,2,\ldots\ldots, n-1,$$

$$u_n' = -a_n(t)u_1 - a_{n-1}(t)u_2 - \cdots - -a_1(t)u_n + b(t).$$

When n=3, (1.2.8) takes the form (1.2.7) with

$$x = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3(t) & -a_2(t) & -a_1(t) \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ b(t) \end{bmatrix}.$$

Summary

- The implicit and explicit form of first-order to higher-order differential equations are defined.
- All the types of differential equations with examples are explained.
- Different kinds of solutions are elaborated.
- Discussion on the need of an initial value problem was done.
- System of first-order differential equations are explained
- Conversions relations from nth order differential equation to a system of first-order differential equations are derived.

Keywords

- Implicit form
- Explicit form
- General higher order differential equation
- Initial value problem
- System of first-order differential equations
- Conversion from higher order to system of first order

Self-assessment

Choose the most suitable answer from the options given with each question.

Question 1: The solution of differential equation which includes all the solutions is called

(a)Arbitrary solution

(b) General solution

(c) Singular solution

(d) Particular solution

Question 2: The differential equation $u' = f(t,u), u(t_0) = u_0$ where $f(t,u) = A(t)x$ represent as

(a) Autonomous and homogenous
(b) Autonomous and non-homogenous
(c) Nonautonomous and homogenous
(d) Nonautonomous and non-homogenous

Question 3: If $F = (t,u,u',.........., u^{(n-1)}u^{(n)}) = 0$ is an implicit equation where $u^{(n)}$ is the n-th order derivatives with respect to t then $u^{(n)}$ can be expressed explicit as

(a) $u^{(n)} = g(u,u',.........., u^{(n-1)})$

(b) $u^{(n)} = g(u,u',...............)$

(c) $u^{(n)} = g(t,u,u',.........., u^{(n-1)})$

(d) none of these

Question 4 The solution $u_1(t) = 1 - t$ of equation $u' = (-t + (t^2 + 4u)^{1/2})/2,\ u(2) = -1$ is

(a)General solution

(b) Particular solution

(c) Singular solution

(d) None of these

Question 5: For differential equation $u' = (-t + (t^2 + 4u)^{1/2})/2,\ u(2) = -1$ the function $u_2(t) = -\frac{t^2}{4}$ is

(a) General solution

(b) Particular solution

(c) Singular solution

(d) None of these

Question 6: The initial value problem

$$\frac{d^2u}{dt^2} - 2\frac{du}{dt} + u = t + 1;\ u(1) = 1,\ \left(\frac{du}{dt}\right)_{t=1} = 2.$$

reduces to the system of the differential equation of

(a) first-order linear homogeneous

(b) first-order linear non-homogeneous

(c) first-order nonlinear homogeneous

(d) first-order nonlinear non-homogeneous

Question 7: The first order linear system in the vector-matrix form of the following initial value problem represents as

$$\frac{d^2u}{dt^2}+2\frac{du}{dt}-8u=e^t;\ u(0)=1,\left(\frac{du}{dt}\right)_{t=0}=-4.$$

(a) a square matrix of order 2

(b) a rectangular matrix of order 2 X 3

(c) a square matrix of order 3X2

(d) a square matrix of order 3

Question 8: The m[th] order linear differential equation in u can be written as a system of linear equations in y by using

(a) $y_i' = y_{i+1}$

(b) $y_{i+1}' = y_i$

(c) $y_i' = y_i$

(d) $y_i' = y_{i-1}$

Answers:

1	b	2	c	3	c	4	b
5	c	6	b	7	a	8	a

Review Questions

Q1.Write the following scalar differential equations in the vector matrix form:

$$u''' + 2u'' + 3u' + 7u\sinh t = 0$$

Q2. Write the following scalar differential equations in the vector matrix form:

$$u^{(4)} + u'''\cos t - u'' + u\sin t = 0$$

Q3. Reduce the following differential equation in vector matrix form:

$$e^t y''' - ty'' + y' - e^t y = 0; y(-1) = 1,\ y'(-1) = 0,\ y''(-1) = 1.$$

Q4. Reduce the following differential equation in vector matrix form:

$$3y''' + 2y'' - 4y' + 5y - t^2 + 16t = 0, y(\pi) = -1, y'(\pi) = -2, y''(\pi) = -3.$$

Q5. Express the following system of scalar differential equation in the vector matrix form

$$x'' = 2x' + 5y + 4,\ y' = -x' - 2y,\ x(0) = 0,\ x'(0) = 0, y(0) = 1.$$

Further Readings

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Unit 02: Existence and uniqueness

CONTENTS

Expected Learning Outcomes

After studying this unit, you will be able to

- identify the concept of initial value problem to find the solution.
- understand the continuity concept through Lipschitz condition
- know about the Picard's approximation of solutions
- apply basic theorems on the convergence of solutions of initial value problems.
- find the condition of existence and uniqueness of solution for an IVP

Introduction

A scientist and an engineer can use differential equations in his work more confidently if he is conversant with the theory of existence, uniqueness, and continuation of solutions. Similarly, a mathematician who is familiar with these properties of solutions is better equipped to develop further mathematical methods for examining the behavior of solutions of differential equations.

This unit introduces the existence, uniqueness, and continuation of solutions. Besides the classical methods, fixed-point techniques are employed in proving some of the existence and uniqueness of theorems.

The questions that now arise are: Does there exist a solution to the initial value problem of the form (1.1.3), (1.1.7)? If yes, is the solution unique? The answers to these questions are provided by the existence and uniqueness theorems.

2.1 Initial value problem for first-order linear differential equation

We now state a fundamental theorem giving sufficient conditions for the existence and uniqueness of solutions of initial value problems for first-order linear differential equations.

Theorem 2.1.1: Let $a(t)$ and $b(t)$ be continues on the intervals I and let $t_0 \in I$. Then, there exists a unique solution $u(t)$ to the initial value problem (1.1.4), (1.1.7) i.e. $u' = a(t)u + b(t),\ u(t_0) = u_0$ on I.

Proof Let u(t) be a function defined by

$$u(t) = K(t)\left[c + \int_{t_0}^{t} b(s)\exp\left(-\int_{t_0}^{s} a(\tau)d\tau\right)ds\right], \qquad (2.1.1)$$

Notes

where $K(t) = \exp\,[\int_{t_0}^{t} a(s)ds]$

and c is an arbitrary constant. Since $a(t)$ is continuous on I,

$\exp\,[\int_{t_0}^{t} a(s)ds]$

is a nonzero differentiable function on I. Thus, the equation $u' = a(t)u + b(t)$, can be written as

$$\left[u\, exp\left(-\int_{t_0}^{t} a(s)ds\right)\right]' = b(t)\exp\left[-\int_{t_0}^{t} a(s)ds\right]. \qquad (2.1.2)$$

Since b(t) and

$$\exp\left[-\int_{t_0}^{t} a(s)ds\right]$$

are continuous, the right-hand side of equation (2.1.2) is integrable; hence, equation (2.1.1) follows. The existence of a solution $u(t)$ of (1.1.3) can be verified by substituting (2.1.1) in (1.1.3). Finally, the initial condition (1.1.7) determines the constant c uniquely.

<u>Remark 2.1.2</u> The fundamental theorem guarantees not only the existence of a unique solution of the given initial value problem but also the validity of this solution on the whole interval $I$where the function $a(t)$ and $b(t)$ are continuous.

The example that follows illustrates another important feature of initial value problems for linear equations.

<u>Example 2.1.3</u>: let us consider the initial value problem

$$u' = -\frac{u}{t} + 2,\ t > 0, \qquad (2.1.3)$$

$$u(1) = 2 \qquad (2.1.4)$$

and look for a solution in the interval containing t=1. Now, since the coefficients in equation (2.1.3) are continuous, except at t=0, theorem (2.1.1) guarantees the existence of a unique solution of (2.1.3) (2.1.4) at least in the interval$0 < t < \infty$. The general solution of (2.1.3) is $u(t) = t + c/t$, where c is an arbitrary constant. Thus, the solution of the initial value problem (2.1.3) (2.1.4) is $u(t) = t + 1/t$. It should be noted that this solution becomes infinite as $t \to 0$. This is not unusual since $t = 0$ is a point discontinuity of $a(t)$. On the other hand, if we slightly change the initial condition to $u(t) = t$ and it behaves properly as $t \to 0$.

We thus conclude that the solutions of the initial value problem (1.1.3), (1.1.7) are not necessarily discontinuous, i.e., they do not necessarily break down, at those points where the functions $a(t)$ and $b(t)$ are discontinuous. But if at all the solutions break down, this would be only at those points where $a(t)$ and $b(t)$ are discontinuous and not at the points where these functions are continuous. Therefore, the qualitative behavior of the solutions can be assessed to a certain extent by a mere identification of the point of discontinuity, if any, of $a(t)$ and $b(t)$.

2.2 <u>Initial value problems for first-order non-linear differential equation</u>

It should be noted that the linear value problem (1.1.4), (1.1.7) has s unique solution on the whole interval $|t - t_0| \le a$ where the functions $a(t)$ and $b(t)$ are continuous whereas the nonlinear initial value problem (1.1.3), (1.1.7) has a unique solution only in the interval $|t - t_0| \le h$. In other words, there is no apparent relationship between the region where the function $g(t, u)$ is continuous and the interval of existence of the solution. This is illustrated by the well-known example that follows.

<u>Example 2.2.1:</u> the function $u(t) = {}^{1}\!/_{1\,-\,t}$ is the solution of the non-linear initial value problem $u' = u_2, u(0) = 1$.

Obviously, the solution becomes infinite at $t = 1$, and hence is valid for $-\infty < t < 1$. Thus the right-hand side of the differential equation does not indicate the interval of existence of the solution.

Notes

Further, if we modify the initial condition to $u(0) = 2$, then the solution is $u(t) = {}^{2}/_{1-2t}$, that is, the solution becomes infinite at $t = \frac{1}{2}$. Therefore, the points of discontinuity of the solution may move about depending upon the initial condition. This behavior of the solutions of nonlinear initial value problems is also peculiar.

We now state a theorem giving sufficient conditions for the existence and uniqueness of solutions of initial value problems for first-order non-linear differential equations.

Theorem 2.2.2: If $g(t,u)$and $\frac{\partial g}{\partial u}$ are continuous functions of t and u in the region$R(a,b)\colon |t - t_0| \le a, |u - u_0| \le b, a > 0, b > 0$, then there exists a unique solution $u(t)$ to the initial value problem (1.3.1), (1.3.7) on some interval $|t - t_0| \le h \le a$.

Before proving Theorem 2.2.2, we give certain facts which we shall use subsequently.

Lemma 2.2.3: If a function $g(t,u)$ is continuous in $R(a,b)$, then the initial value problem (1.1.3), (1.1.7) is equivalent to the integral equation

$u(t) = u_0 + \int_{t_0}^{t} g(s,u(s))ds$ for t in $|t - t_0| \le a.$ (2.2.1)

Proof: If $u(t)$ is a solution of (1.1.3) i.e. $u' = g(t,u)$ satisfying (1.1.7) i.e. $u(t_0) = u_0$, then by integrating (1.1.3) between the limits t_0 and t, we obtain (2.3.1). Conversely, let $u(t)$ be a solution of (2.2.1). By setting. $t = t_0$ in (2.3.1), we obtain $u(t_0) = u_0$. Further, since $g(t,u)$ is continuous, the right-hand side of (2.2.1) is differentiable; hence, by differentiating (2.2.1), we get $u' = g(t,u(t))$.

Definition 2.2.4: A real-valued continuous function $u(t)$ defined on the interval $|t - t_0| \le a$ if the points $(t,u(t)) \in R(a,b)$ for all t in $|t - t_0| \le a$ and in $u(t)$ satisfies (2.2.1) on $|t - t_0| \le a$.

Definition 2.2.5: If $g(t,u(t))$ is continuous in the closed, bounded region $R(a,b)$, then g is bounded there. That is, there exists a positive number M such that $|g(t,u)| \le M$ for $(t,u) \in R(a,b)$.

Now, let $h = \min\,(a, b/M)$ and consider the interval $J\colon |t - t_0| \le h$ and a smaller rectangle D: $|t - t_0| \le h, |u - u_0| \le b$.

Lipschitz conditions

Lemma 2.2.6: If $\frac{\partial g}{\partial u}$ is continuous in D, then there exists a positive constant K such that

$|g(t,u_1) - g(t,u_2)| \le K|u_1 - u_2|, \quad (t,u_1),(t,u_2) \in D.$ (2.2.2)

Proof: Assume that $(u_1 > u_2)$. From the mean value theorem, it follows that

$$g(t,u_1) - g(t,u_2) = \frac{\partial g(t,u^*)}{\partial u}(u_1 - u_2),$$

Where u^* lies in the interval $u_2 < u^* < u_1$. Taking absolute value on both sides of the equation, we have

$|g(t,u_1) - g(t,u_2)| = \left|\frac{\partial g(t,u^*)}{\partial u}\right| |u_1 - u_2|.$

Since $\frac{\partial g(t,u)}{\partial u}$ is continuous, and hence bounded, in If D, there exists a positive number K such that $\left|\frac{\partial g(t,u^*)}{\partial u}\right| \le K$. Thus,

$|g(t,u_1) - g(t,u_2)| \le K\,|u_1 - u_2|$ for $(t,u_1),(t,u_2) \in D.$

Definition 2.2.7: A function g satisfying inequality (2.2.2) for all $(t,u_1),(t,u_2)$ in the region, D is said to satisfy a Lipschitz condition in D, and K is called the Lipschitz constant.

Example 2.2.8: The function $g(t,u) = tu^2$ satisfies a Lipschitz condition on the rectangle R given by $|t| \le 1, |u| \le 1$ with Lipschitz constant 2, since, for all (t,u) in R, we have

$$\left|\frac{\partial g}{\partial u}\right| = 2|t||u|^2 \le 2.$$

Whereas, g does not satisfy a Lipschitz condition on the strip $|t| \le 1, |u| \le \infty$.

For this case $\left|\frac{\partial g}{\partial u}\right| = 2|t||u|^2 \to \infty$ as $|u| \le \infty$.

Notes

From this example, we observe that the satisfaction of the Lipschitz condition depends not only on the rule of the function but also on the domain where it is defined.

Example 2.2.9: The function $g(t,u) = u^{\frac{2}{3}}$ does not satisfy a Lipschitz condition on the rectangle R, given by $|t| \le 1, |u| \le 1$.

Since for all (t,u)in R, we have

$\left|\frac{\partial g}{\partial u}\right| = \frac{2}{3}\left|u^{-\frac{1}{3}}\right| \to \infty$ as $u \to 0$.

Exercise 1

Q1. By computing appropriate Lipschitz constants, show that the following functions satisfy Lipschitz condition on the sets D indicated:

(a) $g(t,u) = 4t^2 + u^2$, on $D: |t| \le 1, |u| \le 1$.
(b) $g(t,u) = t^2 \cos^2 u + u\, sin^2 t$, on $D: |t| \le 1, |u| < \infty$.
(c) $g(t,u) = t^3 e^{-tu^2}$, on $D: |t| \le 1, |u| < \infty$ and on $D: 0 \le t \le a, |u| < \infty$. (here a>0 is a constant)

Q2. Show that the following functions do not satisfy the Lipschitz condition in the region indicated

(a) $g(t,u) =, {}^{\sin u}/_{t}\,, g(0,u) = 0$ on $D: |t| \le 1, |u| < \infty$.
(b) $g(t,u) = \frac{e^t}{u^2}, g(t,0) = 0$,on $D: |t| \le 2, |u| \le \frac{1}{2}$.

It is integrating to know that if $\frac{\partial g}{\partial u}$ is continuous in D, then g satisfies the Lipschitz condition in D, but the converse is not true. That is, there are certain function g satisfies the Lipschitz condition in a region but do not have continuous partial derivatives with respect to u in that region. For example, the function $g(t,u) = t|u|$ satisfies the Lipschitz condition in a region containing (0,0), but its partial derivatives with respect to u does not exist for $u = 0$.

Successive approximation or method of iteration or Picard's method

In what follows, we introduce a technique, called the method of *successive approximation* (also known as the *method of iteration* or *Picard's method*), which is helpful in constructing a solution of the integral equation (2.2.1). This method requires a sequence of functions defined as

$u_0(t) = u_0$

$u_1(t) = u_0 + \int_{t_0}^{t} g(s, u_0(s))ds$

. (2.2.3)

.

$u_n(t) = u_0 + \int_{t_0}^{t} g(s, u_{n-1}(s))ds$

Where the functions , $u_0(t), u_1(t), \dots \dots, u_n(t)$ are the *successive approximation* to a solution of (2.2.1)., and thus to a solution of the initial value problem (1.1.3), (1.1.7).

Example 2.2.10: Consider the IVP $u' = u^2, u(0) = 1$. The equation is equivalent to integral equation $u(t) = 1 + \int_{t_0}^{t} s^2 ds$.

The first approximation is $u_0(t) = 1$. Now

$u_1(t) = 1 + \int_{t_0}^{t} u_0^2(s)ds = 1 + \int_0^t 1ds = 1 + t,$

$u_2(t) = 1 + \int_{t_0}^{t}(1+s)^2 ds = 1 + \int_0^t (1+s)^2 ds = 1 + t + t^2 + \frac{t^3}{3},$

$u_3(t) = 1 + \int_{t_0}^{t}\left(1 + s + s^2 + \frac{s^3}{3}\right)^2 ds = 1 + t + t^2 + t^3 + \frac{2t^4}{3} + \frac{t^5}{3} + \frac{t^6}{9} + \frac{t^7}{63}.$

All
$u_n(t)$, n=0,1,2,….are polynomials.

Observe that the IVP can be solved explicitly by the method of separation of variables. Here

$u(t) = \frac{1}{1-t}$ is a solution existing on $-\infty < t < 1$.

Example 2.2.11: Compute the first five approximations of IVP

$u' = tu$, $u(0) = 1$ and find the limit of the successive approximations.

In this problem, the initial point is $t_0 = 0$, the initial value $u_0 = 1$, and the rule of the function g is $g(t, u) = tu$. Hence the integral equation corresponding to this IVP is given by

$$u = 1 + \int_0^t suds.$$

Therefore, the successive approximations of this problem are defined by

$u_0(0) = 1, u_{k+1}(t) = 1 + \int_0^t su_k(s)ds,\ k = 0,1,2, \dots \dots \dots$

Putting k=0,1,2,……, in the preceding relations, we obtain

$u_1(t) = 1 + \int_0^t su_0(s)ds = 1 + \int_0^t sds = 1 + \frac{t^2}{2}$

$u_2(t) = 1 + \int_0^t su_1(s)ds = 1 + \int_0^t s(1 + \frac{s^2}{2})ds = 1 + \frac{t^2}{2} + \frac{t^4}{8} = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2$

$$u_3(t) = 1 + \int_0^t su_2(s)ds = 1 + \int_0^t s(1 + \frac{s^2}{2} + \frac{s^4}{8})ds = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3$$

In general, for k=0,1,2,……, it can be established by mathematical induction that $u_k(t)$ is given by

$$u_k(t) = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \cdots \dots \dots .. + \frac{1}{k!}\left(\frac{x^2}{2}\right)^k$$

We may recognize $u_k(t)$ as the kth partial sum for the series expansion of the function

$$u(t) = e^{\frac{t^2}{2}}$$

We know that this series converges for all real t. This means that

$u_k(t) \to u(t) = e^{\frac{t^2}{2}}$ as $k \to \infty$.

Also, it is easy to see that the function

$u(t) = e^{\frac{t^2}{2}}$ is a solution to the given IVP.

Exercise 2

Q1. Compute the first three successive approximations for the solution of the following equations

(i) $u' = tu,\ u(0) = 1;$ (iii) $u' = u/1 + t^2$, $u(0) = 1;$

(ii) $u' = e^u,\ u(0) = 0;$

We will now prove that the sequence of functions $\{u_n(t)\}$ defined by (2.2.3) converges on J to a limit function $u(t)$ which represents a solution of (2.2.1). To do this, we need the result that follows.

Lemma 2.2.12: Assume that g satisfy is continuous on D. Then, the successive approximations $\{u_n(t)\}$ defined by (2.2.3) exist as continuous functions on J and $(t, u_n(t)) \in D$ for $t \in J$ given by $|t - t_0| \le h = \min(a, b/M)$.

Proof: Let $t \in J$. Since $u_0(t) = u_0$, it is obvious that $u_0(t)$ exists and is continuous on J. In view of (2.2.3) and the continuity of g on D, it follows that all the successive approximations $u_1(t), u_2(t), \dots \dots \dots, u_n(t)$ exist and are continuous on J. We now show that $(t, u_n(t)) \in D$. For $\in J$, we have

Notes

$|u_1(t) - u_0| = \left|u_0 + \int_{t_0}^{t} g(s, u_0(s))ds - u_0\right| \leq \int_{t_0}^{t}\left|g\big(s, u(s)\big)\right|ds$

$\leq M\,|t - t_0| \leq Mh \leq b.$

The remaining proof runs by induction. Let us assume, for $t \in J$, that

$$|u_k(t) - u_0| \leq b, \quad k = 1,2,3, \ldots \ldots, n-1.$$

This implies $(t, u_{n-1}(t)) \in D$, and hence $\left|g\big(t, u_{n-1}(t)\big)\right| \leq M$. Therefore, for $t \in J$, we obtain

$|u_n(t) - u_0| = \left|u_0 + \int_{t_0}^{t} g(s, u_{n-1}(s))ds - u_0\right| \leq \int_{t_0}^{t}\left|g\big(s, u_{n-1}(s)\big)\right|ds$

$\leq M\,|t - t_0| \leq Mh \leq b,$

$$|t - t_0| \leq \frac{b}{M},$$

that is,

$(t, u_n(t)) \in D$ for $t \in J$.

Hence $\big(t, u_n(t)\big)$ will be in D if $|t - t_0| \leq a$ and $|t - t_0| \leq \frac{b}{M}$., that is, if

$$|t - t_0| \leq h = \min\left(a, \frac{b}{M}\right).$$

<u>**Remark 2.2.13:**</u> In proving Lemma 2.2.11, a somewhat stronger result, namely,

$$|u_n(t) - u_0| \leq M\,|t - t_0|,$$

has been obtained. Geometrically, this means that the graph of each function $u_n(t)$ lies in two triangular regions.

Picard's existence theorem

In the course of proving the Theorem 2.2.2, we actually need the Lipschitz condition on g and not the strong property, that is, the continuity of $\frac{\partial g}{\partial u}$. Therefore, the condition of Theorem 2.2.2 can be relaxed as follows.

<u>**Theorem 2.2.14:**</u> If $g(t, u)$ is a continuous function of t and u in a closed, bounded region $R(a, b)$ and satisfies the Lipschitz condition in R, then there exists a unique solution $u(t)$ to the initial value problem (1.1.3), (1.1.7) defined on the interval J.

Proof: To establish the convergence of the sequence of functions $\{u_n(t)\}$, we shall estimate the difference between the successive approximations. Let t lie in the interval $[t_0, t_0 + h]$.

Set $v_n(t) = \big(u_n(t) - u_{n-1}(t)\big)$.

For $t \in [t_0, t_0 + h]$, we have Lemma 2.2.12,

$|v_1(t)| = |u_1(t) - u_0| \leq M(t - t_0)$ and $\big(t, u_0(t)\big), \big(t, u_1(t)\big) \in D$.

Since g satisfy the Lipschitz condition in D, it follows that

$|v_2(t)| = |u_2(t) - u_1(t)| = \left|\int_{t_0}^{t}\left[g\big(s, u_1(s)\big) - g(s, u_0(s))\right]ds\right|$

$\leq \int_{t_0}^{t}\left|g\big(s, u_1(s)\big) - g(s, u_0(s))\right|ds$

$\leq K\int_{t_0}^{t}|u_1(s) - u_0(s)|ds = K\int_{t_0}^{t}|v_1(s)|ds$

$\leq K\int_{t_0}^{t}(s - t_0)ds = KM\frac{(t-t_0)^2}{2!}$

Similarly,

$$|v_3(t)| \leq K^2 M\frac{(t - t_0)^3}{3!}$$

A simple induction argument shows that, in general, for $t \in [t_0, t_0 + h]$,

$|v_m(t)| \leq K^{m-1}M\frac{(t-t_0)^m}{m!}, m = 1,2, \ldots \ldots.., n.$ (2.2.4)

To see this, assume, for $t \in [t_0, t_0 + h]$, that

$|v_{m-1}(t)| \leq K^{m-2} M \frac{(t-t_0)^{m-1}}{(m-1)!}, m = 1,2, \dots \dots .., n.$

Then, for $t \in [t_0, t_0 + h]$, we have

$$|v_n(t)| = |u_n(t) - u_{n-1}(t)| = \left|\int_{t_0}^{t} [g(s, u_{n-1}(s)) - g(s, u_{n-2}(s))] ds\right|$$

$$\leq \int_{t_0}^{t} |g(s, u_{n-1}(s)) - g(s, u_{n-2}(s))| ds$$

$$\leq K \int_{t_0}^{t} |u_{n-1}(s) - u_{n-2}(s)| ds = K \int_{t_0}^{t} |v_{n-1}(s)| ds$$

$$\leq M\, K^{n-1} \int_{t_0}^{t} \frac{(s-t_0)^{n-1}}{(n-1)!} ds = K^{n-1} M \frac{(t-t_0)^n}{n!}.$$

This establishes relation (2.2.4). The proof for $t \in [t_0 - h, t_0]$ is similar to that for $t \in [t_0, t_0 + h]$. Hence, for $t \in J$, we have

$$|v_n(t)| \leq K^{n-1} M \frac{|t-t_0|^n}{n!} \leq M K^{n-1} \frac{h^n}{n!}. \qquad (2.2.5)$$

Now, consider an infinite series of the form

$$u_0 + v_1(t) + v_2(t) + \cdots \dots .. + v_n(t) + \cdots .... \qquad (2.2.6)$$

The nth partial sum of this series is $u_n(t)$, that is,

$$u_n(t) = u_0 + \sum_{m=1}^{\infty} v_m(t). \qquad (2.2.7)$$

Therefore, the sequence $\{ u_n(t)\}$ converges if and only if (2.2.6) also converges.

From inequality (2.3.5), we have

$$u_0 + \sum_{m=1}^{\infty} |v_m(t)| \leq u_0 + M \sum_{m=1}^{\infty} \frac{K^{m-1} h^m}{m!} \qquad (2.3.8)$$

It follows from the ration test that the series on the right-hand side of (2.2.8) converges, and hence, by the comparison test, series (2.2.6) also converges (in fact, uniformly), on the interval J. Let the sum of series (2.2.6) be $u(t)$. The relation (2.2.7) gives

$$\lim_{n \to \infty} u^n(t) = u(t).$$

Finally, we show that the limit function $u(t)$ satisfies (2.2.1). Since

$$u_n(t) = u_0 + \int_{t_0}^{t} g(s, u_{n-1}(s)) ds.$$

It follows that

$$u(t) = u_0 + \lim_{n \to \infty} \int_{t_0}^{t} g(s, u_{n-1}(s)) ds.$$

From the uniform convergence of $u_n(t)$ to $u(t)$ and continuity of the function $g(t, u)$, we obtain

$$u(t) = u_0 + \int_{t_0}^{t} g(s, u(s)) ds.$$

This completes the proof of the existence of the solution $u(t)$.

In order to ensure that this solution of the initial value problem (1.1.3), (1.1.7). Then, the nonnegative function $w(t) = |u(t) - v(t)|$ satisfies $w(t_0) = 0$, and

$$w(t) \leq \int_{t_0}^{t} |g(s, u(s)) - g(s, v(s))| ds \leq K \int_{t_0}^{t} w(s) ds$$

or $\frac{d}{dt}\left[e^{-K(t-t_0)} \int_{t_0}^{t} w(s) ds\right] \leq 0.$

Integrating this inequality from t_0 to t, we obtain $w(t) \leq 0$.

This is incompatible with $w(t) \geq 0$ unless $w(t) = 0$ on J.

Notes

2.3 Existence in the large and uniqueness of the solutions with examples

Example 2.2.15: consider the IVP $u' = u^2 + \cos^2 t, u(0) = 0$. We try to determine the largest interval of existence of its solution. Let R be the rectangle containing (0,0)

$$R: \{(t,u); 0 \le t \le a, |u| \le b, a \ge \frac{1}{2}, b > 0.\}$$

Clearly, $|g(t,u)| = |u^2 + \cos^2 t| \le 1 + b^2 = M$.

The function $g(t,u) = u^2 + \cos^2 t$ satisfies Lipschitz condition on R, Since $\left|\frac{\partial g}{\partial u}\right| = |2u| \le 2b = K$.

We find that $u(t)$ exists for $0 \le t \le h = min\left(a, \frac{b}{1+b^2}\right)$.

Observe that the maximum values of $\frac{b}{1+b^2}$ is $\frac{1}{2}$. Hence $h = \frac{1}{2}$, i.e. $u(t)$ exists on the interval $0 \le t \le \frac{1}{2}$.

Example 2.2.16: Consider the IVP $u' = u^2, u(0) = 2$. Let R be the rectangle

$$R: \{(t,u); |t| < a, |u-2| \le b, a > 0, b > 0\}.$$

In R, $|g(t,u)| = |t^2| \le (b+2)^2 = M$ and the interval of existence of a solution is $|t| \le h$, where $h = \min\left(a, \frac{b}{(b+2)^2}\right) = \frac{1}{8}$. Hence the solution of the IVP exists on the t-interval $-\frac{1}{8} \le t \le \frac{1}{8}$.

However, we observe that this IVP can be explicitly solved. Its solution is $u(t) = \frac{2}{1-2t}$.

We find that $u(t)$ exists on $-\infty < t < \frac{1}{2}$. This interval of existence is much larger than that obtained by the application of Picard's method.

Example 2.2.17: Consider the IVP $u' = t(1+u), u(0) = -1$. Let R be the rectangle

$$R: \{(t,u); |t| < a, |u+1| \le b, a > 0, b > 0\}.$$

In R $|g(t,u)| = |t(1+u)| \le a(b+2) = M$ and the interval of existence of a solution is $|t| \le h$, where $h = \min\left(a, \frac{b}{a(b+2)}\right)$. Hence the solution of the IVP exists on the t-interval $|t| \le h$.

However, we observe that this IVP can be explicitly solved. Its solution is $u(t) = -1$ which is independent of t.

We find that $u(t)$ exists on $|t| \le h$.

The Picard's theorem assumes the Lipschitz condition. Can drop this condition?

The answer is no. The following examples illustrate this point.

Example 2.2.18: In the case of IVP

$$u' = \begin{cases} \frac{2u}{t}, t > 0; u(0) = 0 \\ 0, t = 0. \end{cases}$$

Here $g(t,u) = 2u/t$ does not satisfy Lipschitz condition in any closed rectangle containing (0,0).

The method of successive approximations shows that $u_n(t) = 0$ for $n = 0,1,2, \ldots..$

Hence $u(t) = \lim_{n\to\infty} u_n(t) = 0$. Yet the given equation possesses another solution $u(t) = t^2$existing on $t > 0$.

Example 2.2.19: In the case of IVP

$$u' = 4u^{\frac{3}{4}}, u(0) = 0, t \ge 0,$$

Again $g(t,u) = 4u^{\frac{3}{4}}$ fails to satisfy Lipschitz condition. Each successive approximation $u_n(t) = 0$ and hence $u(t) = 0$ on $[0, \infty]$.

We observe that $u(t) = t^4$ is yet another solution of the given IVP. In fact,

$$u_c(t) = \begin{cases} 0, 0 \le t \le c; \\ (t-c)^4, c \le t < \infty; \end{cases}$$

For each real value of c is a solution of the given IVP. Thus, we get uncountable solutions to the IVP.

Summary

- The concept of the initial value problem to find the solutions are discussed.
- Lipschitz condition is derived and elaborated with suitable examples.
- Determine the Picard's approximation of solutions and examples are solved.
- The convergence of solutions of initial value problems was discussed.
- The condition of existence and uniqueness of solution on an IVP is derived with examples.

Keywords

- Linear first-order differential equation
- Non-Linear first-order differential equation
- Lipschitz condition
- Picard's approximation
- Existence and uniqueness of a solution

Self-assessment

Choose the most suitable answer from the options given with each question.

Q1. The solution y(x) of the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ exists if

a) $f(x, y)$ is bounded and continuous in a closed region
b) $f(x, y)$ is bounded only
c) $f(x, y)$ is continuous only in a closed region
d) None of the above

Q2. The solution y(x) of an initial value problem $y' = f(x, y),\ y(x_0) = y_0$ unique if

a) $f(x, y)$ is continuous in R
b) $f(x, y)$ satisfy the Lipschitz condition in R
c) $f(x, y)$ is continuous and satisfy the Lipschitz condition in R
d) None of the above

Q3. For the initial value problem $y' = \sqrt{|y|},\ y(0) = 0$ over the rectangle $|x|<1,\ |y|<1$

a) At least one solution exists for all x in (-1,1)
b) Only one solution exists for all x in (-1,1)
c) No solution exists for all x in (-1,1)
d) One solution exists over R

Q4. For the initial value problem $y' = y, y(0) = 1,\ R: |x| \le 1,\ |y-1| \le 1$

a) A solution exists and unique in at least $|x| \le 1/2$
b) A solution exists and unique in at least $|x| < 1/2$
c) A solution exists and unique in at least $|x| > 1/2$
d) A solution exists and unique in at least $|x| \ge 1/2$

Notes

Q5. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous in close region R, then initial value problem

$y' = f(x, y), y(x_0) = y_0$ has

a) A unique solution in $[x_0\text{-h}, x_0\text{+h})$ where h is a positive number
b) A unique solution in $[x_0\text{-h}, x_0\text{+h}]$ where h is a positive number
c) A unique solution in $(x_0\text{-h}, x_0\text{+h}]$ where h is a positive number
d) A unique solution in $(x_0\text{-h}, x_0\text{+h})$ where h is a positive number

Q6. A real value function $u(t) \in I$ be the solution of IVP $u' = f(t,u),\ u(t_0) = u_0$ if

a) u'(t) exits for $t \in I$
b) the points $(t, u(t)) \in I \times R$
c) Both (i) and (ii)
d) None of these

Q7. If $\partial g / \partial u$ is continuous in $|t - t_0| \le a, |u - u_0| \le b$, then there exists a positive constant K such that

a) $|g(t,u_1) - g(t,u_2)| \ge K|u_1 - u_2|$
b) $|g(t,u_1) - g(t,u_2)| \le K|u_1 - u_2|$
c) $|g(t,u_1) - g(t,u_2)| < K|u_1 - u_2|$
d) $|g(t,u_1) - g(t,u_2)| > K|u_1 - u_2|$

Q8. The value of Lipschitz constant of initial value problem $u' = t^2 u + u^2,\ u(0) = 1$ on R: $|t| \le 2$, $|u-2| \le 2$ is

a)12 (b) 10 (c) 8 (d) 14

Q9. The maximum value of g of initial value problem $u' = t^2 u + u^2,\ u(0) = 1$ on R: $|t| \le 2$, $|u-2| \le 2$ is.

(a) 23 (b) 32 (c) 10 (d)8

Q10. The value of Lipschitz condition on rectangle R indicated for

$f(t,x) = (x + x^2)\frac{\cos t}{2};\ R: |t-1| \le \frac{1}{2}, |x| \le 1$

a) 3/2
b) ½
c) 5/2
d) 2

Q11. Using Picard's approximation method, the first iterates of the initial value problem

$y' = y + y^2,\ \ y(0) = 1$ is

(a) $1 + 2t$

(b) $1 - 2t$

(c) $1 + t$

(d) $1 - t$

Q12. Using Picard's approximation method first iterates of the initial value problem

$y' = 1 + y^2,\ \ y(0) = 0$ is

(a) $1 + t$

(b) $1 - t$

(c) t

(d) $1-2t$

Q13. The first approximate solution of IVP $y' = x + y,\ y(0) = 1$ by using Picard's iteration method is

(a) $y_1 = 1 + x + x^2$

(b) $y_1 = 1 + x - x^2$

(c) $y_1 = 1 - x - \frac{x^2}{2}$

(d) $y_1 = 1 + x + \frac{x^2}{2}$

Q14. The first Picard solution of IVP $y' = x - y,\ y(0) = 1$ is

(a) $y_1 = 1 - x + x^2$

(b) $y_1 = 1 - x - x^2$

(c) $y_1 = 1 - x + \frac{x^2}{2}$

(d) $y_1 = 1 + x - \frac{x^2}{2}$

Q15. The function $g(t,u) = t\,|\,u\,|$ satisfies Lipschitz condition in region containing (0,0) but

(a) Partial derivative of $g(t,u) = t\,|\,u\,|$ exist for $u = 0$

(b) Partial derivative of $g(t,u) = t\,|\,u\,|$ does not exist for $u = 0$

(c) Partial derivative of $g(t,u) = t\,|\,u\,|$ exist for $u \in R$

(d) None of these

Answers:

1	a	6	c	11	a
2	c	7	b	12	c
3	a	8	a	13	d
4	b	9	b	14	c
5	b	10	a	15	b

Review Questions

1. Apply the method of iteration to the initial value problem $u' = t^2 + u^2,\ u(0) = 1$ and compute the first two approximations.
2. Show that the function $f(t, y) = t^2 + y^2$, defined in the rectangle $R : |\,t\,| \le a, |\,y\,| \le b$ satisfies the Lipschitz condition. Find the Lipschitz constant.
3. Show that the function $f(t, y) = y^2$, defined in the rectangle $R : |\,t\,| \le a, |\,y\,| \le b$ satisfies the Lipschitz condition. Find the Lipschitz constant.
4. Show that the function $f(t, y)$ satisfies the Lipschitz condition on region R indicated and find the Lipschitz constant

Notes

$f(t,y) = t\sin y + y\cos t;\ R:|t| \le a, |y| \le b,$ where a and b are real positive constants.

5. Show that the function $f(t,x)$ satisfies the Lipschitz condition on rectangle R indicated and find the Lipschitz constant

   $f(t,x) = e^t Sinx;\ R:|t| \le 1, |x| \le 2\pi,$

   $f(t,x) = t^2\cos^2 x + x\sin^2 t;\ R:|t| \le 1, |x| < \infty$

6. The function f is given by $f(x,y) = x^2|y|$

   (i) Show that function f satisfies a Lipschitz condition on rectangle $|x|<1,\ |y|<1$.

   (ii) Show that $\dfrac{\partial f}{\partial y}$ does not exist at (x,0) if $x \neq 0$.

7. Study the existence uniqueness of solutions of the following initial value problem

   $y' = (1+2x+3y)/(2+x^2+y^2),\ y(0)=0,\ R:|x| \le 2, |y| \le 1.$

   where a and b are real positive constants.

8. Show that the function $f(t,x)$ satisfies the Lipschitz condition on region R indicated and find the Lipschitz constant $f(t,x) = 1+3tx^2;\ R:|t| \le a, |y| \le b,$

9. Show that the function $f(t,x) = \dfrac{3x^3e^t}{1+x^2} + 2t^2\cos x$ satisfies the Lipschitz condition on strip $S_a:|t| \le a, |x| \le \infty$ and find the Lipschitz constant

10. Show that the existence of a solution for the initial value problem $x' = -x/t + 2,\ t > 0;$ $x(1) = 2.$

11. Study the existence of solutions to the initial value problem

    $y' = 2x^2 + 3y^2,\ y(0) = 1$ over the rectangle $|x| \le 1,\ |y-1| \le 1$.

Further Readings

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Unit 03: Peano's Existence Theorem

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of existence only but not uniqueness.
- understand the concept of existence in a large interval.
- know about the Asscoli-Arzela theorem.
- apply basic theorems on the convergence of solutions of initial value problems.
- find the condition of existence and uniqueness of solution on an IVP in a large interval.

Introduction

The theorem 2.2.14 is a local existence theorem, that is, it asserts the existence of a unique solution only on a sufficiently small interval$|t - t_0| \leq h$. Also, its proof demands the Lipschitz condition on g even when only the existence of solutions without uniqueness is required. However, if g does not satisfy the Lipschitz condition, it would still be possible to obtain the existence of solutions without uniqueness as shown by the results that follow.

The Lipschitz condition is a sufficient, but not a necessary, condition for the uniqueness of solutions. For example, $u(t) \equiv 0$ is the unique solution of the real-valued scalar differential equation $u' = -tu^{\frac{1}{3}}$ passing through the point (0,0). Obviously $g(t,u) = -tu^{\frac{1}{3}}$ does not satisfy the Lipschitz condition at any point where u=0.

3.1 Existence theorem

It is also not possible to prove the theorem for existence only not uniqueness by the method of iteration or Picard's method as the successive approximations may not converge.

This is probably so because the continuity of g alone is not sufficient for the convergence of the approximations, as indicated by the familiar example as follows

Example 3.1.1: Consider a function g defined on the region $D_1 = -\infty < t \leq 1, -\infty < u < \infty$ by

$$g(t,u) = \begin{cases} 0 & for -\infty < t \leq 0, -\infty < u < \infty \\ 2t & for\ 0 \leq t \leq 1, -\infty < u < 0 \\ 2t - \frac{4u}{t} & for\ 0 < t \leq 1, 0 \leq u < t^2 \\ -2t & for\ 0 < t \leq 1, t^2 < u < \infty \end{cases}$$

The function g is continuous and bounded by 2, on D_1. Further, the successive approximations to the solution u of $u' = g(t,u)$ through the initial point (0,0), for $0 \le t \le 1$, are given by

$u_0(t) = 0,$

$u_{2k-1}(t) = t^2, \quad k = 1,2, \dots \dots \dots,$

$$u_{2k}(t) = -t^2, \quad k = 1,2, \dots \dots \dots.$$

Thus, the successive approximations alternate between $t^2$and $-t^2$, and hence do not converge.

Since the function g in an example, 3.1.1 guarantees the existence of a solution through the point (0,0).

Moreover, as g is monotonically non-increasing in u for each fixed t yields the uniqueness of the solution starting at (0,0) and proceeding to the right of the origin. However, it is clear that the method of iteration cannot be used to obtain this solution as the successive approximations do not converge. This illustrates that the continuity of g plus the uniqueness do not imply the convergence of the successive approximations. On the other hand, it is also true that the convergence of the successive approximations does not imply uniqueness, as the following example shows.

Example 3.1.2:consider the initial value problem

$$u' = 4u^{\frac{3}{4}}, \quad u(0) = 0. \tag{3.1.1}$$

Here, the successive approximations are all zero functions, and hence converge to the identically zero solution, i.e. $u(t) \equiv 0$. The function $u(t) = t^4$ is also a solution of (3.1.1). Thus, this problem does not have a unique solution. Of course, this fact does not contradict Theorem (2.2.2) since $\frac{\partial g}{\partial u} = 3u^{-\frac{1}{4}}$ is not continuous or even defined at any point where $u = 0$. Also, we can see that g does not satisfy the Lipschitz condition when $u = 0$. Thus, Theorem (2.2.14) to this initial value problem since g is continuous in the whole (t,u) −plane.

Remark 3.1.3: The problem of uniqueness and convergence of successive approximations are logically independent. Nevertheless, the hypotheses of Theorem (2.2.2) are sufficient for proving this convergence.

To prove the next theorem, we need the concept of the equicontinuous family of functions.

Definition 3.1.4:A family of functions $F = \{f\}$defined on a real interval I issaid to be equicontinuous on Iif, for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ independent of $f \in F$ and also of $t, t_1 \in I$ such that $|f(t) - f(t_1)| < \varepsilon$ whenever $|t - t_1| < \delta$.

Remarks 3.1.5: In the view of definition 3.1.4, it is true that

(i) Any subset of an equicontinuous family is also equicontinuous;

(ii) Each member of an equicontinuous family is a continuous function;

(iii) A family of differentiable functions is equicontinuous at every point of the interval I if their derivatives are uniformly bounded on I (follow from the mean value theorem).

Another important property of equicontinuous functions can be expressed as in the following lemma.

Ascoli-Arzela Theorem

Lemma 3.1.6: If F be a family of function bounded and equicontinuous at every point of an interval I. Then, every sequence of functions $\{f_n\}$ in F contains a subsequence uniformly convergent on every compact subinterval of I.

Proof: Let $I = [a,b] \subset R$ be a closed and bounded interval. If F is an infinite set of function $f: I \to R$

Which is uniformly bounded and equicontinuous, then there is a sequence f_n of the element of F such that $f_n$converges uniformly on I.

Fix an enumeration $\{x_i\}_{i\in N}$ of rational number in I. Since F is uniformly bounded, the set of points $\{f(x_i)\}_{f\in F}$ is bounded, and hence by the Bolzano-Weierstrass theorem, there is a sequence $\{f_{n_1}\}$

of distinct functions in F such that $\{f_{n_1}(x_1)\}$ converges. Repeating the same argument for the sequence of points $\{f_{n_2}(x_2)\}$, there is a subsequence $\{f_{n_2}\}$ of $\{f_{n_1}\}$ such that $\{f_{n_2}(x_2)\}$ converges.

By induction, this process can be continued forever, and so there is a chain of sub-sequences $\{f_{n_1}\} \supseteq \{f_{n_2}\} \supseteq \ldots\ldots$ such that for each $k = 1,2,3,\ldots\ldots$, the sub-sequence $\{f_{n_k}\}$ converges at $x_{n_1}, x_{n_2}, \ldots\ldots\ldots, x_{n_k}$.

Now from the diagonal subsequence $\{f\}$ whose m^{th}term f_m is the m^{th}term in the m^{th}sequence $\{f_{n_m}\}$. By construction, f_m converges at every rational point of I. Therefore, given any $\epsilon > 0$ and rational x_k in I, there is an integers $N = N(\epsilon, x_k)$ such that

$|f_n(x_k) - f_m(x_k)| < \frac{\epsilon}{3}, \;\; n,m \geq N.$

Since the family F equicontinuous for the fixed ϵ and for every $x \in I$, there is an open interval U_x containing x such that

$|f(s) - f(t)| < \frac{\epsilon}{3}$, for all $f \in F$ and all s,t in I such that $s,t \in U_x$.

The collection interval U_x, $x \in I$ forms an open cover ofI. Since I is compact bt Heine-Borel Theorem, this covering admits a finite sub covers $U_1, U_2, \ldots\ldots, U_j$. There exists an integer k such that open each interval U_j, $1 \leq j \leq J$, contains a rational x_k with $1 \leq k \leq K$.

Finally, for any $t \in I$, there are j and k so that t and $x_k$belong to the same interval U_j. For this choice of k,

$$\begin{aligned}|f_n(t) - f_m(t)| &= |f_{n(t)} - f_n(x_k) + f_n(x_k) - f_m(x_k) + f_m(x_k) - f_m(t)| \\ &\leq |f_{n(t)} - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(t)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for all } n,m > N = \max\big(N(\epsilon_1, x_1), N(\epsilon_1, x_2), \ldots\ldots. N(\epsilon_1, x_k)\big).\end{aligned}$$

Consequently sequence, $\{f_n\}$ converges t continuous functions. This claim the proof.

Peano's Existence Theorem

Theorem 3.1.7: Let the function $g(t,u)$ be continuous and bounded in the strip $S: t_0 \leq t \leq t_0 + a, |u| < \infty$. Then, the initial value problem $u' = g(t,u), u(t_0) = u_0$, has at least one solution $u(t)$ defined on the interval $[t_0, t_0 + \alpha]$.

Proof: We define a sequence of functions $\{u_n(t)\}$ by

$$u_n(t) = u_0 \;\; for \;\; t_0 \leq t \leq t_0 + \frac{a}{n}, \tag{3.1.2a}$$

$$u_{n(t)} = u_0 + \int_{t_0}^{t-\frac{a}{n}} g\big(s, u_n(s)\big)ds \;\; for \;\; t_0 + \frac{Ka}{n} \leq t \leq t_0 + \frac{(K+1)a}{n} \;\; (K = 1,2,3,\ldots\ldots\ldots, n-1). \tag{3.1.2b}$$

Clearly, equation(3.1.2a) defines the function $u_n(t)$ on the interval $\left[t_0, t_0 + \frac{a}{n}\right]$; equation (3.1.2b) defines the function $u_n(t)$first on the interval $\left[t_0 + \frac{a}{n}, t_0 + \frac{2a}{n}\right]$, then on the interval $\left[t_0 + \frac{2a}{n}, t_0 + \frac{3a}{n}\right]$, and so on. Since g is bounded on the strip, there exists a positive number M such that

$|g(t,u)| \leq M \;\; for \;\; (t,u) \in S.$

Therefore, from the relations (3.1.2), we have for $t, t_1 \in [t_0, t_0 + a]$,

$|u_n(t) - u_n(t_1)| \leq M|t - t_1|.$

Thus, the sequence$\{u_n(t)\}$ is equicontinuous on the interval $[t_0, t_0 + a]$.

Now, since $u_n(t) = u_0$ in the interval $[t_0, t_0 + \frac{a}{n}]$, it is clearly bounded on this interval. Also, for $t \in (t_0 + \frac{a}{n}, t_0 + a]$, we obtain

$$\begin{aligned}|u_n(t)| &\leq |u_0| + \int_{t_0}^{t-\frac{a}{n}} \big|g\big(s, u_n(s)\big)\big|\, ds \\ &\leq |u_0| + M\left(t - t_0 - \frac{a}{n}\right) \leq |u_0| + Ma.\end{aligned}$$

This implies the uniform boundedness of the sequence $\{u_n(t)\}$ on the interval $[t_0, t_0 + a]$.

Hence, the application of Lemma 3.1.6 shows that the sequence $\{u_n(t)\}$ contains a subsequence $\{u_{n_k}(t)\}$ which converges uniformly on the interval $[t_0, t_0 + a]$ to a continuous function $u(t)$. We shall now show that the limit function $u(t)$ satisfies the integral equation

$u(t) = u_0 + \int_{t_0}^{t} g(s, u(s))\, ds$ for t in $|t - t_0| \leq a$.

Let $k \to \infty$ in

$u_{n_k}(t) = u_0 + \int_{t_0}^{t} g\left(s, u_{n_k}(s)\right) ds - \int_{t-\frac{a}{n_k}}^{t} g\left(s, u_{n_k}(s)\right) ds.$

Then, for the first integral on the right- hand side of this relation, we can proceed to the limit under the integral sign since g is continuous and the convergence is uniform; the second integral tens to zero since it does not exceed Ma/n_k in absolute value. Thus, we obtain

$u(t) = u_0 + \int_{t_0}^{t} g(s, u(s))\, ds, \quad t \in [t_0, t_0 + a].$

Remarks 3.1.8: The continuity of g alone is sufficient in Theorem 3.1.7 if we replace S by closed, bounded region $R_1: t_0 \leq t \leq t_0 + a, |u - u_0| \leq b$.

Corollary 3.1.9: If, in addition to the assumptions of Theorem 3.1.7, g is monotonically non-increasing in u for each fixed t on S, then the initial value problem $u' = g(t, u), u(t_0) = u_0$, has a unique solution on the interval $[t_0, t_0 + a]$.

Proof: Let $u(t)$ and $v(t)$ be any two solutions of the initial value problem $u' = g(t, u), u(t_0) = u_0$, in the interval $[t_0, t_0 + a]$. We claim that

$u(t) \equiv v(t)$ on $[t_0, t_0 + a]$.

Suppose this is not true. Then, there exists a $t_1 \in [t_0, t_0 + a)$ such that

$u(t) \equiv v(t)$ on $t_0 \leq t \leq t_1$

and, for some $\alpha > 0$,

$u(t) > v(t)$ on $t_1 < t < t_1 + \alpha \leq t_0 + a$. (3.1.3)

Since g is monotonically non-increasing in u for each fixed t, it follows that $g\big(t, v(t)\big) \geq g\big(t, u(t)\big)$.

Further, since both $u(t)$ and $v(t)$ are the solutions of $u' = g(t, u), u(t_0) = u_0$, we have

$v'(t) \geq u'(t)$ on $[t_1, t_0 + \alpha]$.

Therefore, the function $w(t) = v(t) - u(t)$ has a nonnegative derivative on $[t_1, t_0 + \alpha]$.

That is,

$w'(t) = v'(t) - u'(t) \geq 0.$

By integrating this relation between t_1 and t, we get $w(t) \geq w(t_1) = 0$ which implies $v(t) \geq u(t)$.

This contradicts (3.1.3). Hence, $u(t) \equiv v(t)$ on $[t_0, t_0 + a]$.

Summary

- The concept existence of solution of aninitial value problem in the large interval is discussed.
- Ascoli and Arzela lemma is derived
- The convergence of solutions of initial value problems was discussed.
- The condition of existence and uniqueness in a large interval of solution on an IVP is derived with examples.

Keywords

- Existence of a solution in large interval
- Ascoli-Arzela Lemma

- Peano's Existence theorem
- Condition of uniqueness in the large interval

Self-Assessment

Choose the most suitable answer from the options given with each question.

1. The initial value problem $u'^2 + u^2 = 0,\ u(0) = 0$ has

A. One solution
B. More than one solution
C. No real solution
D. None of these

2. The function f(t,u) be bounded and continuous in the strip $S : t_0 \le t \le t_0 + a, |u| < \infty$. Then IVP $u' = f(t,u),\ u(t_0) = u_0$

A. has at most one solution in $[t_0, t_0+a]$
B. has at least one solution in $[t_0, t_0+a]$.
C. only one solution in $[t_0, t_0+a]$.
D. No solution in $[t_0, t_0+a]$.

3. The IVP $u' = f(t,u),\ u(t_0) = u_0$ defined in the strip $S : t_0 \le t \le t_0 + a, |u| < \infty$ has a unique solution. If

A. f(t,u) is monotonically non-increasing in u
B. f(t,u) is monotonically increasing in u
C. f(t,u) is non-increasing in u
D. f(t,u) is increasing in u.

4. A family of uniformly bounded and equicontinuous functions on B has a uniformly convergent subsequence then

A. B is a closed set
B. B is a compact set
C. A subset of B is compact
D. None of these

5. Every sequence of an equicontinuous and bounded family of function has a

A. convergent sequence in the whole defined interval
B. convergent subsequence in the given interval
C. convergent subsequence in the subinterval
D. None of these

6.Which one is true?

A. Any subset of an equicontinuous family is also equicontinuous.
B. Each member of an equicontinuous family is a continuous function.
C. A family of differentiable functions is equicontinuous
D. All of above

7. Peano's existence theorem shows the

A. Local existence on a sufficiently small interval.
B. Local existence on a sufficiently large interval
C. Global existence on a Large interval

D. None of these

8. The continuity of g alone in the differential equation $u' = g(t,u)$ is

A. not sufficient for the convergence of the approximations.
B. the sufficient for the convergence of the approximations.
C. the necessary condition for the convergence of the approximations.
D. None of these

Answers for Self Assessment

1.	C	2.	B	3.	A	4.	C	5.	C
6.	D	7.	C	8.	A				

Review Questions

Q1. Consider the IVP $x'(t) = \frac{1}{1+x^2}, x(0) = 0,\ t \geq 0,\ |x| < \infty.$

Show that IVP has a unique non-local solution on (0, ∞).

Solve the above equation by the method of separation of variables and them show that the solution x(t), with x(0) =0 satisfies $1/3x^3(t) + x(t) - t = 0, t \geq 0.$

Q2. Study that the IVP $y' = \frac{\cos y}{1-t^2}; y(0) = y_0, |t| < 1, |y_0| < \infty$, has a solution.

Q3. Establish that the solution of IVPs $x'(t) = e^{-t}\cos x,\ x(0) = 0, |t| \geq 0, |x| < \infty,$ exists non-locally and uniquely.

Q4. If $g(t,u)$ be continuous function and bounded in the strip $S: t_0 \leq t \leq t_0 + a, |u| < \infty$, then the initial value problem $u'(t) = g(t,u), u(t_0) = u_0$ has at least one solution u(t) defined on the interval $[t_0, t_0 + a]$.

Q5. If in a compact x-set $B_0 \subset R^n$, let $\{f_n(x)\}$, n=1,2,........., be a uniformly bounded and equicontinuous sequence of functions. Then prove that there exists a subsequence $\{f_{n_k}(x)\}$, uniformly convergent on B_0.

Further Readings

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Unit 04: Existence Theorem of System of Differential Equations

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of initial value problem to the system of differential equations.
- understand the continuity concept through Lipschitz condition
- apply basic theorems on the convergence of solutions.
- find the condition of existence and uniqueness of solution.
- drive the Picard's Lindelof and Peano's existence theorem.

Introduction

We shall now extend the results of a differential equation to system of differential equations with initial conditions. It should be noted that, because of the equivalence of a single scalar differential equation of the n-th order and a system of n first order differential equations, the results we establish also hold for the n-th order scalar differential equation and, in general, for a system of differential equations of any order.

4.1 The System of Differential Equations

Let us consider an initial value problem for the system of differential equations

$$x_i' = f_i(t, x_1, x_2, \dots \dots, x_n), x_i(t_0) = x_{i0},$$

where $i = 1,2, \dots., n$. In vector notation, these equations can be written as

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{4.1.1}$$

where $x = (x_1, x_2, \dots \dots., x_n)$, $f = (f_1, f_2, \dots \dots., f_n)$,

and $x_0 = (x_{10}, x_{20}, \dots\dots, x_{n0})$

are vectors in R^n. We shall assume that $f \in C[\Omega, R^n]$, where Ω is an open (t, x) −set of R^{n+1}.

A solution $x(t)$ of the initial value problem (4.1.1) is a differentiable function of t such that, for a t −interval J containing t_0, $x(t_0) = x_0$, $(t, x(t)) \in \Omega$, and

$x' = f(t, x(t))$.

It is easy to verify that the differentiable function $x(t)$ is a solution of (4.1.1) on J if and only if it is a solution of the Volterra integral equation

$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in J.$ (4.1.2)

Definition 4.1.1: If $f(t, x)$ is continuously differentiable with respect to t and the components of x in Ω, then we write $f \in C^1(\Omega)$. Suppose there exists a positive constant L such that $\frac{\partial f}{\partial x_i}(i = 1,2, \dots\dots, n)$ satisfy

$\left\|\frac{\partial f}{\partial x_i}\right\| \le L$ for $(t, x) \in \Omega.$ (4.1.3)

We should note that inequality (4.1.3) is automatically satisfied if $f \in C^1(B_0)$, where B_0 is any closed, bounded set in R^{n+1}. By applying the mean value theorem to each variable separately and then using (4.1.3), we get

$\|f(t, x) - f(t, y)\| \le L\|x - y\|$ for $(t, x), (t, y) \in \Omega.$ (4.1.4)

An alternative proof that gives result (4.1.4) follows. We define a function G by

$G(\sigma) = f(t, y + \sigma(x - y)), \quad 0 \le \sigma \le 1.$

Then, it is clear that

$f(t, x) - f(t, y) = G(1) - G(0) = \int_0^1 G'(\sigma)d\sigma.$

Let $f_{x_i} = \frac{\partial f}{\partial x_i}(i = 1,2,3, \dots\dots, n).$

From the chain rule, it follows that

$G'(\sigma) = f_{x_1}(t, y + \sigma(x - y))(x_1 - y_1) + f_{x_2}(t, y + \sigma(x - y))(x_2 - y_2) + \cdots \dots \dots \dots \dots + f_{x_n}(t, y + \sigma(x - y))(x_n - y_n).$

Using estimate (4.1.3), we have

$$\begin{aligned}\|f(t, x) - f(t, y)\| &\le \int_0^1 \|G'(\sigma)\| d\sigma \\ &\le L(|x_1 - y_1| + |x_2 - y_2| + \cdots \dots + |x_n - y_n|) \\ &= L\|x - y\|.\end{aligned}$$

A function f satisfying an inequality of the form (4.1.4) for $(t, x), (t, y) \in \Omega$ is said to satisfy the *Lipschitz condition* in Ω with the Lipschitz constant L.

However, a function f satisfying (4.1.4) need not belong to the class C^1, and all the remarks made on the scalar function g for a differential equation are valid for the vector valued function f.

We shall now give the various types of existence proofs for the initial value problem (4.1.1). to begin with, we state a fundamental results.

Remarks 4.1.2: The proof of the Theorem 4.1.4 is similar to that of Theorem 2.2.14, except that the absolute value is replaced by norm of the vector-valued function $x \in R^n$. However, for the sake of completeness, we shall give the proof of this theorem. The norm $\|.\|$ can be any convenient norm in R^n, not necessarily norm or the Euclidean norm.

Remarks 4.1.3: In view of the proofs of Lemma 2.2.12 and Theorem 2.2.14 the choice of $\alpha = \min(a, b/M)$ is natural.

4.2 Picard-Lindel of Theorem

Theorem 4.1.4: If $f(t, x)$ is continuous on $B_0: t_0 \le t \le t_0 + a, \|x - x_0\| \le b_0$, where a and b are positive real numbers, satisfies the Lipschitz condition (4.1.4) in B_0. Let

$M = \max_{(t,x) \in B_0} \|f(t, x)\|, \quad \alpha = \min(a, b/M).$

Then, the initial value problem (4.1.1) has a unique solution $x(t)$ on $[t_0, t_0] = \alpha$.

Proof: Define a sequence of functions

$x_0(t) = x\_0,$

$x_k(t) = x_0 + \int_{t_0}^{t} f(s, x_{k-1}(s))ds, \ k = 1,2,3, \dots \dots \dots, n,$ (4.1.5)

For $t_0 \le t \le t_0 = \alpha$. Since $f(t, x(t))$ is continuous on $[t_0, t_0 + \alpha]$, it is clear that the functions $x_0(t), x_1(t), \dots\dots\dots, x_n(t)$ are defined and continuous on $[t_0, t_0 + \alpha]$. Obviously, $(t, x_0(t)) \in B_0$. Therefore, we have

$\|x_1(t) - x_0\| \le M(t - t_0) \le M\alpha \le b,$

and hence $(t, x_1(t)) \in B_0$. Further, it can be easily shown, by induction, that

$\|x_k(t) - x_0\| \le b$, and therefore $(t, x_k(t)) \in B_0, k = 2,3, \dots\dots\dots, n.$

Set

$z_n(t) = x_n(t) - x_{n-1}(t).$

Since f satisfies the Lipschitz condition (4.1.4) in B_0, it follows that

$$\|z_2(t)\| \le \int_{t_0}^{t} \|f(s, x_1(s)) - f(s, x_0(s))\| ds \le L \int_{t_0}^{t} \|x_1(s) - x_0(s)\|\, ds$$

$$\le L \int_{t_0}^{t} M(s - t_0) ds = LM \frac{(t-t_0)^2}{2!}.$$

A simple induction argument, for $t \in [t_0, t_0 + \alpha]$,yields

$$\|z_k(t)\| \le ML^{k-1} \frac{(t-t_0)^k}{k!}, \quad k = 1,2, \dots\dots, n. \tag{4.1.6}$$

To prove this, assume, for $t \in [t_0, t_0 + \alpha], that$

$$\|z_{k-1}(t)\| \le ML^{k-2} \frac{(t-t_0)^{k-1}}{(k-1)!}, \quad k = 2,3, \dots\dots, n.$$

Then, for $t \in [t_0, t_0 + \alpha]$, we obtain

$$\|z_n(t)\| \le \int_{t_0}^{t} \|f(s, x_{n-1}(s)) - f(s, x_{n-2}(s))\| ds$$

$$\le L \int_{t_0}^{t} \|x_{n-1}(s) - x_{n-2}(s)\|\, ds = L \int_{t_0}^{t} \|z_{n-1}(s)\|\, ds$$

$$\le L^{n-1} M \int_{t_0}^{t} \frac{(s-t_0)^{n-1}}{(n-1)!} ds = L^{n-1} M \frac{(t-t_0)^n}{n!}.$$

This establishes inequality (4.1.6). Now, consider the infinite series

$$x_0 + \sum_{k=1}^{\infty} z_k(t). \tag{4.1.7}$$

The n-th partial sum of this series is $x_n(t)$, that is,

$x_n(t) = x_0 + \sum_{k=1}^{n} z_k(t).$

Therefore, the sequence $\{x_n(t)\}$ converges if and only if series (4.1.7) does so. In the view of (4.1.6), it follows series (1.4.7) is uniformly convergent on $[t_0, t_0 + \alpha]$.

Let the sum of series (4.1.7) be $x(t)$. Thus, we have

$\lim_{n \to \infty} x_n(t) = x(t).$

From the uniform convergence of $x_n(t)$ to $x(t)$ and the continuity of the function $f(t, x)$ on B_0, it follows that $f(t, x_n(t)) \to f(t, x(t))$ uniformly on $[t_0, t_0 + \alpha]$ as $n \to \infty$. Therefore, the term-by-term integration is valid for the integrals in (4.1.5) and yields

$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds.$

Hence, $x(t)$ is a solution of (4.1.1). in order to prove the uniqueness, let $y(t)$ be any other solution of (4.1.1) on $[t_0, t_0 + \alpha]$. Then, we have

$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s)) ds.$

Therefore, the nonnegative function $z(t) = \|x(t) - y(t)\|$ satisfies $z(t_0) = 0$, and

$z(t) \le \int_{t_0}^{t} \|f(s, x(s)) - f(s, y(s))\|\, ds \le L \int_{t_0}^{t} z(s)\, ds.$

This implies $z(t) \le 0$ which is incompatible with $z(t) \ge 0$ unless $z(t) \equiv 0$.

Remarks 4.1.5: A simple induction argument using (4.1.5), for $t \in [t_0, t_0 + \alpha]$, yields

$\|x_i(t) - y(t)\| \le \frac{ML^i(t-t_0)^{i+1}}{(i+1)!}, \quad i = 0,1,2, \dots \dots \dots, n.$

Since $x(t) \equiv y(t)$, this inequality gives an estimate of the error of approximation

$\|x_n(t) - x(t)\| \le \frac{ML^n(t-t_0)^{n+1}}{(n+1)!}$ on $[t_0, t_0 + \alpha]$.

In the theorem that we now give, we shall drop the assumption of the Lipschitz condition and the assertion of uniqueness. To prove such an existence theorem as similar done earlier in previous unit, we need the generalization of Lemma 3.1.6 whose lengthy and intricate proof is omitted.

4.3 Peano's Existence Theorem (Vector Case)

Theorem 4.1.6: Let $f \in C[B_0, R^n]$, where B_0 is the set $\{(t,x) \in \Omega: t_0 \le t \le t_0 + a, \|x - x_0\| \le b\}$, and let $\|f(t,x)\| \le M$ on B_0. Then, the initial value problem (4.1.1) possesses at least one solution $x(t)$ on $t_0 \le t \le t_0 + \alpha$, where $\alpha = \min(a, \frac{b}{M})$.

Proof: Let $x_0(t)$ be a continuously differentiable function on $[t_0 - \eta, t_0]$, $\eta > 0$, satisfying $x_0(t_0) = x_0, x_0'(t) = f(t, x_0(t)), \|x_0(t) - x_0\| \le b$, *and* $\|x_0'(t) \le M\|$.

For $0 < \epsilon \le \eta$, we define a function $x_\epsilon(t)$ o $[t_0 - \eta, t_0 + \alpha]$ by

$$x_\epsilon(t) = \begin{bmatrix} x_0(t) \;\; on \;\;\; [t_0 - \eta, t_0] \\ x_0 + \int_{t_0}^{t} f(s, x_\epsilon(s - \epsilon))ds \end{bmatrix}. \tag{4.1.8}$$

It should be observed that relations (1.4.8) define $x_\epsilon(t)$ as a C^1 –function on$[t_0 - \eta, t_0 + \alpha_1], \alpha_1 = \min\ (\alpha, \epsilon)$, and on this interval

$$\|x_\epsilon(t) - x_0\| \le b. \tag{4.1.9}$$

If $\alpha_1 < \alpha$, we can use (4.1.8) to extend $x_\epsilon(t)$ as a C^1 –function over the interval $[t_0 - \eta, t_0 + \alpha_2], \alpha_2 = \min(\alpha, 2\epsilon)$, such that inequality (1.4.9) holds. By continuing this process, we can define$x_\epsilon(t)$ as a C^1 –function over the interval $[t_0 - \eta, t_0 + \alpha]$ so as to satisfy (4.1.9) on the same interval. Moreover,

$$\|x_\epsilon'(t)\| \le M \; on \; [t_0 - \eta, t_0 + \alpha],$$

And hence the sequence $\{x_\epsilon(t)\}$, $0 < \epsilon \le \eta$, forms a family of equicontinuous and uniformly bounded functions. Thus, the application of Lemma 3.2.6 shows the existence of sequence $\{\epsilon_n\}$ as $n \to \infty$, and

$$\lim_{n\to\infty} x_{\epsilon_n}(t) = x(t)$$

Exists uniformly on $[t_0 - \eta, t_0 + \alpha]$. Since this convergence is uniform, the continuity of $f$on B_0

Implies that $f(t, x_{\epsilon_n}(t - \epsilon_n))$ converges uniformly to $f(t, x(t))$ as $n \to \infty$. Hence, the term by term integration of (4.1.8) with $\epsilon = \epsilon_n$ yields

$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$

This shows that $x(t)$ is a solution of (4.1.1).

4.4 Extension theorem

Theorem 4.1.8 is analogous to the theorems on the continuation of solutions given in previous chapters. To prove these results, the next lemma is needed.

Lemma 4.1.7: Let $f \in C[\Omega, R^n]$ and $x(t)$ be a solution of (4.1.1) on the interval $[t_0, t_0 + a), a < \infty$.

Assume that there is a sequence $\{t_k\}$ such that $t_k \to t_0 + a \; as \; k \to \infty$ and that $\lambda = \lim_{k\to\infty} x(t_k)$ exists. If $f(t,x)$ is bounded on the intersection of Ω and a neighbourhood of $(t_0 + a, \lambda)$, then

$$\lim_{t\to t_0+a} x(t) = \lambda. \tag{4.1.10}$$

If, in addition, $f(t_0 + a, \lambda)$ is defined such that $f(t,x)$ is continuous at $(t_0 + a, \lambda)$, then $x(t) \in C^1[t_0, t_0 + a]$ and $x(t)$ is a solution of (4.1.1) on $[t_0, t_0 + a]$.

Proof: Let $\varepsilon > 0$ be sufficiently small. Consider the set

$$B_1 = \{(t,x) \in \Omega: 0 \le t_0 + a - t \le \epsilon, \|x - \lambda\| \le \epsilon\}.$$

Choose$M = M(\varepsilon) > 1$ so large that $\|f(t,x)\| \leq M$ for $(t,x) \in \omega \cap B_1$. For k sufficiently large, if $0 < t_0 + a - t_k \leq \frac{\epsilon}{2M}$ and $\|x(t_k) - \lambda\| \leq \frac{\epsilon}{2}$, we claim that

$$\|x(t) - x(t_k)\| < M(t_0 + a - t_k) \leq \epsilon/2 \text{ for } t_k \leq t < t_0 + a. \tag{4.1.11}$$

Suppose this is not true. Then, there exists the smallest $t_1 \in (t_k, t_0 + a)$ such that

$$\|x(t_1) - x(t_k)\| = M(t_0 + a - t_k). \tag{4.1.12}$$

Since t_1 is the smallest, we have

$\|x(t) - x(t_k)\| < M(t_0 + a - t_k) \leq \epsilon/2$ for $t_k \leq t < t_1$.

It follows therefore that

$$\|x(t) - \lambda\| \leq \|x(t) - x(t_k)\| + \|x(t_k) - \lambda\|$$
$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } t_k \leq t < t_1.$$

This implies $\|x'(t)\| \leq M$ for $t_k \leq t \leq t_1$.

Thus, we obtain

$$\|x(t_1) - x(t_k)\| \leq M(t_1 - t_k) < M(t_0 + a - t_k).$$

But this contradicts relations (4.1.12). Hence, inequality (4.1.11) holds, and this in turn implies limit (4.1.10). The second part of the lemma follows from the fact that

$x'(t) = f\big(t, x(t)\big) \to f(t_0 + a, \lambda) \ as \ t \to t_0 + a.$

Theorem 4.1.8: Assume that $f \in C[\Omega, R^n]$ and that $x(t)$ is a solution of (4.1.1) on $t_0 \leq t \leq t_0 + b_0$. Then, $x(t)$ can be extended as a solution of (4.1.1) to the boundary of Ω.

Proof: Let $\Omega_1, \Omega_2, \Omega_3, \ldots \ldots$ be the open sets of Ω such that $\Omega = \cup \Omega_n$, the closures $\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3, \ldots \ldots$ are compact, and $\bar{\Omega}_n \subset \Omega_{n+1}$. Then, it follows that there exists an $\epsilon_n > 0$ such that, if $(t_0, x_0) \in \bar{\Omega}_n$, all the solutions of (4.1.1) exists on$t_0 \leq t \leq t_0 + \varepsilon_n$. Now, select an integers $\eta_1$so large that $\big(t_0 + b_0, x(t_0 + b_0)\big) \in \bar{\Omega}_{\eta_1}$. Then, the solution $x(t)$ can be extended over an interval $[t_0 + b_0, t_0 + b_0 + \varepsilon_{\eta_1}]$. Similarly, if $\big(t_0 + b_0 + \varepsilon_{\eta_1}, x(t_0 + b_0 + \varepsilon_{\eta_1})\big) \in \bar{\Omega}_{\eta_1}$, the solution $x(t)$ can be extended over an interval $[t_0 + b_0, t_0 + b_0 + 2\varepsilon_{\eta_1}]$.

We repeat this argument until the solution $x(t)$ is extended over the interval$[t_0, t_0 + b_1]$, where $b_1 = b_0 + N_1 \epsilon_{\eta_1}, N_1$ being an integers ≥ 1, such that $\big(t_0 + b_1, x(t_0 + b_1)\big) \notin \bar{\Omega}_{\eta_1}$. Again, select a sufficiently large integer η_2 such that $\big(t_0 + b_1, x(t_0 + b_1)\big) \in \bar{\Omega}_{\eta_2}$. Then, an argument similar to the just given leads us to conclude that the solution $x(t)$ can be extended over the interval $[t_0, t_0 + b_2]$, where, $b_2 = b_1 + N_2 \epsilon_{\eta_2}, N_2$ being an integers ≥ 1 such that$\big(t_0 + b_2, x(t_0 + b_2)\big) \notin \bar{\Omega}_{\eta_2}$. Proceeding in this way, we obtain a sequence of integers $\eta_1 < \eta_2 < \eta_3 < \cdots \ldots$ and real numbers $b_0 < b_1 < b_2 < \cdots$..so that $x(t)$ has a continuation over $[t_0, t_0 + b]$, where $b = \lim_{k \to \infty} b_k$ such that $\big(t_0 + b_k, x(t_0 + b_k)\big) \in \bar{\Omega}_{\eta_k}$. Thus, the sequence $\{\big(t_0 + b_k, x(t_0 + b_k)\big)\}$ is either unbounded or has a limit point on the boundary of Ω. If it is unbounded, then our assertion follows immediately. If it has a limit point on the boundary of Ω, then by the lemma 4.1.7, we can conclude that the solution $x(t)$ tends to the boundary of Ω as $t \to t_0 + b$.

Summary

- The system of first order differential equations is defined.
- The concept of the initial value problem to the system of differential equations is discussed.
- Lipschitz condition is elaborated.
- The convergence of solutions system of initial value problems was discussed.
- The condition of existence and uniqueness of solution on system of IVP is derived with examples.

Keywords

- Linear first-order system of differential equation
- Non-Linear first-order system of differential equation

- Lipschitz condition
- Existence and uniqueness of a solution
- Picard's-Lindel of theorem
- Peano's existence theorem

Self-assessment

1. The Volterra integral equation is

a) $x(t) = x_0 - \int_{t_0}^{t} f(s, x(s))\, ds, \;\; t \in J$

b) $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \;\; t \in J$

c) $x(t) = \int_{t_0}^{t} f(s, x(s))\, ds, \;\; t \in J$

d) None of these

2. The system of first order_________ initial value problem is defined as $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x}), \boldsymbol{x}(\boldsymbol{t_0}) = \boldsymbol{x_0}$, where $\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{x_0}$ are vectors in $\boldsymbol{R^n}$.

a) Homogeneous

b) Non-homogeneous

c) Autonomous

d) None of these

3. If $\boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x}) \in \boldsymbol{C^1}(\boldsymbol{\Omega})$ is continuously differentiable in $\boldsymbol{\Omega}$, then there exists a positive constant $\boldsymbol{L}$ such that

a) $\left\| \frac{\partial f}{\partial x_i} \right\| < L \text{for } (t, x) \in \Omega$

b) $\left\| \frac{\partial f}{\partial x_i} \right\| \le \boldsymbol{L} \textbf{for} (\boldsymbol{t}, \boldsymbol{x}) \in \boldsymbol{\Omega}$

c) $\left\| \frac{\partial f}{\partial x_i} \right\| = L \text{for } (t, x) \in \Omega$

d) None of these

4. The Lipschitz condition for the system of differentiable equations is defined as

a) $\|f(t, x) - f(t, y)\| < L\|x - y\|$

b) $\|f(t, x) - f(t, y)\| = L\|x - y\|$

c) $\|\boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{y})\| \le \boldsymbol{L}\|\boldsymbol{x} - \boldsymbol{y}\|$

d) None of these

5. The solution of $x' = f(t, x),\ x(t_0) = x_0$, $B_0 : t_0 \le t \le t_0 + a, \| x - x_0 \| \le b$ exists and unique in $[t_0, t_0 + \alpha]$ implies

a) $M = \min_{(t,x) \in B_0} \| f(t, x) \|, \;\; \alpha = \min(a, b/M)$

b) $M = \max_{(t,x) \in B_0} \| f(t, x) \|, \;\; \alpha = \max(a, b/M)$

c) $M = \max_{(t,x) \in B_0} \| f(t, x) \|, \;\; \alpha = \min(a, b/M)$

d) $M = \max_{(t,x) \in B_0} \| f(t, x) \|, \;\; \alpha = \min(a, b)$

6. The absolute value replaced with norm values in the theorem is

a) Picrad's theorem

b) Picard's Lindel of theorem
c) Peanos's theorem
d) None of these

7. The Lipschitz condition is a
a) Necessary condition
b) Sufficient condition
c) Both necessary and sufficient
d) None of these

8. The system of equations given by $x' = f(t,x),\ x(t_0) = x_0$, on $B_0 : t_0 \leq t \leq t_0 + a, \| x - x_0 \| \leq b$ has
a) A unique solution
b) At least one solution
c) Many solutions
d) None of these

9. The solution x(t) of an IVP $x'(t) = f(t,x),\ x(t_0) = x_0$ on $t_0 \leq t \leq t_0 + b_0,\ f \in (\Omega, R^n)$
a) The solution can extended to the boundary of the Ω.
b) The solution can extended anywhere in Ω.
c) The solution cannot be extended to the boundary on Ω.
d) The solution will not remain same.

10. The IVP $u' = f(t,u),\ u(t_0) = u_0$ has a maximal solution r(t) on the interval I. Then every solution existing on I,
a) The inequality u(t) < r(t) holds for t in I
b) The inequality u(t) $\geq$r(t) holds for t in I
c) The inequality u(t) > r(t) holds for t in I
d) The inequality u(t) $\leq$ r(t) holds for t in I

11. The IVP $u' = f(t,u),\ u(t_0) = u_0$ defined in $S : t_0 \leq t \leq t_0 + a, | u | < \infty$ has unique solution. If
a) f(t, u) is monotonically non increasing in u
b) f(t, u) is monotonically increasing in u
c) f(t, u) is non increasing in u
d) f(t, u) is increasing in u

12. The function f(t,u) be bounded and continuous in $S : t_0 \leq t \leq t_0 + a, | u | < \infty.$ Then IVP $u' = f(t,u),\ u(t_0) = u_0$
a) has at most one solution in [t_0, t_0+a]
b) has at least one solution in [t_0, t_0+a]
c) only one solution in [t_0, t_0+a]
d) No solution in [t_0, t_0+a]

13. A family of uniformly bounded and equicontinuous function on B has uniformly convergent subsequence then
a) B is a closed set
b) B is a compact set

c) A subset of B is compact

d) None of these

Answer of Self-Assessment

1	B	2	B	3	B	4	C	5	C
6	B	7	A	8	A	9	A	10	D
11	A	12	B	13	C				

Review Questions

Q1. Establish that the solution ofthe following IVPs exists non-locally and uniquely.

$$\left.\begin{aligned} x'(t) &= \frac{\sin y}{1+t^2}, \quad x(0)=1 \\ y'(t) &= e^{-t}\cos x, \; y(0)=0 \end{aligned}\right\} |t| \geq 0, |x| < \infty, |y| < \infty.$$

Q2.If $f \in C[\Omega, R^n]$ and that $x(t)$ is a solution of systems of equations $x' = f(t,x),\ x(t_0) = x_0$, on $t_0 \leq t \leq t_0 + b_0$ Then, $x(t)$ can be extended as a solution of $x' = f(t,x),\ x(t_0) = x_0$, to the boundary of Ω.

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Unit 05: Differential Inequality

CONTENTS

Objective

After studying this unit, you will be able to

- identify the concept of need of differential inequality.
- understand the concept of upper solution and lower solution.
- know about the maximal and minimal solution.
- apply basic theorems on existence of solutions of differential inequality.

Introduction

The differential inequality occupies a very privileged position in the theory of differential equations. In recent years, these inequalities have been greatly enriched by the recognition of their potential and intrinsic worth in many applications of the applied sciences. The theory of such inequalities depends essentially upon the integration of differential inequalities which is usually known as the general comparison principle. In this section, we shall introduce some basic inequalities of this type along with their applications.

5.1 Differential Inequalities

A function $u(t)$ is said to be a solution of the differential inequality

$u'(t) > g(t,u)$ or $u'(t) \geq g(t,u)$

on an interval I if it is differentiable and satisfies

$u'(t) > g(t,u(t))$ or $u'(t) \geq g(t,u(t))$,

Respectively, for every t in I. For example, the function $u(t) = \tan t$ is a solution of the differential inequality $u'(t) > u^2$ on the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ since $u'(t) = 1 = \tan^2 t$.

5.2 Dini's Derivation

In the view of the application of the inequalities, it would be useful in the foregoing definitions are extended. For any scalar function$u(t)$, the upper and lower-right derivative, D^+u and $D_+u$are defined by

$D^+u(t) = \lim_{h\to 0^+} sup\left(\frac{u(t+h)-u(t)}{h}\right)$,

$D_+u(t) = \lim_{h\to 0^+} inf\left(\frac{u(t+h)-u(t)}{h}\right)$

Similarly, the upper- and lower-left derivative of $u(t)$, $D^-u(t)$ and $D_-u(t)$, are given by

$D^-u(t) = \lim_{h\to 0^+} sup\left(\frac{u(t+h)-u(t)}{h}\right)$

$D_-u(t) = \lim_{h\to 0^+} sup\left(\frac{u(t+h)-u(t)}{h}\right)$

These derivatives are usually referred to as Dini's derivatives.

A function $u(t)$ is said to be a solution of, let us say, the differential inequality $D^+u(t) > g(t, u(t))$ for every t in I. When $D^+u(t) = D_+u(t)$, the right-hand derivative of $u(t)$ exists and is often denoted by $u'(t)$ represents the left-hand derivative of $u(t)$.

Consider the initial value problem

$u'(t) = g(t,u), u(t_0) = u_0,$ (5.1.1)

Where $g \in C[\Omega, R], \Omega$ being an open set in R^2. Let $J_1 = [t_0, t_0 + a), a > 0$.

5.3 Upper and Lower function

The following results shows that any solution of (5.1.1) can be bracketed between the lower and upper functions of this initial value problem.

Definition 5.2.1: A real valued function $v(t)$ which has continuous first derivative on the interval I is aid to be a lower solution of IVP (5.1.1) if for all t in$I$, the following relations hold:

$$v' \le g(t, v); v(t_0) \le u_0$$

A real valued function $w(t)$ which has continuous first derivative on the interval I is aid to be a upper solution of IVP (5.1.1) if for all t in$I$, the following relations hold:

$$w' \ge g(t, w); w(t_0) \ge u_0$$

Definition 5.2.2: A function $v \in C[J_1, R]$ is said to be an upper [or lower] function with respect to (5.1.1) if $v'_+(t)$ exists and satisfies the differential inequality $v'_+(t) > g(t, v(t))$ [or $v'_+(t) < g(t, v(t))$] on J_1.

We shall now derive some basic results for differential inequalities.

Theorem 5.2.3: Let $v, w \in C[J_1, R]$ and satisfy the inequalities

$D_-v(t) \le g(t, v(t)),$ (5.2.1)

$D_-w(t) > g(t, w(t))$ (5.2.2)

with $(t, v(t)), (t, w(t)) \in \Omega$ for $t \in J_1$. Then,

$v(t_0) < w(t_0)$ (5.2.3)

implies

$v(t) < w(t), t \in J_1.$ (5.2.4)

Proof: We claim that inequality (5.2.4) holds. Suppose it does not. Then, there exists a $t_1 > t_0$ such that

$v(t_1) = w(t_1),$ (5.2.5)

$v(t) < w(t),\ t \in [t_0, t_1).$ (5.2.6)

For a sufficiently small $h < 0$, it follows, from the relations (1.5.6) and (1.5.7), that

$D_-v(t_1) > D_-w(t_1)$ (5.2.7)

Since $v(t_1 + h) < w(t_1 + h)$. Therefore, from inequalities (5.2.1), (5.2.2), and (5.2.7), we obtain

$$g(t_1, v(t_1)) > g(t_1, w(t_1)).$$

This contradicts (5.1.6). Hence, our claim is true. Thus, (5.2.4) holds on J_1.

Remarks 5.2.4: Assertion (5.2.4) is true even if we replace " $\leq$ " by " $\geq$ " in (5.2.1) and (5.2.2), respectively.

Example 5.2.5: For the IVP $u' = u^2; u(0) = u_0 = -1$

the function v and w defined by

$v(t) = -\frac{2}{t+1}$ and $w(t) == -\frac{1}{2(t+1)}$ are lower and upper solutions, respectively, on $-\infty < x < \infty$. For, in this case, we have $g(t,u) = u^2, t_0 = 0, u_0 = -1$, and

$$v' = \frac{2}{(t+1)^2} < \frac{4}{(t+1)^2} = v^2 = g(t,v), v(0) = -2 < -1 = u_0$$

And

$w' = \frac{1}{2(t+1)^2} > \frac{1}{4(t+1)^2} = w^2 = g(w,t), w(0) = -\frac{1}{2} > -1 = u_0$.

Also, for any $n > 1$, it is easy to see that the functions v_n and w_n defined on $-\infty < t < \infty$ by

$v_n(t) = -\frac{n}{t+1}$ and $w_n(t) = -\frac{1}{n(t+1)}$

are lower and upper solutions, respectively, of the given IVP. Because we have

$$v_n' = \frac{n}{(t+1)^2} < \frac{n^2}{(t+1)^2} = v_n^2 = g(t,v_n), v_n(0) = -n < -1 = u_0$$

And

$w_n' = \frac{1}{n(t+1)^2} > \frac{1}{n^2(t+1)^2} = w_n^2 = g(t,w_n), w_n(0) = -\frac{1}{n} > -1 = u_0$.

Theorem 5.2.6: Let $v(t)$ and $w(t)$, respectively, be the lower and upper functions with respect to (5.1.1) on J_1. Let $u(t)$ be any solution of (5.1.1) such that

$v(t_0) = u_0 = w(t_0)$. (5.2.8)

Then, the inequality

$v(t) < u(t) < w(t)$ on $(t_0, t_0 + a)$ (5.2.9)

holds.

Proof: We shall first prove the right half of assertion (5.2.9). Set $m(t) = w(t) - u(t)$. Then, from conditions (5.2.8), it is clear that $m_+'(t_0) > 0$. This implies that $m(t)$ is increasing to the right of t_0 in a sufficiently small interval$[t_0, t_0 + \delta]$. Therefore, we have $u(t_0 + \delta) < w(t_0 + \delta)$. Since $w(t)$is the upper function and $u(t)$ is any solution of (5.1.1), it follows, from definition 5.1.1, that, for $t \in [t_0 + \delta, t_0 = a]$,

$u'(t) = g(t, u(t)),\ \ w_+'(t) > g(t, w(t))$.

Thus, the application of Theorem 5.1.2 yields

$u(t) < w(t)$ for $t \in (t_0, t_0 = a)$.

The proof for the left half of (5.2.6) is similar.

Example 5.2.7: Consider the initial value problem

$u' = u^2 - t,\ \ u(0) = 1$. (5.2.10)

It should be noted that, in (5.2.10), the differential equation (a special case of the Riccati equation) is not integrable in elementary terms. However, we observe that

$u - t \leq u^2 - t$ and $u^2 \geq u^2 - t$ for all $t \geq 0$ and $|u| \geq 1$.

Therefore by solving the initial value problem $v' = v - t,\ v(0) = 1,$ we obtain a lower function $v(t) = 1 + t$ with respect to (5.2.10) and, from the initial value problem $w' = w^2, w(0) = 1$, we get an upper function $w(t) = \frac{1}{1-t}, t \neq 1$, with respect to (5.2.10). Hence, we have

$$1 + t < u(t) < \frac{1}{1-t} \quad for \;\; t \in (0,1),$$

where u(t) is the solution of (5.2.10).

Corollary 5.2.8: Let $g_1, g_2 \in C[\Omega, R]$ and satisfy the inequality

$g_1(t, u_1) < g(t, u_2)$ for $(t, u_1), (t, u_2) \in \Omega$.

If $u_1(t)$ and $u_2(t)$ are, respectively, the solutions of

$u_i' = g_i(t, u_i), i = 1{,}2$, existing on the interval J_1 such that $u_1(t_0) < u_2(t_0)$, then the inequality $u_1(t) < u_2(t)$ holds on J_1.

The proof for this corollary is similar to that for Theorem 5.2.6.

5.4 Maximal and Minimal Solution

We shall now use Theorem 5.2.3 to prove a result on the existence of the maximal solution of (5.1.1), the hypothesis being a in Theorem 3.1.7.

Definition 5.3.1: Let $r(t)$ be any solution of (5.1.1) on the interval I. Then, $r(t)$ is said to be the maximal solution of (5.1.1) if, for every solution $u(t)$ of (5.1.1) existing on I, the inequality $u(t) \leq r(t)$ holds for $t \in I$.

Definition 5.3.2: Let $\rho(t)$ be any solution of (5.1.1) on the interval I. Then, $\rho(t)$ is said to be the minimal solution of (5.1.1) if, for every solution $u(t)$ of (5.1.1) existing on I, the inequality $\rho(t) \leq u(t)$ holds for $t \in I$.

Remarks 5.3.3: The maximal solution $r(t)$ and minimal solution $\rho(t)$, if they exists, are unique.

Theorem 5.3.4: Let $g(t, u)$ be continuous in a closed, bounded region $R(a, b)$: $t_0 \leq t \leq t_0 + a, |u - u_0| \leq b$. Then, there exists a maximal solution and a minimal solution of (5.1.1) on $[t_0, t_0 + \alpha]$ for some positive α.

Proof: We shall first prove the existence of the maximal solution. Let $\epsilon > 0$ be given such that $0 < \epsilon \leq b/2$. Since g is continuous in $R(a, b)$, there exists a positive constant M such that

$|g(t, u)| \leq M$ for $(t, u) \in R(a, b)$.

Consider the initial value problem

$u' = g_\epsilon(t, u), \quad u_\epsilon(t_0) = u_0 + \epsilon,$ (5.3.1)

Where

$g_\epsilon(t, u) = g(t, u) + \epsilon.$ (5.3.2)

Clearly, the function $g_\epsilon(t, u)$ is defined and continuous in the closed, bounded region

$R(a, b, \epsilon)$: $t_0 \leq t \leq t_0 + a, |u - u_0 - \epsilon| \leq \frac{b}{2}$.

Moreover, we have $R(a, b, \epsilon) \subset R(a, b)$ and

$|g_\epsilon(t, u)| \leq M + \frac{b}{2}$ for $(t, u) \in R(a, b, \epsilon)$.

Therefore, from Peano's Existence Theorem $n = 1$, it follows that (5.1.1) has a solution $u_\epsilon(t)$ on the interval $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, \frac{b}{2M+b})$.

Let ϵ_1 and ϵ_2 such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$. Then, from (5.3.1) and the relation (5.3.2), we have

$$u_{\epsilon_2}(t_0) < u_{\epsilon_1}(t_0),$$

$$u'_{\epsilon_2}(t) = g\left(t, u_{\epsilon_2}(t)\right) + \epsilon_2,$$

$$u'_{\epsilon_1}(t) > g\left(t, u_{\epsilon_1}(t)\right) + \epsilon_1.$$

The application of Theorem 5.2.3 yields

$u_{\epsilon_2}(t) < u_{\epsilon_1}(t)$ for $t \in [t_0, t_0 + \alpha]$.

From the hypothesis, it follows (see Theorem 3.1.7) that the family of functions $u_\epsilon(t)$ is equicontinuous and uniformly bounded on $[t_0, t_0+\alpha]$. Hence, by Lemma 3.1.6, there exists a decreasing sequence $\{\epsilon_n\}$such that $\epsilon_n \to 0$ as $n \to \infty$ and

$\lim_{n\to\infty} u_{\epsilon_n}(t)$

exists uniformly in $t \in [t_0, t_0+\alpha]$; we denote this limiting value by $r(t)$. Obviously, $r(t_0) = u_0$. Also, the uniform continuity of g and

$$u_{\epsilon_n}(t) = u_0 + \epsilon_n + \int_{t_0}^{t} g\big(s, u_n(s)\big)ds$$

yields $r(t)$ as a solution of (5.2.3). Finally, we show that the solution $r(t)$ is the maximal solution of (5.1.1). To do this, let $u(t)$ be any solution of (5.1.1) existing on the interval $[t_0, t_0+\alpha]$. Then,

$u(t_0) = u_0 < u_0 + \epsilon = u_\epsilon(t_0)$.

Further, for $t \in [t_0, t_0+\alpha]$, we have

$u'(t) < g\big(t, u(t)\big) + \epsilon, u'_\epsilon(t) = g\big(t, u(t)\big) + \epsilon.$

By remarks 5.2.4, we have

$u(t) < u_\epsilon(t)$for $t \in [t_0, t_0+\alpha]$.

Since the maximal solution is unique, it is clear that $u_\epsilon(t)$ tends to $r(t)$ uniformly in $t \in [t_0, t_0+\alpha]$ as $\epsilon \to 0$. A similar argument holds for the minimal solution.

Summary

- The concept differential inequality is discussed.
- Dini's derivatives are derived.
- The upper and lower function elaborated with examples.
- The existence of solution of differential inequalities proved.
- The condition of existence and uniqueness of maximal and minimal solution is derived.

Keywords

- Differential inequality
- Dini's derivative
- Upper and lower function
- Minimal and maximal solution

Self Assessment

1. The upper-right Dini's derivative is given by

$$D^+u(t) = \lim_{h\to 0^+} \sup\left(\frac{u(t+h)-u(t)}{h}\right)$$

A.

$$D^-u(t) = \lim_{h\to 0^+} \sup\left(\frac{u(t+h)-u(t)}{h}\right)$$

B.

$$D_+u(t) = \lim_{h\to 0^+} \inf\left(\frac{u(t-h)-u(t)}{h}\right)$$

C.

$$D\_u(t) = \lim_{h\to 0^+} \inf\left(\frac{u(t+h)+u(t)}{h}\right)$$

D.

2. If $u(t) = \tan t$ is a solution of $u' > u^2 \; on (-\pi/2, \pi/2)$ then

A. $u'(t) = \tan t$

B. $u'(t) = 1 + \tan t$

C. $u'(t) = 1 - \tan t$

D. None of these

3. The solution of Initial value problem $u' = u^2 - t, u(0) = 1$ lies in

A. $1 + t \le u(t) < \frac{1}{1-t}$

B. $1 + t < u(t) \le \frac{1}{1-t}$

C. $1 + t < u(t) < \frac{1}{1-t}$

D. $1 + t \le u(t) \le \frac{1}{1-t}$

4. The upper solution for initial value problem $y' = y^2 - t^2, \;\; y(0) = 1$

A. $y(t) < \frac{1}{1-t}$

B. $y(t) \le \frac{1}{1-t}$

C. $y(t) < \frac{1}{1+t}$

D. $y(t) \le \frac{1}{1+t}$

5. The lower solution of IVP $u' = u^2, \; u(0) = -1$ is

A. $\frac{2}{t+1}$

B. $\frac{-2}{t-1}$

C. $\frac{-2}{t+1}$

D. $\frac{1}{2(t-1)}$

6. The upper solution of IVP $y' = y^2,\ y(0) = -1$ is

A. $\frac{1}{2(x+1)}$

B. $\frac{-1}{2(x+1)}$

C. $\frac{-2}{x+1}$

D. $\frac{1}{2(x-1)}$

7. The Lower-left Dini's derivative is given by

A. $D^{+}u(t) = \lim_{h\to 0^{+}} \sup\left(\frac{u(t+h)-u(t)}{h}\right)$

B. $D^{-}u(t) = \lim_{h\to 0^{-}} \sup\left(\frac{u(t+h)-u(t)}{h}\right)$

C. $D_{+}u(t) = \lim_{h\to 0^{+}} \inf\left(\frac{u(t-h)-u(t)}{h}\right)$

D. $D_{-}u(t) = \lim_{h\to 0^{-}} \inf\left(\frac{u(t+h)-u(t)}{h}\right)$

8. The IVP has a minimal solution r(t) on the interval I. Then every solution existing on I,

A. The inequality u(t) < r(t) holds for t in I

B. The inequality u(t) $\geq$ r(t) holds for t in I

C. The inequality u(t) > r(t) holds for t in I

D. The inequality u(t) $\leq$ r(t) holds for t in I

9. Which of the following relation is true

A. $D^{+}u(t) > D^{-}u(t)$

B. $D^{+}u(t) \geq D^{-}u(t)$

C. $D^{+}u(t) < D^{-}u(t)$

D. $D^{+}u(t) \leq D^{-}u(t)$

10. If $D_{-}v(t) \leq g(t, v(t)),\ D_{-}w(t) > g(t.w(t))\ and\ v(t_0) < w(t_0)$ implies

A. $v(t) \leq w(t)$

B. $v(t) > w(t)$

C. $v(t) < w(t)$

D. $v(t) \geq w(t)$

11. For initial value problem there exist a
A. Only maximal solution
B. Only minimal solution
C. Maximal and minimal solution
D. None of these

12. Let $v(t)$ and $w(t)$, respectively be the lower and upper functions and u(t) be the solution of with respect to $u^{'} = g(t,u),\ u(t_0) = u_0$, where $g \in C[\Omega, R],\ \Omega$ being an open set in R2 such that $v(t_0) = u_0 = w(t_0)$. Then
A. inequality $v(t) < u(t) \le w(t)\ on\ (t_0, t_0 + a)$ holds.
B. inequality $v(t) < u(t) < w(t)\ on\ (t_0, t_0 + a)$ holds.
C. inequality $v(t) \le u(t) < w(t)\ on\ (t_0, t_0 + a)$ holds.
D. inequality $v(t) \le u(t) \le w(t)\ on\ (t_0, t_0 + a)$ holds.

13. The IVP has a maximal solution r(t) on the interval I. Then every solution existing on I,
A. The inequality u(t) < r(t) holds for t in I
B. The inequality u(t) $\ge$r(t) holds for t in I
C. The inequality u(t) > r(t) holds for t in I
D. The inequality u(t) $\le$ r(t) holds for t in I

Answers for Self Assessment

1. A	2. B	3. C	4. A	5. C
6. B	7. D	8. B	9. B	10. C
11. C	12. D	13. D		

Review Questions

1. Define external solutions of initial value problem.
2. If $u(t)$ be the solution of initial value problem $u^{'} = u^2 - t,\ u(0) = 1$, then prove that inequality $1 + t < u(t) < \dfrac{1}{1-t}, t \in (0,1)$ holds.
3. If $v, w \in C[J_1, R]$ and satisfy the inequalities
$D\_v(t) \le g(t, v(t)),\ D\_w(t) > g(t, w(t)),$ with $(t, v(t)), (t, w(t)) \in \Omega$ for $t \in J_1$.
Then prove that $v(t_0) < w(t_0)$ implies $v(t) < w(t),\ t \in J_1$.
4. Find lower and upper solution of IVP $u^{'} = u^2 - t,\ u(0) = 1$.

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Unit 06: Integral Inequality

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of need of integral inequality.
- understand the concept of Gronwall-Reid-Bellman inequality.
- know the properties of integral inequality.
- apply basic theorems on existence of solutions of integral inequality.

Introduction

We shall now give some of the important results involving the integral inequalities that are useful in studying the qualitative properties of solutions of ordinary differential equations.

6.1 Gronwall-Reid-Bellman Inequality

***Theorem 6.1.1*:** Let c be a non-negative constant and let u and v be nonnegative continuous functions on some interval $t_0 \le t \le t_0 + a$ satisfying

$$u(t) \le c + \int_{t_0}^{t} u(s)v(s)ds, \quad t \in [t_0, t_0 + a]. \tag{6.1.1}$$

Then, the inequality

$$u(t) \le c \exp \int_{t_0}^{t} v(s)ds, \quad t \in [t_0, t_0 + a], \text{holds.} \tag{6.1.2}$$

Proof: Let $w(t) = c + \int_{t_0}^{t} u(s)v(s)ds$.

Clearly, $w(t_0) = c$. Then, by the hypothesis, $u(t) \le w(t)$.Since$u(t)$ and $v(t)$ are nonnegative continuous functions, it follows that

$w'(t) = u(t)v(t) \le w(t)v(t), \quad t \in [t_0, t_0 + a]$.

Multiplying this inequality by $\exp\left(-\int_{t_0}^{t} v(s)ds\right)$, we obtain

$\frac{d}{dt}[w(t)\exp(-\int_{t_0}^{t} v(s)ds)] \le 0$

Integrating this inequality from t_0 to t, we get

$$w(t)\exp(-\int_{t_0}^{t} v(s)ds) - w(t_0) \le 0.$$

Since $u(t) \le w(t)$ and $w(t_0) = c$, we have

$$u(t) \le c \exp\left[\int_{t_0}^{t} v(s)ds\right], \quad t \in [t_0, t_0 + a].$$

The generalization of Theorem 6.1.1 which we shall now give are useful in applications.

Theorem 6.1.2: Let u and v be nonnegative continuous functions on some interval $t_0 \le t \le t_0 + a$. Also, let the function $f(t)$ be positive, continuous, and monotonically non-decreasing on $t_0 \le t \le t_0 + a$ and satisfy the in-equality

$$u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds, \quad t \in [t_0, t_0 + a]. \tag{6.1.3}$$

Then, we have

$$u(t) \le f(t) exp \int_{t_0}^{t} v(s)ds, \quad t \in [t_0, t_0 + a].$$

Proof: Since $f(t)$ is positive, from (6.1.3), it follows that

$$\frac{u(t)}{f(t)} \le 1 + \int_{t_0}^{t} \frac{u(s)v(s)}{f(t)} ds, \quad t \in [t_0, t_0 + a].$$

Further, since f is monotonically non decreasing, we have $\frac{1}{f(t)} \le \frac{1}{f(s)}$, and hence

$\frac{u(t)}{f(t)} \le 1 + \int_{t_0}^{t} \frac{u(s)v(s)}{f(s)} ds.$

Thus, by setting $K(t) = \frac{u(t)}{f(t)}$ and applying Theorem 6.1.1, we get

$$K(t) \le \exp\, [\int_{t_0}^{t} v(s)ds,$$

And hence

$$u(t) \le f(t)\exp\, [\int_{t_0}^{t} v(s)ds, \quad t \in [t_0, t_0 + a].$$

Theorem 6.1.3: Let $f(t)$ be a continuous function and $v(t)$ a nonnegative continuous function on the interval $t_0 \le t \le t_0 + a$.If continuous function $u(t)$ has the property

$$u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds \quad for \quad t \in [t_0, t_0 + a], \tag{6.1.4}$$

then

$$u(t) \le f(t) + \int_{t_0}^{t} f(s)v(s) \exp\left[\int_{s}^{t} v(\tau)d\tau\right] ds \quad for \quad t \in [t_0, t_0 + a]. \tag{6.1.5}$$

Proof: Put $w(t) = \int_{t_0}^{t} v(s)u(s)ds$.

Then, w is differentiable and

$$w'^{(t)} - v(t)w(t) \le f(t)v(t).$$

If we now put

$$K(t) = w(t) \exp\left[-\int_{t_0}^{t} v(s)ds\right],$$

Then the forgoing inequality is equivalent to

$$K'(t) \le f(t)v(t)\exp\, [-\int_{t_0}^{t} v(\tau)d\tau].$$

Since $K(t_0) = 0$, on integrating this inequality between $t_0$and t, we get

$$K(t) \leq \int_{t_0}^{t} f(s)v(s)\exp[-\int_{t_0}^{s} v(\tau)d\tau]\,ds,$$

That is, by the definition of $K(t)$,

$$w(t) \leq \int_{t_0}^{t} f(s)v(s)\exp[\int_{s}^{t} v(\tau)d\tau]\,ds.$$

Since $u(t) \leq f(t) + w(t)$, the result follows.

Theorem 6.1.4: Assume that

(i) $g(t,u)$ is continuous in the region $t_0 \leq t \leq t_0 + a, |u| < \infty$;

(ii) $g(t,u)$ is non-decreasing in u for each fixed t;

(iii) The maximal solution $r(t)$ of (5.1.1) exists on the interval $t_0 \leq t < t_0 + a$; and

(iv) $m(t)$ is a continuous function satisfying the integral inequality

$m(t) \leq m(t_0) + \int_{t_0}^{t} g\big(s, m(s)\big)ds, \quad t \in [t_0, t_0 + a].$ (6.1.6)

Then, the inequality

$m(t) \leq r(t), t \in [t_0, t_0 + a]$, holds. (6.1.7)

Proof: Let $v(t)$ be a function defined by the right-hand side of (6.1.6). That is,

$$v(t) = m(t_0) + \int_{t_0}^{t} g\big(s, m(s)\big)ds.$$

Then, we have

$$m(t) \leq v(t),$$

$v'(t) = g(t, m(t)).$ (6.1.8)

From the non-decreasing property on g, it follows that

$v'(t) \leq g(t, v(t)), \quad t \in [t_0, t_0 + a].$

Thus, from the comparison principle implies

$v(t) \leq r(t)$for$t \in [t_0, t_0 + a]$.

Hence, (6.1.7) follows from inequality (6.1.8).

Corollary 6.1.5: Let $m(t)$ and $v(t)$ be nonnegative continuous functions on $t_0 \leq t \leq t_0 + a$. Let $g \in C[R^+, R^+], g(0) = 0, g(u) > 0 \ for \ u > 0, g(u)$ be non-decreasing in u, and k be a non-negative constant. Then, if inequality

$m(t) \leq k + \int_{t_0}^{t} v(s)g\big(m(s)\big)\,ds, \quad t \in [t_0, t_0 + a],$ (6.1.9)

holds, the inequality

$$m(t) \leq w^{-1}\left[w(k) + \int_{t_0}^{t} v(s)ds\right] \quad for \quad t \in [t_0, t_0 + a]$$

Remain valid as long as

$$w(k) + \int_{t_0}^{t} v(s)ds$$

Lies in the domain of the definition of w^{-1}, where the function w, is given by

$$w(u) = \int_{\epsilon}^{u} \frac{d\tau}{g(\tau)}, \epsilon > 0,$$

w^{-1}being the inverse mapping of w.

Proof: Denoting the right-hand side of (6.1.9) by $\eta(t)$, we have $m(t) \leq \eta(t)$. Since g is an increasing function of u and v is a non-negative function, we have

$\frac{g(m(t))v(t)}{g(\eta(t))} \leq v(t).$

But $\eta'(t) = g\big(m(t)\big)v(t)$. Hence, by the definition of w, we have

$\frac{d}{dt} w\big(\eta(t)\big) \leq v(t).$

Integrating this inequality between $t_0$and t, we get

$$w\big(\eta(t)\big) \leq w(k) + \int_{t_0}^{t} v(s)ds.$$

Since w^{-1} is also is an increasing function, we finally have

$$\eta \leq w^{-1}[w(k) + \int_{t_0}^{t} v(s)ds].$$

Corollary 6.1.6: let the assumption of Theorem 6.1.4 be satisfied, except for (6.1.6) which is replaced by

$$m(t) \leq f(t) + \int_{a}^{t} g\big(s, m(s)\big)ds, \quad t \in [t_0, t_0 + a],$$

Where f is continuous on $[t_0, t_0 + a]$. Then, (6.1.7) takes the form

$$m(t) \leq f(t) + r(t), \quad t \in [t_0, t_0 + a],$$

where$r(t)$ is the maximal solution of

$$u' = g(t, f(t) + u), \ u(t_0) = 0,$$

existing on $[t_0, t_0 + a]$.

Summary

- The concept of the integral differential equations is discussed.
- The Gronwall-Reid-Bellman inequality is derived.
- The properties of integral equation were discussed.
- The condition of existence of solution of integral equation is elaborated.

Keywords

- Integral equations
- Integro-differential equations
- Gronwall-Reid-Bellman inequality
- Properties of solutions

Self Assessment

1. The basic condition for the Gronwall-Reid-Bellman inequality that

A. The functions must be nonnegative on closed interval

B. The function must be continuous on closed interval

C. The function must be nonnegative continuous on closed interval

D. None of these

2. The integral inequality helps to understand the _________of solution of ordinary differential equations.

A. Qualitative property

B. Quantitative property

C. Both Qualitative and quantitative

D. None of these

3. Let c be a nonnegative constant and let u and v be nonnegative functions on some interval $t_0 \le t \le t_0 + a$ satisfying $u(t) \le c + \int_{t_0}^{t} u(s)v(s)ds,\ t \in [t_0, t_0+a]$. Then following inequality holds

A. $u(t) \le c\int_{t_0}^{t} v(s)ds,\ t \in [t_0, t_0+a]$

B. $u(t) \le c\int_{t_0}^{t} u(s)ds,\ t \in [t_0, t_0+a]$

C. $u(t) \le c\exp\{\int_{t_0}^{t} v(s)ds\},\ t \in [t_0, t_0+a]$

D. $u(t) \le c\exp\int_{t_0}^{t} u(s)v(s)ds,\ t \in [t_0, t_0+a]$

4. Let $f(t)$ be a continuous function and v(t) be a nonnegative continous functions on the interval. If $u(t)$ has the property $u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds,\ t \in [t_0, t_0+a]$. Then following inequality holds

A. $u(t) \le f(t) + \int_{t_0}^{t} f(s)v(s)\exp\left[\int_{s}^{t} v(\tau)d\tau\right]ds,\ t \in [t_0, t_0+a]$

B. $u(t) < f(t) + \int_{t_0}^{t} f(s)v(s)\exp\left[\int_{s}^{t} v(\tau)d\tau\right]ds,\ t \in [t_0, t_0+a]$

C. $u(t) > f(t) + \int_{t_0}^{t} f(s)v(s)\exp\left[\int_{s}^{t} v(\tau)d\tau\right]ds,\ t \in [t_0, t_0+a]$

D. $u(t) \le f(t) - \int_{t_0}^{t} f(s)v(s)\exp\left[\int_{s}^{t} v(\tau)d\tau\right]ds,\ t \in [t_0, t_0+a]$

5. The qualitative property discussed for first integral inequality is named as

A. Grownall-Reid-Bellman inequality

B. Dini's inequality

C. Volterra inequality

D. None of these

6. Let u and v be nonnegative continuous functions on some interval $t_0 \le t \le t_0 + a$. Also, let the function $\boldsymbol{f(t)}$ be positive, continuous, and monotonically non decreasing and satisfy the inequality $u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds, \ t \in [t_0, t_0+a]$. Then following inequality holds

A. $u(t) \le f(t)\int_{t_0}^{t} v(s)ds, \ t \in [t_0, t_0+a]$

B. $u(t) \le c\int_{t_0}^{t} u(s)ds, \ t \in [t_0, t_0+a]$

C. $u(t) \le f(t)\exp\{\int_{t_0}^{t} v(s)ds\}, \ t \in [t_0, t_0+a]$

D. $u(t) \le c\exp\int_{t_0}^{t} u(s)v(s)ds, \ t \in [t_0, t_0+a]$

Answers for Self Assessment:

1.	**C**	**2.**	**A**	**3.**	**C**	**4.**	**A**	**5.**	**A**
6.	**C**								

Review Questions

1. If f(t) be a continuous function and v(t) a nonnegative continuous function on the interval $t_0 \le t \le t_0+a$. If a continuous function u(t) has property

$$u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds \quad for \ t \in [t_0, t_0 + a],$$

then

$$u(t) \le f(t) + \int_{t_0}^{t} f(s)v(s)\exp[\int_{s}^{t} v(\tau)d\tau]ds \quad for \quad t \in [t_0, t_0 + a].$$

2. If u(t) and v(t) be nonnegative continuous function on some interval $t_0 \le t \le t_0+a$. Also, let the function f(t) be positive, continuous, and monotonically non decreasing on $t_0 \le t \le t_0+a$ and satisfy the inequality If a continuous function u(t) has property

$$u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds \quad for \ t \in [t_0, t_0 + a],$$

then, prove that

$$u(t) \le f(t)\exp[\int_{t_0}^{t} v(s)ds] \quad for \quad t \in [t_0, t_0 + a].$$

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Web links

https://nptel.ac.in/courses/111/106/111106100/

https://nptel.ac.in/content/storage2/courses/111104031/lectures.pdf

Unit 07: More Theorem of Integral Inequality

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of integral inequality in uniqueness.
- understand the concept of convergence theorem
- know about the uniqueness of solutions
- apply basic theorems on the convergence of solutions.
- find the condition of existence and uniqueness of maximal solution.

Introduction

One of the principal objectives in studying the differential and integral inequalities is to prove the uniqueness theorems. A solution of the differential equation in (7.1.4) continuously depends upon the initial conditions and we find that the initial value problem has at most one solution. An important and useful technique giving the uniqueness of solutions follows.

7.1 Kameke's Convergence Theorem

Theorem 7.1.1: Let $g(t,u), g(t,0)=0$, be a non-negative continuous scalar function defined on

$$R_0: t_0 < t \leq t_0 + a, 0 \leq u \leq 2b.$$

Assume that the only solution of

$$u' = g(t,u) \tag{7.1.1}$$

on any interval $(t_0, t_0+\epsilon], \epsilon > 0$, satisfying

$$u(t) \to 0, \frac{u(t)}{t-t_0} \to 0 \quad as \quad t \to t_0 + 0 \tag{7.1.2}$$

is$u(t) \equiv 0$. Further, let $f(t,x)$ be a continuous vector-valued function defined on $R: t_0 \leq t \leq t_0 + a, ||x - x_0|| \leq b$ satisfying

$$\|f(t,x_1) - f(t,x_2)\| \leq g(t, \|x_1 - x_2\|), \quad (t,x_1),(t,x_2) \in R \tag{7.1.3}$$

whenever $t(t,0) => t_0$. Then, the initial value problem

$x' = f(t,x), \quad x(t_0) = x_0, \quad t_0 \in J,$ (7.1.4)

Has at most one solution on $[t_0, t_0 + \epsilon]$.

Proof: Suppose $x_1(t)$ and $x_2(t)$ are any two distinct solutions of (7.1.4) on $t_0 \le t \le t_0 + \epsilon$ for some $\epsilon > 0$. Let $m(t) = \|x_1(t) - x_2(t)\|$. Clearly. $m(t_0) = m'(t_0) = 0$. Without any loss of generality, we can assume that $m(t_0 + \epsilon) \neq 0$. Then, we have $0 < m(t_0 + \epsilon) < 2b$. Using inequality (7.1.3), we obtain

$m'(t) \le g\big(t, m(t)\big), \; t \in (t_0, t_0 + \epsilon).$ (7.1.5)

Let $\rho(t)$ be the minimal solution of the initial value problem $u' = g(t,u),\; u(t_0 + \epsilon) = m(t_0 + \epsilon)$, on $(t_0, t_0 + \epsilon]$.From (7.1.5) and from the differential inequality, we conclude that

$m(t) \ge \rho(t), \quad t \in (t_0, t_0 + \epsilon]$ (7.1.6)

Since $g(t,0) \equiv 0$, it follows, from inequality (7.1.5), that

$$\rho(t) \to 0, \frac{\rho(t)}{t - t_0} \to 0 \quad as \quad t \to t_0 + 0.$$

Therefore, from the hypothesis on equation (7.1.1), it is clear that $\rho(t) \equiv 0$. This contradicts our supposition because $\rho(t_0 + \epsilon) = m(t_0 + \epsilon) \neq 0$. Hence, the assertion of the theorem follows.

Corollary 7.1.2: (Nagumo's criterion): The assertion of Theorem 7.1.1 still holds if condition (7.1.2) is replaced by

$\|f(t,x_1) - f(t,x_2)\| \le \frac{\|x_1 - x_2\|}{t - t_0}, \quad (t,x_1),(t,x_2) \in R$, whenever $t > t_0$.

7.2 Kneser's Theorem: (statement only)

Let $f(t,y) \in C$ on a rectangle $R: |t - t_0| \le a, |y - y_0| \le b$ and

$|f(t,y)| \le M; \alpha = Min\left\{a, \frac{b}{M}\right\}$ on R and $t_0 < c \le t_0 + \alpha$. Finally, let S_c be the set of points ϕ_c for which there is a solution $y = \phi(t)$ of $y' = f(t,y),\; y(t_0) = y_0 \; on \; [t_0, c]$ such that $\phi(c) = \phi_c$ i.e. $\phi_c \in S_c$ means that ϕ_c is a point reached at $t = c$ by some solution of $I.V.P.$ then S_c is a continuum i.e. a closed connected set.

This theorem is about the case of non-unique solutions of initial value problems.

7.3 Theorem of Wintner

Let $F(t,y)$ be continuous on $[t_0, t_0 + a], y \ge 0$ and let the maximal solution of $y' = F(t,y); y(t_0) = y_0 \ge 0$ exists on $[t_0, t_0 + a\,]$.

Let $F(t,y) = w(y)$ where $w(y)$ is positive continuous function on $y \ge 0$ such that

$\int_{y_0}^{\infty} \frac{dy}{w(y)} = \infty$ (7.3.1)

Let $f(t,z)$ be continuous on the strip $t_0 \le t \le t_0 + a$, where z is arbitrary vector function and satisfy

$|f(t,z)| \le F(t,|z|)$ (7.3.2)

(where | | denotes the norm of the functions).

Then the maximal interval of existence of solutions of

$z' = f(t,z); z(t_0) = z_0$ (7.3.3)

Where $|z_0| \le y_0$ is $[t_0, t_0 + a]$.

Proof: From (7.3.2) we have $|z'(t)| \le F(t, |z(t)|)$ (7.3.4)

On any interval on which $z(t)$ exists.

We know that if $\phi_M(t)$ be maximal solution of $y' = F(t, y); y(t_0) = y_0$

and $z = z(t)$ be a C^1 vector values function on $[t_0, t_0 + a]$ such that

$$|z(t_0)| \le \phi_M(t); (t, |z(t)| \in D$$

and $|z'(t)| \le F(t, |z(t)|)$

on any interval $[t_0, t_0 + a]$ on which $z(t)$ exists then

$|z(t)| \le \phi_M(t)$ (7.3.5)

holds on any common interval of existence of $\phi_M(t)$ and z(t).

[Also we know that if $z = z(t)$ is a solution of differential equation

$$y' = F(t, y); y(t_0) = y_0 \ge 0$$

On the right maximal interval J and $F(t, z)$ be continuous on a strip $t_0 \le t \le t_0 + a$ and it is arbitrary then

either $J = [t_0, t_0 + a]$ or $J = [t_0, \delta)$

where $\delta \le t_0 + a$ and $|z(t)| \to \infty$ as $t \to t_0 + a]$ (7.3.6)

To prove this theorem it has to be shown that the function $F(t, y) = w(y)$ satisfies the condition of maximal solution of

$y' = w(y); y(t_0) = y_0 \ge 0$ (7.3.7)

exists on $J = [t_0, t_0 + a]$ by virtue of (7.3.1).

Since $w(y) > 0$, so (7.3.7) implies for any solution $y = y(t)$

$t - t_0 = \int_{t_0}^{t} \frac{y'(s)}{w(y(s)} ds = \int_{y(t_0)}^{y(t)} \frac{dy}{w(y)}$ (7.3.8)

By taking $dy = \frac{dy}{ds}, ds = y'ds$

Note that $w > 0$ implies that $y'(t) > 0$ and $y(t) > 0$ for $t > t_0$.

By result (7.3.6), the solution $y(t)$ can fail to exists on $[t_0, t_0 + a]$ only if it exists on some interval $[t_0, \delta), \delta \le t_0 + a$ and satisfies $y(t) \to \infty$ as $t \to \delta$.

By virtue of (7.3.8) as $t \to \delta$,

$\delta - t_0 = \lim_{t \to \delta} \int_{y_0}^{y(t)} \frac{dy}{w(y)} = \int_{y_0}^{\infty} \frac{dy}{w(y)} = \infty,$ using (7.3.1),

which is contradiction for left side tends to $\delta - t_0$ and right side tends to ∞.

Hence $y(t)$ does not exists on $[t, \delta), \delta \le t_0 + a$.

Thus $y(t)$ must exists on $[t_0, t_0 + a]$.

⇒ The maximal interval of existence of solution of (7.3.3) is $[t_0, t_0 + a]$.

Summary

- The concept of integral inequality in uniqueness is discussed.
- The concept of convergence theorem elaborated.
- know about the theorem of non- uniqueness of solutions.
- Kameke's convergence theorem derived.
- The theorem of Wintner for maximal interval is proved.

Keywords

- Integral inequality
- Convergence
- Uniqueness
- Kameke's convergence
- Kneser's theorem
- Theorem of Wintner

Self-assessment

1. Kamke's convergence theorem holds good

A. for two different solutions

B. at least one solution

C. at most one solution

D. none of these

2. If $g(t,u),\ g(t,0)=0$ be a continuous scalar function on $R_0 : t_0 < t \le t_0 + a, 0 < u \le 2b$ then the $u(t) \equiv 0$ be only solution of $u' = g(t,u)\, on\, (t_0, t_0 + \varepsilon], \varepsilon > 0$ Satisfying

A. $u(t) \to 0, \dfrac{u(t)}{t} \to 0\, as\ \ t \to t_0 + 0$

B. $u(t) \to 0, \dfrac{u(t)}{t + t_0} \to 0\, as\ \ t \to t_0 + 0$

C. $u(t) \to 0, \dfrac{u(t)}{t - t_0} \to 0\, as\ \ t \to t_0 + 0$

D. $u(t) \to 0, \dfrac{u(t)}{t_0} \to 0\, as\ \ t \to t_0 + 0$

3. The continuum set is always

A. A Closed connected set

B. A Open connected set

C. A Semi open connected set

D. A Semi closed connected set

4. The $\boldsymbol{S_c}$be the set of points $\boldsymbol{\phi_c}$ for which there is a solution $\boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{t})$of IVP $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{y}), \boldsymbol{y}(\boldsymbol{t_0}) = \boldsymbol{y_0} \boldsymbol{on}\ [\boldsymbol{t_0}, \boldsymbol{c}]$ then

A. S_c is a continuum

B. S_c is a continuous

C. S_c is a continuum and continuous

D. None of these

5. The maximal interval of existence to an IVP in closed, bounded domain is given by
A. Kneser's theorem
B. Kamekeconvergence theorem
C. Thorem of Wintner
D. None of theses

6. The assumption for the Wintner theorem is
A. The minimal solution exists
B. The maximal solution exists
C. Both maximal and minimal solution exits
D. None of these

7. The assumption for the Wintner theorem with existence of maximal solution is
A. The IVP is continuous non-autonomous
B. The IVP is continuous autonomous
C. The IVP is non-continuous autonomous
D. None of these

8. Let $U(t,u)$ be continuous for $t_0 \le t \le t_0 + a, u \ge 0$ and let the maximal solution of $U' = U(t,u)$, where $u \ge 0,$ *exist on* $[t_0, t_0 + a]$, Let $U(t,u) = \psi(u)$ is a positive, continuous function on u ≥ 0 such that $\int^{\infty} du/\psi(u) = \infty$. Let $f(t, y)$ be continuous on $t_0 \le t \le t_0 + a,$ and $| f(t, y) | \le U(t, | y |)$.
A. Then minimal interval of existence of solution of $y' = f(t, y), y(t_0) = y_0$ where $| y_0 | < u_0$.
B. Then maximal interval of existence of solution of $y' = f(t, y), y(t_0) = y_0$ where $| y_0 | \le u_0$.
C. Then maximal interval of existence of solution of $y' = f(t, y), y(t_0) = y_0$ where $| y_0 | < u_0$.
D. None of these

Answer for Self Assessment

1.	C	2.	C	3.	A	4.	A	5.	C
6.	B	7.	B	8.	B				

Review Questions

1. If $g(t,u), g(t,0) = 0$, be a non-negative continuous scalar function defined on

$$R_0: t_0 < t \le t_0 + a, 0 \le u \le 2b.$$

Assume that the only solution of

$u' = g(t,u)$on any interval $(t_0, t_0 + \epsilon], \epsilon > 0$, satisfying

$$u(t) \to 0, \frac{u(t)}{t - t_0} \to 0 \quad as \quad t \to t_0 + 0$$

is$u(t) \equiv 0$. Further, let $f(t,x)$ be a continuous vector-valued function defined on $R: t_0 \leq t \leq t_0 + a, ||x - x_0|| \leq b$ satisfying$\|f(t,x_1) - f(t,x_2)\| \leq \frac{\|x_1 - x_2\|}{t - t_0}$, $(t,x_1),(t,x_2) \in R$, whenever $t > t_0$

Further Readings

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Unit 08: Linear Systems

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of system of linear differential equation.
- understand the properties of homogeneous linear system.
- know about the fundamental systems of solution.
- apply basic theorems to solve the linear system of differential equation.
- find the condition of uniqueness of solution for linear homogeneous and non-homogeneous system of differential equations.

Introduction

In this chapter, we shall study the various properties of the solutions of linear systems. The results we obtain will often provide a background for treating non-linear systems in subsequent chapters.

8.1 Linear System of Differential Equation

Consider the n-dimensional first order systems of linear equations

$x_i' = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t),\ i = 1,2,3,\ldots\ldots\ldots,n.$ (8.1.1)

Where $x(t) = (x_1(t), x_2(t), \ldots\ldots..x_n(t))$ is an unknown vector function and $a_{ij}(t)$ and $b_i(t), i, j = 1,2,3 \ldots\ldots.., n$, are given continuous functions on $r_1 < t < r_2$. It can be written in the vector-matrix form as

$x' = A(t)x + B(t),$ (8.1.2)

Where $A(t)$ is the $n \times n$ matrix $(a_{ij}(t))$ and $B(t)$ is the $n -$ vector $(b_1(t), b_2(t), \ldots\ldots\ldots..b_n(t))$. For the existence and uniqueness of the solutions of (8.1.2) see chapter 4.

Superposition principle

An important feature of linear systems of the type (8.1.2) is that their solutions can be linearly superposed to obtain new solutions. More specifically, let $x_1(t)$ be a solution of (8.1.2) when $B(t) = B_1(t)$, and let $x_2(t)$ be a solution of this system when $B(t) = B_2(t)$. If c_1 and c_2 are given scalars, then $x(t) = c_1x(t) + c_2x(t)$ is a solution of (8.1.2) when $B(t) = c_1B_1(t) + c_2B_2(t)$. To see this,

$x'(t) = c_1x_1'(t) + c_2x_2'(t)$

$= c_1(t)(A(t)x_1(t) + B_1(t)) + c_2(t)(A(t)x_2(t) + B_2(t))$

$= A(t)(c_1x_1(t) + c_2x_2(t)) + c_1B_1(t) + c_2B_2(t)$

$= A(t)x(t) + B(t).$

In particular, if both $x_1(t)$ and $x_2(t)$ are the solutions of the linear homogeneous system

$x' = A(t)x,$ (8.1.3)

then$c_1x_1(t) + c_2x_2(t)$ is also a solution of (8.1.3). Further, if $x_1(t)$ is a solution of (8.1.2), then $x_2(t)$ is a solution of (8.1.2) if and only if $x_1(t) - x_2(t)$ is a solution of (8.1.3). This important property of the linear superposition principle may be stated in a more abstract way as follows. Suppose the vector function $B(t)$ in (8.1.2) is allowed to vary over a linear space L of functions. Also, assume that, for each $B(t)$ in L, system (8.1.2) has a unique solution$x(t)$. Then, the set $Y$of the solutions of (8.1.2) form a linear space.

8.2 Properties of Linear Homogeneous Systems

We shall now consider the fundamental properties of the solutions of the first order linear homogeneous system

$x' = A(t)x,$ (8.1.4)

Where ,$x(t) = (x_1(t), x_2(t), \dots\dots. x_n(t))$ is an unknown n dimensional vector function and

$A(t) = (a_{ij}(t))$is an $n \times n$ continuous matrix on $r_1 < t < r_2$. Equation (8.1.4) is called linear homogeneous because any linear combination of the solutions of (8.1.4) is also a solution of (8.1.4). More specifically, let

$\phi_j(t) = \left(\phi_{1j}(t), \phi_{2j}, \dots\dots\dots \phi_{nj}(t)\right), \quad j = 1,2,3, \dots\dots.., n,$

Be the solutions of (8.1.4) and let $c_j(j = 1,2, \dots\dots.., n)$ be the arbitrary constants. Also, let

$\phi(t) = \sum_{j=1}^{n} c_j\phi_j(t).$

Then,

$\phi'(t) = \sum_{j=1}^{n} c_j\phi_j'(t).$

Since $\phi_j(t)$ are the solutions of (8.1.4), we have

$\phi'(t) = \sum_{j=1}^{n} c_jA(t)\phi_j(t).$

From the properties of matrix-vector multiplication, it follows that

$\phi'(t) = A(t)\sum_{j=1}^{n} c_j\phi_j(t) = A(t)\phi(t).$

This implies that $\phi(t)$also is a solution of (8.1.4).

It should be noted that $\phi(t) = 0, t \in (r_1, r_2)$, is a solution of (8.1.4); in fact it is the only solution satisfying $\phi(t_0) = 0$ for $t \in (r_1, r_2)$, as the following results shows.

Lemma 8.1.1: Let $t_0 \in (r_1, r_2)$ and $\phi(t)$be a solution of (8.1.4); in fact, it is the only solution satisfying $\phi(t_0) = 0$for $t \in (r_1, r_2)$, as the following results shows.

Proof: The function $x(t) = 0, t \in (r_1, r_2)$, satisfies (8.1.4) together with the initial condition $x(t_0) \equiv \phi(t)$ since all the solutions are defined on $r_1 < t < r_2$.

Example 8.1.2: The vector differential equation corresponding to a linear homogeneous system

$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3$

$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3$ (8.1.5)

$\frac{dx_3}{dt} = -2x_1 + x_2 - x_3$

is

$$\frac{dx}{dt} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & 12 \\ -2 & 1 & -1 \end{bmatrix} x \qquad (8.1.6)$$

Where

$$x = [x_1 \quad x_2 \quad x_3]^T$$

The column vector function

$$\phi(t) = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$$

is a solution of the vector differential equation (8.1.5) on every interval $r_1 < t < r_2$as $u = \phi(t)$satisfies (8.1.6) identically on $r_1 < t < r_2$ i.e.

$$\begin{bmatrix} 3e^{3t} \\ -6e^{3t} \\ -3e^{3t} \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & 12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3e^{3t} \\ -6e^{3t} \\ -3e^{3t} \end{bmatrix}$$

$$\Rightarrow \qquad x_1 = e^{3t}, x_2 = -2e^{3t}, x_3 = -e^{3t} \qquad (8.1.7)$$

Simultaneously satisfy all the three equations of the system (8.1.6) for $r_1 < t < r_2$. So we call (8.1.7) a solution of the system (8.1.6).

We now introduce the concept of linear independence of a set of scalar or vector-valued functions.

Definition 8.1.3: A set of vector-valued functions $v_1(t), v_2(t), \dots\dots.. v_n(t)$ is linearly independent on an interval I (where these functions are defined) if and only if there exists no constants $c_1, c_2, \dots\dots.., c_n$, not all zero, such that

$$\sum_{i=1}^{n} c_i v_i(t) \equiv 0 \quad on \quad I.$$

A set of vector- valued functions is linearly dependent on I if it is not linearly independent on I.

Example 8.1.4 (i) The set of scalar functions $e^t, e^{-t}, sin\, t, \cos t$ is linearly independent on $-\infty < t < \infty$.

(ii) The set of n-dimensional unit vectors

$$e_j = (0,0, \dots \dots ,0,1,0, \dots \dots .,0), \qquad j = 1,2, \dots \dots \dots ., n.$$

is linearly independent in the space R^n.

(iii) The set of vector-valued function $v_1(t) = (1,0,0), v_2(t) = (t^2, e^t, 0), v_3(t) = (t^4, e^{-t}, 0)$

is linearly independent on $-\infty < t < \infty$.

(iv) The set of scalar functions $1, t, 2t$ is linear dependent on $-\infty < t < \infty$.

(v) The set of vectors $v_1 = (1,2,3)$, $v_2 = (2,3,5), v_3 = (2,3,7)$, $v_4 = (0,8,-20)$ is linearly dependent in the space R^3.

Theorem 8.1.5: If $\phi_1(t), \dots \dots, \phi_n(t)$, $r_1 < t < r_2$, is a set of linearly independent solutions of (8.1.4), then linear combination

$$\sum_{j=1}^{n} c_j \phi_j(t)$$

never vanishes on $r_1 < t < r_2$unless $c_1 = c_2 = \cdots \dots . = c_n = 0$.

Proof: Let $\phi(t) = \sum_{j=1}^{n} c_j \phi_j(t)$.

Then, from the linearity of the homogeneous differential equation (8.1.4), $\phi(t)$ is a solution of (8.1.4). If $\phi(t_0) = 0$ for some $t_0 \in (r_1, r_2)$ and $c_j$are not all zero, then, by Lemma 8.1.1, $\phi(t)$ is identically zero. This is a contradiction. Hence, the linear combination never vanishes on $r_1 < t < r_2$.

Definition 8.1.6: A set $\phi_1(t), \phi_2(t), \dots \dots . \phi_n(t), r_1 < t < r_2$, of the solutions of (8.1.4) is called a fundamental system of solutions of (8.1.4) if the set is linearly independent on (r_1, r_2).

Remark 8.1.7: Any solution of (8.1.4) can be expressed in terms of a fundamental systems of solutions of (8.1.4). Thus, the problem of finding any solution of (8.1.4) entails finding n linearly independent solutions of (8.1.4). Evidently then, for determining the behavior of any solution of (8.1.4), we need only the properties of a fundamental system of solutions of (8.1.4).

Theorem 8.1.8: A fundamental system of solutions of (8.1.4) exists.

Proof: Let $\phi_1(t), \dots \dots . ., \phi_n(t)$ be the solutions of (8.1.4) defined on the interval $r_1 < t < r_2$ with the initial conditions

$$\phi_j(t_0) = e_j, \;\; j = 1,2,3, \dots \dots , n,$$

For $t_0 \in (r_1, r_2)$, where $e_1, e_2, \dots \dots . e_n$ are the n-dimensional unit vectors. These solutions are distinct since they satisfy distinct initial conditions. We claim that the solutions are linearly independent on (r_1, r_2). Suppose this is not true. Then, there eist some constants $c_j (j = 1,2,3, \dots \dots , n)$, not all zero, such that

$\phi(t) = \sum_{j=1}^{n} c_j \phi_j(t) \equiv 0 \quad on \quad (r_1, r_2).$

Thus, we have

$$\phi(t_0) = \sum_{j=1}^{n} c_j \phi_j(t_0) = \sum_{j=1}^{n} c_j e_j = (c_1, c_2, \dots \dots . ., c_n) = 0$$

This implies $c_1 = c_2 = \cdots . . = c_n = 0$. But this is a contradiction. Therefore, $\phi_1(t), \dots \dots . ., \phi_n(t)$ are linearly independent. Since these are the solutions of (8.1.4), they form a fundamental system of solutions of (8.1.4).

Corollary 8.1.9: Every solution of (8.1.4) can be expressed as a linear combination of the elements of a fundamental system of solutions of (8.1.4).

Proof: Let $x(t)$ be a solution of (8.1.4) defined for $r_1 < t < r_2$ such that $x(t_0) = x_0 = (x_{10}, x_{20}, \dots \dots . , x_{n0}),\; t_0 \in (r_1, r_2)$, Also, let $\phi_1(t), \dots \dots \dots . \phi_n(t)$ be a fundamental system of solutions of (8.1.4) satisfying $\phi_j(t_0) = e_j (j = 1,2, \dots \dots , n)$. Set

$\phi(t) = \sum_{j=1}^{n} x_{j0} \phi_j(t).$

Clearly, $\phi(t)$ is a solution of (8.1.4) and, moreover,

$\phi(t_0) = \sum_{j=1}^{n} x_{j0} e_j = (x_{10}, x_{20}, \dots \dots . . x_{n0}) = x_0.$

Since the solutions of (8.1.4) are unique, it follows that

$\phi(t) \equiv x(t)$for $r_1 < t < r_2$.

Remarks 8.1.10: The foregoing results shows that the space X of all the solutions of (8.1.4) is linear and has the dimension n.

We now introduce the $n \times n$ matrix $\Phi(t)$ whose $j - th$ column is $\phi_j(t) = (\phi_{1j}, \dots \dots , \phi_{nj})$ such that $\phi_j(t_0) = e_j$, that is,

$$\Phi(t) = \begin{pmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \vdots & \ddots & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{pmatrix}.$$

Obviously, $\Phi(t_0) = I$ is the identity matrix.

Definition 8.1.11: Let $\phi_1(t), \phi_2(t), \dots \dots . , \phi_n(t)$ be the solution of (8.1.4), where $\phi_j(t) = \left(\phi_{1j}(t), \dots \dots , \phi_{nj}(t)\right)$. Then, the scalar function $W(t) = \det \Phi(t)$is called the Wroskian of $\phi_1(t), \dots \dots . ., \phi_n(t)$.

Definition8.1.12: If $\phi_1(t), \phi_2(t), \dots \dots , \phi_n(t)$ is a fundamental system of solutions of (8.1.4), then $\Phi(t)$ is called the fundamental matrix of (8.1.4).

Corollary 8.1.13: The fundamental matrix $\Phi(t)$ is the solution of the matrix differential equation

$$\Phi'(t) = A(t)\Phi(t) \tag{8.1.8}$$

and satisfy $\Phi(t_0) = I$. Further, the solution $x(t)$ of (8.1.4), we have $\phi_j'(t) = A(t)\phi_j(t)$. This implies that $\Phi(t)$ satisfies the equation $\Phi' = A(t)\Phi$. Let $x_0 = (x_{10}, x_{20}, \dots\dots, x_{n0})$. Then, by Corollary 8.1.1, we have

$x(t) = \sum_{j=1}^{n} x_{j0}\phi_j(t)$.

Hence, using matrix- vector multiplication, we get

$x(t) = \Phi(t)x_0$.

The following result expression a relation between the Wronskian $W(t)$ and matrix $A(t)$.

8.3 Abel-Liouville Formula

Theorem 8.1.14: Let $\Phi(t)$ be a fundamental matrix of (8.1.4) and let $t_0 \in (r_1, r_2)$.Then, $W(t) = W(t_0)\exp\left[\int_{t_0}^{t} Tr\, A(s)ds\right] for\ \ t \in (r_1, r_2)$.

Proof: Since $\Phi(t)$ is a fundamental matrix of (8.1.4), it satisfies the matrix differential equation $\Phi' = A(t)\Phi(t)$. Therefore, we have

$$\phi_{ij}' = \sum_{k=1}^{n} a_{ik}(t)\phi_{kj}(t), \quad i, j = 1,2, \dots\dots, n,$$

Where $\Phi(t) = (\phi_{ij}(t))$and $A(t) = (a_{ij}(t))$. We now consider the derivative formula

$$W'(t) = \begin{vmatrix} \phi_{11}' \dots\dots \phi_{1j}' \dots\dots \phi_{1n}' \\ \phi_{21} \dots\dots \phi_{2j} \dots\dots \phi_{2n} \\ \vdots \qquad \vdots \qquad \vdots \\ \phi_{n1}\phi_{nj}\phi_{nn} \end{vmatrix} + \begin{vmatrix} \phi_{11}' \dots\dots \phi_{1j}' \dots\dots \phi_{1n}' \\ \phi_{21} \dots\dots \phi_{2j} \dots\dots \phi_{2n} \\ \vdots \qquad \vdots \qquad \vdots \\ \phi_{n1}\phi_{nj}\phi_{nn} \end{vmatrix} + \dots\dots\dots\dots + \begin{vmatrix} \phi_{11}' \dots\dots \phi_{1j}' \dots\dots \phi_{1n}' \\ \phi_{21} \dots\dots \phi_{2j} \dots\dots \phi_{2n} \\ \vdots \qquad \vdots \qquad \vdots \\ \phi_{n1}\phi_{nj}\phi_{nn} \end{vmatrix} \tag{8.1.9}$$

Set

$$\psi_i(t) = \begin{vmatrix} \phi_{11} \dots\dots \phi_{1j} \dots\dots \phi_{1n} \\ \phi_{i1}' \dots\dots \phi_{ij}' \dots\dots \phi_{in}' \\ \vdots \qquad \vdots \qquad \vdots \\ \phi_{n1} \dots\dots \phi_{nj} \dots\dots \phi_{nn} \end{vmatrix}$$

Hence it follows that

$$\psi_i(t) = \begin{vmatrix} \phi_{11} \dots\dots\dots\dots \phi_{1j} \dots\dots\dots\dots \phi_{1n} \\ \sum_{k=1}^{n} a_{ik}\phi_{k1} \dots\dots \sum_{k=1}^{n} a_{ik}\,\phi_{kj} \dots\dots \sum_{k=1}^{n} a_{ik}\phi_{kn} \\ \vdots \qquad \vdots \qquad \vdots \\ \phi_{n1} \dots\dots\dots\dots \phi_{nj} \dots\dots\dots\dots \phi_{nn} \end{vmatrix}$$

In this determinant, multiplying the first row by a_{i1}, the second by a_{i2}, and so on, except the i-th row, and subtracting their sum from the i-th row, we get

$\psi_i(t) = a_{ii}W(t)$.

This relation is true for $i = 1,2, \dots\dots\dots, n$. Therefore, from (8.1.9), we obtain

$$W'(t) = \sum_{i=1}^{n} a_{ii}(t)W(t).$$

This implies

$W'(t) = (Tr\,(A(t))W(t)$.

Integrating this equation between $t_0$and t, we have

$$W(t) = W(t_0)\exp\left[\int_{t_0}^{t} Tr\, A(s)ds\right] \quad for \quad t \in (r_1, r_2).$$

In particular, if $A(t) = A$ is a constant matrix, then the foregoing results shows that

$W(t) = W(t_0)\exp[(t - t_0)Tr\ A] \quad for\ t \in (r_1, r_2).$

Remarks 8.1.15: Since $\exp\left[\int_{t_0}^{t} Tr\ A(s)ds\right]$ never vanishes, Theorem 8.1.14 implies that if $W(t_0) = 0$ for some $t_0 \in (r_1, r_2)$, then $W(t) = 0$ for all $t \in (r_1, r_2)$.

We now prove a general result which characterizes a fundamental system of solution of (8.1.4).

Theorem 8.1.16: A necessary and sufficient condition for a matrix solution $\Phi(t)$of (8.1.8) to be a fundamental matrix of (8.1.4) is

$W(t) \neq 0$for $t \in (r_1, r_2)$.

Proof: Suppose $\Phi(t)$is a fundamental matrix of (8.1.4) with the column vectors $\phi_1(t), \dots\dots, \phi_n(t)$, and let $\phi(t)$ be any solution of (8.1.4). Then, there exist constants $c_j (j = 1,2, \dots\dots, n)$, not all zero, such that

$\phi(t) = \Phi(t)c.$

For any$t \in (r_1, r_2)$, this relation represents a system of n linear algebraic equations in the n-unknown $c_1, c_2, \dots\dots\dots, c_n$ and has a unique solution. This implies that the determinant of $\Phi(t)$ is not equal to zero. Therefore, $W(t) \neq 0$. Conversely, if $W(t) \neq 0$for $r_1 < t < r_2$, then the column vectors $\phi_1(t), \dots\dots, \phi_n(t)$ of $\Phi(t)$ are linearly independent for $r_1 < t < r_2$. Since these are the solutions of (8.1.4), they form a fundamental system of solutions.

Remarks 8.1.17: The determinant of a matrix of column vectors may be identically zero on an interval $I = (r_1, r_2)$ even when these vectors are linearly independent on I. For example, suppose the matrix $\Phi(t)$ is defined by

$$\Phi(t) = \begin{pmatrix} 1 & t^2 & t^4 \\ 0 & e^t & e^{-t} \\ 0 & 0 & 0 \end{pmatrix} \text{for } t \in I.$$

Here, it is clear that $\det\Phi(t) = 0$ on I; also, the column vectors of $\Phi(t)$ are linearly independent on I. Since this is not true of column vectors that are the solutions of (8.1.4), the assertion of Theorem 8.1.16 is not contradicted.

Example 8.1.18: Consider the homogeneous linear vector differential equation

$$\frac{dx}{dt} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & 12 \\ -2 & 1 & -1 \end{bmatrix} u \qquad (8.1.6)$$

Where

$$x = [x_1 \quad x_2 \quad x_3]^T$$

on every interval $r_1 < t < r_2$.

Solution: It is easy to verify that the vector functions $\phi_1(t)$, $\phi_2(t)$ *and* $\phi_3(t)$ defined by

$$\phi_1(t) = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix};\ \phi_2(t) = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix};\ \phi_1(t) = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

are all solutions of the given homogeneous linear vector differential equation and

$$W(\phi_1, \phi_2, \phi_3)(t) = \begin{vmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{vmatrix} = -e^{10t} \neq 0 \text{for all real } t \in [r_1, r_2].$$

The solutions defined by ϕ_1, ϕ_2 *and* ϕ_3 of our equation are linearly independent on every real interval $[r_1, r_2]$.

Therefore, the fundamental matrix of the given linear vector differential equation is

$$\begin{bmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{bmatrix}.$$

Theorem 8.1.19: The unique solution ϕ of the homogeneous linear vector differential equation (8.1.4) that satisfies the initial condition $\phi(t_0) = x_0,\ t_0 \in (r_1, r_2)$, can be expressed in the form

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0$$

Where $\Phi(t)$ is an arbitrary fundamental matrix of differential equation (8.1.4).

Proof: Let $\Phi(t)$ is an arbitrary fundamental matrix of differential equation (8.1.4) and $\phi(t)$ be any solution of (8.1.4) be any solution of (8.1.4) then there exists a constant vector c such that $\phi(t) = \Phi c$.

The initial condition $\phi(t_0) = x_0$ we get $x_0 = \Phi(t_0)c$.

The determinant $|\Phi(t)|$is the Wronskian of n linearly independent solutions of (8.1.4) and constitute the individual columns of fundamental matrix$\Phi(t)$. As the n columns of $\Phi(t)$ are linearly independent, we have $\Phi(t_0) \neq 0$ and so $\Phi(t_0)$ is a non-singular and its inverse matrix $\Phi^{-1}(t_0)$ exists. Thus we find

$$\Phi^{-1}(t_0)x_0 = \Phi^{-1}(t_0)\Phi(t_0)c = Ic = c$$

Putting this value of c in$\phi(t) = \Phi c$, we get

$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0$.

Example 8.1.20: Find the unique solution of the differential equation

$$x' = \frac{dx}{dt} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} x \text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

that satisfy the initial condition $\phi(0) = x_0 = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$.

Solution: According to theorem 8.1.19, we know that the required solution is given by $\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0$ where $\Phi(t)$ is a fundamental matrix of given differential equation. In example 8.1.18, we have shown

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{bmatrix}$$

is a fundamental matrix. After performing the required calculations, we find

$$\Phi^{-1}(t) = \begin{bmatrix} 2e^{-2t} & e^{-2t} & 0 \\ -4e^{-3t} & -e^{-3t} & -3e^{-3t} \\ e^{-5t} & 0 & e^{-5t} \end{bmatrix}$$

$$\text{Which gives } \Phi^{-1}(0) = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & -3 \\ 1 & 0 & 1 \end{bmatrix}.$$

Making use of $\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0$, we follow

$$\phi(t) = \begin{bmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & -3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} - 6e^{3t} + 3e^{5t} \\ -2e^{2t} + 12e^{3t} - 6e^{5t} \\ -2e^{2t} + 6e^{3t} - 2e^{5t} \end{bmatrix}$$

$$x_1 = 2e^{2t} - 6e^{3t} + 3e^{5t}$$

$$x_2 = -2e^{2t} + 12e^{3t} - 6e^{5t}$$

$$x_3 = -2e^{2t} + 6e^{3t} - 2e^{5t}$$

which is the required solution of given differential equation.

8.4 Non-Homogeneous System of Differnetial Equation

Consider a differential system of the form

$x' = A(t)x + B(t),$ (8.2.1)

Where x is an$n$ −vector, $A(t)$ is an $n \times n$ continuous matrix on $r_1 < t < r_2$, and $B(t)$ a continuous $n - vector$ on $r_1 < t < r_2$. System (8.2.1) is called the inhomogeneous (or non-homogeneous) linear system of $n - th$ order.

If the elements of $A(t)$ and B(t) are continuous or just measurable and majorized by integrable functions on $r_1 < t < r_2$, then there exists a unique solution ϕ of (8.2.1) satisfying $\phi(t_0) = x_0$, where $t_0 \in (r_1, r_2)$and $\|x_0\| < \infty$.

If a fundamental matrix Φ of (8.2.1) is known, then the solution of (8.2.1) can be calculated by using a simple formula which we shall now derive

If a fundamental matrix Φ of (8.2.1) is known, then the solution of (8.2.1) can be calculated by using a simple formula which we shall now derive

Variation of Constants Formula

Theorem 8.2.1: The solution $x(t)$ of (8.2.1) satisfying $x(t_0) = x_0, t_0 \in (r_1, r_2)$,is given by

$x(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds,\ r_1 < t < r_2,$ (8.2.2)

where$\Phi(t)$ is a fundamental matrix of (8.1.4) satisfying $\Phi(t_0) = I$.

Proof: We know that the solution $y(t)$ of (8.1.4) with $y(t_0) = x_0$ can be written as $y(t) = \Phi(t)x_0$. The method we apply in our proof entails considering the constant vectors c as a function or parameter on (r_1, r_2) and determining c (if it exists) so that the function

$x(t) = \Phi(t)c(t),$

Where $c(t) = \big(c_1(t), c_2(t), \ldots \ldots \ldots c_n(t)\big), c(t_0) = x_0,$

is a solution of (8.2.1). Let

$$x(t) = \Phi(t)c(t), c(t_0) = x_0,$$

be a solution (8.2.1). Then,

$$x'(t) = \Phi'(t)c(t) + \Phi(t)c'(t).$$

Since $x(t)$ is a solution of (8.2.1) and $\Phi(t)$ satisfies (8.1.8), we have

$$A(t)x(t) + B(t) = A(t)\Phi(t)c(t) + \Phi(t)c'(t)$$

or

$B(t) = \Phi(t)c'(t).$

Because $\Phi$is a fundamental matrix of (8.1.4), its inverse exists, and hence

$$c'(t) = \Phi^{-1}(t)B(t), c(t_0) = x_0.$$

Therefore, the solution of this equation is

$$c(t) = x_0 + \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds,\quad r_1 < t < r_2,$$

Which implies

$$x(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds,\ r_1 < t < r_2.$$

Remarks 8.2.2: The use of (8.2.2) for obtaining the explicit solutions of (8.2.1) when $n > 3$ is very limited. This is because (8.2.2) involves the fundamental matrix Φ and its inverse Φ^{-1}. Even when $n = 3$, finding the fundamental matrix $\Phi(t)$ may turn out to be difficult, if not impossible. However, the importance of (8.2.2) should be clear from the fact that knowledge of the properties of the fundamental matrix $\Phi(t)$ and the behavior of $B(t)$ are sufficient for deriving considerable information about the solution $x(t)$ of (8.2.1).

Lemma 8.2.3: If $\Phi(t)$ is a fundamental matrix of

$x' = Ax,$ (8.2.3)

Where $x \in R^n, A$ is an $n \times n$ constant matrix, and $\Phi(0) = I$, then

$\Phi(t)\Phi^{-1}(\alpha) = \Phi(t - \alpha)$ (8.2.4)
for every α.

Proof: For a real number α, let $\Omega_1(t) = \Phi(t)\Phi^{-1}(\alpha)$. Since $\Phi(t)$ satisfies $\Phi' = A\Phi, \Omega_1(t)$ also satisfies it, the initial condition being $\Omega_1(\alpha) = I$. Similarly, $\Omega_2(t) = \Phi(t - \alpha)$ satisfies

$$\Omega_2(\alpha) = \Phi(0) = I$$

And also

$\Omega_2'(t) = A\Phi(t - \alpha) = A\Omega_2(t)$.

Hence, from uniqueness, we must have $\Omega_2(t) \equiv \Omega_1(t)$.

In view of Lemma 8.2.3, representation (8.2.2) for the solutions of

$x' = Ax + B(t),$ (8.2.5)

Where $A = (a_{ij})$ is a constant matrix and $B(t)$ is continuous on $r_1 < t < r_2$, where $\Phi(t)$ is a fundamental matrix of (8.2.3) satisfying

Theorem 8.2.4: The solution$x(t)$ of equation (8.2.5) satisfying $x(0) = x_0$ for $r_1 < t < r_2$ is

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - s)B(s)ds, r_1 < t < r_2,$$

where$\Phi(t)$ is a fundamental matrix of (8.2.3) with $\Phi(0) = I$.

Proof: Same as Theorem 8.2.1 with Lemma 8.2.3.

Lemma 8.2.5: If $\Phi(t)$ is a fundamental matrix of (8.1.4) satisfying (8.1.8) with $\Phi(t_0) \neq I$, then any solution $x(t)$ of (8.1.4) satisfying $x(t_0) = x_0, r_1 < t < r_2$, can be written as

$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0$,and$\Omega(t) = \Phi(t)\Phi^{-1}(t_0)$ is a fundamental matrix of (8.1.4) satisfying $\Omega(t_0) = I$.

Proof: For the fundamental matrix $\Phi(t)$, the solution $x(t)$ of (8.1.4) can be written as $x(t) = \Phi(t)c$ for some constant vector c. Therefore, $x(t_0) = \Phi(t_0)c$. Since Φ is the fundamental matrix, its inverse exists, and hence $c = \Phi^{-1}(t_0)x_0$. We now prove that $\Omega(t)$ is a fundamental matrix of (8.1.4). As we know, $\det(\Omega(t)) \neq 0$ for $r_1 < t < r_2$since $\det(\Phi(t)) \neq 0$. Hence, the column of $\Omega(t)$ are linearly independent. Since the column vectors of $\Omega(t)$ are the solutions of (8.1.4), $\Omega(t)$ is a fundamental matrix of (8.1.4) with $\Omega(t_0) = I$.

Example 8.2.5: Find the solution of the non-homogeneous differential equation

$x' = \frac{dx}{dt} = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} e^{5t} \\ 4 \end{bmatrix}$where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: Here $A(t) = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$ and $B(t) = \begin{bmatrix} e^{5t} \\ 4 \end{bmatrix}$ and the corresponding homogeneous differential equation is

$$x' = \frac{dx}{dt} = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} x$$

Whose two simultaneous differential equations are$x_1' = 6x_1 - 3x_2; x_2' = 2x_1 + x_2$.

On solving them, we find

$$x_1 = \frac{3c_1}{2}e^{4t} + c_2e^{3t} = 3ke^{4t} + c_2e^{3t}$$

$$x_2 = c_1e^{4t} + c_2e^{3t} = 3ke^{4t} + c_2e^{3t} \quad (c_1 = 2k)$$

$$\Rightarrow \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3ke^{4t} + c_2e^{3t} \\ 2ke^{4t} + c_2e^{3t} \end{bmatrix}$$

$\Rightarrow \phi_1(t) = \begin{bmatrix} 3e^{4t} \\ 2e^{4t} \end{bmatrix}and\phi_2(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$

Which constitute a fundamental set of solutions of homogeneous differential equation. Thus fundamental matrix $\Phi(t)$ is given by

$$\Phi(t) = \begin{bmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{bmatrix} \tag{8.2.6}$$

We know that any solution of our non-homogeneous differential equation is given by

$$\phi_0(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds$$

for any real number t_0. For convenience, let $t_0 = 0$, then

$$\phi_0(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds$$

where$\Phi(t)$ is given by (8.2.6) and

$$B(s) = \begin{bmatrix} e^{5s} \\ 4 \end{bmatrix}$$

Now $\Phi^{-1}(s) = \begin{bmatrix} e^{-4s} & -e^{-4s} \\ -2e^{-3s} & 3e^{-3s} \end{bmatrix}$

Hence $\phi_0(t) = \begin{bmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-4s} & -e^{-4s} \\ -2e^{-3s} & 3e^{-3s} \end{bmatrix} \begin{bmatrix} e^{5s} \\ 4 \end{bmatrix} ds$

$$= \begin{bmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{s} - 4e^{-4s} \\ -2e^{2s} + 12e^{-3s} \end{bmatrix} ds$$

$$\begin{bmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{bmatrix} \begin{bmatrix} e^{t} + e^{-4t} - 2 \\ -2e^{2t} - 4e^{-3t} + 5 \end{bmatrix}$$

$$\begin{bmatrix} -6e^{4t} + 5e^{3t} + 2e^{5t} - 1 \\ -4e^{4t} + 5e^{3t} + e^{5t} - 2 \end{bmatrix}$$

$x_1 = -6e^{4t} + 5e^{3t} + 2e^{5t} - 1, x_2 = -4e^{4t} + 5e^{3t} + e^{5t} - 2$which is required solution of given system of non-homogeneous differential equation. Obviously, we follow

$$\phi_0(0) = \begin{bmatrix} -6 + 5 + 2 - 1 \\ -4 + 5 + 1 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \boldsymbol{o}$$

Further, we observe that this solution can also be expressed in the form

$\phi_0(t) = -2\phi_1(t) + 5\phi_2(t) + \phi_0^*(t)$

Where$\phi_0^*(t) = \begin{bmatrix} 2e^{5t} - 1 \\ e^{5t} - 2 \end{bmatrix}$ is a solution of given non homogeneous differential equation.

Summary

- The properties of system of linear homogeneous and non-homogeneous differential equations are discussed.
- Fundamental solution is derived and elaborated with suitable examples.
- The Abel-Liouville formula is derived to find the Wronskian.
- The variation of constant formula was discussed to solve homogeneous linear system of differential equation.
- The condition of uniqueness of solution of boundary value problem with an example.

Keywords

- Linear homogeneous first order system
- Linear non- homogeneous first order system
- Wronskian
- Fundamental matrix
- Abel-Liouville formula
- Variation of constant formula

Self Assessment

1. The linear homogenous system $x' = A(t)x,$ where $x = (x_1(t), \ldots\ldots\ldots, x_n(t)),$ is an unknown n-dimensional vector function and $A(t) = (a_{ij}(t))_{n\times n}$ matrix on $r_1 < t < r_2$ .Then

A. Any Linear combination of the solution is also a solution.
B. All the solutions are linearly independent.
C. Both (a) and (b).
D. None of these.

2. Let $t_0 \in (r_1, r_2)$ and $\phi(t)$ be the solution of linear homogenous system $x' = A(t)x,$ is satisfying $\phi(t_0) = 0$. Then,

A. $\phi(t)$ is identically zero on $r_1 < t < r_2.$
B. $\phi(t)$ is identically zero on R .
C. $\phi(t)$ never zero on $r_1 < t < r_2.$
D. None of these.

3. A set of vector- valued functions $\boldsymbol{v_1(t)}, \boldsymbol{v_2(t)}, \ldots\ldots \boldsymbol{v_n(t)}$ is linearly independent on an interval I if and only if

A. There exists constant not all zero
B. There exists constant all zero to vanish the linear combination
C. There exists no constant not all zero to vanish the linear combination
D. None of these

4. The set of scalar functions $1, t, 2t$ is

A. Linearly dependent
B. Linearly independent
C. Completely constant
D. None of these

5. A set of solutions of linear homogenous system $x' = A(t)x$ is called Fundamental system if

A. The set is linearly dependent.
B. The set is linearly independent.
C. The set contains zero solution.
D. The set is empty.

6. Every solution of linear homogeneous system can be expressed as
A. Linear combination of the elements of a fundamental system of solutions
B. Linear combination of another solution
C. Zero solution
D. None of these

7. If $\boldsymbol{\phi_1(t)}, \boldsymbol{\phi_2(t)}, \ldots \ldots \boldsymbol{\phi_n(t)}, \boldsymbol{r_1} < t < \boldsymbol{r_2}$, is a linearly independent solutions, then
A. The linear combination vanishes
B. The linear combination never vanishes
C. It is constant every where
D. None of these

8. $\Phi(t)$ be the fundamental matrix of linear homogenous system $x' = A(t)x,$ on $r_1 < t < r_2$ .Then the scalar function $W(t)$ be the Wronskian defined as
A. $W(t) = -\det \Phi'(t)$
B. $W(t) = \Phi'(t)$
C. $W(t) = \det \Phi(t)$
D. $W'(t) = \Phi(t)$

9. If Φ be a fundamental matrix of the homogenous linear vector differential equation $\frac{dx}{dt} = A(t)x$ where $x = (x_1(t), x_2(t), \ldots\ldots x_n(t))$ is n-dimensional vector function and A(t) = (a_{ij}(t)) is a continuous square matrix of order n on $r_1<t<r_2$. C is any constant non-singular matrix then
A. ΦC is also fundamental matrix
B. $\Phi^{-1}C$ is also fundamental matrix
C. ΦC^{-1} is also fundamental matrix
D. $\Phi^{-1}C^{-1}$ is also fundamental matrix

10. The fundamental matrix $\boldsymbol{\phi(t)}$ for the linear homogenous system satisfies $\boldsymbol{\phi(t_0)} = \boldsymbol{I}$, then the solution $\boldsymbol{x(t)}$ can be written as
A. $x(t) = \phi(t)^{-1}x_0$
B. $x(t) = \phi(t)x_0$
C. $x^{-1}(t) = \phi(t)x_0$
D. None of these

11. The formula $W(t) = W(t\_0)exp\left[\int_{t_0}^{t} TrA(s)ds\right]$ is known as

A. Variation of constant

B. Abel-Liouville formula

C. Periodic formula

D. None of these

12. The necessary and sufficient condition for fundamental matrix is

A. $W(t) = 0$

B. $W(t) \neq 0$

C. $W(t) =$ constant

D. None of these

13. $x(t)$ is the solution of linear inhomogeneous system $x' = A(t)x + B(t)$ and $x(t_0) = x_0$ on $r_1 < t_0 < r_2$. $\Phi(t)$ be the fundamental matrix then

A. $x(t) = \Phi'(t)x_0 + \int \Phi(t)B(s)ds$

B. $x(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds$

C. $x(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(t)B(s)ds$

D. $x(t) = \Phi(t)x_0 + \int \Phi(t)B(s)ds$

14. A fundamental matrix of linear homogenous system $\frac{dy}{dt} = A(t)y$ with constant coefficients is given by $\Phi(t) = e^{tA}, |t| < \infty$ and the solution φ of above equation with initial condition $\varphi(t_0) = y_0 \; (|t_0| < \infty, |y_0 < \infty|)$ is given by

A. $\varphi(t) = y_0 e^{tA} \; (|t_0| < \infty)$

B. $\varphi(t) = -y_0 e^{(t+t_0)A} \; (|t_0| < \infty)$

C. $\varphi(t) = -y_0 e^{(t_0-t)A} \; (|t_0| < \infty)$

D. $\varphi(t) = y_0 e^{(t-t_0)A} \; (|t_0| < \infty)$

15. If $\boldsymbol{\phi(t)}$ is a fundamental matrix of $\boldsymbol{x'(t) = A(t)x(t)}$, and $\boldsymbol{\phi(0) = I}$, then

A. $\phi(t)\phi^{-1}(\alpha) = \phi(t+\alpha)$

B. $\phi(t)\phi^{-1}(\alpha) = \phi(t\alpha)$

C. $\phi(t)\phi^{-1}(\alpha) = \phi(t-\alpha)$

D. None of these

16. The formula $x(t) = \Phi'(t)x_0 + \int \Phi(t)B(s)ds$ is named as

A. Abel-Liouville formula
B. Variation of constant formula
C. Wronskian formula
D. None of these

17. The solution of linear homogeneous system of first order $y' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} y$ is

A. $y_1 = c_1e^{4t} + c_2e^{-3t}, y_2 = 3c_1e^{4t} + c_2e^{-3t}$

B. $y_1 = c_1e^{-t} - c_2e^{4t}, y_2 = 2c_1e^{-t} + c_2e^{4t}$

C. $y_1 = 2c_1e^{4t} + c_2e^{-t}, y_2 = 3c_1e^{4t} - c_2e^{-t}$

D. $y_1 = 3c_1e^{-t} - c_2e^{-3t}, y_2 = 2c_1e^{-t} + 3c_2e^{-3t}$

18. The eigen values corresponding to linear non homogeneous differential system $y' = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} y + \begin{bmatrix} e^{5t} \\ 4 \end{bmatrix}$ are

A. 3,4
B. -3,4
C. 3,-4
D. -3,-4

19. The eigen values corresponding to linear non homogeneous differential system $y' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} y + \begin{bmatrix} 2e^{2t} \\ -2 \end{bmatrix}$ are

A. 1,2
B. 1,-2
C. 1,1
D. -1,2

20. The fundamental matrix corresponding to linear homogeneous $y' = \begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} y$ is

A. $\Phi(t) = \begin{bmatrix} e^{t} & e^{-3t} \\ -2e^{t} & e^{-3t} \end{bmatrix}$

B. $\Phi(t)=\begin{bmatrix} e^{-t} & e^{3t} \\ e^{-t} & 3e^{3t} \end{bmatrix}$

C. $\Phi(t)=\begin{bmatrix} e^{-t} & e^{3t} \\ e^{t} & e^{-3t} \end{bmatrix}$

D. $\Phi(t)=\begin{bmatrix} -4e^{-3t} & 2e^{3t} \\ e^{-3t} & e^{3t} \end{bmatrix}$

21. The characteristic roots corresponding to linear homogeneous differential system

$$y'=\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}y \text{ are}$$

A. 2,3,-2
B. 2,2,3
C. 1,2,3
D. 1,-2,3

22. The solution of linear homogeneous system of first order $y'=\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}y$ is

A. $y_1=c_1e^{t}+c_2e^{-3t}, y_2=-2c_1e^{t}+c_2e^{-3t}$

B. $y_1=c_1e^{-t}-c_2e^{3t}, y_2=2c_1e^{-t}+c_2e^{3t}$

C. $y_1=c_1e^{t}+c_2e^{3t}, y_2=2c_1e^{t}+c_2e^{3t}$

D. $y_1=c_1e^{-t}-c_2e^{-3t}, y_2=2c_1e^{-t}-c_2e^{-3t}$

Answers for Self Assessment

1.	C	2.	A	3.	C	4.	A	5.	B
6.	A	7.	B	8.	C	9.	A	10.	B
11.	B	12.	B	13.	B	14.	D	15.	C
16.	B	17.	C	18.	A	19.	C	20.	D
21.	A	22.	C						

Review Questions

Find the solution of the following non-homogeneous systems:

1. $\frac{dx}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 2e^{2t} \\ -2 \end{bmatrix}$

2. $\frac{dx}{dt} = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} -2\sin t \\ 6\cos t \end{bmatrix}$

3. $\frac{dx}{dt} = \begin{bmatrix} -10 & 6 \\ -12 & 7 \end{bmatrix} x + \begin{bmatrix} 10e^{-3t} \\ 18e^{-3t} \end{bmatrix}; \phi(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4. $\frac{dx}{dt} = \begin{bmatrix} -1 & 1 \\ -12 & 6 \end{bmatrix} x + \begin{bmatrix} 3e^{4t} \\ 8e^{4t} \end{bmatrix}; \phi(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Further Readings

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Unit 09: Periodic and Adjoint Linear System

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of periodic linear system of differential equation.
- understand the more properties of fundamental matrix of homogeneous linear system.
- know about the Floquet's theorem for periodic linear differential system.
- apply basic theorems to adjoint of the linear system of differential equation.
- find the condition of adjoint for linear homogeneous system of differential equations.

Introduction

In this chapter, we will learn about the more properties of linear homogenous differential systems like periodic and adjoint system which helps to understand some dynamical behavior of delay differential and impulsive differential equations

9.1 Periodic Linear Systems

Consider the linear homogeneous system

$$x' = A(t)x, \tag{9.1.1}$$

Where $A(t)$ is an $n \times n$ continuous matrix on the interval $-\infty < t < \infty$ and

$$A(t+\omega) = A(t) \tag{9.1.2}$$

for some constant $\omega \neq 0$. Then, (9.1.2) is called periodic system, and ω is the period of A. A basic results of periodic system is the representation of a fundamental matrix as the product of a periodic matrix (with the same period as that of the fundamental matrix) and a solution matrix of a system with constant coefficients.

Theorem 9.1.1: If Φ is a fundamental matrix of (9.1.1) and C is any constant nonsingular matrix, then ΦCalso is a fundamental matrix of(9.1.1). Moreover, every fundamental matrix of (9.1.1) is of the form ΦC for some constant nonsingular matrix C.

Proof: If $\Phi(t)$ is a fundamental matrix of (9.1.1), then it follows that

$$\Phi'(t)C = A(t)\Phi(t)C, \quad r_1 < t < r_2,$$

that is

$$(\Phi(t)C)' = A(t)(\Phi(t)C),$$

And hence $\Phi(t)C$ is a solution of $\Phi'(t) = A(t)\Phi(t)$. (9.1.3)

Since $\det(\Phi C) = \det(\Phi)\det(C) \neq 0$,

It is clear that ΦC is a fundamental matrix of (9.1.1).

Conversely, if Φ_1 and Φ_2 are the fundamental matrices of (9.1.1), then $\Phi_2 = \Phi_1 C$ for some constant nonsingular matrix of C.

To show this, let $\Phi_1^{-1}\Phi_2 = \psi$. Then, we have $\Phi_2 = \Phi_1\psi$. Differentiating both sides of this equations, we get $\Phi_2' = \Phi_1\psi' + \Phi_1'\psi$.

Using (9.1.3), we obtain $A\Phi_2 = \Phi_1\psi' + A\Phi_1\psi$or $\Phi_1\psi' = 0$,

Thus, $\psi' = 0$, rendering $\psi = C$ a constant matrix; further, C is nonsingular since Φ_1 and Φ_2 are nonsingular.

Remarks 9.1.2: If only Φ_2 is required to be solution of (9.1.2), then$C$ need to be non singular. Further, if Φ is a fundamental matrix of (9.1.1) and C any constant nonsingular matrix, then $C\Phi$ is not, in general, a fundamental matrix of (9.1.1). Moreover, two different homogeneous systems cannot have the same fundamental matrix. Hence, Φ determines A uniquely, although the converse is not true.

9.2 Floquet's Theorem

Theorem 9.1.3: If Φ is a fundamental matrix of (9.1.1), then so is ψ, where

$\psi(t) = \Phi(t+\omega),\ -\infty < t < \infty$.

Corresponding to every such Φ, there exist a periodic nonsingular matrix P with the period ω and a constant matrix R such that

$\Phi(t) = P(t)e^{tR}$. (9.1.4)

Proof: Since Φ is a fundamental matrix of (9.1.1), we have $\Phi'(t) = A(t)\Phi(t)$.

From the relation (9.1.2), it follows that

$\psi'(t) = \Phi'(t+\omega) = A(t+\omega)\Phi(t+\omega) = A(t)\Phi(t+\omega)$.

Thus, ψ is a solution matrix of (9.1.1); it is also a fundamental matrix of (9.1.1) since $\det(\psi(t)) = \det(\Phi(t+\omega)) \neq 0$for $-\infty < t < \infty$. Therefore, there exists a constant nonsingular matrix C (see Theorem (9.1.1) such that

$\Phi(t+\omega) = \Phi(t)C$ (9.1.5)

and there exists also a constant matrix R such that

$C = e^{\omega R}$. (9.1.6)

From relation (9.1.5) and (9.1.6), we obtain

$\Phi(t+\omega) = \Phi(t)e^{\omega R}$. (9.1.7)

Let $P(t)$ be a matrix defined by the relation

$P(t) = \Phi(t)e^{-tR}$. (9.1.8)

Then, we have

$$P(t+\omega) = \Phi(t+\omega)e^{-(t+\omega)R} = \Phi(t)e^{\omega R}e^{-(t+\omega)R} = \Phi(t)e^{-tR} = P(t)$$

Since $\Phi(t)$ and e^{-tR} are nonsingular for $-\infty < t < \infty$, so too is$P(t)$.

Remarks 9.1.4: The practical utility of the Theorem 9.1.3 is that the fundamental matrix $\Phi(t)$ of (9.1.1) can be determined over the entire interval $-\infty < t < \infty$ once $\Phi(t)$ is given over an interval of length $\omega(i.e. 0 \le t \le \omega)$.

To justify the foregoing statement, we proceed as follows. From (9.1.5), we know $C = \Phi^{-1}(0)\Phi(\omega)$, and hence $R = (\log C)/\omega$. Thus, (9.1.8) gives $P(t)$ over the interval $0 \le t \le \omega$. Since $P(t)$ is periodic with the period ω, it can be determined over the interval $-\infty < t < \infty$. Hence, we derive the result from (9.1.4).

Remarks 9.1.5: If Φ_1 is another fundamental matrix of (9.1.1) where (9.1.2) holds, then $\Phi = \Phi_1 M$ for some constant nonsingular matrix M.

Adjoint Systems

If Φ is a fundamental matrix of (9.1.1), then $\Phi\Phi^{-1} = I$ yields

$$(\Phi^{-1})' = -\Phi^{-1}\Phi'\Phi^{-1} == -\Phi^{-1}A\Phi\Phi^{-1} = -\Phi^{-1}A$$

or

$\left(\Phi^{T^{-1}}\right)' = -A^T\Phi^{T^{-1}}$.

Therefore, $\Phi^{T^{-1}}$ is a fundamental matrix of the system

$x' = -A^T(t)x,$ (9.1.9)

And the matrix equation

$X' = -A^T(t)X, \quad t \in (r_1, r_2),$ (9.1.10)

is called the adjoint system to (9.1.3). This relationship is symmetric in the sense that (9.1.1) and (9.1.3) are the adjoint systems to (9.1.6), respectively, and vice versa.

Theorem 9.1.6: If Φ is a fundamental matrix of (9.1.1), then ψ is a fundamental matrix of its adjoint system (9.1.9) if and only if

$\psi^T\Phi = \mathrm{C},$ (9.1.11)

Where C is a constant nonsingular matrix.

Proof: If Φ is a fundamental matrix of (9.1.1), and ψ is a fundamental matrix of (9.1.9), then

$$\psi = \Phi^{\mathrm{T}^{-1}}D$$

for some constant nonsingular matrix D (see Theorem 9.1.1) since $\Phi^{\mathrm{T}^{-1}}$ is a fundamental matrix of (9.1.9). Therefore, $\Phi^{\mathrm{T}}\psi = D$, and hence $\psi^T\Phi = \mathrm{C}$, where $C = D^T$.

To prove the converse, let Φbe a fundamental matrix of (9.1.1) satisfying (9.1.11). Then, we have

$\psi^{\mathrm{T}} = \mathrm{C}\Phi^{-1}$,i.e. $\psi = \Phi^{\mathrm{T}^{-1}}C^T$.

Hence, by Theorem 9.1.1, ψ is a fundamental matrix of the adjoint system (9.1.9).

Remarks 9.1.7: If $\mathrm{A} = -\mathrm{A}^{\mathrm{T}}$, then $\Phi^{\mathrm{T}^{-1}}$ being a fundamental matrix of (9.1.9) is a fundamental matrix of (9.1.1) too.

In view of the forgoing statement and Theorem 9.1.1,

$\Phi = \Phi^{\mathrm{T}^{-1}}C,$

that is,

$\Phi^T\Phi = C,$ (9.1.12)

Where C is a constant nonsingular matrix. Equation (9.1.12) implies, in particular, that the Euclidean length of any solution vector ϕ of (9.1.1) is constant.

Summary

- The more properties of fundamental matrix of system of linear homogeneous differential equations are discussed.
- Periodic linear system is elaborated and Floquet's theorem for periodic system is derived.

- The adjoint of system of linear homogeneous differential equation is explained.
- The relation between the fundamental matrix of it's adjoint system is derived.

Keywords

- Linear homogeneous first order system
- Properties of fundamental matrix
- Periodic linear system
- Floquet's theorem
- Adjoint system

Self Assessment

1. The linear homogeneous system $x' = A(t)x$ is called periodic with period $\boldsymbol{\omega}$ then

A. $A(t/\omega) = A(t)$

B. $A(t+\omega) = A(t)$

C. $A(t\omega) = A(t)$

D. None of these

2. The periodic fundamental matrix represented as

A. Abel-Liouville theorem

B. Floquet's theorem

C. Fundamental theorem

D. None of these

3. The fundamental matrix for the periodic systems is the product of a periodic matrix and

A. the solution of system of matrix with variable coefficients.

B. the solution of system of matrix with constant coefficients.

C. the solution of system of matrix with zero coefficients.

D. none of these.

4. If Φ is a fundamental matrix of the periodic system $\dfrac{dy}{dt} = A(t)y$ with $A(t+\omega) = A(t)$ then

A. $\Phi(t) = -P(t)e^{tR}$, where P is periodic nonsingular matrix with period ω.

B. $\Phi(t) = P(t)e^{-tR}$, where P is periodic nonsingular matrix with period ω.

C. $\Phi(t) = P(t)e^{tR}$, where P is periodic nonsingular matrix with period ω.

D. None of these

5. The adjoint for the linear homogeneous system $x' = A(t)x$, is given by

A. $x' = -A(t)x$

B. $x' = A^{Y}(t)x$

C. $x' = -A^{T}(t)x$

D. None of these

6. If $\boldsymbol{A} = -\boldsymbol{A}^{\boldsymbol{T}}$, then

A. $\phi\phi^T = -C$

B. $\phi\phi^T = C$

C. $\phi = \phi^T C$

D. None on these

7. If Φ be the fundamental matrix of linear homogenous system $x' = A(t)x,$ on $r_1 < t < r_2$, then ψ is fundamental matrix of its adjoint and C is a constant non-singular matrix if and only if

A. $\psi^T \Phi = C$

B. $\psi^T = C\Phi$

C. $\psi^T = -C\Phi$

D. $\psi^T \Phi = -C$

8. If Φ be a fundamental matrix of the homogenous linear vector differential equation $\dfrac{dx}{dt} = A(t)x$ where $x = (x_1(t), x_2(t), ...x_n(t))$ is n-dimensional vector function and A(t) = (a_{ij}(t)) is a continuous square matrix of order n on $r_1<t<r_2$. C is any constant non-singular matrix then

A. ΦC is also fundamental matrix.

B. $\Phi^{-1} C$ is also fundamental matrix.

C. ΦC^{-1} is also fundamental matrix.

D. $\Phi^{-1} C^{-1}$ is also fundamental matrix.

Answers for self Assessment

1.	B	2.	B	3.	B	4.	B	5.	C
6.	B	7.	A	8.	A				

Review Questions

Find the adjoint of the following homogeneous linear systems:

1. $\frac{dx}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} x$

2. $\frac{dx}{dt} = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix} x$

3. $\frac{dx}{dt} = \begin{bmatrix} -10 & 6 \\ -12 & 7 \end{bmatrix} x$

4. $\frac{dx}{dt} = \begin{bmatrix} -1 & 1 \\ -12 & 6 \end{bmatrix} x$

Further Readings

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Unit 10: Liapunov Function

CONTENTS

Objective

After studying this unit, you will be able to

- identify the concept of an autonomous system of differential equations.
- understand the stability analysis using Liapunov's second method.
- know about the Liapunov function for autonomous function.
- Apply definite properties of functions and connect with the Liapunov function.
- determine the stability behavior of solutions of linear and non-linear systems.

Introduction

In the previous unit, we analyzed the stability behavior of solutions of linear and weakly nonlinear systems, using the techniques of the variation of constant formula and integral inequalities. As a result, this analysis was conformed to a small neighborhood of the operating point, i.e., to stability in the small or local stability. Further, the techniques used require, in the case of linear systems, some explicit knowledge of solutions and, in the case of weakly nonlinear systems, a complete grasp of the solutions on the corresponding linear systems. These curbs apparently limit the applications of the techniques when investigating the stability behaviour of a physical system.

In this chapter, we shall introduce a completely different technique, known as *Liapunov's second method*, to determine the stability behaviour of solutions of linear and non-linear systems. The major advantage of this method is that stability in the large can be obtained without any prior knowledge of solutions. Although A.M. Liapunov, who introduced this method in 1892, used it only to establish simple stability theorems; his basic ideas have during the last 40 years been extensively exploited and effectively applied to entirely new problems in physics and engineering. Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time lag, power system analysis, time varying nonlinear feedback systems, and so on. Its chief characteristic is the construction of a scalar function, namely, the *Liapunov function*. Unfortunately, it is sometimes very difficult to find a proper Liapunov function for a given system. However, we shall indicate through examples and remarks, the limitations of this method, particularly in constructing a Liapunov function. Because the method yields stability information directly, i.e., without solving the differential equation, it is also known as *Liapunov's direct method.* In this chapter, we shall merely emphasize the basic ideas of the method and its applications to stability criteria of solutions of ordinary differential equations.

10.1 Autonomous systems

In this section, we shall consider an autonomous differential system of the form

$$x' = f(x), \tag{10.1.1}$$

Where $f \in C[R^n, R^n]$. Assume that f is smooth enough to ensure the existence and uniqueness of the solutions of (10.1.1). Let $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$ in some neighbourhood of the origin so that (10.1.1) admits the so-called zero solution ($x \equiv 0$) and the origin is an isolated critical point of (10.1.1).

Let Ω be an open set in R^n containing the origin. Suppose $V(x)$ is a scalar continuous function (that is, a real-valued continuous function in the variables $x_1, x_2, x_3, \ldots\ldots, x_n$) defined on Ω. For the sake of easy geometrical interpretation, we shall use the Euclidean norm

$$\|x\|_e^2 = x_1^2 + x_2^2 + \cdots \ldots\ldots + x_n^2$$

in our discussion. For convenience, we shall drop the subscript e.

The theory developed here is equally valid for the norm of a vector function $x \in R^n$ defined by $\|x\| = \sum_{i=1}^{n} |x_i|$

and call it the norm of x.

Definition 10.1.1: A scalar function $V(x)$ is said to be *positive definite* on the set Ω if and only if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and $x \in \Omega$.

Definition 10.1.2: A scalar function $V(x)$ is called to be *positive semi definite* on the set Ω when V has the positive sign throughout Ω, except at certain points (including the origin) where it is zero.

Definition 10.1.3: A scalar function $V(x)$ is called to be *negative definite* (negative semi definite) on the set Ω if and only if $-V(x)$ is positive definite (positive semi definite) on Ω.

Example 10.1.4 : The function is

(i) $V(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ is positive definite on R^3;

(ii) $V(x_1, x_2) = \frac{x_1^2}{x_1^4+1} + x_2^2$ is positive definite on R^2;

(iii) $V(x_1, x_2, x_3) = x_1^2 + (x_2 + x_3)^2$ is positive semi definite because it vanishes not only at the origin but also on the line $x_2 = -x_3, x_1 = 0$;

(iv) $V(x_1, x_2, x_3) = x_1^2 + x_2^2$ is positive definite in the plane, and positive semi definite on R^3 since it vanishes on the x_3 axis;

(v) $V(x_1, x_2) = x_1^2 + x_2^2 - (x_1^4 + x_2^4)$ is positive definite in the interior of the unit circle, clearly, $V \geq \|x\|^2 - \|x\|^4, \|x\| < 1$;

(vi) $V(x_1, x_2) = x_1^4 + x_2^4$ is positive definite since $V \geq \frac{1}{2} r^4$, where $r = (x_1^2 + x_2^2)^{1/2}$.

10.2 Quadratic forms

Let $V(x) = x^T B x = \sum_{i,j=1}^{n} b_{ij} x_i x_j$ (10.1.2)

be a quadratic form with the symmetric matrix $B = (b_{ij})$, that is, $b_{ij} = b_{ji}$.

To test the positive definiteness of $V(x)$ in (10.1.2), we can apply the Sylvester criterion which asserts that a necessary and sufficient condition for $V(x)$ in (10.1.2) to be positive definite is that determinants of all the successive principal minors of the asymmetric matrix $B = (b_{ij})$ be positive, that is,

$$b_{11} > 0, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \ldots\ldots, \begin{vmatrix} b_{11} & b_{12} & \ldots\ldots\ldots & b_{1n} \\ b_{21} & b_{22} & \ldots\ldots\ldots & b_{2n} \\ . & . & .. & . \\ b_{n1} & b_{n2} & & b_{nn} \end{vmatrix} > 0.$$

The derivative of V with respect to (10.1.10 is scalar product

$V^*(x) = grad\ V(x).f(x)$

$= \frac{\partial V}{\partial x_1} f_1(x) \frac{\partial V}{\partial x_2} f_2(x) + \cdots \ldots\ldots + \frac{\partial V}{\partial x_n} f_n(x)$ (10.1.3)

It should be noted that if $x = x(t)$ is any solution of (10.1.1),then by the chain rule and from (10.1.3), we can obtain

$\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x_1} x_1'(t) + \ldots\ldots\ldots\ldots \frac{\partial V}{\partial n} x_n'(t)$

$= \sum_{i=1}^{n} \quad \frac{\partial V}{\partial x_i} f_i(x(t)) = V^*(x(t))$. (10.1.4)

Here, $dV(x(t))/dt$ can be computed directly from (10.1.1).

Remarks 10.1.5: If there exists a positive definite scalar function $V(x)$ such that $V^*(x) \le 0$ [i.e. negative definite or derivative (10.13) w.r.t (10.1.1) is non-positive] then the zero solution (10.1.1) is stable.

Example 10.1.6 : (i) Consider the two-dimensional system

$x_1' = -x_2 + x_1(r^2 - x_1^2 - x_2^2),$ (10.1.5)

$x_2' = x_1 + x_2(r^2 - x_1^2 - x_2^2).$

Choose a positive definite function $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ on R^2. A simple computation gives

$V^*(x_1, x_2) = -(x_1^2 + x_2^2)(x_1^2 + x_2^2 - r^2).$

Obviously, V^* is negative definite when $r = 0$, and hence the zero solution of (10.1.5) is asymptotically stable. On the other hand, when $r \neq 0, V^*$ is positive definite in the region $x_1^2 + x_2^2 < r^2$. Therefore, the zero solution of (10.1.5) is unstable.

(ii) Consider the two-dimensional system

$x_1' = x_2 + x_1(x_1^2 + x_2^2)^{1/2}(x_1^2 + x_2^2 - 1)^2,$ (10.1.6)

$x_2' = -x_1 + x_2(x_1^2 + x_2^2)^{\frac{1}{2}}(x_1^2 + x_2^2 - 1)^2.$

Select a positive definite function $V(x_1, x_2) = (x_1^2 + x_2^2)^{\frac{3}{2}}$ on R^2. The derivative of V along the solutions of (10.1.6) is given by

$V^*(x_1, x_2) = 3(x_1^2 + x_2^2)^2(x_1^2 + x_2^2 - 1)^2.$

Clearly, V^* is positive definite on R^2, and hence the zero solution of (10.1.6) is asymptotically unstable.

(iii) Consider the system

$x_1' = -x_1 + x_2^2,$ (10.1.7)

$x_2' = -x_2 - x_1 x_2.$

Choose a positive definite function $V(x_1, x_2) = x_1^2 + x_2^2$ on R^2. Then, we have $V^* = -2V$; integrating this equation, we obtain

$V(x_1(t), x_2(t)) = V(x_1(0), x_2(0))e^{-2t}.$

Therefore, the zero solution of (10.1.7) is exponentially asymptotically stable.

(iv) Consider the two-dimensional system

$x_1' = -6x_2 - \frac{1}{4}x_1 x_2^2,$ (10.1.8)

$x_2' = 4x_1 - \frac{1}{6}x_2.$

Select a positive definite function $V(x_1, x_2) = 2x_1^2 + 3x_2^2$ on R^2. Then, we have $V^*(x_1, x_2) = -x_2^2(1 + x_1^2)$ is negative semi-definite (i.e. it vanishes on the x_1-axis) on R^2. Therefore, the zero solution of (10.1.8) is exponentially stable.

(v) Consider the second order differential equation

$$u'' + cu' + sin\, sin\, u = 0,$$

where c is a positive constant, Set $u = x_1$ and $u' = x_2$. Then, this equation is equivalent to the system

$x_1' = x_2, \qquad x_2' = -\, sin\, sin\, x_1 - cx_2$.

Now, select a scalar function

$$V(x_1, x_2) = \frac{1}{2}\left(\sqrt{c}x_1 + \frac{1}{\sqrt{c}}x_2\right)^2 + \frac{1}{c}\left(x_1^2 + \frac{1}{2}x_2^2\right).$$

This is clearly positive definite on R^2. After a little computation, we obtain

$$V^*(x_1, x_2) = -(x_1^2 + x_2^2) - (x_1 + \frac{2}{c}x_2)(sin\, sin\, x_1 - x_1).$$

It is easy to verify that V^* is negative definite in a sufficiently small neighbourhood of the origin. Hence, the zero solution of the given equation is asymptotically stable.

(vii) For the system

$$x_1' = 3x_1 + x_2^3, \qquad x_2' = -4x_2 + x_1^3,$$

Select a scalar function $V(x_1, x_2) = 4x_1^2 - 3x_2^2$. Then, V, together with its first partial derivatives, is continuous, $V(0,0) = 0$, and V has positive as well as negative values in any neighbourhood of the origin. Further,

$$V^*(x_1, x_2) = 24(x_1^2 + x_2^2) + (8x_1x_2^2 - 6x_1^3x_2).$$

Here, if $|x_1|$ and $|x_2|$ are sufficiently small, then the definiteness of V^* depends upon the first term within parentheses on the right-hand side. Since $V^*(0,0) = 0, V^*$ is positive definite in a small neighbourhood of the origin. Therefore, the critical point (0,0) of the given system is unstable.

Remarks 10.1.7: In (i)-(vi) of example Example (10.1.6), we picked up the Liapunov functions arbitrarily. Once a Liapunov function has been found in some region around the origin, it becomes possible to test the stability or asymptotic stability or instability of the zero solution of a given system. The failure to find such a Liapunov function does not of course mean that the stability cannot be determined. An important question therefore is the procedure to be adopted in selecting or constructing a Liapunov function. Though, in general, no satisfactory technique that provides the answer is known, particularly for nonlinear non autonomous systems, we shall nevertheless consider some of the methods applicable to linear systems and nonlinear autonomous systems.

10.3 Krasovskii's Method

Consider an autonomous differential system of the form

$$x' = f(x) \tag{10.1.9}$$

Where $f: R^n \to R^n, f(0) = 0, f(x) \neq 0$ for $x \neq 0$ in some neighbourhood of the origin, and $f(x)$ is differentiable with respect to $x_i(i = 1,2, \dots \dots, n)$. The real symmetric $n \times n$ matrix $B = (b_{ij})$ is said to be positive definite if and only if the quadratic form $x^T Bx$ is *positive definite*. It is well known (see the Sylvester criterion) that the real symmetric $n \times n$ matrix $B = (b_{ij})$ is positive definite if and only if

$$det\, det\, B_j = det\, det \begin{pmatrix} b_{11} & b_{12} & \dots\dots\dots & b_{1n} \\ b_{21} & b_{22} & \dots\dots\dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots\dots\dots & b_{n3} \end{pmatrix} > 0, \; j = 1,2, \dots\dots, n$$

where $det\, det\, B_j(j = 1,2, \dots, n)$ are the principal minors of $det\, det\, B$. The real symmetric $n \times n$ matrix B is called *negative definite* if and only if $-B$ is positive definite.

The Jacobian matrix of (10.1.9) is given by

$$J(x) = \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots\dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots\dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots\dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Define a matrix $M(x) = J^T(x) + J(x)$, where J^T is the transpose of J. A suitable Liapunov function for (10.1.9) is $V(x) = f^T(x)f(x)$. Clearly V is positive definite in some neighbourhood of the origin. If the matrix $M(x)$ is negative definite in some neighbourhood of the origin, then the zero solution of (10.1.9) is asymptotically stable. It should be observed that

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} = J(x)f(x).$

Then,

$$V^*(x) = f'^T f + f^T f' = f^T J^T f + f^T J f = f^T(J^T + J)f = f^T M f.$$

If $M(x)$ is negative definite in some neighbourhood of the origin, then V^* too is so, and hence the zero solution of (10.1.9) is asymptotically stable.

Example 10.1.7 : Determine the stability of the zero solution of

$x_1' = -x_1 - x_2 - x_1^3,$

$x_2' = x_1 - x_2 - x_2^3.$

For this system,

$$x = (x_1\ x_2\ ), \quad f = (f_1\ f_2\ ),$$

where $f_1 = = -x_1 - x_2 - x_1^3$ and $f_2 = x_1 - x_2 - x_2^3$.

Therefore,

$$J(x) = \ (-1 - 3x_1^2\ \ -1\ 1\ \ -1 - 3x_2^2\ ),$$

and hence

$M(x) = \ (-2 - 6x_1^2\ 0\ 0\ \ -2 - 6x_2^2\ ).$

Since $M(x)$ is negative definite for all $x \in R^2$, Krasovskii's method ensures that the zero solution of the given system is asymptotically stable.

Remarks 10.1.8: Krasovskii's method guarantees the asymptotic stability of the zero solution of a given system if $M(x)$ is negative definite, but does not lead to any answer when $M(x)$ is not negative definite.

Remarks 10.1.9: The negative definiteness of $M(x)$ requires that this matrix have nonzero elements on its main diagonal.

The application of Krasovskii's method fails if $f_i(x)$ does not involve x_i. For example, the method does not cover the n-th order $(n \geq 2)$ differential equation

$u^{(n)} + g\left(u, u', \dots\dots, u^{(n-1)}\right) = 0.$

10.4 Construction of A Liapunov Function for Linear Systems with constant Coefficients

The method we now give is very helpful when studying the stability of perturbed linear systems.

Consider the differential system

$$x' = Ax, \tag{10.2.1}$$

Where x is an n −vector and A is a real $n \times n$ constant matrix. Let the characteristic roots of A be distinct; we shall denote the real characteristic roots by $\lambda_1, \lambda_2, \dots\dots, \lambda_r$ and the complex characteristic roots by $u_1, u_2, \dots\dots . u_m$, such that

$u_i = \underline{u}_{i-1}, \quad i = 2,4, \dots\dots\dots, m,$

where $r + m = n$. Define the real numbers α_k and β_k such that $u_{k-1} = \alpha_{k-1} + i\beta_{k-1,}\ k = 2,4, \dots\dots.., m$. We now find a non-singular constant matrix $T^{-1}AT$ is of the form

$T^{-1}AT = (\lambda_1\ 0\ \cdots\ 0\ \ 0\ \ \ldots\ \ldots\ \cdots\ 0\ 0\ 0\ \lambda_2\ \ \ddots\ \ \alpha_1\ \ \beta_1\ \ -\beta_1\ \ \alpha_1\ \ \ \alpha_2\ \beta_2\ \ -\beta_2\ \alpha_2\ \ \ddots\ \ 0\ )$.

To do this, consider the transformation $x = Ty$. Now, (10.2.1) reduces to

$$y' = Dy \qquad (10.2.2)$$

where $D = T^{-1}AT$. For system (10.2.2) to be asymptotically stable, we require all the diagonal elements of D to be negative. Select a Liapunov function

$$V(y) = (y, By), \qquad (10.2.3)$$

where (,) is the usual inner product and B a real $n \times n$ constant symmetric matrix. Then,

$$V^*(y) = (y', By) + (y, By') = (Dy, By) + (y, BDy)$$
$$= (y, (D^T B + BD)y), \qquad (10.2.4)$$

where $D^T$is the transpose of D. In order to ensure that V^* is negative definite, we require

$$V^*(y) = -(y, y) = -\sum_{j=1}^{n} y_j^2, \qquad (10.2.5)$$

where y_j are the components of y. To see this compatibility of relations (10.2.4) and (10.2.5), we assume that the condition

$$D^T B + BD = -I, \qquad (10.2.6)$$

where I is the identity matrix, holds. After a little computation, the matrix equation (10.2.6) yields

$$B = \Big(-\frac{1}{2\lambda_1}\quad 0 \quad 0\ 0 \quad -\frac{1}{2\lambda_2} \quad 0 \quad 0 \quad -\frac{1}{2\lambda_3} \quad \ddots \quad -\frac{1}{2\lambda_r} \quad \ddots \quad 0 \quad -\frac{1}{2\alpha_1} \quad \ddots \Big)$$

Let us assume that the matrix A is stable so that all λ_i and α_i are negative, and hence all the diagonal elements of B positive. Then, from (10.2.3), it follows that V is positive definite. In fact, $V(y)$ takes the form

$$V(y) = -\frac{1}{2\lambda_1}y_1^2 - \frac{1}{2\lambda_2}y_2^2 - \cdots \ldots \ldots . -\frac{1}{2\lambda_r}y_r^2 - \frac{1}{2\alpha_1}(y_{r+1}^2 + y_{r+2}^2) - \cdots \ldots \ldots$$

Example 10.1.7 : Construct a Liapunov function for the three-dimensional system

$$x' = Ax,$$

where

$$A = (0\ 1\ 0\ 0\ 0\ 1\ -12\ -20\ -9\ )$$

The characteristic equation $det\ det\ (A - \lambda I) = 0$ has the roots $\lambda_1 = -1, \lambda_2 = -2, and\ \lambda_3 = -6$. Then, it follows that

$$V_1 = (1\ -1\ 1\ ),\ V_2 = (1\ -2\ 4\ ),\quad V_1 = (1\ -6\ 36\ ).$$

Therefore,

$$T = (1\ 1\ 1\ -1\ -2\ -6\ 1\ 4\ 36\ )$$

It can be easily shown that

$T^{-1}AT = (-1\ 0\ 0\ 0\ \ -2\ \ 0\ 1\ 4\ -6\ )$.

The transformation $x = Ty$ reduces the given system to $y' = Dy$,
(10.2.7) where $D = T^{-1}A\,T$.

To find a Liapunov function for system (10.2.7), we look for a matrix B such that $D^T B + BD = -I$. After a simple computation, we get

$$B = \left(\frac{1}{2}\ 0\ 0\ 0\ \frac{1}{4}\ \ 0\ 0\ 0\ \frac{1}{12}\right).$$

Thus, the Liapunov function for (10.2.7) is

$Vy = (y, By) = \frac{1}{2}y_1^2 + \frac{1}{4}y_2^2 + \frac{1}{12}y_3^2.$

To get a Liapunov function for given system, we transform the variable y back into the variable x.

Summary

- The definite properties of function are discussed.
- Stability analysis using Liapunov's second method is elaborated with the help of examples.
- The construction of Liapunov's function is discussed and suitable examples are solved.
- Determine the stability behavior of solutions of linear and non-linear systems.

Keywords

- Autonomous linear system
- Liapunov function
- Stability
- Liapunov second method
- Krasovskii's method

Self Assessment

1. The differential system of the form $x' = f(x)$ where $f \in C[R^n, R^n]$ is

A. An Autonomous system

B. Non Autonomous system

C. Non homogeneous system

D. None of these

2. The differential system of the form $x' = f(x)$ where $f \in C[R^n, R^n]$ has

A. Number of critical point

B. An isolated critical point on origin

C. No critical point on origin

D. None of these

3. The function $V(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ is

A. Positive definite on R^3.

B. Negative definite on R^3.

C. Both (a) and (b)

D. None of these

4. The function $V(x_1, x_2, x_3) = x_1^2 + (x_2 + x_3)^2$ is

A. Positive definite

B. Negative definite

C. Positive semi definite

D. Negative semi definite.

5. The V(x) be a Liapunov function if V(x) is positive definite and V*(x) is negative definite where

A. V*(x) = grad V(x)/ f(x)
B. V*(x) = grad V(x). f(x)
C. V*(x) = -grad V(x). f(x)
D. V*(x) = -grad V(x)/ f(x)

6. A scalar function $V(x)$ is positive definite on the set Ω if and only if

A. $V(x) \geq 0 \;\; for \;\; x \in \Omega$

B. $V(x) \leq 0 \;\; for \; x \in \Omega$

C. $V(0) = 0 \; and \; V(x) < 0 \;\; for \;\; x \neq 0 \; and \;\; x \in \Omega$

D. $V(0) = 0 \; and \; V(x) > 0 \;\; for \;\; x \neq 0 \; and \;\; x \in \Omega$

7. If D is diagonalizable form for linear homogeneous system $x' = Ax$, then Liapunov function is

A. $V(y) = (y, By)$ where $D^T B = -I$

B. $V(y) = (y, By)$ where $BD = -I$

C. $V(y) = (y, By)$ **where** $D^T B + BD = -I$

D. $V(y) = (y, By)$ where $D^T B + BD = I$

8. The function $3x_1^2 + x_2^2 + 4x_3^2$ is

A. positive definite
B. negative definite
C. semi positive definite
D. semi negative definite

9. For the system of differential equation
$$x_1' = -x_1 + x_2^2, \quad x_2' = -x_2 - x_1 x_2$$

A. The zero solution is asymptotically unstable
B. The zero solution is asymptotically stable
C. The zero solution is neither stable nor unstable
D. None of these

10. For the system of differential equation
$$x_1' = 3x_1 + x_2^3, \quad x_2' = -4x_2 + x_1^3$$

A. The zero solution is asymptotically unstable
B. The zero solution is asymptotically stable
C. The zero solution is neither stable nor unstable
D. None of these

11. The two dimensional system
$$x_1' = -6x_2 - x_1 x_2^2 / 4, \quad x_2' = 4x_1 - x_2 / 6$$

A. The zero solution is unstable

B. The zero solution is stable

C. The zero solution is neither stable nor unstable

D. None of these

12. The Liapunov function for the $\begin{aligned} x_1' &= -6x_2 - x_1x_2^2/4, \\ x_2' &= 4x_1 - x_2/6 \end{aligned}$ is

A. $V(x_1, x_2) = x_1^2 + x_2^2$

B. $V(x_1, x_2) = 2x_1^2 + 3x_2^2$

C. $V(x_1, x_2) = x_1^2 + 2x_2^2$

D. *None of these*

Answers for Self Assessment

1.	A	2.	B	3.	A	4.	C	5.	B
6.	D	7.	C	8.	A	9.	B	10.	B
11.	B	12.	B						

Review Questions

1. Construct a Liapunov function for the three-dimensional system

$$x' = Ax,$$

where, $A = (0\ 1\ 0\ 0\ 0\ 1\ -8\ -14\ -7\ )$.

2. Construct a Liapunov function for the three-dimensional system

$$x' = Ax,$$

where, $A = (0\ 1\ 0\ 0\ 0\ 1\ -6\ -11\ -6\ )$.

3. Use Karsovskii's method to determine the asymptotic stability of the zero solution of

$x_1' = -x_1$

$x_2' = x_1 - x_2 - x_2^3.$

Further Readings

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Unit 11: Linear Second Order Differential Equation

CONTENTS

Objectives

After studying this unit, you will be able to

- identify the concept of second order linear differential equation.
- understand about the adjoint and self adjoint equation.
- know about the conversion of differential equation into self adjoint form.
- apply the Abel's formula to know the behaviour of solutions.
- determine solutions of linear differential equation using Abel's formula.

Introduction

In the previous chapters, we discussed in detail the qualitative behavior of solutions of general linear and non-linear differential systems. We shall now confirm ourselves to second order differential equations which fid ample applications in many scientific investigations of practical importance. In particular, we shall concentrate on general second order differential equations of the type

$$(r(t)x')' + p(t)x = 0 \qquad (11.0.1)$$

$$x'' + g(x, x')x' + h(x) = e(t) = 0 \qquad (11.0.2)$$

It should be observed that both (11.0.1) and (11.0.2) are time varying; however, the former is linear and the latter nonlinear. Such equations are frequently encountered as mathematical models of most dynamic processes in electromechanical systems. Since these equations are only of the second order, we would naturally be inclined to compute their solutions explicitly or numerically. However, as we know from practice, there are very few such equations, e.g. linear equations with constant coefficients, for which this can be effectively done. In most instances, this can be accomplished only under very restrictive conditions. The problem therefore is to find those techniques that will be used in obtaining some qualitative information about the elusive solutions of the equations of the type (11.0.1) and (11.0.2).

11.1 Linear Differential Equations

By a linear differential equation of order n, we shall mean an equation of the form

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ \ldots\ldots\ldots..+a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = f(t) \qquad (11.1.1)$$

where $a_0(t) \neq 0$. Of these, second order linear differential equations are of special theoretical and practical interest.

The general second order linear differential equation is

$$a_0(t)\frac{d^2 x}{dt^2} + a_1(t)\frac{dx}{dt} + \ \ldots\ldots.+a_2(t)x = f(t) \qquad (11.1.2)$$

or $a_0(t)x'' + a_1(t)x' + \ \ldots\ldots.+a_2(t)x = f(t)$

(prime 'denote the differentiation with respect to 't')

in which $a_0(t) \neq 0$ and $a_0(t)$, $a_1(t)$, $a_2(t)$ are real functions of t alone. Without any loss of generality, we may assume that the leading coefficient $a_0(t)$ to be 1 since this can always be accomplished by division. It should be noted that most of the ideas and procedures we discuss can be generalized at once to linear equations of higher order with no change in the underlying principle. Just to attain as much simplicity as possible, without distorting the main ideas, we prefer to limit ourselves to second order equation like

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = f(t) \qquad (11.1.3)$$

We further restrict ourselves for detailed consideration of actual methods for solving (11.1.3) and assume ourselves that equation (11.1.3) really has a solution.

11.2 Adjoint Equation

Consider the second order homogeneous linear differential equation

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0 \qquad (11.2.1)$$

in which $a_0(t)x'' \neq 0, a_1(t) and\ a_2(t)$ are continuous functions of t and $a_1(t)$ is differentiable on $a \leq t \leq b$.

The adjoint equation to (11.2.1) is

$$(a_0(t)x)'' - (a_1(t)x)' + a_2(t)x = 0$$

Or $a_0(t)x'' + 2a_0'(t)x' + a_0''(t)x - a_1'(t)x - a_1(t)x' + a_2(t)x = 0$

$$a_0(t)x'' + [2a_0'(t) - a_1(t)]x' + [a_0''(t) - a_1'(t) + a_2(t)]x = 0 \qquad (11.2.2)$$

11.2.1: It should be noted the adjoint equation of the adjoint equation (11.2.2) is always the original equation (11.2.1) itself.

11.2.2: Find the adjoint equation to each of the following equations:

(a) $t^2x'' + 3tx' + 3x = 0$
(b) $(2t+1)x'' + t^3x' + x = 0$
(c) $t^2x'' + 7tx' + 8x = 0$

Solution: For equation (a), we have $a_0(t) = t^2; a_1(t) = 3t; a_2(t) = 3$

$$a_0'(t) = 2t,\ a_0''(t) = 2, a_1'(t) = 3$$

Therefore, required adjoint equation to (a) becomes

$$t^2x'' + [4t - 3t]x' + [2 - 3 + 3]x = 0$$
$$t^2x'' + tx' + 2x = 0$$

For equation (b), we have $a_0(t) = 2t + 1; a_1(t) = t^3; a_2(t) = 1$

$$a_0'(t) = 2,\ a_0''(t) = 0, a_1'(t) = 3t^2$$

Therefore, required adjoint equation to (b) becomes

$$(2t+1)x'' + [4 - t^3]x' + [1 - 3t^2]x = 0$$

For equation (c), we have $a_0(t) = t^2; a_1(t) = 7t; a_2(t) = 8$

$$a_0'(t) = 2t,\ a_0''(t) = 2, a_1'(t) = 7$$

Therefore, required adjoint equation to (b) becomes

$$t^2x'' - 3tx' + 3x = 0$$

11.3 Self Adjoint Equation

The second order homogeneous linear differential equation (11.2.1) i.e.

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0$$

is called self-adjoint if it is identical with its adjoint equation (11.2.2) i.e.

$$a_0(t)x'' + [2a_0'(t) - a_1(t)]x' + [a_0''(t) - a_1'(t) + a_2(t)]x = 0$$

Here we observe that if $a_1(t) = a_0'(t)$ then $a_1'(t) = a_0''(t)$.

As a result of which equation (11.2.2) becomes identically (11.2.1). Hence we are in position to define the self-adjoint equation in other way as given below.

The differential equation (11.2.1) i.e.,

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0$$

is said to be in self adjoint form if $a_1(t) = a_0'(t)$ and in that case equation (11.2.1) may be written in a particular form

$$(a_0(t)x')' + a_2(t)x = 0 \qquad (11.3.1)$$

Theorem 11.3.1: Consider a second order linear differential equation (11.2.1) i.e.

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0$$

where (i) $a_0(t)$ has a continuous second order derivative, $a_0(t) \neq 0$

(ii) $a_1(t)$ has a continuous first order derivative

and (iii) $a_2(t)$ has a continuous on $a \leq t \leq b$.

The necessary and sufficient condition for (11.2.1) to be self-adjoint equation is that

$a_1(t) = a_0'(t) \; on \; a \leq t \leq b$.

Proof: First we suppose that condition $a_1(t) = a_0'(t)$ is true for equation (11.2.1)

$a_1(t) = a_0'(t) \quad \Rightarrow a_1'(t) = a_0''(t)$

Making these substitutions in (11.2.2), we get

$$a_0(t)x'' + [2a_1(t) - a_1(t)]x' + [a_1'(t) - a_1'(t) + a_2(t)]x = 0$$

Or

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0$$

which is equation (11.2.1).

⇒ Equation (11.2.1) and (11.2.2) are identical.

⇒ Equation (11.2.1) is self adjoint.

Conversely, we suppose that equation (11.2.1) is elf adjoint.

i.e. Equation (11.2.1) and (11.2.2) are identical

Equating the coefficient of x' and x in (11.2.1) and (11.2.2), we get

$$2a_0'(t) - a_1(t) = a_1(t); \quad a_0''(t) - a_1'(t) + a_2(t) = a_2(t)$$

Out of them, second relation is $a''_0(t) = a_1'(t)$

which on integration gives

$a_0'(t) = a_1(t) + c$ [c is arbitrary constant]

The constant c vanishes on account of the first of these two relations. Hence $a_0'(t) = a_1(t)$ which completes the proof.

11.3.2: Prove that the Legendre's equation

$(1 - t^2)x'' - 2tx' + n(n+1)x = 0$ is self adjoint.

Solution: Here $a_0(t) = (1-t^2),\ a_1(t) = -2t,\ a_2(t) = n(n+1)$

and we observe that $a_0'(t) = -2t = a_1(t)$.

⇒ Legendre's Equation is self adjoint and can also be written in the form

$[(1-t^2)x']' + n(n+1)x = 0$.

11.3.3: Check the self adjoint character of the following equations

(a) $t^3x'' + 3t^2x' + x = 0$
(b) $sint\ x'' + \cos t\, x' + 2x = 0$.

(c) Solution: Equation (a) $a_0(t) = t^3 \Rightarrow a_0'(t) = 3t^2 = a_1(t)$

and for equation (b) $a_0(t) = \sin t \Rightarrow a_0'(t) = \cos t = a_1(t)$

Equations (a) and (b) are self adjoint.

Theorem 11.3.4: Let the coefficient $a_0(t),\ a_1(t)$ and $a_2(t)$ appearing in the differential equation (11.2.1) i.e.

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0$$

are continuous on $a \le t \le b$ and $a_0(t) \ne 0$ on $a \le t \le b$ then this equation can be transformed into the equivalent self-adjoint equation

$$\frac{d}{dx}[r(t)x'] + p(t)x = 0 \qquad (11.3.2)$$

on $a \le t \le b$ where

$$r(t) = e^{\int \frac{a_1(t)}{a_0(t)}dt} \text{ and } p(t) = \left(\frac{a_2(t)}{a_0(t)}\right) r(t) \qquad (11.3.3)$$

By multiplication throughout by the factor

$\frac{1}{a_0(t)} e^{\int \frac{a_1(t)}{a_0(t)}dt}$.

Proof: In order to get the form (11.3.2) of equation (11.2.1) we have to multiply equation (11.2.1) throughout by a suitable factor $H(t)$ obtained by a quadrature.

Consider the function $H(t)$ determined by the equation

$$\frac{d}{dt}[a_0(t)H(t)] = a_1(t)H(t)$$

$$a_0(t)H'(t) + a_0'H(t) = a_1(t)H(t)$$

Division throughout by $a_0(t)H(t)$ gives

$$\frac{H'(t)}{H(t)} = -\frac{a_0'(t)}{a_0(t)} + \frac{a_1(t)}{a_0(t)}$$

which on integration gives

$$\log H(t) = -\log a_0(t) + \int \frac{a_1(t)}{a_0(t)} dt$$

so that H(t) = $\frac{1}{a_0(t)} e^{\int \frac{a_1(t)}{a_0(t)}dt}$

Now multiplying the equation (11.2.1) by $H(t)$, we follow

$$e^{\int \frac{a_1(t)}{a_0(t)}dt} x'' + \frac{a_1(t)}{a_0(t)} e^{\int \frac{a_1(t)}{a_0(t)}dt} x' + \frac{a_2(t)}{a_0(t)} e^{\int \frac{a_1(t)}{a_0(t)}dt} x = 0$$

$r(t) = e^{\int \frac{a_1(t)}{a_0(t)}dt}$ and $p(t) = \left(\frac{a_2(t)}{a_0(t)}\right) r(t)$

we get

$$\frac{d}{dt}[r(t)x'] + p(t)x = 0$$

Thus from (11.3.2) of equation (11.2.1) (in which r(t)>0) is extremely useful and plays a central role in the calculus of variations. It arises very frequently in mechanics. In order to study the behaviour of the solutions of the linear differential equation of the second order, we shall take of this form throughout the present chapter.

We shall be concerned, henceforth, with equation (11.3.2) where $r(t) > 0$ and $r(t)$ and $p(t)$ are continuous on some interval (a, b). The theorem and the proofs which follow will not require the existence of the derivative of $r(t)$.

11.3.4: Transform each of the following equations into an equivalent self adjoint equation

(a) $t^2x'' + tx' + x = 0$
(b) $f(t)x'' + g(t)x' = 0$
(c) $t^2x'' - 2tx' + 2x = 0$

Solution: For equation (a), we have $a_0(t) = t^2, a_1(t) = t, a_2(t) = 1$

Multiplication factor = $\frac{1}{a_0(t)}e^{\int\frac{a_1(t)}{a_0(t)}dt} = \frac{1}{t^2}e^{\int\frac{t}{t^2}dt} = \frac{1}{t^2}e^{\int\frac{1}{t}dt} = \frac{1}{t^2}e^{\log t} = \frac{1}{t^2}t = \frac{1}{t}$.

Multiplying the equation (a) by $\frac{1}{t}$ on any interval $a \le t \le b$, excluding the point $t = 0$,

we get $tx'' + x' + \frac{1}{t}x = 0$.

This equation is self-adjoint equation and may be written in the form

$[tx']' + \frac{1}{t}x = 0$.

For equation (b), we have $a_0(t) = f(t), a_1(t) = g(t), a_2(t) = 0$

Multiplication factor = $\frac{1}{a_0(t)}e^{\int\frac{a_1(t)}{a_0(t)}dt} = \frac{1}{f(t)}e^{\int\frac{g(t)}{f(t)}dt}$.

Multiplying the equation (b) by $\frac{1}{f(t)}e^{\int\frac{g(t)}{f(t)}dt}$ on any interval $a \le t \le b$, we get

$e^{\int\frac{g(t)}{f(t)}dt}x'' + \frac{g(t)}{f(t)}e^{\int\frac{g(t)}{f(t)}dt}x' = 0$.

This equation is self-adjoint equation and may be written in the form

$\left[e^{\int\frac{g(t)}{f(t)}dt}.x'\right]' = 0$.

For equation (c), we have $a_0(t) = t^2, a_1(t) = -2t, a_2(t) = 2$

Multiplication factor = $\frac{1}{a_0(t)}e^{\int\frac{a_1(t)}{a_0(t)}dt} = \frac{1}{t^2}e^{\int\frac{-2t}{t^2}dt} = \frac{1}{t^2}e^{\int\frac{-2}{t}dt}$

$= \frac{1}{t^2}e^{-2\log t} = \frac{1}{t^2}\frac{1}{t^2} = \frac{1}{t^4}$.

Multiplying the equation (b) by $\frac{1}{t^4}$ on any interval $a \le t \le b$, excluding the point $t = 0$, we get

$\frac{1}{t^2}x'' - \frac{2}{t^3}x' + \frac{2}{t^4} = 0$.

This equation is self-adjoint equation and may be written in the form

$\left[\left(\frac{1}{t^2}\right).x'\right]' + \left(\frac{2}{t^4}\right)x = 0$.

11.4 Abel's Formula

Theorem 11.4.1: If $u(t)$ and $v(t)$ are any two solutions of self adjoint equation (11.3.2) then

$r(t)[u(t)v'(t) - u'(t)v(t)] \equiv k, a\ constant$ (11.4.1)

Proof: Since $u(t)$ and $v(t)$ are solutions of equation (11.3.2), we have

$[r(t)u'(t)]' + p(t)u(t) = 0$ (11.4.2)

and

$[r(t)v'(t)]' + p(t)v(t) = 0$ (11.4.3)

Multiplying (11.4.2) by $-v(t)$, (11.4.1) by $u(t)$ and then adding, we get

$$u(t)[r(t)v'(t)]' - v(t)[r(t)u'(t)]' = 0$$

which on integration by parts under limits a to t, gives

$r(t)[u(t)v'(t) - u'(t)v(t)] \equiv r(a)[u(a)v'(a) - u'(a)v(a)]$ (11.4.4)

$\Rightarrow$ The right hand member of (11.4.4) is constant

$r(t)[u(t)v'(t) - u'(t)v(t)] \equiv k$, a constant

11.4.2: Formula (11.4.1), known as Abel's formula, is useful in finding all solutions of a linear differential equation of order two, when one of its solutions is known. The same is proved in the following theorem.

The expression inside the bracket of left hand side is nothing but the Wronskian W of two solutions of $u(t)$ and $v(t)$ of equation (11.3.2)

i.e. $W = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}$

It means that the constant k appearing in Abel's formula is zero if and only if the two solutions $u(t)$ and $v(t)$ of equation (11.3.2) are linearly dependent.

Theorem 11.4.2: Let $u(t)$ be a solution of a linear differential equation of order two represented in self adjoint form (11.3.2) i.e.

$[r(t)u'(t)]' + p(t)u(t) = 0$

Where $r(t) \neq 0$ and p are continuous functions on an interval I. Then a second linearly independent solution $v(t)$ of equation (11.3.2) is given by

$v(t) = u(t)\int \frac{dt}{r(t)u_1^2(t)}$.

Proof: Since $u(t) \neq 0$ is a solution of equation (11.3.2), there exists a second order linearly independent solution $v(t)$ of equation (11.3.2), satisfying Abel's formula (11.4.1) such that, for all

t in I

$$r[uv' - u'v] \equiv 1$$

$$v' - \frac{u'}{u}v = \frac{1}{ru}$$

This is linear differential equation of order one in v. Solving this equation, we obtain

$v(t) = u(t)\int \frac{dt}{r(t)u_1^2(t)}$.

Summary

- The second order linear differential equation and its behaviour is discussed.
- The adjoint and self adjoint equations are elaborated with the help of examples.
- The conversion of differential equation into self adjoint form is derived and examples are solved.
- Determined the Abel's formula for self adjoint equation solutions and linear dependent or independent behaviour of solution is also discussed.

Keywords

- Linear differential equation
- Second order differential equation
- Adjoint equation
- Self adjoint equation
- Abel formula
- Linear dependent and independent solutions

Self Assessment

1. The equation $t^2x'' - 2tx' + 2x = 0$ reduces into an equivalent self adjoint equation as
A. $t^{-2}x'' - 4tx' + 2x = 0$
B. $2t^{-2}x'' - 4t^{-3}x' + t^{-4}x = 0$
C. $2t^{-2}x'' + 2t^{-3}x' - t^{-4}x = 0$
D. $2t^{-2}x'' + 4t^{-3}x' - 2t^{-4}x = 0$

2. The adjoint of the differential equation $t^2x'' + 3tx' + 3x = 0$ is
A. $t^2x'' + tx' + 3x = 0$
B. $t^2x'' - tx' - 3x = 0$
C. $t^2x'' + tx' + 5x = 0$
D. $t^2x'' + 2tx' + 3x = 0$

3. The coefficients $a_0(t),\ a_1(t)\ \&\ a_2(t)$ appearing in the differential equation $a_0(t)x'' + a_1(t)x' + a_2(t)x = 0$ are continuous on $a \le t \le b$ and $a_0(t) \ne 0$ on $a \le t \le b$ then this equation can be transformed into the equivalent self-adjoint equation by multiplying
A. $e^{\int \frac{a_1(t)}{a_0(t)}dt}$
B. $\frac{a_2(t)}{a_0(t)} e^{\int \frac{a_1(t)}{a_0(t)}dt}$
C. $-\frac{a_2(t)}{a_0(t)} e^{\int \frac{a_1(t)}{a_0(t)}dt}$
D. $e^{-\int \frac{a_1(t)}{a_0(t)}dt}$

4. Which of the following is second order linear differential equation?
A. $u'' + g(t)u' + f(t)u^2 = h(t)$
B. $(p(t)u')' + q(t)u = h(t)$
C. $(p(t)u)' + q(t)u = h(t)$
D. None of these

5. The equation $\boldsymbol{a_0(t)u'' + a_1(t)u' + a_2(t)u = 0}$ is self adjoint if
A. $a_0'(t) = -a_1(t)$
B. $a_0'(t) = a_1(t)$
C. $a_0(t) = a_1'(t)$
D. None of these

6. The adjoint of the equation $\boldsymbol{a_0(t)u'' + a_1(t)u' + a_2(t)u = 0}$ is
A. $[a_0(t)u]'' + [a_1(t)u]' + a_2(t)u = 0$
B. $[a_0(t)u]'' - [a_1(t)u]' - a_2(t)u = 0$
C. $[a_0(t)u]'' - [a_1(t)u]' + a_2(t)u = 0$
D. None of these

7. Every homogeneous second order linear differential equation can be reduced to
A. Adjoint form

B. Self-adjoint form
C. Both adjoint and self adjoint form
D. None of these

8. The equation $t^2x'' + tx' + x = 0$ reduces into an equivalent self adjoint equation as
A. $(tx')' - \frac{1}{t}x = 0$
B. $(tx')' + \frac{1}{t}x = 0$
C. $(tx')' + tx = 0$
D. None of these

9. If u and v be any two solutions of a self adjoint equation of order two of the form $[\boldsymbol{r(t)u'}]' + \boldsymbol{p(t)u} = \boldsymbol{0}$, where $\boldsymbol{r(t)} \neq \boldsymbol{0}$ and $\boldsymbol{p}, \boldsymbol{r'}$ are continuous functions. Then
A. $r(t)[u(t)v'(t) - v(t)u'(t)] = 0$
B. $r(t)[u(t)v'(t) - v(t)u'(t)] = k$
C. $r(t)[u(t)v'(t) + v(t)u'(t)] = -k$
D. None of these

10. If u and v be any two solutions of a self adjoint equation of order two of the form $[\boldsymbol{r(t)u'}]' + \boldsymbol{p(t)u} = \boldsymbol{0}$, where $\boldsymbol{r(t)} \neq \boldsymbol{0}$ and $\boldsymbol{p}, \boldsymbol{r'}$ are continuous functions. Then $\boldsymbol{r(t)}[\boldsymbol{u(t)v'(t)} - \boldsymbol{v(t)u'(t)}] = \boldsymbol{k}$ is known as
A. Ricatti formula
B. Variational formula
C. Abel's formula
D. None of these

11. If the formula $\boldsymbol{r(t)}[\boldsymbol{u(t)v'(t)} - \boldsymbol{v(t)u'(t)}] = \boldsymbol{k}$ then the u and v be
A. Linear independent
B. Linear dependent
C. Constant
D. None of these

12. If u_1 be the one solution of second order linear differential equation in self adjoint form $[\boldsymbol{r(t)u'}]' + \boldsymbol{p(t)u} = \boldsymbol{0}$. Then another solution
A. $u_2 = -u_1 \int \frac{dt}{r(t)u_1^2(t)}$
B. $u_2 = \int \frac{dt}{r(t)u_1^2(t)}$
C. $u_2 = u_1 \int \frac{dt}{r(t)u_1^2(t)}$
D. None of these

13. If in the formula $\boldsymbol{r(t)}[\boldsymbol{u(t)v'(t)} - \boldsymbol{v(t)u'(t)}] = \boldsymbol{k}, \boldsymbol{k} = \boldsymbol{0}$ then the u and v be
A. Linear independent
B. Linear dependent
C. Constant
D. None of these

Answer for Self Assessment

1.	B	2.	C	3.	B	4.	B	5.	B
6.	C	7.	C	8.	B	9.	B	10.	C
11.	A	12.	C	13.	B				

Review Questions

1. Show that every homogeneous linear differential equation of order two can be reduced to self-adjoint form.
2. Reduce the following equation to self adjoint form:
   (i) $tx'' - x' + t^2x = 0$
   (ii) $x'' - 3x' + 2x = 0$
   (iii) $t^2x'' + tx' + (t^2 - n^2)x = 0$
   (iv) $(1 - t^2)x'' - 2tx' + (n^2 + n)x = 0$

Further Reading

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Unit 12: The Sturmian Theory

CONTENTS

Objective:

After studying this unit, you will be able to

- identify the concept of common zero of solutions of second order linear differential equations.
- understand about the Sturm separation and comparison theorem
- know about the separation of zeros of solutions of self adjoint equation.
- determine the comparison of solutions of two self adjoint equations.

Introduction

In this chapter, more theory related to the solution of second order self adjoint equation of the form

$$\frac{d}{dt}[r(t)x'] + p(t)x = 0 \qquad (12.0.1)$$

will be discussed and more properties of solutions will be determined.

12.1 Sturm Separation Theorem

Lemma 12.1.1: If the two solutions $u(t)$ and $v(t)$ have common zero, they are linearly dependent. Conversely, if $u(t)$ and $v(t)$ are linearly dependent solutions, neither identically zero, and if one of them vanishes at $t = t_0$, so does other.

Proof: First, we assume that $t = t_0$ is the common zero of two solutions $u(t)$ and $v(t)$ i.e. $u(t_0) = 0$, $v(t_0) = 0$.

In accordance with Abel's formula, we have

$r(t)[u(t)v'(t) - u'(t)v(t)] \equiv k, a\ constant.$

Replacing t by t_0 in this formula, we follow $k = 0$.

$\Rightarrow$ Two solutions $u(t)$ and $v(t)$ are linearly dependent.

Conversely, we assume that the two solutions $u(t)$ and $v(t)$ are linearly dependent.

$\Rightarrow$ There exist constants c_1 and c_2 (not both zero) such that

$c_1 u(t) + c_2 v(t) = 0.$

As neither $u(t)$ nor $v(t)$ is identically zero, we follow that both c_1 and c_2 are different from zero.

Hence if $u(t_0) = 0$ then $v(t_0) = 0$

And if $v(t_0) = 0$ then $u(t_0) = 0$.

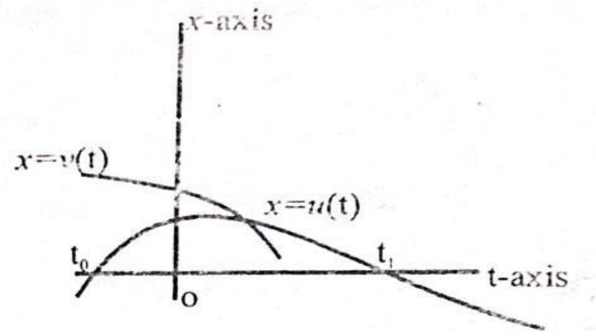
i.e. $u(t)$ and $v(t)$ have a common zero.

Theorem 12.1.2: If $u(t)$ and $v(t)$ are linearly independent solutions of equation (12.0.1) i.e. $\frac{d}{dt}[r(t)x'] + p(t)x = 0$, then between two consecutive zeros of $u(t)$ there will be precisely one zero of $v(t)$.

Proof: let us suppose that the two consecutive zeros of the solution of u(t) are $t = t_0$ and $t = t_1 (t_0 < t_1)$, whose existence is being supposed by the theorem. Without any loss of generality, we may assume that

$u(t) > 0$ in the range $t_0 < t < t_1$

and $u'(t_0) > 0$ and $u'(t_1) < 0$.



$u(t) > 0$ in the range $t_0 < t < t_1$ $\Rightarrow$ The graph of the curve $x = u(t)$ in the range $t_0 < t < t_1$ lies above the t-axis.

$u'(t_0) > 0 \Rightarrow$ The tangent to the curve $x = u(t)$ at $t = t_0$ makes an angle which is less than $\frac{1}{2}\pi$ with the positive direction of the t-axis.

$u'(t_1) < 0 \Rightarrow$ The tangent to the curve $x = u(t)$ at $t = t_1$ makes an angle which is greater than $\frac{1}{2}\pi$ with the positive direction of the t-axis.

Two solutions $u(t)$ and $v(t)$ are given to be linearly independent.

$\Rightarrow$ They cannot have a common zero. As $u(t_0)$=0 we follow $v(t_0) \neq 0$.

We may assume $v(t_0) > 0$ without any loss to generality. $v(t_0) > 0$ without any loss to generality. $v(t_0) > 0$ means that the graph of the curve $x = v(t)$ at t_0 is above t-axis.

Taking $t = t_0$ in Abel's formula, we get

$r(t_0)[u(t_0)v'(t_0) - u'(t_0)v(t_0)] = k$, a constant

$k < 0, r(t_0) > 0, u(t_0) = 0, u'(t_0) > 0, v(t_0) > 0.$

$r(t_1)[u(t_1)v'(t_1) - u'(t_1)v(t_1)] < 0$

$v(t_1) < 0$ as $r(t_0) > 0, u_1(t_\cdot) = 0, u'(t_1) < 0$.

Now $v(t_0) > 0$ and $v(t_1) < 0$ and $v(t)$ is continuous functions.

$\Rightarrow$ $v(t)$ must have at least one zero in between t_0 and t_1.

Further, there cannot, however, be more than zero because the argument above, with the roles of $u(t)$ and $v(t)$ reserved, shows that between two consecutive zeros of $v(t)$ there must be at least one zero of $u(t)$. It completes the proof.

Example 12.1.3: Show that the differential equation $x'' + x = 0$ has common zeros and no others zero.

Solution: Given differential equation $x'' + x = 0$ which of the form

$$\frac{d}{dt}[r(t)x'] + p(t)x = 0$$

Here $r(t) = 1, p(t) = 1$ on every interval $a \leq t \leq b$.

The linearly dependent solutions $A\,sint$ and $B\,sin\,sin\,t$ have the common zeros $t = \pm n\pi(n = 0,1,2, \dots \dots)$ and no other zeros.

Example 12.1.4: Use the Sturm separation theorem to show that between any two consecutive zeros of Here $sin\,sin\,2t + cos\,cos\,2t$ there is precisely one zero of $sin\,sin\,2t - cos\,cos\,2t$.

Solution: Let $u(t) = sin\,sin\,2t + cos\,cos\,2t$ and $v(t) = sin\,sin\,2t - cos\,cos\,2t$

Differentiating twice, we get

$$u''(t) = -4(sin\,sin\,2t + cos\,cos\,2t\ ) = -4(u(t))$$

and $v''(t) = -4(sin\,sin\,2t - cos\,cos\,2t\ ) = -4(v(t))$.

This shows that $u(t)$ and $v(t)$ are the solutions of

$$x'' + 4x = 0$$

Which is a self adjoint equation.

Further, the Wronksian of these functions is

$$W(sin\,sin\,2t + cos\,cos\,2t, \qquad sin2t - cos\,cos\,2t) =$$

$$= |sin\,sin\,2t + cos\,cos\,2t \quad sin\,sin\,2t - cos\,cos\,2t \quad 2t - 2t \quad 2\,cos\,cos\,2t\ + 2\,sin\,sin\,2t\ |$$

$$= 22t\ + 22t + 22t\ + 22t\ - (2sin2tcos\,2t - 22t - 22t + 2\,sin\,sin\,2t\,cos\,2t)$$

$$2t$$

$$2t\,) = 4 \neq 0$$

Hence the two solutions are linearly independent.

Therefore, $u(t)$ and $v(t)$ defined earlier satisfy the conditions of Sturm's separation theorem is precisely one zero of $sin\,sin\,2t - cos\,cos\,2t$ between two consecutive zeros of the function $sin\,sin\,2t + cos\,cos\,2t$.

Example1 2.1.5: Show that the roots of the equation $x'' + x = 0$ form separation for one another.

Solution: The roots of the real solution are

$$u(t) = sint$$

$$v(t) = cost$$

More generally, the roots of any two real solutions are

$$u(t) = A\,sint + B\,cost\,t$$

$$v(t) = Csint + Dcos\,t$$

$$u'(t) = Acos\,t - Bsint$$

$$v'(t) = C\,cos\,t - D\,sint$$

Separate one another provided that $W(t)$ of $u(t)$ and $v(t)$does not vanish

i.e. $uv' - u'v \neq 0$

or $AC\,sint\,cos\,t - AD\,sin^2t + BC\,cos^2t - BD\,sint\,cos\,cos\,t - AC\,sint\,cos\,cos\,t + BC\,sin^2t - DAt + DB\,sint\,cos\,cos\,t$

$AD - BC \neq 0.$

This is merely the condition that these two solutions are linearly independent.

12.2 Sturm Comparison Theorem

The Sturm Comparison Theorem, which follows, compares the rates of oscillation of the solutions of two equations.

$[r(t)x']' + p(t)x = 0$ (12.2.1)

$[r(t)z']' + p_1(t)z = 0$ (12.2.2)

where $r(t) > 0$; $r(t)$, $p(t)$ and $p_1(t)$ are continuous on $a \le t \le b$ and

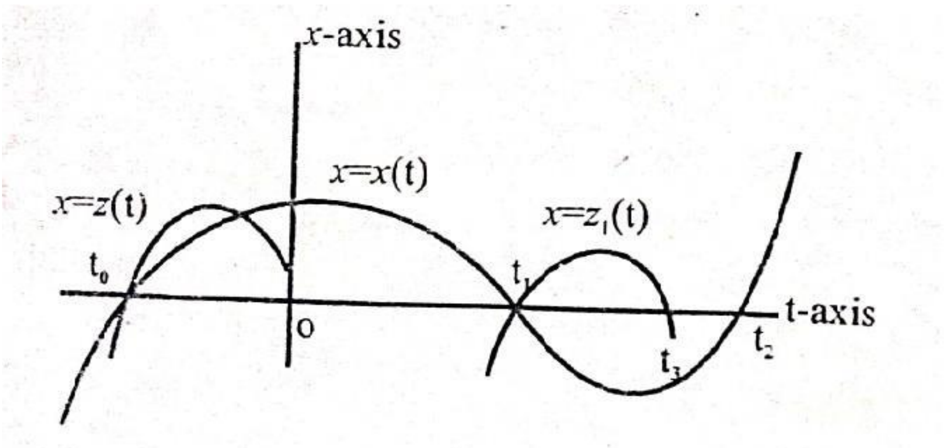
$p_1(t) > p(t)$

with strict inequality holding for at least one point of the interval.

Theorem 12.2.1: If the solution $x(t)$ of equation (12.2.1) has consecutive zeros at, $t = t_0$ and $t = t_1(t_0 < t_1)$, a solution, $z(t)$ of equation (12.2.2) which vanishes at $t = t_0$, will vanish again on the interval $t_0 < t < t_1$.

Proof: Without any loss of generality, we may assume $x(t) > 0$ on the interval $t_0 < t < t_1$

and $x'(t_0) > 0, x'(t_1) < 0$ and $z'(t_0) > 0$.



As $x(t)$ and $z(t)$ are solutions of equation (12.2.1) and (12.2.2) respectively, we have

$[r(t)x']' + p(t)x = 0$ (12.2.3)

and $[r(t)z']' + p_1(t)z = 0$ (12.2.4)

Multiplying (12.2.3) by $-z(t)$ and (12.2.4) by $x(t)$ and then adding, we get

$x(t)[r(t)z'(t)]' - z(t)[r(t)x'(t)]' + \{p_1(t) - p(t)\}x(t)z(t) = 0$ (12.2.5)

Integration of the identity (12.2.5) over the interval $t_0 \le t \le t_1$, gives

$$[r(t)\{x(t)z'(t) - x'(t)z(t)\}]_{t_0}^{t_1} + \int_{t_0}^{t_1} [p_1(t) - p(t)]x(t)z(t)dt \equiv 0$$

$$r(t)\{x(t_1)z'(t_1) - x'(t_1)z(t_1)\} + \int_{t_0}^{t_1} [p_1(t) - p(t)]x(t)z(t)dt \equiv 0$$

$x'(t_1)z(t_1)r(t) = \int_{t_0}^{t_1} [p_1(t) - p(t)]x(t)z(t)dt$, as $x(t_1) = 0$ (12.2.6)

If we suppose $z(t) > 0$ on $t_0 < t < t_1$ then the left hand member of (12.2.6) is negative and the right hand member of (12.2.6) is positive which is absurd.

$\Rightarrow$ $z(t)$ will not remain $+ve$ on the whole interval $t_0 < t < t_1$.

$\Rightarrow$ $z(t)$ will vanish somewhere within this interval $t_0 < t < t_1$.

We further notice that $x(t)$ again vanishes at $t = t_2 > t_1$ and if $z_1(t)$ is the second solution of equation (12.2.4) such that

$z_1(t_1) = 0$ and $z_1'(t_1) > 0$

then $z_1(t)$ will have a zero t_3 on $t_1 < t < t_2$

Now, using Sturm Separation Theorem, we follow

$Z(t)$ has a zero on $t_1 < t < t_3$

i.e. $Z(t)$ has a zero on $t_1 < t < t_3 < t_2$

i.e. $Z(t)$ has a zero on $t_1 < t < t_2$ which completes the proof.

Example: 12.2.2: Verify the Sturm's comparison theorem for the differential equations

$x'' + A^2x = 0$ and $x'' + B^2x = 0$

Where A and B are constants such that $B > A$.

Solutions: Let us take

$u(t) = Sin\ At$ and $v(t) = sin\ sin\ Bt$

which are the solutions of the given equations.

The two consecutive zeros of $u(t) = Sin\ At$ are $u(t) = \frac{n\pi}{A}$ and $\frac{(n+1)\pi}{A}, n = 0, \pm 1, \pm 2, \dots ....$

In particular for $n = 0$,0 and $\frac{\pi}{A}$ are the two consecutive zeros.

Now, the zero of $v(t) = Sin\ Bt$ are $\frac{n\pi}{A}$ and $\frac{(n+1)\pi}{A}$, and for $n = 0, \pm 1, \pm 2, \dots ....$ And for $n = 0$, the two consecutive zeros are 0 and $\frac{\pi}{B}$.

Both these solutions have one of their zero at $x = 0$.

The next zero of $v(t) = sin\ sin\ Bt$ is at $\frac{\pi}{B}$.

Since $B > A$, therefore $\frac{B}{\pi} > \frac{A}{\pi}$ or $\frac{\pi}{B} < \frac{\pi}{A}$.

Hence the next zero of $v(t) = sin\ sin\ Bt$ is before the next zero of $u(t) = sin\ sin\ At$.

Example: 12.2.3: Consider the equation $x'' + q(t)x = 0$ where $q(t) > 0$ on $a \le t \le b$.

Let q_m denote the minimum value of $q(t)$ on $a \le t \le b$. Show that if $q_m > \frac{k^2\pi^2}{(b-a)^2}$, then every real solution of the given equation has at least k zeros on $a \le t \le b$ where

$k = \left(\frac{b-a}{\pi}\right)\sqrt{q_m}$.

Solution: Consider the differential equation $x'' + \frac{k^2\pi^2}{(b-a)^2}x = 0$ which has a solution

$$x = sin\ sin\ \frac{k\pi}{b-a}x \qquad (12.2.7)$$

The zeros of this solution are at $x = \frac{n\pi(b-a)}{k\pi} = \frac{n}{k}(b - a), n = 0, \pm 1, \pm 2, \dots \dots ..$

In particular they are $x = 0, \frac{1}{k}(b - a), n = 0, \pm 1, \pm 2, \dots ..$

which are $k + 1$ in number in the interval $a < t < b$.

Now, consider the differential equation

$x'' + q(t)x = 0$

Since q_m is the minimum value of $q(t)$, therefore

$$q_m(t) > q_m > \frac{k^2\pi^2}{(b-a)^2} \qquad (12.2.8)$$

Hence by Sturm's comparison theorem, the zeros of the solution of (12.2.8) are at least one between the consecutive zeros of the solution of (12.2.1).

Hence in the interval $a \le t \le b$, the numbers of zeros of the solution of (12.2.8) are at least k in number.

Summary

- The zeros of solutions of self adjoint equation is discussed.
- The condition for common zeros of linearly dependent and independent is derived
- The separation of zeros of solutions of self adjoint equation is derived and examples are solved.
- The comparison theorem proved for two self adjoint equations and elaborated with examples.

Keywords

- Zeros of solution
- Sturm separation
- Sturm comparison
- Abel's formula
- Linear dependent and independent solutions

Self Assessment

Choose the most suitable answer from the options given with each question.

1. For any two linearly independent solution of equation $(r(x)y')' + p(x)y = 0$, then between any two consecutive zeros

A. There is precisely one zero

B. There is no zero

C. There are number of zeros

D. There is one zero

2. By change of variables $t = e^s$, $u = t^{1/2}z$, the equation $u'' + q(t)u = 0$ transform to

A. $\frac{d^2z}{ds^2} - t^2\left[q(t) - \frac{1}{4t^2}\right]z = 0, \ where \ \ t = e^s$

B. $\frac{d^2z}{ds^2} + t^2\left[q(t) - \frac{1}{4t^2}\right]z = 0, \ where \ \ t = e^s$

C. $\frac{d^2z}{ds^2} + t^2\left[q(t) + \frac{1}{4t^2}\right]z = 0, \ where \ \ t = e^s$

D. None of these

3. The Lagrange integral identity of pair of equations $(pu')' + qu = f$, $(pv')' + qv = g$ is

A. $[p(uv' - u'v)]_a^t = \int_a^t (gu + fv)ds$

B. $$[p(uv'+u'v)]_a^t = \int_a^t (gu - fv)ds$$

C. $$[p(uv'+u'v)]_a^t = \int_a^t (gu + fv)ds$$

D. $$[p(uv'-u'v)]_a^t = \int_a^t (gu - fv)ds$$

4. If two solution have common zero, then solutions are
A. Linearly dependent
B. Linearly independent
C. Constant
D. None of these

5. The differential equation $y'' + y = 0$ has
A. The non-common zero
B. The common zero and no other zero
C. The constant zero
D. None of these

6. The roots of the equation $y'' + y = 0$ form
A. Separation from one another
B. Non separation from one another
C. Linear dependent
D. None of these

7. The two consecutive zeros of $sin\ 2x + cos2x$ separated by
A. Precisely two zero of $sin\ 2x - cos2x$.
B. Precisely no zero of $sin\ 2x - cos2x$.
C. Precisely one zero of $sin\ 2x - cos2x$.
D. None of these

8. For any two linearly dependent nontrivial solutions of equation, if one of them vanishes, then
A. Other also vanish
B. Other is non-vanish
C. Non constant
D. None of these

9. The solution of two equations $[r(t)x']' + p(t)x = 0$ and $[r(t)z]' + p_1(t)z = 0$ where $r(t), p(t)\ and\ p_1(t)$ are continuous on [a, b] then the inequality one point of the interval hold
A. $p_1(t) < p(t)$
B. $p_1(t) \leq p(t)$
C. $p_1(t) > p(t)$
D. $p_1(t) \geq p(t)$

10. The solution $x(t)$ of equations $[r(t)x']' + p(t)x = 0$ has consecutive zeros at $t = t_0$ and $t = t_1$, a solution $z(t)$ of equations $[r(t)z']' + p_1(t)z = 0$ which vanishes $t = t_0$,

A. It will not vanishes again on $t_0 < t < t_1$
B. It will vanishes again on $t = t_1$
C. It will vanishes again on $t_0 < t < t_1$.
D. None of these

11. The equations $x'' + A^2x = 0$ and $x'' + B^2x = 0$ where A and B are constants such that $B > A$ verify the

A. Sturm separation theorem
B. Sturm comparison theorem
C. Sturm Liouville problem
D. None of these

12. The y and z are nontrivial solutions of $x^2y'' + xy' + (x^2 - 1)y = 0$ and $x^2z'' + xz' + x^2z = 0$, respectively, such that both vanish at $x = 1$

A. The solution $x(t)$ will vanish faster than 1
B. The solution $z(t)$ will vanish faster than 1
C. The solution $x(t)$ and $z(t)$ will vanish faster than 1
D. None of these

13. The equations $y'' + y = 0$ and $z'' - z = 0$ for $t \geq 0$ hold that

A. Between two consecutive zero of $z(t)$, there is a zero of y(t).
B. Between two consecutive zero of $y(t)$, there is a zero of z(t).
C. Zero of $x(t)$ and $y(t)$ are common.
D. None of these

14. The equations $x'' + x = 0$ and $y'' + 4y = 0$ hold that

A. Between two consecutive zero of $y(t)$, there is a zero of x(t).
B. Between two consecutive zero of $x(t)$, there is a zero of y(t).
C. Zero of $x(t)$ and $y(t)$ are common.
D. None of these.

15. The Sturm's comparison theorem asserts is that if a solution of the first differential equation has a certain property P then

A. The solutions of the second differential equation do not have the same or some related property.
B. The solution of the second differential equation is not comparable.
C. The solution of the second differential equation has the same or some related property P.
D. None of these

Answers for Self Assessment

1.	A	2.	B	3.	D	4.	A	5.	B
6.	A	7.	D	8.	A	9.	D	10.	C
11.	B	12.	B	13.	A	14.	B	15.	C

Review Questions

1. Show that between any pair of consecutive zero of $sin\, t$, there is exactly one zero of $sin\, t + cos\, cos\, t$.
2. Show that between any pair of consecutive zero of $sin\, (log\, log\, t\,)$, there is exactly one zero of $cos\, cos\, (log\, log\, t)$.
3. Let $r(t)$ be a continuous function (for $t \geq 0$) such that $r(t) > m^2 > 0$, where m is an integer. Consider the equations $x'' + m^2 x = 0, t > 0$

$$y'' + r(t)y = 0, t > 0.$$

If $y(t)$ is any solution of the second equation, prove that $y(t)$ must vanish in any interval of length π/m.

Further Readings

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Unit 13: Sturm Boundary Value Problem

CONTENTS

Objective:

After studying this unit, you will be able to

- identify the concept of boundary value problem
- understand about the Sturm problems
- know about the eigen values and eigen functions.
- determine about the trivial and nontrivial of solutions
- apply the oscillatory theory to the zero of solutions

Introduction

In this chapter, we shall consider boundary value problems. For the sake of convenience, we recall the definition of boundary value problem.

Consider a linear differential of order n of the form

$$L(x) = a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \cdots .. a_n(t)x = b(x) \qquad (13.0.1)$$

where$a_0(t) \neq 0, a_1, \dots \dots a_n$ and b are real or complex-valued continuous functions defined on an interval $I = [c, d]$.

A boundary condition is a condition imposed on the solutions of equation (13.0.1) at two or more points of the interval I. The points of I (denoted by $t_0, t_1, \dots \dots$) where the conditions are imposed, are known as boundary points and the value of $x, x' \dots \dots ., x^{(n-1)}$at the boundary points are known as boundary values.

A differential equation with some boundary conditions is known as boundary value problem (denoted in short by BVP).

There are several forms of boundary conditions. We define some important forms of boundary conditions for equation (13.0.1).

For equation (13.0.1), the boundary conditions of the form

$$x(c) = x(d), x'(c) = x'(d), \dots \dots .., x^{(n-1)}(c) = y^{(n-1)}(d)$$

Are called periodic boundary conditions stated at $x = c$and $x = d$.

13.1 Sturm-Lioville's Problem

Now consider a differential equation of the type

$[r(t)x']' + \{p(t) + \lambda q(t)\}x = 0$ (13.1.1)

in which $r(t) > 0$ and $p(t) > 0$ and $r(t), p(t), q(t)$ are continuous real functions of t on $a \le t \le b$ and the constant λ is the parameter independent of t.

This equation is called Sturm-Liouville's Equation.

Equation (13.1.1) is considered on a closed interval $a \le t \le b$ and subject to the boundary conditions (at the end points) by

$a_1 x(a) + a_2 x'(a) = 0$ (13.1.2a)

$b_1 x(b) + b_2 x'(b) = 0$ (13.1.2b)

wherea_1, a_2, b_1 and b_2 are constants and neither a_1, a_2 are both zero nor b_1, b_2 are both zero together. The problem of finding the solution of (13.1.1) subject to the boundary conditions (13.1.2a) and (13.1.2b) is called Sturm Liouville's Boundary Value Problem. The trivial solution $x = 0$, for every value of parameter λ,is of no practical use. The nontrivial solutions of Sturm Liouville's boundary value problem are called characteristic functions(or eigen functions) and all the value of λ for which such solutions exist are called characteristic values(or eigen values) of the problem.

It should be noted that the Sturm Liouville's problem are linear boundary value problems of order 2. A boundary value problem of order 2 is said to be non linear boundary value problem of order 2 if the differential equation is non-linear.

The boundary value problem $x'' + |x| = 0; x(0) = 0 = x(\pi); 0 \le t \le \pi$ is nonlinear boundary value problem of order 2. Here the non linearity in this equation is due to $f(x) = |x|$.

Example 13.1.1: Find the eigen values and the corresponding eigen functions of $X'' + \lambda X = 0$ when $X(0) = 0$and $X'(L) = 0$.

Solution: The given equation is $X'' + \lambda X = 0$ (13.1.3)

And the boundary conditions are $X(0) = 0$and $X'(L) = 0$.

Case I$\lambda = 0$

Equation (13.1.3) becomes$X'' = 0$ which gives $X(t) = At + B \;\Rightarrow\; X'(t)$=A .

Using boundary conditions $0 = X(0) = B; 0 = X'(L) = A \Rightarrow A = B = 0$.

$\therefore X(t) = 0 \;\Rightarrow \lambda = 0$ gives the trivial solution.

$\Rightarrow \lambda$is not the eigen value and there is no eigen function corresponding to $\lambda = 0$.

Case II$\lambda = -ve$say $\lambda = -\mu^2$ $(\mu \ne 0)$.

In this case equation (13.1.3) becomes $X'' - \mu^2 X = 0$.

$$\Rightarrow X(t) = Ae^{\mu t} + Be^{-\mu t}$$

and $X'(t) = A\mu e^{\mu t} - B\mu e^{-\mu t}$.

Using boundary conditions, we follow

$0 = X(0) = A + B$and $0 = X'(L) = A\mu e^{\mu L} - B\mu e^{-\mu L}$.

$\Rightarrow A + B = 0$or$B = -A \Rightarrow A\mu(e^{\mu L} + e^{-\mu L}) = 0 \;\therefore A = 0, B = 0$.

It again gives the trivial solution $X(t) = 0$.

There is no eigen function when $\lambda < 0$.

Case III$\lambda = +ve$say $\lambda = \mu^2$ $(\mu \ne 0)$.

In this case equation (13.1.3) becomes $X'' + \mu^2 X = 0$ whose solution is

$$X(t) = A\,cos\,\mu t + B\sin\mu t \quad \Rightarrow X'(t) = -A\mu\sin\mu t + B\mu\cos\mu t.$$

Using initial conditions, we get $0 = X(0) = A; 0 = X'(L) = B\mu\cos\mu L$

$\Rightarrow A = 0$and $\Rightarrow Bcos\,\mu L = 0$.

$\Rightarrow Either\ A = 0, B = 0orA = 0, \cos\mu L = 0$

$A = 0, B = 0$there exist no eigen function.

Let us deal with the subcase $B \neq 0$ which gives

$\cos \mu L = 0$or$\mu L = \frac{1}{2}(2n-1)\pi$

or$\mu = \frac{(2n-1)\pi}{2L}$ for $n = 1,2,3, \dots ..$

Using $A = 0$, solution becomes $X(t) = Bsin\frac{(2n-1)\pi}{2L}t$

Or $X(t) = sin\frac{(2n-1)\pi}{2L}t$ (Taking B=1) for $n = 1,2,3, \dots ..$

are eigen functions and the corresponding eigen values are

$$\lambda = \mu^2 = \frac{(2n-1)^2\pi^2}{4L^2}$$

where$n = 1,2,3, \dots \dots$

Example 13.1.2: Find the eigen values and the corresponding eigen functions of $X'' + \lambda X = 0$ when $X'(0) = 0$and $X'(L) = 0$.

Solution: Sturm Liouville's Problem is to solve $X'' + \lambda X = 0$ …..(13.1.4)

when boundary conditions are $X'(0) = 0$ and $X'(L) = 0$.

Case I$\lambda = 0$

Equation (13.1.4) becomes $X'' = 0$ which gives $X(t) = At + B \quad \Rightarrow \; X'(t)$=A .

Using boundary conditions $0 = X'(0) = A;$ $and\ B$ is arbitrary.

Taking $B = 1$ we get

$\therefore X(t) = 1$ which is the nontrivial solution.

$\Rightarrow X(t) = 1$ is an eigen function and $\lambda = 0$ is the eigen value.

Case II $\lambda = -ve$say $\lambda = -\mu^2$ $(\mu \neq 0)$.

In this case equation (13.1.4) becomes $X'' - \mu^2 X = 0$.

$$\Rightarrow X(t) = Ae^{\mu t} + Be^{-\mu t}$$

and hence$X'(t) = A\mu e^{\mu t} - B\mu e^{-\mu t}$.

Using boundary conditions, we follow

$0 = X'(0) = A - B$and $0 = X'(L) = A\mu e^{\mu L} - B\mu e^{-\mu L}$.

$\Rightarrow A = B$and$A\mu(e^{\mu L} - e^{-\mu L}) = 0 \;\; \therefore A = 0\ as\ \mu \neq 0$.

$A = 0$gives $B = 0$.

$\therefore X(t) = 0$ which is trivial solution .

Case III$\lambda = +ve$say $\lambda = \mu^2$ $(\mu \neq 0)$.

In this case equation (13.1.4) becomes $X'' + \mu^2 X = 0$ whose solution is

$$X(t) = A\ cos\ \mu t + B \sin \mu t \quad \Rightarrow X'(t) = -A\mu \sin \mu t + B\mu \cos \mu t.$$

Using boundary conditions, we get $0 = X'(0) = B\mu;$ $0 = X'(L) = -A\mu \sin \mu L + B\mu \cos \mu L$

$\Rightarrow B = 0$as$\mu \neq 0$and $\Rightarrow Asin\ \mu L = 0$.

$\Rightarrow B = 0 \quad and \quad Either\ A = 0$ or$\sin \mu L = 0$

$A = 0, B = 0$ or$B = 0, \sin \mu L = 0$.

$A = 0, B = 0$ gives$X(t) = 0$ is the trivial solution.

and$B = 0, \sin \mu L = 0$ gives $\mu L = n\pi$ for $n = 1,2,3, \dots ..$

$A$is arbitrary, let $A = 1$.

$\therefore X(t) = cos\frac{n\pi}{L}t$ for $n = 1,2,3, \ldots..$

are eigen functions and the corresponding eigen values are

$\lambda = \mu^2 = \frac{n^2\pi^2}{L^2}$ for $n = 1,2,3, \ldots..$

Example 13.1.3: Find all the eigen values and eigen functions of the Sturm Lioville's problem of $X'' + \lambda X = 0$ with $X(0) + X'(0) = 0$ and $X(1) + X'(1) = 0$.

Solution: Given $X'' + \lambda X = 0$ (13.1.5)

With the boundary conditions are

$X(0) + X'(0) = 0$ (13.1.6)

and $X(1) + X'(1) = 0$. (13.1.7)

Case I When $\lambda = 0$, our equation (13.1.5) becomes $X'' = 0$ whose solution is given by $X(t) = At + B$ and hence $X'(t) = A \Rightarrow X(t) + X'(t) = A + At + B$.

Using boundary conditions (13.1.6), we find

$0 = X(0) + X'(0) = A + B \Rightarrow A + B = 0.$

Using boundary conditions (13.1.7), we find

$0 = X(1) + X'(1) = 2A + B \Rightarrow 2A + B = 0.$

Now $A + B = 0; \Rightarrow 2A + B = 0$ gives $\Rightarrow A = B = 0$ and solution becomes $X(t) = 0$, the trivial solution $\Rightarrow \lambda = 0$ is not the eigen value.

Case II When $\lambda < 0$, let $\lambda = -\mu^2$ where $\mu \neq 0$.

In this case, our equation (13.1.5) becomes $X'' - \mu^2 X = 0$ whose solution is given by $X(t) = Ae^{\mu t} + Be^{-\mu t}$ and hence $X'(t) = A\mu e^{\mu t} - B\mu e^{-\mu t} \Rightarrow X(t) = X'(t) = A(1+\mu)e^{\mu t} + B(1-\mu)e^{-\mu t}$.

Using boundary conditions (13.1.6), we find

$0 = X(0) + X'(0) = A(1+\mu) + B(1-\mu)$ (13.1.8)

And using boundary conditions (13.1.7), we find

$0 = X(1) + X'(1) = A(1+\mu)e^{\mu} + B(1-\mu)e^{-\mu}$ (13.1.9)

For nontrivial solution of (13.1.8) and (13.1.9), we have

$$\begin{vmatrix} 1+\mu & 1-\mu \\ (1+\mu)e^{\mu} & (1-\mu)e^{-\mu} \end{vmatrix} = 0 \Rightarrow (1+\mu)(1-\mu)(e^{-\mu} - e^{\mu}) = 0 \Rightarrow \mu = 1, -1.$$

$\mu = 1$ gives $2A = 0$ ($from$ (13.1.8)) $\therefore A = 0$ while B is arbitrary.

$\therefore$ Solution becomes $X(t) = Be - t$ which is eigen corresponding to eigen value $\lambda = -\mu^2 = -1$.

$\mu = 1$ gives $B = 0$ ($from$ (13.1.8)) while A is arbitrary and $X(t) = Ae^{-t}$ is eigen function and corresponding eigen value is $\lambda = -\mu^2 = -(-1)^2 = -1$.

Hence, taking $A = B = 1$, we follow $X(t) = e^{-t}$ is an eigen function and corresponding eigen values is -1.

Case III $\lambda > 0$, let $\lambda = \mu^2$ where $\mu \neq 0$.

In this case equation (13.1.5) becomes $X'' + \mu^2 X = 0$ whose solution is given by

$X(t) = A\cos\mu t + Bsin\,\mu t$ and hence $X'(t) = -A\mu\sin\mu t + B\mu\cos\mu t$

$$\Rightarrow X(t) + X'(t) = (A + B\mu)cos\mu t + (B - A\mu)\sin\mu t$$

Boundary condition (13.1.6) gives

$0 = X(0) + X'(0) = A + B\mu$ (13.1.10)

Boundary condition (13.1.7) gives

$0 = X(1) + X'(1) = (A + B\mu)\cos\mu + (B - A\mu)\sin\mu$ (13.1.11)

From (13.1.10) $A = -B\mu$ using it in (13.1.11), we get

$B(1+\mu^2)\sin\mu = 0$ $or\ Bsin\ \mu = 0$as $\mu^2+1 \neq 0$.

$\Rightarrow B = 0$ or$Sin\ \mu = 0 \Rightarrow B = 0$ gives $A = 0$ (from (13.1.6)).

$A = B = 0$gives the trivial solution $y(x) = 0$

$\sin\mu = 0$gives$\mu = n\pi$ for $n = 1,2,3, \dots ....$

$A = -B\mu \Rightarrow A = -Bn\pi$and hence $X(t) = -Bn\pi\cos n\pi t + B\sin n\pi t$

or$x(t) = B(\sin n\pi t - n\pi\cos n\pi t)$ for $n = 1,2,3, \dots ....$

Taking $B = 1$, Eigen function are

$X(t) = \sin n\pi t - n\pi\cos n\pi tforn = 1,2,3, \dots ....$

and the corresponding eigen value are

$\lambda = \mu^2 = n^2\pi^2forn = 1,2,3, \dots ....$

13.2 Oscillation Theory

When the solution of the differential equation cannot be found explicitly, we have to resort to the study of qualitative properties of its solutions. One such qualitative property, which has wide applications, is the oscillation of solutions.

In this chapter, we shall study the oscillation properties of solutions of linear differential equations of order two, but the theory developed hold good for linear differential equation of order higher than two. Since every linear differential equation of order two can be put in self –adjoint form, it suffices to study the oscillation properties of solution of self-adjoint linear differential equations of order two of the form

$[r(t)x']' + p(t)x = 0$ (13.2.1)

Where $r, r'$and p are continuous functions and $r(t) > 0$ on the interval $I = [0,\infty)$. We observe that identically zero function $x(t) \equiv 0$on I, is solution of equation (13.2.1). This solution is known as trivial (or zero) solution and any other solution $x(t) \neq 0$ is known as nontrivial solution of equation (13.2.1). In this chapter, a solution x of a equation (13.2.1) means a nontrivial solution, unless it is mentioned otherwise.

A point $t^* \geq 0$ is called a zero of solution $x(t)$ of equation (13.2.1) if $x(t^*) = 0$. A solution of equation (13.2.1) is called an oscillation (or oscillatory) if it has infinitely many zero in the interval $[0,\infty\ )$. Equation (13.2.1) is called oscillatory, if every solution of it is an oscillation.

Example 13.2.1: The functions $x_1(t) = sint$ and $x_2(t) = cost$ are non trivial solution of

$x'' + x = 0$, which vanish infinitely often on the interval$[0,\infty)$. Hence, $sint$ and $cost$ are oscillations of $x'' + x = 0$. Since any solution of $x'' + x = 0$ is a linear combination of $sint$and $cost$, all the solutions of $x'' + x = 0$ vanishes infinitely often on the interval $[0,\infty)$. Hence, of $x'' + x = 0$ is oscillation on the interval $[0,\infty)$.

13.3 Number of Zeros in a Finite Interval

Theorem 13.3.1: If a solution$x$of equation (13.2.1) vanishes infinitely often on I, then thee exists a finite subinterval of I, on which x vanishes identically.

Proof: Let x be solution of (13.2.1) with infinitely many zeros in I. By Bolzano-Weierstrass theorem, there is a limit point t^* in I for the set of zero of x. Since $t^*$is a limit point for the set of zeros of x, there is a sequence $\{t_n\}$ of zero of x such that $t_n \to t^*$. This means that, for given $\varepsilon > 0$, we can find an N such that $n > N \Rightarrow |t_n - t^*| < \varepsilon$. That is, for all $n > N$, we have

$t^* - \varepsilon < t_n < t^* + \varepsilon$.

This shows that there is a finite subinterval of I containing infinitely many zeros of x.

Further, since x is continuous function of t, we must have

$$\lim_{t\to t^*} x(t) = x(t^*)$$

Hence, if $t \to t^*$ through the sequence $\{t_n\}$ of zeros of x, we must have

$x(t^*)$=$\lim_{t\to t^*} x(t_n) = 0$

This shows that t^* is a zero of x. That is, any limit point of set of zeros of x is a zero of x. Since each point in the interval $(x^* - \varepsilon, x^* + \varepsilon)$ is a limit point for the set of zeros of x we must have $x(t) \equiv 0$, identically on $(x^* - \varepsilon, x^* + \varepsilon)$. This shows that there is a finite subinterval of I on which x vanishes identically.

Remarks 13.3.2: A consequence of Theorem 13.3.1 is that any finite subinterval of I contains at most a finite number of zeros of a solution x of equation (13.2.1) unless $x(t) \equiv 0$, identically on that subinterval of I.

Remarks 13.3.3: If a solution x of equation (13.2.1) has infinitely many zeros in I and if nth derivative $x^{(n)}$ of x exists and continuous on I, then there is a point t^*in I satisfying

$$x^{(k)}(t^*) = 0, k = 0,1,2, \dots \dots, n.$$

Proof: Let x be a solution of equation (13.2.1) with infinitely many zeros in I. Let t^* be a limit point for the set of zeros of x. Let $\{t_n\}$ be a sequence of zeros of x such that $t_n \to t^*$. Since x is continuous on Iwe must have

$$\lim_{t \to t^*} x(t) = x(t^*).$$

Letting $t \to t^*$ through the sequence $\{t_n\}$ of zeros of x, we find that

$$x(t^*) = \lim_{t \to t^*} x(t_n) = 0$$

Again, since x' is continuous on I, we have

$$x'(t^*) = \lim_{t_n \to t^*} \frac{x(t_n) - x(t^*)}{t_n - t^*} = \frac{0 - 0}{t_n - t^*} = 0.$$

In the same way, it can be shown that

$$x^{(k)}(t^*) = 0, k = 0,1,2, \dots \dots, n.$$

Theorem 13.3.4: The zeros of any nontrivial solution of equation (13.2.1) are isolated.

Proof: Let x be a nontrivial solution of equation (13.2.1). Let $t = t_0$ be a zero of x, that is, $x(t_0) = 0$. Then $x'(t) \neq 0$ on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$, otherwise, $x(t) = 0$ on $[0, \infty)$, a contradiction to the fact that x is a nontrivial solution of equation (13.2.1).

There arise two cases:

Case I $x'(t_0) > 0$

Since x' is continuous and positive at $t = t_0$, it follow that the function x is strictly increasing in some neighbourhood of t_0, which means that $t = t_0$, is the only zero of x in that neighborhood. This shows that $t = t_0$ is an isolated zero of x.

Case II $x'(t_0) < 0$

Since x' is continuous and negative at $t = t_0$, it follow that the function x is strictly decreasing in some neighbourhood of t_0, which means that $t = t_0$, is the only zero of x in that neighborhood. This shows that $t = t_0$ is an isolated zero of x.

Summary

- The Sturm boundary value problem discussed.
- The eigen value and eigen function are determined for boundary value problem.
- The zeros of solutions of self adjoint equation is discussed.
- The trivial and nontrivial solutions are explained.

  The oscillatory behaviour of second order differential equation elaborated with examples.

Keywords

- Boundary value problem
- Zeros of solution

- Sturm BVP
- Eigenvalue
- Eigen functions
- Oscillatory behaviour

Self Assessment

1. $u = \sin(n+1)t$ be the solution of $u''+\lambda u = 0,\ u(0) = u(\pi) = 0$ if

A. $\lambda = (n+1)^2$ for n = 0, 1, 2.......

B. $\lambda = (n-1)^2$ for n = 0, 1, 2.......

C. $\lambda = (n+1)$ for n = 0, 1, 2.......

D. $\lambda = (n-1)$ for n = 0, 1, 2.......

2. The eigen value of $X''+\lambda X = 0,\ X(0) = X'(L) = 0$ are

A. $(2n-1)^2 \pi^2 / 4L^2, \quad where \quad n = 1,2,3,......$

B. $(2n+1)^2 \pi^2 / 4L^2, \quad where \quad n = 1,2,3,......$

C. $(n-1)^2 \pi^2 / 4L^2, \quad where \quad n = 1,2,3,......$

D. None of these

3. The non-trivial solutions of Sturm Liouville Boundary Value Problem are called

A. Particular function

B. General function

C. Eigen function

D. None of these

4. The value of the parameter for which the function exists as solution of boundary value problem is called

A. Constant value

B. Particular value

C. Eigen value

D. None of these

5. A boundary value problem of order of 2 is said to be non-linear if the differential equation is

A. Linear

B. Non-linear

C. Constant

D. None of these

6. The eigen function for the boundary value problem $X''+\lambda X = 0,\ X'(0) = X'(L) = 0$ is

A. $X(t) = \cos\frac{n\pi}{L}t \quad for \quad n = 1,2,3,.........$

B. $X(t) = \sin\frac{n\pi}{L}t \quad for \quad n = 1,2,3,.........$

C. $X(t) = \cos ec\frac{n\pi}{L}t \quad for \quad n = 1,2,3,.........$

D. None of these

7. The eigen function for the boundary value problem $X''+\lambda X = 0,\ X'(0) = X'(L) = 0$ is

A. $X(t) = \cos\frac{n\pi}{L}t \quad for \quad n = 1,2,3,.........$

B. $X(t) = \sin\frac{n\pi}{L}t \quad for \quad n = 1,2,3,.........$

C. $X(t) = \cos ec\frac{n\pi}{L}t \quad for \quad n = 1,2,3,.........$

D. None of these

8. The zeros of any nontrivial solution of equation $[\boldsymbol{r}(\boldsymbol{x})\boldsymbol{y}']' + \boldsymbol{p}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{0}$ are
A. Isolated
B. Non-isolated
C. Constant
D. None of these

9. The solution $\boldsymbol{y}(\boldsymbol{x}) \neq \boldsymbol{0}$ of $[\boldsymbol{r}(\boldsymbol{x})\boldsymbol{y}']' + \boldsymbol{p}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{0}$ is known as
A. Trivial solution
B. Non trivial solution
C. Constant
D. None of these

10. A solution of equation $[\boldsymbol{r}(\boldsymbol{x})\boldsymbol{y}']' + \boldsymbol{p}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{0}$ is called oscillatory if
A. It has infinitely many zeros
B. It has finitely many zeros
C. No zero
D. None of these

11. The solution $\boldsymbol{y}(\boldsymbol{x})$ of $[\boldsymbol{r}(\boldsymbol{x})\boldsymbol{y}']' + \boldsymbol{p}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{0}$ vanishes infinitely often on I, then
A. There exists finite subinterval on which y vanishes identically
B. There exists infinite subinterval on which y vanishes identically
C. No subinterval on which y vanishes identically
D. None of these

12. The non-trivial solution of $\boldsymbol{x'' + \phi(t)x = 0}$ are oscillatory if

A. $\phi(t) \le m^2$for all t.

B. $\phi(t) \ge m^2$for all t.

C. $\phi(t) = m^2 = 0$for all t.

D. None of these

13. The equation $\boldsymbol{x'' + x = 0}$ is

A. Oscillatory

B. Non –oscillatory

C. Nonlinear

D. None of these

14. The equation $\boldsymbol{x'' - 9x = 0}$ is

A. Oscillatory

B. Linear

C. Nonlinear

D. None of these

15. A point $\boldsymbol{t = t^* \ge 0}$ is called zero of a solution $\boldsymbol{x}$ of the equation $\boldsymbol{x'' = f(t, x, x')},\ \boldsymbol{t \ge 0}$ if

A. $x(t^*) < 0$

B. $x(t^*) > 0$

C. $x(t^*) = 0$

D. None of these

Answers for Self Assessment

1.	A	2.	A	3.	C	4.	C	5.	B
6.	A	7.	A	8.	A	9.	B	10.	A
11.	A	12.	B	13.	A	14.	B	15.	C

Review Questions

1. Consider the equation $x'' + \lambda x = 0, 0 \le x \le \pi$.
   Find the eigenvalues and eigenfunctions in the following form
   (i) $x'(0) = x'(\pi) = 0$.
   (ii) $x(0) = x(\pi) = 0$.
   (iii) $x(0) = x'(\pi) = 0$
   (iv) $x'(0) = x(\pi) = 0$
2. Prove that the non-trivial solution of $x'' + [1 = f(t)]x = 0$, where$\lim_{t\to\infty} f(t) = 0$ are oscillatory.
3. If $\lim_{t\to\infty} a(t) = \infty$monotonically, then prove that all the non-trivial solution of $x'' + a(t)x = 0$ are oscillatory.

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Unit 14: Non Oscillatory and Principal Solution

CONTENTS

Objective

After studying this unit, you will be able to

- identify the concept of non-oscillatory solutions.
- understand about zeros of solutions.
- know about the disconjugate property.
- determine about principal solutions of non –oscillatory equations.

Introduction

In this chapter, more behavior about the oscillatory and non-oscillatory equations with principal solutions will be discussed.

14.1 Non-Oscillatory Equations

An equation $x'' + a(t)x = 0$ (14.1.1)

Where $a(t)$ is a real valued continuous function on $t_0 \le t < \infty$, is called non oscillatory if all the nontrivial solutions of (14.1.1) has at most a finite number of zeros on $t_0 \le t < \infty$. On other hand, if all the non-trivial solutions of (14.1.1) have an infinite number of zeros on $t_0 \le t < \infty$. In this case, the non-trivial solutions are called oscillatory solutions.

The differential equation $x''' - x'' + 11x' - 4x = 0$ is non-oscillatory because its one non trivial solution e^t has no zero in any interval $t_0 \le t < \infty$.

Theorem 14.1.1: let $x(t)$ be a solution of (14.1.1) existing on $(0, \infty)$. If $a(t) < 0$on $(0, \infty)$, then $x(t)$ has at most one zero.

Proof: Let t_0 be a zero of $x(t)$. It is clear that $x'(t_0)$ is not zero for $x(t) \neq 0$. Without loss of generality it may be assumed that $x'(t_0) > 0$ so that $x(t)$ is positive in some interval to right of t_0. Now $a(t) < 0$ implies that x'' is positive on the same interval in which in turn implies that x' is an increasing function, and so, x does not vanish to right of t_0. A similar argument shows that x has no zero to the left of t_0. Thus x has at most one zero.

Remark: 14.1.2: Theorem 14.1.1 can also be seen as a corollary of Sturm's Comparison theorem. Consider the equation $x'' = 0$. It is known that any nonzero constant function $x(t) = k$ is a solution. Thus, if this equation is compared with the Equation (14.1.1) (observed that all the hypothesis of Sturm Comparison theorem are satisfied) then $x(t)$ vanishes at most once, for otherwise $x(t)$ vanishes twice and $x(t)$ necessarily vanishes at least once by Sturm Comparison theorem. So $x(t)$ canot have more than one zero.

From Theorem 14.1.1 the question arises, if $a(t) > 0$ on $(0, \infty)$ is the equation (14.1.1) oscillatory?

Theorem 14.1.3: Let $a(t)$ be continuous and positive on $(0, \infty)$ with

$\int_1^\infty a(s)ds = \infty.$ (14.1.2)

Also assume that $x(t)$ is any solution of (14.1.1) existing for$(t) \geq 0$. Then, $x(t)$ has infinite zeros in $(0, \infty)$.

Proof: Assume, on the contrary, that $x(t)$ has only a finite number of zeros in $(0, \infty)$. Then there exists a point $t_0 > 1$ such that $x(t)$ does not vanish on $[t_0, \infty)$.

Without loss of generality it can be assumed that $x(t) > 0$ for all $t \geq t_0$. Thus,

$$v(t) = +\frac{x'(t)}{x(t)}, t \geq t_0$$

is well defined. It now follow that

$$v'(t) = -a(t) - v^2(t).$$

Integration of the above leads to

$$v(t) - v(t_0) = -\int_{t_0}^{t} a(s)\, ds - \int_{t_0}^{t} v^2(s) ds.$$

The condition (14.1.2) now implies that there exists two constants $A \ and \ T$ such that $v(t) < A(< 0)$ if $t \geq T$ since $v^2(t)$ is always non-negative and

$$v(t) < v(t_0) - \int_{t_0}^{t} a(s) ds.$$

This means that $x'(t)$ is negative for large t. Let $T \geq t_0$ be so large that $x'(T) < 0$. Then on $[T, \infty]$, notice that $x(t) > 0, x'(t) < 0$ and $x''(t) < 0$. But

$$\int_T^t x''(s) ds = x'(t) - x'(T).$$

Integrating once again it is seen that

$x(t) - x(T) = x'(T)(t - T), t \geq T \geq t_0.$ (14.1.3)

Since $x'(T)$ is negative, the right hand side of (14.1.3) tens to $-\infty$ ans $t \to \infty$ while the left hand side of (14.1.3) either tends to finite limit (because $x(t)$ is finite) or tends to $+\infty$ (in case $x(t) \to \infty$, as $t \to \infty$). Thus, in either case a contradiction is reached. So the assumption that $x(t)$ has a finite number of zero in $(0, \infty)$ is false. Hence $x(t)$ has finite number of zeros in $(0, \infty)$, which completes the proof.

Theorem 14.1.4: If all non-trivial solutions of (14.1.1) are oscillatory and $b(t)$ is continuous with $b(t) > a(t)$ on $t_0 \leq t < \infty$, then all the nontrivial solutions of

$y'' + b(t)y = 0$ (14.1.4)

are oscillatory. On the other hand if some of the trivial solutions of (14.1.4) are non-oscillatory and

$b(t) \geq a(t)$, then some non-trivial solutions of (14.1.1) must be non-oscillatory.

Proof: Let $x(t)$ and $y(t)$ be non-trivial solution of (14.1.1) and (14.1.4) respectively. Also suppose $x(t)$ is an oscillatory solution of (14.1.1) and $t_1$and t_2 be any two consecutive zeros of $x(t)$, then by Sturm Comparison theorem there exists one zero t_3 on y(t) between $t_1$and t_2. Also $x(t)$ is an oscillatory solution of (14.1.1).

$\Rightarrow x(t)$has infinite number of zeros on $t_0 \leq t < \infty$. These two assertion implies that $y(t)$ has infinitely many zero on $t_0 \leq t < \infty$.Hence $y(t)$ is oscillatory solution of (14.1.4) while $x(t)$ and $y(t)$ are arbitrary solutions of (14.1.1) and (14.1.4) respectively. This proves the first part of the theorem.

For the second part, let $y(t)$ is non-oscillatory solution of (14.1.1), then $y(t)$ has finite number of zeros on $t_0 \leq t < \infty$. But by Sturm Comparison theorem, we know that between any two consecutive zeros of $x(t)$ there exists a zero of $y(t)$. This implies that $x(t)$ has finite number of zero on $t_0 \leq t < \infty$ i.e. $x(t)$ is non-oscillatory solution of (14.1.1). This proves the second part of the theorem.

Example14.1.5: Show that non trivial solution of $x'' + [1 + f(t)]x = 0$, where $\lim_{t \to \infty} f(t) = 0$ are oscillatory or non-oscillatory.

Proof: Given $\lim_{t \to \infty} f(t) = 0$ this implies that for sufficiently large t_0, we have

$|f(t)| \leq \varepsilon$ for all $t \geq t_0$

$$\Rightarrow -\varepsilon \leq f(t) \leq \varepsilon \Rightarrow 1 - \varepsilon \leq f(t) \leq 1 + \varepsilon$$

Choose $\varepsilon = \frac{1}{2}$, we get $1 + f(t) \geq \frac{1}{2}$ for all $t \geq t\_0$.

Note that all the solutions of the differential equation $x'' + \frac{1}{2}x = 0$ be oscillatory.

Thus by special case of Strum Comparison theorem, all the nontrivial solutions of $x'' + [1 + f(t)]x = 0$ must be oscillatory.

14.2 Number of zeros

Now we discuss the problems to determine the number of zero of a nontrivial solution of the general second order differential equation

$$[p(t)x']' + q(t)x = 0 \tag{14.2.1}$$

Where $p(t)$ and $q(t)$are continuous functions on some interval [a,b].

14.3 Prufer's transformation

Theorem 14.2.1: Let $x(t)$ be a non-trivial solution of the general second order differential equation (14.2.1) on $[a, b]$. If we use the transformation

$$\rho^2 = x^2 + p^2 x'^2; \phi = \tan^{-1} \frac{x}{px'} \tag{14.2.2}$$

Then the equation (14.2.1) reduces to

$$\phi' = \frac{1}{p(t)} \cos^2 \phi + q(t) \sin^2 \phi \tag{14.2.3}$$

$$\rho' = -\left(q(t) - \frac{1}{p(t)}\right) \rho \, sin\phi \, cos\phi \tag{14.2.4}$$

Proof: By simplifying relation (14.2.2), we get

$$\rho^2 = x^2 + x^2 \cot^2 \phi \Longrightarrow x = \rho \sin \phi \tag{14.2.5}$$

and

$$px' = xcot \, \phi \Longrightarrow px' = \rho \cos \phi \tag{14.2.6}$$

Now differentiating (14.2.2) with respect t, we get

$$\phi' = \frac{1}{1 + \frac{x^2}{(px')^2}} \cdot \frac{px'^2 - x(px')'}{(px')^2} = \frac{px'^2 - xq(t)x}{(px')^2 + x^2}$$

$= \frac{px'^2 - q(t)x^2}{\rho^2}$, Using (14.2.1) and (14.2.2)

$= \frac{1}{p(t)} \cos^2 \phi + q(t) \sin^2 \phi$, Using (14.2.5) and (14.2.6)

and$\rho' = \frac{1}{2} \frac{[2xx' + 2px'(px')']}{\sqrt{x^2 + p^2 x'^2}} = \frac{1}{\rho}\left[x \frac{\rho}{p(t)} \cos \phi - \rho \cos \phi q(t) \, x\right]$using (14.2.1) and (14.2.6)

$$= -\left(q(t) - \frac{1}{p(t)}\right)\rho\, sin\phi\, cos\phi$$

Hence the result gets proved.

Theorem 14.2.2: Suppose in differential equation (14.2.1),$p(t) > 0$ and$q(t)$ are continuous on $[a, b]$ and $x(t)$ is a non-trivial solution of (14.2.1). Also let $x(t)$ has exactly n zero at

$t = t_1, t_2, \dots \dots \dots, t_n$ Where $t_1 < t_2 < t_3 \; \dots \dots \dots < t_n$ on [a,b].

If $\phi(t)$ is a function defined by (14.2.1), then

$\phi(t_k) = k\pi$ and $\phi(t)\begin{cases} > k\pi \;\; if \;\; t_k < t \le b \\ < k\pi \;\; if \;\; a < t \le t_k \end{cases}$

Proof: If $t = t_1, t_2, \dots \dots \dots, t_n$ are the zeros of $x(t)$, then from $\phi = \tan^{-1}\frac{x}{px'}$

It follows that

$\phi(t) = 0\;(mod\;\pi)$at$t = t_k(k = 1{,}2{,}3, \dots \dots \dots .., n)$.

Thus for these values of t, we have

$\phi' = \frac{1}{p(t)} > 0$ (14.2.7)

From continuity of $\phi(t)$, equation (14.2.7) implies that $\phi(t)$ is increasing in some neighbourhood of the points $t = t_k(k = 1{,}2{,}3, \dots \dots ., n)$.

Hence if $\phi(t^*) \ge n\pi$ for some $t^* \in [a, b]$, it follows that $\phi(t) \ge n\pi$ for all $t \in (t^*, b]$.

Also if $\phi(t^*) \le n\pi$, then $\phi(t) \le n\pi$ for all $t \in [a, t^*)$. This proves the theorem.

14.4 Principal solutions

A homogeneous, linear second order equation with real-valued coefficient functions defined on an interval J is said to be oscillatory on J if one (and/or every) real-valued solution (≠0) has infinitely many zeros on J. Conversely, when every solution (≠0) has at most a finite number of zeros on J, it is said to be non-oscillatory on J. In the latter case, the equation is said to be disconjugate on J if every solution ((≠0)) has at most one zero on J. If $(t = \omega)$ is a (possibly infinite) endpoint of J which does not belong to $(t = \omega)$, then the equation is said to be oscillatory at $(t = \omega)$ if one (and/or every) real-valued solution (≠0) has an infinite sequence of zeros if one (and/or every) real-valued solution (≠0) has infinitely many zeros on J. Conversely, when every solution (≠0) has at most a finite number of zeros on J, it is said to be non-oscillatory on J. In the latter case, the equation is said to be disconjugate on J if every solution (≠0) has at most one zero on J. If $(t = \omega)$ is a (possibly infinite) endpoint of J which does not belong to J, then the equation is said to be oscillatory at $(t = \omega)$ if one (and/or every) real-valued solution (≠0) has an infinite sequence of zeros clustering at $t = \omega$. Otherwise it is called non-oscillatory at $t = \omega$.

Theorem14.3.1: Let p(t)> 0, q(t) be real-valued, continuous functions on a t-interval J. Then

$[p(t)x']' + q(t)x = 0$ (14.3.1)

is disconjugate on J if and only if, for every pair of distinct points$t_1, t_2 \in J$and arbitrary numbers u_1, u_2 there exists a unique solution $u = u^*(t)of$ of (14.3.1) satisfying

$u^*(t_1) = u_1$and$u^*(t_2) = u_2$; (14.3.2)

or, equivalently, if and only if every pair of linearly independent solutions $u(t)$& $v(t)$of (14.3.1) satisfy

$u(t_1)v(t_2) - u(t_2)v(t_1) \neq 0$ (14.3.3)

for distinct points $t_1, t_2 \in J$.

Proof: Let $u(t), v(t)$be a pair of linearly independent solutions of (14.3.1). Then any solution $u^*(t)$) is of the form $u^* = c_1u(t) + c_2u(t)$.This solution satisfies (14.3.2) if and only if

$c_1u(t_1) + c_2v(t_1) = u_1, c_1u(t_2) + c_2v(t_2) = u_2$.

These linear equations for c_1, c_2have a solution for all u, u_2if and only if (14.3..3) holds. In addition, they have a solution for all u, u_2if and only if the only solution of

$$c_1u(t_1) + c_2v(t_1) = 0, c_1u(t_2) + c_2v(t_2) = 0$$

is $c_1, = c_2 = 0$; i.e., if and only if the only solution $u^*(t)$ of (14.3.1) with two zeros $t = t_1, t_2$ is $u^*(t) = 0$.

Definition 14.3.2: Let p(t) > 0, q(t) be real-valued and continuous on an interval J. Then(14.3.1) is non- oscillatory on J if and only if every pair of linearly independent solutions u(t) and v(t) of (14.3.1) satisfy

$$\int \frac{dt}{p(t)(|u|^2 + |v|^2)} < \infty$$

Furthermore, (14.3.1) is disconjugate on J if and only if

$$|c| \int_a^b \frac{dt}{p(t)(u^2 + v^2)} < \pi$$

for every pair of real-valued solutions u(t), v(t) satisfyingp(u′v — uv′) ≠ 0and every interval $[a, b] \subset J$.

If J is a half-open interval, say J = a ≤ t < $\omega(\leq \omega)$ and (14.3.1) is non-oscillatory at t = ω, then (14.3.1) has real-valued solutions u(t) for which$\int_0^\infty dt/pu^2$ is convergent and solutions for which it is divergent. The latter type of solution will be called a principal solution of (14.3.1) at t = ω.

Theorem 14.3.3: Let p(t) > 0, q(t) be real-valued and continuous on an interval J = a ≤ t < $\omega(\leq \omega)$. And such that (14.3.1) is non- oscillatory at t = ω.then there exists a real-valued solution u = u_0(t) of (14.3.1)which is uniquely determined up to a constant factor by any one of the following conditions in which $u_1(t)$(0 denotes an arbitrary real-valued solution linearly independentof u_0(t):

(i) u_0, u_1 satisfy

$$\frac{u_0(t)}{u_1(t)} \to 0 \text{ as } t \to \omega \tag{14.3.4}$$

(ii) u_0, u_1 satisfy

$$\int \frac{dt}{p(t)u_0^2(t)} < \infty \text{ and} |c| \int_a^b \frac{dt}{p(t)u_1^2(t)} < \infty \tag{14.3.5}$$

(iii) if a$T \in J$ exceeds the largest zero, if any, ofu_0 (t)and ifu_1(T) ≠ 0 thenu_1 (t) has no zero on T < $t < \omega$ according as

$$\frac{u_1'}{u_1} < \frac{u_0'}{u_0} \text{or} \frac{u_1'}{u_1} > \frac{u_0'}{u_0} \tag{14.3.6}$$

holds at $t = T$; in particular (14.3.6) holds for all t ∈ J near ω.

It is understood that in (14.3.4) and (14.3.5) only t-values exceeding the largest zeros, if any,u_0, u_1 are considered. A solution $u_0(t)$ satisfying one (and/or) all of the conditions (i), (ii), (iii) will be called a *principal solution* of (14.3.1) (at t = ω). A solution *u(t)* linearly independent of $u_0(t)$ will be termed a *nonprincipal solution* of (6.1) (at t = ω). In view of (14.3.4), (14.3.5), the terms "principal" and "nonprincipal" might well be replaced by "small" and "large." The expressions "small," "large" will not be used in this context because of the relative nature of these terms. Consider, e.g., the. equations $u'' — u = 0$, $u'' = 0$ and at $u'' + \frac{u}{4t^2} = 0 \; for \; t \geq 1$. Examples of principal and nonprincipal solutions at att = ∞ for the first equation are $u = e^{-t}$and $u = e^t$; for the second, u =1 and $u = t$; for the third $u = t^{\frac{1}{2}}$and $u = t^{\frac{1}{2}} \log t$. The proof of (ii) will lead to the following:

Corollary 14.3.4: Assume the conditions of Theorem 14.3.2. Let$u = u(t) \neq 0$ be any real-valued solution of (14.3.1) and let t = T exceed its last zero. Then

$$u_1(t) = u(t) \int_T^t \frac{ds}{p(s)u^2(s)} \tag{14.3.7}$$

is a nonprincipal solution of (14.3.1) on $T \leq t < \omega$. If, in addition, u(t) is anonprincipal solution of (14.3.1), then

$$u_0(t) = u(t) \int_t^{\omega} \frac{ds}{p(s)u^2(s)} \tag{14.3.8}$$

is a principal solution on $T \leq t < \omega$.

Proof of Theorem 14.3.3 and Corollary 14.3.4

On (i). Let u(t), v(t) be a pair of real-valued linearly independent solutions of (14.3.1) such that

$$p(u'v - uv') = c \neq 0. \tag{14.3.9}$$

If Texceeds the largest zero, if any, of v(t), then (14.3.9) is equivalent to

$$\left(\frac{u}{v}\right)' = \frac{c}{pv^2} \neq 0, \tag{14.3.10}$$

for $T \leq t < \omega$. Hence u/v is monotone on this t-range and so

$$C = \lim_{t \to \omega} \frac{u(t)}{v(t)} \tag{14.3.11}$$

Exists if $C = \pm\infty =$ is allowed.

It will be shown that u, v can be chosen so that C = 0 in (14.3.11). If thisis granted and if u(t) is called$u_0(t)$, then (i) holds. In fact, a solution$u_1(t)$is linearly independent of $u_0(t)$ if and only if it is of the form $u_1(t) = c_0u_0(t) + c_1v(t)$and $c_1 \neq 0$ in which case, C =0 implies that $[C_1 + O(1)]v(t) = c_0u(t) + c_1v(t)$ thus $u_0(t) = O(u_1)$ as $t \to \omega$.
If $C = \pm\infty$ in (14.3.11) and if u,v are interchanged, then (14.3.11) holdswith C = 0. If $|C| < \infty$ and if u(t) $-$ Cv(t) is renamed u(t), then (14.3.9)still holds and (14.3.11) holds with C = 0. This proves (i).

On (ii). Note that (14.3.10), (14.3.11) give

$$C = \frac{u(t)}{v(t)} + c\int_T^{\infty} \frac{ds}{p(s)v^2(s)}$$

Whether or not |C|= ∞ or |C| < ∞. If u, v is a pair u_0, u_1, so thatC = 0, then first part of (14.3.5) holds. If u, v is a pair u_0, u_1, so thatC = $\pm\infty$, then second part of (14.3.5) holds.

On Corollary 14.3.4: Note that if u(t) is a solution of (14.3.1) and u(t)$\neq$ 0forT $\leq t < \omega$, then (14.3.7) defines a solution $u_1(t)$ linearly independent ofand that the same is true of (14.3.6) when the integral is convergent.

On (iii). Since $u_0(t), u_1(t)$can be replaced by $-u_0(t), -u_1(t)$respectively, without affecting the zeros of $u_1(t)$or the inequalities (14.3.6), it can be supposed that

$$u_0(t) > 0 \text{for} T \leq t < \omega \text{ and } u_1(t) > 0. \tag{14.3.12}$$

Multiplying (14.3.6) by $u_0(T)u_1(T) > 0$ shows that the case (14.3.9), where (u, v) = (u_1, v_1) holds with c < 0 or c > 0 according as (14.3.6) holds.

Hence $\frac{u_1(T)}{u_0(T)} \to \pm\infty$ as $t \to \infty$according as (14.3.6) holds. Since $u_1(T)/u_0(T) > 0$ and, by the Sturm separation theorem, u_1 has at most one zero on $T < t < \omega$, the statement concerning the zeros of $u_1$on $T < t < \omega$ follows.

It remains to show that property (iii) is characteristic of a principal solution; i.e., if $u_0(t)$has the property (iii) for every solution $u_1(t)$linearly independent of $u_0(t)$, then $u_0(t)$ is a principal solution. In particular (14.3.6) holds for $t(\in J)$near ω. Consequently $|u_0(t)| \leq const. |u_1(t)|$ for $t \to \omega$.This is a contradiction if $u_0(t)$is not a principal solution and $u_1(t)$is chosen to be a principal solution.

Summary

- The behaviour ofnon-oscillatory solutions arediscussed.
- The zeros of the solutions are determined with the help of Puffer's transformation.
- The relation between the zeros of solution and oscillatory behaviour.

- The discongugate properties are explained.
- The Principal solutions of non-oscillatory equation elaborated with the help of derived results.

Keywords

- Zeros of solution
- Non oscillatory equation
- Oscillatory equation
- Discongugate
- Principal solutions

Self Assessment

1. The differential equation $\boldsymbol{x''' - x'' + 11x' - 4x = 0}$ is
A. Oscillatory equation
B. Non-oscillatory equation
C. Non-linear equation
D. None of these

2. The differential equation $\boldsymbol{x'' - 9x = 0}$ is
A. Oscillatory equation
B. Non-oscillatory equation
C. Non-linear equation
D. None of these

3. The differential equation $\boldsymbol{x'' + 4x = 0}$ is
A. Linear equation
B. Non-oscillatory equation
C. Non-linear equation
D. None of these

4. If some of trivial solution of $\boldsymbol{x'' + a(t)x = 0}$ are non-oscillatory and $\boldsymbol{b(t) \geq a(t)}$, then some non-trivial solutions of $\boldsymbol{x'' + b(t)x = 0}$ must be
A. Oscillatory equation
B. Non-oscillatory equation
C. Non-linear equation
D. None of these

5. Which of the following is non-oscillatory equation?
A. $\boldsymbol{x''} + e^{t}\boldsymbol{x} = 0$
B. $x'' + (t + e^{-2t})x = 0$
C. $x'' - (t - \sin t)x = 0, t \geq 0$
D. None of these

6. If $\boldsymbol{q(t)} < 0$ and $\boldsymbol{u(t)}$ is a non-trivial solution of equation $\boldsymbol{u''(t) + q(t)u(t) = 0}$ then
A. $u(t)$ has at most one zero
B. $\boldsymbol{u(t)}$ has at least one zero
C. $\boldsymbol{u(t)}$ has more one zero
D. None of these

7. If some of non-trivial solution of $\boldsymbol{x'' + a(t)x = 0}$ are oscillatory and $\boldsymbol{b(t) \geq a(t)}$, then some non-trivial solutions of $\boldsymbol{x'' + b(t)x = 0}$ must be
A. Oscillatory equation
B. Non-oscillatory equation

C. Non-linear equation
D. None of these

8. The non-trivial solution of the differential equation $\boldsymbol{x'' + k(t)x = 0}$ can vanish more than once in the interval [a,b] if
A. $k(t) \geq 0$
B. $k(t) \leq 0$
C. $k(t) = 0$
D. None of these

9. The Euler equation $\boldsymbol{x'' + \frac{k}{t^2}x = 0}$ is non –oscillatory if
A. $k \leq \frac{1}{4}$
B. $k \geq \frac{1}{4}$
C. $k = \frac{1}{4}$
D. None of these

10. The equation $\boldsymbol{x'' - \frac{t}{logt}x = 0,\ t \geq 1}$ is
A. Oscillatory equation
B. Non-oscillatory equation
C. Linear equation
D. None of these

11. The equation is called disconjugate if
A. Every solution $(\neq 0)$ has at least one zero.
B. Every solution $(\neq 0)$ has at most one zero.
C. Every solution $(\neq 0)$ has no zero.
D. None of these

12. The non-oscillatory equation is known as
A. Conjugate
B. Disconjugate
C. Stable
D. None of these

13. The equation $\boldsymbol{(p(t)u')' + q(t)u = 0}$ is discojugate if for every pair of distinct point $\boldsymbol{t_1, t_2}$ has
A. Every pair of linearly independent solution
B. Every pair of linearly dependent solution
C. Every pair of constant solution
D. None of these

14. A pair of solutions $\boldsymbol{u\ and\ v}$ on distinct point $\boldsymbol{t_1, t_2}$are L.I if
A. $u(t_1)v(t_2) + v(t_1)u(t_2) \neq 0$
B. $u(t_1)v(t_2) - v(t_1)u(t_2) = 0$
C. $u(t_1)v(t_2) - v(t_1)u(t_2) \neq 0$
D. None of these

15. The non-oscillatory divergent solution is called
A. Non principal solution
B. Principal solution
C. Permanent solution
D. None of these

Answers for Self Assessment

1.	B	2.	B	3.	A	4.	B	5.	C
6.	A	7.	A	8.	B	9.	A	10.	B
11.	B	12.	A	13.	C	14.	B	15.	B

Review Questions

1. Check for the following equations are oscillatory and non-oscillatory :

   (i) $tx' + \frac{x}{t} = 0$

   (ii) $x'' + x'/t + x = 0$

   (iii) $tx'' + (1-t)x' + nx = 0, n$ is a constant (Laguerre's equation)

   (iv) $x'' - 2tx' + 2nx = 0, n$ is a constant (Hermite's equation)

   (v) $tx'' + (2n-1)x' + tx = 0, n$ is a constant

   (vi) $t^2x'' + ktx' + nx = 0, k, n,$ are constants.

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