Real Analysis I

DEMTH515

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LOVELY PROFESSIONAL UNIVERSITY

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CONTENT

Unit 01: The Riemann-Stieltjes integral

CONTENTS

Objectives

After studying this unit, students will be able to:

- define upper and lower Riemann-Stieltjes integral
- describe the condition of Riemann-Stieltjes integrability in terms of upper and lower Riemann-Stieltjes integral
- establish relation between Riemann-Stieltjes integral and Riemann integral
- define necessary and sufficient condition for Riemann-Stieltjes integrability
- understand theorems related to Riemann-Stieltjes integrability

Introduction

Riemann-Stieltjes integral is the generalization of Riemann integral. It is based on the definition of Riemann integral. For the sake of convenience, we are giving the definition and preliminaries of Riemann integral.

1.1 Definition and Existence of the Integral

Definition: Let $f:[a,b]\to\mathbb{R}$ be a bounded function and

$$P=\{a=x_0,x_1,x_2,\dots,x_n\}$$

be the partition of [a, b] such that $x_0\le x_1\le x_2\le\cdots\le x_n$.

If P and P^* be the partition of the interval [a, b] such that $P^*\supset P$ then P^* is known as the refinement of P.

If $P_1\cup P_2=P$ then P is the common refinement for P_1 and P_2.

We write

$m_i=\inf f(x), x_{i-1}\le x\le x_i,$

$M_i=\sup f(x), x_{i-1}\le x\le x_i,$

$$m \le f(x) \le M, x \in [a,b], \text{and}$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\Rightarrow \Delta x_i \ge 0$$

$$\Rightarrow \sum_{i=1}^{n} \Delta x_i = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$$

$$= x_n - x_0$$

Thus, we get,

$$\sum_{i=1}^{n} \Delta x_i = b - a$$

Put

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

and finally

$$\inf_P U(P,f) = \overline{\int_a^b} f(x)\,dx$$

$$\sup_P L(P,f) = \underline{\int_a^b} f(x)\,dx$$

If

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

Then f is Riemann integrable and we write $f \in \mathcal{R}$ and the common value is written as

$$\int_a^b f(x)\,dx\ .$$

Now,

$$m \le m_i \le M_i \le M$$

$$\Rightarrow \sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le \sum_{i=1}^{n} M \Delta x_i$$

$$\Rightarrow m \sum_{i=1}^{n} \Delta x_i \le L(P,f) \le U(P,f) \le M \sum_{i=1}^{n} \Delta x_i$$

$$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Thus, the numbers $L(P,f)$ and $U(P,f)$ form a bounded set which shows that upper and lower integrals are defined for every bounded function$f$. Under what circumstances these two integrals are equal? This is a delicate issue. Instead of handling it separately for Riemann integral, we will consider now the more general case.

1.2 The Riemann -Stieltjes Integral

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and α is a monotonically increasing function. Corresponding to the partition $P=\{a=x_0,x_1,x_2,\dots,x_n\}$ of [a, b] such that $x_0\le x_1\le x_2\le\cdots\le x_n$, we define

$$\Delta\alpha_i=\alpha(x_i)-\alpha(x_{i-1})\ \forall i$$

$$\Rightarrow \Delta\alpha_i\ge 0 \text{ as } \alpha \text{ is monotonically increasing.}$$

Now

$$\sum_{i=1}^{n}\alpha_i=\Delta\alpha_1+\Delta\alpha_2+\cdots+\Delta\alpha_n$$

$$=[\alpha(x_1)-\alpha(x_0)]+[\alpha(x_2)-\alpha(x_1)]+\cdots+[\alpha(x_n)-\alpha(x_{n-1})]$$

$$=\alpha(x_n)-\alpha(x_0)$$

$$=\alpha(b)-\alpha(a).$$

Further, we put

$$L(P,f,\alpha)=\sum_{i=1}^{n}m_i\Delta\alpha_i\,,\ m_i=\inf f(x),\ x_{i-1}\le x\le x_i$$

$$U(P,f,\alpha)=\sum_{i=1}^{n}M_i\Delta\alpha_i\,,M_i=\sup f(x),\ x_{i-1}\le x\le x_i$$

$$\Rightarrow L(P,f,\alpha)\le U(P,f,\alpha).$$

We define,

$$\sup_P L(P,f,\alpha)=\int_{\underline{a}}^{b}f\,d\alpha$$

$$\Rightarrow \int_{\underline{a}}^{b}f\,d\alpha\ge L(P,f,\alpha)$$

and

$$\inf_P U(P,f,\alpha)=\int_{a}^{\bar{b}}f\,d\alpha$$

$$\Rightarrow \int_{a}^{\bar{b}}f\,d\alpha\le U(P,f,\alpha)$$

Now,

$$m\le m_i\le M_i\le M$$

$$\Rightarrow \sum_{i=1}^{n}m\Delta\alpha_i\le\sum_{i=1}^{n}m_i\Delta\alpha_i\le\sum_{i=1}^{n}M_i\Delta\alpha_i\le\sum_{i=1}^{n}M\Delta\alpha_i$$

$$\Rightarrow m\sum_{i=1}^{n}\Delta\alpha_i\le L(P,f,\alpha)\le U(P,f,\alpha)\le M\sum_{i=1}^{n}\Delta\alpha_i$$

$$\Rightarrow m[\alpha(b)-\alpha(a)]\le L(P,f,\alpha)\le U(P,f,\alpha\le M[\alpha(b)-\alpha(a)].$$

Now if

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha,$$

then we denote their common value by

$$\int_a^b f\,d\alpha$$

or

$$\int_a^b f(x)\,d\alpha(x).$$

This is the Riemann-Stieltjes integral or simply the Stieltjes integral. Here we say that f is integrable with respect to α, in the Riemann sense and write, $f \in \mathcal{R}(\alpha)$.

If $\alpha(x) = x$ then Riemann -Stieltjes integral becomes Riemann integral. Or we can say that Riemann integral is the special case of Riemann -Stieltjes integral.

Theorem 1.2.1: If P^* is a refinement of P then

$(i)\, U(P^*, f, \alpha) \le U(P, f, \alpha)$

$(ii)\, L(P, f, \alpha) \le L(P^*, f, \alpha)$.

Proof: Let $P = \{a = x_0, x_1, x_2, \dots, x_{j-1}, x_j, \dots, x_n\}$

and $P^* = \{a = x_0, x_1, x_2, \dots, x_{j-1}, y, x_j, \dots, x_n\}$

$m_i = \inf f(x),\ x_{i-1} \le x \le x_i$

$M_i = \sup f(x),\ x_{i-1} \le x \le x_i$

W_1 *and* w_1 be the supremum and infimum of $f(x), x \in [x_{j-1}, y]$

W_2 *and* w_2 be the supremum and infimum of $f(x), x \in [y, x_j]$

Since $\sup A \le \sup B$ and $\inf A \ge \inf B$ whenever $A \subseteq B$.
Here we have,

$[x_{j-1}, y] \subseteq [x_{j-1}, x_j]$ *and*

$[y, x_j] \subseteq [x_{j-1}, x_j]$

Therefore $W_1 \le M_j, W_2 \le M_j, and\ w_1 \ge m_j, w_2 \ge m_j \quad \dots (1)$

Since $U(P^*, f, \alpha) = \sum_{i=1}^{j-1} M_i \Delta\alpha_i + W_1[\alpha(y) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(y)] + \sum_{i=j+1}^{n} M_i \Delta\alpha_i$

$$U(P, f, \alpha) = \sum_{i=1}^{j-1} M_i \Delta\alpha_i + M_j \Delta\alpha_j + \sum_{i=j+1}^{n} M_i \Delta\alpha_i$$

$\therefore U(P^*, f, \alpha) - U(P, f, \alpha) = W_1[\alpha(y) - \alpha(x_{j-1})] + W_2[\alpha(x_j) - \alpha(y)] - M_j \Delta\alpha_j$

$$\le M_j[\alpha(x_j) - \alpha(x_{j-1})] - M_j \Delta\alpha_j \qquad \{by (1)\}$$

$$= M_j \Delta\alpha_j - M_j \Delta\alpha_j$$

$$= 0$$

$\Rightarrow U(P^*, f, \alpha) - U(P, f, \alpha) = 0$

$U(P^*, f, \alpha) \leq U(P, f, \alpha)$

This completes the proof of the first part.

Now we consider,

$$L(P^*, f, \alpha) - L(P, f, \alpha) = w_1[\alpha(y) - \alpha(x_{j-1})] + w_2[\alpha(x_j) - \alpha(y)] - m_j\Delta\alpha_j$$

$$\geq m_j[\alpha(y) - \alpha(x_{j-1}) + \alpha(x_j) - \alpha(y)] - m_j\Delta\alpha_j$$

$$= m_j\Delta\alpha_j - m_j\Delta\alpha_j$$

$$= 0$$

$\Rightarrow L(P^*, f, \alpha) - L(P, f, \alpha) \geq 0$

$$\Rightarrow L(P, f, \alpha) \leq L(P^*, f, \alpha).$$

This completes the proof of the second part.

Cor 1: $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$

Proof: Let $P^* = P_1 \cup P_2$

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$

$\Rightarrow L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

Keeping P_2 fixed and taking supremum over all partitions P_1, we get

$$\underline{\int_a^b} f\,d\alpha \leq U(P_2, f, \alpha)$$

Now by taking infimum over all partitionsP_2, we get

$$\Rightarrow \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha.$$

Cor 2: $L(P, f, \alpha) \leq \underline{\int_a^b} f\,d\alpha$ and

$$U(P, f, \alpha) \geq \overline{\int_a^b} f\,d\alpha$$

Also, we have

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

Therefore,

$$L(P, f, \alpha) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P, f, \alpha)$$

If $f \in \mathcal{R}(\alpha)$ then $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$

So

$$L(P, f, \alpha) \le \int_a^b f d\alpha \le U(P, f, \alpha)$$

Cor 3: If $f \in \mathcal{R}(\alpha)$ on [a, b] and m, M are the lower and upper bounds of f defined on $[a, b]$ then

$$m[\alpha(b) - \alpha(a)] \le \int_a^b f d\alpha \le M[\alpha(b) - \alpha(a)]$$

Since,

$$m[\alpha(b) - \alpha(a)] \le L(P, f, \alpha) \le U(P, f, \alpha) \le M[\alpha(b) - \alpha(a)]$$

and

$$L(P, f, \alpha) \le \int_a^b f d\alpha \le U(P, f, \alpha)$$

$$\therefore m[\alpha(b) - \alpha(a)] \le L(P, f, \alpha) \le \int_a^b f d\alpha \le U(P, f, \alpha) \le M[\alpha(b) - \alpha(a)]$$

$$\Rightarrow m[\alpha(b) - \alpha(a)] \le \int_a^b f d\alpha \le M[\alpha(b) - \alpha(a)].$$

Theorem 1.2.2: Let f and α be bounded functions on $[a, b]$ and α be monotonically increasing on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ if and only if for every $\in > 0$, there exists a partition of $[a, b]$such that $U(P, f, \alpha) - L(P, f, \alpha) < \in$.

Proof: Suppose $f \in \mathcal{R}(\alpha)$.

$$\therefore \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$$

Let $\in > 0$ be any number.

Since $\inf_P U(P, f, \alpha) = \overline{\int_a^b} f d\alpha$

$\Rightarrow \overline{\int_a^b} f d\alpha + \frac{\in}{2}$ is not the lower bound of this set.

So there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f, \alpha) < \overline{\int_a^b} f d\alpha + \frac{\in}{2}$$

$$= \int_a^b f d\alpha + \frac{\in}{2} \qquad \because f \in \mathcal{R}(\alpha)$$

$$\therefore U(P_1, f, \alpha) < \int_a^b f d\alpha + \frac{\in}{2} \qquad \ldots(1)$$

Further,

$$\sup_P L(P,f,\alpha) = \underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - \frac{\epsilon}{2} \text{ can not be the upper bound of this set.}$$

$\therefore$ there exists a partition P_2 of $[a, b]$ such that

$$L(P_2,f,\alpha) > \underline{\int_a^b} f\,d\alpha - \frac{\epsilon}{2}$$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha < L(P_2,f,\alpha) + \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b f\,d\alpha < L(P_2,f,\alpha) + \frac{\epsilon}{2} \qquad \ldots.(2)$$

Let $P = P_1 \cup P_2$

Now,

$$U(P,f,\alpha) \le U(P_1,f,\alpha)$$

$$< \int_a^b f\,d\alpha + \frac{\epsilon}{2} \qquad \{by(1)\}$$

$$< L(P_2,f,\alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \{by(2)\}$$

$$\le L(P,f,\alpha) + \epsilon \text{ as } P \text{ is the refinement of } P_2$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

Conversely, let $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

We also have

$$L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \int_a^{\bar{b}} f\,d\alpha \le U(P,f,\alpha)$$

$$\Rightarrow \int_a^{\bar{b}} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha)$$

$< \epsilon$

$$\Rightarrow 0 \le \int_a^{\bar{b}} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le \epsilon$$

$$\Rightarrow \int_a^{\bar{b}} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow f \in \mathcal{R}(\alpha)$$

This completes the proof.

Cor: If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ for some partition P then this result holds for every refinement P^* of P.

Proof: We have,

$U(P^*, f, \alpha) \leq U(P, f, \alpha)$

and

$L^*(P, f, \alpha) \geq L(P, f, \alpha)$

$$\Rightarrow -L(P^*, f, \alpha) \leq -L(P, f, \alpha)$$

Therefore,

$$U(P^*, f, \alpha) - L^*(P, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$$

$< \in$

Thus, we get

$$U(P^*, f, \alpha) - L^*(P, f, \alpha) < \in$$

Theorem 1.2.3: If f is a constant function defined by $f(x) = k \ \forall x \in [a, b]$ and α is monotonically increasing function on $[a, b]$ then $\int_a^b f d\alpha$ exists and is equal to $k[\alpha(b) - \alpha(b)]$.

Proof: Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be the partition of [a, b] such that $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$,

$m_i = \inf f(x), x_{i-1} \leq x \leq x_i$,

$M_i = \sup f(x), x_{i-1} \leq x \leq x_i$

Since $f(x) = k \ \forall x \in [a, b]$.

Therefore, $m_i = M_i = k \ \ \forall i$

Now $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$$= k \sum_{i=1}^{n} \Delta\alpha_i$$

$$= k[\alpha(b) - \alpha(a)]$$

and $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

$$= k \sum_{i=1}^{n} \Delta\alpha_i$$

$$= k[\alpha(b) - \alpha(a)]$$

$$\underline{\int_a^b} f d\alpha = \sup_P L(P, f, \alpha)$$

$$= \sup_P k[\alpha(b) - \alpha(a)]$$

$$= k[\alpha(b) - \alpha(a)]$$

$$\overline{\int_a^b} f d\alpha = \inf_P U(P, f, \alpha)$$

$$= \inf_P k[\alpha(b) - \alpha(a)]$$

$= k[\alpha(b) - \alpha(a)]$

Thus, we get

$$\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha = k[\alpha(b) - \alpha(a)]$$

$$\Rightarrow \int_a^b f\, d\alpha = k[\alpha(b) - \alpha(a)]$$

This completes the proof.

Theorem 1.2.4: If $U(P, f, \alpha) - L(P, f, \alpha) < \in$ for a partition P of $[a, b]$ and if s_i and t_i are arbitrary points of $[x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\, \Delta\alpha_i < \in.$$

Proof: Let $m_i = \inf f(x), x_{i-1} \le x \le x_i,$

$M_i = \sup f(x), x_{i-1} \le x \le x_i$

Since $s_i, t_i \in [x_{i-1}, x_i]$

Therefore,

$$f(s_i), f(t_i) \in [m_i, M_i]$$

$$\Rightarrow |f(s_i) - f(t_i)| \le M_i - m_i$$

$$\Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \le \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$\le \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= U(P, f, \alpha) - L(P, f, \alpha)$$

$< \in$

Thus, we get

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\, \Delta\alpha_i < \in.$$

This completes the proof.

Theorem 1.2.5: If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ then for every $\in > 0$, there exists a partition

$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ such that

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\, d\alpha \right| < \in, t_i \in [x_{i-1}, x_i].$$

Proof: Since $f \in \mathcal{R}(\alpha)$ on $[a, b]$, therefore for given $\in > 0$, there exists a partition P of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \in.$

Let $m_i = \inf f(x), x_{i-1} \le x \le x_i,$

$M_i = \sup f(x), x_{i-1} \le x \le x_i$

Since$t_i \in [x_{i-1}, x_i]$

$\Rightarrow f(t_i) \in [m_i, M_i]$

$\Rightarrow m_i \le f(t_i) \le M_i$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le U(P, f, \alpha) \quad \dots(1)$$

We also have,

$$L(P, f, \alpha) \le \int_a^b f d\alpha \le U(P, f, \alpha) \quad \dots(2)$$

Thus, using relations (1) and (2), we get

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < U(P, f, \alpha) - L(P, f, \alpha)$$

$< \in.$

Thus, we get

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \in, t_i \in [x_{i-1}, x_i].$$

This completes the proof.

Theorem1.2.6: If f is a continuous function on $[a, b]$ then $f \in \mathcal{R}(\alpha), \alpha$ is monotonically increasing function on $[a, b]$.

Proof: Let $\epsilon > 0$ and choose $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$.

Since f is continuous on $[a, b]$ therefore f is uniformly continuous on $[a, b]$.

For above $\eta > 0$, there exits $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x - y| < \delta, \quad \forall x, y \in [a, b]$

Consider partition P of $[a, b]$ such that $\|P\| < \delta$

$\Rightarrow \Delta x_i < \delta, \qquad \forall i$

$\|P\| = Max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$

f is continuous on $[a, b]$, therefore f is bounded and attains its bound on $[a, b]$.

Therefore, there existnumbers$c_i, d_i \in [x_{i-1}, x_i]$ such that $f(c_i) = m_i = \inf f(x)$

$f(d_i) = M_i = \sup f(x)$

$M_i - m_i = |M_i - m_i|$

$= |f(d_i) - f(c_i)|$

$< \eta$

$\because |d_i - c_i| \le |x_i - x_{i-1}| < \delta,$

Thus $|d_i - c_i| < \delta \Rightarrow |f(d_i) - f(c_i)| < \eta$.

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i$$

$$< \sum_{i=1}^{n} \eta\,\Delta\alpha_i$$

$$= \eta \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \eta[\alpha(b) - \alpha(a)] < \epsilon$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha)$$

This completes the proof.

Theorem1.2.7: If f is monotonic on $[a,b]$ and α is continuous and monotonically increasing then $f \in \mathcal{R}(\alpha)$.

Proof: Since α is a continuous and monotonically increasing function.

∴ for every positive integer n, we can find a partition P such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}, i = 1,2,\dots n.$$

Define

$$m_i = \inf f(x),\ \ x_{i-1} \le x \le x_i$$

$$= f(x_{i-1})$$

$$M_i = \sup f(x),\ \ x_{i-1} \le x \le x_i$$

$$= f(x_i).$$

Now,

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i)$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \big(f(b) - f(a)\big)$$

$< \epsilon$ when $n \to \infty$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \ \Rightarrow\ f \in \mathcal{R}(\alpha)$

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$$

If f is continuous on $[0, n]$ where n is a positive integer then

$$\int_0^n f(x)\,d[x] = f(1) + f(2) + \cdots + f(n).$$

Example: If $f(x) = x^2$ and $\alpha(x) = x^3$ then evaluate

$$\int_a^b f\,d\alpha,\ a = 0, b = 1.$$

Solution: $\int_a^b f\,d\alpha$

$$= \int_0^1 x^2\,d(x^3)$$

$$= \int_0^1 x^2\,3x^2\,dx$$

$$= 3\int_0^1 x^4\,dx$$

$$= 3\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{5}$$

Example: Evaluate

$$\int_0^3 x\,d\{[x] - x\}$$

Solution: We have

$$\int_0^3 x\,d\{[x] - x\}$$

$$= \int_0^3 x\,d[x] - \int_0^3 x\,dx$$

$$= 1 + 2 + 3 - \left[\frac{x^2}{2}\right]_0^3$$

$$= 6 - \frac{9}{2} = \frac{12 - 9}{2} = \frac{3}{2}$$

Example: Evaluate

$$\int_0^3 e^x \, d\{x - [x]\}$$

Solution: We have

$$\int_0^3 e^x \, d\{x - [x]\}$$

$$= \int_0^3 e^x \, dx - \int_0^3 e^x \, d[x]$$

$$= [e^x]_0^3 - (e + e^2 + e^3)$$

$$= e^3 - 1 - e - e^2 - e^3$$

$$= -(1 + e + e^2)$$

Example: Evaluate

$$\int_0^6 \{x^2 + [x]\} \, d(|3 - x|)$$

Solution: We have

$$|3 - x| = \begin{cases} 3 - x & if\ 3 - x \geq 0 \ i.e.\ x \leq 3 \\ -(3 - x) & if\ 3 - x < 0 \ i.e.\ x > 3 \end{cases}$$

$$\therefore \int_0^6 \{x^2 + [x]\} \, d(|3 - x|) = \int_0^3 \{x^2 + [x]\} \, d(3 - x) + \int_3^6 \{x^2 + [x]\} \, d(x - 3)$$

$$= -\int_0^3 \{x^2 + [x]\} \, dx + \int_3^6 \{x^2 + [x]\} \, dx$$

$$= -\left[\int_0^3 x^2 \, dx + \int_0^3 [x] \, dx\right] + \int_3^6 x^2 \, dx + \int_3^6 [x] \, dx$$

$$= -\left[\int_0^3 x^2 \, dx + \int_0^1 [x] \, dx + \int_1^2 [x] \, dx + \int_2^3 [x] \, dx\right] + \int_3^6 x^2 \, dx + \int_3^4 [x] \, dx + \int_4^5 [x] \, dx$$
$$+ \int_5^6 [x] \, dx$$

$$= -\left\{\left[\frac{x^3}{3}\right]_0^3 + 1 + 2\right\} + \left[\frac{x^3}{6}\right]_3^6 + 3 + 4 + 5 = 63$$

Example: Evaluate

$$\int_\pi^{2\pi} \sin x \, d(\cos x)$$

Solution: We have,

$$\int_{\pi}^{2\pi} \sin x \; d(\cos x)$$

$$= \int_{\pi}^{2\pi} \sin x \,(-\sin x)\; dx$$

$$= -\int_{\pi}^{2\pi} \sin^2 x \; dx$$

$$= -\int_{\pi}^{2\pi} \frac{1-\cos 2x}{2}\; dx$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} (\cos 2x - 1\,)dx$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - x\right]_{\pi}^{2\pi}$$

$$= -\frac{\pi}{2}$$

Theorem1.2.8: Let $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $m \le f(x) \le M\ ,\forall\ x \in [a,b]$.Let$\Phi$: $[m,M] \to \mathbb{R}$ be continuous function. Then $h = \Phi \circ f \in \mathcal{R}(\alpha)on[a,b]$.

Proof: Let $\epsilon > 0$ be given. AsΦiscontinuous in $[m,M]$ therefore ϕ is uniformly continuous in $[m,M]$.

So for given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$|\Phi(s) - \Phi(t)| < \epsilon \quad \forall\ s,t \in [m,M]$ whenever $|s-t| < \delta \qquad \ldots(1)$

We assume $\delta < \epsilon$.

Since $f \in \mathcal{R}(\alpha)$ on$[a,b]$.

$\therefore$for given$\delta^2 > 0, \exists$ a partition

$P = \{a = x_0,\ x_1, x_2, \ldots x_n = b\}$ of $[a,b]$ such that

$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2 \qquad \ldots(2)$

Let

$$m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\}$$

$$M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\}$$

$$m_i' = \inf\{h(x): x \in [x_{i-1}, x_i]\}$$

$$M_i' = \sup\{h(x): x \in [x_{i-1}, x_i]\}$$

Divide the numbers $i = 1,2,\ldots,n$ into two classes A and B where

$A = \{i: M_i - m_i < \delta\}$and $B = \{i: M_i - m_i \ge \delta\}$

$\therefore$ when $i \in A$ and $x_{i-1} \le x < y \le x_i$,we have

$|f(x)-f(y)| \leq M_i - m_i < \delta$

$\Rightarrow |\Phi(f(x)) - \Phi(f(y))| < \epsilon \quad (by\ (1))$

$\Rightarrow |h(x)-h(y)| < \epsilon$

$\Rightarrow |M_i' - m_i'| < \epsilon$

$i.e. M_i' - m_i' < \epsilon$ when $i \in A$

$\therefore \sum_{i\in A}(M_i' - m_i')\Delta\alpha_i < \sum_{i\in A}\epsilon\,\Delta\alpha_i$

$= \epsilon \sum_{i\in A}\Delta\alpha_i$

$= \epsilon\,[\,\alpha(b) - \alpha(a)]$

$i.e. \sum_{i\in A}(M_i' - m_i')\Delta\alpha_i < \epsilon\,[\,\alpha(b) - \alpha(a)]$

Now

$\delta \sum_{i\in B}\Delta\alpha_i \leq \sum_{i\in A}(M_i - m_i)\Delta\alpha_i$

$\leq \sum_{i}(M_i - m_i)\Delta\alpha_i$

$= U(P,f,\alpha) - L(P,f,\alpha)$

$< \delta^2$ (Using (2))

$\therefore \sum_{i\in B}\Delta\alpha_i < \delta < \epsilon \quad \ldots(4)$

Also, for $i \in B$, we have

$M_i' - m_i' = |M_i' - m_i'|$

$\leq |M_i'| + |m_i'|$

$\leq k + k = 2k \quad \ldots(5)$

Where $k = \sup|\Phi(t)|,\ \ m \leq t \leq M$

$\therefore\ U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i}(M_i' - m_i')\Delta\alpha_i$

$= \sum_{i\in A}(M_i' - m_i')\Delta\alpha_i + \sum_{i\in B}(M_i' - m_i')\Delta\alpha_i$

$< \epsilon[\alpha(b) - \alpha(a)] + 2k\epsilon \quad (by\ (3),(4),(5))$

$= [\alpha(b) - \alpha(a) + 2k]\epsilon = \epsilon'$

$\Rightarrow\ U(P,h,\alpha) - L(P,h,\alpha) < \epsilon'$

$\Rightarrow h \in \mathcal{R}(\alpha)$ on $[a,b]$.

This completes the proof.

Cor: If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f^2 \in \mathcal{R}(\alpha)$ and $|f| \in \mathcal{R}(\alpha)$.

Let $\Phi(t) = t^2$, so that Φ is continuous on $[m, M]$ then $\Phi \circ f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Now

$$(\Phi \circ f)(x) = \Phi(f(x))$$

$$= [f(x)]^2$$

$\Rightarrow f^2 \in \mathcal{R}(\alpha)$on $[a, b]$.

Again let,

$\Phi(t) = |t|$ so that Φ is continuous on $[m, M]$ then $\Phi \circ f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Now

$$(\Phi \circ f)(x) = \Phi(f(x)) = |f(x)|$$

$\therefore |f| \in \mathcal{R}(\alpha)$on $[a, b]$.

Summary

- Let $f: [a, b] \to \mathbb{R}$ be a bounded function and α is a monotonically increasing function. Corresponding to the partition $P = \{a = x_0, x_1, x_2, \dots, x_n\}$ of [a, b] such that $x_0 \le x_1 \le x_2 \le \cdots \le x_n$, we define $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})\ \forall i$.Further, we put $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$, $m_i = \inf f(x)$, $x_{i-1} \le x \le x_i$and $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$,$M_i = \sup f(x)$, $x_{i-1} \le x \le x_i$.
- We define, $\sup_P L(P, f, \alpha) = \underline{\int_a^b} f d\alpha$ and $\inf_P U(P, f, \alpha) = \overline{\int_a^b} f d\alpha$ If$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$, then we denote their common value by $\int_a^b f d\alpha$ .This is the Riemann-Stieltjes integral or simply the Stieltjes integral. Here we say that f is integrable with respect to α, in the Riemann sense and write, $f \in \mathcal{R}(\alpha)$.
- If $\alpha(x) = x$ then Riemann -Stieltjes integral becomes Riemann integral. Or we can say that Riemann integral is the special case of Riemann -Stieltjes integral.
- If P^* is a refinement of P then

$(i) U(P^*, f, \alpha) \le U(P, f, \alpha)$

$(ii) L(P, f, \alpha) \le L(P^*, f, \alpha)$.

- $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$
- If $f \in \mathcal{R}(\alpha)$then $L(P, f, \alpha) \le \int_a^b f d\alpha \le U(P, f, \alpha)$
- If $f \in \mathcal{R}(\alpha)$ on [a, b] and m, M are the lower and upper bounds of f defined on $[a, b]$ then $m[\alpha(b) - \alpha(a)] \le \int^b f d\alpha \le M[\alpha(b) - \alpha(a)]$
- Let f and α be bounded functions on $[a, b]$ and α be monotonically increasing on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.
- If f is a continuous function on $[a, b]$ then $f \in \mathcal{R}(\alpha)$, α is monotonically increasing function on $[a, b]$.
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ then for every $\epsilon > 0$, there exists a partition

$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ such that $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \epsilon, t_i \in [x_{i-1}, x_i]$.

- If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ for a partition P of $[a, b]$ and if s_i and t_i are arbitrary points of $[x_{i-1}, x_i]$ then $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$.
- If f is monotonic on $[a, b]$ and α is continuous and monotonically increasing then $f \in \mathcal{R}(\alpha)$.
- $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$
- If f is continuous on $[0, n]$ where n is a positive integer then $\int^{n} f(x) d[x] = f(1) + f(2) + \cdots + f(n)$.
- Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $m \le f(x) \le M$, $\forall x \in [a, b]$. Let $\Phi: [m, M] \to \mathbb{R}$ be continuous function. Then $h = \Phi \circ f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Keywords

Partition of an interval: By a partition P of $[a, b]$ we mean a finite set of points $x_0, x_1, x_2, \dots, x_n$ where $a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$.

Refinement of a partition: If P and P^* be the partition of the interval [a, b] such that $P^* \supset P$ then P^* is known as the refinement of P.

Upper Riemann-Stieltjes

Integral: $\inf_P U(P, f, \alpha) = \overline{\int_a^b} f d\alpha$, $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$, $M_i = \sup f(x)$, $x_{i-1} \le x \le x_i$.

Lower Riemann-Stieltjes

Integral: $\sup_P L(P, f, \alpha) = \underline{\int_a^b} f d\alpha$, $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$, $m_i = \inf f(x)$, $x_{i-1} \le x \le x_i$.

Riemann-Stieltjes Integral: If $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$, then we denote their common value by $\int_a^b f d\alpha$ or $\int_a^b f(x) d\alpha(x)$. This is the Riemann-Stieltjes integral or simply the Stieltjes integral. Here we say that f is integrable with respect to α, in the Riemann sense and write, $f \in \mathcal{R}(\alpha)$.

Necessary and sufficient condition: Let f and α be bounded functions on $[a, b]$ and α be monotonically increasing on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Self-Assessment

Let f be a real-valued bounded function defined on [a, b] and let α be a real-valued monotonically increasing function defined on [a, b]. Further, let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Suppose $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be the partition of [a, b] and $m_i = inf\{f(x): x \in [x_{i-1}, x_i]\}$, $M_i = sup\{f(x): x \in [x_{i-1}, x_i]\}$, $m = inf\{f(x): x \in [a, b]\}$, $M = sup\{f(x): x \in [a, b]\}$.

1) Consider the following statements:

(I) If P^* be the refinement of the partition P then $P \subset P^*$.

(II) If $P = P_1 \cup P_2$ then P is the common refinement of P_1 and P_2

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

2) Riemann-Stieltjes integral becomes Riemann integral if:

A. $\alpha(x) = x^2$

B. $\alpha(x) = 2x$

C. $\alpha(x) = x$

D. none of these

3) $\sum_{i=1}^{n} \Delta\alpha_i =$

A. $\alpha(a) - \alpha(b)$

B. $\alpha(b) - \alpha(a)$

C. $\alpha(a) + \alpha(b)$

D. none of these

4) Select the correct option:

$a)\ L(P,f,\alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i \text{and } U(P,f,\alpha) = \sum_{i=1}^{n} M\,\Delta\alpha_i$

$b)\ L(P,f,\alpha) = \sum_{i=1}^{n} m\,\Delta\alpha_i \text{and } U(P,f,\alpha) = \sum_{i=1}^{n} M\,\Delta\alpha_i$

$c)\ L(P,f,\alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i \text{and } U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$

$d)\ L(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i \text{ and } U(P,f,\alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i$

5) $L(P,f,\alpha) \geq U(P,f,\alpha)$

A. True

B. False

6) Select the correct option:

A. $m_i \leq m \leq M_i \leq M$

B. $m \leq m_i \leq M_i \leq M$

C. $m_i \leq m \leq M \leq M_i$

D. $m_i \leq M_i \leq m \leq M$

7) Select the correct option:

$a)\ m[\alpha(b)-\alpha(a)] \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M[\alpha(b)-\alpha(a)]$

$b)\ m[\alpha(a)-\alpha(b)] \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M[\alpha(a)-\alpha(b)]$

$c)\ m[\alpha(b)-\alpha(a)] \leq U(P,f,\alpha) \leq L(P,f,\alpha) \leq M[\alpha(b)-\alpha(a)]$

d) none of these

8) Select the correct option:

$a)\ \underline{\int_a^b} f\,d\alpha = \inf_P L(P,f,\alpha)$ and $\overline{\int_a^b} f\,d\alpha = \sup_P U(P,f,\alpha)$

$b)\ \underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha)$ and $\overline{\int_a^b} f\,d\alpha = \inf_P L(P,f,\alpha)$

$c)\ \underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha)$ and $\overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha)$

$d)\ \underline{\int_a^b} f\,d\alpha = L(P,f,\alpha)$ and $\overline{\int_a^b} f\,d\alpha = U(P,f,\alpha)$

9) Consider the following statements:

(I) $\underline{\int_a^b} f\,d\alpha \geq \overline{\int_a^b} f\,d\alpha$

(II) If $f \in \mathcal{R}(\alpha)$ on [a, b] then $f^4 \in \mathcal{R}(\alpha)$ on [a, b]. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

10) If f is Riemann-Stieltjes integral then

$a)\ \underline{\int_a^b} f\,d\alpha \geq \overline{\int_a^b} f\,d\alpha$

$b)\ \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$

$c)\ \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

d) none of these

11) For the refinement $P^*$of P, select the correct option:

a)$U(P,f,\alpha) \leq U(P^*,f,\alpha)$ and $L(P,f,\alpha) \leq L(P^*,f,\alpha)$

b) $U(P,f,\alpha) \leq U(P^*,f,\alpha)$ and $L^*(P,f,\alpha) \leq L(P,f,\alpha)$

c)$U(P^*,f,\alpha) \leq U(P,f,\alpha)$ and $L(P,f,\alpha) \leq L(P^*,f,\alpha)$

d)$U(P^*,f,\alpha) \leq U(P,f,\alpha)$ and $L^*(P,f,\alpha) \leq L(P,f,\alpha)$

12) Select the correct option:

a)$L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha$ and $U(P,f,\alpha) \leq \overline{\int_a^b} f\,d\alpha$

b)$L(P,f,\alpha) \geq \underline{\int_a^b} f\,d\alpha$ and $U(P,f,\alpha) \leq \overline{\int_a^b} f\,d\alpha$

c) $L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha$ and $U(P,f,\alpha) \geq \overline{\int_a^b} f\,d\alpha$

d) $L(P,f,\alpha) \geq \underline{\int_a^b} f\,d\alpha$ and $U(P,f,\alpha) \leq \overline{\int_a^b} f\,d\alpha$

13) Select the correct option:

a) $L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha) \leq \overline{\int_a^b} f\,d\alpha$

b) $L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$

c) $L(P,f,\alpha) \leq \overline{\int_a^b} f\,d\alpha \leq \underline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$

d) none of these

14) For $f \in \mathcal{R}(\alpha)$, select the correct option:

a) $L(P,f,\alpha) \leq \int_a^b f\,d\alpha \leq U(P,f,\alpha)$

b) $U(P,f,\alpha) \leq \int_a^b f\,d\alpha \leq L(P,f,\alpha)$

c) $\int_a^b f\,d\alpha \leq L(P,f,\alpha) \leq U(P,f,\alpha)$

d) None of these

15) For $f \in \mathcal{R}(\alpha)$ on [a, b], select the correct option:

a) $m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$

b) $m[\alpha(a) - \alpha(b)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$

c) $m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(a) - \alpha(b)]$

d) None of these

16) If $\overline{\int_a^b} f d\alpha = \inf\{U(P, f, \alpha)\text{: P is the partition of [a,b]}\}$ and

$\underline{\int_a^b} f d\alpha = \sup\{L(P, f, \alpha)\text{: P is partition of [a,b]}\}$.

Then consider the following statements:

(I) For given $\eta > 0$, there exists a partition P_1 of [a, b] such that $U(P_1, f, \alpha) < \overline{\int_a^b} f d\alpha + \eta$.

(II) For given $\eta > 0$, there exists a partition P_2 of [a, b] such that $L(P_2, f, \alpha) > \underline{\int_a^b} f d\alpha - \eta$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

17) Select the correct option.

a) $\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = U(P, f, \alpha) - L(P, f, \alpha)$

b) $\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = L(P, f, \alpha)$

c) $\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = U(P, f, \alpha)$

d) None of these

18) Consider the following statements:

(I) If f is continuous on [a, b] then $f \in \mathcal{R}(\alpha)$ on [a, b].

(II) If f is continuous on [a, b] then f is bounded and attains its bounds on [a, b]. Then

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

19) $\int_0^2 x^2 d(x^2) =$

A. 4

B. 8

C. 16

D. 2

20) $\int_{\pi}^{2\pi} sinx d(cosx) =$

a) $\frac{-\pi}{2}$

b) $\frac{\pi}{2}$

c) $\frac{-3\pi}{2}$

d. 0

21) $\int_0^2 [x] d(x^2) =$

A. 0

B. 1

C. 2

D. 3

22) $\int_{-1}^{2} x^5 d(|x|^3) =$

A. 71/8

B. 177/8

C. 711/8

D. 771/8

23) Consider the following statements:

(I) If a function is uniformly continuous then it must be continuous on any given interval.

(II) If f is continuous on any finite closed interval then the function f doesn't need to be uniformly continuous on that interval.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

24) Consider the following statements:

(I) If $f \in \mathcal{R}(\alpha)$ on [a, b] then for given $\eta > 0, U(P, f, \alpha) - L(P, f, \alpha) > \eta$.

$$(II) \sum_{i=1}^{n} \Delta\alpha_i = \alpha(a) - \alpha(b)$$

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	C	2.	C	3.	B	4.	C	5.	B
6.	B	7.	A	8.	C	9.	B	10.	C
11.	C	12.	C	13.	B	14.	A	15.	A
16.	C	17.	A	18.	C	19.	B	20.	A
21.	D	22.	D	23.	A	24.	D		

Review Questions

1) Evaluate:

$$\int_{\pi}^{2\pi} cosx d(sinx).$$

2) Evaluate:

$$\int_{0}^{2} x d\alpha, \quad \alpha(x) = \begin{cases} x; 0 \leq x \leq 1 \\ 2 + x; 1 \leq x \leq 2 \end{cases}$$

3) Evaluate:

$$\int_{\pi}^{2\pi} [x] d(e^x), [.] \text{denotes greater integer function.}$$

4) Evaluate:

$$\int_{0}^{4} x d[x], [.] \text{denotes greater integer function.}$$

5) Evaluate:

$$\int_0^2 x^2 d\alpha, \quad \alpha(x) = \begin{cases} x; 0 \le x \le 1 \\ 5 + x; 1 \le x \le 2 \end{cases}$$

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

Shanti Narayan, Elements of Real Analysis

Web Links

https://nptel.ac.in/courses/111/105/111105069/

https://www.youtube.com/watch?v=DO0Dzz07DNI

https://www.youtube.com/watch?v=YLB1wLkPbeI

Monika Arora, Lovely Professional University

Unit 02: Properties of the Riemann- Stieltjes Integral

CONTENTS

Objectives

After studying this unit, students will be able to:

- understand various properties of Riemann-Stieltjes integral
- define the relation of linearity in terms of Riemann-Stieltjes integral
- establish the relation of monotonicity in terms of Riemann-Stieltjes integral
- describe Riemann-Stieltjes sum
- express Riemann-Stieltjes integral in terms of Riemann-Stieltjes sum

Introduction

In the last unit, the concept of Riemann-Stieltjes integral has been discussed in detail with the proof of the related theorems. In this unit, we discuss the properties of Riemann-Stieltjes integral and their proof.

2.1 Properties of Riemann Stieltjes Integral

Theorem 2.1.1: If $f \in R(\alpha)$ on $[a, b]$ and $c \in \mathbb{R}$ then $cf \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha$$

Proof: Case 1: For $c = 0$, the result is obvious.

Let

$$M_i(cf) = Sup(cf), x \in [x_{i-1}, x_i]$$

$$m_i(cf) = Inf(cf), x \in [x_{i-1}, x_i]$$

$$M_i(cf) = c\, M_i(f)$$

$$m_i(cf) = c\, m_i(f)$$

Case 2: If $c > 0$

Consider

$$U(P, cf, \alpha) - L(P, cf, \alpha)$$

$$= \sum_{i=1}^{n} M_i(cf)\Delta\alpha_i - \sum_{i=1}^{n} m_i(cf)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} c\, M_i(f)\Delta\alpha_i - \sum_{i=1}^{n} c\, m_i(f)\Delta\alpha_i$$

$$= c\left[\sum_{i=1}^{n}[M_i(f) - m_i(f)]\Delta\alpha_i\right]$$

$$= c\,[U(P,f,\alpha) - L(P,f,\alpha)]$$

$$= c\,\epsilon = \epsilon'(say)$$

$$\Rightarrow cf \in \mathcal{R}(\alpha)$$

$$\Rightarrow \underline{\int_a^b} cf\,d\alpha = \overline{\int_a^b} cf\,d\alpha = \int_a^b cf\,d\alpha$$

Now,

$$\underline{\int_a^b} cf\,d\alpha = \sup_P L(P,cf,\alpha)$$

$$= \sup_P c\,L(P,f,\alpha)$$

$$= c\sup_P L(P,f,\alpha)$$

Thus, we get,

$$\underline{\int_a^b} cf\,d\alpha = c\underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow \int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$$

Case 3: If $c < 0$

Here $M_i(cf) = c\,m_i(f)$ and $m_i(cf) = c\,M_i(f)$

Consider,

$$U(P,cf,\alpha) - L(P,cf,\alpha)$$

$$= \sum_{i=1}^{n}[M_i(cf) - m_i(cf)]\,\Delta\alpha_i$$

$$= \sum_{i=1}^{n}[c\,m_i(f) - c\,M_i(f)]\,\Delta\alpha_i$$

$$= -c\sum_{i=1}^{n}(M_i - m_i)\,\Delta\alpha_i$$

$$= -c\,[U(P,f,\alpha) - L(P,f,\alpha)]$$

$$= -c\,\epsilon = \epsilon'(say) > 0$$

$$\Rightarrow U(P,cf,\alpha) - L(P,cf,\alpha) < \epsilon'$$

$\Rightarrow cf \in \mathcal{R}(\alpha)$.

Since,

$$\underline{\int_a^b} cf\,d\alpha$$

$$= \sup_P L(P,cf,\alpha)$$

$$= \sup_P c\,U(P,f,\alpha)$$

$$= c \inf_P U(P, f, \alpha)$$

$$\Rightarrow \underline{\int_a^b} cf\, d\alpha = c \overline{\int_a^b} f\, d\alpha$$

Since $cf \in R(\alpha)$ and $f \in R(\alpha)$

$$\Rightarrow \int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha.$$

This completes the proof.

Theorem 2.1.2: If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Proof: Let $\epsilon > 0$ be given and let $f = f_1 + f_2$.

Since $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$

$\therefore \exists$ partitions P_1 and P_2 for f_1 and f_2 respectively such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2}$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$

Therefore,

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{\epsilon}{2}$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\epsilon}{2}$$

Let m_i, M_i, m_i', M_i' and m_i'', M_i'' be the supremum and infimum of f, f_1 and f_2 on $[x_{i-1}, x_i]$ respectively. Then $m_i \geq m_i' + m_i''$ *and* $M_i \leq M_i' + M_i''$.

$$\Rightarrow \sum_{i=1}^{n} m_i\, \Delta\alpha_i \geq \sum_{i=1}^{n} m_i'\, \Delta\alpha_i + \sum_{i=1}^{n} m_i''\, \Delta\alpha_i \text{and} \sum_{i=1}^{n} M_i\, \Delta\alpha_i \leq \sum_{i=1}^{n} M_i'\, \Delta\alpha_i + \sum_{i=1}^{n} M_i''\, \Delta\alpha_i$$

$\Rightarrow L(P, f, \alpha) \geq L(P, f_1, \alpha) + L(P, f_2, \alpha)$ and $U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$

$\Rightarrow -L(P, f, \alpha) \leq -L(P, f_1, \alpha) - L(P, f_2, \alpha)$ and $U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$

$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) \leq U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

Thus, we get

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha)$$

$$i.e.\ f_1 + f_2 \in \mathcal{R}(\alpha)$$

Now we will show that

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$$

Since $f \in \mathcal{R}(\alpha)$, therefore

$$\int_a^b f\, d\alpha = \overline{\int_a^b} f\, d\alpha$$

Since,

$$\int_a^b f\,d\alpha \le U(P,f,\alpha)$$

$$\le U(P,f_1,\alpha) + U(P,f_2,\alpha)$$

$$< L(P,f_1,\alpha) + \frac{\epsilon}{2} + L(P,f_2,\alpha) + \frac{\epsilon}{2}$$

$$= L(P,f_1,\alpha) + L(P,f_2,\alpha) + \epsilon$$

$$\le \underline{\int_a^b} f_1\,d\alpha + \underline{\int_a^b} f_2\,d\alpha + \epsilon$$

$$\Rightarrow \int_a^b f\,d\alpha < \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha + \epsilon$$

$$\Rightarrow \int_a^b f\,d\alpha \le \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha. \quad \ldots(1)$$

Now, since $f, f_1, f_2 \in \mathcal{R}(\alpha)$.

$\therefore\ -f, -f_1, -f_2 \in \mathcal{R}(\alpha)$

$$\Rightarrow \int_a^b -f\,d\alpha \le \int_a^b (-f_1)\,d\alpha + \int_a^b (-f_2)\,d\alpha$$

$$\Rightarrow -\int_a^b f\,d\alpha \le -\int_a^b f_1\,d\alpha - \int_a^b f_2\,d\alpha$$

$$\Rightarrow \int_a^b f\,d\alpha \ge \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \quad \ldots(2)$$

From (1) and (2), we get

$$\int_a^b f\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha.$$

This completes the proof.

Theorem2.1.3: If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ then $f^2 \in \mathcal{R}(\alpha)$ on $[a,b]$.

Proof: Since $f \in \mathcal{R}(\alpha)$ on $[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow \exists\ 0 < k \in \mathbb{R}$ such that $|f(x)| \le k\ \ \forall x \in [a,b]$.

If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ then for given $\epsilon > 0, \exists\ P$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{2k}$$

Let m_i, M_i and m_i', M_i' be the bounds of f and f^2 respectively on $[x_{i-1}, x_i]$.

Now let $t_1, t_2 \in [x_{i-1}, x_i]$.

Then

$$|f^2(t_1) - f^2(t_2)| = |f(t_1) - f(t_2)||f(t_1) + f(t_2)|$$

$$\Rightarrow |f^2(t_1) - f^2(t_2)| \le [|f(t_1)| + |f(t_2)|]|f(t_1) - f(t_2)|$$

$$\le (k+k)|f(t_1) - f(t_2)|$$

$$= 2k\,|f(t_1) - f(t_2)|$$

This relation must hold for m_i', M_i' and m_i, M_i.

$\therefore |M_i' - m_i'| \le 2k\,|M_i - m_i|$

$\Rightarrow (M_i' - m_i') \le 2k(M_i - m_i)$

$\Rightarrow \sum_{i=1}^{n} (M_i' - m_i')\Delta\alpha_i \le \sum_{i=1}^{n} 2k(M_i - m_{i_i}')\Delta\alpha_i$

$\Rightarrow\ U(P, f^2, \alpha) - L(P, f^2, \alpha) \le 2k\,[U(P, f, \alpha) - L(P, f, \alpha)]$

$$< \frac{\epsilon}{2k}\ 2k = \epsilon$$

Thus, we get,

$$U(P, f^2, \alpha) - L(P, f^2, \alpha) < \epsilon$$

$$\Rightarrow f^2 \in \mathcal{R}(\alpha).$$

This completes the proof.

Theorem 2.1.4: If $f \in \mathcal{R}(\alpha)on[a, b]$and $g \in \mathcal{R}(\alpha)on[a, b]$then $fg \in \mathcal{R}(\alpha)on[a, b]$.

Proof:Since $f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha)$

$\therefore f + g, f - g \in \mathcal{R}(\alpha)$

$\Rightarrow (f + g)^2, (f - g)^2 \in \mathcal{R}(\alpha)$

$\Rightarrow (f + g)^2 - (f - g)^2 \in \mathcal{R}(\alpha)$

$\Rightarrow 4fg \in \mathcal{R}(\alpha)$

$\Rightarrow \frac{1}{4}(4fg) \in \mathcal{R}(\alpha)$

$\Rightarrow fg \in \mathcal{R}(\alpha)$on $[a, b]$

This completes the proof.

Theorem 2.1.5: If $f \in \mathcal{R}(\alpha)$on $[a, b]$ then $|f| \in \mathcal{R}(\alpha)$on $[a, b]$ and

$$\left| \int_a^b f\, d\alpha \right| \le \int_a^b |f|\, d\alpha$$

Proof: Since $f \in \mathcal{R}(\alpha)$on $[a, b]$ therefore for given $\epsilon > 0, \exists$ partitionof $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Let m_i, M_iandm_i', M_i'be the supremum and infimum of f and $|f|$respectively on $[x_{i-1}, x_i]$.

Now let $t_1, t_2 \in [x_{i-1}, x_i]$.

$\therefore\ \big||f(t_1)| - |f(t_2)|\big| \le |f(t_1) - f(t_2)|$ ... (1)

$\Rightarrow M_i' - m_i' \le M_i - m_i$

$\Rightarrow \sum_{i=1}^{n} (M_i' - m_i')\Delta\alpha_i \le \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$

$\Rightarrow U(P, |f|, \alpha) - L(P, |f|, \alpha) \le U(P, f, \alpha) - L(P, f, \alpha)$

$< \epsilon$

Thus, we get

$$U(P, |f|, \alpha) - L(P, |f|, \alpha) < \epsilon$$

$$\Rightarrow\ |f| \in \mathcal{R}(\alpha)\text{on}[a, b].$$

Next, we show that

$$\left| \int_a^b f\, d\alpha \right| \le \int_a^b |f|\, d\alpha$$

Since $f \in \mathcal{R}(\alpha)$

$\therefore -f \in \mathcal{R}(\alpha)$ and$|f| \in \mathcal{R}(\alpha)$

$$\Rightarrow \int_a^b f\,d\alpha, \int_a^b -f\,d\alpha \text{ and} \int_a^b |f|\,d\alpha \text{ exists.}$$

Now,

$$-f \leq |f| \text{and } f \leq |f|$$

$$\Rightarrow \int_a^b -f\,d\alpha \leq \int_a^b |f|\,d\alpha \text{ and} \int_a^b f\,d\alpha \leq \int_a^b |f|\,d\alpha$$

$$\Rightarrow -\int_a^b f\,d\alpha \leq \int_a^b |f|\,d\alpha \text{ and} \int_a^b f\,d\alpha \leq \int_a^b |f|\,d\alpha$$

$$\Rightarrow Max \left\{-\int_a^b f\,d\alpha, \int_a^b f\,d\alpha\right\} \leq \int_a^b |f|\,d\alpha$$

$$\Rightarrow \left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

This completes the proof.

Theorem 2.1.6: If $f \in \mathcal{R}(\alpha)$on $[a,b]$ and $a < c < b$ then $f \in \mathcal{R}(\alpha)$on $[a,c]$ and $f \in \mathcal{R}(\alpha)$on $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

Proof:$f \in \mathcal{R}(\alpha)on[a,b]$.

$\therefore$ for given $\epsilon > 0, \exists$ partition of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

Let $P^* = P \cup \{c\}$.

Then P^* is the refinement of P.

$\Rightarrow U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$ ... (1)

Let P_1 and P_2 be the set of points of P^*which constitute the partitions for $[a,c]$ and $[c,b]$

$$U(P_1,f,\alpha) + U(P_2,f,\alpha) = U(P^*,f,\alpha) \quad \text{... (2)}$$

$$L(P_1,f,\alpha) + L(P_2,f,\alpha) = L(P^*,f,\alpha) \quad \text{... (3)}$$

From (2) and (3)

$$[U(P_1,f,\alpha) - L(P_1,f,\alpha)] + [U(P_2,f,\alpha) - L(P_2,f,\alpha)] = U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$$

$$\Rightarrow [U(P_1,f,\alpha) - L(P_1,f,\alpha)] + [U(P_2,f,\alpha) - L(P_2,f,\alpha)] < \epsilon$$

$$\Rightarrow U(P_1,f,\alpha) - L(P_1,f,\alpha) < \epsilon \text{ and } U(P_2,f,\alpha) - L(P_2,f,\alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha)\text{on } [a,c] \text{and } f \in \mathcal{R}(\alpha)\text{on } [c,b].$$

Now$U(P_1,f,\alpha) + U(P_2,f,\alpha) = U(P^*,f,\alpha)$

$$\Rightarrow U(P_1,f,\alpha) + U(P_2,f,\alpha) \geq \int_a^b f\,d\alpha$$

Keeping P_2, taking infimum over all P_1, we get

$$\int_a^{\bar{c}} f\,d\alpha + U(P_2,f,\alpha) \geq \int_a^b f\,d\alpha$$

Taking infimum over all P_2 and using the fact that $f \in \mathcal{R}(\alpha)$on $[a,c]$, and$f \in \mathcal{R}(\alpha)$on $[c,b]$, we get

$$\Rightarrow \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \geq \int_a^b f\,d\alpha \qquad \dots(4)$$

Now consider,

$$L(P_1, f, \alpha) + L(P_2, f, \alpha) = L(P^*, f, \alpha)$$

$$\leq \int_a^b f\,d\alpha$$

$$\Rightarrow \int_{\underline{a}}^c f\,d\alpha + L(P_2, f, \alpha) \leq \int_a^b f\,d\alpha$$

$$\Rightarrow \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \leq \int_a^b f\,d\alpha \qquad \dots(5)$$

From (4) and (5), we get

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha$$

This completes the proof.

Theorem 2.1.7: If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\,d(\alpha_1 + \alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

Proof: For given $\epsilon > 0, \exists\ P_1$ and P_2 such that

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\epsilon}{2} \qquad \dots(1)$$

and

$$U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\epsilon}{2} \qquad \dots(2)$$

Let $P = P_1 \cup P_2$, then

$$U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{\epsilon}{2} \qquad \dots(3)$$

and

$$U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{\epsilon}{2} \qquad \dots(4)$$

Let $\alpha = \alpha_1 + \alpha_2$.

Then

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_i[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= \sum_{i=1}^{n} M_i[(\alpha_1 + \alpha_2)(x_i) - (\alpha_1 + \alpha_2)(x_{i-1})]$$

$$\Rightarrow U(P, f, \alpha) = \sum_{i=1}^{n} M_i[\alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})]$$

$$= \sum_{i=1}^{n} M_i[\alpha_1(x_i) - \alpha_1(x_{i-1})] + \sum_{i=1}^{n} M_i[\alpha_2(x_i) - \alpha_2(x_{i-1})]$$

$$\Rightarrow U(P,f,\alpha) = U(P,f,\alpha_1) + U(P,f,\alpha_2) \quad \dots(5)$$

Similarly, we get

$$L(P,f,\alpha) = L(P,f,\alpha_1) + L(P,f,\alpha_2) \quad \dots(6)$$

From (5) and (6), we get

$$U(P,f,\alpha) - L(P,f,\alpha) = [U(P,f,\alpha_1) - L(P,f,\alpha_1)] + [U(P,f,\alpha_2) - L(P,f,\alpha_2)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha)$$

$i.e.$ $f \in \mathcal{R}(\alpha_1 + \alpha_2)$.

Next, we show that

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

Since

$$\int_a^b f\, d\alpha \le U(P,f,\alpha)$$

$$< L(P,f,\alpha) + \epsilon$$

$$= L(P,f,\alpha_1) + L(P,f,\alpha_2) + \epsilon \qquad \{by(6)\}$$

$$\le \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 + \epsilon$$

$$\Rightarrow \int_a^b f\, d\alpha < \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 + \epsilon$$

$$\Rightarrow \int_a^b f\, d\alpha \le \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 \quad \dots(7)$$

$$\int_a^b f\, d\alpha \ge L(P,f,\alpha)$$

$$> U(P,f,\alpha) - \epsilon$$

$$= U(P,f,\alpha_1) + U(P,f,\alpha_2) - \epsilon \qquad \{by(5)\}$$

$$\ge \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 - \epsilon$$

$$\Rightarrow \int_a^b f\, d\alpha \ge \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 \quad \dots(8)$$

From (7) and (8) we get

$$\int_a^b f\, d\alpha = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

$$\Rightarrow \int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

This completes the proof.

Theorem 2.1.8: If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $c > 0$ be any real number then $f \in \mathcal{R}(c\alpha)$ and

$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$$

Proof: Since $f \in \mathcal{R}(\alpha)$ on $[a,b]$ therefore for given $\epsilon > 0, \exists\, P$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{c} \qquad \dots(1)$$

$\Delta(c\alpha_i) = (c\alpha)x_i - (c\alpha)x_{i-1}$

$= c\alpha(x_i) - c\alpha(x_{i-1})$

$= c[\alpha(x_i) - \alpha(x_{i-1})]$

$\Rightarrow \Delta(c\alpha_i) = c\,\Delta(\alpha_i)$

Now,

$$U(P,f,c\alpha) = \sum_{i=1}^{n} M_i \Delta(c\alpha_i)$$

$$= \sum_{i=1}^{n} M_i\, c\, \Delta\alpha_i$$

$$= c\sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$\Rightarrow U(P,f,c\alpha) = c\, U(P,f,\alpha)$$

Similarly, we get

$$L(P,f,c\alpha) = c\, L(P,f,\alpha)$$

$\therefore U(P,f,c\alpha) - L(P,f,c\alpha)$

$= c\, U(P,f,\alpha) - c\, L(P,f,\alpha)$

$= c\,[U(P,f,\alpha) - L(P,f,\alpha)]$

$< c\left(\frac{\epsilon}{c}\right) = \epsilon$

Thus, we get

$$U(P,f,c\alpha) - L(P,f,c\alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(c\alpha).$$

Next, we show that

$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$$

Since

$$\int_a^b f\, d(c\alpha) = \int_a^{\bar{b}} f\, d(c\alpha)$$

$$= \inf_P U(P,f,c\alpha)$$

$$= \inf_P c\, U(P,f,\alpha)$$

$$= c\inf_P U(P,f,\alpha)$$

$$= c\int_a^{\bar{b}} f\, d\alpha$$

$$\Rightarrow \int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha$$

This completes the proof.

Theorem 2.1.9: If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f(x) \geq 0, \forall x \in [a,b]$ then

$$\int_a^b f\, d\alpha \geq 0$$

Proof: Let

$$m = \inf_{x\in[a,b]} f(x) \text{ and } M = \sup_{x\in[a,b]} f(x)$$

$$\therefore m[\alpha(b) - \alpha(a)] \leq \int_a^b f\, d\alpha \leq M[\alpha(b) - \alpha(a)] \qquad \dots(1)$$

Since $f(x) \geq 0, \forall x \in [a,b]$

$\therefore m, M \geq 0.$

Also $\alpha(b) - \alpha(a) \geq 0$ as α is increasing

$$\therefore \ \mathrm{m}[\alpha(b) - \alpha(a)] \geq 0$$

$$\Rightarrow \int_a^b f\, d\alpha \geq 0 \qquad \{by(1)\}$$

This completes the proof.

Theorem 2.1.10: If $f_1 \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f_1 \leq f_2, \forall x \in [a,b]$ then

$$\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$$

Proof: Since $f_1 \leq f_2 \forall x \in [a,b]$

$$\Rightarrow f_2 - f_1 \geq 0 \ \ \forall x \in [a,b]$$

$$\Rightarrow \int_a^b (f_2 - f_1)\, d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_2\, d\alpha - \int_a^b f_1\, d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_2\, d\alpha \geq \int_a^b f_1\, d\alpha$$

$$i.e. \int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha .$$

This completes the proof.

Theorem 2.1.11: Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$ and α is monotonically increasing function continuous at all those points where f is continuous then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Proof: Let $\epsilon > 0$ be given and let $E = \{y_1, y_2, \dots, y_p\}$ be an ordered set of finite number of points at which f is discontinuous in $[a,b]$.

Since E is finite and α is continuous at every point of E. Therefore, we can cover E by finitely many disjoint intervals $[u_i, v_i] \subseteq [a,b]$ and place these intervals in such a way that every point of E lies in the interior of some $[u_i, v_i]$ such that

$$\sum_{i=1}^{p}[\alpha(v_i)-\alpha(u_i)] < \frac{\epsilon}{2(M-m)} \qquad \dots (1)$$

where m, M are infimum and supremum of f on $[a, b]$.

Let m_i, M_i be the infimum and supremum of f on $[u_i, v_i]$ $(i = 1,2, \dots, p)$ then

$$m \le m_i \le M_i \le M \qquad \dots (2)$$

$$\Rightarrow M_i - m_i \le M - m, \qquad i = 1, 2, \dots, p$$

$$\Rightarrow \sum_{i=1}^{p}(M_i - m_i)\Delta\alpha_i \le \sum_{i=1}^{p}(M-m)\Delta\alpha_i$$

$$= (M-m)\sum_{i=1}^{p}\Delta\alpha_i$$

$$< (M-m)\frac{\epsilon}{2(M-m)}) \qquad \{\text{by } (1)\}$$

$$\Rightarrow \sum_{i=1}^{p}(M_i - m_i)\Delta\alpha_i \le \frac{\epsilon}{2} \qquad \dots (3)$$

If we remove the segments $(u_i, v_i)$from $[a, b]$, then remaining $(p+1)$ subintervals of $[a, b]$ are

$$[a, u_i], [v_1, u_2], [v_2, u_3], \dots, [v_p, b]$$

Since f is continuous on each of above $(p+1)$ sub intervals and α is monotonically increasing.

$\therefore f \in \mathcal{R}(\alpha)$on each these $(p+1)$ sub intervals.

$\Rightarrow \exists$ Partitions $P_1, P_2, \dots, P_{p+1}$ of above $(p+1)$ sub intervals such that

$$U(P_r, f, \alpha) - L(P_r, f, \alpha) < \frac{\epsilon}{2(p+1)}, \qquad r = 1,2, \dots, p+1 \qquad \dots (4)$$

Now we form a partition P of $[a, b]$ as follows:

Each u_i occurs in P, each v_i occurs in P, no point of any segment (u_i, v_i) occurs in P i.e.

$$P = \{a, \dots, u_1, v_1, \dots, u_2, v_2, \dots, u_p, v_p, \dots, b\}$$

Then

$$U(P, f, \alpha) = \sum_{i=1}^{p} M_i[\alpha(v_i) - \alpha(u_i)] + \sum_{r=1}^{p+1} U(P_r, f, \alpha)$$

and

$$L(P, f, \alpha) = \sum_{i=1}^{p} m_i[\alpha(v_i) - \alpha(u_i)] + \sum_{r=1}^{p+1} L(P_r, f, \alpha)$$

$$\therefore\ U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{p}(M_i - m_i)[\alpha(v_i) - \alpha(u_i)] + \sum_{r=1}^{p+1}[U(P_r, f, \alpha) - L(P_r, f, \alpha)]$$

$$< \sum_{i=1}^{p}(M_i - m_i)[\alpha(v_i) - \alpha(u_i)] + \sum_{r=1}^{p+1}\frac{\epsilon}{2(p+1)} \qquad \{by\ (4)\}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2(p+1)}\sum_{r=1}^{p+1} 1 = \epsilon$$

Thus, we get

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha) \text{on } [a, b]$$

This completes the proof.

Example: Let α be monotonically increasing function defined on $[a,b]$ which is continuous at $x' \in [a,b]$, and let f be a function defined on $[a,b]$ by

$$f(x) = \begin{cases} 0 \,; x \neq x', \\ 1 \,; x = x'. \end{cases}$$

Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = 0$.

Solution: Since f is discontinuous at $x = x'$ and α is continuous at x'. Therefore, by using the preceding theorem we get $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

We have $L(P,f,\alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i\,, \qquad m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$

Here $m_i = 0\ \ \forall i$,

$$\Rightarrow\ L(P,f,\alpha) = 0$$

$$\Rightarrow\ \sup_P L(P,f,\alpha) = 0$$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha = 0$$

$$\Rightarrow \int_a^b f\,d\alpha = 0$$

This completes the proof.

Unit Step Function: The unit step function I is defined by

$$\mathrm{I(x)} = \begin{cases} 0 & if\quad x \le 0, \\ 1 & if\quad x > 0. \end{cases}$$

Theorem 2.1.12: If $a < s < b, f$ is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then

$$\int_a^b f\,d\alpha = f(s).$$

Proof: Since

$$\alpha(x) = I(x-s)$$

$$= \begin{cases} 0 \text{ if } x-s \le 0 \ \Rightarrow\ x \le s\ \Rightarrow\ x \le x_1 \\ 1 \text{ if } x-s > 0\ \Rightarrow\ x > s\ \Rightarrow\ x > x_1 \end{cases}$$

$$= \begin{cases} 0 \text{ if } x \le x_1 \\ 1 \text{ if }\ x > x_1 \end{cases}$$

Let $P = \{a = x_0, x_1 = s,\ x_2, x_3 = b\}$ be the partition of $[a,b], m_i, M_i$ be the infimum and supremum of $f(x)$, $x_{i-1} \le x \le x_{i.}\ i = 1,2,3$

Then

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$$

$$= \sum_{i=1}^{3} M_i\,\Delta\alpha_i$$

$$= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3$$

$$= M_1[\alpha(x_1) - \alpha(x_0)] + M_2[\alpha(x_2) - \alpha(x_1)] + M_3[\alpha(x_3) - \alpha(x_2)]$$

By using definition of $\alpha(x)$, we get

$$U(P,f,\alpha) = M_2.$$

Similarly, we get

$$L(P, f, \alpha) = m_2$$

Since f is continuous at s, $a < s < b$ therefore $M_2 \to f(s), m_2 \to f(s)$ as $x_2 \to s$

$$\therefore \inf U(P, f, \alpha) = \sup L(P, f, \alpha) = f(s)$$

$$\Rightarrow \overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha = f(s)$$

$$\Rightarrow \int_a^b f\, d\alpha = f(s).$$

This completes the proof.

Theorem 2.1.13: Let f be a continuous function on $[a, b]$ and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)\,, c_n \geq 0, \forall n,$$

where $\sum_{n=1}^{\infty} c_n$is convergent and$\{s_n\}$ is the sequence of distinct points in (a, b) then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

Proof: We have

$$I(x - s_n) = \begin{cases} 0 \text{ if } x - s_n \leq 0 \; i.e.\; x \leq s_n \\ 1 \text{ if } x - s_n > 0 \; i.e.\; x > s_n \end{cases}$$

$$= \begin{cases} 0 \;\; if \; x \leq s_n \\ 1 \; if \;\; x > s_n \end{cases}$$

$$\Rightarrow I(x - s_n) \leq 1$$

$$\Rightarrow c_n\, I(x - s_n) \leq c_n$$

Since $\sum_{n=1}^{\infty} c_n$ is convergent

$\therefore$ by comparison test $\alpha(x) = \sum_{n=1}^{\infty} c_n \;\; I(x - s_n)$is convergent $\forall x \in [a, b]$.

Now let$x_1, x_2 \in [a, b]$such that $x_1 < x_2$, then

$$I(x_1 - s_n) \leq I(x_2 - s_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n \;\; I(x_1 - s_n) \leq \sum_{n=1}^{\infty} c_n \;\; I(x_2 - s_n)$$

$$\Rightarrow \alpha(x_1) \leq \alpha(x_2).$$

Thus α is monotonically increasing function on $[a, b]$.

Let $\epsilon > 0$ be given.Since $\sum_{n=1}^{\infty} c_n$ is convergent,so we can choose $n \in \mathbb{N}$ such that

$$\sum_{n=m+1}^{\infty} c_n < \epsilon \qquad \ldots (1)$$

Let

$$\alpha_1(x) = \sum_{n=1}^{m} c_n \;\; I(x - s_n)$$

and

$$\alpha_2(x) = \sum_{n=m+1}^{\infty} c_n \;\; I(x - s_n).$$

So

$$\alpha(x) = \alpha_1(x) + \alpha_2(x)$$

$$i.e.\; \alpha = \alpha_1 + \alpha_2$$

Then

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2.$$

We know if f is continuous at s and $\alpha(x) = I(x-s)$ then $\int_a^b f\,d\alpha = f(s)$.

Therefore,

$$\int_a^b f\,d\alpha_1 = \int_a^b f\,d\left[\sum_{n=1}^{m} c_n \quad I(x-s_n)\right]$$

$$\Rightarrow \int_a^b f\,d\alpha_1 = \sum_{n=1}^{m} c_n \quad f(s_n)$$

Now

$$\alpha_2(b) - \alpha_2(a) = \sum_{n=m+1}^{\infty} c_n \quad I(b-s_n) - \sum_{n=m+1}^{\infty} c_n \quad I(a-s_n)$$

$$= \sum_{n=m+1}^{\infty} c_n \qquad \{\text{by definition of } I(x-s_n)\}$$

$$\Rightarrow \alpha_2(b) - \alpha_2(a) < \epsilon \qquad \{by(1)\}$$

Since f is continuous on $[a,b]$.

$\Rightarrow f$ is bounded on $[a,b]$.

Therefore $\exists\ 0 < k \in \mathbb{R}$ such that

$$|f(x)| \le k, \forall x \in [a,b]$$

$$\Rightarrow \left|\int_a^b f\,d\alpha_2\right| \le k[\alpha_2(b) - \alpha_2(a)]$$

$$< k\epsilon.$$

Since $\alpha = \alpha_1 + \alpha_2$, therefore,

$$\Rightarrow \left|\int_a^b f\,d\alpha - \int_a^b f\,d\alpha_1\right| < k\epsilon$$

$$\Rightarrow \left|\int_a^b f\,d\alpha - \sum_{n=1}^{m} c_n \quad f(s_n)\right| < k\epsilon$$

Letting $m \to \infty$, we get

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n \quad f(s_n)$$

This completes the proof.

Theorem 2.1.14: (Change of Variable)

Let Φ be strictly increasing continuous function that maps $[A,B]$ onto $[a,b], \alpha$ is monotonically increasing on $[a,b], f \in \mathcal{R}(\alpha)$on $[a,b]$. Define β, g on $[A,B]$ such that

$$\beta(y) = \alpha(\Phi(y)), \qquad g(y) = f(\Phi(y))$$

Then $g \in \mathcal{R}(B)$and

$$\int_a^b f\,d\alpha = \int_A^B g\,d\beta$$

Proof: $\Phi: [A, B] \to [a, b]$ is strictly increasing, continuous and onto.

$\therefore$ Φ is one-one and onto

$\Rightarrow$ Φ is invertible.

Therefore, corresponding to each partition

$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}of[a, b]$, there existspartition $Q = \{A = y_0, y_1, \ldots, y_n = B\}of[A, B]$ such that $\Phi^{-1}(x_i) = y_i, \forall i$.

Let m_i, M_i be the infimum and supremum of f respectively on $[x_{i-1}, x_i]$ and let m_i', M_i' be the infimum and supremum of g respectively on $[y_{i-1}, y_i]$.

Now,

$$m_i' = \inf\{ g(y) : y \in [y_{i-1}, y_i]\}$$
$$= \inf\{ f(\Phi(y)) : y \in [y_{i-1}, y_i]\}$$
$$= \inf\{ f(x) : x \in [x_{i-1}, x_i]\}$$
$$\Rightarrow m_i' = m_i$$

Similarly, we can get $M_i' = M_i$.

Now,

$$L(\Phi, g, \beta) = \sum_{i=1}^{n} m_i' \, \Delta\beta_i$$
$$= \sum_{i=1}^{n} m_i \, [\beta(y_i) - \beta(y_{i-1})]$$
$$= \sum_{i=1}^{n} m_i \, [\alpha(\Phi(y_i)) - \alpha(\Phi(y_{i-1}))]$$
$$= \sum_{i=1}^{n} m_i \, [\alpha(x_i) - \alpha(x_{i-1})]$$
$$= \sum_{i=1}^{n} m_i \, \Delta\alpha_i$$
$$\Rightarrow L(\Phi, g, \beta) = L(P, f, \alpha).$$

Similarly, we can get $U(\Phi, g, \beta) = U(P, f, \alpha)$.

Thus

$$\sup_Q L(Q, g, \beta) = \sup_P L(P, f, \alpha) \text{and} \inf_Q U(Q, g, \beta) = \inf_P U(P, f, \alpha)$$
$$\Rightarrow \underline{\int_A^B} g \, d\beta = \underline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha \text{ and } \overline{\int_A^B} g \, d\beta = \overline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha$$
$$\Rightarrow \underline{\int_A^B} g \, d\beta = \overline{\int_A^B} g \, d\beta = \int_a^b f \, d\alpha$$
$$\Rightarrow g \in \mathcal{R}(B) \text{and} \int_A^B g \, d\beta = \int_a^b f \, d\alpha$$

This completes the proof.

Theorem 2.1.15: Let f be a bounded function on $[a, b]$, α is monotonically increasing function on $[a, b]$ such that α' is R-integrableon $[a, b]$. Then f is Riemann Stieltjes integrable if and only if $f\alpha'$is R-integrable and

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx.$$

Proof: Let $\epsilon > 0$ be given.

Since α' is R-integrable therefore $\exists$ partition P of $[a,b]$ such that

$$U(P,\alpha') - L(P,\alpha') < \epsilon$$

α is derivable in$[a,b]$

$\Rightarrow \alpha$ is derivable in$[x_{i-1},x_i]$

So, by mean value theorem, for $t_i \in [x_{i-1},x_i], i = 1,2,\dots,n,$ we have

$$\frac{\alpha(x_i)-\alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i),$$

$$\Rightarrow \frac{\Delta\alpha_i}{\Delta x_i} = \alpha'(t_i)$$

$$\Rightarrow \Delta\alpha_i = \alpha'(t_i)\,\Delta x_i \qquad \dots(1)$$

Since f is bounded on $[a,b]$, $\therefore \exists\ 0 < k \in \mathbb{R}$ such that

$$|f(x)| \le k, \forall x \in [a,b]. \qquad \dots(2)$$

Let $M_i = \sup \alpha'(x), x \in [x_{i-1},x_i]$, $\quad m_i = \inf \alpha'(x), x \in [x_{i-1},x_i]$

Let $s_i,\ t_i \in [x_{i-1},x_i]$.

Therefore,

$$\sum_{i=1}^{n}[\alpha'(s_i)-\alpha'(t_i)]\ \Delta x_i \le \sum_{i=1}^{n}[M_i - m_i]\ \Delta x_i$$

$$\therefore \sum_{i=1}^{n}[\alpha'(s_i)-\alpha'(t_i)]\ \Delta x_i \le U(P,\alpha') - L(P,\alpha') < \epsilon$$

$$\Rightarrow \sum_{i=1}^{n}[\alpha'(s_i)-\alpha'(t_i)]\,\Delta x_i < \epsilon \qquad \dots(3)$$

Now,

$$\left|\sum_{i=1}^{n} f(s_i)\ \Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\,\alpha'(s_i)\,\Delta x_i\right|$$

$$= \left|\sum_{i=1}^{n} f(s_i)\,\alpha'(t_i)\,\Delta x_i - \sum_{i=1}^{n} f(s_i)\,\alpha'(s_i)\,\Delta x_i\right| \qquad \{by(1)\}$$

$$= \left|\sum_{i=1}^{n} f(s_i)\,[\alpha'(t_i)-\alpha'(s_i)]\,\Delta x_i\right|$$

$$\le \sum_{i=1}^{n}|f(s_i)|\,|\alpha'(t_i)-\alpha'(s_i)|\,\Delta x_i$$

$$\le \sum_{i=1}^{n} k\,|\alpha'(t_i)-\alpha'(s_i)|\,\Delta x_i \qquad \{by(2)\}$$

$$= k\sum_{i=1}^{n}|\alpha'(t_i)-\alpha'(s_i)|\,\Delta x_i$$

$$\Rightarrow \left|\sum_{i=1}^{n} f(s_i)\ \Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\,\alpha'(s_i)\,\Delta x_i\right| < k\,\epsilon \qquad \dots(4)\quad \{by(3)\}$$

$$\Rightarrow \sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i < \sum_{i=1}^{n} f(s_i)\,\alpha'(s_i)\Delta x_i + k\,\epsilon$$

$$\Rightarrow \sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i < U(P, f\,\alpha') + k\,\epsilon$$

$$\Rightarrow U(P, f, \alpha) < U(P, f\alpha') + k\,\epsilon \qquad \ldots (5)$$

Again from (4), we have

$$\sum_{i=1}^{n} f(s_i)\,\alpha'(s_i)\,\Delta x_i < \sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i + k\,\epsilon$$

$$< U(P, f, \alpha) + k\epsilon$$

$\Rightarrow\ U(P, f\alpha') < U(P, f, \alpha) + k\epsilon$ ... (6)

From (5) and (6), we get

$$|\,U(P, f, \alpha) - U(P, f\alpha')| < k\epsilon \qquad \ldots (7)$$

As $U(P, \alpha') - L(P, \alpha') < \epsilon$ remains true if P is replaced by any refinement. Hence (7) also remains true. We conclude that

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\,\alpha'(x)\,dx \right| < k\epsilon.$$

But ϵ is arbitrary. Hence

$$\overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\,\alpha'(x)\,dx \qquad \ldots (8)$$

Similarly

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\,\alpha'(x)\,dx \qquad \ldots (9)$$

From (8) and (9), we get f is Riemann Stieltjes integrable if and only if $f\alpha'$ is R-integrable and then

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx.$$

This completes the proof.

Example: Let

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ -1 \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases} \text{ on } [0,1].$$

Show that f is not Riemann integrable.

Solution: Let $P = \{x_0, x_1, x_2, \ldots, x_n = 1\}$, be the partition of $[a, b]$, $m_i = \inf f(x)$, $M_i = \sup f(x)$, $x \in [x_{i-1}, x_i]$

$\therefore\ m_i = -1$ and $M_i = 1$.

Now

$$L(P, f) = \sum_{i=1}^{n} m_i\ \Delta x_i$$

$$= -1 \sum_{i=1}^{n} \Delta x_i$$

$$= -1\,[x_n - x_0]$$

$$= -1\,[1 - 0] = -1$$

and

$$U(P,f) = \sum_{i=1}^{n} M_i \,\Delta x_i$$

$$= 1 \sum_{i=1}^{n} \Delta x_i$$

$$= 1\,[x_n - x_0]$$

$$= 1\,[1-0] = 1$$

$\therefore \sup_P L(P,f) = -1$

$\Rightarrow \underline{\int_a^b} f\,dx = -1$ and

$\inf_P U(P,f) = 1$

$\Rightarrow \overline{\int_a^b} f\,dx = 1$

Thus, we get

$$\underline{\int_a^b} f\,dx \neq \overline{\int_a^b} f\,dx$$

f is not integrable.

Example: Let

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ -1 \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases} \text{ on } [0,1] \;.$$

Show that $|\,f\,|$ is Riemann integrable.

Solution: Let $P = \{x_0, x_1, x_2, \dots, x_n = 1\}$ be the partition of $[a,b]$,

$$m_i' = \inf|f|\,, x \in [x_{i-1}, x_i], \qquad M_i = \sup|f|\,, x \in [x_{i-1}, x_i]$$

$\Rightarrow m_i' = 1, M_i' = 1$

$$\therefore U(P,|f|) = \sum_{i=1}^{n} M_i' \,\Delta x_i$$

$$= \sum_{i=1}^{n} \Delta x_i = 1$$

Similarly, $L(P,|f|) = 1$.

So, we have

$\sup_P \big(L(P,|f|)\big) = 1,\ \inf_P \big(U(P,|f|)\big) = 1$

$$\Rightarrow \underline{\int_a^b} |f|\,dx = \overline{\int_a^b} |f|\,dx$$

$$\Rightarrow |f| \in \mathcal{R} \text{ on } [a,b].$$

2.2 Riemann Stieltjes Sum:

Let f be a bounded real function on $[a,b]$, α be monotonically increasing function defined on $[a,b]$.

Let $P = \{a = x_0, x_1, \dots, x_n\}$ be the partition of $[a,b]$.The Riemann Stieltjessum is denoted by $S(P,f,\alpha)$and is defined as

$$\sum_{i=1}^{n} f(t_i)\Delta\alpha_i \,, t_i \in [x_{i-1}, x_i]$$

$$i.e.\ S(P,f,\alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \,, t_i \in [x_{i-1}, x_i]$$

where $m_i = inf\, f(x)\,, x \in [x_{i-1}, x_i]$, $\quad M_i = sup\, f(x)\,, x \in [x_{i-1}, x_i]$

Then

$$m_i \le f(t_i) \le M_i, t_i \in [x_{i-1}, x_i]$$

$$\Rightarrow \sum_{i=1}^{n} m_i\, \Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i\, \Delta\alpha_i$$

$$\Rightarrow L(P,f,\alpha) \le S(P,f,\alpha) \le U(P,f,\alpha).$$

$S(P,f,\alpha) \to A$ as $\|P\| \to 0$ if for given $\epsilon > 0, \exists\delta > 0$ such that

$$|S(P,f,\alpha) - A| < \epsilon, \text{with} \|P\| < \delta.$$

Theorem 2.2.1: If $\lim_{\|P\|\to 0} S(P,f,\alpha)$exists as $\|P\| \to 0$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and

$$\lim_{\|P\|\to 0} S(P,f,\alpha) = \int_a^b f\, d\alpha.$$

Proof: Since $\lim_{\|P\|\to 0} S(P,f,\alpha)$ exists so $\lim_{\|P\|\to 0} S(P,f,\alpha) = A\ (say)$

$\therefore$ for given $\epsilon > 0,\ \exists\delta > 0$ such that

$$|S(P,f,\alpha) - A| < \frac{\epsilon}{4} \text{with} \|P\| < \delta \qquad \dots(1)$$

$$\Rightarrow A - \frac{\epsilon}{4} < S(P,f,\alpha) < A + \frac{\epsilon}{4} \qquad \dots(2)$$

Let $\alpha(b) - \alpha(a) = k$,

$m_i = \inf f(x)\,, x \in [x_{i-1}, x_i]$, $\quad M_i = \sup f(x)\,, x \in [x_{i-1}, x_i]$.

$\therefore \exists\ s_i, t_i \in [x_{i-1}, x_i]$such that$M_i - \frac{\epsilon}{4k} < f(t_i)andm_i + \frac{\epsilon}{4k} > f(s_i)$

$$\Rightarrow M_i < f(t_i) + \frac{\epsilon}{4k} \text{ and} \qquad \dots(3)$$

$$m_i > f(s_i) - \frac{\epsilon}{4k} \qquad \dots(4)$$

Consider

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i$$

$$< \sum_{i=1}^{n} \left(f(t_i) + \frac{\epsilon}{4k}\right)\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i \qquad \{by(3)\}$$

$$< \sum_{i=1}^{n} \left(f(t_i) + \frac{\epsilon}{4k}\right)\Delta\alpha_i - \sum_{i=1}^{n} \left(f(s_i) - \frac{\epsilon}{4k}\right)\Delta\alpha_i \qquad \{by(4)\}$$

$$= \sum_{i=1}^{n} f(t_i)\Delta\alpha_i + \sum_{i=1}^{n} \frac{\epsilon}{4k}\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + \sum_{i=1}^{n} \frac{\epsilon}{4k}\Delta\alpha_i$$

$$= S(P,f,\alpha) + 2\left(\frac{\epsilon}{4k}\right)\sum_{i=1}^{n} \Delta\alpha_i - S(P,f,\alpha)$$

$$= S(P,f,\alpha) + \left(\frac{\epsilon}{2k}\right)[\alpha(b) - \alpha(a)] - S(P,f,\alpha)$$

$$= S(P,f,\alpha) + \frac{\epsilon}{2} - S(P,f,\alpha)$$

$$= [S(P,f,\alpha) - A] + \frac{\epsilon}{2} + [A - S(P,f,\alpha)]$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon. \qquad \{by(1)\}$$

Thus, we get,

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha) \text{on} [a,b].$$

Now we show that

$$\lim_{\|P\|\to 0} S(P,f,\alpha) = \int_a^b f\, d\alpha$$

As $f \in \mathcal{R}(\alpha)on[a,b]$ therefore

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{4} \qquad \dots(5)$$

Also, we have

$$L(P,f,\alpha) \leq \int_a^b f\, d\alpha \leq U(P,f,\alpha)$$

$$\Rightarrow L(P,f,\alpha) \leq \int_a^b f\, d\alpha \leq U(P,f,\alpha) < L(P,f,\alpha) + \frac{\epsilon}{4} \qquad \dots(6)$$

Also

$$L(P,f,\alpha) \leq S(P,f,\alpha) \leq U(P,f,\alpha) < L(P,f,\alpha) + \frac{\epsilon}{4} \qquad \dots(7)$$

Now

$$\left|A - \int_a^b f\, d\alpha\right| = \left|A - S(P,f,\alpha) + S(P,f,\alpha) - L(P,f,\alpha) + L(P,f,\alpha) - \int_a^b f\, d\alpha\right|$$

$$\leq |A - S(P,f,\alpha)| + |S(P,f,\alpha) - L(P,f,\alpha)| + \left|L(P,f,\alpha) - \int_a^b f\, d\alpha\right|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{4} < \epsilon \qquad \{by\ (1),(6),(7)\}$$

$$\Rightarrow \left|A - \int_a^b f\, d\alpha\right| < \epsilon \Rightarrow\ A = \int_a^b f\, d\alpha$$

$$\Rightarrow \lim_{\|P\|\to 0} S(P,f,\alpha) = \int_a^b f\, d\alpha$$

This completes the proof.

Theorem 2.2.2.: If f is continuous on $[a,b]$ and α has a continuous derivative on $[a,b]$ then

$$\int_a^b f\, d\alpha = \int_a^b f\alpha'\, dx$$

Proof: Since f is continuous on $[a,b]$ and α has a continuous derivative on $[a,b]$ therefore both $\int_a^b f\, d\alpha$ and $\int_a^b f\alpha'\, dx$ exists.

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be the partition of $[a,b]$.

Since α has continuous derivative on $[a,b]$

$\Rightarrow \alpha$ has continuous derivative on $[x_{i-1}, x_i]$so by mean value theorem, we have

$$\frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i), \forall i$$

$$\Rightarrow \frac{\Delta\alpha_i}{\Delta x_i} = \alpha'(t_i)$$

$$\Rightarrow \Delta\alpha_i = \alpha'(t_i)\Delta x_i \qquad \dots(1)$$

Now

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} f(t_i)\alpha'(t_i)\Delta x_i \qquad \{by(1)\}$$

$$= \sum_{i=1}^{n} (f\alpha')(t_i)\Delta x_i$$

$$= S(P, f\alpha')$$

$$\Rightarrow \lim_{\|P\|\to 0} S(P, f, \alpha) = \lim_{\|P\|\to 0} S(P, f\alpha')$$

$$\Rightarrow \int_a^b f\, d\alpha = \int_a^b f\alpha'\, dx$$

This completes the proof.

Summary

- If $f \in R(\alpha)$ on $[a, b]$ and $c \in \mathbb{R}$ then $cf \in R(\alpha)$ on $[a, b]$ and $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$.
- If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f^2 \in \mathcal{R}(\alpha)$ on $[a, b]$.
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$ then $fg \in \mathcal{R}(\alpha)$ on $[a, b]$.
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ then $|f| \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$.
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $a < c < b$ then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and $f \in \mathcal{R}(\alpha)$ on $[c, b]$ and $\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$.
- If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and $\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$.
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $c > 0$ be any real number then $f \in \mathcal{R}(c\alpha)$ and $\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$.
- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $f(x) \ge 0, \forall x \in [a, b]$ then $\int_a^b f\, d\alpha \ge 0$.
- If $f_1 \in \mathcal{R}(\alpha)$ on $[a, b]$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ and $f_1 \le f_2, \forall x \in [a, b]$ then $\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$.
- Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is monotonically increasing function continuous at all those points where f is continuous then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.
- If $a < s < b, f$ is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then $\int_a^b f\, d\alpha = f(s)$.
- Let f be a continuous function on $[a, b]$ and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n), c_n \ge 0, \forall n$, where $\sum_{n=1}^{\infty} c_n$ is convergent and $\{s_n\}$ is the sequence of distinct points in (a, b) then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

- Let Φ be strictly increasing continuous function that maps $[A,B]$ onto $[a,b], \alpha$ is monotonically increasing on $[a,b], f \in \mathcal{R}(\alpha)$on $[a,b]$. Define β, g on $[A,B]$ such that $\beta(y) = \alpha(\Phi(y)),\ \ g(y) = f(\Phi(y))$. Then $g \in \mathcal{R}(B)$and $\int_a^b f\,d\alpha = \int_A^B g\,d\beta$.
- Let f be a bounded function on $[a,b]$, α is monotonically increasing function on $[a,b]$ such that α' is R-integrable on $[a,b]$. Then f is Riemann Stieltjes integrable if and only if $f\alpha'$is R-integrable and $\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx$.
- If $\lim_{\|P\|\to 0} S(P,f,\alpha)$exists as $\|P\| \to 0$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\lim_{\|P\|\to 0} S(P,f,\alpha) = \int_a^b f\,d\alpha$.
- If f is continuous on $[a,b]$ and α has a continuous derivative on $[a,b]$ then $\int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$.

Keywords

Riemann Stieltjes Sum: Let f be a bounded real function on $[a,b], \alpha$ be monotonically increasing function defined on $[a,b]$.Let $P = \{a = x_0, x_1, \dots, x_n\}$ be the partition of $[a,b]$. The Riemann Stieltjes sum is denoted by $S(P,f,\alpha)$ and is defined as

$$\sum_{i=1}^{n} f(t_i)\Delta\alpha_i\,, t_i \in [x_{i-1}, x_i].$$

$$i.e.\ S(P,f,\alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i\,, t_i \in [x_{i-1}, x_i].$$

Unit Step Function: The unit step function I is defined by

$$\mathrm{I(x)} = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Self-Assessment

1) Consider the following statements:

(I)If $f \in \mathcal{R}(\alpha)$ on [a, b] and c is any constant, then $cf \in \mathcal{R}(\alpha)$ on [a, b].

(II) If $f \in \mathcal{R}(\alpha)$ on [a, b] and c is any constatnt, then $\int_a^b cf d\alpha = c\int_a^b f d\alpha$ only if c>0.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

Let$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$be any partition of [a, b] and let m_i, M_i, m_i', M_i' be bounds of f and cf in $[x_{i-1}, x_i], i = 1,2,\dots,n$. Then select the correct option in Q (2-5).

2) $\boldsymbol{m_i'} = \begin{cases} cm_i, c < 0 \\ cM_i, c > 0 \end{cases}$

A. True
B. False

3) $M_i' = \begin{cases} cM_i, c > 0 \\ cm_i, c < 0 \end{cases}$

A. True
B. False

4) $L(P, cf, \alpha) = \begin{cases} cL(P, f, \alpha), c > 0 \\ cU(P, f, \alpha), c < 0 \end{cases}$

A. True
B. False

5) $U(P, cf, \alpha) = \begin{cases} cL(P, f, \alpha), c > 0 \\ cU(P, f, \alpha), c < 0 \end{cases}$

A. True
B. False

6) Consider the following statements:

(I) $\left| \int_a^b f d\alpha \right| \leq \int_a^b |f|\, d\alpha.$

(II) $Max\, \{-\int_a^b f d\alpha\, , \int_a^b f d\alpha\} \leq \int_a^b |f|\, d\alpha.$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

7) $|f(a) - f(b)| \leq \big||f(a)| - |f(b)|\big|$

A. True
B. False

8) Consider the following statements:

(I) $f \in \mathcal{R}(\alpha) \Rightarrow -f \in \mathcal{R}(\alpha).$

(II) $f \in \mathcal{R}(\alpha) \nRightarrow |f| \in \mathcal{R}(\alpha).$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

9) $-f \leq |f|$ and $f \geq |f|$

A. True
B. False

10) For $\alpha = \alpha_1 + \alpha_2$, consider the following statements:

(I) $U(P, f, \alpha) = U(P, f, \alpha_1) + U(P, f, \alpha_2)$

(II) $L(P, f, \alpha) < L(P, f, \alpha_1) + L(P, f, \alpha_2)$

A. only (I) is correct

B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

11) Consider the following statements:

(I) If $f \in \mathcal{R}(\alpha_1), f \in \mathcal{R}(\alpha_2)$on [a, b] then$f \notin \mathcal{R}(\alpha_1 + \alpha_2)$

(II) If $f \in \mathcal{R}(\alpha_1), f \in \mathcal{R}(\alpha_2)$on [a, b] then

$$\int_a^b f d(\alpha_1 + \alpha_2) < \int_a^b f d(\alpha_1) + \int_a^b f d(\alpha_2)$$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

12) Let I be a unit step function then select the INCORRECT option.

A. $I(0) = 0$
B. $I(1) = 1$
C. $I(2) = 2$
D. none of these

13) Let I be a unit step function then

A. $I(x-s) = \begin{cases} -1, x < s \\ 0, x = s \\ 1, x > s \end{cases}$

B. $I(x-s) = \begin{cases} 0, x \leq s \\ 1, x > s \end{cases}$

C. $I(x-s) = \begin{cases} 0, x < s \\ 1, x \geq s \end{cases}$

D. $I(x-s) = 1 \; \forall x$

14) If f is continuous on [a, b] and α is monotonically increasing in [a, b] then $f \in \mathcal{R}(\alpha)$ on [a, b].

A. True
B. False

15) Consider the following statements:

(I)Let $0 \leq a_n \leq b_n$ then $\sum a_n$ is convergent if $\sum b_n$ is convergent.

(II)Let $0 \leq a_n \leq b_n$ then $\sum b_n$ is divergent if $\sum a_n$ is divergent.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

16) If f is continuous on [a, b] then there exists a positive real number k such that $|f(x)| \leq k, \forall x \in [a, b]$.

A. True

B. False

17) Consider the following statements:

(I) Let $f: A \to B$then for f to be invertible it must be one-one and onto.

(II) Let $f: A \to B$, if f is one-one and onto then it is invertible.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

18) If f is a bounded function on [a, b], f is continuous at $s \in (a, b)$ and $\alpha(x) = I(x - s)$ then select the INCORRECT option.

$A. \int_{\underline{a}}^{b} f d\alpha = f(s)$

$B. \int_{a}^{\bar{b}} f d\alpha = f(s)$

$C. \int_{a}^{b} f d\alpha = f(s)$

D .All are incorrect

19) Consider the following statements:

(I) Every bounded function is integrable.

(II) If $|f|$ is integrable then f must be integrable.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

20) If f is Riemann integrable on [a, b] then for given $\in > 0, L(P, f) + \in > U(P, f)$

A. True
B. False

21) Select the correct option:

A. $L(P, f, \alpha) \leq U(P, f, \alpha) \leq S(P, f, \alpha)$
B. $L(P, f, \alpha) \leq S(P, f, \alpha) \leq U(P, f, \alpha)$
C. $U(P, f, \alpha) \leq S(P, f, \alpha) \leq L(P, f, \alpha)$
D. $U(P, f, \alpha) \leq L(P, f, \alpha) \leq S(P, f, \alpha)$

22) The Riemann-Stieltjes sum $S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i) \Delta\alpha_i , t_i \in [x_{i-1,} x_i]$

A. True
B. False

Answers for Self Assessment

1.	A	2.	B	3.	A	4.	A	5.	B
6.	C	7.	B	8.	A	9.	B	10.	A
11.	D	12.	C	13.	B	14.	A	15.	C
16.	A	17.	C	18.	D	19.	D	20.	A
21.	B	22.	A						

Review Questions

1) Show with the help of an example that every bounded function need not be integrable.

2) Show with the help of an example that if $|f|$ is integrable then it is not necessary that f is integrable.

3) Evaluate:

$$\int_0^3 (x^2+1)d[x].$$

4) Evaluate:

$$\int_0^1 xd(e^{2x}).$$

5) Evaluate:

$$\int_0^2 [x]dx^2, [.]\text{denotes greater integer function.}$$

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

Shanti Narayan, Elements of Real Analysis

Web Links

https://nptel.ac.in/courses/111/105/111105069/

https://www.youtube.com/watch?v=DO0Dzz07DNI

https://www.youtube.com/watch?v=YLB1wLkPbeI

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Unit 03: The fundamental theorem of calculus and mean value theorems for the Riemann-Stieltjes integral

CONTENTS

Objectives

After studying this unit, students will be able to:

- discuss fundamental theorem of calculus
- establish the relationship between differentiation and integration
- describe the first mean value theorem
- explain the second mean value theorem

Introduction

Differentiation and integration are related to each other in the sense that they are inverse operations of each other. This fact is established with the help of the fundamental theorem of calculus. In various problems, we can see the occurrence of integrals but there are very few cases in which integral value is explicitly obtained. However, it is often sufficient to have an estimated value of the integral rather than its exact value. The mean value theorems here are especially useful in making such estimates.

3.1 Fundamental Theorem of Calculus

Statement: If f is Riemann integrable on [a, b] and if there is a differentiable function F on [a, b] such that $F' = f$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof: Since f is Riemann integrable on [a, b]. Therefore, for given $\in > 0$, there exists a partition

$P = [a = x_0, x_1, x_2, \ldots, x_n = b]$ of [a, b] such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \in \quad \ldots (1)$$

Since F is differentiable on [a, b].

$\Rightarrow F$ is differentiable on $[x_{i-1}, x_i], i = 1,2, \ldots, n$.

$\Rightarrow F$ is continuous on $[x_{i-1}, x_i], i = 1,2, \ldots, n$.

By Lagrange mean value theorem, there exists $c_i \in (x_{i-1}, x_i)$ such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i)$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = F'(c_i)\Delta x_i$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$$

$$\because F' = f$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$$

$$\Rightarrow \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$\Rightarrow F(b) - F(b) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

$$\Rightarrow F(b) - F(b) = S(P, f) \quad \dots (2)$$

We know that

$$L(P,f) \le S(P,f) \le U(P,f)$$

and

$$L(P,f) \le \int_a^b f(x)\,dx \le U(P,f)$$

$$\Rightarrow \left| S(P,f) - \int_a^b f(x)\,dx \right| \le U(P,f) - L(P,f)$$

$$< \in \quad by\ (1)$$

$$\Rightarrow \left| \int_a^b f(x)\,dx - [F(b) - F(a)] \right| < \in \quad by\ (2)$$

But $\in$ is arbitrarily small, so let $\in \to 0$, we get

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This completes the proof.

Example3.1.1: Evaluate the integral:

$$\int_0^2 |x^2 + 2x - 3|\,dx$$

Solution: Let

$$I = \int_0^2 |x^2 + 2x - 3|\,dx$$

We have

$$(x^2 + 2x - 3) = (x + 3)(x - 1)$$

Therefore,

$$|x^2 + 2x - 3| = \begin{cases} -(x^2 + 2x - 3), 0 \le x \le 1 \\ (x^2 + 2x - 3), 1 \le x \le 2 \end{cases}$$

$$\Rightarrow I = \int_0^2 |x^2 + 2x - 3| dx$$

$$= \int_0^1 |x^2 + 2x - 3| dx + \int_1^2 |x^2 + 2x - 3| dx$$

$$= \int_0^1 -(x^2 + 2x - 3) dx + \int_1^2 (x^2 + 2x - 3) dx$$

$$= -\left[\frac{x^3}{3} + x^2 - 3x\right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2$$

$$= 4$$

Example 3.1.1: Evaluate the integral:

$$\int_0^3 [x] dx$$

Solution: We have

$$\int_0^3 [x] dx$$

$$= \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx$$

$$= \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx$$

$$= 0 + [x]_0^2 + [2x]_2^3$$

$$= 3.$$

Evaluate:

$$\int_{-1}^1 e^{|x|} dx$$

Evaluate:

$$\int_0^1 |5x - 3| dx$$

3.2 First Mean Value Theorem for Riemann-Stieltjes Integral

Statement: Assume that α is monotonically increasing and let $f \in \mathcal{R}(\alpha)$ on [a, b]. Let M and m denote respectively, the supremum and infimum of the set $\{f(x): x \in [a, b]\}$. Then there exists a real number c satisfying $m \leq c \leq M$ such that

$$\int_a^b f(x) d\alpha(x) = c \int_a^b d\alpha$$

$$i.e. \int_a^b f(x)d\alpha(x) = c[\alpha(b) - \alpha(a)]$$

In particular, if f is continuous on [a, b] then $c = f(x_0)$ for some x_0 in [a, b].

Proof: First of all we will show that if $f \in \mathcal{R}(\alpha)$ on [a, b], then

$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$$

where m and M are the bounds of f on [a, b].

Since $f \in \mathcal{R}(\alpha)$ on [a, b], therefore

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha.$$

Let $P = [a = x_0, x_1, x_2, \ldots, x_n = b]$ be any partition of [a, b] and m_i, M_i be bounds of f in $[x_{i-1}, x_i]$.

Then

$$m \leq m_i \leq M_i \leq M, i = 1,2, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{n} m\Delta\alpha_i \leq \sum_{i=1}^{n} m_i\Delta\alpha_i \leq \sum_{i=1}^{n} M_i\Delta\alpha_i \leq \sum_{i=1}^{n} M\Delta\alpha_i$$

$$\Rightarrow m\sum_{i=1}^{n} \Delta\alpha_i \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M\sum_{i=1}^{n} \Delta\alpha_i$$

$$\Rightarrow m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)] \quad \ldots (1)$$

Also, we know

$$L(P, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha)$$

Since $f \in \mathcal{R}(\alpha)$, therefore

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha) \quad \ldots (2)$$

Using (1) and (2), we get

$$\Rightarrow m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

$$\Rightarrow m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)].$$

Now we have

$$\Rightarrow m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)].$$

Therefore, there exists $c \in [m, M]$ such that

$$\int_a^b f d\alpha = c[\alpha(b) - \alpha(a)].$$

When f is continuous on [a, b], it takes all values between m and M over the interval [a, b].

Since $c \in [m, M]$, therefore thereexists some $x_0 \in [a, b]$ such that $c = f(x_0)$.

Therefore,

$$\int_a^b f(x)d\alpha(x) = f(x_0)[\alpha(b) - \alpha(a)].$$

This completes the proof.

3.3 Second Mean Value Theorem for Riemann-Stieltjes Integral

Statement: Assume that α is continuous and that f is monotonically increasing on [a, b]. Then there exists a point x_0 in [a, b] such that

$$\int_a^b f(x)d\alpha(x) = f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)].$$

$$i.e.\ \int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

Proof: Given α is continuous and f is monotonically increasing on [a, b].

$\Rightarrow \alpha \in \mathcal{R}(f)$ on [a, b].

$\Rightarrow f \in \mathcal{R}(\alpha)$ on [a, b]

and

$$\int_a^b f(x)d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)df(x) \quad \ldots (1)$$

Since α is continuous on [a, b].

Therefore, by the First mean value theorem, there exists $x_0 \in [a,b]$ such that

$$\int_a^b \alpha(x)df(x) = \alpha(x_0)[f(b) - f(a)] \quad \ldots (2)$$

Therefore, from (1) and (2) we get,

$$\int_a^b f(x)d\alpha(x) = f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)].$$

$$i.e.\ \int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

This completes the proof.

Summary

- Fundamental Theorem of Calculus: If f is Riemann integrable on [a, b] and if there is a differentiable function F on [a, b] such that $F' = f$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

- First Mean Value Theorem for Riemann-Stieltjes Integral: Assume that α is monotonically increasing and let $f \in \mathcal{R}(\alpha)$ on [a, b]. Let M and m denote respectively, the supremum and infimum of the set $\{f(x): x \in [a,b]\}$. Then there exists a real number c satisfying $m \le c \le M$ such that

$$\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha$$

$$i.e. \int_a^b f(x)d\alpha(x) = c[\alpha(b) - \alpha(a)]$$

- Second Mean Value Theorem for Riemann-Stieltjes Integral: Assume that α is continuous and that f is monotonically increasing on [a, b]. Then there exists a point x_0 in [a, b] such that

$$\int_a^b f(x)d\alpha(x) = f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)].$$

$$i.e. \int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

Keywords

Fundamental Theorem of Calculus: If f is Riemann integrable on [a, b] and if there is a differentiable function F on [a, b] such that $F' = f$ then $\int_a^b f(x)dx = F(b) - F(a)$.

First Mean Value Theorem for Riemann-Stieltjes Integral: Assume that α is monotonically increasing and let $f \in \mathcal{R}(\alpha)$ on [a, b]. Let M and m denote respectively, the supremum and infimum of the set $\{f(x): x \in [a,b]\}$. Then there exists a real number c satisfying $m \leq c \leq M$ such that $\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha$

Second Mean Value Theorem for Riemann-Stieltjes Integral: Assume that α is continuous and that f is monotonically increasing on [a, b]. Then there exists a point x_0 in [a, b] such that $\int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$

Self-Assessment

Let g be a real-valued bounded function defined on [s, t] and let α be a real-valued monotonically increasing function defined on [s, t]. Further, let

$\Delta\alpha_k = \alpha(y_k) - \alpha(y_{k-1})$. Suppose $P^* = \{s = y_0, y_1, y_2, \dots, y_p = t\}$ be the partition of [s, t] and

$m_k = inf\{g(y): y \in [y_{k-1,} y_k]\}$,

$M_k = sup\{g(y): y \in [y_{k-1,} y_k]\}$,

$m = inf\{g(y): y \in [s,t]\}$,

$M = sup\{g(y): y \in [s,t]\}$.

Then select the correct option in Q (1-10).

1) $\sum_{k=1}^{p}[\alpha(y_k) - \alpha(y_{k-1})] = \alpha(s) - \alpha(t)$

A. True

B. False

2) $L(P^*, g, \alpha) = \sum_{k=1}^{p} M_k \,\Delta\alpha_k$

A. True

B. False

3) $U(P^*, g, \alpha) = \sum_{k=1}^{p} M_k \,\Delta\alpha_k$

A. True

B. False

4) $U(P^*, g, \alpha) \geq L(P^*, g, \alpha)$

A. True

B. False

5) $m_k \leq m \leq M_k \leq M$

A. True

B. False

6) $m[\alpha(t) - \alpha(s)] \leq L(P^*, g, \alpha) \leq U(P^*, g, \alpha) \leq M[\alpha(t) - \alpha(s)]$

A. True

B. False

7) If $g \in \mathcal{R}(\alpha)$ then $\underline{\int_s^t} g d\alpha = \int_s^t g d\alpha$ but $\overline{\int_s^t} g d\alpha \neq \int_s^t g d\alpha$

A. True

B. False

8) $L(P^*, g, \alpha) \leq \underline{\int_s^t} g d\alpha \leq U(P^*, g, \alpha) \leq \overline{\int_s^t} g d\alpha$

A. True

B. False

9) If $g \in \mathcal{R}(\alpha)$ then $m[\alpha(s) - \alpha(t)] \geq \int_t^s g d\alpha \geq M[\alpha(s) - \alpha(t)]$

A. True

a. False

10) If $g \in \mathcal{R}(\alpha)$ then $L(P^*, g, \alpha) \leq \int_s^t g d\alpha \leq U(P^*, g, \alpha)$

A. True

B. False

11) Let f be a function defined on [a, b] such that f is continuous on [a, b] and f is differentiable in (a, b), then $\frac{f(b)-f(a)}{b-a}$ is the value of the derivative of f at some point in (a, b).

A. True

B. False

12) If $f \in \mathcal{R}(\alpha)$ then for given $\in > 0, U(P, f, \alpha) - L(P, f, \alpha) < \in$ but the converse does not hold good.

A. True

B. False

13) If f is continuous on [a, b] and α is monotonically increasing function on [a, b] then there exists $\xi \in [a, b]$ such that $\int_a^b f d\alpha = f(\xi)[\alpha(a) - \alpha(b)]$.

A. True

B. False

14) If f is continuous on [a, b] and α is monotonically increasing function on [a, b] then there exists $\xi \in [a, b]$ such that $\int_a^b f d\alpha = f(a)[\alpha(\xi) + \alpha(a)] + f(b)[\alpha(b) + \alpha(\xi)]$.

A. True

B. False

15) Let f be a Riemann integrable on [a, b]. If there is a differentiable function g on [a, b] such that $g' = f$ then $\int_a^b f dx = g(b) - g(a)$.

A. True

B. False

Answers for Self Assessment

1.	B	2.	B	3.	A	4.	A	5.	B
6.	A	7.	B	8.	B	9.	A	10.	A
11.	A	12.	B	13.	B	14.	B	15.	A

Review Questions

1) Evaluate:

$$\int_{-1}^{1} f(x)dx,$$

where

$$f(x) = \begin{cases} 1 - 2x, x \leq 0 \\ 1 + 2x, x \geq 0 \end{cases}$$

2) Evaluate:

$$\int_{1}^{4} f(x)dx,$$

where

$$f(x) = \begin{cases} 2x + 8, & 1 \leq x \leq 2 \\ 6x, & 2 \leq x \leq 4 \end{cases}$$

3)Evaluate:

$$\int_{1}^{4} (|x-1| + |x-2| + |x-3|)dx$$

4)Evaluate:

$$\int_{1}^{4} (|x-1| + |x-2| + |x-3|)dx$$

5) Evaluate:

$$\int_{0}^{1.5} [x^2]dx$$

6) Evaluate:

$$\int_{0}^{\frac{\pi}{2}} \log tanx\, dx$$

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

Shanti Narayan, Elements of Real Analysis

Web Links

https://nptel.ac.in/courses/111/105/111105069/

https://www.youtube.com/watch?v=OR27vq-iJS8

Unit 04: Integration and Differentiation

CONTENTS

Objectives

After studying this unit, students will be able to:

- establish the relationship between differentiation and integration
- understand theorem on differentiation and integration
- describe integration by parts
- solve integrals using integration by parts

Introduction

In the previous unit, we have established the relationship between differentiation and integration withthe fact that, if f is Riemann integrable on [a, b] and if there is a differentiable function F on [a, b] such that $F' = f$ then $\int_a^b f(x)dx = F(b) - F(a)$, known as the fundamental theorem of calculus. In this unit, we shall show more results exhibiting the relation between differentiation and integration.

4.1 Integration and Differentiation

Theorem 4.1.1: Suppose f is a Rieman integrable function on [a, b] *i.e.*,$f \in \mathcal{R}$ on [a, b].

For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on [a, b]; furthermore, if f is continuous at a point x_0 of [a, b], then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Proof: Since $f \in \mathcal{R}$ on [a, b].

$\Rightarrow f$ is bounded on [a, b].

Therefore, there exists a positive real number M such that

$$|f(t)| \leq M \; \forall t \in [a, b]$$

Let $a \leq x < y \leq b$, then

$$|F(y) - F(x)| = \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right|$$

$$= \left| \int_a^y f(t)dt + \int_x^a f(t)dt \right|$$

$$= \left| \int_x^y f(t)dt \right|$$

$$\leq \int_x^y |f(t)|\,dt$$

$$\leq M \int_x^y 1\,dt$$

$$= M[t]_x^y$$

$$= M(y-x)$$

$$< \epsilon$$

whenever $y - x < \frac{\epsilon}{M} = \delta$.

Thus,

$|F(y) - F(x)| < \epsilon$ whenever $|y - x| < \delta, \forall x, y \in [a, b]$.

$\Rightarrow F$ is uniformly continuous on [a, b].

$\Rightarrow F$ is continuous on [a, b].

Now suppose f is continuous at x_0.

Therefore, for given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon$$

whenever $|x - x_0| < \delta$.

If $a \leq x_0 < x \leq b$,

then consider

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{F(x) - F(x_0) - (x - x_0)f(x_0)}{x - x_0} \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_a^x f(t)dt - \int_a^{x_0} f(t)dt - \int_{x_0}^x f(x_0)dt \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_a^x f(t)dt + \int_{x_0}^a f(t)dt - \int_{x_0}^x f(x_0)dt \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(t)dt - \int_{x_0}^x f(x_0)dt \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_{x_0}^x [f(t) - f(x_0)]dt \right|$$

$$\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)|dt$$

$$< \frac{1}{|x - x_0|} \int_{x_0}^x \epsilon\,dt$$

$$= \frac{1}{|x - x_0|} \epsilon\,[x - x_0]$$

$$=\in$$

Thus, we have

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|<\in \textit{ whenever } 0<|x-x_0|<\delta$$

$$\Rightarrow \lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0)$$

$$\Rightarrow F'(x_0)=f(x_0).$$

This completes the proof.

Theorem 4.1.2: (Integration by parts)

Let F and G be differentiable functions on [a, b] such that

$$F'=f\in\mathcal{R}$$

and

$$G'=g\in\mathcal{R}$$

then

$$\int_a^b F(x)g(x)dx=F(b)G(b)-F(a)G(a)-\int_a^b f(x)G(x)dx.$$

Proof: We have

$$[F(x)G(x)]'=F'(x)G(x)+F(x)G'(x)$$
$$=f(x)G(x)+F(x)g(x)$$

Therefore, by the fundamental theorem of calculus, we get

$$\int_a^b [f(x)G(x)+F(x)g(x)]dx=F(b)G(b)-F(a)G(a)$$

$$\Rightarrow \int_a^b f(x)G(x)dx+\int_a^b F(x)g(x)dx=F(b)G(b)-F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx=F(b)G(b)-F(a)G(a)-\int_a^b f(x)G(x)dx.$$

This completes the proof.

Example 4.1.3: Evaluate the integral:

$$\int_0^{\frac{\pi}{6}}(2+3x^2)\cos 3x\,dx$$

Solution: We have,

$$\int_0^{\frac{\pi}{6}}(2+3x^2)\cos 3x\,dx$$

On integrating by parts, we get

$$=\left[\frac{(2+3x^2)}{3}\sin 3x\right]_0^{\pi/6}-\int_0^{\frac{\pi}{6}}6x\frac{\sin 3x}{3}dx$$

$$= \left[\frac{(2+3x^2)}{3}\sin 3x\right]_0^{\pi/6} - 2\int_0^{\frac{\pi}{6}} x\sin 3x dx$$

$$= \left[\frac{(2+3x^2)}{3}\sin 3x\right]_0^{\pi/6} - 2\left[\left[\frac{-x\cos 3x}{3}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{-\cos 3x}{3} dx\right]$$

$$= \left[\frac{(2+3x^2)}{3}\sin 3x\right]_0^{\pi/6} - 2\left[\left[\frac{-x\cos 3x}{3}\right]_0^{\frac{\pi}{6}} + \frac{1}{9}[\sin 3x]_0^{\frac{\pi}{6}}\right]$$

$$= \frac{1}{36}(\pi^2 + 16).$$

Example 4.1.4: Evaluate the integral:

$$I = \int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx.$$

Solution: We have,

$$I = \int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$$

On integrating by parts, we get

$$I = \left[\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$$

$$\Rightarrow I = \left[\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} - \frac{1}{2}\left[\left[\cos\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx\right]$$

$$\Rightarrow I = \left[\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} - \frac{1}{2}\left[\left[\cos\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} + \frac{1}{2}I\right]$$

$$\Rightarrow I + \frac{1}{4}I = \left[\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} - \frac{1}{2}\left[\cos\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi}$$

$$\Rightarrow \frac{5}{4}I = \left[\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi} - \frac{1}{2}\left[\left[\cos\left(\frac{\pi}{4} + \frac{x}{2}\right) e^x\right]_0^{2\pi}\right]$$

$$\Rightarrow \frac{5}{4}I = -\frac{(e^{2\pi} + 1)}{2\sqrt{2}}$$

$$\Rightarrow I = -\frac{\sqrt{2}}{5}(e^{2\pi} + 1).$$

Example 4.1.5: Evaluate the integral:

$$I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx.$$

Solution: We have,

$$I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx$$

$$\Rightarrow I = \int_0^1 xe^x dx + \int_0^1 \sin\frac{\pi x}{4} dx$$

$$\Rightarrow I = [xe^x]_0^1 - \int_0^1 1.e^x dx - \frac{4}{\pi}\left[\cos\frac{\pi x}{4}\right]_0^1$$

$$\Rightarrow I = [xe^x]_0^1 - [e^x]_0^1 - \frac{4}{\pi}\left[\cos\frac{\pi x}{4}\right]_0^1$$

$$\Rightarrow I = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}.$$

Evaluate the integral:

$$\int_0^1 xe^x dx$$

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} x^2 \cos 2x \, dx$$

Summary

- Suppose f is a Rieman integrable function on [a, b] *i.e.*, $f \in \mathcal{R}$ on [a, b].
  For $a \le x \le b$, put
  $$F(x) = \int_a^x f(t)dt.$$
  Then F is continuous on [a, b]; furthermore, if f is continuous at a point x_0 of [a, b], then F is differentiable at x_0, and
  $$F'(x_0) = f(x_0).$$

- **Integration by parts:** Let F and G be differentiable functions on [a, b] such that
  $$F' = f \in \mathcal{R}$$

and

$$G' = g \in \mathcal{R}$$

then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Keywords

Integration by parts: Let F and G be differentiable functions on [a, b] such that

$F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$ then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Self Assessment

1)If f is Riemann integrable on [a, b] then f is bounded.

A. True
B. False

2) Let $f:[a,b] \to \mathbb{R}$. If there exists a positive real number k such that $|f(t)| \leq k$, then f is said to be bounded above but not bounded below.

A. True
B. False

3) If $g(x) = \int_c^x f(t)dt$ then $|g(y) - g(x)| = \left|\int_x^y f(t)dt\right|$.

A. True
B. False

4) $\left|\int f dx\right| \geq \int |f| dx$

A. True
B. False

5) Let $f:[a,b] \to \mathbb{R}$. If for every $\in > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \in \forall x, y \in [a,b]$ for which $|y - x| < \delta$ then f is said to be uniformly continuous.

A. True
B. False

6) Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

A. True
B. False

7) Let $f:[a,b] \to \mathbb{R}$. Then f is said to be continuous at x_0 if for every $\in > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| > \in \forall x \in [a,b]$ for which $|x - x_0| < \delta$.

A. True
B. False

8) Let $f:[a,b] \to \mathbb{R}$. Then $f(x) \to q$ as $x \to p$ if for at least one $\in > 0$ there exists a $\delta > 0$ such that $|f(x) - q| < \in \forall x \in [a,b]$ for which $0 < |x - p| < \delta$.

A. True
B. False

9) Select the correct option.

A. a) $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$
B. b) $f'(c) = \lim_{x \to 0} \frac{f(x)-f(c)}{x-c}$
C. c) $f'(c) = \lim_{x \to \infty} \frac{f(x)-f(c)}{x-c}$
D. d) None of these

10) Suppose F and G are differentiable functions on [a, b], $F' = f$, and $G' = g$, where f, g are Riemann integrable functions. Then

A. $\int_a^b F(x)g(x)dx = F(b)G(b) + F(a)G(a) + \int_a^b f(x)G(x)dx$.
B. $\int_a^b F(x)g(x)dx = F(a)G(b) + F(b)G(a) + \int_a^b f(x)G(x)dx$.
C. $\int_a^b F(x)g(x)dx = F(a)G(b) - F(b)G(a) - \int_a^b f(x)G(x)dx$.
D. $\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$.

11) Let f be a Riemann integrable on [a, b]. If there is a differentiable function F on [a, b] such that $F' = f$, then $\int_a^b f dx = F(b) + F(a)$.

A. True
B. False

12) $\int_{0}^{\frac{\pi}{2}} xcosx =$

A. $\frac{\pi}{2}$
B. 1
C. $\frac{\pi}{2} - 1$
D. $1 - \frac{\pi}{2}$

13) $\int_{0}^{1} \left(xe^{2x} + sin\frac{\pi x}{2}\right) dx =$

A. $\frac{\pi}{4} - 1$
B. $\frac{\pi}{4} - 2\pi$
C. $\frac{1-2\sqrt{2}}{\pi}$
D. None of these

14) $\int_{0}^{1} \left(xe^{x} + cos\frac{\pi x}{4}\right) dx =$

A. $\frac{\pi}{2}$
B. $\frac{\pi}{2} - 1$
C. $\frac{1-2\sqrt{2}}{\pi}$
D. $\frac{1+2\sqrt{2}}{\pi}$

15) $\int_{0}^{\frac{\pi}{2}} x^2 \cos x \, dx =$

A. $\frac{\pi}{2}$
B. 1
C. $\frac{\pi}{2} - 1$
D. None of these

16) $\int_{1}^{e} \frac{logx}{x} dx =$

A. $\frac{1}{2}$
B. 1
C. e
D. None of these

Answers for Self Assessment

1.	A	2.	B	3.	A	4.	B	5.	A
6.	A	7.	B	8.	B	9.	A	10.	D
11.	B	12.	C	13.	D	14.	D	15.	D
16.	A								

Review Questions

1) Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} xsinx\,dx.$$

2) Evaluate the integral:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos2x\,log\,sinx\,dx.$$

3) Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} \frac{\log x}{x^2} dx$$

4) Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$$

5) Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} x^2 cos\,x\,dx$$

6) Evaluate the integral:

$$\int_1^{e} \frac{e^x}{x}(1 + xlogx)\,dx$$

7) Evaluate the integral:

$$\int_1^{e} \left(\frac{x-1}{x^2}\right) e^x\,dx$$

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

Shanti Narayan, Elements of Real Analysis

Web Links

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Monika Arora, Lovely Professional University

Unit 05: Integration of Vector-Valued Functions and Rectifiable Curves

CONTENTS

Objectives

After studying this unit, students will be able to:

- understand the integration of vector-valued functions
- define curves in $\mathbb{R}^k$
- define the concept of a closed curve in reference to curves in $\mathbb{R}^k$
- demonstrate the concept of arc in $\mathbb{R}^k$
- describe rectifiable curves

Introduction

In the previous units, we have discussed Riemann-Stieltjes integral and its properties for a real-valued function. In this unit, we are going to study the Riemann-Stieltjes integral of a vector-valued function.

We will also learn the concepts of curve and arc in $\mathbb{R}^k$ to understand the rectifiable curves.

5.1 Integration of Vector-Valued Functions

Definition 5.1.1: R-S Integral of a Vector-Valued Function: Let $f_1, f_2 \dots, f_k$ be real-valued functions defined on [a, b] and let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as

$$f(x) = \left(f_1(x), f_2(x), \dots, f_k(x)\right) \cdot$$

If $\alpha: [a, b] \to R$ is a monotonically increasing function; then we say $f \in R(\alpha)$ on [a, b] if and only if each $f_i \in R(\alpha)$ on [a, b] for $i = 1, 2, 3, \dots, k$ and in that case

$$\int_a^b f \, d\alpha = \left[\int_a^b f_1 \, d\alpha, \int_a^b f_2 \, d\alpha, \cdots, \int_a^b f_k \, d\alpha\right]$$

Notes

Most of the results which hold for real-valued functions $f_1, f_2, \cdots, f_k$ also, hold for vector-valued function f; we can apply the earlier results to each coordinate.

Theorem 5.1.2: Let $f: [a, b] \to \mathbb{R}^k$ be a function and $f \in R(\alpha)$, where $\alpha: [a, b] \to \mathbb{R}$ is a monotonically increasing function. Then $|f| \in R(\alpha)$ and

$$\left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha.$$

Proof: Let $f = (f_1, f_2, \ldots, f_k)$ so that

$$|f| = \sqrt{f_1^2 + f_2^2 + \cdots + f_k^2}\,\cdot$$

Since $f \in R(\alpha)$ on [a, b], therefore each of $f_i \in R(\alpha)$ and hence $f_i^2 \in R(\alpha)$ on [a, b]

$\Rightarrow f_1^2 + f_2^2 + \cdots f_k^2 \in R(\alpha)$ on [a, b].

Also, as x^2 is continuous functions of x, the square root function (the inverse of square function) is continuous on [0, M] $\forall M \in \mathbb{R}$, therefore the composite function of $f_1^2 + f_2^2 + \cdots f_k^2$ and the square root function, i.e., |f| also $\in R(\alpha)$ on [a, b].

Now,

Let $y = (y_1, y_2, \cdots y_k)$,

where

$$y_i = \int_a^b f_i\, \mathrm{d}\alpha\,\cdot$$

Then

$$y = \int_a^b f\, d\alpha$$

and

$$|y|^2 = \sum_{i=1}^{k} y_i^2$$

$$= \sum_{i=1}^{k} y_i \int_a^b f_i\, d\alpha$$

$$= \int_a^b \left(\sum_{i=1}^{k} y_i f_i\right) d\alpha \qquad \ldots (1)$$

By Schwarz inequality, we have

$$\left(\sum_{i=1}^{k} y_i f_i(t)\right)^2 \leq \sum_{i=1}^{k} y\, i^2 \sum_{i=1}^{k} \big(f_i(t)\big)^2$$

$$\Rightarrow \sum_{i=1}^{k} y_i f_i(t) \leq |y||f(t)|$$

$$\int_a^b \left(\sum_{i=1}^{k} y_i f_i(t)\right) \mathrm{d}\alpha \le \int_a^b |y||f(t)|\,\mathrm{d}\alpha$$

$$= |y| \int_a^b |f(t)|\,\mathrm{d}\alpha$$

$$\therefore |y|^2 \le |y| \int_a^b |f|\,\mathrm{d}\alpha$$

[Using (1)]

$$\Rightarrow |y| \le \int_a^b |f|\,\mathrm{d}\alpha$$

[for y=0, the theorem holds trivially]

$$\Rightarrow \left|\int_a^b f\,\mathrm{d}\alpha\right| \le \int_a^b |f|\,\mathrm{d}\alpha.$$

Example 5.1.3: Evaluate the integral:

$$\int (\sec^2 t, \ln t) dt$$

Solution:

$$\int (\sec^2 t, \ln t) dt = \left(\int \sec^2 t dt, \int \ln t\, dt\right)$$

Now,

$$\int \sec^2 t dt = tant + c_1$$

and

$$\int \ln t\, dt = \mathrm{t} \ln t - \int \left(\frac{1}{t}\right) t dt$$

$$= t \ln t - t + c_2$$

Therefore,

$$\int (\sec^2 t, \ln t) dt = \ (tant + c_1, t \ln t - t + c_2)$$

$$\Rightarrow \int (\sec^2 t, \ln t) dt = \ (\tan t\,, t \ln t - t) + (c_1, c_2)$$

Example 5.1.4: Evaluate the integral:

$$\int \left(\frac{1}{t}, 4t^3, \sqrt{t}\right) dt$$

Solution:

$$\int \left(\frac{1}{t}, 4t^3, \sqrt{t}\right) dt = \left(\int \frac{1}{t} dt, \int 4t^3 dt, \int \sqrt{t}\, dt\right)$$

Now,

$$\int \frac{1}{t} dt = \ln t + c_1,$$

$$\int 4t^3 dt = t^4 + c_2$$

and

$$\int \sqrt{t}\, dt = \frac{2}{3} t^{\frac{3}{2}} + c_3$$

Therefore,

$$\int \left(\frac{1}{t}, 4t^3, \sqrt{t}\right) dt = \left(\ln t + c_1, t^4 + c_2, \frac{2}{3} t^{\frac{3}{2}} + c_3\right)$$

$$\Rightarrow \int \left(\frac{1}{t}, 4t^3, \sqrt{t}\right) dt = \left(\ln t, t^4, \frac{2}{3} t^{\frac{3}{2}}\right) + (c_1, c_2, c_3)$$

Evaluate the integral:

$$\int (\frac{1}{t^2}, \frac{1}{t^3}, t) dt$$

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} (sint, 2cost, 1)\, dt$$

5.2 Rectifiable Curves

Definition 5.2.1: Curve in R^k: A continuous function γ: [a, b] → $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$ on [a, b].

If γ(a)= $\gamma(b)$, then γ is said to be a closed curve.

Definition 5.2.2: Arc in R^k: If γ: [a, b] → R^k is a 1-1 function, then γ is called an arc.

Definition: 5.2.3: Length of a Curve: Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of [a, b].

Let ∧(P, γ)= $\sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$, i.e., the sum of distances between points $\gamma(x_{i-1})$ and $\gamma(x_i)$, so that ∧(P,γ) is a length of a polygonal path with vertices $\gamma(x_0)$, $\gamma(x_1)$,…, $\gamma(x_n)$.

As we go on refining P, the polygon approaches the range of γ more and more closely, so that length of γ may be defined as

$$\wedge (\nu) = \sup_P \wedge (P, \nu)$$

Definition 5.2.4: Rectifiable Curve: If $\wedge (\gamma) < \infty$, then γ is said to be a rectifiable curve.

Theorem 5.2.5: If γ : [a, b] ⟶ R^k is a curve such that γ' is continuous on [a, b], then γ is rectifiable and

$$\wedge (\gamma) = \int_a^b |\gamma'(t)|\, dt$$

.

Proof. Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of [a, b].

Then

$$|\gamma(x_i) - \gamma(x_{i-1}) = \left| \int_{x_{i-1}}^{x_i} \nu'(t) dt \right|$$

$$\le \int_{x_{i-1}}^{x_i} |\nu'(t)| dt$$

Putting i = 1, 2, …, n and adding, we get,

$$\sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1}) \leq \int_a^b |\nu'(t)|dt$$

$$\wedge (P, \nu) \leq \int_a^b |\nu'(t)|dt \qquad \forall P$$

$$\Rightarrow \quad \wedge (\nu) = \sup_P \wedge (P, \gamma) \leq \int_a^b |\gamma'(t)|dt. \qquad \dots (1)$$

To prove the opposite inequality, let $\varepsilon > 0$ be given.

Since γ' is continuous on [a, b], therefore γ' is uniformly continuous on [a, b]

$\therefore$ For given $\varepsilon > 0$ there exist $\delta > 0$ such that

$|\gamma'(s)-\gamma'(t)| < \varepsilon$ whenever $|s-t| < \delta$.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of [a, b] with $\| P \| < \delta$.

Then $|\gamma'(t)- \gamma'(x_i)| < \varepsilon$ for $x_{i-1} \leq t \leq x_i$

$\therefore |\nu'(t)| - |\gamma'(x_i)| \leq |\gamma'(t) - \gamma'(x_i)| < \varepsilon$

$\Rightarrow |\gamma'(t)| < |\gamma'(x_i)| + \varepsilon$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |\nu'(t)|dt \leq |\nu'(x_i)|\Delta x_i + \in \Delta x_i$$

$$= \left|\int_{x_{i-1}}^{x_i} [\nu'(t) + \nu'(x_i) - \nu'(t)]dt\right| + \in \Delta x_i$$

$$\leq \left|\int_{x_{i-1}}^{x_i} \nu'(t)dt\right| + \left|\int_{x_{i-1}}^{x_i} [\nu'(x_i) - \nu'(t)]dt\right| + \in \Delta x_i$$

$$\leq |\gamma'(x_i) - \gamma'(x_{i-1})| + \left|\int_{x_{i-1}}^{x_i} \in dt\right| + \in \Delta x_i$$

$$\leq |\nu(x_i) - \nu(x_{i-1})| + \left|\int_{x_{i-1}}^{x_i} \in dt\right| + \in \Delta x_i$$

$$= |\nu(x_i) - \nu(x_{i-1})| + \epsilon|x_i - x_{i-1}| + \in \Delta x_i$$

$$= |\nu(x_i) - \nu(x_{i-1})| + 2\epsilon\Delta x_i$$

Putting $i = 1, 2, 3, \dots, n$ and adding, we have

$$\int_a^b |\nu'(t)|dt \leq \sum_{i=1}^{n} |\nu(x_i) - \nu(x_{i-1})| + 2\epsilon(b-a)$$

$$i.e. \int_a^b |\nu'(t)|dt \leq \wedge (P, \nu) + 2\epsilon(b-a) \quad \forall P$$

$$\Rightarrow \int_a^b |\nu'(t)|dt \leq \sup_P \wedge (P, \nu) + 2\epsilon(b-a)$$

$$= \wedge (\nu) + 2\epsilon(b-a)$$

$$\Rightarrow \int_a^b |\nu'(t)|dt \leq \wedge (\nu) \qquad \dots (2)$$

From (1) and (2), we get

$$\Rightarrow \wedge (\nu) = \int_a^b |\nu'(t)|dt < \infty$$

$\Rightarrow \nu$ is rectifiable.

Notes

Example 5.2.6: If $\nu: [0, 2\pi] \to \mathbb{R}^2$ is defined by $\nu(t) = (cost, sint)$, show that ν is rectifiable and find its length.

Solution: We have, $\nu: [0, 2\pi] \to \mathbb{R}^2$ defined by

$\nu(t) = (cost, sint)$

$\Rightarrow \nu'(t) = (-sint, cost)$

which is continuous in $[0,2\pi]$.

$\therefore$ ν is rectifiable, and its length is given as

$$\wedge(\nu) = \int_0^{2\pi} |\nu'(t)| dt$$

$$= \int_0^{2\pi} |\nu'(t)| dt$$

$$= \int_0^{2\pi} \sqrt{(-sint)^2 + (cost)^2} dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + cos^2 t}\, dt$$

$$= \int_0^{2\pi} 1 dt$$

$$= 2\pi$$

Self-Assessment

1) $\wedge(\nu) =$

(a) $\sup_P \wedge(P, \nu)$

(b) $\inf_P \wedge(P, \nu)$

(c) $\wedge(P, \nu)$

(d) none of these

2) State true or false:

If γ: [a, b] $\to R^k$ is a function, then γ is called an arc.

3) State true or false:

If $\wedge(\gamma) = \infty$, then γ is said to be a rectifiable curve.

4) State true or false:

A continuous function γ: [a, b] $\to \mathbb{R}^k$ is called a curve in $\mathbb{R}^k$ on [a, b].

5) State true or false:

If γ(a)= $\gamma(b)$, then γ is said to be a closed curve.

6) State true or false:

If $\wedge(\gamma) < \infty$, then γ is said to be a rectifiable curve.

7) Choose the correct option:

$(a) \left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha$

$(b) \left|\int_a^b f\, d\alpha\right| \geq \int_a^b |f|\, d\alpha$

$(c) \left|\int_a^b f\, d\alpha\right| = \int_a^b |f|\, d\alpha$

(d) none of these.

Notes

8) State true or false:

Let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$ and

if $\alpha: [a, b] \to R$ is a monotonically increasing function then f is Riemann-Stieltjes integrable with respect to α if at most one $f_i \in R(\alpha)$ on [a, b] for $i = 1, 2, 3, \dots, k$.

9) State True or false:

Let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$. If $\alpha: [a, b] \to R$ is a monotonically increasing function then

$$\int_a^b f \, d\alpha = \left[\int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + \dots + \int_a^b f_k \, d\alpha \right].$$

10) State true or false:

Let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$ and

if $\alpha: [a, b] \to R$ is a monotonically increasing function then f is Riemann-Stieltjes integrable with respect to α if at least one $f_i \in R(\alpha)$ on [a, b] for $i = 1, 2, 3, \dots, k$.

11) Choose the correct option:

Let $f = (f_1, f_2, \dots, f_k)$ then

$(a) |f| = (f_1^2 + f_2^2 + \dots + f_k^2)^2$

$(b) |f| = f_1^2 + f_2^2 + \dots + f_k^2$

$(c) |f| = \sqrt{f_1^2 + f_2^2 + \dots + f_k^2}$

(d) none of these

12)State true or false:

Let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$ and

if $\alpha: [a, b] \to R$ is a monotonically increasing function then f is Riemann-Stieltjes integrable with respect to α if each $f_i \in R(\alpha)$ on [a, b] for $i = 1, 2, 3, \dots, k$.

13) State True or false:

Let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$. If $\alpha: [a, b] \to R$ is a monotonically increasing function then

$$\int_a^b f \, d\alpha = \left[\int_a^b f_1 \, d\alpha, \int_a^b f_2 \, d\alpha, \dots, \int_a^b f_k \, d\alpha \right].$$

14) State true or false:

Let γ be a curve such that γ' is continuous on [a, b], then γ is non-rectifiable curve.

15)State true or false:

If $\gamma : [a, b] \longrightarrow R^k$ is a curve such that γ' is continuous on [a, b] then

$$\wedge (\gamma) < \int_a^b |\gamma'(t)| \, dt$$

16) State true or false:

If $\gamma : [a, b] \longrightarrow R^k$ is a curve such that γ' is continuous on [a, b] then

$$\wedge (\gamma) = \int_a^b |\gamma'(t)| \, dt$$

Answer: Self-Assessment

1	a	5	True	9	False	13	True
2	False	6	True	10	False	14	False
3	False	7	a	11	c	15	False
4	True	8	False	12	True	16	True

Summary

- Let $f_1, f_2 \dots, f_k$ be real-valued functions defined on [a, b] and let $f: [a, b] \to \mathbb{R}^k$ be a vector-valued function defined as $f(x) = \big(f_1(x), f_2(x), \dots, f_k(x)\big)\cdot$
- If $\alpha: [a, b] \to R$ is a monotonically increasing function; then we say $f \in R(\alpha)$ on [a, b] if and only if each $f_i \in R(\alpha)$ on [a, b] for $i = 1, 2, 3, \dots, k$ and in that case

$$\int_a^b f \, d\alpha = \left[\int_a^b f_1 \, d\alpha, \int_a^b f_2 \, d\alpha, \cdots, \int_a^b f_k \, d\alpha\right]\cdot$$

- A continuous function γ: [a, b] → $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$ on [a, b].
- If γ(a)= $\gamma(b)$, then γ is said to be a closed curve.
- If γ: [a, b] → R^k is a 1-1 function, then γ is called an arc.
- Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of [a, b]. Let ∧(P, γ)= $\sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$, i.e., the sum of distances between points $\gamma(x_{i-1})$ and $\gamma(x_i)$, so that ∧(P,γ) is a length of a polygonal path with vertices $\gamma(x_0)$, $\gamma(x_1)$,… , $\gamma(x_n)$.
  As we go on refining P, the polygon approaches the range of γ more and more closely, so that length of γ may be defined as
  $$\wedge(\nu) = \sup_P \wedge(P, \nu)$$
- If $\wedge(\gamma) < \infty$, then γ is said to be a rectifiable curve.

Keywords

Curve in R^k: A continuous function γ: [a, b] → $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$ on [a, b]. If γ(a)= $\gamma(b)$, then γ is said to be a closed curve.

Arc in R^k: If γ: [a, b] → R^k is a 1-1 function, then γ is called an arc.

Length of a Curve: Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of [a, b].

Rectifiable Curve: If $\wedge(\gamma) < \infty$, then γ is said to be a rectifiable curve.

Review Questions

1)Find the integral:

$$\int_0^{\frac{\pi}{6}} (2cost, \sin 2t) \, dt.$$

2) Evaluate the integral:

$$\int (4cos2t, 4te^{t^2}, 2t + 3t^2) \, dt.$$

3) Find $R(t)$ if:

$$R'(t) = (1 + 2t, 2e^{2t}), \qquad R(0) = (1, 3).$$

4) Find $R(t)$ if:

$$R'(t) = \left(\sin\frac{t}{3}, \cos\frac{t}{2}\right), \qquad R(\pi) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

5) Compute the integral:

$$\int \left(\frac{2t}{1+t^2}, \frac{2}{1+t^2}\right) dt.$$

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

https://nptel.ac.in/courses/111/106/111106053/

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Unit 06: Pointwise and Uniform Convergence

CONTENTS

Objectives

After studying this unit, students will be able to:

- define sequence and series of real-valued functions
- understand the pointwise convergence of sequence and series of functions
- define the uniform convergence of sequence and series of functions
- differentiate between pointwise convergence and uniform convergence
- demonstrate the effect of uniform convergence on the limit function

Introduction

Sequence of real-valued functions: Let X be a metric space and $E \subseteq X$. Let f_n be a real-valued function defined on E for each $n \in \mathbb{N}$. Then, $\{f_1, f_2, f_3, \dots, f_n, \dots\}$ is called a sequence of real-valued functions on E. It is denoted by $\{f_n\}$ or $\langle f_n \rangle$.

e.g., If f_n is a real-valued function defined by

$$f_n(x) = \frac{cosnx}{n^2}, 0 \leq x \leq 1,$$

then

$$\{f_1, f_2, f_3, \dots, f_n, \dots\} = \left\{\frac{cosx}{1^2}, \frac{cos2x}{2^2}, \frac{cos3x}{3^2}, \dots\right\}$$

is a sequence of real valued functions on [0, 1].

Series of real-valued functions: If $\{f_n\}$ is a sequence of real-valued functions defined on a set E, then

$$f_1 + f_2 + f_3 + \cdots + f_n + \cdots$$

is called a series of real-valued functions defined on E. It is denoted by $\sum_{n=1}^{\infty} f_n$

e.g., If the sequence $=\{f_n\}$ is defined by

$$f_n(x) = \frac{cosnx}{\sqrt{n}}, x \in \mathbb{R},$$

then the series is

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots + f_n + \cdots$$

$$= cosx + \frac{cos2x}{\sqrt{2}} + \frac{cos3x}{\sqrt{3}} + \cdots + \frac{cosnx}{\sqrt{n}} + \cdots$$

6.1 Pointwise and Uniform Convergence of Sequence and Series of Functions

Definition 6.1.1: Pointwise Convergence: Let X be a metric space and $E \subseteq X$. Let $f_n(x)$ be a sequence of functions defined on E. Then to each point $a \in E$, there corresponds a sequence of numbers $\{f_n(a)\}$ with terms $f_1(a), f_2(a), f_3(a), \ldots$

Further, let the sequence of numbers $\{f_n(a)\}$ converges to $f(a)$ (say). In this way, let the sequences $\{f_n(a)\}, \{f_n(b)\}, \{f_n(c)\}, \ldots$ at the points $a, b, c, \ldots$ of E converge to $f(a), f(b), f(c), \ldots$ respectively i.e., all the sequences of numbers $\{f_n(x)\}$ converge $\forall x \in E$. Then we can define a function $f(x)$ with domain E and range $\{f(a), f(b), f(c), \ldots\}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x) \; \forall x \in E$$

In this case, we say $\{f_n(x)\}$ converges to f pointwise on E and f is called the pointwise limit function of sequence $\{f_n(x)\}$.

Thus, a sequence $\{f_n\}$ of functions defined on E is said to converge pointwise to a function f on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ and x) such that

$|f_n(x) - f(x)| < \epsilon \; \forall n \geq m.$

Further, let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions defined on E.

Let $f_1 = u_1, f_2 = u_1 + u_2, \ldots, f_n = u_1 + u_2 + \cdots + u_n$.

Then the sequence $\{f_n(x)\}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n(x)$. If the sequence $\{f_n\}$ converges pointwise to the function f on E, the series $\sum_{n=1}^{\infty} u_n(x)$ is said to converge pointwise to f on E. The limit function f of $\{f_n\}$ is called the pointwise sum or simply the sum of the series $\sum u_n$ and we write

$$\sum_{n=1}^{\infty} u_n(x) = f(x), \quad \forall x \in E.$$

Definition 6.1.2: Uniform Convergence: A sequence of functions $\{f_n(x)\}$ is said to converge uniformly to a function $f(x)$ on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ only) such that $|f_n(x) - f(x)| < \epsilon \; \forall n \geq m$

Here, the function f is called the uniform limit function of sequence $\{f_n(x)\}$.

Similarly, the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E if the sequence $\{f_n(x)\}$ of partial sums defined by

$$f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = \sum_{i=1}^{n} u_i(x)$$

converges uniformly on E.

Uniform convergence of $\{f_n(x)\}$ on E implies pointwise convergence but not vice versa. However non-pointwise convergence of $\{f_n(x)\}$ on E implies non-uniform convergence of $\{f_n(x)\}$ on E.

If a sequence is uniformly convergent, then the uniform limit function is the same as the pointwise limit function.

Definition 6.1.3 Pointwise Bounded Sequence: Let $\{f_n\}$ be a sequence of functions defined on a set E. The sequence $\{f_n\}$ is said to be pointwise bounded on E if the sequence $\{f_n\}$ is bounded for every $x \in E$. i.e., $\{f_n\}$ is pointwise bounded if there exists a finite valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \; \forall x \in E, n \in \mathbb{N}.$$

Definition 6.1.4: Uniformly Bounded Sequence: A sequence of functions $\{f_n\}$ defined on a set E is said to be uniformly bounded on E if there exists $0 < M \in \mathbb{R}$ such that $|f_n(x)| < M \; \forall x \in E, n \in \mathbb{N}$.

Example 6.1.5: By definition show that sequence of functions $\{f_n(x)\}$, where $f_n(x) = \frac{x^n}{n}, x \in [0,1]$ converges uniformly to 0.

Solution: We have, $f_n(x) = \frac{x^n}{n}$

Therefore,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{x^n}{n}$$

$$= 0, 0 \le x \le 1$$

Let $\in > 0$ be given, then

$$|f_n(x) - f(x)| < \in$$

$$\Rightarrow \left|\frac{x^n}{n} - 0\right| < \in$$

$$\Rightarrow \frac{x^n}{n} < \epsilon$$

$$\Rightarrow n > \frac{x^n}{\in}$$

$$\Rightarrow n > \frac{1}{\in}, 0 \le x^n \le 1, \forall n$$

Therefore, if m is a positive integer greater than $\frac{1}{\in}$, then

$|f_n(x) - f(x)| < \in \forall n \ge m \text{ and } \forall\, x \in [0,1]$

Hence $\{f_n\}$ converges uniformly to 0.

Example 6.1.6: Show that the sequence $\{f_n(x)\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}, x \in \mathbb{R}$ is not uniformly convergent in any interval that contains zero.

Solution: We have, $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{nx}{1+n^2x^2}$$

$$= 0, x \in \mathbb{R}$$

Let $\in > 0$ be given, then

$$|f_n(x) - f(x)| < \in$$

$$\Rightarrow \left|\frac{nx}{1+n^2x^2}\right| < \in$$

$$\Rightarrow n > \frac{1 + \sqrt{1 - 4\in^2}}{2|x| \in}$$

Now when $x \to 0, n \to \infty$, therefore it is not possible to choose positive integer m such that

$$|f_n(x) - f(x)| < \in \forall n \ge m \text{ and } \forall\, x \in [0,1].$$

Hence $\{f_n(x)\}$ is not uniformly convergent in any interval that contains zero.

Example 6.1.7: Show that the sequence $\{f_n\}$ defined on [0, 1] by $f_n(x) = x^n$ is uniformly convergent on [0, k], (k<1) and only pointwise convergent on [0, 1].

Notes

Solution: Here, the limit function f is given by

$$f(x) = \lim_{n\to\infty} x^n$$
$$= \begin{cases} 0; 0 \le x < 1 \\ 1; x = 1 \end{cases}$$

Now let $\in > 0$ be given, then

$$|f_n(x) - f(x)| < \in$$
$$\Rightarrow |x^n - 0| < \in$$
$$\Rightarrow log\left(\frac{1}{x}\right)^n > log\frac{1}{\in}$$
$$\Rightarrow n > \frac{log\frac{1}{\in}}{log\frac{1}{x}}$$

Now, when $x \to 1$, then $n \to \infty$, so that it is not possible to find a positive integer m such that $|f_n(x) - f(x)| < \in \forall n \ge m \; and \; \forall \; x \in [0,1]$.

Hence $\{f_n\}$ is not uniformly convergent in [0, 1] and $x = 1$ is a point of non-uniform convergence.

However, if we consider the interval $0 \le x \le k$, where $0 < k < 1$, we see that the maximum value of $\frac{log\frac{1}{\in}}{log\frac{1}{x}}$ is $\frac{log\frac{1}{\in}}{log\frac{1}{k}}$, so that if we choose a positive integer $m \ge \frac{log\frac{1}{\in}}{log\frac{1}{k}}$, then we have $|f_n(x) - f(x)| < \in \forall n \ge m \; and \; \forall \; x \in [0,k]$.

Hence $\{f_n\}$ converges uniformly on [0, k].

Example 6.1.8: Test the sequence $\{f_n\}$ for uniform convergence, where $f_n(x) = \frac{1}{x+n}, x \in [0,a]$

Solution: We have, $f_n(x) = \frac{1}{x+n}, x \in [0,a]$.

Therefore,

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \lim_{n\to\infty} \frac{1}{x+n}$$
$$= 0, 0 \le x \le a$$

Let $\in > 0$ be given, then

$$|f_n(x) - f(x)| < \in$$
$$\Rightarrow \left|\frac{1}{x+n} - 0\right| < \in$$
$$\Rightarrow x + n > \frac{1}{\in}$$
$$\Rightarrow n > \frac{1}{\in} - x$$

Now $\frac{1}{\in} - x$ decreases as x increases and its maximum value is $\frac{1}{\in}$. Therefore, if we choose a positive integer m greater than $\frac{1}{\in}$, then we have $|f_n(x) - f(x)| < \in \; \forall n \ge m \; and \; \forall \; x \in [0,a]$.

Hence $\{f_n\}$ converges uniformly on [0, a]

Example 6.1.9: Discuss the uniform convergence of the series in [0,1]:

$$\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right]$$

Solution: Let the given series be $\sum_{n=1}^{\infty} u_n(x)$, so that

$$u_1(x) = \frac{x}{1+x^2} - 0$$

$$u_2(x) = \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$$

$$u_3(x) = \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$$

Therefore,

$$f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$

$$= \frac{nx}{1+n^2x^2}$$

Now sequence $\{f_n(x)\}$ does not converge uniformly on $[0,1]$. (See Example 2)

Therefore, series $\sum_{n=1}^{\infty} u_n(x)$ does not converge on $[0,1]$.

Example 6.1.10: Show that $x = 0$ is a point of non-uniform convergence of the series:

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

Solution: We have,

$$f_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots \frac{x^2}{(1+x^2)^{n-1}}$$

which forms a G.P.

Therefore,

$$f_n(x) = \frac{x^2\left[1-\frac{1}{(1+x^2)^n}\right]}{1-\frac{1}{1+x^2}}$$

$$= (1+x^2)\left[1-\left(\frac{1}{1+x^2}\right)^n\right]$$

Now,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \begin{cases} 1+x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Let $\epsilon > 0$ be given, then for $x \neq 0$, we have

$$|f_n(x) - f(x)| < \epsilon$$

$$\Rightarrow \left|1 + x^2 - \frac{I}{(1+x^2)^{n-1}} - (1+x^2)\right| < \epsilon$$

$$(n-1)\log(1+x^2) > \log\frac{1}{\epsilon}$$

Notes

$$\Rightarrow n > 1 + \frac{\log \frac{1}{\epsilon}}{\log(1 + x^2)}$$

This shows that if $x \to 0$, then $n \to \infty$, so that $x = 0$ is a point of non-uniform convergence of $\{f_n\}$ and hence of the given series.

Prove that the series of functions

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \cdots, x \geq 0$$

is convergent on $[0, \infty)$ but the convergence is not uniform on $[0, \infty)$.

Theorem 6.1.11: **(Cauchy Criterion for Uniform Convergence of a Sequence)**

The sequence of functions $\{f_n(x)\}$ defined on E converges uniformly on E if and only if for every $\epsilon > 0$, there exists a positive integer t such that

$$|f_n(x) - f_m(x)| < \epsilon \ \forall n, m \geq t, \forall x \in E.$$

Proof: Let $\{f_n(x)\}$ converges uniformly to $f(x)$ on E.

Then for given $\epsilon > 0$ and for all $x \in E$, there exists a positive integer t such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \ \forall n \geq t$$

Therefore for $m \geq t, n \geq t, \forall x \in E$,

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$$
$$\leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Conversely, let $|f_n(x) - f_m(x)| < \epsilon \ \forall n, m \geq t, \forall x \in E$.

Therefore, by Cauchy general principle of convergence of the sequence of real numbers, $\{f_n(x)\}$ converges to a limit $f(x)$ (say) for each $x \in E$.

Thus, the sequence $\{f_n(x)\}$ converges pointwise to $f(x)$. Now we shall prove that this convergence is uniform.

Since for given $\epsilon > 0$ and for all $x \in E$, there exists a positive integer t such that

$$|f_n(x) - f_m(x)| < \epsilon \ \ \forall n \geq t$$

Now, fixing n and letting $m \to \infty$, we get

$$|f_n(x) - f(x)| < \epsilon \ \ \forall n \geq t, \forall x \in E$$

$\Rightarrow \{f_n(x)\}$ converges uniformly to f on E.

Another form of Cauchy Criteria:

The sequence of functions $\{f_n(x)\}$ defined on E converges uniformly on E if and only if for every $\epsilon > 0$, there exists a positive integer t such that

$|f_{n+p}(x) - f_n(x)| < \epsilon \ \ \forall n \geq t, p \in \mathbb{N}, \forall x \in E.$

Theorem 6.1.12: **(Cauchy Criterion for Uniform Convergence of a Series)**

A series of functions $\sum_{n=1}^{\infty} u_n(x)$ defined on E converges uniformly on E if and only if for given $\epsilon > 0$ and $\forall x \in E$, there exists a positive integer t such that

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \epsilon \ \forall n \geq t, p \in \mathbb{N}$$

Proof: Let $\{f_n(x)\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n(x)$ defined by $f_n(x) = \sum_{i=1}^{n} u_i(x)$.

Now $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E if and only if $\{f_n(x)\}$ converges uniformly on E, i.e., if and only if for every $\epsilon > 0$, there exists a positive integer t such that $|f_{n+p}(x) - f_n(x)| < \epsilon \ \forall n \geq t, p \in$

$\mathbb{N}, \forall x \in E$. (by Cauchy criterion for uniform convergence of the sequence of functions), i.e., if and only if

$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \in \forall n \geq t, p \in \mathbb{N}, \forall x \in E$.

Theorem 6.1.13: **(M_n-Test)**

Let $\{f_n(x)\}$ be a sequence of functions on E such that $f(x) = \lim_{n\to\infty} f_n(x)\ \forall x \in E$ and $M_n = \sup_{x\in E}|f_n(x) - f(x)|$, then $\{f_n(x)\} \to f(x)$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof: Let $\{f_n(x)\} \to f(x)$ uniformly on E, then for given $\in > 0$, there exists a positive integer m (independent of x) such that

$$|f_n(x) - f(x)| < \in \quad \forall n \geq m, \forall x \in E$$

$$\Rightarrow M_n < \in \ \forall n \geq m$$

$$\Rightarrow |M_n - 0| < \in \ \forall n \geq m$$

$$\Rightarrow M_n \to 0\ as\ n \to \infty$$

Conversely, let $M_n \to 0\ as\ n \to \infty$. Therefore, for given $\in > 0$, there exists a positive integer m such that

$$|M_n - 0| < \in \ \forall n \geq m$$

$$\Rightarrow M_n < \in \ \forall n \geq m$$

$$\Rightarrow |f_n(x) - f(x)| < \in \quad \forall n \geq m, \forall x \in E$$

$\therefore \{f_n(x)\} \to f(x)$ uniformly on E.

Theorem 6.1.14: **(Weierstrass M-Test)**

The series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on a set $E \subseteq \mathbb{R}$ if there exists a convergence series $\sum_{n=1}^{\infty} M_n$ of non-negative real numbers such that $|u_n(x)| \leq M_n\ \forall n \in \mathbb{N},\ \forall x \in E$.

Proof: Since $\sum_{n=1}^{\infty} M_n$ is convergent, therefore by Cauchy criterion for convergence of a series of real numbers, for given $\in > 0$, there exists a positive integer m such that

$$|M_{n+1}(x) + M_{n+2}(x) + \cdots + M_{n+p}(x)| < \in \forall n \geq m, p \in \mathbb{N}$$

$$\Rightarrow M_{n+1}(x) + M_{n+2}(x) + \cdots + M_{n+p}(x) < \in \forall n \geq m, p \in \mathbb{N}$$

Now,

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots + |u_{n+p}(x)|$$

$$\leq M_{n+1}(x) + M_{n+2}(x) + \cdots + M_{n+p}(x)$$

$$< \in \forall n \geq m, p \in \mathbb{N}, \forall x \in E$$

Therefore, by Cauchy criteria for uniform convergence of series of functions, the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E.

Example 6.1.15: Show that the sequence $\{f_n(x)\}$, where $f_n(x) = nx(1-x)^n$ does not converge uniformly on $[0, 1]$.

Solution: Here,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} nx(1-x)^n$$

$$= 0$$

Therefore, $f(x) = 0\ \forall x \in [0,1]$.

Now,

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$

$$= sup\{nx(1-x)^n : x \in [0,1]\}$$

$$\geq n\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^n \quad \left(taking\ x = \frac{1}{n}\right)$$

Notes

$$= \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e} \ as\ n \to \infty$$

i.e., M_n does not converge to 0 as $n \to \infty$.

Hence by M_n-test $\{f_n\}$ does not converge uniformly on [0,1].

Example 6.1.16: Show that the sequence $\{f_n(x)\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, does not converge uniformly on $[0, 1]$.

Solution: Here,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{nx}{1 + n^2x^2}$$

$$= 0$$

$$\therefore f(x) = 0\ \forall x \in [0, 1].$$

Now,

$$M_n = \sup \{|f_n(x) - f(x)|: x \in [0, 1]\}$$

$$= \sup \left\{\left|\frac{nx}{1 + n^2x^2}\right| : x \in [0, 1]\right\}$$

$$\geq \frac{n\left(\frac{1}{n}\right)}{1 + n^2\left(\frac{1}{n^2}\right)}$$

$$= \frac{1}{2}$$

Therefore, M_n does not converge to 0 as $n \to \infty$.

Hence by M_n-test $\{f_n\}$ does not converge uniformly on [0,1].

Example 6.1.17: Show that the sequence $\{f_n(x)\}$, where $f_n(x) = nxe^{-nx^2}, x \in \mathbb{R}$, does not converge uniformly on $\mathbb{R}$

Solution: Here,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} nxe^{-nx^2}$$

$$= 0$$

$$\therefore f(x) = 0\ \forall x \in \mathbb{R}.$$

Now,

$$M_n = \sup \{|f_n(x) - f(x)|: x \in \mathbb{R}\}$$

$$= \sup \{n|x|e^{-nx^2} : x \in \mathbb{R}\}$$

$$\geq n\frac{1}{\sqrt{n}}e^{-n\left(\frac{1}{n}\right)}$$

$$= \frac{\sqrt{n}}{e} \to \infty \ as\ n \to \infty$$

Therefore, M_n does not converge to 0 as $n \to \infty$.

Hence by M_n-test $\{f_n\}$ does not converge uniformly on $\mathbb{R}$.

Example 6.1.18: Show that the series whose sum to n terms is $f_n(x) = \frac{n^2x}{1+n^4x^2}$ does not converge uniformly on $[0, 1]$.

Solution: Here,

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2}$$
$$= 0$$
$$\therefore f(x) = 0 \; \forall x \in [0,1].$$

Now,

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$
$$= \sup\{\frac{n^2x}{1+n^4x^2} : x \in [0,1]\}$$
$$\geq \frac{n^2\left(\frac{1}{n^2}\right)}{1+n^4\left(\frac{1}{n^4}\right)} = \frac{1}{2} \quad \left(Taking\ x = \frac{1}{n^2}\right)$$

Therefore, M_n does not converge to 0 as $n \to \infty$.

Hence by M_n-test $\{f_n\}$ does not converge uniformly on $[0,1]$.

Example 6.1.19: Test for uniform convergence of the series:

$$\sum_{n=1}^{\infty} \left\{\frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}\right\}, x \in \mathbb{R}$$

Solution: Let the given series be $\sum_{n=1}^{\infty} u_n(x)$, so that

$$u_1(x) = \frac{2x^2}{e^{x^2}} - 0$$
$$u_2(x) = \frac{2.2^2x^2}{e^{2^2x^2}} - \frac{2x^2}{e^{x^2}}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$u_n(x) = \frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

Therefore,

$$f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$
$$= \frac{2n^2x^2}{e^{n^2x^2}}.$$
$$\therefore f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \lim_{n\to\infty} \frac{2n^2x^2}{e^{n^2x^2}}$$
$$= 0.$$
$$\therefore f(x) = 0 \; \forall x \in \mathbb{R}.$$

Now,

$$M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\}$$
$$= sup\left\{\frac{2n^2x^2}{e^{n^2x^2}} : x \in \mathbb{R}\right\}$$

$$\geq \frac{2n^2\left(\frac{1}{n^2}\right)}{e^{n^2\left(\frac{1}{n^2}\right)}} = \frac{2}{e}, \quad \left(taking\ x = \frac{1}{n}\right)$$

Therefore, M_n does not converge to 0 as $n \to \infty$.

Hence by M_n-test $\{f_n\}$ does not converge uniformly on $\mathbb{R}$.

Example 6.1.20: Show that the series

$$\frac{x}{1+x^2} + \left(\frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}\right) + \left(\frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}\right) + \cdots$$

does not converge uniformly on $[0, 1]$.

Solution: Let the given series be $\sum_{n=1}^{\infty} u_n(x)$, so that

$$u_1(x) = \frac{x}{1+x^2}$$

$$u_2(x) = \frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}$$

$$u_3(x) = \frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$u_n(x) = \frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}$$

Therefore,

$$f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$

$$= \frac{n^2x}{1+n^3x^2}.$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{n^2x}{1+n^3x^2}$$

$$= 0.$$

$$\therefore f(x) = 0\ \forall x \in \mathbb{R}.$$

Now,

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [0, 1]\}$$

$$= sup\left\{\frac{n^2x}{1+n^3x^2} : x \in [0, 1]\right\}$$

$$\geq \frac{n^2\left(\frac{1}{n^{\frac{3}{2}}}\right)}{e^{n^3\left(\frac{1}{n^3}\right)}} \quad \left(taking\ x = \frac{1}{n^{\frac{3}{2}}}\right)$$

$$= \frac{\sqrt{n}}{2} \to \infty\ as\ n \to \infty$$

Therefore, M_n does not converge to 0 as $n \to \infty$.

Hence by M_n-test $\{f_n\}$ does not converge uniformly on $[0, 1]$.

Example 6.1.21: Show that the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly on $[1, \infty)$.

Solution: Let the given series be $\sum_{n=1}^{\infty} u_n(x)$, so that $u_n(x) = \frac{1}{1+n^2x}$.

Then

$$|u_n(x)| = \left|\frac{1}{1+n^2x}\right|$$
$$\leq \frac{1}{1+n^2}$$
$$< \frac{1}{n^2}, \forall x \in [1,\infty).$$

Let $M_n = \frac{1}{n^2}$ so that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-test.

∴ by Weierstrass M-test, the given series converges uniformly on $[1,\infty)$.

Example 6.2.22: Show that if $0 < r < 1$, then $\sum_{n=1}^{\infty} r^n cosnx$ is uniformly convergent on ℝ.

Solution: Let the given series be $\sum_{n=1}^{\infty} u_n(x)$, where $u_n(x) = r^n cosnx$.

Then

$$|u_n(x)| = |r^n cosnx|$$
$$\leq r^n |cosnx|$$
$$\leq r^n \ \forall x \in \mathbb{R}$$

and $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$ is a geometric series with common ratio r $(0 < r < 1)$, therefore it is convergent.

Hence by Weierstrass M-test, the given series is uniformly convergent on ℝ.

Example 6.1.23: Show that the series $\sum_{n=1}^{\infty} n^2 x^n$ is uniformly convergent in [-a, a], $o < a <$ 1.

Solution: Let the given series be $\sum_{n=1}^{\infty} u_n(x)$, where $u_n(x) = n^2 x^n$.

Then

$$|u_n(x)| = |n^2 x^n|$$
$$\leq n^2 |x^n|$$
$$\leq n^2 a^n = M_n (say)$$

Now

$$\frac{M_n}{M_{n+1}} = \frac{n^2 a^n}{(n+1)^2 a^{n+1}}$$
$$= \left(\frac{n}{n+1}\right)^2 \frac{1}{a}$$
$$= \left[1 - \frac{1}{n+1}\right]^2 \frac{1}{a}$$
$$= \frac{1}{a} \quad as\ n \to \infty$$

Therefore,

$$\frac{M_n}{M_{n+1}} > 1 \text{ as } 0 < a < 1$$

which shows $\sum_{n=1}^{\infty} M_n$ is convergent (by ratio test).

Hence by Weierstrass M-test, the given series is uniformly convergent.

Example 6.1.24: Show that the sequence $\{f_n(x)\}$, where $f_n(x) = x^{n-1}(1-x)$, converges uniformly on [0, 1].

Solution: Here,

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \lim_{n\to\infty} x^{n-1}(1-x)$$
$$= 0$$
$$\therefore f(x) = 0 \;\forall x \in [0, 1].$$

Now,

$$M_n = \sup\{|f_n(x) - f(x)|: x \in [0, 1]\}$$
$$= \sup\{x^{n-1}(1-x): x \in [0, 1]\}$$

Let $y = x^{n-1}(1-x) = x^{n-1} - x^n$.

Then

$$\frac{dy}{dx} = (n-1)x^{n-2} - nx^{n-1}$$

and

$$\frac{dy}{dx} = 0$$
$$\Rightarrow x = 0 \; or \; \frac{n-1}{n}$$

Now,

$$\frac{d^2y}{dx^2} = (n-1)x^{n-3}(n-2-nx)$$

and

$$\frac{d^2y}{dx^2} < 0 \; at \; x = \frac{n-1}{n}$$

$\therefore$ y is maximum at $x = \frac{n-1}{n}$

and

$$M_n = Max\; y$$
$$= \left(1-\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{-1} \frac{1}{n} \to 0 \; as \; n \to \infty$$

Hence by M_n-test $\{f_n\}$ does not converge uniformly on [0,1].

Show that the sequence of functions $\{f_n(x)\}$, where $f_n(x) = \frac{x}{(n+x^2)^2}$ is uniformly convergent for $x \geq 0$.

Show that if $0 < r < 1$, then $\sum_{n=1}^{\infty} r^n sin a^n x$ is uniformly convergent on $\mathbb{R}$.

Theorem 6.1.25: **(Abel's Test):**

Let (i) the series of functions $\sum_{n=1}^{\infty} u_n(x)$ be uniformly convergent on [a, b] and

(ii) the sequence of functions $\{v_n\}$ be monotonic for every x belongs to $[a, b]$ and uniformly bounded on [a, b],

then the series $u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$ is uniformly convergent on [a, b].

Proof: Since the sequence $\{v_n\}$ is uniformly bounded on [a, b], therefore there exists a real number B such that $|v_n(x)| < B$ for all $x \in [a, b]$ and for all $n \in \mathbb{N}$.

Let us choose $\in > 0$. Since the series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on [a, b], therefore there exists a natural number k such that for all $x \in [a, b]$,

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \frac{\in}{3B} \forall n \geq k, p \in \mathbb{N}$$

Next, we put

$$R_{n,p}(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)$$

Then,

$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)|$

$= |R_{n,1}(x)v_{n+1}(x) + \{R_{n,2} - R_{n,1}(x)\}v_{n+2}(x) + \cdots + \{R_{n,p}(x) - R_{n,p-1}(x)\}v_{n+p}(x)|$

$= |R_{n,1}(x)\{v_{n+1}(x) - v_{n+2}(x)\} + R_{n,2}(x)\{v_{n+2}(x) - v_{n+2}(x)\} + \cdots + R_{n,p-1}(x)\{v_{n+p-1}(x) - v_{n+p}(x) + R_{n,p}(x)v_{n+p}(x)|$

$\leq |R_{n,1}(x)|\,|v_{n+1}(x) - v_{n+2}(x)| + |R_{n,2}(x)|\,|v_{n+2}(x) - v_{n+3}(x)| + \cdots + |R_{n,p-1}(x)|\,|v_{n+p-1}(x) - v_{n+p}(x)| + |R_{n,p}(x)|\,|v_{n+p}(x)|$

$\leq \frac{\in}{3B}\left[|v_{n+1}(x) - v_{n+2}(x)| + \cdots + |v_{n+p-1}(x) - v_{n+p}(x)| + |v_{n+p}(x)|\right]$ $for\ all\ n \geq k, p = 1,2,3, \ldots$

Since $\{v_n\}$ is monotonic for every $x \in [a,b]$, therefore

$|v_{n+1}(x) - v_{n+2}(x)| + |v_{n+2}(x) - v_{n+3}(x)| \cdots + |v_{n+p-1}(x) - v_{n+p}(x)|$

$= |v_{n+1}(x) - v_{n+p}(x)|$

$\leq |v_{n+1}(x)| + |v_{n+p}(x)|$

Therefore,

$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)|$

$< \frac{\in}{3B}2B + \frac{\in}{3B}B = \in$ $for\ all\ n \geq k, p = 1,2,3, \ldots$

Thus, for all $x \in [a,b]$ we have,

$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)| < \in$ $for\ all\ n \geq k, p = 1,2,3, \ldots$

$u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$is uniformly convergent on [a, b].

Theorem 6.1.26: **(Dirichlet's Test):**

Let (i) the sequence of partial sums $\{s_n\}$ of the series of functions $u_1(x) + u_2(x) + u_3(x) \cdots$ be uniformly bounded on [a, b],

(ii) the sequence of functions $\{v_n\}$ be monotonic for every $x \in [a,b]$,

(iii) the sequence $\{v_n\}$ be uniformly convergent to 0 on [a, b],

then the series $u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$is uniformly convergent on [a, b].

Proof: Since the sequence $\{s_n\}$ is uniformly bounded on {a, b}, therefore there exists a positive real number B such that for all $x \in [a,b]$, $|s_n(x)| < B \ \forall n \in \mathbb{N}$.

Let us choose $\in > 0$. Since the sequence $\{v_n\}$ converges uniformly to 0 on [a, b], therefore there exists a natural number k such that for all $x \in [a,b]$,

$$|v_n(x)| < \frac{\in}{4B} \ \forall n \geq k.$$

Now,

$u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)$

$= \{s_{n+1} - s_n(x)\}v_{n+1}(x) + \{s_{n+2} - s_{n+1}(x)\}v_{n+2}(x) + \cdots + \{s_{n+p} - s_{n+p-1}(x)\}v_{n+p}(x)$

$= s_{n+1}(x)\{v_{n+1}(x) - v_{n+2}(x)\} + \cdots + s_{n+p-1}(x)\{v_{n+p-1}(x) - v_{n+p}(x)\} + s_{n+p}(x)v_{n+p}(x) - s_n(x)v_{n+1}(x)$

Therefore, for all $x \in [a,b]$,

$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)|$

$\leq |s_{n+1}(x)||v_{n+1}(x) - v_{n+2}(x)| + \cdots + |s_{n+p-1}(x)||v_{n+p-1}(x) - v_{n+p}(x)| + |s_{n+p}(x)||v_{n+p}(x)| + |s_n(x)||v_{n+1}(x)|$

$< B\left[|v_{n+1}(x) - v_{n+2}(x)| + |v_{n+2}(x) - v_{n+3}(x)| \cdots + |v_{n+p-1}(x) - v_{n+p}(x)| + |v_{n+p}(x)| + |v_{n+1}(x)|\right]$

Since $\{v_n\}$ is monotonic for every $x \in [a,b]$, therefore,

$|v_{n+1}(x) - v_{n+2}(x)| + |v_{n+2}(x) - v_{n+3}(x)| \cdots + |v_{n+p-1}(x) - v_{n+p}(x)|$

$= |v_{n+1}(x) - v_{n+p}(x)|$

$\leq |v_{n+1}(x)| + |v_{n+p}(x)|$

$< \frac{\in}{2B}$ $for\ all\ n \geq k, p = 1,2,3, \ldots$

Therefore, for all $x \in [a, b]$,

$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)|$

$< \frac{\in}{3B} 2B + B\left[\frac{\in}{4B} + \frac{\in}{4B}\right] = \in$ $for\ all\ n \geq k, p = 1,2,3, \ldots$

$\Rightarrow u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$is uniformly convergent on [a, b].

Example 6.1.27: Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ is uniformly convergent on $[0, 1]$.

Solution: The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \sum_{n=}^{\infty} u_n(x)v_n(x)$, where $u_n(x) = \frac{(-1)^{n-1}}{n}$ and $v_n(x) = x^n$.

Let $a_n = \frac{1}{n}$.

Then $a_n > 0\ \forall n$

and

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} < 0\ \ \forall n$$

Therefore, $\{a_n\}$ is monotonically decreasing sequence.

Also, $\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 1}{n} = 0$

$\therefore$ by Leibnitz test, the series $\sum_{n=1}^{\infty} u_n(x)$ is convergent.

Since each $u_n(x)$ is independent of x, therefore $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on $[0, 1]$.

Now,

$$v_{n+1}(x) - v_n(x) = x^{n+1} - x^n = x^n(x-1) \leq 0\ \ \forall x \in [0,1]$$

$\Rightarrow v_{n+1}(x) \leq v_n(x) \forall x \in [0,1]$.

Therefore, the sequence $\{v_n(x)\}$ is monotonically decreasing $\forall x \in [0, 1]$.

Also$|v_n(x)| = |x^n| \leq 1\ \forall n \in \mathbb{N}\ and\ \forall x \in [0,1]$

i.e., $\{v_n(x)\}$ is uniformly bounded on [0,1].

Hence by Abel's test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \sum_{n=1}^{\infty} u_n(x)v_n(x)$ is uniformly convergent on $[0, 1]$.

Example 6.1.28: Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent for all values of x.

Solution: The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2} = \sum_{n=1}^{\infty} u_n(x)v_n(x)$, where $u_n(x) = (-1)^{n-1}$ and $v_n(x) = \frac{1}{n+x^2}$.

$$Let\ f_n(x) = \sum_{i=1}^{n} u_i(x)$$
$$= \sum_{i=1}^{\infty} (-1)^{i-1}$$
$$= 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$$
$$= \begin{cases} 1 & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$
$$\Rightarrow |f_n(x)| \leq 1\ \forall x \in \mathbb{R}\ and\ \forall n \in \mathbb{N}$$

$\Rightarrow \sum_{n=1}^{\infty} u_n(x)$ is uniformly bounded on $\mathbb{R}$.

$Now\ v_n(x) = \dfrac{1}{n + x^2} > 0\ \forall x \in \mathbb{R}$

Since $\dfrac{1}{n + 1 + x^2} < \dfrac{1}{n + x^2}$

Therefore, $v_{n+1}(x) \leq v_n(x)\ \forall n$ and $x \in \mathbb{R}$.

i.e., the sequence $\{v_n(x)\}$ is monotonically decreasing $\forall x \in \mathbb{R}$.

Now we will show that $\{v_n(x)\}$ is uniformly convergent on $\mathbb{R}$.

For this let

$$v(x) = \lim_{n\to\infty} v_n(x)$$
$$= \lim_{n\to\infty} \frac{1}{n + x^2}$$
$$= 0 \quad \forall x \in \mathbb{R}$$

$\therefore |v_{n+1}(x) - v_n(x)| = \frac{1}{n+x^2}$

$Let\ y = \dfrac{1}{n + x^2}$

$Then\ \dfrac{dy}{dx} = \dfrac{-2x}{(n + x^2)^2}$

For maxima or minima, we put

$\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{-2x}{(n + x^2)^2} = 0$

$\Rightarrow x = 0$

When $x < 0$, $\frac{dy}{dx}$ is positive and when $x > 0$, $\frac{dy}{dx}$ is negative.

$\Rightarrow y$ is maximum at $x = 0$ and maximum value of $y = \frac{1}{n}$

$\therefore\ M_n = \sup\ \{|v_n(x) - v(x)| : x \in \mathbb{R}\}$

$= \dfrac{1}{n} \to 0\ as\ n \to \infty.$

$\Rightarrow \{v_n(x)\}$ is uniformly converges to 0 on $\mathbb{R}$, by M_n –test.

Thus $\{v_n(x)\}$ is monotonic decreasing sequence converging uniformly to 0 for all $x \in \mathbb{R}$.

Hence by Dirichlet's test, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent for all values of x.

Notes

Example 6.1.29: Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n(x^2+n)}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Solution: Consider the given series $\sum_{n=1}^{\infty} \frac{(-1)^n(x^2+n)}{n^2}$ on [a, b].

Let $u_n(x) = (-1)^n$ and $v_n(x) = \frac{x^2+n}{n^2}$, $x \in [a,b]$.

$$Let\ f_n(x) = \sum_{i=1}^{n} u_i(x)$$

$$= \sum_{i=1}^{\infty} (-1)^i$$

$$= -1 + 1 - 1 + 1 + \cdots + (-1)^n$$

$$= \begin{cases} -1 & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

$$\Rightarrow |f_n(x)| \le 1\ \forall x \in [a,b] and\ \forall n \in \mathbb{N}$$

$\Rightarrow \sum_{n=1}^{\infty} u_n(x)$ is uniformly bounded on [a, b].

Now, $v_n(x) = \dfrac{x^2+n}{n^2} > 0\ \forall x \in [a,b]$

and $\dfrac{d}{dn} v_n(x) = \dfrac{n^2(1) - (x^2+n)2n}{n^4}$

$-\left(\dfrac{2x^2+n}{n^3}\right) < 0\ \forall x \in [a,b]$

$\therefore \{v_n(x)\}$ is monotonically decreasing sequence $\forall x \in [a,b]$.

Now we will show that $\{v_n(x)\}$ is uniformly convergent on [a, b].

For this let

$$v(x) = \lim_{n\to\infty} v_n(x)$$

$$= \lim_{n\to\infty} \frac{x^2+n}{n^2}$$

$$= 0\ \forall x \in [a,b]$$

$\therefore |v_{n+1}(x) - v_n(x)| = \dfrac{x^2+n}{n^2}$

$< \dfrac{k^2+n}{n^2}\ \forall x \in [a,b]$, where $k = \max\{|a|,|b|\}$

$\therefore\ M_n = \sup\{|v_n(x) - v(x)| : x \in [a,b]\}$

$= \dfrac{k^2+n}{n^2}$

$\Rightarrow M_n \to 0\ as\ n \to \infty$.

$\Rightarrow \{v_n(x)\}$ is uniformly converges to 0 on [a, b], by M_n –test.

Thus $\{v_n(x)\}$ is monotonic decreasing sequence converging uniformly to 0 for all $x \in [a,b]$.

Hence by Dirichlet's test, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(x^2+n)}{n^2}$ is uniformly convergent on [a, b].

Now we consider

Notes

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n(x^2+n)}{n^2}\right|$$

$$=\sum_{n=1}^{\infty}\frac{(x^2+n)}{n^2}$$

$$=\sum_{n=1}^{\infty} w_n(x)\ \ (say)$$

$$w_n(x)=\frac{(x^2+n)}{n^2}$$

$$=\frac{x^2}{n^2}+\frac{1}{n}$$

$$>\frac{1}{n}$$

i.e., $w_n(x) > \frac{1}{n}\ x \in \mathbb{R}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent by p-test. Therefore, by comparison test, $\sum_{n=1}^{\infty} w_n(x)$ is also divergent for all x.

Hence the given series is not absolutely convergent for any value of x.

Example 6.1.30: Show that the series $\sum_{n=1}^{\infty}\frac{cosnx}{n}$ converges uniformly in $(0,2\pi)$.

Solution: The given series is $\sum_{n=1}^{\infty}\frac{cosnx}{n}=\sum_{n=1}^{\infty}u_n(x)v_n(x)$, where $u_n(x)=cosnx$ and $v_n(x)=\frac{1}{n}$.

$$Let\ f_n(x)=\sum_{i=1}^{n}u_i(x)$$

$$=\cos x+\cos 2x+\cos 3x+\cdots+\cos nx$$

$$=\frac{\cos\left(\frac{x+nx}{2}\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

Therefore,

$$|f_n(x)|=\left|\frac{\cos\left(\frac{x+nx}{2}\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right|$$

$$=\frac{\left|\cos\left(\frac{x+nx}{2}\right)\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|}$$

$$\le\frac{1}{\left|\sin\frac{x}{2}\right|}$$

$$=\left|\operatorname{cosec}\frac{x}{2}\right|$$

$$< k, \text{for some } k\in\mathbb{R}$$

$\Rightarrow \{f_n(x)$ is uniformly bounded on $(0,2\pi)$.

Also $\{v_n(x)\}=\left\{\frac{1}{n}\right\}$ is monotonic decreasing sequence converging uniformly to 0 for all $x\in(0,2\pi)$.

Hence by Dirichlet's test, the series $\sum_{n=1}^{\infty}u_n(x)v_n(x)=\sum_{n=1}^{\infty}\frac{cosnx}{n}$ converges uniformly in $(0,2\pi)$.

Notes

Prove that the series $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$ is uniformly convergent on any closed interval [a, b] contained in the open interval $(0, 2\pi)$.

Prove that the series $\sum(-1)^n x^n(1-x)$ converges uniformly on $[0, 1]$ but the series $\sum x^n(1-x)$ is not uniformly convergent on $[0, 1]$.

Self-Assessment

State true or false for $f_n(x) = x^{\frac{1}{n}}$ for $x \in [0,1]$.

1) $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$.

2) $\lim\limits_{n\to\infty} f_n(x)$ defines a continuous function

3) $\{f_n\}$ converges uniformly on [0,1].

4) $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

State true or false for $f_n(x) = (2-x)^n$ for $x \in [1,2]$ and let $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

5) $\lim\limits_{n\to\infty} f_n(x) = 0$ exists for all $x \in [1,2]$

6) $f(x)$ is a continuous function

7) $\{f_n\}$ is not uniformly convergent on [1,2].

State true or false for $f_n(x) = \frac{1}{1+n^2x^2}$ for n ∈ N, x ∈ R.

8) $\{f_n\}$ converges pointwise on [0,1] to a continuous function.

9) $\{f_n\}$ converges uniformly on [0,1].

10) $\{f_n\}$ converges uniformly on $[\frac{1}{2}, 1]$.

11) Let $\{f_n(x)\} = \{\tan^{-1} nx\}$. Consider the following statements:

(1) $\{f_n\}$ converges pointwise on [0,∞] .

(2) $\{f_n\}$ converges uniformly on [a,∞], $a > 0$. Then

a) only (1) is correct

b) only (2) is correct

c) both (1) and (2) are correct

d) both (1) and (2) are incorrect

12) Let $f_n(x) = \frac{x}{x+n}$. Consider the following statements:

(1) $\{f_n\}$ converges uniformly on [0, ∞).

(2) $\{f_n\}$ converges uniformly on [0,a], $a > 0$. Then

a) only (1) is correct

b) only (2) is correct

c) both (1) and (2) are correct

d) both (1) and (2) are incorrect

State true or false for $f_n(x) = x^n$ for $x \in [0,1]$.

13) $\lim\limits_{n\to\infty} f_n(x)$ defines a continuous function for all $x \in [0,1]$.

14) $\{f_n\}$ is uniformly convergent on $[0, k]$, $k < 1$

15)$\{f_n\}$ is uniformly convergent on [0,1]

16)Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for n ∈ N, x ∈ R. Consider the following statements:
(1)$\{f_n\}$ converges pointwise on [0,1] to a continuous function.

(2) $\{f_n\}$ converges uniformly on [0,1]. Then
a)only (1) is correct

b)only (2) is correct

c)both (1) and (2) are correct

d)both (1) and (2) are incorrect

17) Consider the following statements:
1) $a_0 + a_1 + a_2 + \ldots$be a convergent series of real numbers then the series $a_0 + a_1x + a_2x^2 + \ldots$is uniformly convergent on [0, 1].

2) $a_1 + a_2 + a_3 + \cdots$be a convergent series of real numbers then the series $a_1 + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots$is uniformly convergent on [0, ∞). Then
a)only (1) is correct

b)only (2) is correct

c)both (1) and (2) are correct

d)both (1) and (2) are incorrect

18) Consider the following statements:
1) The series $e^{-x} - \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} - \frac{e^{-4x}}{4} + \cdots$is uniformly convergent on [0, 1].

2) The series $\sum \frac{(-1)^{n-1}x^n}{n(1+x^n)}$ is not uniformly convergent on [0, 1].
a)only (1) is correct

b)only (2) is correct

c)both (1) and (2) are correct

d)both (1) and (2) are incorrect

19)State true or false:

The series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{2^n}$ is uniformly convergent on [0, 1].

20)State true or false:

The series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$ is uniformly convergent on [0, 1].

Answer: Self-Assessment

1	True	6	False	11	c	16	a
2	False	7	True	12	b	17	c
3	False	8	False	13	False	18	a
4	False	9	False	14	True	19	True
5	False	10	True	15	False	20	False

Summary

- Let X be a metric space and $E \subseteq X$. Let f_n be a real-valued function defined on E for each $n \in \mathbb{N}$. Then, $\{f_1, f_2, f_3, \ldots, f_n, \ldots\}$ is called a sequence of real-valued functions on E. It is denoted by $\{f_n\}$ or $\langle f_n \rangle$. If $\{f_n\}$ is a sequence of real-valued functions defined on a set E, then $f_1 + f_2 + f_3 + \cdots + f_n + \cdots$ is called a series of real-valued functions defined on E. It is denoted by $\sum_{n=1}^{\infty} f_n$.

- A sequence $\{f_n\}$ of functions defined on E is said to converge pointwise to a function f on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ and x) such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$.
- A sequence of functions $\{f_n(x)\}$ is said to converge uniformly to a function $f(x)$ on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ only) such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$.
- $\{f_n\}$ is pointwise bounded if there exists a finite valued function ϕ defined on E such that $|f_n(x)| < \phi(x) \ \forall x \in E, n \in \mathbb{N}$.
- A sequence of functions $\{f_n\}$ defined on a set E is said to be uniformly bounded on E if there exists $0 < M \in \mathbb{R}$ such that $|f_n(x)| < M \ \forall x \in E, n \in \mathbb{N}$.
- The sequence of functions $\{f_n(x)\}$ defined on E converges uniformly on E if and only if for every $\in > 0$, there exists a positive integer t such that $|f_n(x) - f_m(x)| < \epsilon \ \forall n, m \geq t, \forall x \in E$.
- A series of functions $\sum_{n=1}^{\infty} u_n(x)$ defined on E converges uniformly on E if and only if for given $\in > 0$ and $\forall x \in E$, there exists a positive integer t such that

  $|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \in \ \forall n \geq t, p \in \mathbb{N}$.

- Let $\{f_n(x)\}$ be a sequence of functions on E such that $f(x) = \lim_{n\to\infty} f_n(x) \ \forall x \in E$ and $M_n = \sup_{x \in E} |f_n(x) - f(x)|$, then $\{f_n(x)\} \to f(x)$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.
- The series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on a set $E \subseteq \mathbb{R}$ if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of non-negative real numbers such that $|u_n(x)| \leq M_n \ \forall n \in \mathbb{N}, \ \forall x \in E$.
- Let the series of functions $\sum_{n=1}^{\infty} u_n(x)$ be uniformly convergent on [a, b] and the sequence of functions $\{v_n\}$ be monotonic for every x belongs to $[a, b]$ and uniformly bounded on [a, b], then the series $u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$ is uniformly convergent on [a, b].
- Let the sequence of partial sums $\{s_n\}$ of the series of functions $u_1(x) + u_2(x) + u_3(x) \cdots$ be uniformly bounded on [a, b], the sequence of functions $\{v_n\}$ be monotonic for every $x \in [a, b]$, the sequence $\{v_n\}$ be uniformly convergent to 0 on [a, b], then the series $u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$ is uniformly convergent on [a, b].

Keywords

Pointwise Convergence: A sequence $\{f_n\}$ of functions defined on E is said to converge pointwise to a function f on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ and x) such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$.

Uniform Convergence: A sequence of functions $\{f_n(x)\}$ is said to converge uniformly to a function $f(x)$ on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ only) such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$.

Pointwise Bounded Sequence: $\{f_n\}$ is pointwise bounded if there exists a finite valued function ϕ defined on E such that $|f_n(x)| < \phi(x) \ \forall x \in E, n \in \mathbb{N}$.

Uniformly Bounded Sequence: A sequence of functions $\{f_n\}$ defined on a set E is said to be uniformly bounded on E if there exists $0 < M \in \mathbb{R}$ such that $|f_n(x)| < M \ \forall x \in E, n \in \mathbb{N}$.

Review Questions

1) Prove that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots, x \in [0,1]$ is not uniformly convergent on [0,1].

2) Prove that the series $\sum \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent for all real x.

3) Prove that the series $\sum \frac{x}{n + n^2 x^2}$ is uniformly convergent for all real x.

Notes

4) Show that the series $1 - \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} - \frac{e^{-6k}}{6^2-1} + \cdots$converges uniformly for all $x \geq 0$.

5) Show that the series $(1-x) + x(1-x) + x^2(1-x) + \cdots$is not uniformly convergent on $[0,1]$.

6) A sequence of functions $\{f_n\}$ is defined on $[0,1]$ by $f_n(x) = 1 - \frac{x^n}{n}, x \in [0,1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

7) Prove that the sequence of functions $\{f_n\}$, where $f_n(x) = \frac{x^n}{1+x^n}, x \in [0,2]$ is not uniformly convergent on $[0,2]$.

8)Let g be continuous on $[0,1]$ and $f_n(x) = g(x)x^n, x \in [0,1]$. Prove that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$ if and only if $g(1) = 0$.

9) A sequence of functions $\{f_n\}$ is defined on $\mathbb{R}$ by $f_n(x) = \frac{x}{n}. x \in \mathbb{R}$. Prove that the convergence of the sequence $\{f_n\}$ is not uniform on $[0,\infty)$; but the convergence is uniform on $[0, a]$ if $a > 0$.

10)A sequence of functions $\{f_n\}$ is defined on $[0,1]$ by $f_n(x) = \frac{x}{1+nx^2}. x \in [0,1]$. Show that the sequence $\{f_n\}$ converges uniformly on $[0,1]$.

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

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Unit 07: Uniform Convergence and Continuity

CONTENTS

Objectives

After studying this unit, students will be able to:

- understand explicitly the concept of uniform convergence of sequence and series of functions
- able to check the uniform convergence using continuity of the limit function
- identify that uniform convergence preserves continuity
- demonstrate the effect of uniform convergence on the limit function

Introduction

In the previous unit, we have studied the concepts of pointwise convergence and uniform convergence of sequence and series of functions. Uniform convergence of $\{f_n\}$ on E implies pointwise convergence but not vice versa. However non-pointwise convergence of $\{f_n\}$ on E implies non-uniform convergence of $\{f_n\}$ on E. If a sequence is uniformly convergent, then the uniform limit function is the same as the pointwise limit function. Thus, uniform convergence is a stronger concept than pointwise convergence. In this unit we will discuss that uniform convergence preserves continuity.

7.1 Uniform Convergence and Continuity

Theorem 7.1.1: Let $\{f_n(x)\}$ be a sequence of continuous functions defined on a compact set K. If $\{f_n\}$ converges pointwise to a continuous function f on K and $f_n(x) \geq f_{n+1}(x) \forall x \in K$ and $n = 1, 2, 3, \ldots$, then $\{f_n\}$ converges uniformly to f on K.

Proof: Let $g_n = f_n - f$.

Since $f_n \to f$ pointwise, therefore $g_n \to 0$ pointwise.

Also, we have

$$f_n(x) \geq f_{n+1}(x)$$

$$\Rightarrow f_n - f \geq f_{n+1} - f$$

$$\Rightarrow g_n \geq g_{n+1}$$

Since f and f_n are continuous, therefore g_n is also continuous.

Now we will prove that $g_n \to 0$ uniformly on K.

Let $\in > 0$ be given.

Let $K_n = \{x \in k : g_n(x) \geq \epsilon\}$

$= g_n^{-1}[\in, \infty)$

Since $[\in, \infty)$ is closed and g_n is a continuous function, therefore $K_n = g_n^{-1}[\in, \infty)$ is a closed subset of K. But K is compact.

$\Rightarrow$each K_n is compact. … (1)

Now,

$$x \in K_{n+1}$$
$$\Rightarrow g_{n+1}(x) \geq \in$$
$$\Rightarrow g_n(x) \geq g_{n+1}(x) \geq \epsilon$$
$$\Rightarrow x \in K_n$$
$$\therefore k_{n+1} \subseteq k_n \forall n \ldots (2)$$

Now we show that $\bigcap_{n=1}^{\infty} K_n = \phi$.

If possible, let $x \in \bigcap_{n=1}^{\infty} K_n$

$\Rightarrow x \in K_n \ \forall n$

$\Rightarrow g_n(x) \geq \in \ \forall n$

$\Rightarrow \lim_{n\to\infty} g_n(x) \geq \in$

$\Rightarrow 0 \geq \in$, which is not true

$\therefore \bigcap_{n=1}^{\infty} K_n = \phi. \ldots (3)$

From (1), (2), and (3), there exists $m \in \mathbb{N}$ such that $K_m = \phi$ because if $\{K_n\}$ is a sequence of non-empty compact sets such that $K_{n+1} \subseteq K_n, n = 1, 2, 3, \ldots,$

then $\bigcap_{n=1}^{\infty} K_n \neq \phi$

$\Rightarrow K_n = \phi \ \forall n \geq m$

$\Rightarrow g_n(x) < \in \ \forall n \geq m, \forall x \in K$

$\Rightarrow |g_n(x) - 0| < \in \ \forall n \geq m, \forall x \in K$

$\Rightarrow g_n \to 0$ uniformly on K.

$\Rightarrow f_n \to f$ uniformly on K.

The condition of compactness of K in the above theorem cannot be dropped.

Counter-Example: Let $f_n(x) = \frac{1}{nx+1}, 0 < x < 1$

$\Rightarrow$ each $f_n(x)$ is continuous in $(0, 1)$ and $\lim_{n\to\infty} f_n(x) = 0$

Here we see, $f_{n+1} \leq f_n \ \forall x, n = 1, 2, 3, \ldots$

Now,

$$|f_n(x) - 0| < \in$$
$$\Rightarrow \frac{1}{nx+1} < \in$$
$$\Rightarrow nx + 1 > \frac{1}{\in}$$

$$\Rightarrow n > \frac{1}{x}\left(\frac{1}{\epsilon} - 1\right)$$

$$\rightarrow \infty \; as \; x \rightarrow 0$$

Therefore, it is not possible to find $m \in \mathbb{N}$ such that $|f_n(x) - 0| < \in, \forall n \geq m$.

$\Rightarrow \{f_n(x)\}$ does not converge uniformly on $(0, 1)$.

Theorem 7.1.2: Let (X, d) be a metric space and $E \subseteq X$. Let a sequence of functions $\{f_n(x)\}$ converges uniformly to $f(x)$ on E and c be a limit point of E such that $\lim_{x \to c} f_n(x) = A_n, n = 1, 2, 3, ...$ Then $\{A_n\}$ is convergent and $\lim_{x \to c} f(x) = \lim_{n \to \infty} A_n$.

Proof: Since $f_n \rightarrow f$ uniformly on E, therefore by Cauchy's criterion, for given $\in > 0$, there exists a positive integer t such that

$|f_n(x) - f_n(x)| < \in \;\; \forall n, m \geq t, \forall x \in E$.

Letting $x \rightarrow c$, we get

$|A_n - A_m| < \in \;\; \forall n, m \geq t$

$\Rightarrow \{A_n\}$ is a Cauchy sequence in $\mathbb{R}$ and hence is convergent.

Let $\{A_n\} \rightarrow A$ as $n \rightarrow \infty$.

Now we shall prove that $\lim_{x \to c} f(x) = A$.

Since the sequence $\{f_n\}$ converges uniformly to $f(x)$ on E, therefore for given $\in > 0$, there exists $n_1 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\in}{3} \;\; \forall n \geq n_1$. ... (1)

Also, $\lim_{n \to \infty} A_n = A$.

$\therefore$ for given $\in > 0$, there exists $n_2 \in \mathbb{N}$ such that $|A_n - A| < \frac{\in}{3} \;\; \forall n \geq n_2$ ... (2)

Let $p = \max\{n_1, n_2\}$.

Then from (1) and (2), we have

$|f_n(x) - f(x)| < \frac{\in}{3}$ and $|A_n - A| < \frac{\in}{3} \;\; \forall n \geq p$ ... (3)

Again since $\lim_{x \to c} f_n(x) = A_n$, therefore for given $\in > 0$, there exists $\delta > 0$ such that

$|f_n(x) - An| < \frac{\in}{3}, 0 < |x - c| < \delta, x \in E$ ... (4)

Therefore, using (3) and (4), we have

$$\begin{aligned} |f(x) - A| &= |f(x) + f_n(x) - f_n(x) + A_n - A_n - A| \\ &\leq |f_n(x) - f(x)| + |A_n - A| + |f_n(x) - A_n| \\ &< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} \\ &= \in, 0 < |x - c| < \delta, x \in E \end{aligned}$$

$\Rightarrow \lim_{x \to c} f(x) = A$

$\lim_{x \to c} f(x) = \lim_{n \to \infty} A_n$.

Corollary: Let $\sum_{x=1}^{\infty} u_n(x)$ be a series of functions defined on $E \subseteq X$ such that $\lim_{x \to c} u_n(x)$ exists for all $n \in \mathbb{N}$, where c is a limit point of E. If the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E, then

$$\lim_{x \to c} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to c} u_n(x).$$

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$.

Since the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E, therefore $f_n \rightarrow f$ uniformly on E where $f(x) = \sum_{n=1}^{\infty} u_n(x)$.

$$\therefore \lim_{x\to c}\sum_{n=1}^{\infty} u_n(x) = \lim_{x\to c} f(x)$$

$$= \lim_{x\to c}\ \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty}\ \lim_{x\to c} f_n(x)$$

$$= \lim_{n\to\infty}\ \lim_{x\to c}\left[\sum_{i=1}^{n} u_i(x)\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\lim_{x\to c} u_i(x)$$

$$= \sum_{i=1}^{\infty}\lim_{x\to c} u_i(x)$$

$$i.e., \lim_{x\to c}\sum_{n=1}^{\infty} u_n(x) = \sum_{i=1}^{\infty}\lim_{x\to c} u_i(x)$$

Theorem 7.1.3: Let $\{f_n\}$ be a sequence of functions which converges uniformly to f on E. If each f_n is continuous on E, then f is also continuous on E.

Or

If $\{f_n\}$ is a sequence of continuous functions of E such that $f_n \to f$ uniformly on E, then f is continuous on E.

Proof: Let $\epsilon > 0$ be given.

Since $\{f_n\}$ converges uniformly to f on E, therefore there exists $m \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \forall n \geq m,\ \ \forall x \in E. \qquad \text{... (1)}$$

Let a be any point of E, then from (1), in particular, we have

$$|f_n(a) - f(a)| < \frac{\epsilon}{3} \quad \forall n \geq m. \qquad \text{... (2)}$$

Since f_n is continuous on E for each $n \in \mathbb{N}$, therefore f_n is continuous at $a \in E$.

$\therefore$ there exists $\delta > 0$ such that

$$|f_n(x) - f_n(a)| < \frac{\epsilon}{3}, \qquad |x - a| < \delta \qquad \text{... (3)}$$

Now,

$$|f(x) - f(a)| = |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, for given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$, $|x - a| < \delta$.

$\Rightarrow f(x)$ is continuous at a.

But a is an arbitrary point of E.

Hence $f(x)$ is continuous on E.

If the convergence is only pointwise then the above result may not hold.

Counter-Example: Let $f_n(x) = x^n, x \in [0, 1]$.

Then,

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \begin{cases} 0; 0 \le x < 1 \\ 1; x = 1 \end{cases}$$

is not continuous at $x = 1$ but each $f_n(x)$ is continuous at $x = 1$ as $\{f_n\}$ is not uniformly convergent on [0, 1].

The converse of the above theorem is not true, that is, a sequence of continuous functions may converge to a continuous function although the convergence is not uniform.

Counter-Example: Let

$$f_n(x) = \frac{nx}{1 + n^2x^2}, x \in \mathbb{R}$$

Then,

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= 0 \ \forall x \in \mathbb{R}$$

$\Rightarrow f(x)$ is continuous on $\mathbb{R}$

Also, each $f_n(x)$ is continuous on $\mathbb{R}$ but $f_n(x)$ does not converge uniformly in any interval that contains zero.

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real-valued continuous functions which converges uniformly to $f(x)$ on E, then $f(x)$ is also continuous on E, that is, the sum function of a uniformly convergent series of continuous functions is continuous.

Example 7.1.4: Test the uniform convergence for the series $\{f_n(x)\}$ where

$$f_n(x) = \frac{1}{1 + nx}, 0 \le x \le 1.$$

Solution: Let

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \lim_{n\to\infty} \frac{1}{1 + nx}$$
$$= \begin{cases} 0, & 0 < x \le 1 \\ 1, & x = 0 \end{cases}$$

$\Rightarrow f$ is discontinuous at $x = 0$ and hence f is discontinuous on [0, 1].

Now we see $\{f_n(x)\}$ is a sequence of continuous functions and its limit function $f(x)$ is discontinuous on [0, 1].

Therefore, sequence $\{f_n(x)\}$ is not uniformly convergent on [0, 1]. {see Example 2 of Unit 06}

Example 7.1.5: Examine the series $\sum_{n=0}^{\infty} x\, e^{-nx}$ for uniform convergence near $x = 0$.

Solution: We have,

$$f_n(x) = x + xe^{-x} + xe^{-2x} + \cdots + xe^{-(n-1)x}$$

which forms a G.P.

Therefore,

$$f_n(x) = \frac{x[1 - e^{-nx}]}{1 - e^{-x}}$$

Now,

$$f(x) = \lim_{n\to\infty} f_n(x)$$
$$= \begin{cases} x/(1 - e^{-x}), x \ne 0 \\ 0, \quad x = 0 \end{cases}$$

$$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{1-e^{-x}}$$

$$= \lim_{x\to 0} \frac{1}{e^{-x}}$$

$$= 1$$

and

$$f(0) = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) \neq f(0)$$

$$\Rightarrow f(x) \text{ is discontinuous at } x = 0$$

Therefore, the series is not uniformly convergent in any interval which includes 0.

Example 7.1.6: Show that the series:

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots$$

is not uniformly convergent in [0, 1].

Solution: The given series is:

$$\sum_{n=J}^{\infty} u_n(x)\,,\ where\ u_n(x) = \frac{x^4}{(1+x^4)^{n-1}}$$

Now,

$$f_n(x) = x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots + \frac{x^4}{(1+x^4)^{n-1}}$$

$$= \frac{x^4\left[1 - \frac{1}{(1+x^4)^n}\right]}{1 - \frac{1}{1+x^4}}$$

$$= \frac{x^4\left[1 - \frac{1}{(1+x^4)^n}\right]}{\frac{x^4}{1+x^4}}$$

$$= (1+x^4)\left[1 - \left(\frac{1}{1+x^4}\right)^n\right]$$

Therefore,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \begin{cases} 1+x^4, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Now,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+x^4)$$

$$= 1$$

and,

$$f(0) = 0$$

Therefore,

$$\lim_{x\to 0} f(x) \neq f(0)$$

$$\Rightarrow f(x) \text{ is discontinuous at } x = 0$$

Hence the series is not uniformly convergent in any interval which includes [0, 1].

Self-Assessment

1) For $f_n(y) = \dfrac{ny}{1+n^2y^2}$ for $n \in \mathbb{N}, y \in \mathbb{R}$, select the correct statements:

$(a)\{f_n\}$ converges pointwise on $[0,1]$ to a continuous function.

$(b)\{f_n\}$ converges uniformly on $[0,1]$.

2) State true or false: Let $f_n(x) = x^{1/n}$ for $x \in [0,1]$, then $\lim_{n\to\infty} f_n(x)$ defines a continuous function.

3) Let $\{f_n\}$ be the sequence of real-valued continuous functions and let $f(x) = \lim_{n\to\infty} f_n(x)$. Select the correct statements:

(a)If $f_n \to f$ uniformly then f is continuous.

(b)If f is continuous then f_n converges uniformly.

4) State true or false:

Let $f_n(y) = (2-y)^n$ for $y \in [1,2]$. Let $f(y) = \lim_{n\to\infty} f_n(y)$ then $f(y)$ is a continuous function.

5) State true or false:

Let $f_n(s) = \frac{1}{1+n^2s^2}$ for n $\in \mathbb{N}$, s $\in \mathbb{R}$, then $\{f_n\}$ converges pointwise on $[0,1]$ to a continuous function.

6) State true or false:

Let $f_n(x) = x^n$ for $x \in [0,1]$, then $\lim_{n\to\infty} f_n(x)$ defines a continuous function for all $x \in [0,1]$.

7)Select the correct statement:

(a)The series $e^{-t} - \frac{e^{-2t}}{2} + \frac{e^{-3t}}{3} - \frac{e^{-4t}}{4} + \cdots$is uniformly convergent on $[0, 1]$.

(b) The series $e^{-t} - \frac{e^{-2t}}{2} + \frac{e^{-3t}}{3} - \frac{e^{-4t}}{4} + \cdots$is only pointwise convergent on $[0, 1]$.

8) State true or false:

The series $\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} \frac{\sin nt}{2^n}$ is not uniformly convergent on $[0, 1]$.

9) State true or false:

The series $\sum_{n=1}^{\infty} f_n(s) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4s^2}$ is uniformly convergent on $[0, 1]$.

10) State true or false:

The series $\sum \frac{(-1)^{n-1}x^n}{n(1+x^n)}$ is uniformly convergent on $[0, 1]$.

11) Select the correct statement:

(a) $t_0 + t_1 + t_2 + \ldots$be a convergent series of real numbers then the series $t_0 + t_1x + t_2x^2 + \ldots$is uniformly convergent on $[0, 1]$.

(b) $t_0 + t_1 + t_2 + \ldots$be a convergent series of real numbers then the series $t_0 + t_1x + t_2x^2 + \ldots$need not be uniformly convergent on $[0, 1]$.

12) Select the correct statement:

(a)$b_1 + b_2 + b_3 + \cdots$be a convergent series of real numbers then the series $b_1 + \frac{b_2}{2^x} + \frac{b_3}{3^x} + \cdots$is uniformly convergent on $[0, \infty)$.

(b)$b_1 + b_2 + b_3 + \cdots$be a convergent series of real numbers then the series $b_1 + \frac{b_2}{2^x} + \frac{b_3}{3^x} + \cdots$is only pointwise convergent on $[0, \infty)$.

13)State true or false:

Notes

Let $f_n(x) = x^n$, then $\{ f_n\}$ is uniformly convergent on $[0, k]$, $k < 1$

14) State true or false:

Let $f_n(x) = (2-x)^n$ for $x \in [1,2]$ and let $f(x) = \lim_{n\to\infty} f_n(x)$, then $\{ f_n\}$ is not uniformly convergent on [1,2].

15) State true or false:

$(1-x) + x(1-x) + x^2(1-x) + \cdots$ is uniformly convergent on [0, 1].

Answers: Self-Assessment

1	a	6	False	11	a
2	False	7	a	12	a
3	a	8	False	13	True
4	False	9	True	14	True
5	False	10	True	15	False

Summary

- Let $\{f_n(x)\}$ be a sequence of continuous functions defined on a compact set K. If $\{f_n(x)\}$ converges pointwise to a continuous function f on K and $f_n(x) \geq f_{n+1}(x) \forall x \in K$ and $n = 1, 2, 3, \ldots$, then $\{f_n(x)\}$ converges uniformly to f on K.
- Let (X, d) be a metric space and $E \subseteq X$. Let a sequence of functions $\{f_n(x)\}$ converges uniformly to $f(x)$ on E and c be a limit point of E such that $\lim_{x\to c} f_n(x) = A_n, n = 1, 2, 3, \ldots$. Then $\{A_n\}$ is convergent and $\lim_{x\to c} f(x) = \lim_{n\to\infty} A_n$.
- Let $\sum_{x=1}^{\infty} u_n(x)$ be a series of functions defined on $E \subseteq X$ such that $\lim_{x\to c} u_n(x)$ exists for all $n \in \mathbb{N}$, where c is a limit point of E. If the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E, then
$$\lim_{x\to c} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x\to c} u_n(x).$$
- If $\{f_n\}$ is a sequence of continuous functions of E such that $f_n \to f$ uniformly on E, then f is continuous on E.
- The sum function of a uniformly convergent series of continuous functions is continuous.

Keywords

Uniform convergence: A sequence of functions $\{f_n(x)\}$ is said to converge uniformly to a function $f(x)$ on E if for given $\in > 0$ and for all $x \in E$, there exists a positive integer m (depending upon $\in$ only) such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$.

Uniform convergence preserves continuity: If $\{f_n\}$ is a sequence of continuous functions of E such that $f_n \to f$ uniformly on E, then f is continuous on E.

Review Questions

1)Test the continuity of the sum function of the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$. Also, comment on the uniform convergence of the given series.

2) For $f_n(x) = nx(1-x)^n, 0 \leq x \leq 1$, check the continuity of limit function. Also, test the uniform convergence of the given sequence.

3) Show that the sum function of the series

$$\sum \left(\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \right)$$

is continuous for all x although zero is a point of non-uniform convergence of the series.

4) Let the sequence $\{f_n(x)\}$ defined on [0, 1] by $f_n(x) = x^n$. Check the continuity of its limit function and uniform convergence of the given sequence.

5) Test the continuity of the sum function and uniform convergence of the series for which

$$f_n(x) = \frac{x^2\left[1 - \frac{1}{(1+x^2)^n}\right]}{1 - \frac{1}{1+x^2}}$$

6) Let $f_n(x) = \tan^{-1} nx, x \in [0,1]$. Prove that the sequence of functions $\{f_n\}$ is not uniformly convergent on $[0,1]$.

7) Prove that the sequence of functions $\{f_n\}$ where $f_n(x) = \frac{x^n}{1+x^n}, x \in [0,2]$ is not uniformly convergent on $[0,2]$.

Further Readings

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Unit 08: Uniform Convergence and Integration

CONTENTS

Objectives

After studying this unit, students will be able to:

- understand explicitly the concept of uniformconvergence of sequence and series of functions
- discuss the uniform convergence and integration
- identify that uniform convergence preserves term by term integration of the series of functions
- demonstrate the effect of uniform convergence on the integration of limit function

Introduction

In the previous unit, we have studied the concept of uniform convergence and continuity of sequence and series of functions. We have studied that if $\{f_n\}$ is a sequence of continuous functions of E such that $f_n \to f$ uniformly on E, then f is continuous on E.In this unit, we will discuss uniform convergence and integration.We will study that uniform convergence of series is only a sufficient condition but not a necessary condition for term-by-term integration.

8.1 Uniform Convergence and Integration

Theorem 8.1.1: Let α be monotonically increasing on [a, b], $f_n \in R(\alpha)$ on [a, b], n=1, 2, 3…and $\{f_n\}$ converges uniformly to f on [a, b]. Then $f \in R(\alpha)$ on [a, b] and

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

$$i.e. \int_a^b \lim_{n\to\infty} f_n\, d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

Proof: Let $M_n = \sup_{a\le x\le b} |f_n(x) - f(x)|$

Then $|f_n(x) - f(x)| \le M_n$

$\Rightarrow f_n(x) - M_n \le f(x) \le f_n(x) + M_n$

$$\Rightarrow \int_a^b f_n\, d\alpha - M_n \int_a^b d\alpha \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le \int_a^b f_n\, d\alpha + M_n \int_a^b d\alpha$$

$$\Rightarrow \int_a^b f_n\, d\alpha - M_n[\alpha(b) - \alpha(a)] \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le \int_a^b f_n\, d\alpha + M_n[\alpha(b) - \alpha(a)] \quad \dots (1)$$

$$\Rightarrow 0 \le \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \le 2M_n[\alpha(b) - \alpha(a)]$$

Now, as $\{f_n\}$ converges uniformly to f on [a, b] so by Weierstrass M_n test $M_n \to 0$ as $n \to \infty$

$$\therefore \underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha$$

$\Rightarrow f \in R(\alpha)$ on $[a, b]$.

Now from (1), we have

$$\Rightarrow \int_a^b f_n\, d\alpha - M_n[\alpha(b) - \alpha(a)] \le \int_a^b f\, d\alpha \le \int_a^b f_n\, d\alpha + M_n[\alpha(b) - \alpha(a)]$$

$$\Rightarrow \left| \int_a^b f\, d\alpha - \int_a^b f_n\, d\alpha \right| \le M_n[\alpha(b) - \alpha(a)]$$

$$\therefore \int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

[$\because M_n \to 0$ as $n \to \infty$]

Cor. (Term by Term Integration)

If $u_n \in R(\alpha)$ on [a, b] for all n and if $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ on [a, b], then $f \in R(\alpha)$ on [a, b] and

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b u_n d\alpha$$

$$i.e. \int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] d\alpha = \sum_{n=1}^{\infty} \int_a^b u_n(x) d\alpha.$$

Proof: Let

$$f_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$$

Since $u_n(x) \in R(\alpha)$ on [a, b], $n \in \mathbb{N}$

Therefore $f_n(x) \in R(\alpha)$ on [a, b], $n \in \mathbb{N}$

Also, $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ on [a, b]

Therefore, $f_n \to f$ uniformly on [a, b].

$\therefore$ by above theorem, $f \in R(\alpha)$ and

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] d\alpha = \int_a^b f d\alpha$$

$$= \lim_{n\to\infty} \int_a^b f_n d\alpha$$

$$= \lim_{n\to\infty} \int_a^b \left[\sum_{i=1}^{n} u_i(x)\right] d\alpha$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b u_i(x) d\alpha$$

$$= \sum_{n=1}^{\infty} \int_a^b u_n(x) d\alpha$$

*Example8.1.2:*Examine for term-by-term integration of the series the sum of whose first n terms is $n^2 x(1-x)^n$; $0 \leq x \leq 1$.

Solution: We have,

$$f_n(x) = n^2 x(1-x)^n;\ 0 \leq x \leq 1.$$

Let

$$f(x) = \lim_{n\to\infty} f_n(x)$$

For $0 < x < 1$,

$$f(x) = \lim_{n\to\infty} n^2 x(1-x)^n$$

$$= \lim_{n\to\infty} \frac{n^2 x}{(1-x)^{-n}}$$

$$= \lim_{n\to\infty} \frac{2nx}{-(1-x)^{-n} \log(1-x)}$$

$$= \lim_{n\to\infty} \frac{2x}{(1-x)^{-n}[\log(1-x)]^2}$$

$$= \lim_{n\to\infty} \frac{2x(1-x)^n}{[\log(1-x)]^2}$$

$$= 0$$

Also, when $x = 0$ or 1, then $f_n(x) = 0$

$$\therefore f(x) = 0\ \forall x \in [0,1]$$

Therefore,

$$\int_0^1 f(x)dx = \int_0^1 0dx = 0 \qquad \ldots (1)$$

Now,

$$\int_0^1 f_n(x)dx = \int_0^1 n^2 x(1-x)^n dx$$

$$= \int_0^1 n^2(1-x)[1-(1-x)]^n dx$$

$$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$= \int_0^1 n^2(1-x)x^n dx$$

$$= n^2 \int_0^1 (x^n - x^{n+1})\, dx$$

$$= n^2 \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$= n^2 \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{n^2}{(n+1)(n+2)}$$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\frac{n^2}{(n+1)(n+2)}$$

$$= 1 \qquad \dots (2)$$

From (1) and (2), we get

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx.$$

Therefore, term by term integration of given series is not justified.

*Example8.1.3:*Examine for term-by-term integration of the series the sum of whose first n terms is nxe^{-nx^2}; $0 \le x \le 1$.

Solution: Let

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty}\frac{nx}{e^{nx^2}}$$

$$= \lim_{n\to\infty}\frac{nx}{1+nx^2+\frac{n^2x^4}{2!}+\cdots}$$

$$= 0$$

Thus

$$f(x) = 0\ \forall x \in [0,1]$$

$$\Rightarrow \int_0^1 f(x)dx = 0$$

and

$$\int_0^1 f_n(x)dx = \int_0^1 nxe^{-nx^2}dx$$

$$= \frac{1}{2}[1-e^{-n}]$$

$$= \frac{1}{2} \text{ as } n\to\infty$$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

Hence term by term integration on the interval $[0, 1]$ is not justified here.

*Example 8.1.4:*Examine for term-by-term integration of the series

$$\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}, 0 \le x \le 1.$$

Solution: Let

$$\sum_{n=1}^{\infty}u_n(x) = \sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$$

where $u_n(x) = \frac{x}{(n+x^2)^2}$

Therefore,

$$u_n'(x) = \frac{(n+x^2)^2 - 4x^2(n+x^2)}{(n+x^2)^4}$$

$$= \frac{n-3x^2}{(n+x^2)^3}.$$

Now,

$$u_n'(x) = 0$$

$$\Rightarrow n - 3x^2 = 0$$

$$\Rightarrow x = \sqrt{\frac{n}{3}}$$

$$\left[\because x = -\sqrt{\frac{n}{3}} \notin [0,1]\right]$$

Now,

$$u_n''(x) = \frac{(n+x^2)^3(-6x) - (n-3x^2)3(n+x^2)^2(2x)}{(n+x^2)^6}$$

$$= \frac{-6x(n+x^2) - 6x(n-3x^2)}{(n+x^2)^4}$$

Now,

$$u_n'' < 0 \; at \; x = \sqrt{\frac{n}{3}}$$

$\Rightarrow u_n$ is maximum at $x = \sqrt{\frac{n}{3}}$

and maximum value of $u_n(x)$ is

$$\frac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16n^{3/2}}$$

$$\Rightarrow |u_n(x)| \le \frac{3\sqrt{3}}{16n^{\frac{3}{2}}}$$

$$< \frac{1}{n^{\frac{3}{2}}} = M_n(say) \forall x \in [0,1],$$

and $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent by p-test.

∴ by Weierstrass M-test, the given series is uniformly convergent on [0, 1].

Hence it can be integrated term by term.

Example 8.1.5:Show that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}.$$

Solution: Let

$$f_n(x) = \frac{x^n}{n^2}$$

$$\Rightarrow |f_n(x)| = \frac{x^n}{n^2} \le \frac{1}{n^2}, 0 \le x \le 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$\therefore$ by Weierstrass M-test, the given series is uniformly convergent on $[0, 1]$.

$$\Rightarrow \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2} \right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2} dx$$

$$= \sum_{n=1}^{\infty} \left[\frac{x^{n+1}}{(n+1)n^2} \right]_0^1$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

$Let\ f_n(x) = n^2 x(1-x^2)^n, x \in [0, 1]$. Then find $\lim_{n\to\infty} \int_0^1 f_n(x)\ and\ \int_0^1 f(x)$. Are both the integrals same?

Summary

- Let α be monotonically increasing on [a, b], $f_n \in R(\alpha)$ on [a, b], n=1, 2, 3,… and $\{f_n\}$ converges uniformly to f on [a, b]. Then $f \in R(\alpha)$ on [a, b] and

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

$$i.e. \int_a^b \lim_{n\to\infty} f_n\, d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

- Let α be monotonically increasing on [a, b], $u_n \in R(\alpha)$ on [a, b] for all n and if $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ on [a, b], then $f \in R(\alpha)$ on [a, b] and

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b u_n d\alpha$$

$$i.e. \int_a^b \left[\sum_{n=1}^{\infty} u_n(x) \right] d\alpha = \sum_{n=1}^{\infty} \int_a^b u_n(x) d\alpha.$$

Self-Assessment

1) Select the correct answer:

The sequence of functions $f_n(x) = nx(1-x^2)^n, x \in [0, 1]$. Then

$a)\ \lim_{n\to\infty} \int_0^1 f_n = \frac{1}{2}$

$b)\ \lim_{n\to\infty} \int_0^1 f_n = \frac{1}{3}$

$c)\ \lim_{n\to\infty} \int_0^1 f_n = \frac{1}{4}$

$d)\ \lim_{n\to\infty} \int_0^1 f_n = 0$

2)Let$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\left[n^2 x e^{-n^2x^2} - (n-1)^2 x e^{-(n-1)^2x^2}\right]$, $x \in [0,1]$. Choose the INCORRECT statement.

a) $\sum_{n=1}^{\infty} f_n(x)$ can be integrated term by term.

b) $\sum_{n=1}^{\infty} f_n(x)$is not uniformly convergent.

c) $\int_0^1 \sum_1^{\infty} f_n(x)\,dx \neq \sum_1^{\infty}\int_0^1 f_n(x)dx$

d)none of these

3) Select the correct answer:

For each $n \geq 2$, let

$$f_n(x) = \begin{cases} n^2 x & ; \quad 0 \leq x \leq \frac{1}{n} \\ -n^2 x + 2n; & \frac{1}{n} < x < \frac{2}{n} \\ 0; & \frac{2}{n} \leq x \leq 1 \end{cases}$$

then

a)the sequence $\{f_n\}$is uniformly convergent

b) the sequence $\{f_n\}$is only pointwise convergent

c) the sequence $\{f_n\}$is neither pointwise nor uniformly convergent

d) the sequence $\{f_n\}$is uniformly convergent but not pointwise.

4) State true or false:

The sequence of functions $f_n(x) = nx(1-x^2)^n$,$x \in [0,1]$. Then $\lim_{n\to\infty}\int_0^1 f_n(x) \neq \int_0^1 f(x)$

5) State true or false:

The sequence of functions $f_n(x) = nx(1-x^2)^n$,$x \in [0,1]$. Then $\int_0^1 f = \frac{1}{2}$

6) State true or false:

The sequence of functions $f_n(x) = n^2x(1-x^2)^n$,$x \in [0,1]$. Then $\lim_{n\to\infty}\int_0^1 f_n(x) \neq \int_0^1 f(x)$

7) Let $f_n(x) = \frac{nx}{1+nx}$, $x \in [0,1]$, then

a) the sequence $\{f_n\}$is uniformly convergent

b) the sequence $\{f_n\}$is only pointwise convergent

c) the sequence $\{f_n\}$is neither pointwise nor uniformly convergent

d) the sequence $\{f_n\}$is uniformly convergent but not pointwise.

8)Let $f_n(x) = nx(1-x)^n$, $x \in [0,1]$, Then:

(a) $\lim_{n\to\infty}\int_0^1 f_n(x) \leq \int_0^1 \lim_{n\to\infty} f_n(x)$

(b) $\lim_{n\to\infty}\int_0^1 f_n(x) \neq \int_0^1 \lim_{n\to\infty} f_n(x)$

(c) $\lim_{n\to\infty}\int_0^1 f_n(x) \geq \int_0^1 \lim_{n\to\infty} f_n(x)$

(d) $\lim_{n\to\infty}\int_0^1 f_n(x) = \int_0^1 \lim_{n\to\infty} f_n(x)$

9) The sequence of functions $f_n(x) = n^2x(1-x^2)^n, x \in [0,1]$, then

a) the sequence $\{f_n\}$is uniformly convergent

b) the sequence $\{f_n\}$is only pointwise convergent

c) the sequence $\{f_n\}$is neither pointwise nor uniformly convergent

d) the sequence $\{f_n\}$is uniformly convergent but not pointwise.

10) State true or false:

Let the sequence of functions $f_n(x) = \frac{nx}{1+nx}, x \in [0,1]$. Then $\{f_n\}$ converges uniformly on $[0,1]$

11) Let $f_n(x) = n^2x(1-x^2)^n, x \in [0,1]$. Choose the INCORRECT statement.

a) $\{f_n\}$ converges pointwise on $[0,1]$

$b)$ $\{f_n\}$ converges uniformly on $[0,1]$

c) $\lim_{n\to\infty}\int_0^1 f_n(x) \neq \int_0^1 f(x)$

d) none of these

12. Consider the following statements:

(I) Term by term integration for the series of functions implies uniform convergence.

(II) Uniform convergence of the series of functions implies term by term integration.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

13. Let the sequence of functions $f_n(x) = \frac{nx}{1+nx}, x \in [0,1]$. Let $\lim_{n\to\infty} f_n = f$. Choose the INCORRECT statement.

$a)$ $\lim_{n\to\infty}\int_0^1 f_n(x) = \int_0^1 f(x)$

$b)$ $\{f_n\}$ converges pointwise to some function f on $[0,1]$

$c)$ $\lim_{n\to\infty}\int_0^1 f_n(x) \neq \int_0^1 f(x)$

$d)$ f_n does not converge uniformly on $[0,1]$

14) State true or false:

The series $\sum_{n=1}^{\infty}\left[n^2xe^{-n^2x^2} - (n-1)^2xe^{-(n-1)^2x^2}\right], x \in [0,1]$is uniformly convergent.

15) Let $\sum_{n=1}^{\infty} u_n(x)$ be the series of functions which is uniformly convergent. Then select the correct option:

$(a)\ \lim_{n\to\infty}\int_0^1 f_n(x) \le \int_0^1 \lim_{n\to\infty} f_n(x)$

$(b)\ \lim_{n\to\infty}\int_0^1 f_n(x) \ne \int_0^1 \lim_{n\to\infty} f_n(x)$

$(c)\ \lim_{n\to\infty}\int_0^1 f_n(x) \ge \int_0^1 \lim_{n\to\infty} f_n(x)$

$(d)\ \lim_{n\to\infty}\int_0^1 f_n(x) = \int_0^1 \lim_{n\to\infty} f_n(x)$

16. Let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$, $x \in [0,1]$. Then for $\sum_1^\infty f_n(x)$, choose the INCORRECT statement.

a) $\sum_1^\infty f_n(x)$ can be integrated term by term

$b)\ \sum_1^\infty \left(\int_0^1 f_n(x)\,\mathrm{dx}\right) = \int_0^1 \left(\sum_1^\infty f_n(x)\right) dx$

$c)$ $\sum_1^\infty f_n(x)$ is not uniformly convergent on $[0,1]$

d) none of these

Answer for Self-Assessment

1.	A	2.	A	3.	B	4.	True	5.	False
6.	False	7.	B	8.	D	9.	B	10.	False
11.	B	12.	B	13.	C	14.	False	15.	D
16.	D								

Keywords

Integration for sequence: Let α be monotonically increasing on [a, b], $f_n \in R(\alpha)$ on [a, b], n=1, 2, 3,… and $\{f_n\}$ converges uniformly to f on [a, b]. Then $f \in R(\alpha)$ on [a, b] and $\int_a^b f\,d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha$.

Integration for series: If $u_n \in R(\alpha)$ on [a, b] for all n and if $\sum_{n=1}^\infty u_n(x)$ converges uniformly to $f(x)$ on [a, b], then $f \in R(\alpha)$ on [a, b] and $\int_a^b f\,d\alpha = \sum_{n=1}^\infty \int_a^b u_n d\alpha$

Review Questions

1) Test for uniform convergence and term by term integration of the series

$$\sum_{n=1}^\infty \frac{x}{(n+x^2)^2}, 0 \le x \le 1.$$

2) Show that

$$\int_0^1 \left(\sum_{n=1}^\infty \frac{x^n}{n^2}\right) dx = \sum_{n=1}^\infty \frac{1}{n^2(n+1)}$$

3) Show that the series for which $f_n(x) = \frac{1}{1+nx}$ can be integrated term by term on $[0, 1]$, although it is not uniformly convergent on $[0, 1]$.

4)Show that the series

$$\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$$

can be integrated term by term on $[0, 1]$, although it is not uniformly convergent on $[0, 1]$.

5) Show that the series for which $f_n(x) = nx(1-x)^n$ can be integrated term by term on $[0, 1]$.

Further Readings

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Web Links

https://nptel.ac.in/courses/111/106/111106053/

https://nptel.ac.in/courses/111/101/111101134/

https://doi.org/10.1007/978-1-4419-1296-1_11

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Unit 09: Uniform Convergence and Differentiation

Objectives

After studying this unit, students will be able to:

- understand explicitly the concept of uniform convergence of sequence and series of functions
- discuss the uniform convergence and differentiation
- identify the concept of term-by-term differentiation of the series of functions
- demonstrate the sufficient condition for term-by-term differentiation.

Introduction

In the last two units, we have studied the concept of uniform convergence and continuity and uniform convergence and integration of sequence and series of functions. We have studied that if $\{f_n\}$ is a sequence of continuous functions of E such that $f_n \to f$ uniformly on E, then f is continuous on E. Also we have discussed that uniform convergence of series is only a sufficient condition but not a necessary condition for term-by-term integration. In this unit, we will discuss uniform convergence and differentiation. We will study the sufficient condition for term-by-term differentiation.

9.1 Uniform Convergence and Differentiation

Theorem 9.1.1: Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that

(I) f_n is differentiable on [a, b], n=1,2,3, …

(II) The sequence $\{f_n(d)\}$ converges for some point d of [a, b]

(III) The sequence $\{f_n'\}$ converges uniformly on [a, b].

Then the sequence $\{f_n\}$ converges uniformly to a differentiable function f and

$$\lim_{n\to\infty} f_n'(\mathrm{x}) = f'(x), a \le x \le b$$

$$i.e. \lim_{n\to\infty} \frac{d}{dx}[f_n(x)] = \frac{d}{dx}\left[\lim_{n\to\infty} f_n(x)\right], \forall x \in [a,b]$$

Proof: Let $\in > 0$ be given.

Then by convergence of $\{f_n(d)\}$ and by uniform convergence of $\{f_n'\}$ on [a, b], there exists a positive integer m such that for all $n \ge m, p \ge m$ we have

$$\left|f_n(d) - f_p(d)\right| < \frac{\in}{2} \qquad \text{... (1)}$$

and

$$\left|f_n'(x) - f_p'(x)\right| < \frac{\in}{2(b-a)}, a \le x \le b \qquad \text{... (2)}$$

Applying the mean value theorem to the function $f_n - f_p$ for any two points x and y of [a, b], we have

$$[f_n(x) - f_p(x)] - [f_n(y) - f_p(y)] = (x - y)[f_n'(\zeta) - f_p{}'(\zeta)],$$

where $\zeta \in (x, y)$.

Now for $n \geq m, p \geq m$ and $x, y \in [a, b]$, we have

$$|f_n(x) - f_p(x) - f_n(y) + f_p(y)| = |x - y||f_n'(\zeta) - f_p'(\zeta)|$$

$$< \frac{|x - y| \in}{2(b - a)} \quad [using\ (2)] \ldots (3)$$

$$< \frac{\in}{2} \quad \ldots (4)$$

Therefore, for all $n, p \geq m$ and $x, y \in [a, b]$, we have

$$|f_n(x) - f_p(x)| = |f_n(x) - f_p(x) - f_n(d) + f_p(d) + f_n(d) - f_p(d)|$$

$$\leq |f_n(x) - f_p(x) - f_n(d) + f_p(d)| + |f_n(d) - f_p(d)|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

$\Rightarrow \{f_n\}$ converges uniformly to some function f (say) on [a, b].

$$i.e. \lim_{n\to\infty} f_n(x) = f(x), \forall x \in [a, b]$$

Further, fix a point x in [a, b] and define

$$F_n(y) = \frac{f_n(y) - f_n(x)}{y - x},$$

$$F(y) = \frac{f(y) - f(x)}{y - x}, \quad a \leq y \leq b, y \neq x \quad \ldots (5)$$

Then,

$$\lim_{y\to x} F_n(y) = \lim_{y\to x} \frac{f_n(y) - f_n(x)}{y - x}$$

$$= f_n'(x), n = 1, 2, 3, \ldots \quad \ldots (6)$$

Now for $n \geq m, p \geq m$, we have

$$|F_n(y) - F_p(y)| = \left|\frac{f_n(y) - f_n(x)}{y - x} - \frac{f_p(y) - f_p(x)}{y - x}\right|$$

$$= \left|\frac{f_n(y) - f_n(x) - f_p(y) + f_p(x)}{y - x}\right|$$

$$= \left|\frac{f_n(x) - f_p(x) - f_n(y) + f_p(y)}{x - y}\right|$$

$$< \frac{\epsilon}{2(b - a)} \quad [using\ (3)]$$

$\Rightarrow \{F_n\}$ converges uniformly on [a, b], $y \neq x$.

Since $\{f_n\}$ converges uniformly to f, therefore from (5), we have

$$\lim_{n\to\infty} F_n(y) = \lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x}$$

$$= \frac{f(y) - f(x)}{y - x}$$

$$= F(y) \quad \ldots (7)$$

$\Rightarrow \{F_n(y)\}$ converges uniformly to $F(y)$ for $a \leq y \leq b, y \neq x$

Therefore,

$$\lim_{y\to x} F(y) = \lim_{y\to x} \lim_{n\to\infty} F_n(y)$$

$$= \lim_{n\to\infty} \lim_{y\to x} F_n(y)$$

$$\Rightarrow \lim_{y\to x} F(y) = \lim_{n\to\infty} f_n'(x) \qquad [using\ (6)]$$

$$\Rightarrow \lim_{y\to x} \frac{f(y) - f(x)}{y - x} = \lim_{n\to\infty} f_n'(x)$$

$$\Rightarrow f'(x) = \lim_{n\to\infty} f_n'(x), x \in [a,b]$$

$$\Rightarrow \frac{d}{dx}\left[\lim_{n\to\infty} f_n(x)\right] = \lim_{n\to\infty} \frac{d}{dx}[f_n(x)], x \in [a,b].$$

Cor. (Term by Term Differentiation)

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable functions on [a, b] such that $\sum_{n=1}^{\infty} u_n(d)$ converges for some point d of [a, b] and $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b]. Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on [a, b] to a differentiable function f and

$$f'(x) = \lim_{n\to\infty} \sum_{m=1}^{n} u_m'(x), a \le x \le b.$$

$$i.e \frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty} \left[\frac{d}{dx} u_n(x)\right].$$

Proof: Let

$$f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$

Then

$$f_n'(x) = u_1'(x) + u_2'(x) + \cdots + u_n'(x)$$

Therefore, by above theorem we have,

$$\Rightarrow f'(x) = \lim_{n\to\infty} f_n'(x)$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \lim_{n\to\infty} \sum_{m=1}^{n} u_m'(x)$$

$$= \sum_{n=1}^{\infty} u_n'(x)$$

$$= \sum_{n=1}^{\infty} \left[\frac{d}{dx} u_n(x)\right].$$

Theorem 9.1.2: Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that

(I) f_n is differentiable on [a, b], n=1, 2, 3, …

(II) The sequence $\{f_n\}$ converges to f on [a, b].

(III) The sequence $\{f_n'\}$ converges uniformly on [a, b] to $g(x)$.

(IV) Each f_n' is continuous on [a, b].

Then

$$g(x) = f'(x), a \le x \le b$$

$$i.e. \lim_{n\to\infty} f_n'(x) = f'(x), a \le x \le b.$$

Proof: Since $\{f_n'\}$ is a uniformly convergent sequence of continuous functions, therefore g is continuous on [a, b].

Also $\{f_n'\}$ converges uniformly to g on $[a,x]$, where $x \in [a,b]$.

Therefore,

$$\lim_{n\to\infty}\int_a^x f_n'(t)\,dt = \int_a^x g(t)\,dt \qquad \dots(1)$$

Now, using fundamental theorem of calculus, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$$

Also,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and

$$\lim_{n\to\infty} f_n(a) = f(a)$$

Therefore from (1), we have

$$f(x) - f(a) = \int_a^x g(t)\,dt \quad \forall a \le x \le b$$

$$\Rightarrow f'(x) = g(x)$$

$$i.e. f'(x) = \lim_{n\to\infty} f_n'(x)$$

Cor. (Term by term differentiation)

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions on [a, b] such that

I. $u_n(x)$ is differentiable on [a, b], n=1, 2, 3, …
II. The series $\sum_{n=1}^{\infty} u_n(x)$ converges to f on [a, b].
III. The series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b] to g.
IV. Each u_n' is continuous on [a, b].

Then

$$f'(x) = g(x), a \le x \le b$$

$$i.e. \frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty} u_n'(x)\,, a \le x \le b.$$

The uniform convergence of $\{f_n'\}$ is only a sufficient condition, but not necessary for the validity of the result $g(x) = f'(x)$.

Counter-Example: Let

$$f_n(x) = \frac{1}{2n^2}\log(1 + n^4x^2)\,, x \in [0, 1].$$

Then

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{\log(1 + n^4x^2)}{2n^2}$$

$$= \lim_{x\to\infty} \frac{4n^3x^2}{4n(1 + n^4x^2)}$$

$$= \lim_{n\to\infty} \frac{2nx^2}{4n^3x^2}$$

$$= 0$$

Thus $f(x) = 0, x \in [0, 1]$.

And

$$g(x) = \lim_{n\to\infty} f_n'(x)$$
$$= \lim_{n\to\infty}\left(\frac{1}{2n^2}\right)\frac{2n^4x}{1+n^4x^2}$$
$$= \lim_{n\to\infty}\frac{n^2x}{1+n^4x^2}$$
$$= 0$$

Thus

$$f'(x) = g(x)$$
$$i.e. f'(x) = \lim_{n\to\infty} f_n'(x)$$

Now,

$$|f_n'(x) - g(x)| = \frac{n^2x}{1+n^4x^2}$$

Therefore,

$$M_n = \sup_{x\in[0,1]} |f_n'(x) - g(x)|$$
$$= \sup_{x\in[0,1]}\left|\frac{n^2x}{1+n^4x^2}\right|$$
$$\geq \frac{n^2\left(\frac{1}{n^2}\right)}{1+n^4\left(\frac{1}{n^4}\right)} \qquad \left[taking\ x = \frac{1}{n^2}\right]$$
$$= \frac{1}{2}$$

$\Rightarrow M_n \nrightarrow 0$ as $n \to \infty$ in [0, 1].

Thus by M_n-test $\{f_n'\}$ does not converge uniformly to g on [0, 1].

Let the sequence of function $\{f_n(x)\}$ be given where,

$$f_n(x) = \frac{x}{1+nx^2}.$$

Check whether $\lim_{n\to\infty} f_n'(x) = f'(x)$ or not.

Self-Assessment

1) The series: $cosx + \frac{cos2x}{2^2} + \frac{cos2x}{3^2} + \cdots$ is uniformly convergent $\forall x \in \mathbb{R}$

a) True

b) False

2) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty}\left(\frac{x}{n(n+1)}\right)$ is uniformly convergent in $(0, \infty)$.

(II) The series $\sum_{n=1}^{\infty}\left(\frac{x}{n(n+1)}\right)$ is uniformly convergent in $(0, k), k > 0$. Then

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

3) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(II) The series $\frac{1}{(1+x)^3} + \frac{2}{(2+x)^3} + \frac{3}{(3+x)^3} + \cdots, x \geq 0$ is uniformly convergent.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

4) If series $\sum_{n=1}^{\infty} u_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} u_n \cos nx$ is uniformly convergent $\forall x \in \mathbb{R}$.

a) True

b) False

5) If series $\sum_{n=1}^{\infty} u_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} u_n \sin nx$ need not be uniformly convergent $\forall x \in \mathbb{R}$.

a) True

b) False

6) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2x)}{n(n+2)}$ is uniformly convergent for $x \in [0,1]$ only.

(II) The series $\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2x)}{n(n+2)}$ is uniformly convergent $\forall x \in \mathbb{R}$.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

7) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{\sin(x^2+n^2x)}{n(n+2)}$ is uniformly convergent for $x \in [0,1]$.

(II) The series $\sum_{n=1}^{\infty} \frac{\sin(x^2+n^2x)}{n(n+2)}$ is uniformly convergent $\forall x \in \mathbb{R}$.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

8)Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^4x}$ is uniformly convergent $\forall x \in \mathbb{R}$.

(II) The series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^4x}$ can be differentiated term by term.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

9) Select the correct option for $f_n(x) = \frac{x}{1+nx^2}, x \in [0,1]$.

a) the sequence $\{f_n\}$ is uniformly convergent and converges to 1

b) the sequence $\{f_n\}$ is only pointwise convergent

c) the sequence $\{f_n\}$ is neither pointwise nor uniformly convergent

d) the sequence $\{f_n\}$ is uniformly convergent and converges to 0

10) Consider the following statements for $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^p+n^q x^2}, x \in \mathbb{R}$

(I) The series is uniformly convergent for all $p \in \mathbb{R}$.

(II) The series is uniformly convergent for $p > 1$.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

11) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{sin\, nx}{n^3}$ is uniformly convergent for every x.

(II) The series $\sum_{n=1}^{\infty} \frac{sin\, nx}{n^3}$ can be differentiated term by term.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

12)Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{cos\, nx}{n^3}$ is uniformly convergent for every x.

(II) The series $\sum_{n=1}^{\infty} \frac{cos\, nx}{n^3}$ cannot be differentiated term by term.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

13) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{sin\, nx}{n^5}$ is uniformly convergent for every x.

(II) The series $\sum_{n=1}^{\infty} \frac{cos\, nx}{n^7}$ is not uniformly convergent for every x.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

14) Consider the following statements:

(I) The series $\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$ is uniformly convergent for every x.

(II) The series $\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$ can be differentiated term by term.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

15) Consider the following statements for the series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$

(I) $\sum_{n=1}^{\infty} u_n'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$

(II) $\sum_{n=1}^{\infty} u_n'$ is uniformly convergent for every x.

a) only (I) is correct

b) only (II) is correct

c) both (I) and (II) are correct

d) both (I) and (II) are incorrect

16) The series $\sum_{n=1}^{\infty} \frac{1}{n^p+n^qx^2}, p > 1$ can be differentiated term by term if $q > 3p - 2$

a) True

b) False

Answer: Self-Assessment

1	a	5	b	9	d	13	a
2	b	6	b	10	b	14	c
3	c	7	c	11	c	15	c
4	a	8	c	12	a	16	b

Summary

- Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that f_n is differentiable on [a, b], n=1,2,3, …, the sequence $\{f_n(d)\}$ converges for some point d of [a, b], and the sequence $\{f_n'\}$ converges uniformly on [a, b], then the sequence $\{f_n\}$ converges uniformly to a differentiable function f and

$$\lim_{n\to\infty} f_n'(\mathrm{x}) = f'(x), a \le x \le b$$

$$i.e. \lim_{n\to\infty} \frac{d}{dx}[f_n(x)] = \frac{d}{dx}\left[\lim_{n\to\infty} f_n(x)\right], \forall x \in [a,b]$$

- Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable functions on [a, b] such that $\sum_{n=1}^{\infty} u_n(d)$ converges for some point d of [a, b] and $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b]. Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on [a, b] to a differentiable function f and

$$f'(x) = \lim_{n\to\infty} \sum_{m=1}^{n} u_m'(x), a \le x \le b.$$

$$i.e\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left[\frac{d}{dx}u_n(x)\right].$$

- Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that f_n is differentiable on [a, b], n=1, 2, 3, …, the sequence $\{f_n\}$ converges to f on [a, b], the sequence $\{f_n'\}$ converges uniformly on [a, b] to $g(x)$ and each f_n' is continuous on [a, b]. then

$$g(x) = f'(x), a \le x \le b$$
$$i.e. \lim_{n\to\infty} f_n'(x) = f'(x), a \le x \le b.$$

- Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions on [a, b] such that $u_n(x)$ is differentiable on [a, b], n=1, 2, 3, …, the series $\sum_{n=1}^{\infty} u_n(x)$ converges to f on [a, b], the series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b] to g, each u_n' is continuous on [a, b], then

$$f'(x) = g(x), a \le x \le b$$
$$i.e.\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty} u_n'(x)\,, a \le x \le b.$$

- The uniform convergence of $\{f_n'\}$ is only a sufficient condition, but not necessary for the validity of the result $g(x) = f'(x)$.

Keywords

Differentiation for sequence: Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that f_n is differentiable on [a, b], n=1,2,3, …, the sequence $\{f_n(d)\}$ converges for some point d of [a, b], and the sequence $\{f_n'\}$ converges uniformly on [a, b], then the sequence $\{f_n\}$ converges uniformly to a differentiable function f and

$$\lim_{n\to\infty} f_n'(x) = f'(x), a \le x \le b$$

Differentiation for series: Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable functions on [a, b] such that $\sum_{n=1}^{\infty} u_n(d)$ converges for some point d of [a, b] and $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b]. Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on [a, b] to a differentiable function f and

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left[\frac{d}{dx}u_n(x)\right].$$

Review Questions

1) Find for what values of p, the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p + n^q x^2}$$

is uniformly convergent for all real x. Also find the relation between p and q for which the given series can be differentiated term by term.

2) Only the uniform convergence of the series of functions $f_1(x) + f_2(x) + f_3(x) + \cdots$ on [a, b] is not sufficient to ensure validity of term-by-term differentiation of the series on [a, b]. Give an example in support of this argument.

3) If the series of functions

$$f_1(x) + f_2(x) + f_3(x) + \cdots$$

be convergent, then the uniform convergence of the series

$$f'_1(x) + f_2'(x) + f_3'(x) + \cdots$$

is only a sufficient but not a necessary condition for the validity of term-by-term differentiation of the series

$$f_1(x) + f_2(x) + f_3(x) + \cdots$$

Show this with the help of the following example.

Let the series be

$$f_1(x) + f_2(x) + f_3(x) + \cdots, x \in [0,1]$$

such that

$$s_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots + f_n(x)$$
$$= \frac{\log(1 + n^2x^2)}{2n}, x \in [0,1].$$

4) Let

$$f_n(x) = \frac{nx}{1 + n^2x^2} - \frac{(n-1)x}{1 + (n-1)^2x^2}, x \in [0,1].$$

Show that at $x = 0$,

$$\frac{d}{dx}\sum f_n(x) \neq \sum \frac{d}{dx} f_n(x).$$

5) Let $s_n(x)$ be the sum function for the series

$$\sum \frac{1}{n^3 + n^4x^2}.$$

Verify that $s'(x)$ is obtained by term-by-term differentiation.

Further Readings

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Unit 10: The Weierstrass Approximation Theorem and Equicontinuous Families of Functions

CONTENTS

Objectives

After studying this unit, students will be able to:

- describe Weierstrass approximation theorem
- understand equicontinuous families of functions
- define properties of equicontinuous families of functions
- explain the supremum norm of a function

Introduction

This unit explains Weierstrass approximation theorem and equicontinuous families of functions.

10.1 The Weierstrass Approximation Theorem

Bernstein Polynomial: For every non-negative integer n and any function $f: [0,1] \to R$, we define Bernstein Polynomial as

$$B_n(f,x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x),$$

where

$$p_{nr}(x) = C(n,r)x^r(1-x)^{n-r}.$$

Lemma 10.1.1: For every non-negative integer n and $x \in [0,1]$,

$$(i) \sum_{r=0}^{n} p_{nr}(x) = 1.$$

$$(ii) \sum_{r=0}^{n} r p_{nr}(x) = nx.$$

$$(iii) \sum_{r=0}^{n} r(r-1) p_{nr}(x) = n(n-1)x^2.$$

$$(iv)\sum_{r=0}^{n} (nx-r)^2 p_{nr}(x) = nx(1-x).$$

Proof: (i) Consider

$$\sum_{r=0}^{n} p_{nr}(x)$$

$$= \sum_{r=0}^{n} C(n,r)x^r(1-x)^{n-r}$$

$$= [x+(1-x)]^n$$

$$= 1.$$

(ii) Consider

$$\sum_{r=0}^{n} rp_{nr}(x)$$

$$= \sum_{r=0}^{n} rC(n,r)x^r(1-x)^{n-r}$$

$$= \sum_{r=1}^{n} nC(n-1,r-1)x^r(1-x)^{n-r}$$

$$\{\because rC(n,r) = nC(n-1,r-1)\}$$

$$= nx\sum_{r=1}^{n} C(n-1,r-1)x^{r-1}(1-x)^{n-r}$$

$$= nx\sum_{r=1}^{n} C(n-1,r-1)x^{r-1}(1-x)^{n-1-(r-1)}$$

$$= nx[x+(1-x)]^{n-1}$$

$$= nx.$$

(iii) Consider

$$\sum_{r=0}^{n} r(r-1)p_{nr}(x)$$

$$= \sum_{r=0}^{n} r(r-1)C(n,r)x^r(1-x)^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1)\frac{n!}{r!\,(n-r)!}x^r(1-x)^{n-r}$$

$$= \sum_{r=2}^{n} \frac{n(n-1)(n-2)!}{(r-2)!\,(n-r)!}x^r(1-x)^{n-r}$$

$$= n(n-1)x^2\sum_{r=2}^{n} C(n-2,r-2)x^{r-2}(1-x)^{n-r}$$

$$= n(n-1)x^2\sum_{r=2}^{n} C(n-2,r-2)x^{r-2}(1-x)^{n-2-(r-2)}$$

$$= n(n-1)x^2[x+(1-x)]^{n-2}$$

$$= n(n-1)x^2.$$

(iv) Consider

$$\sum_{r=0}^{n} (nx-r)^2 p_{nr}(x)$$

$$= \sum_{r=0}^{n} (n^2x^2 - 2nrx + r^2) p_{nr}(x)$$

$$= \sum_{r=0}^{n} (n^2x^2 + (1-2nx)r + r(r-1)) p_{nr}(x)$$

$$= \sum_{r=0}^{n} n^2x^2 p_{nr}(x) + \sum_{r=0}^{n} (1-2nx) r p_{nr}(x) + \sum_{r=0}^{n} r(r-1) p_{nr}(x)$$

$$= n^2x^2 \sum_{r=0}^{n} p_{nr}(x) + (1-2nx) \sum_{r=0}^{n} r p_{nr}(x) + \sum_{r=0}^{n} r(r-1) p_{nr}(x)$$

$$= n^2x^2(1) + (1-2nx)nx + n(n-1)x^2$$

$$\because of\ (i), (ii) and\ (iii)$$

$$= n^2x^2 + nx - 2n^2x^2 + n^2x^2 - nx^2$$

$$= nx - nx^2$$

$$= nx(1-x).$$

Theorem 10.1.2: Weierstrass Approximation Theorem

Statement: Let $f(x)$ be a continuous function defined on [a, b]. Then there exists a sequence of polynomials that converges uniformly to f on [a, b].

Proof: Without loss of generality, we may assume that [a, b]=[0, 1].

We shall prove that the sequence of Bernstein Polynomials $\{B_n\}$ is the required sequence.

Since f is continuous on closed interval [0, 1], it is bounded on [0, 1].

Therefore, there exists $0 < M \in R$ such that

$$|f(x)| \leq \frac{M}{2} \forall x \in [0, 1].$$

Since f is continuous on the closed interval [0, 1], therefore f is uniformly continuous on [0, 1], so that for given $\in > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\in}{2}$$

for every $x, y \in [0, 1]$ for which $|x - y| < \delta$.

Now for every $n = 1, 2, 3, \ldots$ and every $x \in [0, 1]$, we have

$$|f(x) - B_n(f, x)|$$

$$= \left| f(x) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x) \right|$$

$$= \left| f(x) \sum_{r=0}^{n} p_{nr}(x) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x) \right|$$

$$\because \left[\sum_{r=0}^{n} p_{nr}(x) = 1 \right]$$

$$= \left| \sum_{r=0}^{n} \left\{ f(x) - f\left(\frac{r}{n}\right) \right\} p_{nr}(x) \right|$$

$$\leq \sum_{r=0}^{n} \left| f(x) - f\left(\frac{r}{n}\right) \right| p_{nr}(x)$$

Now we divide the set $\{0, 1, 2, 3, \ldots, n\}$ as a union of two disjoint sets A and B, given by

$$A = \left\{r \colon \left|x - \frac{r}{n}\right| < \delta\right\}$$

and

$$B = \left\{r \colon \left|x - \frac{r}{n}\right| \geq \delta\right\}$$

Therefore,

$$|f(x) - B_n\,(f, x)| \leq \sum_{r=0}^{n} \left|f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x)$$

$$= \sum_{r \in A} \left|f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x) + \sum_{r \in B} \left|f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x)$$

$$\leq \sum_{r \in A} \left|f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x) + \sum_{r \in B} \left[|f(x)| + \left|f\left(\frac{r}{n}\right)\right|\right] p_{nr}(x)$$

$$= \sum_{r \in A} \left|f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x) + \sum_{r \in B} |f(x)| p_{nr}(x) + \sum_{r \in B} \left|f\left(\frac{r}{n}\right)\right| p_{nr}(x)$$

$$< \frac{\in}{2} \sum_{r \in A} p_{nr}(x) + \frac{M}{2} \sum_{r \in B} p_{nr}(x) + \frac{M}{2} \sum_{r \in B} p_{nr}(x)$$

$$= \frac{\in}{2}.1 + M \sum_{r \in B} p_{nr}(x)$$

$$\because \left[\sum_{r=0}^{n} p_{nr}(x) = 1\right]$$

Now for $r \in B$, we have

$$\left|x - \frac{r}{n}\right| \geq \delta$$

$$\Rightarrow \left|\frac{nx - r}{n}\right| \geq \delta$$

$$\Rightarrow (nx - r)^2 \geq \delta^2 n^2$$

$$\Rightarrow \frac{(nx - r)^2}{\delta^2 n^2} \geq 1$$

$$\Rightarrow p_{nr}(x) \leq \frac{(nx - r)^2}{\delta^2 n^2} p_{nr}(x)$$

Therefore,

$$|f(x) - B_n\,(f, x)| \leq \frac{\in}{2} + M \sum_{r \in B} \frac{(nx - r)^2}{n^2 \delta^2} p_{nr}(x)$$

$$\leq \frac{\in}{2} + \frac{M}{n^2 \delta^2} \sum_{r=0}^{n} (nx - r)^2 p_{nr}(x)$$

$$\leq \frac{\in}{2} + \frac{M}{n^2 \delta^2} nx(1 - x)$$

$$\left[\because \sum_{r=0}^{n} (nx - r)^2 p_{nr}(x) = nx(1 - x)\right]$$

$$= \frac{\in}{2} + \frac{M}{n \delta^2} x(1 - x)$$

$$\leq \frac{\in}{2} + \frac{M}{4n\delta^2}.$$

$$\left[\because x(1 - x) \leq \frac{1}{4} \forall x \in [0, 1]\right]$$

Choosing sufficiently large m such that

$$\frac{M}{4n\delta^2} < \frac{\in}{2} \ \forall n \geq m.$$

Then for every $n \geq m$ and every $x \in [0,1]$, we have

$$|f(x) - B_n(f,x)| < \frac{\in}{2} + \frac{\in}{2}$$
$$= \in$$

Hence the sequence of Bernstein Polynomial $\{B_n\}$ converges uniformly to f on [0, 1].

10.2 Equicontinuous Families of Functions

Definition. Let (X, d) be a metric space. A family of complex functions F defined on a set E in X is said to be equicontinuous on E if for given $\in > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \in$$

whenever $d(x,y) < \delta, x \in E, y \in E$ and $f \in F$.

Every member of an equicontinuous family is uniformly continuous.

Example 10.2.1: Let

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}, x \in [0,1].$$

Show that:

(i) the sequence $\{f_n\}$ is uniformly bounded on [0, 1].

(ii) no subsequence of $\{f_n\}$ can converge uniformly on [0, 1].

(iii) $\{f_n\}$ is not equicontinuous on [0, 1].

Solution: (i) We have

$$x^2 + (1-nx)^2 \geq x^2 \ \forall x \in [0,1]$$
$$\Rightarrow \frac{x^2 + (1-nx)^2}{x^2} \geq 1 \forall x \in [0,1]$$

Therefore,

$$|f_n(x)| = \left|\frac{x^2}{x^2 + (1-nx)^2}\right|$$
$$\leq 1 \forall x \in [0,1].$$

Hence the sequence $\{f_n\}$ is uniformly bounded on [0, 1].

(ii) Let $\{f_{n_k}\}$ be any subsequence of $\{f_n\}$.

Then

$$f_{n_k}(x) = \frac{x^2}{x^2 + (1-n_k x)^2}, x \in [0,1]$$

and

$$f_{n_k}(x) = \lim_{n \to \infty} \frac{x^2}{x^2 + (1-n_k x)^2}$$
$$= 0, x \in [0,1].$$

Therefore, the sequence $\{f_{n_k}\}$ converges pointwise to zero in [0, 1].

Now, for $x = \frac{1}{n_k}$, we have

$$f_{n_k}\left(\frac{1}{n_k}\right) = \frac{x^2}{x^2 + \left(1 - n_k.\frac{1}{n_k}\right)^2}$$

$$= 1 \qquad (k = 1, 2, 3, \ldots)$$

Therefore, for $\in = \frac{1}{2}$ and $x = \frac{1}{n_k} \in [0, 1]$, we have

$$\left|f_{n_k}\left(\frac{1}{n_k}\right) - 0\right| = |1 - 0|$$

$$= 1 > \in.$$

Hence, no subsequence of $\{f_n\}$ can converge uniformly on [0, 1].

(iii) Let

$$\in = \frac{1}{4}, x = \frac{1}{n}, y = \frac{1}{n+1}$$

Then

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right|$$

$$= \frac{1}{n(n+1)}$$

and

$$|f_n(x) - f_n(y)| = \left|f_n\left(\frac{1}{n}\right) - f_n\left(\frac{1}{n+1}\right)\right|$$

$$= \left|1 - \frac{1}{2}\right|$$

$$= \frac{1}{2}$$

Now, if we choose n such that

$$\frac{1}{n(n+1)} < \delta$$

then we have

$$|f_n(x) - f_n(y)| > \in.$$

Hence $\{f_n\}$ is not equicontinuous on [0, 1].

Theorem 10.2.2: Let K be a compact subset of a metric space (X, d) and $\{f_n\}$ be a uniformly convergent sequence of continuous functions defined on K. Then $\{f_n\}$ is equicontinuous on K.

Proof: Let $\in > 0$ be given.

Since $\{f_n\}$ converges uniformly on K.

Therefore, there exists a positive integer N such that

$$|f_n(x) - f_m(x)| < \frac{\in}{3} \ \forall n, m \geq N, x \in K$$

$$\Rightarrow |f_n(x) - f_N(x)| < \frac{\in}{3} \ \forall\, n > N, x \in K \qquad \ldots (1)$$

Since each $f_n, n \in N$ is continuous on compact set K.

Therefore, each $f_n, n \in N$ is uniformly continuous on K.

$\Rightarrow f_n$ is uniformly continuous on K for $1 \leq n \leq N$.

Therefore, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3} \text{ whenever } d(x,y) < \delta, x, y \in K \qquad \ldots(2)$$

Now for $n > N$, we have

$$|f_n(x) - f_n(y)| = |f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)|$$

$$\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

$$[\ by\ (1)\ and\ (2)]$$

$$i.e.\ |f_n(x) - f_n(y)| < \epsilon \text{ whenever } d(x,y) < \delta, x, y \in K \qquad \ldots(3)$$

Combining (2) and (3), we get

$$|f_n(x) - f_n(y)| < \epsilon \text{ whenever } d(x,y) < \delta, x, y \in K, n \in N$$

Thus, the sequence $\{f_n\}$ is equicontinuous on K.

This completes the proof.

Theorem 10.2.3: (i) Show that every uniformly convergent sequence of bounded functions is uniformly bounded.

(ii) If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions such that these sequences converge uniformly on a set E, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof: Let $\{f_n\}$ converges uniformly to $f(x)$ on some set E.

Then for given $\epsilon = 1$ and all $x \in E$, there exists a positive integer such that

$$|f_n(x) - f(x)| < 1 \ \ \forall n \geq t.$$

Since functions of $\{f_n\}$ are bounded, therefore there exists $0 < M \in R$ such that

$$|f_n(x)| < M_n \ \ \forall n \in N.$$

Therefore, for $n \geq t$ and $x \in E$,

$$|f_n(x)| = |\{f_n(x) - f(x)\} + \{f(x) - f_t(x)\} + f_t(x)|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_t(x)| + |f_t(x)|$$

$$< 1 + 1 + M_t$$

$$i.e. \qquad |f_n(x)| < 2 + M_t \ \ \forall n \geq t, x \in E.$$

Let

$$M = \{M_1, M_2, \ldots, M_{t-1}, 2 + M_t\}.$$

Then for any $x \in E$,

$$|f_n(x)| \leq M \ \forall n \in N$$

$\Rightarrow \{f_n\}$ is uniformly bounded.

(ii) Since $\{f_n\}$ and $\{g_n\}$ are uniformly convergent sequences of bounded functions, therefore $\{f_n\}$ and $\{g_n\}$ are uniformly bounded.

Therefore, there exists $0 < M, L \in R$ such that

$$|f_n(x)| \leq M, \forall n \in N, x \in E$$

and

$$|g_n(x)| \leq L, \forall n \in N, x \in E.$$

Let

$$f_n \to f \; uniformly \; on \; E$$

and

$$g_n \to g \; uniformly \; on \; E.$$

Therefore, there exists $t_1, t_2 \in N$ such that

$$|f_n(x) - f(x)| < \frac{\in}{2(L+1)} \quad \forall n \geq t_1, x \in E$$

and

$$|g_n(x) - f(x)| < \frac{\in}{2(M+1)} \quad \forall n \geq t_2, x \in E.$$

Let

$$t = \{t_1, t_2\}.$$

Then for $n \geq t, x \in E$, we have

$$|f_n(x) - f(x)| < \frac{\in}{2(L+1)}$$

and

$$|g_n(x) - f(x)| < \frac{\in}{2(M+1)}.$$

Therefore, for $n \geq t$ and $x \in E$,

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |\{f_n(x)g_n(x) - f_n(x)g(x)\} + \{f_n(x)g(x) - f(x)g(x)\}| \\
&= |f_n(x)\{g_n(x) - g(x)\} + g(x)\{f_n(x) - f(x)\}| \\
&\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&< M\frac{\in}{2(M+1)} + L\frac{\in}{2(L+1)} \\
&< \frac{\in}{2} + \frac{\in}{2} = \in
\end{aligned}$$

Thus $\{f_n g_n\}$ converges uniformly on E.

This completes the proof.

10.3 Supremum Norm of a Function

Definition: Let X be a metric space and $C(X)$ be the set of all complex-valued, continuous and bounded functions with domain X. Then supremum norm denoted by $\|f\|$ of $f \in C(X)$ is defined as

$$\|f\| = |f(x)|.$$

$\|f\| < \infty$ as f is bounded.

If X is compact, then boundedness is redundant, so that $C(X)$ consists of all complex continuous functions on X.

Theorem 10.3.1: $C(X)$ is a complete metric space.

or

If $C(X) =$, then $C(X)$ is a complete metric space under supremum norm.

Proof: Firstly, we show that $C(X)$ is a metric space with the distance between $f, g \in C(X)$ defined as $\|f - g\|$.

(i) $\|f - g\| = |f(x) - g(x)|$

$$\geq 0, \forall f, g \in C(X).$$

(ii) $\|f-g\|=0$

$$\Leftrightarrow |f(x)-g(x)| = 0$$
$$\Leftrightarrow f(x)-g(x)=0 \ \forall x \in X$$
$$\Leftrightarrow f(x)=g(x) \ \forall x \in X$$
$$\Leftrightarrow f=g.$$

(iii) $\|f-g\| = |f(x)-g(x)|$

$$= |g(x)-f(x)|$$
$$= \|g-f\|.$$

(iv) Let $h=f+g$

Then

$$|h(x)| = |f(x)+g(x)|$$
$$\leq |f(x)|+|g(x)|$$
$$\leq \|f\|+\|g\| \ \ \forall x \in X$$

Therefore,

$$\|h\| \leq \|f\|+\|g\|$$
$$i.e. \ \ \|f+g\| \leq \|f\|+\|g\|.$$

Thus $C(X)$ is a metric space.

Now we show that $C(X)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence in $C(X)$.

Then for given $\in > 0$, there exists $n_0 \in N$ such that

$$i.e. \ \ \|f_n - f_m\| < \in \forall n, m \ \geq n_0.$$

Therefore, by Cauchy criterion for uniform convergence, there exists a function f with domain X to which $\{f_n\}$ converges uniformly.

$\Rightarrow f$ is continuous.

$\{\because$ if $\{f_n\}$ is a sequence of continuous functions defined on E such that $f_n \to f$ uniformly on E, then f is continuous on $E.\}$

Also, f is bounded as there exists $n \in N$ such that f_n is bounded and

$$|f(x)-f_n(x)| < 1, x \in X.$$

Thus $\{f_n\}$ converges uniformly to f and $f \in C(X)$.

Hence $C(X)$ is a complete metric space.

Summary

- Bernstein Polynomial: For every non-negative integer n and any function $f:[0,1] \to R$, we define Bernstein Polynomial as

$$B_n(f,x) = \sum_{r=0}^{n} \ \ f\left(\frac{r}{n}\right) p_{nr}(x),$$

where

$$p_{nr}(x) = C(n,r)x^r(1-x)^{n-r}.$$

- For every non-negative integer n and $x \in [0,1]$,

$$(i) \sum_{r=0}^{n} p_{nr}(x) = 1.$$

$$(ii) \sum_{r=0}^{n} r p_{nr}(x) = nx.$$

$$(iii) \sum_{r=0}^{n} r(r-1) p_{nr}(x) = n(n-1)x^2.$$

$$(iv) \sum_{r=0}^{n} (nx-r)^2 p_{nr}(x) = nx(1-x).$$

- Weierstrass Approximation Theorem: Let $f(x)$ be a continuous function defined on [a, b]. Then there exists a sequence of polynomials that converges uniformly to f on [a, b].
- Equicontinuous Families of Functions: Let (X, d) be a metric space. A family of complex functions F defined on a set E in X is said to be equicontinuous on E if for given $\in > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \in$$

whenever $d(x, y) < \delta, x \in E, y \in E$ and $f \in F$.

- Every member of an equicontinuous family is uniformly continuous.
- Let K be a compact subset of a metric space (X, d) and $\{f_n\}$ be a uniformly convergent sequence of continuous functions defined on K. Then $\{f_n\}$ is equicontinuous on K.
- Every uniformly convergent sequence of bounded functions is uniformly bounded.
- If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions such that these sequences converge uniformly on a set E, then $\{f_n g_n\}$ converges uniformly on E.
- Supremum Norm of a Function: Let X be a metric space and $\mathrm{C}(X)$ be the set of all complex-valued, continuous and bounded functions with domain X. Then supremum norm denoted by $\|f\|$ of $f \in \mathrm{C}(X)$ is defined as

$$\|f\| = |f(x)|\,.$$

- $\|f\| < \infty$ as f is bounded.
- If X is compact, then boundedness is redundant, so that $\mathrm{C}(X)$ consists of all complex continuous functions on X.
- If $\mathrm{C}(X) =$, then $\mathrm{C}(X)$ is a complete metric space under supremum norm.

Keywords

Bernstein Polynomial: For every non-negative integer n and any function $f: [0,1] \to R$, we define Bernstein Polynomial as

$$B_n(f, x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x),$$

where

$$p_{nr}(x) = C(n, r) x^r (1-x)^{n-r}.$$

Weierstrass Approximation Theorem: Let $f(x)$ be a continuous function defined on [a, b]. Then there exists a sequence of polynomials that converges uniformly to f on [a, b].

Equicontinuous Families of Functions: Let (X, d) be a metric space. A family of complex functions F defined on a set E in X is said to be equicontinuous on E if for given $\in > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \in$$

whenever $d(x, y) < \delta, x \in E, y \in E$ and $f \in F$.

Supremum Norm of a Function: Let X be a metric space and $C(X)$ be the set of all complex-valued, continuous and bounded functions with domain X. Then supremum norm denoted by $\|f\|$ of $f \in C(X)$ is defined as

$$\|f\| = |f(x)| .$$

Self Assessment

1) Let $p_{nr}(x) = n_{C_r} x^r (1-x)^{n-r}$. For every non-negative integer n and any function $f: [0,1] \to R$, Bernstein Polynomial $B_n(f, x)$ is defined as:

A. $\sum_{r=0}^{n} \quad f\left(\frac{r}{n}\right) p_{nr}(x)$

B. $\sum_{r=0}^{n} \quad f(r) p_{nr}(x)$

C. $\sum_{r=0}^{n} \quad f(n) p_{nr}(x)$

D. None of these

2) For every non-negative integer n and $x \in [0,1]$, $\sum_{r=0}^{n} \quad p_{nr}(x)$ is equal to

A. 0

B. 1

C. 2

D. None of these

3) For every non-negative integer n and $x \in [0,1]$, $\sum_{r=0}^{n} \quad r p_{nr}(x)$ is equal to

A. $n(n-1)x$

B. nx

C. $n(n-1)x^2$

D. $n^2(n-1)x$

4) For every non-negative integer n and $x \in [0,1]$, $\sum_{r=0}^{n} \quad r(r-1) p_{nr}(x)$ is equal to

A. $n(n-1)x$

B. nx

C. $n(n-1)x^2$

D. $n^2(n-1)x$

5) For every non-negative integer n and $x \in [0,1]$, $\sum_{r=0}^{n} \quad (nx - r)^2 p_{nr}(x)$ is equal to

A. $n(n-1)x$

B. nx

C. $nx(1-x)$

D. $n^2(n-1)x$

6) Consider the following statements:

(I) If f is continuous on compact set then f is uniformly continuous on the same.

(II) There exists a sequence of polynomials which converges uniformly to f on [a, b] where $f(x)$ is a function defined on [a, b]. Then

A. only (I) is correct

B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

7) Consider the following statements:

(I)Every member of an equicontinuous family is continuous.

(II)Every member of an equicontinuous family is uniformly continuous.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

8) Let K be a closed and bounded set and f is continuous on K then it is not necessary that f is uniformly continuous on K.

A. True
B. False

9) A sequence of functions $\{f_n\}$ defined on a set K is said to be uniformly bounded on K if there exists a positive real number M such that $|f_n(x)| < M \; \forall x \in K, n \in N$.

A. True
B. False

10) Consider the following statements:

(I)Uniformly convergent sequence of bounded functions is uniformly bounded.

(II)Every member of an equicontinuous family is uniformly continuous.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

11)If $\{s_n\}$ and $\{r_n\}$ are sequences of bounded functions such that these sequences converge uniformly on a set K then $\{s_n r_n\}$ may or may not be uniformly convergent on K.

A. True
B. False

12) Let f be a complex-valued, continuous and bounded function with domain X. Then $\|f\|$ is given as:

A. $a)|f(x)|$
B. $b)|f(x)|$
C. $c)\ |f(x)|$
D. d) None of these

13) Let f be a complex-valued, continuous and bounded function with domain X, then which of the following is INCORRECT.

A. $a)\ \|f\| > \infty$

B. $b)$ $||f|| < \infty$
C. $c)$ $||f|| = |f(x)|$
D. $d)$ $||f||$ is finite

14) Consider the following statements for $C(X) = \{f: X \to C, f$ is continuous and bounded$\}$.
(I) $C(X)$ is not a complete metric space under supremum norm.
(II) Let $\{f_n\}$ be a Cauchy sequence in $C(X)$ then $f_n \to f$ uniformly and $f \in C(X)$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

15) Let $f, g \in C(X)$, $C(X) = \{f: X \to C, f$ is continuous and bounded$\}$, then $||f - g|| \geq 0$.

A. True
B. False

16) Consider the following statements for $C(X) = \{f: X \to C, f$ is continuous and bounded$\}$,
(I) Let $f, g \in C(X)$, then $||f - g|| = 0 \Rightarrow f = g$.
(II) Let $f, g \in C(X)$, then $f = g \nRightarrow ||f - g|| = 0$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

17) consider the following statements for $C(X) = \{f: X \to C, f$ is continuous and bounded$\}$:
(I) Let $f, g \in C(X)$, then $||f|| + ||g|| \leq ||f + g||$
(II) Let $f, g \in C(X)$, then $||f - g|| = ||g - f||$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	A	2.	B	3.	B	4.	C	5.	C
6.	A	7.	C	8.	B	9.	A	10.	C
11.	B	12.	A	13.	A	14.	B	15.	A
16.	A	17.	B						

Review Questions

1) Let K be a compact metric space, $f_n \in C(K)$ $\forall n \in N$ and $\{f_n\}$ be pointwise bounded and equicontinuous on K then $\{f_n\}$ is uniformly bounded on K.

2) Let K be a compact metric space, $f_n \in C(K)$ $\forall n \in N$ and $\{f_n\}$ be pointwise bounded and equicontinuous on K then $\{f_n\}$ contains a uniformly convergent subsequence.

Further Readings

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Shanti Narayan, Elements of Real Analysis

Web Links

https://nptel.ac.in/courses/111/106/111106053/

https://nptel.ac.in/courses/111/101/111101134/

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Unit 11:Power series and uniform convergence, the exponential and logarithmic functions, the trigonometric functions

CONTENTS

Objectives

After studying this unit, students will be able to:

- understand the concept of power series
- define the radius of convergence of power series
- define uniform convergence and related theorems
- describethe exponential and logarithmic functions
- describe the trigonometric functions.

Introduction

A series of the form$a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$where $a_0, a_1, a_2, \ldots$ are real numbers, is called a power series.

The general form of the power series is

$$a_0 + a_1(x - x_0) + a_2(x - x_0\ )^{\ 2} + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

This is called a power series about the point x_0.

To study the nature and properties of power series we will consider the power series about 0 $i.e.$ series of the form $\sum_{n=0}^{\infty} a_nx^n$.

It is a series of functions $\sum_{n=0}^{\infty} f_n(x), f_n(x) = a_nx^n, n = 0,1,2, \ldots, x \in \mathbb{R}$.

Although each function in the series $\sum_{n=0}^{\infty} f_n$is defined for all real x, it is not expected that the series $\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} a_nx^n$will converge for all real x .

For example

1) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges for all real x

2) $1 + x + x^2 + \cdots$ converges only for $x \in (-1, 1)$.

3) $1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$ converges only for $x = 0$.

The power series converges for all $x \in \mathbb{R}$, is called everywhere convergent power series. Some power series converge only for $x = 0$, they are called nowhere convergent power series. Some power series converge for some real x and diverge for the other.

Theorem11.1.1

If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1$then the series converges absolutely for all real x satisfying $|x| < |x_1|$.

Proof: Since the series converges for $x = x_1$.

$\Rightarrow \sum_{n=0}^{\infty} a_n x_1^n$ is convergent.

$\Rightarrow \lim_{n\to\infty} a_n x_1^n = 0$

$\Rightarrow \{a_n x_1^n\}$ is convergent.

$\Rightarrow \{a_n x_1^n\}$ is bounded.

$\therefore$ there exists a positive real number k such that $\Rightarrow |a_n x_1^n| \leq \text{k}\ \forall \text{n} \in \mathbb{N}$.

Now,

$$|a_n x^n| = |a_n x_1^n| \left|\frac{x}{x_1}\right|^n$$

$$\leq k \left|\frac{x}{x_1}\right|^n$$

For all x satisfying $\left|\frac{x}{x_1}\right| < 1, \sum_{n=0}^{\infty} \left|\frac{x}{x_1}\right|^n$ is a convergent series of positive real numbers.

Therefore, by comparison test $\sum_{n=0}^{\infty} |a_n x^n|$is convergent if$|x| < |x_1|$.

$\therefore \sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent if$|x| < |x_1|$.

Theorem 11.1.2

If a power series $\sum_{n=0}^{\infty} a_n x^n$diverges for $x = x_1$ then the series diverges for all real x satisfying $|x| > |x_1|$.

Proof: Let the power series be convergent for $x = c$such that $|c| > |x_1|$.

Since the series converges for $x = c$ and $|x_1| < c$.

$\therefore$ by the previous theorem, the series would be absolutely convergent for $x = x_1$, a contradiction to the given condition.

If the power series $\sum_{n=0}^{\infty} a_n x^n$ be neither nowhere convergent nor everywhere convergent then there exists a positive real number R such that the series converges absolutely for all real x satisfying $|x| < R$ and diverges for all x satisfying $|\ x\ | > R$. R is called the radius of convergence of power series $\sum_{n=0}^{\infty} a_n x^n$.

We define $R = 0$ for a nowhere convergent power series and $R = \infty$ for a series that is everywhere convergent.

Theorem 11.1.3

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with the radius of convergence $R(> 0)$ then the series is uniformly convergent on $[-s, s]$ where $0 < s < R$.

Proof: Let $f_n(x) = a_n x^n, n \geq 0$.

Since R is the radius of convergence of the power series, the series is absolutely convergent for all real x satisfying $|x| < R$.

Since $0 < s < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x satisfying $|x| \leq s < R$.

Therefore, the series $\sum_{n=0}^{\infty} |a_n s^n|$ is convergent. ... (1)

Now, $| f_n(x) | = | a_n x^n | \leq | a_n | s^n$ for all real x satisfying $| x | \leq s$ ... (2)

Let $M_n = | a_n | s^n \ \forall n \in \mathbb{N}$ ... (3)

Then $\sum_{n=0}^{\infty} M_n$ is a convergent series of positive real numbers. $\{by (1)\}$

$and\ \forall n \in \mathbb{N}, | f_n(x) | \leq M_n\ \forall x \in [-s, s]$ ...$\{(by\ (2), (3)\}$

$\therefore$ by Weierstrass M − test, we have, the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent on $[-s, s]$.

i. e. the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-s, s]$.

Cor 1: Let R(> 0) be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, then the series is uniformly convergent on $[-R+\in, R-\in]$, where $\in$ is an arbitrarily small positive number satisfying $R-\in > 0$.

Proof: Since $R-\in > 0$

Let $s = R-\in$

Then $0 < s < R$

$\therefore$ the power series is uniformly convergent on $[-s, s]$ *i.e.*, $[-R+\in, R-\in]$.

Cor 2. Let R(> 0) be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. If $[a, b]$ be any closed interval contained in $(-R, R)$ then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[a, b]$.

Proof: Let us choose positive $\in$ such that $R-\in > 0$ and $-R < -R+\in < a < b < R-\in < R$

Let $R-\in = s$

Then $0 < s < R$

and $-R < -s < a < b < s < R$.

Since the power series is uniformly convergent on [-s, s] and $[a, b] \subset [-s, s]$, therefore the power series is uniformly convergent on $[a, b]$.

11.1 The radius of Convergence of the Power Series

The radius of convergence (RoC) of the power series is given by

$$\frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}$$

$$\text{or} \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$$

Example: Find of the radius of convergence of the following power series:

1) $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots$

Solution: Let

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots = \sum_{n=0}^{\infty} a_n x^n, a_0 = 0, a_n = \frac{n^n}{n!} \forall n \in \mathbb{N}.$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n(n+1)n!}{(n+1)n!\,n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e$$

$$\therefore \text{RoC} = \frac{1}{e}$$

2) $x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \cdots$

Solution: Let

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \cdots \sum_{n=0}^{\infty} a_n x^n, a_0 = 0, a_n = \frac{1}{n^n}$$

Now,

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$$

$$\Rightarrow \text{RoC} = \infty$$

3) $1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$

Solution: Let

$$1 + x + 2!\,x^2 + 3!\,x^3 + \cdots \sum_{n=0}^{\infty} a_n x^n, a_n = n!$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} (n+1)$$

$$= \infty$$

$$\therefore \text{RoC} = 0$$

4) $\frac{1}{2}\,x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots$

Solution: We have

$$\frac{1}{2}\,x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1.3.5\ldots.(2n-1)}{2.5.8\ldots.(3n-1)} x^n$$

$$\text{Let } a_0 = 0, a_n = \frac{1.3.5 \ldots .(2n-1)}{2.5.8 \ldots .(3n-1)}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n+1}{3n+2}$$

$$= \frac{2}{3}$$

$$\therefore \text{RoC} = \frac{3}{2}$$

5) $\sum_{n=0}^{\infty} 5^n x^n$

Solution: Let

$$\sum_{n=0}^{\infty} 5^n x^n = \sum_{n=0}^{\infty} a_n x^n, a_n = 5^n$$

$$\Rightarrow \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 5$$

$$\therefore \text{RoC} = \frac{1}{5}$$

6) $\sum a_n x^n, a_n = \begin{cases} 4^n \text{if } n \text{ is multiple of 4} \\ \frac{1}{4^n} \text{if } n \text{ is not a multiple of 4} \end{cases}$

Solution: We have

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \begin{cases} 4 \text{if } n \text{ is multiple of 4} \\ \frac{1}{4} \text{if } n \text{ is not a multiple of 4} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 4$$

$$\therefore \text{RoC} = \frac{1}{4}$$

7) $\sum x^{2^n}$

Solution: Let

$$\sum x^{2^n} = \sum_{n=0}^{\infty} a_n x^n, a_n = \begin{cases} 1 \text{if } n = 2^{\mathrm{k}} \\ 0 \text{ otherwise} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \begin{cases} 1 \text{if } n = 2^{\mathrm{k}} \\ 0 \text{ otherwise} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 1$$

$$\therefore \text{RoC} = 1$$

8) $\sum_{n=0}^{\infty} 5^n x^{kn}$

Solution: Let

$$\sum_{n=0}^{\infty} 5^n x^{kn} = \sum_{n=0}^{\infty} a_n x^{kn}$$

$$\Rightarrow \lim_{n\to\infty} (a_n)^{\frac{1}{kn}} = \lim_{n\to\infty} (5^n)^{\frac{1}{kn}}$$

$$= \lim_{n\to\infty} 5^{1/k}$$

$$= 5^{1/k}$$

$$\therefore \text{RoC} = \frac{1}{5^{\frac{1}{k}}}$$

Example 9) If RoC of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R then find RoC of the following:

(1) $\sum a_n^2 x^n$

(2) $\sum a_n x^{2n}$

Solution: (1) Since RoC of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R

$\therefore \lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{R}$

Now let

$$b_n = a_n^2$$

$$\therefore \lim_{n\to\infty} b_n^{\frac{1}{n}} = \lim_{n\to\infty} (a_n^2)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} (a_n^{1/n})^2$$

$$= \left(\frac{1}{R}\right)^2$$

$$\therefore \text{RoC} = R^2$$

$$(2) \lim_{n\to\infty} a_n^{\frac{1}{2n}} = \lim_{n\to\infty} \left[a_n^{\frac{1}{n}}\right]^{\frac{1}{2}}$$

$$= \left(\frac{1}{R}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{R}}$$

$$\therefore \text{RoC} = \sqrt{R}.$$

Find the radius of convergence of the power series:

$$\sum a_n x^n, a_n = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \text{ is odd} \\ \left(\frac{1}{2}\right)^n & \text{if } n \text{ even} \end{cases}$$

11.2 The Exponential Function

The power series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ ... (1)

is everywhere convergent for real x.

{We proceed now to examine in detail the function represented by this series.}

The function represented by the power series (1) is called the Exponential function, denoted, provisionally, by $E(x)$.

Thus,

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad ...(2)$$

$$\therefore E(0) = 1 \text{ and}$$

$$E(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \quad ...(3)$$

The series on the right-hand side of (3) converges to a number that lies between 2 and 3.

This number is denoted by e and is the same number as represented by

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Thus $E(1) = e$

The function $E(x)$, defined by (2), is continuous and differentiable any number of times, for every x.

By differentiation, we get

$$E'(x) = E(x)$$

$$E" = E(x)$$

$$\dots\dots\dots$$

$$E^n(x) = E(x)$$

Further, we have

$$E(x_1 + x_2) = E(x_1)E(x_2)$$

This formula is called the addition formula for the exponential function.

It gives further

$$E(x_1 + x_2 + x_3) = E(x_1)E(x_2)E(x_3)$$

and repetition of the process gives, for any positive integer q,

$$E(x_1 + x_2 + \cdots + x_q) = E(x_1)E(x_2)\dots E(x_q) \quad \dots (4)$$

If $x_1 = x_2 = \cdots = x_q = x$, we get

$$E(qx) = \{E(x)\}^q \qquad \dots (5)$$

Hence, for $x = 1, E(q) = \{E(1)\}^q = e^q$, for any positive integer q.

But since $E(0) = 1$, therefore, the above relation holds for $q = 0$ also.

Hence $E(q) = e^q$ holds for all integers greater than equal to zero.

Again, replacing each x by $\frac{p}{q}$ in (5), we get

$$E\left(q.\frac{p}{q}\right) = \left\{E\left(\frac{p}{q}\right)\right\}^q, p, q > 0$$

or

$$E\left(\frac{p}{q}\right) = \{E(p)\}^{1/q} = e^{p/q}$$

$$\because E(p) = e^p$$

Hence $E(m) = e^m$, for all rational numbers $m \geq 0$. ... (6)

Now by addition formula, we have

$$E(x)E(-x) = E(x - x)$$

$$= E(0)$$

$$= 1$$

$$\Rightarrow E(x) \neq 0 \quad \forall x$$

Also,

$$E(-x) = \frac{1}{E(x)}$$

$$\Rightarrow E(-p) = e^{-p}, \text{if p is positive and rational.}$$

Thus relation (6) holds for all rational m.

Now we have $x^y = \sup x^p$, where the supremum is taken over all rational p such that $p < y$, for any real y and $x > 1$.

If we thus define, for any real $x, e^x = \sup e^p$ ($p > x, p$ rational), the continuity and monotonicity properties of E, together with (6) show that $E(x) = e^x$ for all real x.

The notation exp (x) is often used in place of e^x, especially when x is a complicated expression.

Properties of the exponential function:

- e^x is continuous and differentiable for all x.
- $(e^x)' = e^x$
- e^x is a strictly increasing function of x and $e^x > 0$
- $e^{x+y} = e^x e^y$
- $e^x \to +\infty \, as \, x \to +\infty$ and $e^x \to 0 \, as \, x \to -\infty$
- $\lim_{x \to +\infty} x^n e^{-x} = 0$ for every n.

11.3 Logarithmic Function with base e

Since the exponential function E is strictly increasing on the set $\mathbb{R}$ of real numbers $i.e., E: \mathbb{R} \to \mathbb{R}^+$ is 1-1 onto, it has inverse function L or log_e which is also strictly increasing and whose domain of definition is $E(\mathbb{R})$, that is, the set of all positive numbers.

Thus L is defined by

$$E\big(L(y)\big) = y, (y > 0)$$

Or

$$L\big(E(x)\big) = x, (x \in \mathbb{R}) \quad \dots (7)$$

Equivalently, for any real x,

$$E(x) = y \Rightarrow L(y) = x$$

Or

$$e^x = y \Rightarrow \log_e y = x \quad \dots (8)$$

Thus, the logarithmic function $L(or\ log_e)$ is defined for positive values only.

Now we have

$$E(-x) = \frac{1}{y} \Rightarrow L\left(\frac{1}{y}\right) = -x = -L(y)$$

And

$$E(0) = 1 \Rightarrow L(1) = 0 = \log_e 1$$

$$E(1) = e \Rightarrow L(e) = 1 = \log_e e \quad \dots (9)$$

Further, since $E(x) \to +\infty$ as $x \to +\infty$

and $E(x) \to 0$ as $x \to -\infty$

Therefore, $L(x) \to +\infty$ as $x \to +\infty$

and $L(x) \to -\infty$ as $x \to 0$.

Writing $u = E(x_1), v = E(x_2)$

or $L(u) = x_1, L(v) = x_2$ in the relation

$$E(x_1 + x_2) = E(x_1)E(x_2)$$

we get

$$E(x_1 + x_2) = uv$$

$$\Rightarrow L(uv) = x_1 + x_2$$

$$= L(u) + L(v) \ (u > 0, v > 0)$$

This shows that L has the familiar property which makes logarithms useful tools for computation.

Differentiating (7), we get

$$L'(E(x)).E(x) = 1$$

Writing $E(x) = y$, we get

$$L'(y).y = 1$$

$$\Rightarrow L'(y) = \frac{1}{y}$$

$$\Rightarrow L(y) = \int_1^y \frac{dx}{x} \qquad \dots (10)$$

The relation (10) is taken as the starting point of the theory of the logarithmic and the exponential function.

Logarithmic functions with any base:

$$a^x = y \Leftrightarrow \log_a y = x.$$

Since y is always positive, therefore the logarithmic function, log_a, is defined for positive values only of the variable.

Evidently, we have

$$a^{-x} = \frac{1}{y}$$

$$\therefore \log_a \frac{1}{y} = -x = -\log_a y$$

Also,

$$\log_a 1 = 0,$$

$$\log_a a = 1,$$

$$\log_a x + \log_a y = \log_a (xy),$$

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right),$$

$$\log_a x^y = y \log_a x,$$

$$\log_b x . \log_a b = \log_a x,$$

$$\log_b a . \log_a b = 1.$$

11.4 The Trigonometric Functions

Let us define

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)]$$

$$S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

Properties of the functions $C(x), S(x)$:

- $C(x)$ and $S(x)$ coincide with the functions $\cos x$ and $\sin x$
- $C(x)$ and $S(x)$ are real for real x.
- $E(ix) = C(x) + iS(x)$. Thus $C(x)$ and $S(x)$ are the real and imaginary parts, respectively of $E(ix)$, if x is real.
- $C'(x) = -S(x)$ and $S'(x) = C(x)$
- $S(-x) = -S(x)$and $C(-x) = C(x)$
- $C(x_1 + x_2) = C(x_1)C(x_2) - S(x_1)S(x_2)$
- $C(x_1 - x_2) = C(x_1)C(x_2) + S(x_1)S(x_2)$
- $S(x_1 + x_2) = S(x_1)C(x_2) + C(x_1)S(x_2)$

- $S(x_1 - x_2) = S(x_1)C(x_2) - C(x_1)S(x_2)$
- $|S(x)| \leq 1, |C(x)| \leq 1$
- $C(2x) = C^2(x) - S^2(x)$
- $S(2x) = 2S(x)C(x)$

Summary

- A series of the form $a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$ where $a_0, a_1, a_2, \ldots$ are real numbers, is called a Power series.
- The general form of the power series is

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

  This is called a power series about the point x_0.

- The power series converges for all $x \in \mathbb{R}$, is called everywhere convergent power series. Some power series converge only for $x = 0$, they are called nowhere convergent power series. Some power series converge for some real x and diverge for the other.
- If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1$ then the series converges absolutely for all real x.
- If a power series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = x_1$ then the series diverges for all real x satisfying $|x| > |x_1|$
- If the power series $\sum_{n=0}^{\infty} a_n x^n$ be neither nowhere convergent nor everywhere convergent then there exists a positive real number R such that the series converges absolutely for all real x satisfying $|x| < R$ and diverges for all x satisfying $|x| > R$. R is called the radius of convergence of power series $\sum_{n=0}^{\infty} a_n x^n$.
- We define $R = 0$ for a nowhere convergent power series and $R = \infty$ for a series that is everywhere convergent.
- Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with the radius of convergence $R(> 0)$ then the series is uniformly convergent on $[-s, s]$ where $0 < s < R$.
- The radius of convergence (RoC) of the power series is given by

$$\frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \text{ or } \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$$

- The power series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \ldots(1)$

  is everywhere convergent for real x. The function represented by the power series (1) is called the Exponential function, denoted, provisionally, by $E(x)$.
  Thus,

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \ldots(2)$$

$$\therefore E(0) = 1 \text{ and}$$

$$E(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \quad \ldots(3)$$

The series on the right-hand side of (3) converges to a number that lies between 2 and 3. This number is denoted by e. Thus $E(1) = e$. The function $E(x)$, defined by (2), is continuous and differentiable any number of times, for every x.

- Since the exponential function E is strictly increasing on the set $\mathbb{R}$ of real numbers $i.e., E: \mathbb{R} \to \mathbb{R}^+$ is 1-1 onto, it has inverse function L or log_e which is also strictly increasing and whose domain of definition is $E(\mathbb{R})$, that is, the set of all positive numbers.Thus L is defined by

$$E(L(y)) = y, (y > 0)$$

Or

$$L(E(x)) = x, (x \in \mathbb{R})$$

- Logarithmic functions with any base:

$$a^x = y \Leftrightarrow \log_a y = x.$$

Since y is always positive, therefore the logarithmic function, log_a, is defined for positive values only of the variable.

- The Trigonometric Functions

Let us define

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)]$$

$$S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

$C(x)$ and $S(x)$ coincide with the functions $\cos x$ and $\sin x$

Keywords

Power series: A series of the form$a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$where $a_0, a_1, a_2, \ldots$ are real numbers, is called a power series.

Power series about x_0: $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$

The radius of convergence (RoC) of the power series$\sum_{n=0}^{\infty} a_nx^n$:

$$\frac{1}{\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}} \text{ or } \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

Self Assessment

1) If $\sum a_n$ is convergent then

A. $\lim_{n\to\infty} a_n$ must exist and can be equal to any real positive real number.
B. $\lim_{n\to\infty} a_n$ must exist and can be equal to any real number.
C. $\lim_{n\to\infty} a_n$ must exist and is equal to zero.
D. $\lim_{n\to\infty} a_n$ need not exists.

2) Consider the following statements:

(I) Every convergent sequence is bounded.

(II) Every bounded sequence is convergent.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

3) Consider the following statements:

(I) If a power series is everywhere convergent then its radius of convergence is one.

(II)If a power series is nowhere convergent then its radius of convergence is zero.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

4) Consider the following statements:

(I) If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1$ then the series converges absolutely for all real x satisfying $|x| < |x_1|$.

(II) If a power series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = x_1$ then the series diverges for all real x satisfying $|x| > |x_1|$.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

5) $\sum_{n=0}^{\infty} x^n$ converges only for $x \in (-1, 1)$.

A. True
B. False

6) $\sum_{n=0}^{\infty} n!\, x^n$ is everywhere convergent power series.

A. True
B. False

7) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges only for $x = 0$.

A. True
B. False

8) Radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{3^n}{4^n+5^n} x^n$ is:

A. $3/5$
B. $5/3$
C. 0
D. ∞

9) Radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^n}{n^2+n} x^{2n}$ is:

A. $1/\sqrt{2}$
B. $\sqrt{2}$
C. 2
D. ½

10) Radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{4^n n^n} x^{2n}$ is:

A. $\frac{1}{2}$

B. 2

C. 1

D. none of these

11) Radius of convergence of the power series $\sum_{n=0}^{\infty} 2^{-n} x^{kn}$ is:

A. $\sqrt[k]{2}$
B. $1/\sqrt[k]{2}$
C. ∞

D. none of these

12) Radius of convergence of the power series $\sum_{n=0}^{\infty} 7^n x^{3n}$ is:

A. $\sqrt[3]{7}$
B. $\frac{1}{\sqrt[3]{7}}$
C. ∞

D. None of these

13) $(\log_b x)(\log_a b) = \log_a x$

A. True

B. False

14) $(\log_b a)(\log_a b) = 1$

A. True

B. False

15) $e^x = y \Rightarrow \log_e x = y$

A. True

B. False

16) If $E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ then E(x) is continuous and only twice differentiable.

A. True

B. False

17) Consider the following statements:

(I) $e^x \to \infty$ as $x \to \infty$

(II) $e^x \to -\infty$ as $x \to -\infty$

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	C	2.	A	3.	B	4.	C	5.	A
6.	B	7.	B	8.	B	9.	A	10.	D
11.	A	12.	B	13.	A	14.	A	15.	B
16.	B	17.	A						

Review Questions

1) Find the radius of convergence of the power series:

$$\sum \frac{n}{6^n} x^n.$$

2) Find the radius of convergence of the power series:

$$\sum \frac{2^n}{n^2+n} x^{pn}.$$

3) Find the radius of convergence of the power series:

$$\sum x^{n!}.$$

4) Find the radius of convergence of the power series:

$$\sum x^p \text{ , } p \text{ is prime.}$$

5) Find the radius of convergence of the power series:

$$\sum nlog\, nx^n.$$

6) Find the radius of convergence of the power series:

$$\sum \frac{e^{n^2}}{n} x^n.$$

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

Shanti Narayan, Elements of Real Analysis

Web Links

https://nptel.ac.in/courses/111/106/111106142/

Unit 12: Functions of Several Variables

CONTENTS

Objectives

After studying this unit, students will be able to:

- describe space of linear transformation
- define differentiation in $\mathbb{R}^n$
- understand partial derivatives and directional derivatives
- define the concept of contraction principle
- demonstrate fixed point theorem

Introduction

We begin this unit with a discussion on linear transformation and its particular case that is a linear operator.

Linear transformation: Let X and Y be two vector spaces over the same field F and $A: X \to Y$. Then A is said to be a linear transformation if

$$(i)\ A(x_1 + x_2) = Ax_1 + Ax_2\ \forall x_1, x_2 \in X$$
$$(ii) A(cx) = cAx\ \forall x \in X, c \in F.$$

Linear Operator: A linear transformation $A: X \to X$is called linear operator on X.

A linear operator A on a finite-dimensional vector space X is 1-1, if, and only if it is onto.

If $L(X,Y) = \{A | A: X \to Y$ is a linear transformation$\}$then$L(X,Y)$ is also a vector space.

In particular, $L(X) = \{A | A: X \to X$ is a linear operator on X$\}$ is a vector space.

12.1 Space of Linear Transformation on $\mathbb{R}^n$to$\mathbb{R}^m$

Norm of a Linear transformation:

Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$

i.e. A is a linear transformation from $\mathbb{R}^{\boldsymbol{n}}$ **to** $\mathbb{R}^{\boldsymbol{m}}$.

Then norm of A, denoted by$\|A\|$ is defined as

$\|A\| = \sup_{\substack{|x|\leq 1 \\ x\in\mathbb{R}^n}} |Ax|$

Theorem12.1.1: Let $A, B \in L(\mathbb{R}^{\boldsymbol{n}}, \mathbb{R}^{\boldsymbol{m}})$.

Then

$(i)\ |Ax| \leq \|A\|\,|x|\ \forall x \in \mathbb{R}^{\boldsymbol{n}}$

$(ii)\ |Ax| \leq \lambda\,|x|\ \forall x \in \mathbb{R}^{\boldsymbol{n}} \Rightarrow \|A\| \leq \lambda$

$(iii)\|A\| < \infty$

(iv) A is a uniformly continuous mapping from $\mathbb{R}^{\boldsymbol{n}}$ to$\mathbb{R}^{\boldsymbol{m}}$

$(v)\|A + B\| \leq \|A\| + \|B\|$

$(vi)\|cA\| \leq |c|\,\|A\|, c \in \mathbb{R}$

Proof: (i) If $x = 0$ then $|Ax| = 0 = \|A\|\,|x|$.

So the result holds.

Now let $x \neq 0$

Then $\left|\frac{x}{|x|}\right| = 1$

$\therefore$ by definition of $\|A\|$, we have

$\left|A\left(\frac{x}{|x|}\right)\right| \leq \|A\|$

$\Rightarrow \left|\frac{1}{|x|}Ax\right| \leq \|A\|$

$\Rightarrow \frac{1}{|x|}|Ax| \leq \|A\|$

$\Rightarrow |Ax| \leq \|A\|\,|x|$

(ii) We have $|Ax| \leq \lambda\,|x|\ \forall x \in \mathbb{R}^{\boldsymbol{n}}$

$\Rightarrow |Ax| \leq \lambda, \forall x \in \mathbb{R}^{\boldsymbol{n}}$with $|x| \leq 1$

$\Rightarrow \sup_{|x|\leq 1}|Ax| \leq \lambda$

$\Rightarrow \|A\| \leq \lambda.$

(iii) Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$such that $|x| \leq 1$

Then $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ and $|x_i| \leq 1$

$\therefore |Ax| = |A(x_1e_1 + x_2e_2 + \cdots + x_ne_n)|$

$= |x_1Ae_1 + x_2Ae_2 + \cdots + x_2Ae_n|$

$\leq \sum_{i=1}^{n} |x_i||Ae_i|$

$\leq \sum_{i=1}^{n} |Ae_i|$

$\Rightarrow \sup_{|x|\leq 1}|Ax| \leq \sum_{i=1}^{n} |Ae_i| < \infty$

$\Rightarrow \|A\| < \infty$

(iv) Let $\in > 0$ be given and let $x, y \in \mathbb{R}^n$

Then $|Ax - Ay| = |A(x - y)|$

$$\leq \|A\| \mid x-y \mid \quad \{\because \mid Ax \mid \leq \|A\| \mid x \mid \text{ by part(1)}\}$$

$$< \epsilon \text{ whenever } \mid x-y \mid < \frac{\epsilon}{\|A\|+1}$$

Thus we get, for given $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{\|A\|+1} > 0$ such that $\mid Ax - Ay \mid < \epsilon$ whenever $\mid x - y \mid < \delta$.

$\Rightarrow A: \mathbb{R}^n \to \mathbb{R}^m$ is a uniformly continuous mapping.

(v) $\mid (A+B)x \mid = \mid Ax + Bx \mid$

$$\leq \mid Ax \mid + \mid Bx \mid$$

$$\leq \|A\| \mid x \mid + \|B\| \mid x \mid$$

$$\Rightarrow \sup_{|x|\leq 1} \mid (A+B)x \mid \leq \|A\| + \|B\|$$

$$\Rightarrow \|A+B\| \leq \|A\| + \|B\|$$

(vi) $\mid (cA)x \mid = \mid c(Ax) \mid$

$= \mid c \mid\mid Ax \mid$

$\leq \mid c \mid \|A\| \mid x \mid$

$\Rightarrow \sup_{|x|\leq 1} \mid (cA)x \mid \leq \mid c \mid \|A\|$

$\Rightarrow \|cA\| \leq \mid c \mid \|A\|$

Cor: $If A \in L(\mathbb{R}^n, \mathbb{R}^m), B \in L(\mathbb{R}^m, \mathbb{R}^k)$ then $\|BA\| \leq \|B\|\|A\|$.

Proof: $\mid (BA)x \mid = \mid B(Ax) \mid$

$\leq \|B\| \mid Ax \mid$

$\leq \|B\|\|A\| \mid x \mid$

$\Rightarrow \sup_{|x|\leq 1} \mid (BA)x \mid \leq \|B\|\|A\|$

$\Rightarrow \|BA\| \leq \|B\|\|A\|$.

This completes the proof.

Theorem 12.1.2: Prove that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with metric d defined as

$$d(A,B) = \|A-B\| \; \forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m).$$

Proof: We have

$$d(A,B) = \|A-B\| \; \forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$$

(i) From the definition, we have

$$d(A,B) \geq 0 \; \forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$$

(ii) Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ $then\ d(A,B) = 0$

$$\Leftrightarrow \|A-B\| = 0$$

$$\Leftrightarrow \|A-B\| \mid x \mid = 0 \forall x \in \mathbb{R}^n$$

$$\Leftrightarrow \mid (A-B)x \mid = 0 \forall x \in \mathbb{R}^n$$

$$\{\because \mid (A-B)x \mid \leq \|A-B\| \mid x \mid \}$$

$$\Leftrightarrow (A-B)x = 0 \; \forall x \in \mathbb{R}^n$$

$$\Leftrightarrow A - B = O$$

$$\Leftrightarrow A = B$$

(iii) Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$

Then

$$d(A,B) = \|A-B\|$$

$$= \sup_{|x|\leq 1} \mid (A-B)x \mid$$

$$= \sup_{|x|\le 1} |(B-A)x|$$

$$= d(B, A)$$

(iv) Let $A, B, C \in L(\mathbb{R}^n, \mathbb{R}^m)$

Then

$$d(A, C) = \|A - C\|$$
$$= \|(A - B) + (B - C)\|$$
$$\le \|A - B\| + \|B - C\|$$
$$= d(A, B) + d(B, C)$$
$$i.e.\ d(A, C) \le d(A, B) + d(B, C)$$

Hence $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space.

This completes the proof.

Theorem 12.1.3: A linear operator$T$ on a finite-dimensional vector space X is one to one if and only if the range of T is all of X.

Proof: Let $B = \{x_1, x_2, \dots, x_n\}$ be a basis of X.

Let $R(T)$ be the range of T.

First of all, we will show that the set $Q = \{Tx_1, Tx_2, \dots, Tx_n\}$ spans $R(T)$.

Let $y \in R(T)$

$\Rightarrow y = Tx$ for some $x \in X$.

Since B spans X, there exist scalars $c_1, c_2, \dots, c_n$ such that

$$x = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

Now $y = Tx$

$$= T\{c_1x_1 + c_2x_2 + \cdots + c_n\}$$
$$= c_1Tx_1 + c_2Tx_2 + \cdots + c_nTx_n$$

Thus every element of R(T) is a linear combination of elements of Q.

$\Rightarrow$ Set Q spans $R(T)$.

Now $R(T) = X$ if and only if Q is independent.

We have to prove that this happens if and only if T is 1-1.

Let Q be independent and let x be any member of X.

Since B is a basis of X.

$$\therefore x = \sum_{i=1}^{n} c_i x_i \text{ for some scalars} c_i, i = 1,2, \dots, n, then$$

$$Tx = 0$$

$$\Rightarrow T\left(\sum_{i=1}^{n} c_i x_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} c_i T x_i = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0 \quad \because Q \text{ is independent.}$$

$$\Rightarrow x = \sum_{i=1}^{n} c_i x_i = 0$$

$$\text{Thus } Tx = 0 \Rightarrow x = 0 \quad \dots(1)$$

Now, $Tx = Ty$

$\Rightarrow Tx - Ty = 0$

$\Rightarrow T(x - y) = 0$

$\Rightarrow x - y = 0 \; \{by\ (1)\}$

$\Rightarrow x = y$

$\Rightarrow T\ is\ 1 - 1$

Conversely, let T be 1-1. Then

$$\sum_{i=1}^{n} c_i T x_i = 0$$

$$\Rightarrow T\left(\sum_{i=1}^{n} c_i x_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} c_i x_i = 0 \text{ as } T\ is\ 1 - 1$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0, \text{as } B \text{ is independent.}$$

$$\therefore \sum_{i=1}^{n} c_i T x_i = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0.$$

Hence Q is independent.

This completes the proof.

Open Ball: Let (X, d) be a metric space, the open ball of radius $r > 0$ centered at a point a in X, usually denoted by $B_r(a) or\ B(a; r)$ and is defined as

$$B(a; r) = \{x \in X: d(x, a) < r\}.$$

Open Set:A subset E of a metric space (X,d) is open if $\forall x \in E$, there exists an open ball $B(x; r)$ such that $B(x; r) \subseteq E$.

Convex Set:A set $E \subseteq \mathbb{R}^n$ is said to be convex if $x \in E, y \in E \Rightarrow tx + (1 - t)y \in E, \forall t \in [0,1]$.

The set of points $(tx + (1 - t)y: t \in [0,1]\}$ is called the line segment joining the points x, y.

Set E is a convex set if the line segment between two points in E lies in E.

Theorem 12.1.4: Prove that open balls in $\mathbb{R}^n$ are convex**.**

Proof: Let$B(a; r)$ be the open ball and let $x, y \in B(a; r) then$

$$\|x - a\| < r \text{ and } \|y - a\| < r \quad \text{... (1)}$$

Consider

$\|\lambda x + (1 - \lambda)y - a\|, \lambda \in [0,1]$.

$= \|\lambda x + y - \lambda y - a\|$

$= \|\lambda x + y - \lambda y - a + \lambda a - \lambda a\|$

$= \|\lambda(x - a) + (y - a) - \lambda(y - a)\|$

$= \|\lambda(x - a) + (1 - \lambda)(y - a)\|$

$\leq \|\lambda(x - a)\| + \|(1 - \lambda)(y - a)\|$

$\leq |\ \lambda\ |\ \|(x - a)\| + |\ (1 - \lambda)\ |\ \|(y - a)\|$

$= \lambda\|(x - a)\| + (1 - \lambda)\|(y - a)\|$

$< \lambda r + (1 - \lambda)r \qquad \{by\ (1)\}$

$= r$

Thus, $\|\lambda x + (1-\lambda)y - a\| < r$

$\Rightarrow\ \lambda x + (1-\lambda)y \in B(a;r)$

$\Rightarrow$ open ball $B(a;r)$ is convex.

This completes the proof.

Theorem 12.1.5: Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$.

Then

(i)if $A \in \Omega$, $B \in L(\mathbb{R}^n)$and$\|B - A\|\|A^{-1}\| < 1$ then $B \in \Omega$.

$(ii)\Omega$ is an open subset of $L(\mathbb{R}^n)$.

(iii)Mapping $\phi: \Omega \to \Omega$ defined by $\phi(A) = A^{-1}\ \forall A \in \Omega$ is continuous.

Proof: (i) We have $\|B - A\|\|A^{-1}\| < 1$.

$$\text{So if } \alpha = \frac{1}{\|A^{-1}\|} \text{ and } \beta = \|B - A\| \quad \ldots(1)$$

$$\text{then } \beta.\frac{1}{\alpha} < 1.$$

$$\Rightarrow \beta < \alpha$$

$$\Rightarrow \alpha - \beta > 0 \quad \ldots(2)$$

Now to prove $B \in \Omega$ $i.e$, B is invertible, it is sufficient to show that B is 1-1.

For this, let $Bx = 0, x \in \mathbb{R}^n$. Then

$$\begin{aligned}
\alpha \mid x \mid &= \alpha \mid (A^{-1}A)x \mid \\
&= \alpha \mid A^{-1}(Ax) \mid \\
&\leq \alpha\|A^{-1}\| \mid Ax \mid \\
&= \frac{1}{\|A^{-1}\|}\|A^{-1}\| \mid Ax \mid \\
&= \mid Ax \mid \\
&= \mid (A - B)x + Bx \mid \\
&= \mid (A - B)x \mid + \mid Bx \mid \\
&\leq \|(A - B)\| \mid x \mid + \mid Bx \mid
\end{aligned}$$

$$\Rightarrow \alpha \mid x \mid \leq \beta \mid x \mid + \mid Bx \mid$$

$$\Rightarrow \alpha \mid x \mid - \beta \mid x \mid \leq \mid Bx \mid$$

$$\Rightarrow [\alpha - \beta] \mid x \mid \leq \mid Bx \mid \quad \ldots(3)$$

$$\Rightarrow [\alpha - \beta] \mid x \mid \leq 0$$

$$\Rightarrow \mid x \mid \leq 0 \quad \because \alpha - \beta > 0$$

$$\Rightarrow \mid x \mid = 0$$

$$\Rightarrow x = 0$$

Thus $Bx = 0 \Rightarrow x = 0$

$\Rightarrow B\ is\ 1 - 1.$

$\Rightarrow B \in \Omega.$

(ii)Let $A \in \Omega$

Consider an open ball $S\left(A, \frac{1}{\|A^{-1}\|}\right)$.

Claim: $S\left(A, \frac{1}{\|A^{-1}\|}\right) \subseteq \Omega$.

Let

$$B \in S\left(A, \frac{1}{\|A^{-1}\|}\right)$$

$$\Rightarrow d(A,B) < \frac{1}{\|A^{-1}\|}$$

$$\Rightarrow \|B - A\| < \frac{1}{\|A^{-1}\|}$$

$$\Rightarrow \|B - A\|\|A^{-1}\| < 1$$

$$\Rightarrow B \in \Omega \quad \ldots\{\text{by part (i)}\}$$

Thus for all $A \in \Omega$, there exists an open ball $S\left(A, \frac{1}{\|A^{-1}\|}\right)$ such that $S\left(A, \frac{1}{\|A^{-1}\|}\right) \subseteq \Omega$.

$\Rightarrow$ Ω is an open subset of $L(\mathbb{R}^n)$

(iii) Let $A, B \in \Omega$

Then taking $x = B^{-1}y$ in

$(\alpha - \beta) \mid x \mid \leq \mid Bx \mid \forall x \in \mathbb{R}^n$, we get

$(\alpha - \beta) \mid B^{-1}y \mid \leq \mid y \mid$

$$\Rightarrow \mid B^{-1}y \mid \leq (\alpha - \beta)^{-1} \mid y \mid, \forall y \in \mathbb{R}^n$$

$$\Rightarrow \sup_{|y|\leq 1} \mid B^{-1}y \mid \leq (\alpha - \beta)^{-1}$$

$$\Rightarrow \|B^{-1}\| \leq (\alpha - \beta)^{-1} \quad \ldots(4)$$

Now, since $\phi(A) = A^{-1}$, therefore

$$\|\phi(B) - \phi(A)\| = \|B^{-1} - A^{-1}\|$$

$$= \|B^{-1}AA^{-1} - B^{-1}BA^{-1}\|$$

$$\leq \|B^{-1}\|\|A - B\|\|A^{-1}\|$$

$$\leq (\alpha - \beta)^{-1}\beta.\frac{1}{\alpha}$$

$$= \frac{\beta}{\alpha(\alpha - \beta)} \to 0 \; as \; \beta \to 0$$

$$\text{Thus} \|\phi(B) - \phi(A)\| \to 0 \; as \; \beta \to 0$$

$$i.e. \|\phi(B) - \phi(A)\| \to 0 \; as \; \|B - A\| \to 0 \; \forall A, B \in \Omega$$

Hencemapping $\phi: \Omega \to \Omega$ defined by $\phi(A) = A^{-1} \; \forall A \in \Omega$ is continuous.

This completes the proof.

12.2 Differentiation in $\mathbb{R}^n$

Let E be an open subset of $\mathbb{R}^n$ and let $f: E \to \mathbb{R}^n$ be a function. If for $x \in E$, there exists a linear transformation $A: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\mid f(x+h) - f(x) - Ah \mid}{\mid h \mid} = 0,$$

Then we say f is differentiable at x and derivative of f at x is A.

$$i.e. f'(x) = A$$

Let a function f maps $(a,b) \subset \mathbb{R}$ into $\mathbb{R}^m$. Then $f'(x)$ is defined to be a vector $y \in \mathbb{R}^m$ (if there is one) for which

$$\lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} - y \right\} = 0$$

This can be written as

$$f(x+h) - f(x) = hy + r(h)$$

$$\text{where} \frac{r(h)}{h} \to 0 \; as \; h \to 0$$

Hence every $y \in \mathbb{R}^m$ induces a linear transformation of $\mathbb{R}$ into$\mathbb{R}^m$ by associating to each $h \in \mathbb{R}$ the vector $hy \in \mathbb{R}^m$.

This identification of $\mathbb{R}^m$ with $L(\mathbb{R}, \mathbb{R}^m)$ allows us to regard $f'(x)$ as a member of $L(\mathbb{R}, \mathbb{R}^m)$.

$$\therefore f'(x): \mathbb{R} \to \mathbb{R}^m \text{ is a linear transformation and}$$

$$\lim_{h\to 0} \frac{|\, f(x+h) - f(x) - f'(x)h \,|}{|\, h \,|} = 0$$

$$\Rightarrow f(x+h) - f(x) = f'(x)h + r(h)$$

$$\text{where the remainder } r(h) \text{staisfies} \lim_{h\to 0} \frac{r(h)}{h} = 0.$$

If f is differentiable at every $x \in E$ then f is said to be differentiable in E.

Theorem 12.2.1: Let $E \subseteq \mathbb{R}^n$ be an open set and $f: E \to \mathbb{R}^m$. If for $x \in E$, there exists a linear transformation A_1 and A_2 from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{|\, f(x+h) - f(x) - A_1 h \,|}{|\, h \,|} = 0$$

and

$$\lim_{h\to 0} \frac{|\, f(x+h) - f(x) - A_2 h \,|}{|\, h \,|} = 0$$

then $A_1 = A_2$.

Proof: We have

$$\lim_{h\to 0} \frac{|\, f(x+h) - f(x) - A_1 h \,|}{|\, h \,|} = 0, \lim_{h\to 0} \frac{|\, f(x+h) - f(x) - A_2 h \,|}{|\, h \,|} = 0 \quad \dots (1)$$

Let $B = A_1 - A_2$

Then $|\, Bh \,| = |\, (A_1 - A_2)h \,|$

$= |\, A_1 h - A_2 h \,|$

$= |\, A_1 h - f(x+h) + f(x) + f(x+h) - f(x) - A_2 h \,|$

$\Rightarrow |\, Bh \,| \leq |\, f(x+h) - f(x) - A_1 h \,| + |\, f(x+h) - f(x) - A_2 h \,|$

$$\Rightarrow \frac{|\, Bh \,|}{|\, h \,|} \leq \frac{|\, f(x+h) - f(x) - A_1 h \,|}{|\, h \,|} + \frac{|\, f(x+h) - f(x) - A_2 h \,|}{|\, h \,|}$$

$$\Rightarrow \frac{|\, Bh \,|}{|\, h \,|} \to 0 \; as \; h \to 0 \quad \{by (1)\}$$

Now let $h \neq 0$ be a fixed number and t be any real number then

$$\frac{|\, B(th) \,|}{|\, th \,|} = \frac{|\, tBh \,|}{|\, th \,|}$$

$$= \frac{|\, t \,||\, Bh \,|}{|\, t \,||\, h \,|}$$

$$= \frac{|\, Bh \,|}{|\, h \,|}$$

$$\Rightarrow \frac{|\, Bh \,|}{|\, h \,|} = \frac{|\, B(th) \,|}{|\, th \,|} \to 0 \; as \; t \to 0$$

Thus

$Bh = 0 \; \forall \; h \in \mathbb{R}^n$

$\Rightarrow (A_1 - A_2)h = 0 \; \forall \; h \in \mathbb{R}^n$

$\Rightarrow A_1 - A_2 = 0$

$\Rightarrow A_1 = A_2$

This completes the proof.

Theorem 12.2.2: (Chain Rule)

Let E be an open subset of $\mathbb{R}^n, f: E \to \mathbb{R}^m, f$ is a differentiable function at $x_0 \in E$ and $g: G \to \mathbb{R}^k$ be differentiable at $f(x_0)$, where G is an open subset of $\mathbb{R}^m$ containing $f(E)$. Then the function $F: E \to \mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.

Proof: $Let\ f(x_0) = y_0, f'(x_0) = A, g'(y_0) = B.$

Then

$$f(x_0 + h) - f(x_0) = Ah + uh \text{ where } \lim_{h \to 0} \frac{|u(h)|}{|h|} = 0 \quad \dots (1)$$

and

$$g(y_0 + k) - g(y_0) = Bk + vk \text{ where } \lim_{k \to 0} \frac{|v(k)|}{|k|} = 0 \quad \dots (2)$$

Let $k = f(x_0 + h) - f(x_0)$ for given h …(3)

Then

$$|k| = |f(x_0 + h) - f(x_0)|$$

$$= |Ah + u(h)| \qquad \{by(1)\}$$

$$\leq |Ah| + |u(h)|$$

$$\leq \|A\| |h| + |u(h)|$$

$$= \left(\|A\| + \frac{|u(h)|}{|h|}\right)|h| \quad \dots (4)$$

Now

$$|F(x_0 + h) - F(x_0) - BAh|$$

$$= |g(f(x_0 + h)) - g(f(x_0)) - B(Ah)|$$

$$= |g(y_0 + k) - g(y_0) - B(Ah)| \qquad \{by(3)\}$$

$$= |Bk + v(k) - B(Ah)| \qquad \{by(2)\}$$

$$= |B(k - Ah) + v(k)|$$

$$= |B(u(h)) + v(k)| \quad \because u(h) = f(x_0 + h) - f(x_0) - Ah \{by(1)\} = k - Ah \ \{by(3)\}$$

$$\leq |B(u(h))| + \frac{|v(k)|}{|k|}|k|$$

$$\leq \|B\| |u(h)| + \frac{|v(k)|}{|k|}\left[\|A\| + \frac{|u(h)|}{|h|}\right]|h| \qquad \{by(4)\}$$

$$\leq \left\{\|B\| \frac{|u(h)|}{|h|} + \frac{|v(k)|}{|k|}\left[\|A\| + \frac{|u(h)|}{|h|}\right]\right\}|h|$$

$$\to 0 \ as\ h \to 0$$

Thus

$$F'(x_0) = BA$$

$$= g'(y_0)f'(x_0)$$

$$= g'(f(x_0))f'(x_0)$$

This completes the proof.

12.3 Partial Derivatives and Directional Derivatives

Partial Derivatives of a Vector-Valued Function of Several Variables: Let$E$ be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^m$ be a function. Let $\{e_1, e_2, \dots, e_n\}$ and $\{u_1, u_2, \dots, u_m\}$ be standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Let $f(x) = \big(f_1(x), f_2(x), \dots, f_m(x)\big)$

$$= f_1(x)u_1 + f_2(x)u_2 + \cdots + f_m(x)u_m, x \in E$$

Then partial derivatives of f_i with respect to x_j is denoted by $\frac{\partial f_i}{\partial x_j}$ *or* $(D_j f_i)(x)$and is defined as

$\frac{\partial f_i}{\partial x_j} = \lim_{t\to 0} \frac{f_i\big(x + te_j\big) - f_i(x)}{t}$, provided this limit exists.

Theorem 12.3.1: Let E be an open subset of $\mathbb{R}^n$ *and* $f: E \to \mathbb{R}^m$ be a function that is differentiable at $x \in E$. Then $(D_j f_i)(x)$ exists for all $1 \le i \le m, 1 \le j \le n$ and

$$f'(x)e_j = \sum_{i=1}^{m} \big(D_j f_i\big)(x) u_i \text{where}$$

$1 \le j \le n$, $\{e_1, e_2, \dots, e_n\}$and $\{u_1, u_2, \dots, u_m\}$are standard bases of $\mathbb{R}^{\mathbf{n}}$ and $\mathbb{R}^{\mathbf{m}}$ respectively.

Proof: Since f is differentiable at x.

Therefore,

$$f\big(x + te_j\big) - f(x) = f'(x)\big(te_j\big) + r\big(te_j\big)$$

where,

$$\lim_{t\to 0} \frac{|\, r\big(te_j\big) \,|}{|\, te_j \,|} = 0$$

$$\Rightarrow f\big(x + te_j\big) - f(x) = tf'(x)e_j + r(te_j)$$

where, $\lim_{t\to 0} \frac{|\, r\big(te_j\big) \,|}{|\, t \,|} = 0 \quad \because |\, e_j \,| = 1$

$$\Rightarrow \lim_{t\to 0} \frac{f\big(x + te_j\big) - f(x)}{t} = f'(x)e_j$$

$$\Rightarrow \lim_{t\to 0} \frac{\big(f_1\big(x + te_j\big), \dots, f_m(x + te_j)\big) - \big(f_1(x), \dots, f_m(x)\big)}{t} = f'(x)e_j$$

$$\Rightarrow \lim_{t\to 0} \left(\frac{f_1\big(x + te_j\big) - f_1(x)}{t}, \dots, \frac{f_m\big(x + te_j\big) - f_m(x)}{t}\right) = f'(x)e_j$$

$\Rightarrow (D_j f_i)\ (x)$exists, $1 \le i \le m$ and $1 \le j \le n$

and

$$f'(x)e_j = \Big(\big(D_j f_1\big)(x), \dots, \big(D_j f_m\big)(x)\Big)$$

$$= (D_j f_1)(x)u_1 + \cdots + \big(D_j f_m\big)(x)u_m$$

$$= \sum_{i=1}^{m} \big(D_j f_i\big)(x)u_i$$

This completes the proof.

Cor 1: The matrix of a linear transformation$f'(x): \mathbb{R}^n \to \mathbb{R}^m$ withrespect to the standard bases $\{e_1, e_2, \dots, e_n\}$and$\{u_1, u_2, \dots, u_m\}$ of$\mathbb{R}^{\boldsymbol{n}}and\mathbb{R}^{\boldsymbol{m}}$respectively is:

$$[f'(x)] = \begin{bmatrix} (D_1 f_1)(x) & \cdots & (D_n f_1)x \\ \vdots & \ddots & \vdots \\ (D_1 f_m)x & \cdots & (D_n f_m)(x) \end{bmatrix}$$

Cor 2: Let $h = (h_1, h_2, \dots, h_n)$

$= h_1 e_1 + h_2 e_2 + \cdots + h_n e_n$ be any vector in $\mathbb{R}^n$ then

$$f'(x)h = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} (D_j f_i)(x)h_j\right] u_{i.}$$

Proof: $f'(x)h = f'(x)(h_1e_1 + \cdots + h_ne_n)$

$$= \big(f'(x)\big)(h_1e_1) + \big(f'(x)\big)(h_2e_2) + \cdots + \big(f'(x)\big)(h_ne_n)$$

$$= h_1f'(x)e_1 + h_2f'(x)e_2 + \cdots + h_nf'(x)e_n$$

$$= \sum_{j=1}^{n} h_jf'(x)e_j$$

$$= \sum_{j=1}^{n} h_j \sum_{i=1}^{m} (D_jf_i)(x)u_i$$

$$= \sum_{i=1}^{m} \left[\sum_{j=1}^{n} (D_jf_i)(x)h_j\right] u_{i.}$$

Theorem12.3.2: Let E be an open convex subset of $\mathbb{R}^n$ and let $f: E \to \mathbb{R}^m$ be a differentiable function such that $\|f'(x)\| \leq M \forall x \in E$ for some $0 < M \in \mathbb{R}$. Then

$$| f(b) - f(a) | \leq M \mid b - a \mid \forall a, b \in E.$$

Proof: Let $a, b \in E$.

We define a function $\phi: [0,1] \to E$ by

$$\phi(t) = (1 - t)a + tb, t \in [0,1].$$

Since $a, b \in E$ and E is convex.

Therefore, $\phi(t) \in E, t \in [0,1]$.

Let $g(t) = f\big(\phi(t)\big), t \in [0,1]$

Then $g'(t) = f'\big(\phi(t)\big)\phi'(t)$

$$= f'\big(\phi(t)\big)(b - a)$$

$\Rightarrow | g'(t) | = | f'\big(\phi(t)\big) || \ (b - a) |$

$$\leq \|f'\big(\phi(t)\big)\| \mid (b - a) \mid$$

$$\leq M \mid (b - a) \mid$$

$\therefore | g'(t) | \leq M \mid (b - a) \mid \qquad \ldots (1)$

Since ϕ is differentiable on [0,1] and f is differentiable on E.

Therefore, $g = fo\phi: [0,1] \to \mathbb{R}^m$ is also differentiable on [0,1].

Since if $f: [a, b] \to \mathbb{R}^k$ be a continuous function such that f is differentiable on (a,b) then there exists $x \in (a, b)$ such that

$$| f(b) - f(a) | \leq (b - a) \mid f'(x) \mid$$

Thus there exists $x \in (0,1)$such that

$$| g(1) - g(0) | \leq (1 - 0) \mid g'(x) \mid$$

Now since $\ g(1) = f\big(\phi(1)\big) = f(b)$

and $\qquad g(0) = f\big(\phi(0)\big) = f(b)$

$\therefore | f(b) - f(a) | \leq | g'(x) |$

$\Rightarrow | f(b) - f(a) | \leq M \mid b - a \mid \qquad \{by(1)\}$

This completes the proof.

Continuously Differentiable Function: Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^m$ be a function. Then f is said to be continuously differentiable in E if $f': E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function.

If $f: E \to \mathbb{R}^n$ is continuously differentiable function then we say f is C' − mapping or $f \in C'(E)$. Thus if $f \in C'(E)$ then $\forall a \in E, \in > 0$, there exits $\delta > 0$ such that $\|f'(x) - f'(a)\| < \in$ whenever $| x - a | < \delta, x \in E$.

Theorem 12.3.3: Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^m$ be a function. If $f \in C'(E)$

Then partial derivatives $D_j f_i(x)$ exists and are continuous $\forall x \in E, 1 \le i \le m,$ and $1 \le j \le n$.

Proof: Let $f \in C'(E)$

$\Rightarrow f$ is differentiable in E.

$\Rightarrow D_j f_i(x)$ exists and

$$f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i \ \forall x \in E,\ 1 \le i \le m \text{ and } 1 \le j \le n . \qquad \text{... \{by Theorem 12.3.1\}}$$

$$\Rightarrow [f'(x)e_j].u_i = (D_j f_i)(x) \qquad \text{... (1)}$$

Now, $f': E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function.

$\therefore$ for given $\in > 0$ and all $x \in E$, there exists $\delta > 0$, such that

$$\|f'(y) - f'(x)\| < \in, |\ x - y\ | < \delta, y \in E \qquad \text{... (2)}$$

$\therefore |\ (D_j f_i)(y) - (D_j f_i)(x)\ |$

$= |\ [f'(y)e_j].u_i - [f'(x)e_j].u_i\ | \qquad \{by\ (1)\}$

$= |\ f'(y) - f'(x)\ |\ e_j.u_i\ |$

$\le |\ [f'(y) - f'(x)]e_j\ ||\ u_i\ |$ {by Cauchy – Schwarz Inequality}

$= |\ [f'(y) - f'(x)]e_j\ | \qquad \{\because |\ u_i\ | = 1\}$

$\le \|f'(y) - f'(x)\|\ |\ e_j\ |$

$= \|f'(y) - f'(x)\| \qquad \{\because |\ e_j\ | = 1\}$

$i.e. |\ (D_j f_i)(y) - (D_j f_i)(x)\ | < \in$ whenever $|\ x - y\ | < \delta$ {by(2)}

$D_j f_i$ are continuous on $E, 1 \le i \le m,$ and $1 \le j \le n$.

This completes the proof.

Directional Derivative: Directional derivative of f at x in the direction of unit vector u is denoted by $(D_u f)(x)$ and is given as

$$(D_u f)(x) = \lim_{h \to 0} \frac{f(x + uh) - f(x)}{h}$$

Example: If $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}; & (x,y) \ne (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$$

Then find the directional derivative of f at $(0,0)$ in the direction of the vector $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Solution: $(D_u f)(c) = (D_u f)(x) = \lim_{h \to 0} \frac{f(c + uh) - f(c)}{h}$ where

$$u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$c = (0,0)$ and

$$c + uh = \left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right)$$

$$\therefore (D_u f)(c) = \lim_{h \to 0} \frac{f\left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(0,0)}{h}$$

$$= \frac{1}{\sqrt{2}}$$

12.4 The Contraction Principle

Fixed Point: Let X be any non-empty set and$T: X \to X$. A point $x_0 \in X$ is said to be a fixed point of T if $T(x_0) = x_0$.

Example: If $T: \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{x^2 + 12}{7}.$$

Then find fixed points of T.

Solution: We have

$$T(x) = \frac{x^2 + 12}{7}$$

then

$$T(x) = x$$

$$\Rightarrow \frac{x^2 + 12}{7} = x$$

$$\Rightarrow x^2 - 7x + 12 = 0$$

$\Rightarrow 3$ *and* 4 are the fixed points of T .

Contraction Mapping: Let (X, d) be a metric space. A function $f: X \to X$ is said to be a contraction mapping if there exists a real number α with $0 \leq \alpha < 1$ such that

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y) \;\; \forall x, y \in X.$$

Contraction map f is uniformly continuous on X.

Example: If $f(x) = x^2, 0 \leq x < \frac{1}{3}$.

Then show that f is a contraction mapping on $\left[0, \frac{1}{3}\right]$.

Solution:$d\big(f(x), f(y)\big) = d(x^2, y^2)$

$$= \mid x^2 - y^2 \mid$$

$$= \mid x - y \mid\mid x + y \mid$$

$$\leq \mid x - y \mid [\mid x \mid + \mid y \mid]$$

$$= \frac{2}{3} \mid x - y \mid$$

Thus we get,

$$d\big(f(x), f(y)\big) \leq \frac{2}{3} \mid x - y \mid$$

$\Rightarrow f$ is a contraction map.

Theorem 12.4.1: (Fixed Point Theorem)

Let (X, d) be a complete metric space and let ϕ be a contraction mapping on X. Then there exists one and only one $x \in X$ such that $\phi(x) = x$.

Proof: Let x_0 be any element of X and we define a sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = \phi(x_n), n = 0,1,2, \ldots \qquad \ldots (1)$$

We will show that $\{x_n\}$ is a Cauchy sequence.

Since ϕ is a contraction map, there exists a real number α with $0 \leq \alpha < 1$ such that $\forall x, y \in X$, we have $d\big(\phi(x), \phi(y)\big) \leq \alpha d(x, y)$. ... (2)

For $n = 0,1,2, \ldots$, we have

$$d(x_{n+1}, x_n) = d\big(\phi(x_n), \phi(x_{n-1})\big) \qquad \{by(1)\}$$
$$\leq \alpha d(x_n, x_{n-1}) \qquad \{by(2)\}$$
$$= \alpha d\big(\phi(x_{n-1}), \phi(x_{n-2})\big)$$
$$\leq \alpha^2 d(x_{n-1}, x_{n-2})$$
$$\dots\dots\dots$$
$$\leq \alpha^n d(x_1, x_0)$$

Thus we get,

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0) \qquad \dots(3)$$

If n, m are positive integers and $m < n$, it follows that

$$d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$
$$\leq \alpha^m d(x_1, x_0) + \alpha^{m+1} d(x_1, x_0) + \cdots + \alpha^{n-1} d(x_1, x_0) \qquad \{by(3)\}$$
$$= \alpha^m [1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m+1}] d(x_1, x_0)$$
$$\leq \alpha^m [1 + \alpha + \alpha^2 + \cdots] d(x_1, x_0)$$
$$= \frac{\alpha^m}{1-\alpha} d(x_1, x_0)$$
$$\therefore d(x_n, x_m) \leq \frac{\alpha^m}{1-\alpha} d(x_1, x_0), 0 \leq \alpha < 1$$

$\to 0 \; as \; m \to \infty$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete.

$\therefore \lim_{n\to\infty} x_n = x, x \in X.$

Also,ϕ is a contraction map.

$\therefore \phi$ is continuous.

$$\Rightarrow \phi(x) = \lim_{n\to\infty} \phi(x_n)$$
$$= \lim_{n\to\infty} x_{n+1}$$
$$= x$$

Thus we get, $\phi(x) = x$.

Uniqueness: Let $y \in X, y \neq x$ such that $\phi(y) = y$

Then,$d(x, y) = d\big(\phi(x), \phi(y)\big)$

Since ϕ is a contraction map.

$\therefore d(x, y) \leq \alpha d(x, y)$

$\Rightarrow (1 - \alpha) d(x, y) \leq 0$

Since $0 \leq \alpha < 1$.

$\therefore d(x, y) \leq 0$

But $d(x, y) \geq 0$

This is possible only if $d(x, y) = 0 \; i.e. x = y$

This completes the proof.

Summary

- Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$i.e. A is a linear transformation from $\mathbb{R}^n$ **to** $\mathbb{R}^m$.Then norm of A, denotedby $\|A\|$ is defined as

$\|A\| = \sup_{\substack{|x|\leq 1 \\ x\in\mathbb{R}^n}} | Ax |$

- Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$.Then

  $(i)\ | Ax | \leq \|A\|\ | x |\ \forall x \in \mathbb{R}^n$

  $(ii)\ | Ax | \leq \lambda\ | x |\ \forall x \in \mathbb{R}^n \Rightarrow \|A\| \leq \lambda$

  $(iii) \|A\| < \infty$

  (iv) A is a uniformly continuous mapping from $\mathbb{R}^n$ to$\mathbb{R}^m$

  $(v) \|A + B\| \leq \|A\| + \|B\|$

  $(vi) \|cA\| \leq | c |\ \|A\|, c \in \mathbb{R}$

- $If A \in L(\mathbb{R}^n, \mathbb{R}^m), B \in L(\mathbb{R}^m, \mathbb{R}^k)$ then $\|BA\| \leq \|B\|\|A\|$.
- $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with metric d defined as $d(A, B) = \|A - B\|\ \forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$.
- A linear operator T on a finite-dimensional vector space X is one to one if and only if the range of T is all of X.
- Let (X, d) be a metric space, the open ball of radius $r > 0$ centered at a point a in X, usually denoted by $B_r(a) or\ B(a; r)$ and is defined$B(a; r) = \{x \in X: d(x, a) < r\}$.
- A subset E of a metric space (X,d) is open if $\forall x \in E$, there exists an open ball $B(x; r)$ such that $B(x; r) \subseteq E$.
- A set $E \subseteq \mathbb{R}^n$ is said to be convex if $x \in E, y \in E \Rightarrow tx + (1 - t)y \in E, \forall t \in [0,1]$.
- The set of points $(tx + (1 - t)y: t \in [0,1]\}$ is called the line segment joining the points x, y.
- Set E is a convex set if the line segment between two points in E lies in E.
- Open balls in $\mathbb{R}^n$ are convex**.**
- Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. Then

  (i)if $A \in \Omega$, $B \in L(\mathbb{R}^n)$and$\|B - A\|\|A^{-1}\| < 1$ then $B \in \Omega$.

  $(ii)\Omega$ is an open subset of $L(\mathbb{R}^n)$.

  (iii)Mapping $\phi: \Omega \to \Omega$ defined by $\phi(A) = A^{-1}\ \forall A \in \Omega$ is continuous.

- Let E be an open subset of $\mathbb{R}^n$ and let $f: E \to \mathbb{R}^n$ be a function. If for $x \in E$, there exists a linear transformation $A: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{| f(x + h) - f(x) - Ah |}{| h |} = 0,$$

Then we say f is differentiable at x and derivative of f at x is $A i.e. f'(x) = A$.

- If f is differentiable at every $x \in E$ then f is said to be differentiable in E.
- Let $E \subseteq \mathbb{R}^n$ be an open set and $f: E \to \mathbb{R}^m$. If for $x \in E$, there exists a linear transformation A_1 and A_2 from $\mathbb{R}^n\ to\ \mathbb{R}^m$ such that

  $$\lim_{h\to 0} \frac{| f(x + h) - f(x) - A_1 h |}{| h |} = 0 \text{and} \lim_{h\to 0} \frac{| f(x + h) - f(x) - A_2 h |}{| h |} = 0$$

  then $A_1 = A_2$.

- Let E be an open subset of $\mathbb{R}^n, f: E \to \mathbb{R}^m, f$ is a differentiable function at $x_0 \in E$ and $g: G \to \mathbb{R}^k$ be differentiable at $f(x_0)$, where G is an open subset of $\mathbb{R}^m$ containing $f(E)$. Then the function $F: E \to \mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.
- Let$E$ be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^m$ be a function. Let $\{e_1, e_2, \dots, e_n\}$ and $\{u_1, u_2, \dots, u_m\}$ be standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

  Let $f(x) = (f_1(x), f_2(x), \dots, f_m(x)) = f_1(x)u_1 + f_2(x)u_2 + \cdots + f_m(x)u_m, x \in E$

Then partial derivatives of f_i with respect to x_j is denoted by $\frac{\partial f_i}{\partial x_j}$ $or\ (D_j f_i)(x)$and is defined as

$$\frac{\partial f_i}{\partial x_j} = \lim_{t\to 0} \frac{f_i(x + te_j) - f_i(x)}{t}, \text{provided this limit exists.}$$

- Let E be an open subset of $\mathbb{R}^n$ *and* $f: E \to \mathbb{R}^m$ be a function that is differentiable at $x \in E$. Then $(D_j f_i)(x)$ exists for all $1 \leq i \leq m, 1 \leq j \leq n$ and

$$f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i \text{where}$$

$1 \leq j \leq n$, $\{e_1, e_2, \ldots, e_n\}$and $\{u_1, u_2, \ldots, u_m\}$are standard bases of $\mathbb{R}^{\mathbf{n}}$ and $\mathbb{R}^{\mathbf{m}}$ respectively.

- The matrix of a linear transformation$f'(x): \mathbb{R}^n \to \mathbb{R}^m$ with respect to the standard bases $\{e_1, e_2, \ldots, e_n\}and\{u_1, u_2, \ldots, u_m\}$ of$\mathbb{R}^{\boldsymbol{n}}and\mathbb{R}^{\boldsymbol{m}}$respectively is:

$$[f'(x)] = \begin{bmatrix} (D_1 f_1)(x) & \cdots & (D_n f_1)x \\ \vdots & \ddots & \vdots \\ (D_1 f_m)x & \cdots & (D_n f_m)(x) \end{bmatrix}$$

- Let E be an open convex subset of $\mathbb{R}^n$ and let $f: E \to \mathbb{R}^m$ be a differentiable function such that $\|f'(x)\| \leq M \forall x \in E$ for some $0 < M \in \mathbb{R}$. Then$| f(b) - f(a) | \leq M | b - a | \forall a, b \in E$.
- Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^m$ be a function. Then f is said to be continuously differentiable in E if $f': E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function.
- If $f: E \to \mathbb{R}^n$ is continuously differentiable function then we say f is $C' -$ mapping or $f \in C'(E)$. Thus if $f \in C'(E)$ then $\forall a \in E, \in > 0$, there exits $\delta > 0$ such that $\|f'(x) - f'(a)\| < \in$ whenever $| x - a | < \delta, x \in E$.
- Let E be an open subset of$\mathbb{R}^{\boldsymbol{n}}andf: E \to \mathbb{R}^m$be a function**.** If $f \in C'(E)$then partial derivatives $D_j f_i(x)$ exists and are continuous $\forall x \in E, 1 \leq i \leq m,$and $1 \leq j \leq n$.
- Directional derivative of f at x in the direction of unit vector u is denoted by $(D_u f)(x)$ and is given as $(D_u f)(x) = \lim_{h \to 0} \frac{f(x+uh)-f(x)}{h}$
- Let X be any non-empty set and $T: X \to X$. A point $x_0 \in X$ is said to be a fixed point of T if $T(x_0) = x_0$.
- Let (X, d) be a metric space. A function $f: X \to X$ is said to be a contraction mapping if there exists a real number α with $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y) \ \forall x, y \in X$.
- Contraction map f is uniformly continuous on X.
- Let (X, d) be a complete metric space and let ϕ be a contraction mapping on X. Then there exists one and only one $x \in X$ such that $\phi(x) = x$.

Keywords

Norm of a Linear transformation: Let $A \in L(\mathbb{R}^{\boldsymbol{n}}, \mathbb{R}^{\boldsymbol{m}})$i.e. A is a linear transformation from $\mathbb{R}^{\boldsymbol{n}}$ **to** $\mathbb{R}^{\boldsymbol{m}}$.

Then norm of A, denoted by $\|A\|$ is defined as

$$\|A\| = \sup_{\substack{|x| \leq 1 \\ x \in \mathbb{R}^n}} | Ax |$$

Open Ball: Let (X, d) be a metric space, the open ball of radius $r > 0$ centered at a point a in X, usually denoted by $B_r(a) or\ B(a; r)$ and is defined as

$$B(a; r) = \{x \in X: d(x, a) < r\}.$$

Open Set:A subset E of a metric space (X,d) is open if $\forall x \in E$, there exists an open ball $B(x; r)$ such that $B(x; r) \subseteq E$.

Convex Set:A set $E \subseteq \mathbb{R}^n$ is said to be convex if $x \in E, y \in E \Rightarrow tx + (1 - t)y \in E, \forall t \in [0,1]$.

Continuously Differentiable Function: Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^m$ be a function. Then f is said to be continuously differentiable in E if $f': E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function.

Directional Derivative: Directional derivative of f at x in the direction of unit vector u is denoted by $(D_u f)(x)$ and is given as

$$(D_u f)(x) = \lim_{h \to 0} \frac{f(x + uh) - f(x)}{h}$$

Fixed Point: Let X be any non-empty set and $T: X \to X$. A point $x_0 \in X$ is said to be a fixed point of T if $T(x_0) = x_0$.

Contraction Mapping: Let (X, d) be a metric space. A function $f: X \to X$ is said to be a contraction mapping if there exists a real number α with $0 \leq \alpha < 1$ such that

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y) \ \ \forall x, y \in X.$$

Self Assessment

1) Suppose T is a linear operator and is one-one on a finite-dimensional vector space then T is not necessarily onto.

A. True

B. False

2) Suppose $L(X) = \{T | T: X \to X$ is a linear operator$\}$. Then $L(X)$ is a vector space.

A. True

B. False

3) Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, then

A. $\|T\| = \sup_{|x| \geq 1} |Tx|$

B. $\|T\| = \inf_{|x| \leq 1} |Tx|$

C. $\|T\| = \inf_{|x| \geq 1} |Tx|$

D. $\|T\| = \sup_{|x| \leq 1} |Tx|$

4) Let $T_1, T_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$. Consider the following statements:

(I) $\|T_1\| + \|T_2\| \leq \|T_1 + T_2\|$

(II) $\|cT_1\| = c\|T_1\|, c \in \mathbb{R}$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

5) A set $E \subseteq \mathbb{R}^k$ is said to be convex if $\lambda x + (1 - \lambda)y \in E$ for some $x \in E, y \in E and \lambda \in [0, 1]$.

A. True

B. False

6) Open balls are convex.

A. True

B. False

7) Consider the following statements:

(I) Let $T_1 \in L(\mathbb{R}^n, \mathbb{R}^m)$and $T_2 \in L(\mathbb{R}^m, \mathbb{R}^k)$ then $\|T_2 T_1\| \leq \|T_2\| \|T_1\|$.

(II) If $\lambda x + (1 - \lambda)y \in E$ whenever $x \in E, y \in E \ and \ \lambda \in [0, 1]$ then E is said to be convex.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

8) $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, then

A. $|Tx| \leq \|T\||x|$ only if $|x| \leq 1$
B. $|Tx| \leq \|T\||x|$ only if $|x| \geq 1$
C. $|Tx| \leq \|T\||x|$ whenever $x \in \mathbb{R}^n$
D. none of these

9)Suppose f is a differentiable mapping of $(a, b) \subset \mathbb{R}^1$ into $\mathbb{R}^m$, and $x \in [a, b]$. Consider the following statements.

(I) $f'(x)$ is the linear transformation of $\mathbb{R}^1$ into $\mathbb{R}^m$.

(II) $f'(x)$ satisfies $\lim_{h \to 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0$

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

10) Let $f: [a, b] \to \mathbb{R}^k$ be a continuous function such that f is differentiable on (a, b). Then there exists $x \in (a, b)$ such that $\frac{|f(b) - f(a)|}{|b-a|} \leq |f'(x)|$.

A. True
B. False

11) Let E be a convex set. Define $\phi(\lambda) = \lambda a + (1 - \lambda)b,\ \ a \in E, b \in E and \lambda \in [0, 1]$. Then $\phi(\lambda)$ need not be an element of E.

A. True
B. False

12) Let $f: R^2 \to R^2$ be given by $f(x, y) = (x^2, y^2 + \sin x)$. Then the derivative of f at (x, y) is the linear transformation given by:

A. $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$
B. $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$
C. $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$
D. $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

13) Let$T: R \to R$ be a mapping defined by $T(x) = \frac{x^2 - x + 4}{4}$, then fixed points of T are:

A. 1, -4
B. -1, -4
C. -1, 4

D. 1, 4

14) Let$T: R \to R$ be a mapping defined by $T(x) = \frac{x^2+3}{4}$, then fixed points of T are:

A. 1, 3
B. -1, -3
C. -1, 3
D. 1, -3

15) Let$T: R \to R$ be a mapping defined by $T(x) = \frac{x^2+6}{5}$, then fixed points of T are:

A. 2, 3
B. -2, -3
C. -2, 3
D. 2, -3

16) Contraction map $f: X \to X$ is uniformly continuous on X.

A. True
B. False

17)If $f(x) = x^2, 0 \leq x < \frac{1}{2}$. Then f is a contraction mapping on $\left[0, \frac{1}{2}\right]$.

A. True
B. False

Answers for Self Assessment

1.	B	2.	A	3.	D	4.	D	5.	B
6.	A	7.	C	8.	C	9.	C	10.	A
11.	B	12.	A	13.	D	14.	A	15.	A
16.	A	17.	C						

Review Questions

1)Let $f: R^2 \to R^2$ be given by $f(x, y) = (x^2, y^2 + cos\, x)$. Then find the derivative of f at (x, y)in matrix form.

2) Let$T: R \to R$ be a mapping defined by $T(x) = \frac{x^2-x+5}{5}$, then find the fixed points of T.

3) Let f(x,y)= $\log\left(\cos^2\left(e^{x^2}\right)\right) + sin(\, x + y)$. Then find $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y)$.

4) Let f(x,y)= $\log\left(sin\sqrt{1 - x^2}\right) + sin(\, x + y)$. Then find $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y)$.

5) Let $f(x, y) = x^{sin\, x} + e^{x+y}$. Then find $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y)$.

6)Let $f(x,y) = x^{cos^{-1}x} + log(x+y)$. Then find $\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y)$.

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Monika Arora, Lovely Professional University

Unit 13: The Inverse Function Theorem and the Implicit Function Theorem

CONTENTS

Objectives

After studying this unit, students will be able to:

- Discuss inverse function theorem
- discuss elements of $L(\mathbb{R}^{n+m}, \mathbb{R}^n)$
- describe implicit function theorem

Introduction

In this chapter we will discuss Inverse Function Theorem and Implicit Function Theorem. Roughly speaking, the inverse function theorem states that a continuously differentiable mapping f is invertible in neighborhood of any point x at which the linear transformation $f'(x)$ is invertible.

13.1 The Inverse Function Theorem

Theorem 13.1.1 (Inverse Function Theorem) suppose f is a C' −mapping of an open set $E \subset \mathbb{R}^n$ and $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then

(a) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U, b \in V, f$ is 1-1 on U and $f(U) = V$.

(b) if g is the inverse of f, defined in V by $g\big(f(x)\big) = x, x \in U$ then $g \in \mathrm{C}'(V)$.

Proof: (a)We put $f'(a) = A$ such that $A \subset L(\mathbb{R}^n)$ is invertible and let

$$\lambda = \frac{1}{2\|A^{-1}\|} \qquad \text{... (1)}$$

Since $f \in \mathrm{C}'(E)$.

$\Rightarrow f'$ is continuous mapping of E into $L(\mathbb{R}^n)$.

$\Rightarrow f'$ is continuous at a as $a \in E$.

Therefore, for given $\lambda > 0$, there exists an open ball

$$U = B(a, r_1) \subset E$$

such that

$$\|f'(x) - f'(a)\| < \lambda \; \forall x \in U$$

$$\Rightarrow \quad \|f'(x) - A\| < \lambda \;\forall x \in U \qquad \ldots(2)$$

$$[by(1)]$$

Now for each $y \in \mathbb{R}^n$, define a function

$$\phi: E \to \mathbb{R}^n$$

by

$$\phi(x) = x + A^{-1}\big(y - f(x)\big) \qquad \ldots(3)$$

Then

$$\begin{aligned} \phi(x) &= x \\ \Leftrightarrow x + A^{-1}\big(y - f(x)\big) &= x \\ \Leftrightarrow A^{-1}\big(y - f(x)\big) &= 0 \\ \Leftrightarrow y - f(x) &= 0 \\ \Leftrightarrow y &= f(x) \end{aligned}$$

Thus $f(x) = y$ if and only if x is a fixed point of ϕ.… (4)

Now

$$\begin{aligned} \phi'(x) &= I - A^{-1}\big(f'(x)\big) \\ &= A^{-1}A - A^{-1}\big(f'(x)\big) \\ &= A^{-1}[A - f'(x)] \\ \Rightarrow \|\phi'(x)\| &= \|A^{-1}[A - f'(x)]\| \\ &\leq \|A^{-1}\|\|A - f'(x)\| \\ &= \frac{1}{2\lambda}\|A - f'(x)\| \qquad \{by\ (1)\} \\ &= \frac{1}{2\lambda}(\lambda) \qquad \{by\ (2)\} \\ &= \frac{1}{2} \\ \Rightarrow \|\phi'(x)\| &< 1/2, x \in U \end{aligned}$$

Now,U is an open ball.

$\Rightarrow U$ is an open set and hence it is convex also.

We know that if f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E, and there is a real number M such that $\|f'(x)\| \leq M$ for every $x \in E$ then

$$|f(b) - f(a)| \leq M|b - a|, a, b \in E.$$

Therefore,

$$|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2}|x_1 - x_2|, x_1, x_2 \in U \qquad \ldots(5)$$

Now we show that f is one-one in U.

Let

$$f(x_1) = f(x_2) = y\ (say)$$

$$\Rightarrow y = f(x_1), \qquad y = f(x_2)$$

$\Rightarrow x_1, x_2$ are fixed points of ϕ.

$\Rightarrow \phi(x_1) = x_1, \qquad \phi(x_2) = x_2$

Therefore from (5), we have

$$|x_1 - x_2| \leq \frac{1}{2}|x_1 - x_2|$$

$$\Rightarrow \frac{1}{2}|x_1 - x_2| \leq 0$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow f \text{ is one-one in } U.$$

Next, we put $V = f(U)$ and choose $y_0 \in V$ then $y_0 = f(x_0)$ for some $x_0 \in U$.

Since U is an open set and $x_0 \in U$.

Therefore, there exists an open ball $B = B(x_0, r)$ such that $x_0 \in B(x_0, r) \subset U$.

Here we consider r so small that its closure $\bar{B}$ lies in U.

We will prove that $V = f(U)$ is an open set and it will be so if, for each $y \in V$, there exists an open ball contained in V.

Let$y \in B(y_0, \lambda r)$

$\Rightarrow |y - y_0| < \lambda r$

Now,

$$\begin{aligned}
|\phi(x_0) - x_0| &= \left|x_0 + A^{-1}\left(y - f(x_0)\right) - x_0\right| \\
&= \left|A^{-1}\left(y - f(x_0)\right)\right| \\
&\leq \|A^{-1}\||y - f(x_0)| \\
&< \left(\frac{1}{2\lambda}\right)\lambda r \\
&= \frac{r}{2} \qquad \ldots (6)
\end{aligned}$$

Also, for $x \in \bar{B}$, we have

$$\begin{aligned}
|\phi(x) - x_0| &= |\phi(x) - \phi(x_0) + \phi(x_0) - x_0| \\
&\leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| \\
&< \frac{1}{2}|x - x_0| + \frac{r}{2} \\
\Rightarrow |\phi(x) - x_0| &< r \\
\Rightarrow \phi(x) &\in B(x_0, r) \\
\Rightarrow \phi(x) &\in B.
\end{aligned}$$

Now, $B \subseteq \bar{B}$

$\Rightarrow \phi(x) \in \bar{B}\ \forall x \in \bar{B}$.

$\Rightarrow \phi: \bar{B} \to \bar{B}$ is a contraction mapping.

Since $\bar{B}$ is a closed subset of the complete metric space $\mathbb{R}^n$ and a closed subset of complete metric space is complete.

$\Rightarrow \bar{B}$ is complete.

Now fixed-point theorem states that if X is a complete metric space and if ϕ is a contraction mapping of X into X then there exists one and only one $x \in X$ such that $\phi(x) = x$.

Therefore, there exists unique $x \in \bar{B}$ such that $\phi(x) = x$.

By (4), we have x is a fixed point of ϕ if and only if $f(x) = y$.

Therefore,

$$y = f(x) \Rightarrow y \in f(\bar{B}) \subset f(U) = V$$

Thus, $y \in B(y_0, \lambda r) \Rightarrow y \in V$

$\Rightarrow B(y_0, \lambda r) \subseteq V$

Hence for each $y_0 \in V$, there exists an open ball $B(y_0, \lambda r)$ such that $B(y_0, \lambda r) \subseteq V$.

$\Rightarrow V = f(U)$ is an open set.

This proves part (a) of the theorem.

(b) Given $g: V \to U$ is the inverse of f.

We will prove that $g \in C'(V) i.e. g'$ is continuous on V.

Let $y \in V, y + k \in V$.

Then there exists $x \in U, x + h \in U$, so that

$$y = f(x), y + k = f(x + h).$$

Now by (3),

$$\phi(x) = x + A^{-1}(y - f(x))$$

Therefore,

$$\begin{aligned}
\phi(x+h) - \phi(x) &= [x + h + A^{-1}(y - f(x+h))] - [x + A^{-1}(y - f(x))] \\
&= h + A^{-1}[y - f(x+h) - y + f(x)] \\
&= h - A^{-1}[f(x+h) - f(x)] \\
&= h - A^{-1}[y + k - y] \\
\Rightarrow \phi(x+h) - \phi(x) &= h - A^{-1}k. \qquad \ldots (7)
\end{aligned}$$

Now from (5), we have

$$|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2}|x_1 - x_2|, x_1, x_2 \in U$$

$$\Rightarrow |\phi(x+h) - \phi(x)| \leq \frac{1}{2}|x + h - x|$$

$$\Rightarrow |h - A^{-1}k| \leq \frac{1}{2}|h| \{ by\ (7)\}$$

$$\Rightarrow |h| - |A^{-1}k| \leq \frac{1}{2}|h|$$

$$\begin{aligned}
\Rightarrow \frac{1}{2}|h| &\leq |A^{-1}k| \\
&\leq \|A^{-1}\||k| \\
&= \frac{1}{2\lambda}|k|
\end{aligned}$$

$$\Rightarrow |h| \leq \frac{1}{\lambda}|k|.$$

Now from (2),

$$\|f'(x) - A\| < \lambda = \frac{1}{2\|A^{-1}\|}$$

$$\Rightarrow \|f'(x) - A\|\|A^{-1}\| < \frac{1}{2}$$

$$\Rightarrow \|f'(x) - A\|\|A^{-1}\| < 1$$

$\Rightarrow f'(x)$ is invertible since if Ω be the set of all invertible linear operators on $\mathbb{R}^n$ and if $a \in \Omega, B \in L(\mathbb{R}^n)$ and $\|B - A\|\|A^{-1}\| < 1$ then $B \in \Omega$ *i.e.* B is invertible.

Let $\left(f'(x)\right)^{-1} = T$

Now,

$$\begin{aligned} g(y+k) - g(y) - Tk &= x + h - x - Tk \\ &= Ih - Tk \\ &= TT^{-1}h - Tk \\ &= T\left(f'(x)\right)h - Tk \\ &= -T(k - f'(x)h) \\ &= -T(f(x+h) - f(x) - f'(x)h \end{aligned}$$

$$\begin{aligned} \Rightarrow |g(y+k) - g(y) - Tk| &= |-T(f(x+h) - f(x) - f'(x)h| \\ &\leq \|T\||f(x+h) - f(x) - f'(x)h| \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{|g(y+k) - g(y) - Tk|}{|k|} &\leq \frac{\|T\||f(x+h) - f(x) - f'(x)h|}{|k|} \\ &\leq \frac{\|T\||f(x+h) - f(x) - f'(x)h|}{\lambda|h|} \end{aligned}$$

Now, $h \to 0$ as $k \to 0$ and $\dfrac{|f(x+h) - f(x) - f'(x)h|}{|h|} \to 0$ as $h \to 0$

Therefore,

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} \to 0 \text{ as } k \to 0$$

$\Rightarrow g$ is differentiable in V and

$$g'(y) = T = \left(f'(x)\right)^{-1}$$

$$\Rightarrow g'(y) = \left(f'(g(y))\right)^{-1}.$$

Now $g: V \to U$ is differentiable on V.

$\Rightarrow$ It is continuous on V and f' is continuous mapping of U into the set Ω of all invertible elements of $L(\mathbb{R}^n)$ and that inversion is a continuous mapping of Ω onto Ω.

Therefore $g \in C'(V)$.

This completes the proof.

13.2 Notation

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$ then the point

$(x, y) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \in \mathbb{R}^{n+m}$.

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$, then for $(h, k) \in \mathbb{R}^{n+m}$,

$$\begin{aligned} A(h,k) &= A((h,0) + (0,k)) \\ &= A(h,0) + A(0,k). \end{aligned}$$

Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into linear transformations $A_x: \mathbb{R}^n \to \mathbb{R}^n$ and $A_y: \mathbb{R}^m \to \mathbb{R}^n$ defined respectively by

$$A_x h = A(h, 0), A_y k = A(0, k) \text{ for any } h \in \mathbb{R}^n \text{ and } k \in \mathbb{R}^m.$$

Then $A_x \in L(\mathbb{R}^n), A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $A(h, k) = A_x h + A_y k$.

Theorem 13.2.1: If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible, then there corresponds to every$k \in \mathbb{R}^m$,a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.

Proof: $A(h, k) = 0$

$\Rightarrow A_x h + A_y k = 0$

$\Rightarrow A_x h = -A_y k$

$\Rightarrow h = -A_x^{-1} A_y k$.

Fix k. Suppose there exists $h_1, h_2 \in \mathbb{R}^n$ such that

$A(h_1, k) = 0 \text{ } and \text{ } A(h_2, k) = 0$

$\Rightarrow A(h_1, k) = A(h_2, k)$

$\Rightarrow A[(h_1, k) - (h_2, k)] = 0$

$\Rightarrow A[(h_1 - h_2), 0] = 0$

$\Rightarrow A_x(h_1 - h_2) = 0 \quad \{\because A_x h = A(h, 0)\}$

But A_x is invertible.

$\therefore h_1 - h_2 = 0$

$\Rightarrow h_1 = h_2$

Thus $\forall k \in \mathbb{R}^m$, there exists a unique $h = -A_x^{-1} A_y k$such that $A(h, k) = 0$.

Theorem 13.2.2 (Implicit Function Theorem): Let $f: E \to \mathbb{R}^n$ be a C' –mapping, where E is an open subset of $\mathbb{R}^{n+m}$ such that $f(a, b) = 0$ for some point$(a, b) \in E$. Put $A = f'(a, b)$ and assume that A_x is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$ with $(a, b) \in U$ and $b \in W$ such that to every $y \in W$ there corresponds a unique x such that $(x, y) \in U$ and $f(x, y) = 0$.

Further, if this x is defined to be $g(y)$ then g is a C' – mapping of W into $\mathbb{R}^n, g(b) = a, f(g(y), y) = 0, y \in W$ and $g'(b) = -A_x^{-1} A_y$.

Proof: Define $F: E \to \mathbb{R}^{n+m}$ by $F(x, y) = (f(x, y), y), (x, y) \in E$ ... (1)

$\Rightarrow F$ is a C' – mapping of E into $\mathbb{R}^{n+m}$ as $f \in C'(E)$.

We claim that $F'(a, b)$ is an invertible element of $L(\mathbb{R}^{n+m})$.

Since $f'(a, b) = A$

$\therefore f(a + h, b + k) - f(a, b) = A(h, k) + r(h, k)$, where r is the remainder that occurs in the definition of $f'(a, b)$.

$$\Rightarrow f(a + h, b + k) = A(h, k) + r(h, k) \text{ ... (2)}$$

$$\because f(a, b) = 0$$

Now,

$$F(a + h, b + k) - F(a, b) = (f(a + h, b + k), b + k) - (f(a, b), b)$$

$$= (A(h, k) + r(h, k), b + k) - (0, b)$$

$$= (A(h, k) + r(h, k), k)$$

$$F(a + h, b + k) - F(a, b) = (A(h, k), k) + (r(h, k), 0) \quad \text{... (3)}$$

$$\Rightarrow F'(a, b)(h, k) = (A(h, k), k) \quad \text{... (4)}$$

$$\therefore F'(a, b)(h, k) = 0$$

$$\Rightarrow (A(h, k), k) = 0$$

$$\Rightarrow A(h, k) = 0, k = 0$$

$$\Rightarrow A(h,0) = 0, k = 0$$
$$\Rightarrow A_x h = 0$$

But A_x is invertible.

Therefore, $h = 0$ and hence $(h,k) = (0,0)$.

Thus $F'(a,b)$ is invertible.

∴ by inverse function theorem, there exists open subsets U and V of $\mathbb{R}^{n+m}$ such that $(a,b) \in U, (0,b) \in V$ and F is 1-1 mapping of U onto V.

Define $W = \{y \in \mathbb{R}^m : (0,y) \in V\}$.

Since $(a,b) \in U$

$$\Rightarrow F(a,b) \in V$$
$$\Rightarrow (f(a,b),b) \in V$$
$$\Rightarrow (0,b) \in V$$
$$\Rightarrow b \in W.$$

Also, W is open as V is open.

Now, let $y \in W$

$\Rightarrow (0,y) \in V$ and $F: U \to V$ is onto.

∴ there exists some $(x,y) \in U$ such that

$$F(x,y) = (0,y)$$
$$\Rightarrow (f(x,y),y) = (0,y)$$
$$\Rightarrow f(x,y) = 0$$

Thus $\forall y \in W$, there exists $(x,y) \in U$ such that $f(x,y) = 0$.

To prove the uniqueness of x, let there exist x_1, x_2 such that

$$f(x_1,y) = 0 = f(x_2,y)$$
$$\Rightarrow F(x_1,y) = (f(x_1,y),y) = (0,y)$$

and

$$F(x_2,y) = (f(x_2,y),y) = (0,y)$$
$$\Rightarrow F(x_1,y) = F(x_2,y)$$
$$\Rightarrow (x_1,y) = (x_2,y)$$
$$\Rightarrow x_1 = x_2.$$

Hence for all $y \in W$, there exists a unique x such that $(x,y) \in U$ and $f(x,y) = 0$.

Further, let $g: W \to \mathbb{R}^n$ be defined as $g(y) = x$ such that $f(x,y) = 0$.

Since for all $y \in W$, there exists a unique $x \in \mathbb{R}^n$ such that $f(x,y) = 0$

$i.e., g(y) = x$, therefore $g: W \to \mathbb{R}^n$ is a well-defined function.

Now,

$$F(g(y),y) = (f(g(y),y),y) = (f(x,y),y) = (0,y).$$

∴ by inverse function theorem, if $G: V \to U$ is the inverse of $F: U \to V$ then $G \in C'(V)$.

Also, $(g(y),y) = F^{-1}(0,y) = G(0,y)$.

∴ g is also C' −mapping.

Here $g(b) = a$ as $f(a,b) = 0$, and $f(g(y),y) = f(x,y) = 0 \ \forall y \in W$.

Now, we show that $g'(b) = -A_x^{-1} A_y$.

Let

$$\phi(y) = (g(y), y).$$

Then

$$\phi'(y)(k) = (g'(y)k, k) \forall y \in W, k \in \mathbb{R}^m$$

and

$$f(\phi(y)) = f(g(y), y) = 0.$$

∴ by chain rule,

$$f'(\phi(y))\phi'(y) = 0$$

$$\Rightarrow f'(\phi(b))\phi'(b) = 0, y = b$$

$$\Rightarrow f'(g(b), b)\phi'(b) = 0$$

$$\Rightarrow f'(a, b)\phi'(b) = 0 \quad \{\because g(b) = a\}$$

$$\Rightarrow A\phi'(b) = 0$$

$$\Rightarrow A\phi'(b)k = 0 \ \forall k \in \mathbb{R}^m$$

$$\Rightarrow A(g'(b)k, k) = 0 \ \forall k \in \mathbb{R}^m$$

$$\Rightarrow A_x(g'(b)k, k) + A_y(g'(b)k, k) = 0 \ \forall k \in \mathbb{R}^m$$

$$\Rightarrow A_x g'(b) + A_y = 0$$

$$\Rightarrow A_x g'(b) = -A_y$$

$$\Rightarrow g'(b) = -A_x^{-1} A_y.$$

This completes the proof.

Summary

- Suppose f is a C' −mapping of an open set $E \subset \mathbb{R}^n$ and $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then

(a) There exist open sets U and V in $\mathbb{R}^n$ such that $a \in U, b \in V, f$ is 1-1 on U and $f(U) = V$.

(b) If g is the inverse of f, defined in V by $g(f(x)) = x, x \in U$ then $g \in C'(V)$.

- If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible, then there corresponds to every $k \in \mathbb{R}^m$, a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.
- Let $f: E \to \mathbb{R}^n$ be a C' −mapping, where E is an open subset of $\mathbb{R}^{n+m}$ such that $f(a, b) = 0$ for some point$(a, b) \in E$. Put $A = f'(a, b)$ and assume that A_x is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$ with $(a, b) \in U$ and $b \in W$ such that to every $y \in W$ there corresponds a unique x such that $(x, y) \in U$ and $f(x, y) = 0$.
  Further, if this x is defined to be $g(y)$ then g is a C' − mapping of W into $\mathbb{R}^n, g(b) = a, f(g(y), y) = 0, y \in W$ and $g'(b) = -A_x^{-1} A_y$.

Keywords

Inverse Function Theorem

Suppose f is a C' −mapping of an open set $E \subset \mathbb{R}^n$ and $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then

(a) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U, b \in V, f$ is 1-1 on U and $f(U) = V$.

(b) if g is the inverse of f, defined in V by $g(f(x)) = x, x \in U$ then $g \in C'(V)$.

Implicit Function Theorem: Let $f: E \to \mathbb{R}^n$ be a C' −mapping, where E is an open subset of $\mathbb{R}^{n+m}$ such that $f(a, b) = 0$ for some point$(a, b) \in E$. Put $A = f'(a, b)$ and assume that A_x is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$ with $(a,b) \in U$ and $b \in W$ such that to every $y \in W$ there corresponds a unique x such that $(x,y) \in U$ and $f(x,y) = 0$. Further, if this x is defined to be $g(y)$ then g is a C' − mapping of W into $\mathbb{R}^n, g(b) = a, f(g(y),y) = 0, y \in W$ and $g'(b) = -A_x^{-1}A_y$.

Self-Assessment

1) Let A be an open set and $x_0 \in A$. Then there exists an open ball $B(x_0,r)$ such that $x_0 \in B(x_0,r) \subset A$.

A. True

B. False

2) Suppose f is a real function on $(-\infty,\infty)$. Then x is said to be fixed point of f if $f(x) = k, k\ is\ any\ constant$.

A. True

B. False

3) Suppose B is an open ball then $\lambda x + (1-\lambda)y \in B$ whenever $x \in B, y \in B and \lambda \in [0,1]$.

A. True

B. False

4)Let X be a metric space, with metric d. If ϕ maps X into X and if there is a number $c > 1$ such that $d(\phi(x),\phi(y)) \leq cd(x,y) forall\ x,y \in X$, then ϕ is said to be a contraction of X into X.

A. True

B. False

5) Consider the following statements:

(I) Let $\bar{A}$ denotes the closure of A. Then $\bar{A}$ is closed.

(II) Closed subset of a complete metric space is not necessarily complete.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

6) If X is a complete metric space, and if ϕ is a contraction of X into X, then there exists more than one $x \in X$ such that $\phi(x) = x$.

A. True

B. False

7) If $T \in L(\mathbb{R}^n,\mathbb{R}^m)$, then $\|T\||x| \leq |Tx|$ whenever $x \in \mathbb{R}^n$.

A. True

B. False

8) If $\lim\limits_{k\to 0} \frac{|f(x+k)-f(x)-Tk|}{|k|} = 0$ then $f'(x) = T$.

A. True

B. False

9) Let T_1 be an invertible linear operator on $\mathbb{R}^n, T_2$ be a linear operator on $\mathbb{R}^n$, and $\|T_2 - T_1\| \cdot \|T_1^{-1}\| < 1$ then T_2 is also an invertible linear operator on $\mathbb{R}^n$.

A. True

B. False

10) Let $T \in L(\mathbb{R}^{n+m}, \mathbb{R}^n), T_x h = T(h, 0)$, and $T_y k = T(0, k)$ for any $h \in \mathbb{R}^n, k \in \mathbb{R}^m$. Consider the following statements:

(I) T can be split into two linear transformations T_x and T_y.

(II) $T_x \in L(\mathbb{R}^n)$ and $T_y \in L(\mathbb{R}^m)$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

11) Consider the following statements:

(I) A differentiable mapping f of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is said to be continuously differentiable in E if f' is a continuous mapping of E into $L(\mathbb{R}^n, \mathbb{R}^m)$.

(II) Let f be a continuously differentiable mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ then for every $x \in E$ and to every $\in > 0$, there exists a $\delta > 0$ such that $\|f'(y) - f'(x)\| < \in$ whenever $y \in E$ and $|x - y| < \delta$.

A. only (I) is correct

B. only (II) is correct

C. both (I) and (II) are correct

D. both (I) and (II) are incorrect

Answers for Self Assessment

1.	A	2.	B	3.	A	4.	B	5.	A
6.	B	7.	B	8.	A	9.	A	10.	A
11.	C	12.		13.		14.		15.	

Review Questions

1) Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^n$ be a C' − mapping. If $f'(x)$ is invertible for every $x \in E$, then f is an open mapping i.e.,$f(W)$ is open subset of $\mathbb{R}^n$ for every open set W of E.

2) If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible, then there corresponds to every $k \in \mathbb{R}^m$, a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.

Further Readings

Walter Rudin, Principles of Mathematical Analysis (3rd edition), McGraw-Hill International Publishers.

T. M. Apostol, Mathematical Analysis (2nd edition).

S.C. Malik, Mathematical Analysis.

Shanti Narayan, Elements of Real Analysis

Web Links

https://onlinecourses.nptel.ac.in/noc21_ma63/preview

Unit 14: Addition and Multiplication of Series

CONTENTS

Objectives

After studying this unit, students will be able to:

- define addition of two convergent series
- understand Cauchy product of two series
- describe that Cauchy product of two convergent series may be divergent
- understand that Cauchy product of two divergent series may be convergent
- explain various theorems related to Cauchy product

Introduction

If we have given two convergent series then we can add them term by term, and the resulting series converges to the sum of the two series. But in case of multiplication, the situation is little complex. For this, we have to define the product. This can be done in many ways but we will concentrate only one of them, namely "Cauchy product."

Theorem 14.1.1: If $\sum a_n = A$ and $\sum b_n = B$
then

$$\sum a_n + b_n = A + B \text{ and } \sum c a_n = cA \text{ for any fixed } c$$

Proof: Let $A_n = \sum_{k=0}^{n} a_k,\ B_n = \sum_{k=0}^{n} b_k,$

then $A_n + B_n = \sum_{k=0}^{n} (a_k + b_k)$

Since $\lim_{n\to\infty} A_n = A$

and $\lim_{n\to\infty} B_n = B$

Therefore, we have

$$\lim_{n\to\infty} (A_n + B_n) = A + B$$

and

$$\lim_{n\to\infty} c\, A_n = cA$$

$$\Rightarrow \sum a_n + b_n = A + B \text{ and} \sum ca_n = cA$$

Thus, the two convergent series may be added term by term and the resulting series converges to the sum of two series.

14.1 Product of Series

Definition: Given $\sum a_n$ and $\sum b_n$

We put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0,1,2, \dots)$$

and call $\sum c_n$ the product of two given series.

If $\mathrm{A_n} = \sum_{k=0}^{n} a_k,\ \mathrm{B_n} = \sum_{k=0}^{n} b_k,\ \mathrm{C_n} = \sum_{k=0}^{n} c_k$ and $A_n \to A, B_n \to B,$

Then we don't have $C_n \to AB$, since $C_n \neq A_n B_n$.

Theorem 14.1.2: (Abel's Theorem)

Statement: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge to A and B respectively. If their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent then $\sum_{n=0}^{\infty} c_n$ converges to AB.

Proof: Let $\mathrm{A_n} = \sum_{k=0}^{n} a_k,\ \mathrm{B_n} = \sum_{k=0}^{n} b_k$

As $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges to A and B respectively.

$\therefore \lim_{n\to\infty} A_n = A$

and $\lim_{n\to\infty} B_n = B$

Let $\mathrm{C_n} = \sum_{k=0}^{n} c_k$ where $\sum_{n=0}^{\infty} c_n$ is the Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n, c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Now $C_n = c_0 + c_1 + c_2 + \cdots + c_n$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)$$

$$= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + a_2(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_n b_0$$

Therefore,

$$C_n = a_0 B_n + a_1 B_{n-1} + a_2 B_{n-2} + \cdots + a_n B_0$$

Or

$$C_n = a_n B_0 + a_{n-1} B_1 + a_{n-2} B_2 + \cdots + a_1 B_{n-1} + a_0 B_n$$

Here we have,

$$C_0 = a_0 B_0$$

$$C_1 = a_1 B_0 + a_0 B_1$$

$$C_2 = a_2 B_0 + a_1 B_1 + a_0 B_2$$

$$\dots \dots \dots$$

$$C_n = a_n B_0 + a_{n-1} B_1 + \cdots + a_0 B_n.$$

Therefore,

$$C_0 + C_1 + C_2 + \cdots + C_n$$

$$= (a_0 + a_1 + a_2 + \cdots + a_n) B_0 + (a_0 + a_1 + a_2 + \cdots + a_{n-1}) B_1 + \cdots + a_0 B_n$$

$$= A_n B_0 + A_{n-1} B_1 + \cdots + A_0 B_n$$

$$\Rightarrow \frac{C_0 + C_1 + C_2 + \cdots + C_n}{n+1} = \frac{A_n B_0 + A_{n-1} B_1 + \cdots + A_0 B_n}{n+1} \quad \ldots (1)$$

Since $\sum_{n=0}^{\infty} c_n$ is convergent.

So let it converges to C.

Therefore, $\lim_{n\to\infty} C_n = C$ and hence

$$\lim_{n\to\infty} \frac{C_0 + C_1 + C_2 + \cdots + C_n}{n+1} = C \quad \ldots (2)$$

Further,

$$\lim_{n\to\infty} A_n = A$$

$$\text{and } \lim_{n\to\infty} B_n = B$$

Therefore,

$$\lim_{n\to\infty} \frac{A_n B_0 + A_{n-1} B_1 + \cdots + A_0 B_n}{n+1} = AB \quad \ldots (3)$$

Using relation (1), (2) and (3), we get

$$C = AB$$

That is, the Cauchy product $\sum_{n=0}^{\infty} c_n$ converges to AB.

This completes the proof.

Example: Show with the help of examples that

(i) Cauchy product of two convergent series may be divergent and

(ii) Cauchy product of two divergent series may be convergent.

Solution: (i)Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$

Then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

is, by Leibnitz test of alternating series, is a convergent series.

$$\text{Now } c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}}$$

$$= \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}\sqrt{n-k+1}}$$

$$\Rightarrow | c_n |= \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}} \quad \ldots (1)$$

For $0 \le k \le n$;

$$(k+1)(n-k+1) = n(k+1) - k(k+1) + (k+1)$$

$$= nk - k^2 + (n+1)$$

$$= \left(\frac{n^2}{4} + n + 1\right) - \left(\frac{n^2}{4} - nk + k^2\right)$$

$$= \left(\frac{n}{2} + 1\right)^2 - \left(\frac{n}{2} - k\right)^2$$

$$\leq \left(1+\frac{n}{2}\right)^2$$

$$= \left(\frac{2+n}{2}\right)^2$$

$$\Rightarrow \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \frac{2}{2+n} \quad \dots (2)$$

From (1) and (2), we get,

$$|\, c_n \,| \geq \sum_{k=0}^{n} \frac{2}{2+n}$$

$$= \frac{2(n+1)}{2+n}$$

$$\therefore \lim_{n\to\infty} |c_n| \neq 0$$

$$\text{Hence} \sum c_n \text{is divergent.}$$

Thus $\sum a_n$ and $\sum b_n$ are convergent series but Cauchy product $\sum c_n$ is divergent.

(ii) Consider the series

$$\sum_{n=0}^{\infty} a_n \text{, where } a_0 = 1 \text{ and } \mathrm{a_n} = -\left(\frac{3}{2}\right)^n, n \in \mathbb{N}$$

and

$$\sum_{n=0}^{\infty} b_n \text{, where } b_0 = 1 \text{ and } b_n = \left(\frac{3}{2}\right)^{n-1}\left[2^n + \frac{1}{2^{n+1}}\right], n \in \mathbb{N}$$

Now,

$$\lim_{n\to\infty} a_n = -\lim_{n\to\infty} \left(\frac{3}{2}\right)^n \neq 0$$

$$\therefore \sum a_n \text{ is divergent.}$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(\frac{3}{2}\right)^{n-1}\left[2^n + \frac{1}{2^{n+1}}\right] \neq 0$$

$$\therefore \sum b_n \text{ is divergent.}$$

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

$$= (a_n b_0 + a_0 b_n) + \sum_{k=1}^{n-1} a_{n-k} b_k$$

$$= \left[-\left(\frac{3}{2}\right)^n .1 + 1.\left(\frac{3}{2}\right)^{n-1}\left(2^n + \frac{1}{2^{n+1}}\right)\right] - \sum_{k=1}^{n-1}\left(\frac{3}{2}\right)^{n-k}\left(\frac{3}{2}\right)^{k-1}\left(2^k + \frac{1}{2^{k+1}}\right)$$

$$= -\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1}\left(2^n + \frac{1}{2^{n+1}}\right) - \left(\frac{3}{2}\right)^{n-1}\sum_{k=1}^{n-1}\left(2^k + \frac{1}{2^{k+1}}\right)$$

$$= -\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1}\left(2^n + \frac{1}{2^{n+1}}\right) - \left(\frac{3}{2}\right)^{n-1}\left[\frac{2(2^{n-1}-1)}{2-1} + \frac{1}{4}\left(\frac{1-\frac{1}{2^{n-1}}}{1-\frac{1}{2}}\right)\right]$$

$$= -\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1}.2^n + \left(\frac{3}{2}\right)^{n-1}.\frac{1}{2^{n+1}} - \left(\frac{3}{2}\right)^{n-1}.2^n + \left(\frac{3}{2}\right)^{n-1}.2 - \frac{1}{2}\left(\frac{3}{2}\right)^{n-1} + \left(\frac{3}{2}\right)^{n-1}\frac{1}{2^n}$$

$$= -\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1} \cdot \frac{1}{2^n}\left[\frac{1}{2} + 1\right] + \left(\frac{3}{2}\right)^{n-1}\left[2 - \frac{1}{2}\right]$$

$$= -\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1} \cdot \frac{1}{2^n} \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^n$$

$$= \left(\frac{3}{2}\right)^n \left(\frac{1}{2}\right)^n$$

$$= \left(\frac{3}{4}\right)^n$$

$$c_0 = 1 \text{ and } \sum \left(\frac{3}{4}\right)^n \text{ is a convergent series.}$$

Hence $\sum a_n$ and $\sum b_n$ are divergent but $\sum c_n$ is convergent.

Example: Prove that Cauchy product of two series

$$3 + \sum_{n=1}^{\infty} 3^n \; and - 2 + \sum_{n=1}^{\infty} 2^n$$

is absolutely convergent although both the given series are divergent.

Solution: Given series are

$$\sum_{n=0}^{\infty} a_n = 3 + \sum_{n=1}^{\infty} 3^n$$

$$\text{where } a_0 = 3, a_n = 3^n$$

and

$$\sum_{n=0}^{\infty} b_n = -2 + \sum_{n=1}^{\infty} 2^n$$

$$\text{where } b_0 = -2, b_n = 2^n$$

The series

$$\sum_{n=1}^{\infty} 3^n = 3 + 3^2 + 3^3 + \cdots$$

and the series

$$\sum_{n=1}^{\infty} 2^n = 2 + 2^2 + 2^3 + \cdots$$

both are geometric series with common ratio greater than one and hence both are divergent.

$$\therefore \sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n \text{ are divergent series.}$$

Let $\sum_{n=0}^{\infty} c_n$ be the Cauchy product of the given series.

Then

$$c_0 = a_0 b_0 = 3(-2) = -6$$

and

$$\text{for } n \geq 1, c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$= a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

$$= 3(2^n) + 3(2^{n-1}) + 3^2(2^{n-2}) + \cdots + 3^n(-2)$$

$$= 2^n\left[3+\frac{3}{2}+\left(\frac{3}{2}\right)^2+\cdots+\left(\frac{3}{2}\right)^{n-1}\right]-2.3^n$$

$$= 2^n\left[3+\frac{\frac{3}{2}\left\{\left(\frac{3}{2}\right)^{n-1}-1\right\}}{\frac{3}{2}-1}\right]-2.3^n$$

$$= 3.2^n.\left(\frac{3}{2}\right)^{n-1}-2.3^n$$

$$= 2.3^n-2.3^n$$

$$= 0\ \forall n\in\mathbb{N}$$

$$\therefore \sum_{n=0}^{\infty} |\ c_n\ | = |-6| + \sum_{n=1}^{\infty} |\ c_n\ |$$

$$= 6+0=6.$$

Hence Cauchy product $\sum_{n=0}^{\infty} c_n$ converges absolutely.

Theorem 14.1.5: Suppose the series

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n$$

both converge absolutely and converge to the sums A and B respectively. Then their Cauchy product series

$$\sum_{n=0}^{\infty} c_n\ , c_n=\sum_{k=0}^{n} a_k b_{n-k}$$

converge to AB.

Proof: Let $\mathrm{A_n}=\sum_{k=0}^{n} a_k,\ \ \mathrm{B_n}=\sum_{k=0}^{n} b_k$

and $\alpha_\mathrm{n}=\sum_{k=0}^{n} |\ a_k\ |,\ \beta_\mathrm{n}=\sum_{k=0}^{n} |\ b_k\ |.$

Given that the series

$$\sum_{n=0}^{\infty} a_n \text{and} \sum_{n=0}^{\infty} b_n$$

are two absolutely convergent series.

$$\therefore \sum_{n=0}^{\infty} |\ a_n\ | \text{ and } \sum_{n=0}^{\infty} |\ b_n\ |$$

are convergent series and let the series

$$\sum_{n=0}^{\infty} |\ a_n\ | \text{ and } \sum_{n=0}^{\infty} |\ b_n\ |$$

converge to α and β respectively.

We know if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two non-negative term series which converge to A and B respectively, then their Cauchy product $\sum_{n=0}^{\infty} c_n$ converges to AB.

Now, since $\sum_{n=0}^{\infty} |\ a_n\ |\ \ and\ \sum_{n=0}^{\infty} |\ b_n\ |$ are series of positive terms, therefore their product

$$\left(\sum_{n=0}^{\infty} |\ a_n\ |\right)\left(\sum_{n=0}^{\infty} |\ b_n\ |\right)$$

converges to $\alpha\beta$ *i.e.*, their Cauchy product $\sum_{n=0}^{\infty} d_n$ converges to $\alpha\beta$ where $d_n=\sum_{k=0}^{n} |\ a_k\ ||\ b_{n-k}\ |$.

Therefore,

$$d_n = |a_0||b_n| + |a_1||b_{n-1}| + |a_2||b_{n-2}| + \cdots + |a_n||b_0| \quad \ldots(1)$$

Now we shall show that the Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent.

$$|c_n| = \left|\sum_{k=0}^{n} a_k b_{n-k}\right|$$

$$= |a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0|$$

$$\leq |a_0||b_n| + |a_1||b_{n-1}| + |a_2||b_{n-2}| + \cdots + |a_n||b_0|$$

$$\Rightarrow |c_n| \leq d_n \qquad \{by(1)\}$$

$$\text{Now, since } \sum_{n=0}^{\infty} d_n \text{ is convergent.}$$

$$\Rightarrow \sum_{n=0}^{\infty} |c_n| \text{ is convergent.} \quad \{\text{by comparison test}\}.$$

Hence the series $\sum_{n=0}^{\infty} c_n$ is convergent absolutely.

We know that an absolutely convergent series is convergent.

$$\therefore \sum_{n=0}^{\infty} c_n \text{is convergent.} \quad \ldots(2)$$

Further we shall prove that $\sum_{n=0}^{\infty} c_n$ converges to AB.

$$\text{Given } \sum_{n=0}^{\infty} a_n \text{converges to } A \text{ and } \sum_{n=0}^{\infty} b_n \text{ converges to } B$$

$$\therefore \lim_{n\to\infty} A_n = A \; and \; \lim_{n\to\infty} B_n = B$$

$$\Rightarrow \lim_{n\to\infty} A_n B_n = AB$$

$$\Rightarrow \text{for a given } \in > 0, \text{there exists a positive itieger} m_1 \text{ such that}$$

$$|AB - A_n B_n| < \frac{\in}{3} \; \forall n \geq m_1 \quad \ldots(3)$$

$$\text{Also the series } \sum_{n=0}^{\infty} |a_n| \text{ and } \sum_{n=0}^{\infty} |b_n| \text{ converge to } \alpha \text{ and } \beta \text{ respectively.}$$

$$\text{And their Cauchy product } \sum_{n=0}^{\infty} d_n \text{ converges to } \alpha\beta.$$

$$\therefore \lim_{n\to\infty} D_n = \alpha\beta, D_n = \sum_{k=0}^{n} d_k$$

$\Rightarrow$ for a given $\in > 0$, there exists a positive integer m_2 such that $|\alpha\beta - D_n| < \frac{\in}{3} \; \forall \, n \geq m_2 \quad \ldots(4)$

$$\text{Since } \sum_{n=0}^{\infty} |a_n| \; and \; \sum_{n=0}^{\infty} |b_n| \text{ converge to } \alpha \text{ and } \beta \text{ respectively.}$$

$$\therefore \lim_{n\to\infty} \alpha_n = \alpha \text{ and}$$

$$\lim_{n\to\infty} \beta_n = \beta$$

$$\Rightarrow \lim_{n\to\infty} \alpha_n \beta_n = \alpha\beta$$

$\therefore$ for a given $\in > 0$, there exists a positive integer m_3 such that $|\alpha_n\beta_n - \alpha\beta| < \frac{\in}{3} \; \forall \, n \geq m_3 \quad \ldots(5)$

$$\text{Let } C_n = \sum_{k=0}^{n} c_k$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)$$

$$= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0 \quad \ldots (6)$$

$$A_n B_n = \sum_{k=0}^{n} a_k \sum_{k=0}^{n} b_k$$

$$= (a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n)$$

$$= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_n) + \cdots + a_n(b_0 + b_1 + \cdots + b_n) \quad \ldots (7)$$

Now (7)-(6) gives:

$$A_n B_n - C_n = a_1 b_n + a_2(b_{n-1} + b_n) + \cdots + a_n(b_1 + \cdots + b_n)$$

Similarly, by replacing each a_k by $| a_k |$ and b_k by $| b_k |$, we have

$$\alpha_n \beta_n - D_n = | a_1 | |b_n| + |a_2|(|b_{n-1}| + |b_n|) + \cdots + |a_n|(|b_1| + \cdots + |b_n|) \quad \ldots (8)$$

Now,

$$|A_n B_n - C_n| = |a_1 b_n + a_2(b_{n-1} + b_n) + \cdots + a_n(b_1 + \cdots + b_n)|$$

$$\leq | a_1 | |b_n| + |a_2|(|b_{n-1}| + |b_n|) + \cdots + |a_n|(|b_1| + \cdots + |b_n|) \quad \ldots (9)$$

Choose $m = Max(m_1, m_2, m_3)$

Then relations (3), (4) and (5) hold for $n \geq m$.

$$\therefore |\alpha_n \beta_n - D_n| = |\alpha_n \beta_n + \alpha\beta - \alpha\beta - D_n|$$

$$\leq |\alpha_n \beta_n - \alpha\beta| + |\alpha\beta - D_n|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \quad \forall n \geq m \quad \{by (4), (5)\}$$

Therefore, from (8) and (9), we have

$$|A_n B_n - C_n| < \frac{2\epsilon}{3} \; \forall n \geq m \quad \ldots (10)$$

Now,

$$|AB - C_n| = |AB - A_n B_n + A_n B_n - C_n|$$

$$\leq | AB - A_n B_n | + | A_n B_n - C_n |$$

$$< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon \quad \forall n \geq m \quad \{by (3), (10)\}$$

$$\therefore \lim_{n \to \infty} C_n = AB$$

$$\sum_{n=0}^{\infty} c_n \text{ converges to } AB.$$

Theorem 14.1.6: (Merten's Theorem for Cauchy Product)

If(1) $\sum_{n=0}^{\infty} a_n$is absolutely convergent.

(2) $\sum_{n=0}^{\infty} a_n = A$

(3) $\sum_{n=0}^{\infty} b_n = B$

then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = AB$.

Proof: Since $\sum_{n=0}^{\infty} a_n = A$

and $\sum_{n=0}^{\infty} b_n = B$

$$\therefore \lim_{n\to\infty} A_n = A \text{ where} A_n = \sum_{k=0}^{n} a_k$$

$$\text{and } \therefore \lim_{n\to\infty} B_n = B \text{ where } B_n = \sum_{k=0}^{n} b_k$$

$$\text{Put } S_n = B_n - B$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}(B_n - B) = 0 \quad \dots (1)$$

$$\text{In the Cauchy product } \sum_{n=0}^{\infty} c_n \,, c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

Therefore,

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$$

$$\dots \dots \dots$$

$$c_n = c_2 = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$$

$$\sum_{k=0}^{n} c_k = a_0(b_0 + b_1 + \dots + b_n) + a_1(b_0 + b_1 + \dots + b_n) + \dots + a_n b_0$$

$$\Rightarrow C_n = a_0 B_n + a_1 B_{n-1} + a_2 B_{n-2} + \dots + a_n B_0$$

$$\text{where } C_n = \sum_{k=0}^{n} c_k$$

$$= \sum_{k=0}^{n} a_k B_{n-k}$$

$$= \sum_{k=0}^{n} a_k (S_{n-k} + B)$$

$$\because S_{n-k} = B_{n-k} - B$$

$$= \sum_{k=0}^{n} a_k S_{n-k} + B \sum_{k=0}^{n} a_k$$

$$= \sum_{k=0}^{n} a_k S_{n-k} + B A_n$$

$$\text{Since } \lim_{n\to\infty} B A_n = AB$$

Therefore, in order to prove that $C_n \to AB$, it will be sufficient to show that

$$\lambda_n = \sum_{k=0}^{n} a_k S_{n-k}$$

$$= \sum_{k=0}^{n} a_{n-k} S_k \text{converges to } 0 \text{ } as \text{ } n \to \infty$$

$$\text{Now, since } \sum a_n \text{converges absolutely.}$$

$$\text{Therefore, } \sum |\, a_n \,| \text{ converges and let it converges to } k.$$

$$\text{Further, } \because \lim_{n\to\infty} S_n = 0$$

$$\therefore \langle S_n \rangle \text{ converges.}$$

We know every convergent sequence is bounded.

$\Rightarrow \langle S_n \rangle$ is bounded.

$\therefore$ there exists a real number M such that $|S_n| \le M \ \forall n$ ... (2)

Since $\lim_{n\to\infty} S_n = 0$, therefore by the definition of convergence of sequence, for every $\in > 0$, there exists a positive integer m_1 such that

$$|S_n| < \frac{\in}{2k} \ \forall n > m_1 \quad \text{... (3)}$$

$$\text{Also the series } \sum_{n=0}^{\infty} |a_n| \text{ is convergent.}$$

$\therefore$ for a given $\in > 0$, there exists a positive integer m_2 such that

$$\sum_{n=m_2+1}^{\infty} |a_n| < \frac{\in}{2M} \quad \text{... (4)}$$

Let $m = \max\{m_1, m_2\}$

$\therefore$ relation (3) and (4) hold for $n > m$.

Now, whenever $n > 2m$, we have

$$|\lambda_n| = \left|\sum_{k=0}^{n} a_k S_{n-k}\right|$$

$$\le \sum_{k=0}^{n} |a_k S_{n-k}|$$

$$= \sum_{k=0}^{m} |a_k S_{n-k}| + \sum_{k=m+1}^{n} |a_k S_{n-k}|$$

$$\le \frac{\in}{2k} \sum_{k=0}^{m} |a_k| + M \sum_{k=m+1}^{n} |a_k|$$

$$\le \frac{\in}{2k} \sum_{k=0}^{\infty} |a_k| + M \sum_{k=m+1}^{\infty} |a_k|$$

$$< \frac{\in}{2k}.k + M.\frac{\in}{2M}$$

$$= \in$$

$$\therefore |\lambda_n| < \in \ \forall n > 2m$$

$$\Rightarrow \lambda_n \to 0 \ as \ n \to \infty$$

Therefore,

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k S_{n-k} + BA_n\right)$$

$$= \lim_{n\to\infty} \lambda_n + B \lim_{n\to\infty} A_n$$

$$= 0 + BA$$

$$= AB$$

$$\therefore \sum_{n=0}^{\infty} c_n \text{converges to } AB.$$

Theorem 14.1.7: If for $|x| < 1$, the series

$$\sum_{n=0}^{\infty} \alpha_n x^n$$

is absolutely convergent to $A(x)$, then show that

$$(1-x)^{-1}A(x) = \sum_{n=0}^{\infty} S_n x^n \text{ where}$$

$$S_n = \alpha_0 + \alpha_1 + \cdots + \alpha_n.$$

Deduce that

$$\sum_{n=0}^{\infty}(n+1)x^n = (1-x)^{-2}.$$

Proof: The series $\sum_{n=0}^{\infty} a_n = 1 + x + x^2 + \cdots$

$= \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$

and has the sum $\frac{1}{1-x} = (1-x)^{-1}$

Also the series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent to the sum $A(x)$.

$\therefore$ the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely for $|x| < 1$

and $\sum_{n=0}^{\infty} c_n = (1-x)^{-1}A(x)$

where $c_n = \sum_{k=0}^{n} a_{n-k} b_k$

$= \sum_{k=0}^{n} x^{n-k} \alpha_k x^k$

$= x^n \sum_{k=0}^{n} \alpha_k$

$= x^n(\alpha_0 + \alpha_1 + \cdots + \alpha_n)$

Thus $c_n = x^n S_n$

where $S_n = \alpha_0 + \alpha_1 + \cdots + \alpha_n$

$\therefore for\ |x| < 1,$ we have

$(1-x)^{-1}A(x) = \sum_{n=0}^{\infty} x^n S_n.$

Deduction: Put $\alpha_n = 1$ for all $n \geq 0$ then

$$S_n = 1 + 1 + \cdots (n+1) times$$

$$= n + 1$$

$$\text{and } A(x) = \sum_{n=0}^{\infty} \alpha_n x^n$$

$$= \sum_{n=0}^{\infty} x^n$$

$$= \frac{1}{1-x}$$

$$= (1-x)^{-1}$$

$$\therefore (1-x)^{-1}(1-x)^{-1} = \sum_{n=0}^{\infty} x^n (n+1).$$

$$or$$

$$\sum_{n=0}^{\infty}(n+1)x^n = (1-x)^{-2}$$

Example: Given $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^n}{n+1}+\cdots,$

then prove that

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+1).1}+\frac{1}{n.2}+\frac{1}{(n-1).3}+\cdots+\frac{1}{1.(n+1)} = (\log 2)^2.$$

Solution: Let

$$\sum_{n=0}^{\infty}a_n = \log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^n}{n+1}+\cdots$$

$$= \sum_{n=0}^{\infty}b_n$$

$$\because \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$$and \quad \frac{1}{n}-\frac{1}{n+1} = \frac{1}{n(n+1)} > 0$$

$$\Rightarrow \frac{1}{n} > \frac{1}{n+1}$$

$$i.e. \mid a_n \mid > \mid a_{n+1} \mid for\ all\ n.$$

$$\therefore \text{by Leibnitz test,} \sum_{n=0}^{\infty}a_n \text{ and} \sum_{n=0}^{\infty}b_n \text{are convergent series.}$$

$$\text{Let}, \sum_{n=0}^{\infty}a_n = A \text{ and}, \sum_{n=0}^{\infty}b_n = B$$

$$\text{By Abel's theorem} \sum_{n=0}^{\infty}c_n = AB, \text{provided} \sum_{n=0}^{\infty}c_n \text{ converges.} \quad \ldots(1)$$

$$c_n = \sum_{k=0}^{n}a_k b_{n-k}$$

$$= a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$$

$$= 1.\frac{(-1)^n}{n+1}+\left(-\frac{1}{2}\right)\frac{(-1)^{n-1}}{n}+\frac{1}{3}\frac{(-1)^{n-2}}{n-1}+\cdots+\frac{(-1)^n}{n+1}.1$$

$$= (-1)^n\left[\frac{1}{1.(n+1)}+\frac{1}{2.n}+\frac{1}{3(n-1)}+\cdots+\frac{1}{(n+1).1}\right] \quad \ldots(2)$$

$$= \frac{(-1)^n}{n+2}\left[\frac{n+2}{(n+1)}+\frac{n+2}{2.n}+\frac{n+2}{3(n-1)}+\cdots+\frac{n+2}{(n+1)}\right]$$

$$= \frac{(-1)^n.2}{n+2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right] \quad \ldots(3)$$

$$\Rightarrow \mid c_n \mid = \frac{2}{n+2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right]$$

$$= \frac{2(n+1)}{(n+2)}\left[\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}}{n+1}\right]$$

$$= 2\left[1-\frac{1}{n+2}\right]\left[\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}}{n+1}\right] \quad \ldots(4)$$

Now,

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}}{n+1} = 0$$

∴ from relation (4), we get

$$| c_n |= 0 \text{ as } n \to \infty$$

$$\text{Also } | c_{n+1} | - | c_n | = \frac{2}{n+3}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}+\frac{1}{n+2}\right) - \frac{2}{n+2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

$$= \left(\frac{2}{n+3}-\frac{2}{n+2}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)+\frac{2}{(n+3)(n+2)}$$

$$= \frac{-2}{(n+3)(n+2)}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)+\frac{2}{(n+3)(n+2)}$$

$$= \frac{-2}{(n+3)(n+2)}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

$$< 0$$

$$\Rightarrow | c_{n+1} | < | c_n |$$

$$\therefore \text{ by Leibinitz test, } \sum_{n=0}^{\infty} c_n \text{ converges.}$$

So, from (1) and (2), we get

$$\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{1.(n+1)}+\frac{1}{2.n}+\frac{1}{3(n-1)}+\cdots+\frac{1}{(n+1).1}\right]$$

$$= (\log 2)^2$$

$$\because \sum a_n = \sum b_n = \log 2$$

Theorem 14.1.9: Let $\sum c_n$ be the Cauchy product of two convergent series $\sum a_n$ *and* $\sum b_n$. Define

$$S_n = \sum_{k=1}^{n} a_k(b_n + b_{n-1} + \cdots + b_{n-k+1}).$$

Show that $\sum c_n$ is convergent if and only if the sequence $\{S_n\}$ converges to zero.

Proof: Let $\mathrm{A_n} = \sum_{k=0}^{n} a_k$, $\mathrm{B_n} = \sum_{k=0}^{n} b_k$ *and* $\mathrm{C_n} = \sum_{k=0}^{n} c_k$

Since the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent.

So let $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$

$\Rightarrow \lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$ ... (1)

Now $C_n = c_0 + c_1 + c_2 + \cdots + c_n$

$= a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)$

$= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + a_2(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_nb_0$

Therefore,

$$C_n = a_0B_n + a_1B_{n-1} + a_2B_{n-2} + \cdots + a_nB_0 \quad \ldots (2)$$

$$\text{Given } S_n = \sum_{k=1}^{n} a_k(b_n + b_{n-1} + \cdots + b_{n-k+1})$$

$$= a_1b_n + a_2(b_n + b_{n-1}) + \cdots + a_n(b_n + b_{n-1} + \cdots + b_1) \quad \ldots (3)$$

$$C_n + S_n = a_0B_n + a_1B_n + a_2B_n + \cdots + a_nB_n$$

$$= (a_0 + a_1 + a_2 + \cdots + a_n)B_n$$

$$= A_n B_n \quad \dots (4)$$

Suppose the sequence $\{S_n\}$ converges to zero.

$$\Rightarrow \lim_{n\to\infty} S_n = 0 \quad \dots (5)$$

From (4), we have

$$\lim_{n\to\infty}(C_n + S_n) = \lim_{n\to\infty}(A_n B_n)$$

$$\Rightarrow \lim_{n\to\infty} C_n + \lim_{n\to\infty} S_n = \lim_{n\to\infty}(A_n)\lim_{n\to\infty}(B_n)$$

$$\Rightarrow \lim_{n\to\infty} C_n = AB \quad \{by(1),(5)\}$$

$$\Rightarrow \text{Cauchy product} \sum_{n=0}^{\infty} c_n \text{ is convergent and it converges to } AB$$

Conversely, suppose Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent, then by Abel's theorem

$$\sum_{n=0}^{\infty} c_n = AB \quad \dots (6)$$

$$i.e. \lim_{n\to\infty} C_n = \lim_{n\to\infty}(A_n)\lim_{n\to\infty}(B_n) \quad \dots (7)$$

From relation (4), we have

$$\lim_{n\to\infty} C_n + \lim_{n\to\infty} S_n = \lim_{n\to\infty}(A_n)\lim_{n\to\infty}(B_n) \quad \dots (8)$$

From relation (7) and (8), we have,

$$\Rightarrow \lim_{n\to\infty} S_n = 0$$

$$\Rightarrow \{S_n\} \to 0$$

Summary

- Given $\sum a_n$ and $\sum b_n$ we put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0,1,2,\dots)$

  and call $\sum c_n$ the product of two given series.

- Abel's Theorem: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge to A and B respectively. If their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent then $\sum_{n=0}^{\infty} c_n$ converges to AB.

- Merten's Theorem: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = AB$.
- Cauchy product of two convergent series may be divergent.
- Cauchy product of two divergent series may be convergent.
- Suppose the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely and converge to the sums A and B respectively. Then their Cauchy product series $\sum_{n=0}^{\infty} c_n$, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ converge to AB.
- Given $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \cdots$, then $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1).1} + \frac{1}{n.2} + \frac{1}{(n-1).3} + \cdots + \frac{1}{1.(n+1)} = (\log 2)^2$.
- Let $\sum c_n$ be the Cauchy product of two convergent series $\sum a_n$ and $\sum b_n$. If $S_n = \sum_{k=1}^{n} a_k (b_n + b_{n-1} + \cdots + b_{n-k+1})$ $then$ $\sum c_n$ is convergent if and only if the sequence $\{S_n\}$ converges to zero.

Keywords

Product of series: Given $\sum a_n$ and $\sum b_n$

We put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0,1,2,\dots)$$

and call $\sum c_n$ the product of two given series.

Abel's Theorem: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge to A and B respectively. If their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent then $\sum_{n=0}^{\infty} c_n$ converges to AB.

Merten's Theorem:

(1) $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(2) $\sum_{n=0}^{\infty} a_n = A$

(3) $\sum_{n=0}^{\infty} b_n = B$

then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = AB$.

Self Assessment

1) Let $\sum a_n$ and $\sum b_n$ be two series and let $c_n = \sum_{k=0}^{n} a_k b_{nk}$ then $\sum c_n$ is the Cauchy product of the two-given series.

A. True

B. False

2) If the series $\sum a_n, \sum b_n, \sum c_n$ converge to $A, B, C,$ and $c_n = a_0 b_n + \cdots + a_n b_0$ then $C = AB$.

A. True

B. False

3) If $\{a_n\}$ is a sequence of real numbers and $\lim_{n\to\infty} a_n = l$ then

A. $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = \frac{1}{l}$

B. $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 2l$

C. $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = \frac{l}{2}$

D. $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l$

4) If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then

A. $\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = \frac{ab}{2}$

B. $\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = ab$

C. $\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = 2ab$

D. none of these

5) Consider the following statements:

(I) Cauchy product of two divergent series may be convergent.

(II) Cauchy product of two convergent series may be divergent.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

6) Consider the following statements:

(I) If $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

(II) If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ is always convergent.

A. only (I) is correct
B. only (II) is correct
C. both (I) and (II) are correct
D. both (I) and (II) are incorrect

7) $\sum \left(\frac{2}{3}\right)^n$ is a convergent series.

A. True

B. False

8) $\sum 5^n$ is a divergent series.

A. True

B. False

9) If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

A. True

B. False

10) If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ converges non-absolutely.

A. True

B. False

11) The series $\sum \frac{(-1)^n}{n}$ converges absolutely.

A. True

B. False

12) The series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges.

A. True

B. False

13) Suppose $\sum a_n = A, \sum b_n = B, c_n = \sum_{k=0}^{n} a_k b_{n-k}, (n = 0,1,2, \dots)$ then $\sum c_n = \text{AB}$.

A. True

B. False

14) Let $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$, then $\sum a_n$ is divergent series.

A. True

B. False

15) $log2 =$

A. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1} + \cdots$

B. $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \cdots$

C. $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n+1} - \cdots$

D. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \cdots$

Answer for Self Assessment

1.	B	2.	A	3.	D	4.	B	5.	C
6.	A	7.	A	8.	A	9.	A	10.	A
11.	A	12.	A	13.	B	14.	B	15.	D

Review Questions

1) Show that the Cauchy product of the two divergent series

$$\sum_{n=0}^{\infty} a_n = 2 + \sum_{n=1}^{\infty} 2^n \ and \ \sum_{n=0}^{\infty} b_n = -1 + \sum_{n=1}^{\infty} 1^n \ \ is\ convergent.$$

2) Show that the Cauchy product of the convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \ with\ itself\ is\ not\ convergent.$$

3) Give example to show that Cauchy product of two divergent series may be convergent.

4) Check the convergence of the series

$$\sum_{n=0}^{\infty} n!\, x^n$$

5) Check the convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Further Readings

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Web Links

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