Mathematics For Economist

DEQTT501

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LOVELY PROFESSIONAL UNIVERSITY

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CONTENTS

Unit 01: Function

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Objectives

After studying this unit, you will be able to,

- Know the Functions and Related Quantities.
- Determine the Value of Functions.
- Know the Definition of Functions by Mapping.
- Determine the Domain and Range of Functions.
- Understand the use of Functions in Economics.

Introduction

In many practical situations, the value of one quantity may depend on the value of a second. For example, the consumer demand for beef may depend on the current market price; the amount of air pollution in a metropolitan area may depend on the number of cars on the road; or the value of a rare coin may depend on its age. Thus, volume of a cylinder depends on its radius, area of square and volume of a cuboid depend on the length of its arm. The distance covered by a running train in dynamic velocity depends on the time taken. Velocity of a falling particle depends on the distance covered by it. Atmospheric pressure of a certain place depends on the alleviation of its height from sea-coast etc. Such relationships can often be represented mathematically as functions.

Suppose that average weekly household expenditure on food (C), measured in rs., depends on average net household weekly income (Y) according to the relationship

$$C = 42 + 0.2Y$$

For any given value of Y, one can evaluate what C will be. For example

if $Y = 500$

then expenditure on food is $C = 42 + 0.2(500) = 42 + 100 =$ Rs. 142

Whatever value of Y is chosen there will be one unique corresponding value of C. This is an example of a function.

A relationship between the values of two or more variables can be defined as a function when a unique value of one of the variables is determined by the value of the other variable or variables.

If the precise mathematical form of the relationship is not actually known then a function may be written in what is called a general form. For example, a general form demand function is

$Q_d = f(P)$

This just tells us that quantity demanded of a good (Q_d) depends on its price (P). The 'f' is not an algebraic symbol in the usual sense and so f(P) means 'is a function of P' and not 'f multiplied by P'. In this case P is what is known as the 'independent variable' because its value is given and is not dependent on the value of Q_d, i.e. it is exogenously determined. On the other hand, Q_d is the 'dependent variable' because its value depends on the value of P.

Loosely speaking, a function consists of two sets and a rule that associates elements in one set with elements in the other. For instance, suppose you want to determine the effect of price on the number of units of a particular commodity that will be sold at that price. To study this relationship, you need to know the set of admissible prices, the set of possible sales levels, and a rule for associating each price with a particular sales level. Here is the definition of function we shall use.

1.1 Definition of Function

A function is a correspondence between a first set, called the domain, and a second set, called the range, such that each member of the domain corresponds to exactly one member of the range.

A (real-valued) function of a real variable x with domain D is a rule that assigns a unique real number to each real number x in D. As x varies over the whole domain, the set of all possible resulting values f (x) is called the range of f.

Functions are given letter names, such as f, g, F, or ϕ. If f is a function and x is a number in its domain D, then f (x) denotes the number that the function f assigns to x. The symbol f (x) is pronounced "f of x", or often just "f x". It is important to note the difference between f, which is a symbol for the function (the rule), and f (x), which denotes the value of f at x.

If f is a function, we sometimes let y denote the value of f at x, so y = f (x) Then we call x the independent variable, or the argument of f, whereas y is called the dependent variable, because the value y (in general) depends on the value of x. The domain of the function f is then the set of all possible values of the independent variable, whereas the range is the set of corresponding values of the dependent variable. In economics, x is often called the exogenous variable, which is supposed to be fixed outside the economic model, whereas for each given x the equation y = f (x) serves to determine the endogenous variable y inside the economic model.

A function is often defined by a formula such as $y = 8x^2 + 3x + 2$. The function is then the rule $x \rightarrow 8x^2 + 3x + 2$ that assigns the number $8x^2 + 3x + 2$ to each value of x.

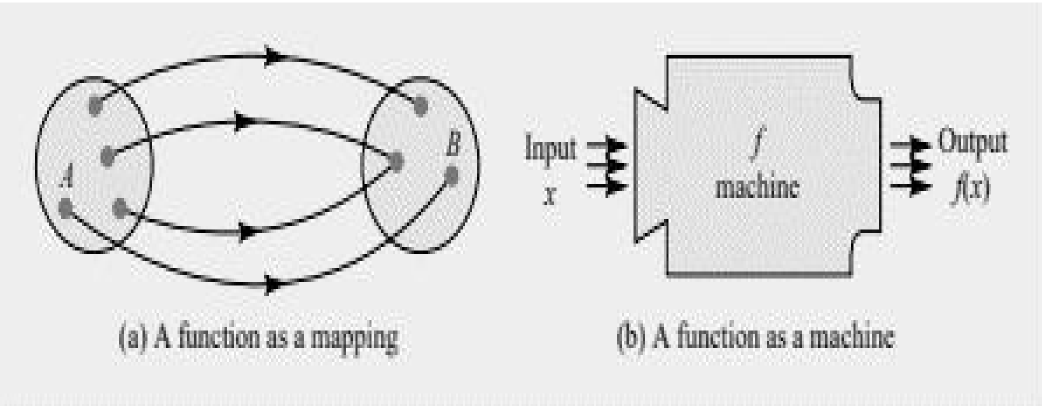


Fig.1: Interpretations of Function

It may help to think of such a function as a "mapping" from numbers in A to numbers in B (Figure 1 a), or as a "machine" that takes a given number from A and converts it into a number in B through a process indicated by the functional rule (Figure 1b). For instance, the function $f(x)= x^2 + 4$ can be thought of as an "f machine" that accepts an input x, then squares it and adds 4 to produce an output $y = x^2 + 4$. No matter how you choose to think of a functional relationship, it is important to remember that a function assigns one and only one number in the range (output) to each number in the domain (input).

Example 1. The total dollar cost of producing x units of a product is given by

$C(x) = 100x\sqrt{x} + 500$

for each nonnegative integer x. Find the cost of producing 16 units.

Solution: The cost of producing 16 units is found by substituting 16 for x in the formula for C(x):
$C(16) = 100 \cdot 16\sqrt{16} + 500 = 100 \cdot 16 \cdot 4 + 500 = 6900$

Example 2. The squaring function is given by

$F(x)=x^2$

Find f(-3), f(1).

Solution: $f(-3)=(-3)^2=9$

$f(1)=1^2=1$

Example 3: A function is given by $f(x)= 3x^2-2x+8$. Find f(0), f(-5) and f(7a).

Solution: One way to find function values when a formula is given is to think of the formula with blanks, or placeholders, as follows:

$f(0)= 3.0^2-2.0+8=8$

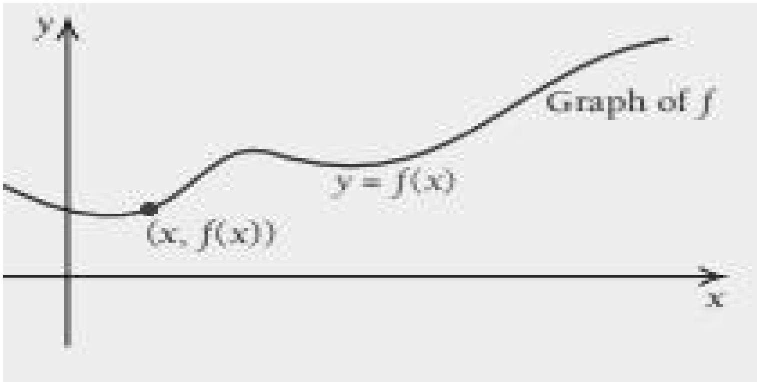
$f(-5)= 3(-5)^2-2(-5)+8= 3.25+10+8=75+10+8=93$

$f(7a)= 3(7a)^2-2(7a)+8= 3.49a^2-14a+8=147a^2-14a+8$

Graphs of Functions

Consider again the squaring function. The input 3 is associated with the output 9. The input–output pair (3, 9) is one point on the graph of this function.

The graph of a function is a drawing that represents all the input–output pairs In, cases where the function is given by an equation, the graph of the function is the graph of the equation y = f(x).



It is customary to locate input values (the domain) on the horizontal axis and output values (the range) on the vertical axis.

1.2 Domain and Range

The definition of a function is not really complete unless its domain is either obvious or specified explicitly. If a function is defined using an algebraic formula, the domain consists of all values of the independent variable for which the formula gives a unique value (unless another domain is explicitly mentioned).

If function f: X→Y is defined, then non-empty set Wis called domain of this mapping or function (f) and non-element set(Y) would be its range or co-domain.

Domain of function (f)= Element of actual values of X for which function is defined.

Range of function (f)= values of elements of Y corresponding to domain of all numbers x.

Let f be a function with domain D. The set of all values f (x) that the function assumes is called the range of f. Often, we denote the domain of f by Df, and the range by Rf. These concepts are illustrated in Fig. 3, using the idea of the graph of a function. (Graphs are discussed in the next section.) Alternatively, we can think of any function f as an engine operating so that if x in the domain is an input, the output is f (x). (See Fig.3.) The range of f is then all the numbers we get as output using all numbers in the domain as inputs. If we try to use as an input a number not in the domain, the engine does not work, and there is no output.

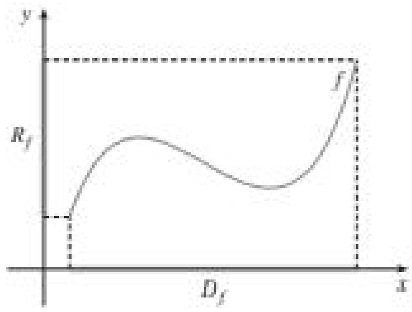


Fig. 3. Domain (D_f) and Range (R_f) of a function

Domain calculation: Steps to be followed

1. Denominator must be non-zero as f(x)=1/(x-2), x≠2
2. Even roots cannot have negative number: f(x)=√x-3, as it makes function undefined.

Example 1: Determine the domain of x−4/ (x^2 −2x−15)

Solution: Set the denominator to zero and solve for x

⇒ $x^2 - 2x - 15 = (x - 5)(x + 3) = 0$

Hence, x = −3, x = 5

For the denominator not to be zero, we need to avoid the numbers −3 and 5. Therefore, the domain is all real numbers except −3 and 5.

Example 2: Find the domain and range of the following function.

f(x) = 2/ (x + 1)

Solution: Set the denominator equal to zero and solve for x.

x + 1 = 0

x= -1

Since the function is undefined when x = -1, the domain is all real numbers except -1.

Example 3: Find the domain of f(x) = 1/√ (x2 -4)

Solution: By factoring the denominator, we get x ≠ (2, – 2).

Test your answer by plugging -3 into the expression within the radical sign.

⇒ (-3)2 – 4 = 5

also try with zero

⇒ 02 – 4 = -4, therefore number between 2 and -2 are invalid

Try number above 2

$\Rightarrow$ 32 – 4 = 5. This one is valid.

Hence, the domain = $(-\infty, -2) \cup (2, \infty)$

Range calculation: Steps to be followed:

1. Put y=f(x)
2. Express x as a function of y
3. Find possible values for y
4. Eliminate values according to definition of range (Applied only in case of underoot).

Example 4: Find the range of function f defined by:

$F(x)=x^2-2$

Solution: The domain of this function is the set of all real numbers. The range is the set of values that f(x) takes as x varies. If x is a real number, x2 is either positive or zero. Hence, we can write the following:

$x^2 \geq 0$

Subtract - 2 to both sides to obtain

$x^2 - 2 \geq -2$

The last inequality indicates that $x^2 - 2$ takes all values greater than or equal to - 2. The range of f is given by

$[-2, +\infty)$

Example 5: Find the Range of function f defined by

f (x) = 4 x + 5

Solution:

Assuming that the domain of the given function is the set of all real numbers R, so that the variable x takes all values in the interval

$(-\infty, +\infty)$

If x takes all values in the interval $(-\infty, +\infty)$ then 4 x + 5 takes all values in the interval $(-\infty, +\infty)$ and the range of the given function is given by the interval $(-\infty, +\infty)$.

Example 6: Find the Range of function f defined by

$f(x) = -2x^2 + 4x - 7$

We first write the given quadratic function in vertex form by completing the square

$f(x) = -2x^2 + 4x - 7 = -2(x\ 2 - 2x) - 7 = -2((x - 1)^2 - 1) - 7 = -2(x - 1)^2 - 5$

The domain of the given function is R with x taking any value in the interval $(-\infty, +\infty)$ hence $(x - 1)^2$ is either zero or positive. We start by writing the inequality

$(x - 1)\ 2 \geq 0$

Multiply both sides of the inequality by - 2 and change the symbol of inequality to obtain

$-2(x - 1)\ 2 \leq 0$

Add - 5 to both sides of the inequality to obtain- $2(x - 1)^2 - 5 \leq -5$ or $f(x) \leq -5$ and hence the range of function f is given by the interval $(-\infty\ -5]$.

Example 7: Find the Range of function f defined by

f (x) = (x - 1) / (x + 2)

Solution: For this rational function, a direct algebraic method similar to those above is not obvious. Let us first find its inverse, the domain of its inverse which give the range of f.

We first prove that f is a one to one function and then find its inverse. For a function to be a one to one, we need to show that

If f(a) = f(b) then a = b.

(a - 1) / (a + 2) = (b - 1) / (b + 2)

Cross multiply, expand and simplify

(a - 1)*(b + 2) = (b - 1)*(a + 2)

a b + 2 a - b - 2 = a b + 2b - a - 2

3a = 3b, which finally gives a = b and proves that f is a one to one function.

Let us find the inverse of f

y = (x - 1) / (x + 2)

Solve for x

x = (2y + 1) / (1 - y)

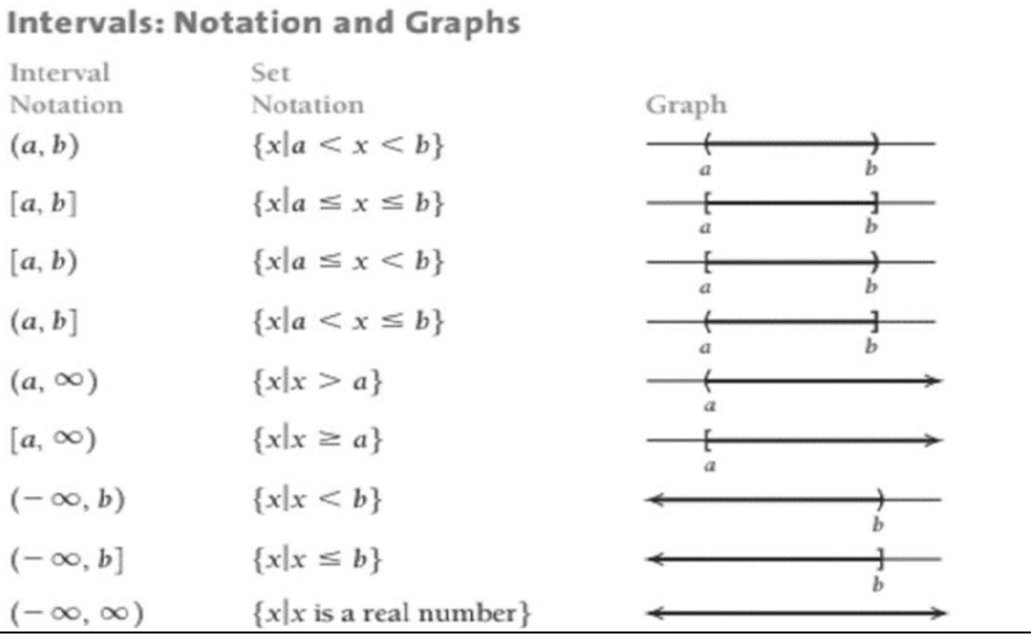
change y into x and x into y and write the inverse function

$f^{-1}(x) = y = (2x + 1) / (1 - x)$

The range of f is given by the domain of f -1 and is therefore given by the interval

$(-\infty, 1) \cup (1, +\infty)$.

Notes:



Intervals: Notation and Graphs

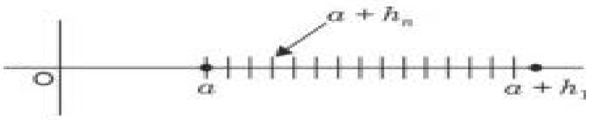
Interval Notation	Set Notation	Graph
(a, b)	$\{x \mid a < x < b\}$	a b
$[a, b]$	$\{x \mid a \le x \le b\}$	a b
$[a, b)$	$\{x \mid a \le x < b\}$	a b
$(a, b]$	$\{x \mid a < x \le b\}$	a b
(a, ∞)	$\{x \mid x > a\}$	a
$[a, \infty)$	$\{x \mid x \ge a\}$	a
$(-\infty, b)$	$\{x \mid x < b\}$	b
$(-\infty, b]$	$\{x \mid x \le b\}$	b
$(-\infty, \infty)$	$\{x \mid x \text{ is a real number}\}$	

1.3 Limits Basics, Inclination and Slope

Assume y-f(x) is a function and $h_1, h_2, \ldots\ldots\ldots h_n, \ldots\ldots\ldots$ is a set of positive numbers, which value is continually decreasing viz

$h_1 > h_2 > h_3 > \ldots\ldots\ldots\ldots h_n \ldots\ldots\ldots > 0$ (1)

And which, choosing n sufficiently greater, can be made smaller as desired. In this state, as the h_n goes down, the value of function decreases

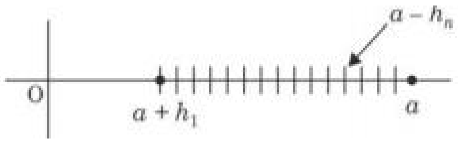


f(a+h_1), f (a+h_2), ………………, f(a+h_n) (2)

if a number tends to A then this number is call right hand limit of function f(x) at x=a or this number A is called the right-hand limit of function f(x), when x tends to a. This can be expressed as:

$$\lim_{x \to a+0} f(x) = A = f(a+0)$$

Here we have considered only those values of x, which is greater than a. (in the figure onlya at the right side)



Now we will consider those values of x, which is smaller (viz in the figure only at the left side of a). As, h_n goes down, the value of function f(a-h_1), f(a-h_2)……………, f(a-h_n)………………tends to B. This number B is called the lefthand limit of function f(x), when x=a.

This is expressed as

$$\lim_{x \to a-0} f(x) = b = f(a-0)$$

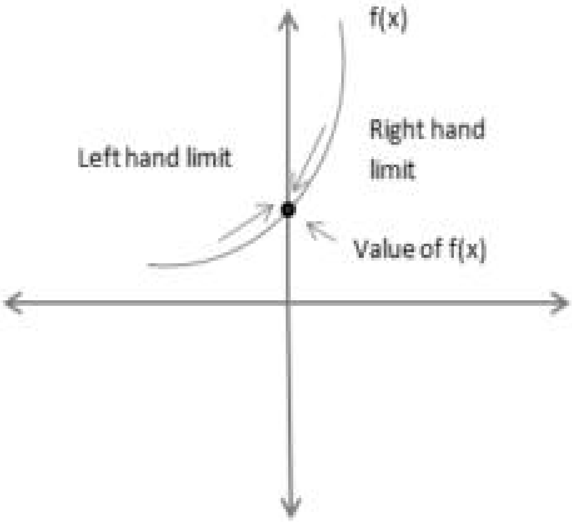
If a=b viz

$$\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x)$$

Then A is called limit of f(x) at x=A

Set h_1,.h_2,……………., h_n………… is a sequence for which limit is 0. Similarly, second makes sequence (2).Here, is to be specially noted that for limit to exist, like sequence (1) f(a+h_n) every type of sequence should tend to A. viz the statistical difference of f(a-hn)-A, choosing hn sufficient smaller, can be reduced as desired. Assigning a+h_n=x, we can define the limit as under

Limits of functions at a point are the common and coincidence value of the left and right-hand limits.



The value of a limit of a function f(x) at a point a i.e., f(a) may vary from the value of f(x) at 'a'.

Left and right limits. When we let "x approacha" we allow x to be both larger or smaller than a, as long as x gets close to a. If we explicitly want to study the behaviour of f(x) as x approaches a through values larger than a, then we write

$\lim_{x\to a+}$ f(x) or $\lim_{x\to a+}$ f(x) or $\lim_{x\to a+0}$ f(x) or $\lim_{x\to a, x>a}$ f(x).

All four notations are in use. Similarly, to designate the value which f(x) approaches as x approaches a through values below a one writes

$\lim_{x\to a+}$ f(x) or $\lim_{x\to a-}$ f(x) or $\lim_{x\to a-0}$ f(x) or $\lim_{x\to a,\ x<a}$ f(x).

Definition of right-limits: Let f be a function. Then

$$\lim_{x\to a} f(x)=L.$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$a < x < a + \delta \Longrightarrow |f(x) - L| < \varepsilon$

holds for all x in the domain of f.

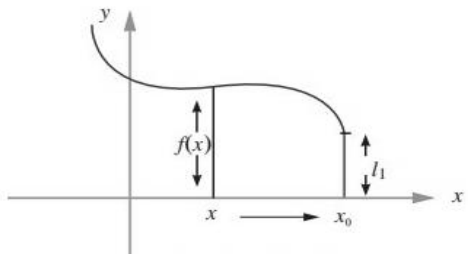
The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way. The following theorem tells you how to use one-sided limits to decide if a function f(x) has a limit at x = a.

Working Rules for Finding Right Hand Limit and Left-Hand Limit

i) To obtain the limit of right andleft hand, replace x variable with (x+h) and (x-h) respectively in the function.

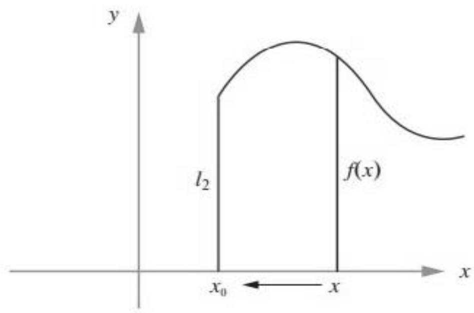
ii) Thus, obtained function x, should be replaced with point (assume a)

iii) Now at h→0 determine the limit of function.



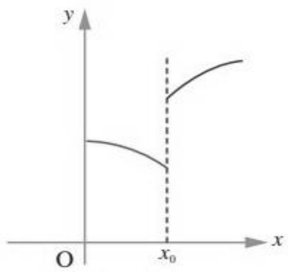
$\lim_{x\to x_0^-} f(x)=l_1=f(x_0^-)$

Fig. 9.6



$\lim_{x\to x_0^+} f(x)=l_2=f(x_0^+)$

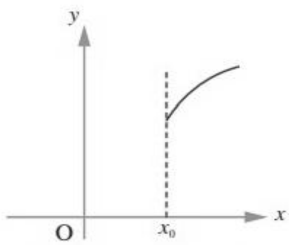
Fig. 9.7



$\lim_{x\to x_0} f(x)$ does not exist

Fig. 9.8

(Different values are obtained as x_0 is approached from the left and from the right)



$\lim_{x\to x_0} f(x)$ does not exist

Fig. 9.9

(Function not defined to the left of x_0)

Example 8: f (x) = {x^3-4 for x<2, 2x for x≥2}

a) What is f (x)?

b) What is f (x)?

c) What is f (x)?

Solution:a) $_{x\to 2-}$f (x) =$_{x\to 2-}$ (x^3 -4) = 23 - 4 = 4.

b) $_{x\to 2+}$f (x) = $_{x\to 2}$+(2x) = 2(2) = 4.

c) Because lim $_{x\to 2-}$f (x) = lim $_{x\to 2+}$ f (x) = 4, f (x) exists and is equal to 4.

Example 9: f (x) =

[x^5-12 for x<2 & $(x+1)^3$-8 for x≥2}

a) What is $_{x\to 2-}$ f (x)?

b) What is$_{x\to 2-}$ f (x)?

c) What is$_{x\to 2-}$ f (x)?

a) lim $_{x\to 2-}$f (x) = $_{x\to 2}$+(x^5 -12) = 25 - 12 = 20.

b) lim $_{x\to 2-}$f (x) = $_{x\to 2}$+[$(x + 1)^3$ -8] = $(2 + 1)^3$ - 8 = 19.

c) Because $_{x\to 2}$f (x)≠f$_{x\to 2+}$ (x),$_{x\to 2}$ f (x) does not exist.

Example 10: Find the limit

lim $_{x\to 2-}$ (x – 1)/ (x^2 − 3x + 2).

Solution: We first evaluate the fraction (x − 1)/(x^2 − 3x + 2) at x = 2. We get 1/0.

This tells us that the limit lim $_{x\to 2-}$ (x – 1)/ (x^2 − 3x + 2) is either +∞, −∞, or it does not exist. We are approaching 2 from the left, which means that x < 2, but really close to 2. In this case x^2 − 3x + 2 is a negative number, since x^2 − 3x + 2 is a parabola that opens up and has x-intercepts at the points 1 and 2. Thus the bottom of the fraction is negative, and the top is close to 1, hence positive. Together the fraction is negative, which gives

lim $_{x\to 2-}$ (x – 1)/ (x^2 − 3x + 2)= −∞

Example 11: Find the limit: lim$_{x\to -3}$(x + 2) /(x + 3)

Solution: Arguing as we did above we can show that lim $_{x\to -3+}$ (x + 2) /(x + 3)= −∞ and lim $_{x\to -3-}$ (x + 2) /(x + 3) = +∞. Since the limit from the right is different than the limit from the left the full limit lim$_{x\to -3}$ (x + 2) /(x + 3) does not exist.

1.4 Logarithmic and Exponential Function

A quantity that increases (or decreases) by a fixed factor per unit of time is said to increase (or decrease) exponentially. If the fixed factor is a, this leads to the exponential function f (t) = Aat (a and A are positive constants) (1)

(It is obvious how to modify the subsequent discussion for the case when A is negative.) Note that if f (t) = Aat , then f (t + 1) = Aat+1 = Aat · a1 = af (t), so the value of f at time t + 1 is a times the value of f at time t. If a > 1, then f is increasing; if 0 <a< 1, then f is decreasing. (See Figs. 1 and 2.) Because f (0) = Aa0 = A, we can write f (t) = f (0)at .

Exponential functions appear in many important economic, social, and physical models. For instance, economic growth, population growth, continuously accumulated interest, radioactive decay, and decreasing illiteracy have all been described by exponential functions. In addition, the exponential function is one of the most important functions in statistics.

A population Q(t) is said to grow exponentially if whenever it is measured at equally spaced time intervals, the population at the end of any particular interval is a fixed multiple (greater than 1) of the population at the end of the previous interval. For instance, according to the United Nations, in the year 2000, the population of the world was 6.1 billion people and was growing at an annual rate of about 1.4%. If this pattern were to continue, then every year, the population would be 1.014 times the population of the previous year. Thus, if P(t) is the world population (in billions) t years after the base year 2000, the population would grow as follows:

2000 $P(0) = 6.1$

2001 $P(1) = 6.1(1.014) = 6.185$

2002 $P(2) = 6.185(1.014) = [6.1(1.014)](1.014) = 6.1(1.014)^2 = 6.272$

2003 $P(3) = 6.272(1.014) = [6.1(1.014)2\ ](1.014) = 6.1(1.014)3 = 6.360$

2000 + t $P(t) = 6.1(1.014)t$

The graph of P(t) is shown in Figure 4.1a. Notice that according to this model, the world population grows gradually at first but doubles after about 50 years (to 12.22 billion in 2050).

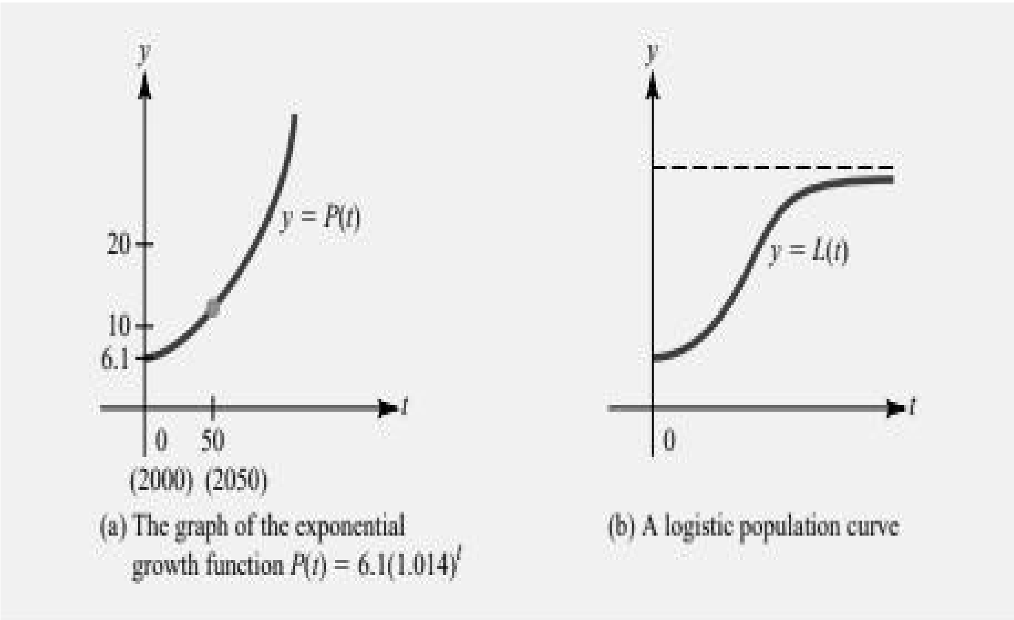


Fig. 1 Two models for population growth.

Exponential population models are sometimes referred to as Malthusian, after Thomas Malthus (1766–1834), a British economist who predicted mass starvation would result if a population grows exponentially while the food supply grows at a constant rate (linearly). Fortunately, world population does not continue to grow exponentially as predicted by Malthus's model, and models that take into account various restrictions on the growth rate actually provide more accurate predictions. The population curve that results from one such model, the so-called logistic model, is shown in Figure 4.1b. Note how the logistic growth curve rises steeply at first, like an exponential curve, but then eventually turns over and flattens out as environmental factors act to break the growth rate.

A function of the general form $f(x)= b^x$, where b is a positive number, is called an exponential function. Such functions can be used to describe exponential and logistic growth and a variety of other important quantities. For instance, exponential functions are used in demography to forecast population size, in finance to calculate the value of investments, in archaeology to date ancient artifacts, in psychology to study learning patterns, and in industry to estimate the reliability of products.

Definition of b^n for Rational Values of n (and $b > 0$) ■ Integer powers: If n is a positive integer,

$$b^n = \underbrace{b \cdot b \cdots b}_{n \text{ factors}}$$

Fractional powers: If n and m are positive integers,

$$b^{n/m} = (\sqrt[m]{b})^n = \sqrt[m]{b^n}$$

where $\sqrt[m]{b}$ denotes the positive mth root.

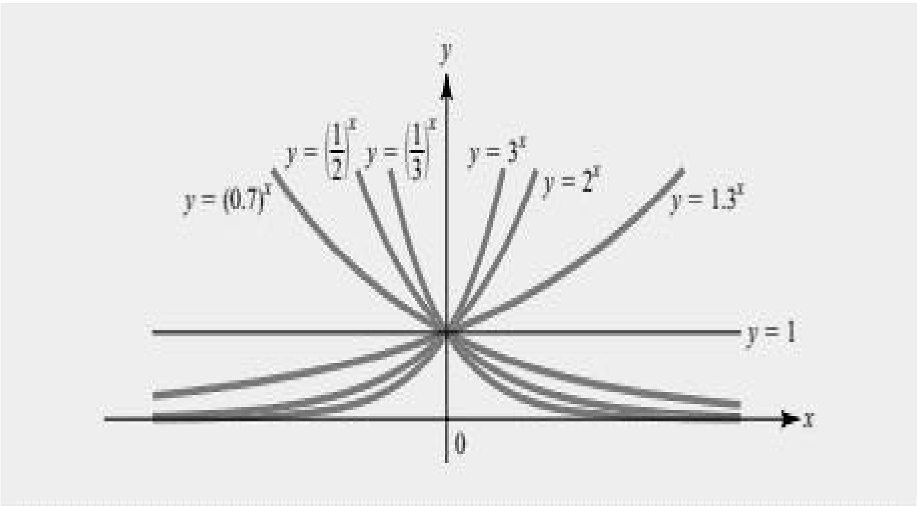
Negative powers: $b^{-n} = \dfrac{1}{b^n}$

Zero power: $b^0 = 1$

Exponential Functions

If b is a positive number other than 1 ($b>0$, $b \neq 1$), there is a unique function called the exponential function with base b that is defined by

$$f(x) = b^x \text{ for every real number } x$$



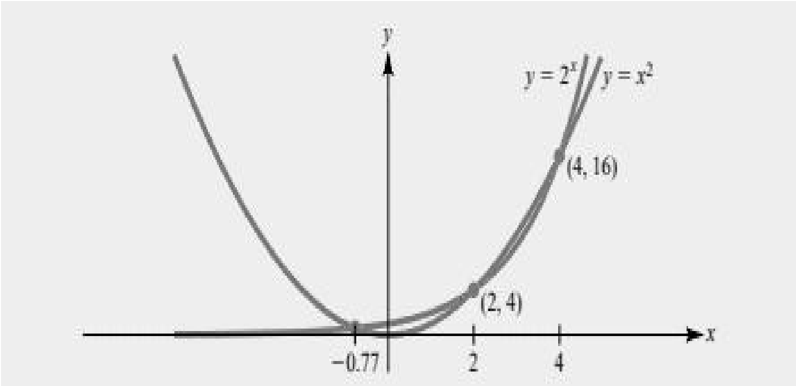
Important graphical and analytical properties of exponential functions are summarized in the following box

Properties of an Exponential Function ■ The exponential function $f(x) = b^x$ for $b > 0$, $b \neq 1$ has these properties:

1. It is defined, continuous, and positive ($b^x > 0$) for all x
2. The x axis is a horizontal asymptote of the graph of f.
3. The y intercept of the graph is (0, 1); there is no x intercept.
4. If $b > 1$, $\lim_{x \to -\infty} b^x = 0$ and $\lim_{x \to +\infty} b^x = +\infty$.

   If $0 < b < 1$, $\lim_{x \to -\infty} b^x = +\infty$ and $\lim_{x \to +\infty} b^x = 0$.
5. For all x, the function is increasing (graph rising) if $b > 1$ and decreasing (graph falling) if $0 < b < 1$.

Notes: Students often confuse the power function $p(x)= x^b$ with the exponential function $f(x)=b^x$. Remember that in x^b, the variable x is the base and the exponent b is constant, while in b^x, the base b is constant and the variable x is the exponent. The graphs of $y= x^2$ and $y = 2^x$ are shown in Figure. Notice that after the crossover point (4, 16), the exponential curve $y= 2^x$ rises much more steeply than the power curve $y = x^2$. For instance, when x = 10, the y value on the power curve is $y = 10^2 = 100$, while the corresponding y value on the exponential curve is $y = 2^{10} = 1{,}024$.



Comparing the power curve $y = x^2$ with the exponential curve $y =2^x$.

Exponential functions obey the same algebraic rules as the rules for exponential These rules are summarized as follows:

Exponential Rules

■ For bases a, b (a > 0, b> 0) and any real numbers x, y, we have

1. The equality rule: $b^x = b^y$ if and only if x = y

2. The product rule: $b^x b^y = b^{x+y}$

3.The quotient rule: $b^x/b^y = b^{x-y}$

4. The power rule: $(bx)^y = b^{xy}$

5. The multiplication rule: $(ab)^x = a^x b^x$

6. The division rule: $(a/b)^x= a^x/b^x$

Example 12: Let $f(x)=5(3)^{x+1.}$ Evaluate f(2) without using a calculator.
Solution:
Follow the order of operations. Be sure to pay attention to the parentheses.

$f(x)=5(3)^{x+1}$	
$f(2)=5(3)^{2+1}$	Substitute x=2.
$=5(3)^3$	Add the exponents.
$=5(27)$	Simplify the power.
$=135$	Multiply.

Example 13: Find dy/dx:

a) $Y=3e^x$

d/dx(3e^x) = 3 d/dx*e^x

=3e^x

b) Y=x^{2}e^x= x^2.e^x + e$^{x.2x}$

= e^x(x^2 +2x)

c) Y=g(x)=(1/2)^x

= (2^{-1})^x

= 2^{-x}

Logarithmic Function

Suppose you invest 1,000 at 8% compounded continuously and wish to know how much time must pass for your investment to double in value to 2,000. According to the formula derived, the value of your account after t years will be 1,000e$^{0.08t}$, so to find the doubling time for your account, you must solve for t in the equation

1,000e$^{0.08t}$ = 2,000

or, by dividing both sides by 1,000,

e$^{0.08t}$ =2

We will answer the question about doubling time. Solving an exponential equation such as this involves using logarithms, which reverse the process of exponentiation. Logarithms play an important role in a variety of applications, such as measuring the capacity of a transmission channel and in the famous Richter scale for measuring earthquake intensity. In this section, we examine the basic properties of logarithmic functions and a few applications. We begin with a definition

A logarithm is defined as follows:

Log$_a$x=y means a^y=x, a>0, a≠1.

The number is the power y to which we raise a to get x. The number a is called the logarithmic base. We read as “the logarithm, base a, of x.

For logarithms with base 10, is the power y such that Therefore, a logarithm can be thought of as an exponent.

Logarithmic Equation	Exponential Equation
$\log_a M = N$	$a^N = M$
$\log_{10} 100 = 2$	$10^2 = 100$
$\log_5 \frac{1}{25} = -2$	$5^{-2} = \frac{1}{25}$
$\log_{49} 7 = \frac{1}{2}$	$49^{1/2} = 7$

In order to graph a logarithmic equation, we can graph its equivalent exponential equation.

Example 14: Graph: y=$_{\log 2}$x.

Solution: We first write the equivalent exponential equation:

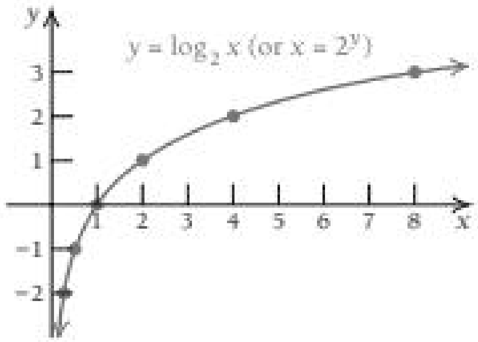
2^y=x

We select values for y and find the corresponding values of 2^y. Then we plot points, remembering that x is still the first coordinate, and connect the points with a smooth curve.

x, or 2^y	y
1	0
2	1
4	2
8	3
$\frac{1}{2}$	−1
$\frac{1}{4}$	−2

(1) Select y.

(2) Compute x.



The graphs of and are shown below on the same set of axes. Note that we can obtain the graph of g by reflecting the graph of f across the line y=x. Functions whose graphs can be obtained in this manner are known as inverses of each other.

Although we do not develop inverses in detail here, it is important to note that they "undo" each other. For example,

$f(3)=2^3=8$,	The input 3 gives the output 8.
and $g(8)= \text{Log}_2 8=3$	The input 8 gets us back to 3

Homogenous Function, Cobb Douglas Production Function, Cost Functions and Production Functions

Homogeneous production functions consist of a broad array of functions with a special characteristic. A production function is said to be homogeneous of degree n if when each input is multiplied by some number t, output increases by the factor tn . Assuming that the time period is sufficiently long such that all inputs can be treated as variables and are included in the production function, n, the degree of homogeneity refers to the returns to scale. Homogeneous production functions are frequently used by agricultural economists to represent a variety of transformations between agricultural inputs and products. A function homogeneous of degree 1 is said to have constant returns to scale, or neither economies or diseconomies of scale. A function homogeneous of a degree greater than 1 is said to have increasing returns to scale or economies of scale. A function homogeneous of degree less than 1 is said to have diminishing returns to scale or diseconomies of scale. While there are many different production functions, only certain kinds of production functions are homogeneous. In general, they are multiplicative rather than additive although a few exceptions exist. The production function

$y = Ax_1^{0.5}x_2^{0.5}$ is homogeneous of degree 1. Multiply x^1 and x^2 by t to get

$A(tx_1)^{0.5}(tx_2)^{0.5} = tAx_1^{0.5}x_2^{0.5} = t^1y$

Thus, the function in equation exhibits constant returns to scale without any economies or diseconomies.

Euler's Theorem

Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted. If every means of production is credited equal to its marginal productivity and total production is liquidated completely. In mathematical formula Euler's Theorem can be indicated. If production, P = f (L, K) is Linear Homogeneous Function:

Euler's theorem states that if $f(x,y)$ is an homogeneous function of degree r, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=rf(x,y).$$

This follows from a simple application of the chain rule since, using the definition of a function that is homogeneous of degree r, we have

$$f(\lambda x,\lambda y)=\lambda^r f(x,y),$$

for any $\lambda\in\mathbb{R}$. As such, differentiating both sides with respect to λ and using the chain rule from (6.3) on the left-hand side, we have

$$\frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}\lambda}+\frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}\lambda}=r\lambda^{r-1}f(x,y),$$

if we think of $f(\lambda x,\lambda y)$ as $f(u,v)$ with $u=\lambda x$ and $v=\lambda y$. This then gives us

$$x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}=r\lambda^{r-1}f(x,y).$$

and, if we now set $\lambda=1$, we get the desired result as we have $u=x$, $v=y$ and $\lambda^{r-1}=1$.

Cobb-Douglas Production Function

The Cobb-Douglas (CD) production functionis an economic production function with two or more variables (inputs) that describes the output of a firm. Typical inputs include labor (L) and capital (K). It is similarly used to describe utility maximization through the following function [U(x)]. However, in this example, we will learn how to answer a minimization problem subject to (s.t.) the CD production function as a constraint.

The functional form of the CD production function:

$$f(L,K)=Y=AL^{\alpha}K^{\beta},$$

where the output Y is a function of labor (L) and capital (K), A is the total factor productivity and is otherwise a constant, L denotes labor, K denotes capital, alpha represents the output elasticity of labor, beta represents the output elasticity of capital, and (alpha + beta = 1) represents the constant returns to scale (CRS). The partial derivative of the CD function with respect to (w.r.t) labor (L) is:

$$\frac{\partial Y}{\partial L}=A\alpha L^{\alpha-1}K^{\beta}$$

$$\frac{\partial Y}{\partial L}=\alpha AL^{\alpha}L^{-1}K^{\beta}$$

$$\frac{\partial Y}{\partial L}=\frac{\alpha AL^{\alpha}K^{\beta}}{L}$$

Recall that quantity produced is based on the labor and capital; therefore, we can solve for alpha:

$$Y=AL^{\alpha}K^{\beta},$$

$$\frac{\partial Y}{\partial L}=\alpha\frac{Y}{L}$$

$$\alpha=\frac{\partial Y}{\partial L}\frac{L}{Y}$$

This will yield the marginal product of labor (L). If alpha = 2, then a 10% increase in labor (L) will result in a 20% increase in output (Y).

The partial derivative of the CD function with respect to (w.r.t) labor (K) is:

$$\beta = \frac{\partial Y}{\partial K}\frac{K}{Y}$$

This will yield the marginal product of capital (K).

The CD production function can be converted to a linear model by taking the logarithm of both sides of the equation:

$$f(L, K) = Y = AL^{\alpha}K^{\beta}$$

$$\log(Y) = \log(A) + \alpha \log(L) + \beta \log(K)$$

We've shown that the Cobb–Douglas function gives diminishing returns to both labor and capital when each factor is varied in isolation. But what happens if we change both K and N in the same proportion?

Suppose an economy in an initial state has inputs K_0 and N_0 and produces output

$$Y_0 = AK^{\alpha}_0 N^{1-\alpha}_0$$

Now suppose we scale the inputs by some common factor λ. (For example, λ = 2 would mean that we double each input.) We'll then have inputs $K_1 = \lambda K_0$ and $N_1 = \lambda N_0$ and will produce output Y_1. The question is, how does Y_1 relate to Y_0? Let's see:

$$Y1 = AK^{\alpha}_1 N^{1-\alpha}_1$$

$$= A(\lambda K_0)^{\alpha} (\lambda N_0)^{1-\alpha}$$

$$= A\lambda^{\alpha} K_0^{\alpha} \lambda^{1-\alpha} N_0^{1-\alpha}$$

$$= \lambda^{\alpha+1-\alpha} AK_0^{\alpha} N_0^{1-\alpha}$$

$$= \lambda Y_0$$

So, if we scale both inputs by a common factor, the effect is to scale the output by that same factor. This is the defining characteristic of constant returns to scale. From the math above we can see that this occurs in the Cobb–Douglas function because the exponents on capital and labor, α and 1 − α, add up to 1.

We could imagine a generalization of Cobb–Douglas in which the exponents on capital and labor are (say) α and β respectively, preserving the requirement that each exponent be a positive fraction (this is needed to give positive but diminishing marginal products) but dropping the requirement that they sum to 1. In that case we'd get increasing returns to scale if $\alpha + \beta > 1$ and decreasing returns to scale if $\alpha + \beta < 1$.

Economic significance of Cobb Douglas Production Function

Cobb Douglas Production Function has a very importance role in economic area. At present many economists are using Cobb Douglas Production Function in various economic areas. The use of this function is day-by-day is increasing especially in various industries and agriculture. This bring important information for these sectors. This also helps in framing various policies. With the help of this function, we can also determine the Marginal Productivity and similarly it helps in determining principle of wages. Production function describes production technique. With the help of this function we can also determine whether any factor is paid the value with respect to its equality with the marginal productivity. In a same fashion it helps in agriculture to find the elasticity of economy. By this function we also display elasticity coefficients. These elasticity coefficients help us in comparing the international and internal areas.

As has already be described when function is linear and homogeneous and $a + b = 1$, then production would be under the constant result, when $a + b > 1$, then increase in production happens, and if $a + b < 1$, then decrease in production happens. This way this function helps us in studying the rules of various results. Besides these it also fetches important information related to substitutability of various factors of production.

In short, this function plays an important role especially in agriculture and industries. This is used in determining the labour policies, inter-area comparison, substitutability of factors and degree of homogeneity.

Limitation of Cobb-Douglas Production Functions:

Although Cobb Douglas Production Function is used widely in economic areas and its use is increasing in especially in various industries and agriculture, but some economists criticize this production function. Among them are Prof K.J. Arrow, H.B. Chenery, B.S. Minhas and R.M. Salow. Their main criticizes are:

1. The main demerit of this function is this that it considers only two factors of production i.e. Capital and Labour, whereas in reality other factors also have important role in production. In other words, this function does not apply to more than two factors. Besides it can be used only in construction industries. This way its use becomes narrow.

2. This function works under the constant result of formula. Rule of increase and decrease in result also apply to production function. But this function does not work under these rules.

3. Function is based on the assumptions that technical knowledge remains constant and no change in techniques happen in production. But the same can change in production. This way assumption of constant technique is irrelevant.

4. Cobb-Douglas Production Functions assume that all inputs are homogeneous. In reality all units of a factors are not homogeneous. For example, some people are skilled and others are not in a labour population.

5. This does not determine any maximum level of production. Prof M. Chand says "Since, this does not ascertain the maximum level of P (Production), it would be practical and convenient not to use this function beyond a certain limit for statistical measurement of its values.

6. a and b of the function reflects the proportion of labour and capital in production. This becomes true only when market has a complete competition. But in case economy has an incomplete competition or monopoly, then above relation cannot be obtained.

7. It takes into account only positive marginal productivity of factors and ignores the negative marginal productivity. Whereas marginal productivity of any factor can be zero or negative.

8. Last, the function is unable to produce information related to inter-relation of factors.

Summary

- If function f: X→Y is defined, then non empty set X is called domain of this mapping or function (f) and non-element set (Y) would be its range or codomain.
- If variable u is dependent on variable x and y, then u=f(x,y) is called functions of many variable.
- Function with different exponential value of x variable, in which factors are certain is called algebraic function.
- Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called rational numbers.
- When the limit of function is obtained from the right hand of the independent variable, then it is called Right Hand Limit(RHL) and applying positive (+) sign for the right side, this can be expressed as under

  Right Hand Limit $= f(a+0)$

  $= \text{Lim}_{x\to a^+} f(X) = l_1$
- When the limit of function is obtained from the right hand of the independent variable, then it is called Left Hand Limit(lHL) and applying positive (-) sign for the right side, this can be expressed as under

Right Hand Limit = f(a-0)

$=\text{Lim}_{x\to a^-} f(X) = l_2$

- A function f(x) is called continuous in an open interval (a,b) if it is continuous for every values of x in this interval (a,b).
- If any such equation exists between x andy such that cannot be solved for y instantaneously then y is said to be the implicit function of x. In contrast if value of y can be found out in terms of x then y is said to be explicit function of x.

  Log(m*n) = logm +logn

  Log(m/n)= logm-logn

  $\text{Log}(m)^n$ =nlogm
- Maximum use of special production is referred to as Homogeneous function.
- Euler's Theorem states that all factors of production are increased in a given proportionresulting output will also increase in the same proportion each factor of production (input) ispaid the value of its marginal product, and the total output is just exhausted.
- Euler's Theorem has an important place in economic area especially in marketing area.
- Cobb Douglas Production Function has a very important role in economic area. At presentmany economists are using Cobb Douglas Production Function in various economic areas.

Keywords

- Domain: Affected area
- Range: Series, limits of variation.
- Sequence: Serial
- Continually: Continuously
- Infinite: which doesn't have an end
- Homogenous: undefined, similar

Self Assessment

1. A function is said to be ______________ if and only if f(a) = f(b) implies that a = b for all a and b in the domain of f.

A. One-to-many

B. One-to-one

C. Many-to-many

D. Many-to-one

2. If a<0, then the function f(x)=ax2+bx+c

A. Maximum value

B. Minimum value

C. Constant value

D. Positive value

3. A parabola graph which opens downward is classified as

A. Concave right

B. Concave left
C. Concave up
D. Concave down

4. What is the domain of a function?
A. the maximal set of numbers for which a function is defined
B. the maximal set of numbers which a function can take values
C. it is a set of natural numbers for which a function is defined
D. none of the mentioned

5. What is range of function f(x) = x-1 which is defined everywhere on its domain?

A. $(-\infty, \infty)$
B. $(-\infty, \infty) - \{0\}$
C. $[0, \infty)$
D. None of the mentioned

6. Codomain is the subset of range.
A. True
B. False
C. May be True or False
D. Can't say

7. The exponential function class f(x)=b^x where 0<b<1, the function is between x and y is classified as

A. x is increasing function of y
B. y is decreasing function of x
C. y is increasing function of x
D. x is decreasing function of y

8. Solve the equation by expressing each side as a power of the same base and then equating exponents: 4logby+6logbz

A. logb y4z6
B. 10logbyz
C. 24logbyz
D. logb(YZ)10

9. Solve the equation by expressing each side as a power of the same base and then equating exponents: e(x+3)= 1/e2

A. {-1}
B. {5}
C. {-5}
D. {1}

10.Log_amn equals to

A. $\text{Log}_a m + \log_a n$

B. $\text{Log}_a m - \log_a n$

C. $\text{Nlog}_a m$

D. $\text{Log}_b n * \log_a b$

11.A significant property of the Cobb-Douglas production function is that the elasticity of substitution between inputs is

A. Equal to 1

B. More than 1

C. Less than 1

D. 0

12. Cobb-Douglas production function $Q = AL\alpha\ K1{-}\alpha$ does not possess the characteristics of

A. Constant Returns to Scale

B. Unit Elasticity of Substitution

C. Variable Elasticity of Substitution

D. Linear homogeneity

13. f (x) is a continuous function and takes only rational values. If f (0) = 3, then f (2) equals

A. 5

B. 0

C. 1

D. None of these

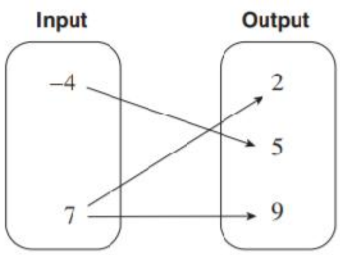
14. If $\lim_{x\to 2} (x-2)(x+2)/(x-2)$, is limit exist at x approaches to 2?

A. True

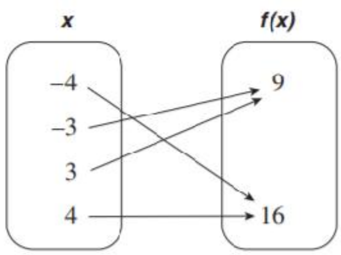
B. False

15.Which of the following relations is not a function?

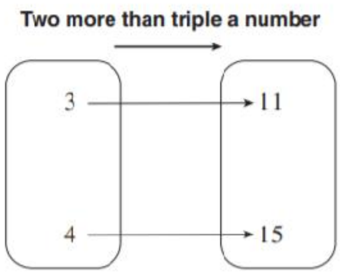
A.



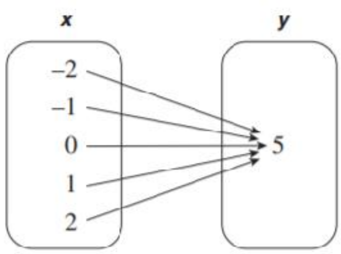
B.



C.



D.



Answers for Self Assessment

1.	B	2.	A	3.	D	4.	A	5.	A
6.	A	7.	B	8.	A	9.	C	10.	C
11.	A	12.	C	13.	D	14.	A	15.	A

Review Questions:

1. If function f: R→R is defined in the following way
   $F(x)= \{3x-1$, if $x>3$, x^2-2, if $2\le x\le 3$ & $2x+3$, if $x<-2\}$
   Then find out the value of f(2) & f(4).
2. Find out the domain and range of function $f(x)=x/1+x^2$. If the function is single?
3. Solve each of the following equations for x:
   a. $Log_4x=1/2$
   b. $Log_{64}16=x$
4. Evaluate the given expressions
   a. $(-128)^{3/7}$
   b. $(X^{2/3})^{-3/4}$
   c. $(2.14)^{x-1}=(2.14)^{1-x}$
   d. $10^{(x2-1)}=10^3$
5. A marketing manager estimates that t days after termination of an advertising campaign, the sales of a new product will be S(t) units, where
   $S(t) =4000\ e^{-0.015t}$
   a. How many units are being sold at the time advertising ends?
   b. How many units will be sold 30 days after the advertising ends? After 60 days?
6. The population density x miles from the center of a city is given by a function of the form
   $Q(x) = Ae^{-kx}$
   Find this function if it is known that the population density at the center of the city is 15,000 people per square mile and the density 10 miles from the center is 9,000 people per square mile.
7. Define homogeneous function with example.
8. Explain Euler's Theorem with realistic example.

Further Readings

- Mathematics for Economics-Council for economic education
- Essential Mathematics for Economists- Nutt Sedester, peter Hawmond, Prentice Hall Publication
- Mathematics for Economists- Carl P Simone, Lawrence Bloom.
- Mathematics for Economist- Simone and Bloom, Viva Publication

Unit 02: Basic Real Analysis

CONTENTS

Objectives

After studying this unit, you will be able to,

- define the continuity of a function at a point of its domain,
- determine whether a given function is continuous or not,
- describe the notion of a neighborhood of a point on the line, and the notion of limit points of a set;
- find the limit points of a set;
- describe when a set is closed and understand the properties of closed sets; and describe when a set is open sets.

Introduction

In this unit, we introduce you to the notion of metric spaces. The metric spaces arose from extending the notions of continuity and convergence on the real line to more abstract spaces. A metric space is just a set which is equipped with a function called metric which measures the distance between the elements of various pairs from the set.

We Begin our study of metric spaces by extending those aspects of real and complex number systems that help us in studying their analytical structure and functions on them. What do you think are the most important concepts in Analysis, then? You would agree that the limit and continuity are the ones which come to your mind. Whatever you studied later had made use of these building blocks. Now, instead of a set of real numbers, if we start with any arbitrary set of objects, how can we introduce limit and continuity on it? Obviously, it would have to have some extra properties. Now, if you go back and recall the definition of a limit, we say the limit of f(x) as x approaches a is l, if f(x) gets arbitrarily close to l as x gets sufficiently close to a. To be precise,

∀c > 0, $\exists \delta > 0$, s.t

$$0 < |x - a| < \delta_1 |f|x| - l| < \varepsilon$$

Here [x – a] and [f(x) – L] denote the distance between x and a and between f(x) and l, respectively.

So, it seems that if we can introduce the notion of 'distance' on our arbitrary set of objects.

Sets form the basis of mathematics, but we cannot do much with them in economics unless we dene an additional structure on them - the notion of distance between two elements. If we can measure the distance between elements in a set, the set is called a metric space. The elements of a metric space are called points.

2.1 Definition

A set X is said to be a metric space if with any two points p and q of X there is associated a real number d(p, q) called the distance from p to q such that

Definition 1: Let X be a non-empty set. A metric d on X is a function d : X x X → R, such that, for x, y, z ε X,

1) d(x,y) >0 and d(x,y) = O if and onlyif x=y,

2) d(x,y) =d(y,x),

3) d(x<y) < d(x<z) + d(z,y), x, y, z εX.

For x, y ε X, d(x, y) is called the distance between x and y.

Any function with these three properties is called a distance function or metric

The most important metric spaces we encounter in economics are the Euclidean spaces R n, in particular the real numbers ℝ and the real plane ℝ^2. In these spaces, the most commonly used distance is the Euclidean distance, which is defined as

$$d(x_3y) = \|x - y\| = \sqrt{\sum_{i=1}^{N}(x_i - yi)^2}$$

Other distances could be street or grid distances. For example, if you are on the south-west corner of a city block and you want to go to the north-east corner of the same block, you must travel east one block and north one block. The grid distance you walked is two blocks, whereas the Euclidean distance is √ 2 blocks. The grid distance can be defined mathematically as

$$d(x, y) == \sum_{i=1}^{N} (x_i - yi)$$

2.2 Some Definitions

Equipped with a distance d we can define the following subsets of a metric space X (you can simply think of d as the Euclidean distance and of X as ℝ^N):

a. Open and Closed Balls: The set B(x, r) = {y ∈ X : d(x, y) < r} is called the open ball B(x, r) with center x and radius r. The set B(x, r) = {y ∈ X : d(x, y) ≤ r} is called the closed ball B(x, r) with center x and radius r.In contrast to an open ball, a closed ball contains the points of the boundary where d(x, y) = r. Sometimes the radius is labeled instead of r and then the ball is also called epsilon ball. Note that in R an open ball is simply an open interval (x − r, x + r), i.e. the set {y ∈R : x − r < y < x + r}, and a closed ball is simply a closed interval (x − r, x + r), i.e. the set {y ∈ R : x − r ≤ y ≤ x + r}.

b. Open and Closed Sets: A set U ⊂ X is open if ∀ x ∈ U there exists r > 0 such that B(x, r) ⊂ U. In English: A set is open if for any point x in the set we can find a small ball around x that is also contained in the set. Basically, an open set is a set that does not contain its boundary since any ball around a point on the boundary will be partially in the set and partially out of the set. For example, the interval (0, 1) is open in ℝ since for any point x in (0, 1), we can find a small interval around x that is also contained in (0, 1).

   A set U ⊂ X is closed if its compliment is open. An equivalent definition is that a set is closed iff∀ sequences $\{x_k\}$ with x_k∈ U ∀ k and $\{x_k\}$ → x, then x ∈ U. Basically a closed set is a set that contains its boundary (since the complement of that set does not contain the boundary and is thus open). The definition using sequences says that if a sequence $\{x_k\}$ gets arbitrarily close to a point x while staying in the closed set then the point x also has to be in the set. For example, the interval [0, 1] is closed in ℝ since its complement, the set (−∞, 0) ∪ (1, ∞), is open. Note that a set can be open (e.g. (0, 1)), closed (e.g. [0, 1]), neither (e.g. (0, 1]) or both ({}, ℝ)!

c. Bounded Set: A set U ⊂ X is bounded if □ r > 0 and x □ X such that U □B(x, r). This is an easy one: A set is bounded if we can fit it into a large enough ball around some point. A set is not bounded if no matter how large we choose the radius of the ball, the set will not be completely contained in it.

The next two definitions concern the Euclidean space R N only.

d. Compact Set: A set U ⊂ R N is compact if it is closed and bounded. So we can think of a compact set in R N as a set that its into a ball and contains its boundary. In a general metric space, the definition of compact set is different, but we do not have to deal with it here.

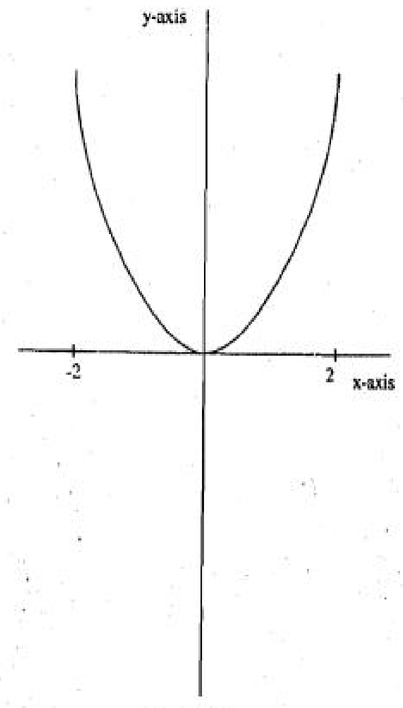
2.3 Continuous Functions, their Optima and their Existence

Suppose that you have functions which are defined on an interval, either opeN; or closed. If you draw the graph of these functions, you will observe that some of these can be sketched down in one smooth 'continuous' sweep of your pen, While others have many breaks or jumps. For example, draw the graph of the following two functions:

A. $f(x) = x^2, x \in [-2, 2]$;

B. . $g(x) = \begin{cases} \frac{1}{x}, x\varepsilon[-2,2] \\ 0, x = 0 \end{cases}$

These are as shown in figures l(a) and l(b).



Pig. 1(a)

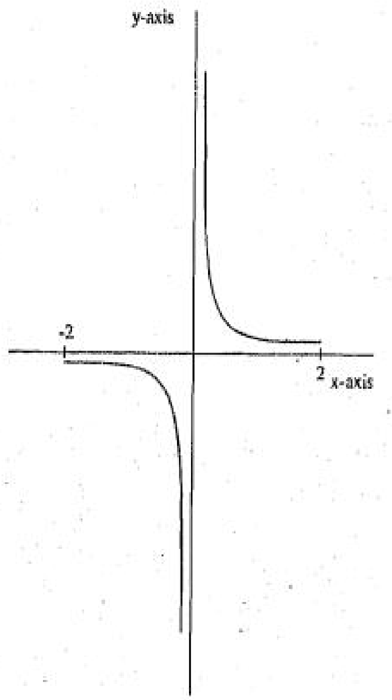


Fig. 1(b)

You can see that while the graph of the first function can be drawn in the 'continuous' motion without lifting the pen from the paper while the graph of the other function cannot lifting &awn in this manner. This is an interesting property of the first function which is nut possessed by the second function. It is, therefore, natural to wonder if it can be given some mathematical meaning. In fact, mathematicians of the past several centuries did confront this question, namely:

"1s there a way to specify those curves which can be drawn with a single stroke of one's pen?"

Continuous Functions

We have seen that the limit of a function f as the variable x approaches a given point a in the domain of'a function f does not depend at all on the value of the function at that point a but it depends only on the values of the function at the points near a. In fact, even if the function f is not defined at a then $\lim_{x \to a} f(x)$ may exist.

For example, $\lim_{x \to a} f(x)$ exists when

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ though f is not defined at } x = 1.$$

We have also seen that $\lim_{x \to a} f(x)$ may exist, still it need not be the same as f(a) when it exists. Naturally, we would like to examine the special case when both $\lim_{x \to a} f(x)$ and f(a) exist and are equal. If a function has these properties, then it is called a continuous function at the point a. We give the precise definition as follows:

DEFINITION 1: Continuity of a Function at a Point

A function f defined on a subset S of the set R is said to be continuous at a point

a S, if

i) lim f(x) exists and is finite

ii) $\lim_{x \to a} f(x)$ = f(a)

Note that in this definition, we assume that S. contains some open interval containing the point a. If we assume that there exists a half open (semi-open) interval [a, c] contained in S for some c E R, then in the above definition, we can replace $\lim_{x \to a} f(x)$ by $\lim_{x \to a+} f(x)$ and say that the function is continuous from the right of a or f is right continuous at a.

iii) Similarly, you can define left continuity at a, replacing the role of by $\lim_{x \to a} f(x)$. Thus, f is continuous from the right at a if and only if $\lim_{x \to a} f(x)$ by $\lim_{x \to a-} f(x)$. Thus, f is continuous from the right at a if and only if

iv) F(a+) =f(a)

It is continuous from the left at a if and only if f(a-) = f(a)

From the definition of continuity of a function f d a point a and properties of limits it follows that f(a+) = f(a-) = f(a) if and only if, f is continuous at a. If a function is both continuous from the right and continuous from the left at a point a, then it is continuous at a and conversely.

The definition 1 is popularly known as the Limit-Definition of Continuity.

Since $\lim_{x \to a} f(x)$ is also defined in terms of E and 6, we have an equivalent x-a formulation of the definition 1 in terms of ε and 6. Note that whenever we talk of continuity of a function f at a in S, we always assume that S contains a neighborhood containing a. Also remember that if there is orie such neighborhood there are infinitely many such neighborhoods. An equivalent definition of continuity in terms of ε and 6 is given as follows:

DEFINITION 2: Definition of Continuity

A fuction f is continuous nt x = a if f is defined in a neighborhood of a and corresponding of a given number E > 0, there exists some number δ> 0 such that (x – a) <δ implies (f(x) -- f(a)<E.

Example 1: Examine the continuity of the function f(x)=x^3 +3x-4 at x=1

Sol.1: Condition 1: For x=1, f(1)= $(1)^3$ +3(1)-4=1+3-4=0

Therefore, the function is defined for x =1.

Condition 2: $\underset{x \to 1}{Lt}(x^3 + 3x - 4) = (1)^3 + 3(1) - 4 = 0$. That is, the limit exists.

Condition 3: Thus, we find that f(1)= : $\underset{x \to 1}{Lt} f(x) = 0$

Since all the conditions are satisfied the function is continuous at x =1.

Example 2: Determine at which values of x , the functions given below are continuous.

a. $f(x) = \frac{x^4 + 3x^2 - 1}{(x-1)(x+2)}$

Solution: $f(x) = \frac{x^4+3x^2-1}{(x-1)(x+2)}$. This is a rational function and hence is continuous at all points except when (x-1)(x-

2) i.e. denominator vanishes (is zero). This gives us two cases:

(i) When x– 1 = 0, x = 1 ,

$\underset{x\to 1}{Lt} f(x) = \infty$i.e. undefined.

(ii) When x +2 = 0, x = −2,

$\underset{x\to -2}{Lt} f(x) = \infty$i.e. undefined.

Other conditions will not be needed. Hence, the function is discontinuous being undefined at x =1and x = −2 .

Continuity of a Function over an Interval

A function y = f (x) is said to be continuous in an interval (a, b), if it is continuous at every value of x in that interval, i.e., it is continuous

1) at x a = 2) at x b = 3) and at any point between a b and .

Example 3: Show that the function 1/ x − 2 is continuous for values of x from

x = −2 to x = −1i.e., in the interval [−2, −1].

Solution: In order to prove that the function is continuous over the range [−2, −1], we will prove that:

1) It is continuous at x = −2
2) It is continuous at x = −1
3) It is continuous for any value between − − 2 and 1, say, 1.5 or 3/2.

Case 1: Let us first find out $\underset{x\to -2}{lim} f(x)$

Assessing Right hand limit

$\underset{x\to -2+}{lim} f(x) = \underset{h\to 0}{lim} f(-2+h) = \underset{h\to 0}{lim} 1/(-2+h-2) =$ -1/4

Assessing Left hand limit

$\underset{x\to -2-}{lim} f(x) = \underset{h\to 0}{lim} f(-2-h) = \underset{h\to 0}{lim} 1/(-2-h-2) =$ -1/4

Since, $\underset{x\to -2+}{lim} f(x) = \underset{x\to -2-}{lim} f(x) = -\frac{1}{4}$.

so the function f (x) is continuous at x = −2

Case 2: Similarly, we can prove that the function is continuous at x = −1. We leave this exercise for the students to try themselves

Case 3: Let us first find out $\underset{x\to -3/2}{lim} f(x)$

Assessing Right hand limit

$\underset{x\to -\frac{3}{2}+}{lim} f(x) = \underset{h\to 0}{lim} f(-3/2+h) = \underset{h\to 0}{lim} 1/(-3/2+h-2) =$ -2/7

Assessing Left hand limit

$\underset{x\to -3/2-}{lim} f(x) = \underset{h\to 0}{lim} f(-3/2-h) = \underset{h\to 0}{lim} 1/(-3/2-h-2) =$ -2/7

Since, $\underset{x\to -3/2+}{lim} f(x) = \underset{x\to -3/2-}{lim} f(x) = -2/7$.

Therefore, we find that the function is continuous at $x = -3/2$ which is a point lying between -2 and -1.

It can easily be checked that the function is continuous at every point between -2 and -1.

Notions of differentiability of mappings between Euclidean spaces, chain rule, higher order derivatives.

A. Euclidean Space

If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is denoted by R n . A = $(a_1, a_2, \ldots,$ an) is called a vector in R^n.

A linear equation in n variables involves a string of n + 1 numbers $(a_1, a_2, \ldots, a_n, b)$. A system of m linear equations in n variables can be defined by m such strings as shown below:

$$\begin{pmatrix} a11 & \cdots & ainb1 \\ \vdots & \ddots & \vdots \\ am1am2 & \cdots & amnbm \end{pmatrix}$$

A close look at the Gaussian elimination method shows that we can reduce any system of linear equations to an equivalent trapezoidal system by using some or all the following operations :

(i) multiply a string by a non-zero number

(ii) interchange two strings

(iii) add two strings.

We also observed that the geometrical analogue, available for n = 2 or 3, is somewhat lost when n > 3. We can indeed introduce some arithmetic in strings of arbitrary number of components and retrieve the geometrical analogue also.

Let $(a_1, a_2, \ldots, a_n)$ be a string, called Ordered-n-tuple , of n components, where a1,a2…………. a_n are real numbers. Let us consider the collection of such n-tuples. This collection is denoted by

R^n = {(a1 , a2 , . . . , a), | a1 , a2 , . .. , a are real numbers}

i.e. R n is the set of all n-tuples of real numbers. Two n-tuples

A = (a1, a2 …………..an) B = (b1,b2…………..bn)

are said to be equal, denoted by A = B, if

ai = bi for i = 1 to n

Thus any two n-truples are equal if their corresponding components are equal.

The sum of A and B, denoted by A + B, is defined as

A + B = (a1 + b1, a2 + b2,…………….. an + bn)

i.e. the sum of any two n-tuples can be obtained by adding the corresponding components of the given n-tuples. The following properties of the sum of n-tuples, as defined above, can be proved easily.

(i) If A R^n and B R^n , then A + B R^n

(ii) A + B = (a1 + b1, a2 + b2,………….. an + bn)

B + A = (b1 + a1, b2 + a2………………bn + an)

 B + A = A + B

v) Let (c1,c2……………cn)

Then A + B + C = ((a1 + b1) + c1, (a2 + b2) + c2,……………..(an + b) + cn)

= (a1+(b1+c1),(a2+(b2+c2),………….,an+(bn+cn))

vi) If we denote the n-tuuple (0, 0, . . . 0) by 0, then A + 0 = 0 + A = A.

vii) Since (a1 , a2 , . . . , an) + (− a1 , − a2 , . . . , − an) = (0, 0, . . . , 0) = 0, we have − A + A = A + (− A) = 0 , where (, , . .. ,) − A =(− a1 − a2 – an).

Let (, , . . . ,) A = a1 a2 an be any n-truple and α be any real number. Then, we define the scalar multiple α A by α (αa1, α a2, ………….. α an), i.e. α A is obtained by multiplying every component of A by α.

This operation is called multiplication of a n-truple by a scalar. The following properties of scalar multiplication along with the properties of real numbers are also easy to observe.

viii) If A □ R^n and α□ R, then α A∈R^n . (A is a vector in R^n)

ix) α(A + B) = $\{(\alpha+\beta)a_1 + (\alpha+\beta)a_{2,\ldots\ldots\ldots}, (\alpha+\beta)an$

$$= (\alpha a_1 + \beta a_1, \alpha a_2 + \beta a2, \ldots\ldots\ldots, \alpha a_n + \beta an)$$

$$= (\alpha a_1, \alpha a, \ldots\ldots\ldots,_2 \alpha a_n) + (\beta a_{1,}\beta_{b_2}, \ldots\ldots\ldots\ldots, \beta_b n)$$

$$= \alpha A + \alpha A$$

$Viii)$ (α + β)A = (α + β)a1 + (α + β)$a2$, ………. . , (α + β)an)

=(α a_1 + β $a_{1,}$ α a_2 + β$a_{2,}$..................... α an + β an)

= (α a1, α a2,…………….., α an)+ (β a1, β b2,………………, β bn)

=

$$\alpha A + \beta A$$

(ix) (α β)A = ((α β) a1 + (α β)a2,……………………,α β a_n)

= (α(β a1), α(β a2),………… α(β an))

= α(β a1, β a2,……………….,β an)

= α (β A)

x) 1A= (1 a1,1 a2 ………………….,1an)

= (a1,a2,………….. an)

=A

The set R ^{n} together with the operation of addition and scalar multiplication, as defined above, is called a Euclidean Space. For n = 2 or 3, the operations that we have defined above coincide with the corresponding operations of vector in 2- or 3-dimensional space. In view of this, the n-tuples (a1,a2,……….an) are also called vectors. Although, we are restricting out attention to n-tuples of real numbers, the generalization to n-tuples of complex numbers is almost identical

B. The Chain Rule

Let f: Df □ R ^{n} → R ^{m} and g: Dg o □ R ^{m} → R p be such that f(D_f) □ D_g. If f is differentiable at x □ D and g is differentiable at f(x), then g ◦ f is differentiable at x and

D f ◦ g (x) = Dg f(x) Df(x)

Moreover, if f and g are continuously differentiable on their respective domain, so is g ◦ f on D_f

We know that

f(y) = f(x) + Df(x)(y − x) + R_1(x,y)

and

g f(y) = g (f(x)) + Dg (f(x))(f(y) − f(x)) + R2(f(x), f(y))

= g f(x) + Dg f(x))[Df(x)(y − x) + R1(x, y) + R2 (f x),f(y)]

Now observe that

|Dg f(x) R1(x, y)| ≤ c|R1(x, y)| = o(|x − y|2) as y →x

and that

|R2 f(x), f(y) | ≤ ε|f(x) − f(y) |

for some δ > 0 and whenever |f(x) − f(y)| ≤ δ since R2(w, z) = o(|w − z|2). Finally f is (locally) Lipschitz continuous at x, since f is differentiable there which gives

|f(x) − f(y)| ≤ c|x − y|2

for $|x - y|2 \leq \delta$ and some (other) $\delta > 0$ which, then, gives

$|R2\ f(x), f(y)\ | \leq c\varepsilon|x - y|2$ for $|x - y|2 \leq \delta$

δ and the claim follows.

It follows that

$\partial j\ (g \circ f)(x) = D(g \circ f)(x)ej = Dg\ f(x)\ \ Df(x)ej = Xm\ k=1\ \partial kg\ f(x)\ \ \partial jf_k(x)$

C. Higher Derivatives

Just as in the case of real functions we now move on to higher order derivatives which, when they exist, give us some information about the convexity properties of functions. The latter make it possible to tell maxima, minima and saddle points apart. Taylor's expansion formula will also be generalized.

C.1: Mixed Partial Derivatives

First we would like to take a closer look at second derivatives. In particularweneedtomakesurethatweunderstandwhatkindofobjectstheyare.On the one hand we could simply say that the are the first derivative of thefirst derivative. This, albeit correct, might, however, conceal their mappingproperties.Letusthereforestartwithasmoothfunction

$f{:}f\,i^{D^{R''}}_{f^{\circ}_{C}} \rightarrow R'''$

Itsderivativeisthemap

$Df{:}D \rightarrow L(R^n, R^m) \triangleq R^{m\times n}, x \mapsto Df(x).$

Takingafurtherderivativeisgoingtogiveusamap

$D^2f = DDf{:}D \rightarrow LR^n, L(R^n, R^m) \triangleq R^{m\times n\times n}$

Since a vector valued map can always be considered component by componentandforthesakeofsimplicityweonlyconsiderthecase$m$=1.Thenthederivativeatapoint

$L(R^n, R)sDf(x) = [\partial_1 f(x) \partial_2 f(x) \ldots \partial_n f(x)]$

is a row vector. It is sometimes useful to think of it as a column vector, inwhich cases we call it the *gradient of f*and denote it by $f\,(x)$.If we nowtake a further derivative of the gradient we obtain the so-called *Hessian* ofthefunction f

$$D(\nabla f) = \begin{bmatrix} D(\partial_1 f) \\ D(\partial_2 f) \\ \vdots \\ D(\partial_n f) \end{bmatrix} = \begin{bmatrix} \partial_1(\partial_1 f)\partial_2(\partial_1 f)\cdots\partial_n(\partial_1 f) \\ \vdots \\ \partial_1(\partial_n f)\partial_2(\partial_n f)\ldots\partial_n(\partial_n f) \end{bmatrix} = [\partial_j \partial_k f]_{1\leq j,k\leq n}$$

Wealsodefinethefunctionspace

$C^2(D, R) := f \square C^1(D, R) \cdot \partial_j f \square C^1(D, R)\}$

$= f \in C(D, R) \cdot \partial_j \partial_k f \in C(D, R)\}$

oftwicecontinuouslydifferentiablefunctions

2.4 Local Extrema

Critical Point) Let f C ¹ (D, R) for some D R ⁿ . A point x D is called critical point if f(x) =0.

Motivation. Since vanishing of the gradient is a necessary condition for a local extremum, we would like to find criteria that would allow us to decide it is a point of minimum, maximum or else.

Assume that f C 2 (D, R) and denote (abusing the notation) its Hessian by D2f. By the previous theorem it is symmetric. Let now x D be a point of minimum for f. Then so is t = 0 for

gu : (−t0, t0) → R , t 7→ f(x +tu)

Observe that g ′u (0) = ∇f(x) | u) = 0 and that

$$g''_u(t) = d/dt = \sum_{j=1}^{n} \partial_j f(x+tu)u_j = \sum_{k=1}^{n} \sum_{j=1}^{n} \partial_R \partial_j f(x+tu)u_k u_j$$

Thus,

$$0 \le g''_u(0) = \sum_{k=1}^{n} \sum_{j=1}^{n} \partial_k \partial_j f(x) u_k u_j = u^T D^2 f(x)$$

and this is valid for any nonzero direction u R^n . We also know that if g 0 u (0) = 0 and g 00 u (0) > 0, then gu has a point of minimum at t = 0. It is therefore legitimate to hope that if

0 <u^T D^2 f(x)u 0 6≠u R^n

we would have that x is a minimum of f.

2.5 Implicit Theorem

Suppose f : $R^n \times R^m \to R^m$ is C^1 on an open set containing (a, b) where a R^n and b R^m.

Let F be a real-valued continuous function defined on some neighborhood N of the point (a, b). If

i) F(a,b)=0
ii) ∂_F/∂_y exists and is continuous o n N, and
iii) $\frac{\partial_F}{\partial_y}(a,b) \neq 0$

then there exists a unique function g defined on some neighborhood N, of a such that

i) G(a)=b
ii) F(x, g(x)) = 0 for each x E N,, and
iii) g, is continuous

Moreover, if $\frac{\partial_F}{\partial_y}$ also exists and is continuous on N, then g is continuously differentiable on ax N, and g' is given by

$$g'(t) = \frac{\frac{\partial F}{\partial x}(t_7 g(t))}{\frac{\partial F}{\partial y}(t, y(t))}, t\varepsilon N_a$$

2.6 Inverse Theorem

Let U be an open set in Rn and let f : U → R^n be C^1

. Let x0 U such that Df(x0) is nonsingular. Then there exists a neighborhood W of x0 such that

i. f : W → f(W) is a bijection;

ii. f(W) is an open set in R^n;

iii. f^{-1}: f(W) → W is C^1 and $Df^{-1}(f(x)) = (Df(x))^{-1}$ for x W.

Example 5: : Consider the transformation

F(x,y)=(2xy,x^2 -y^2)

It maps the whole plane R^2 to R^2. However, it is not 1-1 in the whole plane, since f(1,1) = f(-1, -1) = (2,0). Therefore, it is not invertible. In general. f(p) = f(-p). But, if we take D = { (x, y) x > 0). then f restricted to D is 1-1.

To see this, let f(x,y) = f(a,b). We will prove that x = a and y = b. Thus, we are given that

$2xy = 2ab$ and $x^2 - y^2 = a^2 - b^2$

i.e., $x^2 - y^2 - a^2 + b^2 = 0$

Therefore, since x # 0, y = - and on substituting the value of y in the second equation we X obtain

$0 = x^2 - a^2b^2 - x^2a^2 + x^2b^2$ (because xy = ab)

$= (x^2 + b^2)(x^2 - a^2)$

Thus x^2 - a'= 0 or $x^2 + b^2 = 0$. But $x^2 + b^2$ cannot be zero. Therefore $x^2 = a^2$.

But x > 0 and a > 0 (on D). Therefore we get x = a. Then xy = ab gives that y = b. Thus. f maps the open half plane D into R^2 in a one-one manner.

D*={(u,v) v>0 if u=0}

$=R^2$ negative y-axis

is the range of F.

If u = 0, then y has to be zero, because u = 2xy and x > 0 in D. In that case v = x' > 0. Thus, no point on the negative y axis can be the image of a point of D under f. Notice that for any (x, y) ε D*. an open disc around (x, y) is contained in D* If u ε0. then (u, v) = f(x. y), where

$$X=\left[\frac{v+\sqrt{u^2+v^2}}{2}\right]^{1/2}$$

$$Y= u[\ 2v+2\sqrt{u^2+v^2}]^{1/2}$$

Now f is locally invertible at every point of D. This is because the sets D and D* being open sets, can be regarded as neighborhoods of each of their points. Therefore, for any point of D, both the requirements of local invertibility are satisfied. However, the function f : $R^2 \rightarrow R^2$ as defined above is not locally invertible at (0, 0). The reason for this is that given any neighborhood N of (0,0), we can find x, y E R such that (x, y) E N and (-x, -y) EN. Now since we know that f(x, y) = f(-x. -y), we conclude that f is not 1-1 on N. We leave it as an exercise to you to check the local invertibility off at (x.y) when x < 0.

Summary

- Continuous Variable: If x takes all possible real values from a given number a to another given number b, then x is called a continuous variable.
- Continuity: A function f(x) is continuous provided its graph is continuous, i.e., a continuous function does not have any break at any point of its graph. More formally, a function f(x) is said to be continuous for x = a, provided $\lim_{x\to a} f(x)$ exists, finite and is equal to a.
- Compact sets: A set is said to be compact if every open cover of it admits of a finite subcover of the set.
- Implicit function: Implicit Function Theorem states that at each point of regular curve we can consider y as a function of x or x as a function of y
- Inverse function: An inverse function is a function that returns the original value for which a function has given the output. If f(x) is a function which gives output y, then the inverse function of y, i.e. f-1(y) will return the value x.

Keywords

- Metric spaces: a metric space is a non-empty set together with a metric on the set.
- Euclidean space: A Euclidean space is a finite-dimensional vector space over the reals R, with an inner product h (.,.)

- Continuity: without any break or jerk
- Chain Rule: dzdt=∂z∂x dxdt+∂z∂y dydt
- Higher Derivatives: The derivatives other than the first derivative are called the higher order derivatives.

Self Assessment

1. A metric space (X,d) is compact if and only if;

A. Every open cover of x has a finite subcover.
B. Every infinite subset of x has a limit point in X.
C. If X R then X is closed and bounded.
D. All of the above

2. A set "F" is closed if

A. It contains all of its points of closure
B. It contains all if its accumulation point
C. Both a and b are true
D. First Is true, second is false

3. The set of real numbers R is

A. Uncountable
B. Countable
C. Infinite
D. Bounded

4. The union of a infinite number of closed sets need not be a _________

A. Closed set
B. Open set
C. Both (a) and (b)
D. Union

5. The set is open if and only if its complement is__________

A. Closed
B. Open
C. Compact
D. None of the above

6. The function f is continuous at a € M if limx→a f(x)= --------

A. f(b)
B. f(c)
C. f(a)
D. f(x)

7. What is the mathematical expression for f is right continuous on (a,b)?

A. limx→a+ f(x)=f(a)
B. limx→a+ f(x)=f(b)
C. limx→b+ f(x)=f(a)

D. limx→a- f(x)=f(a)

8. limx→a-f(x)=f(b) then f(x) is left continuous at x = a.

A. False

B. True

9. What is the mathematical expression for f is continuous on (a,b)?

A. limx→c f(x) = f(c) □ c □ a

B. limx→c f(x) = f(c) □ c □ (a,b)

C. limx→c f(x) = f(c) □ c □ b

D. limx→a f(x) = f(c) □ c□ (a,b)

10. What is/are conditions for a function to be continuous on (a,b)?

A. The function is continuous at each point of (a,b)

B. The function is right continuous

C. The function is left continuous

D. Right continuous, left continuous, continuous at each point of (a,b)

11. The two-dimensional Euclidean Plane is known as

A. Euclidean space

B. Dihedral space

C. One dimensional space

D. Zero plane

12. What is an implicit Function?

A. If the value of y cannot be expressed in terms of x

B. If the value of x can be expressed in terms of y

C. If the value of y can be expressed in terms of x

D. If the value of x cannot be expressed in terms of y

13. What will be the chain rule?

A. dy/dx = dy/du.du/dx

B. dy/du = dx/du.du/dx

C. dy/dx = dx/du.du/dx

D. dy/dx = dy/du.dy/dx

14. Find the second derivative of the following function:

A. f(x) = 5x2(x + 47)

B. f “(x) = 30x − 470

C. f “(x) = 30x + 470

D. f “(x) = 15x2 + 235

E. f “(x) = 15x2 + 470x

15. If f(x) = g(u) and u = u(x) then

A. f '(x) = g '(u)
B. f '(x) = g '(u)*u '(x)
C. f '(x) = u '(x)
D. None of the above

Answers for Self Assessment

1.	D	2.	C	3.	A	4.	A	5.	A
6.	C	7.	A	8.	B	9.	B	10.	D
11.	A	12.	A	13.	D	14.	B	15.	B

Review Questions

1. Determine whether the following maps are locally invertible at the given point.
   a) $F(x, y)' = (x^3y + 1, x^2 + yz)$ at (1, 2)
   b) $F(x, y) = (e^{xY}, \text{In } x)$ at (I, 4)

2. $f(x)=x\sin 1/x$ for $x \neq 0$

   0 for $x = 0$

   Examine the continuity of f(x) at x = 0.

3. $f(x)= (x^2-4)/(x-2)$
   What should the value of f(2) be, so that f(x) is continuous at x = 2.
4. Find the second order derivative of $y=e^{3x+2}$
5. If φ is a differentiable function such that y = φ (x) satisfies the equation
   $x^3+y^2+\sin xy=0$,
   Find dy/dx.
6. Find the maximum and minimum values of
   $2(x^2- y^2)-x^4 +y^4$

Further Readings

- Mathematics for Economics-Council for economic education
- Essential Mathematics for Economists- Nutt Sedester, peter Hawmond, Prentice Hall Publication
- Mathematics for Economists- Carl P Simone, Lawrence Bloom.
- Mathematics for Economist- Simone and Bloom, Viva Publication.

Web Links

https://www.math.uci.edu/~gpatrick/source/205c06/chapix.pdf

file:///C:/Users/thaku/Dropbox/My%20PC%20(LAPTOP-J1MRTA18)/Desktop/chapix.pdf

Unit 03: Quadratic Equations

CONTENTS

Objectives

After studying this unit, you will be able to,

- Define Quadratic Equations
- Solve a given quadratic equation, and
- Form a quadratic equation with given roots.

Introduction

Solution of equations lies at the very heart of algebra. It has enormous applications. The importance of equations stems from the fact that they provide a means by which many complicated relationships in real–life problems can be written down in a clear and concise form. One type was the quadratic polynomial of the form $ax^2 + bx + c$, $a \neq 0$. When we equate this polynomial to zero, we get a quadratic equation.

Many people believe that Babylonians were the first to solve quadratic equations. For instance, they knew how to find two positive numbers with a given positive sum and a given positive product, and this problem is equivalent to solving a quadratic equation of the form $x^2 – px + q = 0$. Greek mathematician Euclid developed a geometrical approach for finding out lengths which, in our present day terminology, are solutions of quadratic equations. Solving of quadratic equations, in general form, is often credited to ancient Indian mathematicians. In fact, Brahma gupta (C.E.598–665) gave an explicit formula to solve a quadratic equation of the form ax2 + bx = c. Later, Sridharacharya (C.E. 1025) derived a formula, now known as the quadratic formula, (as quoted by Bhaskara II) for solving a quadratic equation by the method of completing the square. An Arab mathematician Al-Khwarizmi (about C.E. 800) also studied quadratic equations of different types. Abraham bar Hiyya Ha-Nasi, in his book 'Liber embadorum' published in Europe in C.E. 1145 gave complete solutions of different quadratic equations.

3.1 Quadratic Equations

A quadratic equation in the variable x is an equation of the form ax2 + bx + c = 0, where a, b, c are real numbers, $a \neq 0$. For example, $2x^2 + x – 300 = 0$ is a quadratic equation. Similarly, $2x^2 – 3x + 1 = 0$, $4x – 3x^2 + 2 = 0$ and $x – x^2 + 300 = 0$ are also quadratic equations.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation. But when we write the terms of $p(x)$ in descending order of their degrees, then we get the standard form of the equation. This unit is about how to solve quadratic equations.

A quadratic equation is one which must contain a term involving x^2, e.g. $3x^2$, $-5x^2$ or just x^2 on its own. It may also contain terms involving x, e.g. 5x or −7x, or 0.5x. It can also have constant terms - these are just numbers: 6, −7, 1 2. It cannot have terms involving higher powers of x, like x^3. It cannot have terms like 1/x in it. In general, a quadratic equation will take the form

$$ax^2 + bx + c = 0$$

a can be any number excluding zero. b and c can be any numbers including zero. If b or c is zero then these terms will not appear.

Quadratic equations arise in several situations in the world around us and in different fields of mathematics. Let us consider a few examples.

Example 1: Represent the following situations mathematically:

(i) John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 124. We would like to find out how many marbles they had to start with.

(ii) (ii) A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ` 750. We would like to find out the number of toys produced on that day.

Solution: (i) Let the number of marbles John had be x.

Then the number of marbles Jivanti had = 45 – x (Why?).

The number of marbles left with John, when he lost 5 marbles = x – 5

The number of marbles left with Jivanti, when she lost 5 marbles = 45 – x – 5

$$= 40 - x$$

Therefore, their product = $(x - 5)(40 - x)$

$$= 40x - x^2 - 200 + 5x$$

$$= -x^2 + 45x - 200$$

So, $-x^2 + 45x - 200 = 124$ (Given that product = 124)

i.e., $-x^2 + 45x - 324 = 0$

i.e., $x^2 - 45x + 324 = 0$

Therefore, the number of marbles John had, satisfies the quadratic equation

$$x^2 - 45x + 324 = 0$$

which is the required representation of the problem mathematically.

(ii) Let the number of toys produced on that day be x. Therefore, the cost of production (in rupees) of each toy that day = 55 – x So, the total cost of production (in rupees) that day = x (55 – x) Therefore, $x(55 - x) = 750$

i.e., $55x - x^2 = 750$

i.e., $-x^2 + 55x - 750 = 0$

i.e., $x^2 - 55x + 750 = 0$

Therefore, the number of toys produced that day satisfies the quadratic equation

$$x^2 - 55x + 750 = 0$$

which is the required representation of the problem mathematically.

Example 2 : Check whether the following are quadratic equations:

(i) $(x - 2)^2 + 1 = 2x - 3$

(ii) $x(x + 1) + 8 = (x + 2)(x - 2)$

Solution :

(i) LHS = $(x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

Therefore, $(x - 2)^2 + 1 = 2x - 3$ can be rewritten as

$x^2 - 4x + 5 = 2x - 3$

i.e., $x^2 - 6x + 8 = 0$

It is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

(ii) Since $x(x + 1) + 8 = x^2 + x + 8$ and $(x + 2)(x - 2) = x^2 - 4$

Therefore, $x^2 + x + 8 = x^2 - 4$

i.e., $x + 12 = 0$

It is not of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

There are three possible methods one might try to use to solve for the unknown in a quadratic equation:

(i) by plotting a graph

(ii) by factorization

(iii) completing square

(iv) using the quadratic 'formula'

Solving quadratic equations by using graphs

In this section we will see how graphs can be used to solve quadratic equations. If the coefficient of x^2 in the quadratic expression $ax^2 + bx + c$ is positive then a graph of $y = ax^2 + bx + c$ will take the form shown in Figure 1(a). If the coefficient of x^2 is negative the graph will take the form shown in Figure 1(b).

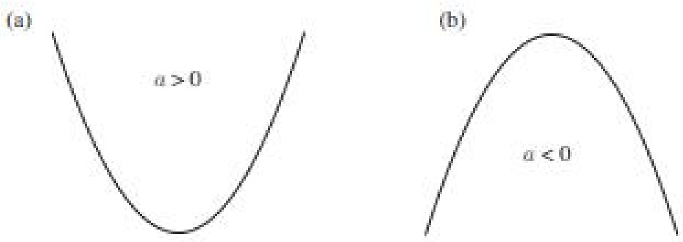


Figure 1. Graphs of $y = ax^2 + bx + c$ have these general shapes

Add x and y axes. Figure 2 shows what can happen when we plot a graph of $y = ax^2 + bx + c$ for the case in which a is positive.

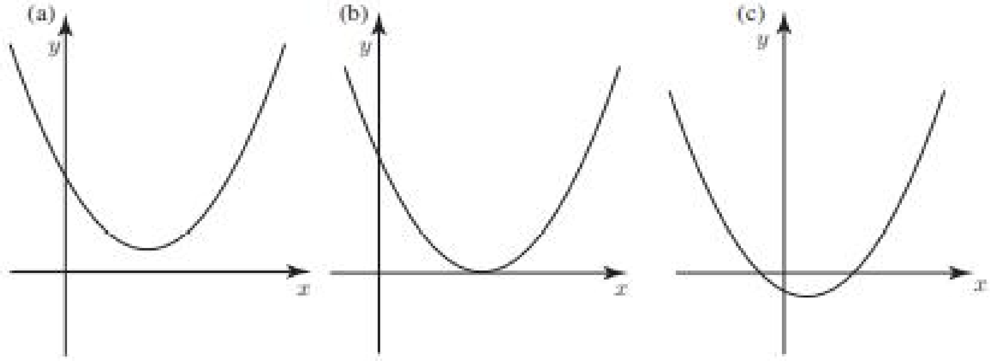


Figure 2. Graphs of $y = ax^2 + bx + c$ when a is positive

The horizontal line, the x axis, corresponds to points on the graph where y = 0. So, points where the graph touches or crosses this axis correspond to solutions of $ax^2 + bx + c = 0$.

In Figure 2, the graph in (a) never cuts or touches the horizontal axis and so this corresponds to a quadratic equation $ax^2 + bx + c = 0$ having no real roots.

The graph in (b) just touches the horizontal axis corresponding to the case in which the quadratic equation has two equal roots, also called 'repeated roots'.

The graph in (c) cuts the horizontal axis twice, corresponding to the case in which the quadratic equation has two different roots.

Figure 2 is the case in which a is positive, the other case is that a is negative. This case is shown in Figure 3.

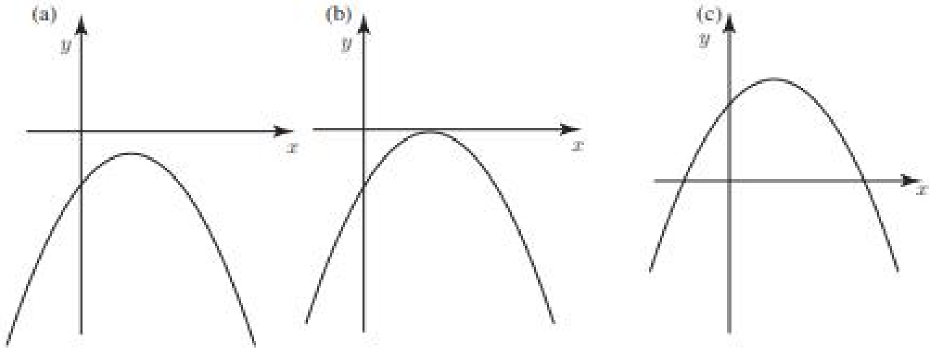


Figure 3. Graphs of $y = ax^2 + bx + c$ when a is negative

Referring to Figure 3:

in case (a) there are no real roots. In case (b) there will be repeated roots. Case (c) corresponds to there being two real roots.

1. Graph: $f(x)=-2x^2+10x-7$

   Solution: We first note that a=-2 and since a<0, the graph will open downward. Let's next find the vertex, or turning point. The x-coordinate of the vertex is

$$X=-b/2a$$
$$= -10/2(-2)$$
$$= 5/2$$

Substituting 5/2 for x in the equation, we find the second coordinate of the vertex:

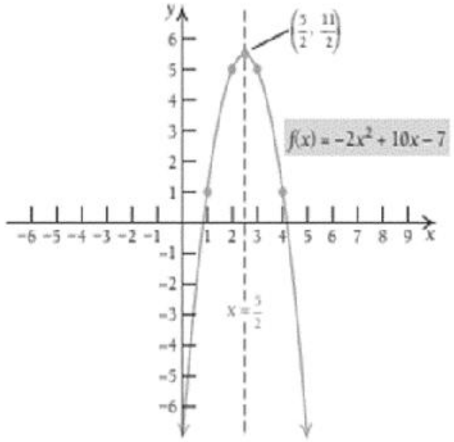
$$Y=f(5/2)$$
$$= -2(5/2)^2 +10(5/2) -7$$

= -2(25/4) +25-7

= 11/2.

The vertex is (5/2, 11/2), and the line of symmetry is x=5/2. We choose some x-values on each side of the vertex, compute y-values, plot the points, and graph the parabola:

x	$f(x)$	
$\frac{5}{2}$	$\frac{11}{2}$	← Vertex
0	−7	
1	1	
2	5	
3	5	
4	1	
5	−7	



2. Show graphically that a solution does exist for the quadratic equation

   $2q^2 - 85q + 200 = 0$

   Solution

We first need to define a function

$$y = 2q^2 - 85q + 200$$

If the graph of this function cuts the q axis then y = 0 and we have a solution to the quadratic equation specified in the question. Next, we calculate a few values of the function to get an approximate idea of its shape.

When q = 0, then y = 200

When q = 1, then y = 2 − 85 + 200 = 117

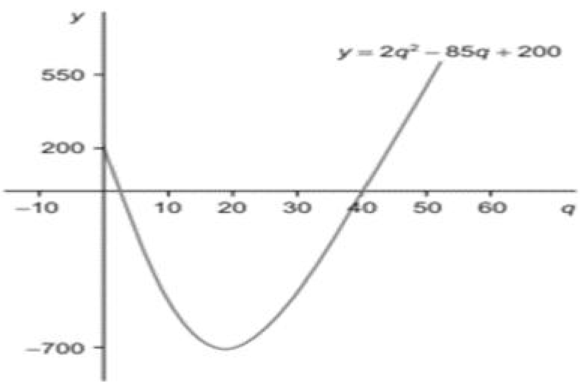
and so the graph initially falls.

When q = 3, then y = 18 − 255 + 200 = −37

and so it must cut the q axis as y has gone from a positive to a negative value.

When q = 50 then y = 5,000 − 4,250 + 200 = 950

and so the value of y rises again and must cut the q axis a second time.



These values indicate that the graph is a U-shape, as shown in Figure 6.1. This cuts the horizontal axis twice and so there are two values of q for which y is zero, which means that there are two solutions to the question.

Features of a quadratic graph

1. The graph of a quadratic function has a characteristic shape called a parabola.

2. This is a curve with a single maximum or a minimum point.

3. The sign of the constant, a, in the quadratic function, indicates whether the parabola has a maximum or a minimum point. For $a > 0$, the parabola has a minimum point and for $a < 0$, the parabola has a maximum point.

4. Its degree of concavity depends on the actual values of the constants a, b and c. The quadratic curve has a single axis of symmetry (a vertical) which passes through the maximum or the minimum point. 5. The quadratic function is not a one to one function. If we draw a horizontal line on the graph, it cuts at two points, except at the maximum or the minimum point.

6. The x-coordinates of the point of intersection of the curve and the x-axis are called the roots or solutions of the quadratic equation $ax^2 + bx + c = 0$.

Solving quadratic equations by Factorization Method

Consider the quadratic equation $2x^2 – 3x + 1 = 0$. If we replace x by 1 on the LHS of this equation, we get $(2 \times 1^2) – (3 \times 1) + 1 = 0$ = RHS of the equation. We say that 1 is a root of the quadratic equation $2x^2 – 3x + 1 = 0$. This also means that 1 is a zero of the quadratic polynomial $2x^2 – 3x + 1$.

In general, a real number α is called a root of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ if $a\alpha^2 + b\alpha + c = 0$. We also say that $x = \alpha$ is a solution of the quadratic equation, or that α satisfies the quadratic equation. Note that the zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.

Steps to be followed while solving quadratic equation by Factorization method

Step 1: Write the equation in the correct form. To be in the correct form, you must remove all parentheses from each side of the equation by distributing, combine all like terms, and finally set the equation equal to zero with the terms written in descending order.

Step 2: Use a factoring strategy to factor the problem.

Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.

Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.

3. Find the roots of the equation 2x 2 – 5x + 3 = 0, by factorization.

   Solution: Let us first split the middle term $– 5x$ as $–2x –3x$ [because $(–2x) * (–3x) = 6x^2 = (2x^2) *3$].

   So, $2x^2 – 5x + 3 = 2x^2 – 2x – 3x + 3 = 2x (x – 1) –3(x – 1) = (2x – 3)(x – 1)$

   Now, $2x^2 – 5x + 3 = 0$ can be rewritten as $(2x – 3)(x – 1) = 0$.

   So, the values of x for which $2x^2 – 5x + 3 = 0$ are the same for which $(2x – 3)(x – 1) = 0$, i.e., either $2x – 3 = 0$ or $x – 1 = 0$.

   Now, $2x – 3 = 0$ gives $x =3/2$ and $x – 1 = 0$ gives $x = 1$.

   So, $x =3/2$ and $x = 1$ are the solutions of the equation.

   In other words, 1 and 3 /2 are the roots of the equation $2x^2 – 5x + 3 = 0$

4. Suppose we wish to solve $3x^2 = 27$. We begin by writing this in the standard form of a quadratic equation by subtracting 27 from each side to give $3x^2 − 27 = 0$.

   Check common factors. By observation there is a common factor of 3 in both terms. This factor is extracted and written outside a pair of brackets. The contents of the brackets are adjusted accordingly:

$$3x^2 − 27 = 3(x^2 − 9) = 0$$

Notice here the difference of two squares which can be factorized as

$$3(x^2 - 9) = 3(x - 3)(x + 3) = 0$$

If two quantities are multiplied together and the result is zero then either or both of the quantities must be zero. So either

$$x - 3 = 0 \text{ or } x + 3 = 0$$

so that $x = 3$ or $x = -3$ These are the two solutions of the equation.

5. Suppose we wish to solve $4x^2 + 9 = 12x$.

   First of all, we write this in the standard form:

$$4x^2 - 12x + 9 = 0$$

   We should look to see if there is a common factor - but there is not. To factorize we seek two numbers which multiply to give 36 (the coefficient of x 2 multiplied by the constant term) and add to give −12. Now, by inspection,

$$-6 \times -6 = 36 \quad \& \quad -6 + -6 = -12$$

   so the two numbers are −6 and −6. We use these two numbers to write −12x as −6x − 6x and proceed to factorise as follows:

$$4x^2 - 12x + 9 = 0$$
$$4x^2 - 6x - 6x + 9 = 0$$
$$2x(2x - 3) - 3(2x - 3) = 0$$
$$(2x - 3)(2x - 3) = 0$$

   from which

$$2x - 3 = 0 \text{ or } 2x - 3 = 0$$

   so that

$$x = 3/2 \text{ or } x = 3/2.$$

   These are the two solutions, but we have obtained the same answer twice.

   So, we can have quadratic equations for which the solution is repeated.

3.2 Solving Quadratic Equations by Completing the Square

The product of Sunita's age (in years) two years ago and her age four years from now is one more than twice her present age. What is her present age?

To answer this, let her present age (in years) be x. Then the product of her ages two years ago and four years from now is $(x - 2)(x + 4)$

Therefore, $\quad (x - 2)(x + 4) = 2x + 1$

i.e., $\quad x^2 + 2x - 8 = 2x + 1$

i.e., $\quad x^2 - 9 = 0$

So, Sunita's present age satisfies the quadratic equation $x^2 - 9 = 0$.

We can write this as $x^2 = 9$. Taking square roots, we get $x = 3$ or $x = -3$. Since the age is a positive number, $x = 3$.

So, Sunita's present age is 3 years.

Now consider the quadratic equation $(x + 2)^2 - 9 = 0$. To solve it, we can write it as $(x + 2)^2 = 9$. Taking square roots, we get $x + 2 = 3$ or $x + 2 = -3$.

Therefore, $x = 1$ or $x = -5$

So, the roots of the equation $(x + 2)^2 - 9 = 0$ are 1 and – 5.

In both the examples above, the term containing x is completely inside a square, and we found the roots easily by taking the square roots. But, what happens if we are asked to solve the equation

$$x^2 + 4x - 5 = 0?$$

We would probably apply factorization to do so, unless we realize (somehow!) that

$x^2 + 4x - 5 = (x + 2)^2 - 9$.

So, solving $x^2 + 4x - 5 = 0$ is equivalent to solving $(x + 2)^2 - 9 = 0$, which we have seen is very quick to do. In fact, we can convert any quadratic equation to the form $(x + a)^2 - b^2 = 0$ and then we can easily find its roots.

Steps to be followed while solving quadratic equation by Complete Square method

Step 1: Write the quadratic in the correct form, it must be in descending order and equal to zero.

Step 2: Fill in the first blank by taking the coefficient (number) from the x-term (middle term) and cutting it in half and squaring it. Fill in the second blank with the same number.

Step 3: Factor the part in parenthesis and combine like terms for the numbers outside of the parenthesis. Note: The part in parenthesis will always factor into half of the coefficient (number) from the x-term (middle term).

Step 4: Now you are done completing the square and it is time to solve the problem.

Step 5: Use the square root property and take the square root of each side, don't forget the plus or minus.

Step 6: Add according to requirement to each side.

Step 7: Check to determine if you can simplify the square root. Thus, final answer will be same.

6. Solve: $x^2 + 8x + 4 = 0$

Step 1: Write the quadratic in the correct form, it must be in descending order and equal to zero.	$(x^2 + 8x + \_\_\_\_\_\_) - \_\_\_\_\_\_ + 4 = 0$
Step 2: Fill in the first blank by taking the coefficient (number) from the x-term (middle term) and cutting it in half and squaring it. Fill in the second blank with the same number. In this case: $\left(\frac{8}{2}\right)^2 = 16$	$(x^2 + 8x + 16) - 16 + 4 = 0$
Step 3: Factor the part in parenthesis and combine like terms for the numbers outside of the parenthesis. **Note:** The part in parenthesis will always factor into half of the coefficient (number) from the x-term (middle term). In this case, half of +8 is +4.	$(x + 4)^2 - 12 = 0$
Step 4: Now you are done completing the square and it is time to solve the problem. First add 12 to both sides.	$(x + 4)^2 = 12$
Step 5: Use the square root property and take the square root of each side, don't forget the plus or minus.	$x + 4 = \pm\sqrt{12}$
Step 6: Subtract 4 from each side.	$x = -4 \pm \sqrt{12}$

Step 7: Check to determine if you can simplify the square root, in this case we can. So the final answer is:	$x = -4 \pm 2\sqrt{3}$

7. Find the roots of the equation $5x^2 – 6x – 2 = 0$ by the method of completing the square.

Solution: Multiplying the equation throughout by 5, we get 25x 2 – 30x – 10 = 0

This is the same as $(5x)^2 – 2 \times (5x) \times 3 + 32 – 32 – 10 = 0$

i.e., $(5x – 3)^2 – 9 – 10 = 0$

i.e., $(5x – 3)^2 – 19 = 0$

i.e., $(5x – 3)^2 = 19$

i.e., $5x – 3 = \pm\sqrt{19}$

i.e., $5x = 3\sqrt{19}$

So, x = (3 ±√19)/5

Therefore, the roots are (3 +√19)/5 and (3 -√19)/5

Method in general:

1. First, move the constant term to the right side of the equal sign:

$$ax^2+bx=-c$$

2. As we want the leading coefficient to equal 1, divide through by *a*:

$$x^2+b/ax=-c/a$$

3. Then, find 1/2 of the middle term, and add $(1/2*b/a)^2=b^2/4a^2$ to both sides of the equal sign:

$$x^2+b/ax+b^2/4a^2=b^2/4a^2-c/a$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$(x+b/2a)^2=(b^2-4ac)/4a^2$$

5. Now, use the square root property, which gives

   $(x+b/2a)=\pm\sqrt{(b^2-4ac)}/2a$

6. Finally, add −b/2a to both sides of the equation and combine the terms on the right side. Thus,

$$x=(-b\pm\sqrt{b^2-4ac})/2a$$

3.3 Solving quadratic equations by using the quadratic 'formula'

Consider the general quadratic equation $ax^2 + bx + c = 0$. There is a formula for solving this:

$x = (-b \pm \sqrt{b^2 - 4ac})/2a$. It is so important that you should learn it.

7. Suppose we wish to solve $x^2 - 3x - 2 = 0$.

   Comparing this with the general form $ax^2 + bx + c = 0$ we see that a = 1, b = −3 and c = −2.

   These values are substituted into the formula.

   $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

   $= (-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)})/2 \times 1$

   $= (3 \pm \sqrt{9 + 8})/2$

   $= (3 \pm \sqrt{17})/2$

   These solutions are exact.

8. Solve: $3x^2 = 5x - 1$.

Solution: First we write this in the standard form as $3x^2 - 5x + 1 = 0$ in order to identify the values of a, b and c.

We see that a = 3, b = −5 and c = 1.

These values are substituted into the formula.

$$x = (-b \pm \sqrt{b^2 - 4ac})/2a$$
$$=(-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1})/2 \times 3$$
$$= (5 \pm \sqrt{25 - 12})/6$$
$$= (5 \pm \sqrt{13})/6$$

3.4 Nature of Roots

Since the solution set for every quadratic equation is

$$\left\{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right\},$$

the solutions can be expressed as

$$\alpha = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} \quad or \; \beta = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$

Discriminant

The expression D = is called discriminant because it determines the nature of the solutions of a quadratic equation.

1. If b^2-4ac=0, then α = β and the equation will have one real root.

2. If b^2-4ac >0, then α and β will be two distinct real numbers, and the equation will have two, unequal real roots.

3. Ifb^2-4ac<0, then α and β will be two distinct complex numbers, and the equation will have no real roots.

If D = b^2-4ac<0, then 4ac –b^2>0 . In this case, the complex number.

Sum and Product of the roots

Let us consider the standard form of a quadratic equation,

ax2 + bx + c = 0

(Here a, b and c are real and rational numbers)

Let α and β be the two zeros of the above quadratic equation. Then the formula to get sum and product of the roots of a quadratic equation is,

$$\text{sum of zeros :} \quad \alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$\text{product of zeros :} \quad \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

9. Find the sum and product of roots of the quadratic equation given below.

   $x^2 - 5x + 6 = 0$

   Solution: Comparing $x^2 - 5x + 6 = 0$

   and $ax^2 + bx + c= 0$

   we get, a= 1, b = -5 and c = 6

   Therefore,

   Sum of the roots = -b/a = -(-5)/1 = 5

   Product of the roots = c/a = 6/1 = 6

10. Find the sum and product of roots of the quadratic equation given below.

    $x^2 - 6 = 0$

    Solution : Comparing, $x^2 - 6 = 0$

    and $ax^2 + bx + c = 0$

    we get, a = 1, b = 0 and c = -6

    Therefore,

    Sum of the roots = -b/a = 0/1 = 0

    Product of the roots = c/a = -6/1 = -6

Summary

1. A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.
2. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. The zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.
3. If we can factorize, $ax^2 + bx + c$, $a \neq 0$, into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.
4. A quadratic equation can also be solved by the method of completing the square.
5. Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $x = (-b \pm \sqrt{b^2 - 4ac})/2a$, provided $b^2 - 4ac \geq 0$.
6. A quadratic equation $ax^2 + bx + c = 0$ has

(i) two distinct real roots, if $b^2 - 4ac > 0$,

(ii) two equal roots (i.e., coincident roots), if $b^2 - 4ac = 0$, and

(iii) no real roots, if $b^2 - 4ac < 0$.

Keywords

- Quadratic: a number multiplied by itself
- Discriminant: It determines the number of times the graph crosses the x-axis.
- Vertex: Highest or lowest point of parabola.
- Roots of quadratic: for which quadratic function hold true.

Self Assessment

1. The two orientation of quadratic equation graph parabola are

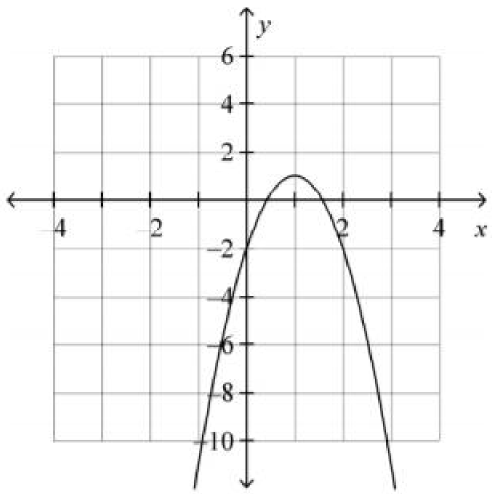
A. concave left

B. concave up

C. concave down

D. both a and b

2. For which discriminant is the graph possible?



A. $b^2 - 4ac = -9$

B. $b^2 - 4ac = 0$

C. $b^2 - 4ac = 4$

D. None of the above

3. An equation $ax^2+bx+c=0$, is called

A. Linear

B. Quadratic

C. Cubic equation

D. None of these

4. If a>0a>0, then the function f(x)=ax2+bx+cf(x)=ax2+bx+c has

A. Maximum value

B. Minimum value

C. Constant value

D. Positive value

5. The graph of quadratic function is

A. Circle
B. Parabola
C. Triangle
D. Rectangle

6. For equal roots of ax2+bx+c=, b2−4ac will be

A. Negative
B. Zero
C. 11
D. 2

7. The roots of the equations ax2+bx+c=0ax2+bx+c=0 are complex or imaginary if
A. b2−4ac≥0
B. b2−4ac>0
C. b2−4ac<0
D. b2−4ac=0

8. For a quadratic equation ax2+bx+c=0ax2+bx+c=0
A. b≠0
B. c≠0
C. a≠0
D. None of these

9. The nature of the roots of quadratic equation depends upon the value of the expression
A. b2+4ac
B. 4ac−b2
C. b2−4ac
D. None of these

10. The quadratic equation has degree
A. 0
B. 1
C. 2
D. 3

11. Number of basic techniques for solving a quadratic equation are

A. Two
B. Three
C. Four
D. None of these

12. A quadratic equation which cannot be solved by factorization, that will be solved by

A. Comparing coefficients
B. Completing square
C. Both A and B
D. None of these

13. If we solve ax2+bx+c=0 by complete square method, we get

A. Cramer's rule
B. De Morgan's Law
C. Quadratic roots
D. None of these

14. The solution set of x2−7x+10=0 is

A. {7,10}
B. {2,5}
C. {5,10}
D. None of these

15. The product of the roots of equation 5x2−x+2=0 is

A. 52
B. −52
C. 25
D. 2

Answers for Self Assessment

1.	D	2.	C	3.	B	4.	A	5.	B
6.	B	7.	C	8.	C	9.	C	10.	C
11.	B	12.	B	13.	C	14.	B	15.	D

Review Questions

1. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:
   (i) $2x^2 - 3x + 5 = 0$
   (ii) $3x^2 - 4 /3 x + 4 = 0$
   (iii) $2x^2 - 6x + 3 = 0$
2. Find the values of k for each of the following quadratic equations, so that they have two equal roots.
   (i) $2x^2 + kx + 3 = 0$
   (ii) $kx (x - 2) + 6 = 0$
3. Find the roots of the following quadratic equations by the factorization method:
   (i) $2x^2 + 5/3 x - 2 = 0$

(ii) $2/5\,x^2 - x - 3/5 = 0$

4. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2? If so, find its length and breadth.
5. Check whether the equation $6x^2 - 7x + 2 = 0$ has real roots, and if it has, find them by the method of completing the squares.
6. Find whether the following equations have real roots. If real roots exist, find them.
   (i) $8x^2 + 2x - 3 = 0$
   (ii) $-2x^2 + 3x + 2 = 0$
   (iii) $5x^2 - 2x - 10 = 0$

Further Readings

- Mathematics for Economics-Council for economic education
- Essential Mathematics for Economists- Nutt Sedester, peter Hawmond, Prentice Hall Publication
- Mathematics for Economists- Carl P Simone, Lawrence Bloom.
- Mathematics for Economist- Simone and Bloom, Viva Publication

Web Links

- https://www.cuemath.com/calculus/quadratic-functions/
- https://tutorial.math.lamar.edu/classes/alg/solvequadraticeqnsi.aspx

Unit 04: Linear Programming

CONTENTS

Objectives

After studying this unit, you will be able to,

- enable you to grasp the basic idea of linear programming principles;
- formulate a management problem as a linear programming problem in suitable cases,
- identify the characteristics of a linear programming problem,
- make a graphical analysis of the problem,
- formulate the dual linear programming problem and analyse the dual variables.

Introduction

Linear programming (LP) is a technique used for deriving optimum use of limited resources. Specifically, it deals with maximizing a linear function of variables subject to linear constraints. Applications range from economic planning and environmental management to the diet problem. In principle, any linear programming problem (often called an LP problem) can be solved numerically, provided that a solution exists. This is because the simplex method introduced by G. B. Dantzig in 1947 provides a very efficient numerical algorithm that finds the solution in a finite number of steps.

4.1 Linear Programming: Basic Concept

Linear programming deals with the optimization of the total effectiveness expressed as a linear function of decision variables, known as the objective function, subject to a set of linear equalities/inequalities known as constraints. Decision variables are the variables in terms of which the problem is defined.

Optimization problems consist of two basic features, viz.,

1. An objective function that you want to minimize or maximize.

2. The objective function describes the behavior of the measure of effectiveness and captures the relationship between that measure and those variables that cause it to change. System variables can be categorized as decision variables and parameters. A decision variable is directly controlled by the decision-maker. Thus, constraints are the relations between decision variables and the parameters.

Thus, any LP problem will have an objective function and a set of constraints. In most cases, constraints come from the nature of problem you work with. For example, if the desirable objective (maximisation or minimisation) in view is of keeping some constraints (i.e., the difficulties, restrictions).

Check the following conditions while formulating a LP problem:

- The objective function must be linear. That is, check if all variables have power of 1 and they are added or subtracted (not divided or multiplied).
- The objective must be either maximisation or minimisation of a linear function. The objective must represent the goal of the decision-maker.
- The constraints must also be linear. Moreover, the constraint must be of the following forms ($\le, \ge$ or $=$, that is, the LP-constraints are always closed) giving the LP in standard format:
- Max $c^T x$ subject to (s.t.) $Ax = b$, for $x \ge 0$

As the above quotation indicates, the simplex method has made linear programming a mathematical technique of immense practical importance. It is reported that when Mobil Oil Company's multimillion-dollar computer system was installed in 1958, it paid off this huge investment in two weeks by doing linear programming. That said, the simplex method will not be discussed in this book. After all, faced with a nontrivial LP problem, it is natural to use one of the great numbers of available LP computer programs to find the solution. In any case, it is probably more important for economists to understand the basic theory of LP than the details of the simplex method.

Indeed, the importance of LP extends even beyond its practical applications. In particular, the duality theory of linear programming is a basis for understanding key theoretical properties of more complicated optimization problems with an even larger range of interesting economic applications.

4.2 Feasible and Optimal Solutions

A solution value for decision variables, where all of the constraints are satisfied, is called a feasible solution. Most solution algorithms proceed by first finding a feasible solution, then seeking to improve upon it, and finally changing the decision variables to move from one feasible solution to another feasible solution. This process is repeated until the objective function has reached its maximum or minimum. This result is called an optimal solution. The basic goal of the optimization process is to find values of the variables that minimize or maximize the objective function while satisfying the constraints. This result is called an optimal solution.

4.3 Formulation: Structure & Variables of Linear Programming

The mathematical formulation of linear programming problem (LPP) is described in the following steps:

1. Identify the decision variables of the problem.

2. Express the objective function, which is to be optimised, i.e., maximised or minimised, as a linear function of the decision variables.

3. Identify the limited available resources, i.e., the constraints and express them as linear inequalities or equalities in terms of decision variables.

4. Since negative values of the decision variables do not have any valid physical interpretation, introduce non-negative restrictions. Let us take an example to illustrate these steps.

Example 1: A small scale industry manufactures two products P and Q which are processed in a machine shop and assembly shop. Product P requires 2 hours of work in a machine shop and 4 hours of work in the assembly shop to manufacture while product Q requires 3 hours of work in machine shop and 2 hours of work in assembly shop. In one day, the industry cannot use more than 16 hours of machine shop and 22 hours of assembly shop. It earns a profit of `3 per unit of product P and `4 per unit of product Q. Give the mathematical formulation of the problem so as to maximise profit.

Solution: Let x and y be the number of units of product P and Q, which are to be produced. Here, x and y are the decision variables. Suppose Z is the profit function.

Since one unit of product P and one unit of product Q gives the profit of 3 and 4, respectively, the objective function is

Maximise: $Z = 3x + 4y$

The requirement and availability in hours of each of the shops for manufacturing the products are tabulated as follows:

	Machine Shop	Assembly	Shop Profit
Product P	2 hours	4 hours	3 per unit
Product Q	3 hours	2 hours	4 per unit
Available hours per day	16 hours	22 hours	

Total hours of machine shop required for both types of product = $2x + 3y$ Total hours of assembly shop required for both types of product = $4x + 2y$ Hence, the constraints as per the limited available resources are:

$$2x + 3y \leq 16 \text{ and}$$

$$4x + 2y \leq 22$$

Since the number of units produced for both P and Q cannot be negative, the non-negative restrictions are:

$$x \geq 0, y \geq 0$$

Thus, the mathematical formulation of the given problem is

Maximise $Z = 3x + 4y$
subject to the constraints
$2x + 3y \leq 16$
$4x + 2y \leq 22$ and
non-negative restrictions
$x \geq 0, v \geq 0$

Example 2: A manufacturer has two types of machines. Her production requires that she must have at least three A type of machines and one B type of machine. The cost of production in type A machine is Rs.1000 while that in type B is Rs.1200 for. The floor areas taken up by each of these machines are $4m^2$ and $5m^2$ respectively. The total cost of production must not exceed Rs. 15000/- and the available floor space is $40m^2$.

Solution: Let x, y be the number of type A and type B machines. Then the equalities will be l00Ox + 1200y ≤15000 for the constraint as regards money.

Such a formulation implies 5x + 6y ≤75 .... (1)

For space constraint we have 4x + 5y ≤ 40 ….... (2)

Again for A type of machine x ≥ 3 . . . . (3)

And for B type of machine x ≥ 1 . . . . (4)

Now, suppose the weekly profit from the output as Rs.120 for each type A and Rs.100 for each type B machine. Our problem is to find the combination of machine use giving maximum profit. If the total profit p = 120x + 100y, then the problems is to find x and y which will maximize the objective function. p = 120x + l00y subject to constraints (I), (2), (3) and (4) above. Here x and y are decision variables.

4.4 Graphic Solution

The graphical method is used to solve linear programming problems having two decision variables. For solving LPPs involving more than two decision variables, we use another method called the Simplex method.

The graphical method of solving a linear programming problem comprises the following steps:

Step 1 : Plot the constraints on the graph paper and find the feasible region.

Step 2: Find the coordinates of that verifies (comer points) of the feasible region.

Step 3: Find the values of the objective function of these comer points.

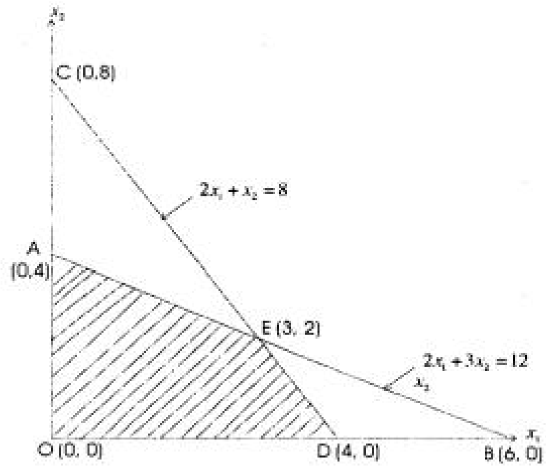
Step 4: Select the comer point at which the value of the objective function is maximum (or minimum as in minimisation problem). This comer point will be the required solution of the given L.P.P.

Example 3: Maximise $z = 6x_1 + 7x_2$

Subject to $2x_1 + 3x_2 \leq 12$

$2x_1 + x_2 \leq 8$

$(x_1, x_2) \geq 0$



The shaded region OAED is the set of points that are feasible under all the constraints it is called the feasible region. The task now is to find which combination of x_1 and x_2 in OAED for which the value of z is maximised.

The task is added by a theorem: If there exists an optimum solution' to the problem, it will be found among the combinations of x_1 and x_2 values represented by the vertices (or the extreme comers) of the feasible solution polygon.

Comer Point	Coordination (x_1, x_2)	Value of z =6x1 + 7x2
O	(0,0)	6(0)+7(0)=0
D	(4,0)	6(4)+7(0)=24
E	(3,2)	6.(3)+7(2)=32
A	(0,4)	6.(0)+7(4)=28

A firm is producing two goods, A and B. It has two factories that jointly produce the two goods in the following quantities (per hour):

The value of z is maximum at E whose coordinates are (3, 2). Hence, the required solution is xl = 3, x2 = 2 and the corresponding maximum value of z =32.

The same process is followed for a minimization process but only then we have to consider the minimum value of z.

Example 3: A firm is producing two goods, A and B. It has two factories that jointly produce the two goods in the following quantities (per hour):

	Factory 1	Factory 2
Good A	10	20
Good B	25	25

The firm receives an order for 300 units of A and 500 units of B. The costs of operating the two factories are 10 000 and 8 000 per hour. Formulate the linear programming problem of minimizing the total cost of meeting this order.

Solution:

Let u_1 and u_2 be the number of hours that the two factories operate to produce the order. Then

$10u_1 + 20u_2$ units of good A are produced, and $25u_1 + 25u_2$ units of good B. Because 300 units of A and 500 units of B are required, u_1 and u_2 must satisfy

$$10u_1 + 20u_2 \geq 300$$

$$25u_1 + 25u_2 \geq 500$$

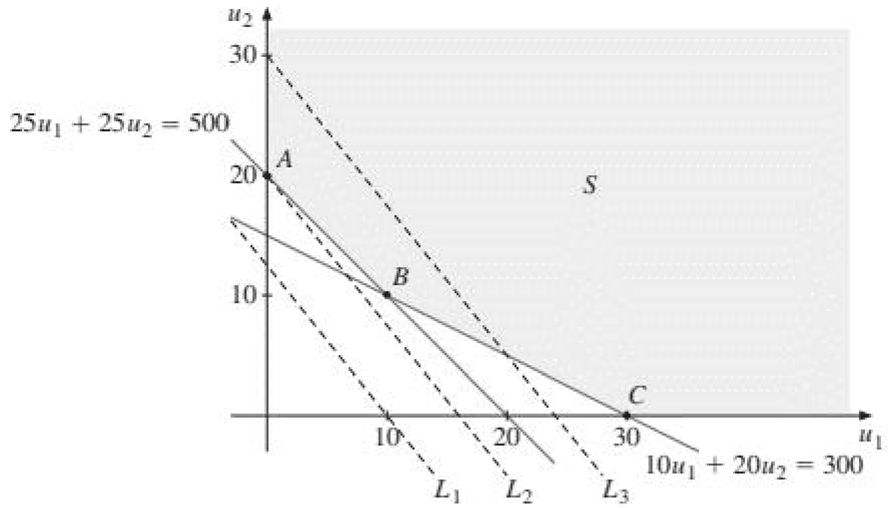
In addition, of course, $u_1 \geq 0$ and $u_2 \geq 0$. The total costs of operating the two factories for u_1 and u_2 hours, respectively, are $10\,000\ u_1 + 8\,000\ u_2$. The problem is, therefore,

min $10\,000\ u_1 + 8\,000\ u_2$ subject to

$$10u_1 + 20u_2 \geq 300$$

$$25u_1 + 25u_2 \geq 500 \qquad u_1 \geq 0,\ u_2 \geq 0$$

The feasible set S is shown in Fig. Because the inequalities in (i) are of the ≥ type and all the coefficients of u_1 and u_2 are positive, the feasible set lies to the north-east. Figure includes three of the level curves $10\,000u_1 + 8\,000u_2 = c$, marked L_1, L_2, and L_3. These three correspond to the values 100 000, 160 000, and 240 000 of the cost level c. As c increases, the level curve moves farther and farther to the north-east.



The solution to the minimization problem is clearly the level curve that touches the feasible set S at point A with coordinates (0, 20). Hence, the optimal solution is to operate factory 2 for 20 hours and not to use factory 1 at all, with minimum cost 160000.

The graphical method of solving linear programming problems works well when there are only two decision variables. One can extend the method to the case with three decision variables. Then the feasible set is a convex polyhedron in 3-space, and the level surfaces of the objective function are planes in 3-space. However, it is not easy to visualize the solution in such cases. For more than three decision variables, no graphical method is available.

4.5 Simplex Method

The simplex method is based on the property that the optimum solution of a L.P. problem if exists can always be found in one of the basic feasible solutions.

Extra Variables

The solution to an LP problem is a set of optimal values for each of the variables. However, the output that comes with the solution to a LP problem usually contains much more information than just this. In addition to the optimal values of the variables, the output will typically include reduced cost values, slack or surplus values, and dual prices (also known as shadow prices).

Reduced Cost

Associated with each variable is a reduced cost value. However, the reduced cost value is only non-zero when the optimal value of a variable is zero. A somewhat intuitive way to think about the reduced cost variable is to think of it as indicating how much the cost of the activity represented by the variable must be reduced before any of that activity will be done. More precisely, ... the reduced cost value indicates how much the objective function coefficient on the corresponding variable must be improved before the value of the variable will be positive in the optimal solution. In the case of a minimization problem, “improved” means “reduced.”

So, in the case of a cost-minimization problem, where the objective function coefficients represent the per-unit cost of the activities represented by the variables, the “reduced cost” coefficients indicate how much each cost coefficient would have to be reduced before the activity represented by the corresponding variable would be cost-effective. In the case of a maximization problem, “improved” means “increased.” In this case, where, for example, the objective function coefficient might represent the net profit per unit of the activity, the reduced cost value indicates how much the profitability of the activity would have to increase in order for the activity to occur in the optimal solution. The units of the reduced cost values are the same as the units of the corresponding objective function coefficients.

If the optimal value of a variable is positive (not zero), then the reduced cost is always zero. If the optimal value of a variable is zero and the reduced cost corresponding to the variable is also zero, then there is at least one other corner that is also in the optimal solution. The value of this variable will be positive at one of the other optimal corners.

Slack or Surplus

A slack or surplus value is reported for each of the constraints. The term "slack" applies to less than or equal constraints, and the term "surplus" applies to greater than or equal constraints. If a constraint is binding, then the corresponding slack or surplus value will equal zero. When a less-than-or-equal constraint is not binding, then there is some unutilized, or slack, resource. The slack value is the amount of a resource, as represented by a less-than-or equal constraint, that is not being used. When a greater-than-or-equal constraint is not binding, then the surplus is the extra amount over the constraint that is being produced or utilized. The units of the slack or surplus values are the same as the units of the corresponding constraints.

Dual Prices (a.k.a. Shadow Prices)

The dual prices are some of the most interesting values in the solution to a linear program. A dual price is reported for each constraint.

The dual price is only positive when a constraint is binding. The dual price gives the improvement in the objective function if the constraint is relaxed by one unit.

In the case of a less-than-or-equal constraint, such as a resource constraint, the dual price gives the value of having one more unit of the resource represented by that constraint. In the case of a greater-than-or-equal constraint, such as a minimum production level constraint, the dual price gives the cost of meeting the last unit of the minimum production target.

The units of the dual prices are the units of the objective function divided by the units of the constraint. Knowing the units of the dual prices can be useful when you are trying to interpret what the dual prices mean.

Standard form of LPP

Standard form of LPP must have following three characteristics:

1. Objective function should be of maximization type

2. All the constraints should of equality type

3. All the decision variables should be nonnegative

The procedure to transform a general form of a LPP to its standard form is discussed below. Let us consider the following example.

Example 4: Maximise $z = x + y$,

subject to $x + y \leq 5$

$x+3y \leq 12,$

$x\geq 0, y\geq 0$

Step 1: Convert the inequalities to equalities by addition of non-negative slack variables. Let s_1 and s_2 be the slack variables, which convert the inequalities into equations. Then

$x + y + s_1 = 5$ and $3x + 2y + s_2 = 10$

where $s_1 \geq 0$ and $s_2 \geq 0$.

The problem can then be written as follows:

maximise $z = 5x + 6y + 0s_1 + 0s_2$

subject to $x + y + s_1 + 0s_2 = 5$ and

$3x + 2y + s_2 + 0s_1 = 12$

$x\geq 0,\ y\geq 0,\ s_1\geq 0,\ s_2 \geq 0$

Step 2: Put the problem in a simplex tableau

			5	6	0	0	
C	Basic variable	Values of the basic variable	x	y	s_1	s_2	Ratio
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	s_1	5	1	1	1	0	5/1 =5
0	s_2	12	2	3	0	1	12/3=4
	z_j	0	0	0	0	0	
	C_j- z_j	-	5	6	0	0	

The 1" column denoted by Cj, is known as the objective column, which represents the coefficients of the objective function of the basic variables listed in Column(2). The 2"d column shows the basic in the solution. The 3rd column represents the values of basic variables listed in Column (2).

Columns (4), (5), (6) and (7) represent the 4 variables x, y, s_1, and s_2 respectively. The row shows the coefficients of the respective variables in the objective function. This row is known as the objective row. The Column (8) known as the ratio column. The elements in zj, row are obtained by multiplying the elements of that column by the compounding elements of the objective column (Cj column) and then added up. Similarly, other elements of the zj row all zero since all the elements in the Cj column are zero. The row Cj- zj is known as the net evaluation row or index row.

Step 3: (a) Calculate the net evaluation of (Cj- zj). To get an element in the net Values row under any column multiply the entries in that column by the corresponding entities in the objective column (Cj) and all them up. Next, subtract this sum from the element in the objective row listed at the top of the table

(b) After examining that the net evaluation row of all the elements are zero or negative, the optimum solution is reached. But if any positive element is present it indicates that a better program can be formulated.

(c) Revise the program.

1) Find the pivot column: The column under which falls the largest positive element of the net evolution row is pivot column.

2) Find the pivot row and the pivot number: Divide the elements of the Application constant column by corresponding non-negative elements of the pivot column to form replacement ratio. The row in which the replacement ratio is the smallest is the pivot row. The number lying at the intersection of the pivot row and the pivot column is the pivot number.

3) Transform the pivot row: Divide all the elements of the pivot row (starting from the constant column) by the pivot number. The resulting numbers will form the corresponding row of the next table.

4) The non-pivot rows are transformed by using the rule

New Number= Old Number - $\frac{Corresponding\ number\ in\ the\ pivot\ row * corresponding\ number\ in\ the\ pivot\ column}{Pivot\ number}$

5) With the results of (3) and (4) form a new table representing a new basic solution. In the new table the variable of the pivot row of previous table will be replaced by the variable of the column of the previous table.

Then steps 3 and 4 are repeated until an optimal column is reached.

Example 5: Maximise $z = 3x_1 + 7x_2 + 6x_3$

Subject to $2x_1 + 2x_2 + 2x_3 \leq 8$

$x_1 + x_2 \leq 3$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Next step is to convert it into equalities by two slack variables s_1 and s_2

Maximise $z = 3x_1 + 7x_2 + 6x_3 + 0s_1 + 0s_2$

Subject to $2x_1 + 2x_2 + 2x_3 + s_1 + 0s_2 = 8$

$x_1 + x_2 + 0s_1 + s_2 = 3$

$(x_1, x_2, x_3, s_1, s_2) \geq 0$

				3	7	6	0	0	
	C_j	Basic variable	Values of the basic variable	x_1	x_2	x_3	s_1	S_2	Ratio
Table 1	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	0	s_1	8	2	2	2	1	0	8/2 =4
	0	s_2	3	1	1	0	0	1	3/1=3
		z_j	0	0	0	0	0	0	
		C_j-z_j	-	3	7	6	0	0	
Table 2	0	s_1	2	0	0	2	1	-2	2/2=1
	7	x_2	3	1	1	0	0	1	
		z_j.	21	7	7	0	0	7	
		C_j-z_j	-	-4	0	0	0	-7	
Table 3	6	x_3	1	0	0	1	½	-1	
	7	x_2	3	1	1	0	0	1	
		zj.	27	7	7	6	3	1	
		C_j-z_j	-	-4	0	0	-3	-1	

In the l[st] program we include only the slack variables s_1= 8, s_2 =3, x_1 =0, x_2 = 0 and x_3 = 0. In this solution, we find that the value of the objective function is zero. Further, from the Cj-zj row it is found that there are positive elements. Hence, a better program can be formulated. The highest positive element in the net evaluation row is 7, which lies in the x_2 column. Then the x_2 column is the pivot column and in the next program x_2 has to be included as one of the basic variables. Now dividing the elements of the constant column by the nonnegative elements of the pivot column, we get the replacement ratios of column (9). The lowest ratio appears to be in the second column (s_2) column, which is now the pivot column. Hence in the next program s_2 will be replaced by x_2. The pivot number is 1, which lies at the intersection of the pivot row and pivot column. The 2nd program is formulated in Table 2, which is given below Table 1. In Table 2 the basic variables are s_1 and s_2. The 2[nd] row (x_2 row) of Table 2 has been obtained by dividing the elements of the 2nd row of Table 1 by the pivot number, 1. The elements of the first now (s_1 row) of Table 2 have been obtained by following the rule for the transformation of the non-pivot row. The calculations are shown below:

$8-\frac{3\times 2}{1}=2 \qquad 2-\frac{1\times 2}{1}=0$

$2-\frac{1\times 2}{1}=0 \qquad 2-\frac{0\times 2}{1}=0$

$1-\frac{0\times 2}{1}=1 \qquad 0-\frac{1\times 2}{1}=-2$

Thus the 2nd table represents a basic solution where $x_1 = 0$, $x_2 = 3$, $x_3 = 0$, $s_1 = 2$, $s_2 = 0$ and where the value of the objective function is 21. Inspecting the net evaluation row (Cj-zj) it is found that one element remains positive. The highest positive number in the net evaluation row lies in the x_3 column, which is now the pivot column. The variable x_3 will have to be introduced as a basic variable in the next programme. The smallest replacement ratio in Table 2 falls in the s_1 row.

Therefore, this row is the pivot row and in the new program s_l will be replace by x_3. The $s_l$row in Table 2 is the pivot row and 2 is the pivot number.

The 3rd program is shown in Table 3, which is placed immediately below Table 2. By diving all the elements of the sl row of Table 2 by 2, the pivot number we get the corresponding elements of the first row (x_3 row) in Table 3. The elements of the second row (x_2 row) of Table 3 have been obtained from the corresponding elements of the 2" row (x_2 row) of Table 2 by using the rule for transformation of the non-pivot row(s). These calculations are shown below:

$$3-\frac{2\times 0}{2}=3 \qquad 1-\frac{0\times 0}{2}=1$$

$$1-\frac{0\times 0}{2}=1 \qquad 0-\frac{2\times 0}{2}=0$$

$$0-\frac{1\times 0}{2}=0 \qquad 1-\frac{(-2)\times 0}{2}=1$$

In the third the basic solution is given by $x_l = 0$, $x_2 = 3$, $x_3 =1$, $s_1 = 0$, $s_2 = 0$. In the net evaluation row of the table it is found that all the elements are either zero or negative. This means that the optimum program has been attained and there is no scope for further improvement. Hence, the required optimum solution is $x_1= 0$, $x_2 = 3$ and $x_3 = 1$. The corresponding value of $z = 27$.

4.6 Duality of Linear Programming

Every L.P. problem is intimately related another called its "dual". For purposes of identification, the original problem is called the primal problem. The relationship between the primal problem and its dual can be summarized as follows:

1) The dual has as many variables as there are constraints in the original problem.

2) The dual has as many constrains as there are variables in the original problem.

3) The dual of a maximization problem is a minimization problem and vice versa.

4) The coefficients of the objective function of the original problem appear as the constant terms of the constraints of the dual and the constant terms of the original constraints are the coefficients of the objective function of the dual.

5) The coefficients of a single variable in the original constraints become the coefficients of a single constraint in the dual. Stated visually, each column of coefficients in the constraints of the original problem becomes a row of coefficients in the dual.

6) The sense of the inequalities, the dual is the reverse of the inequalities in the original problem, except that the inequalities restricting the variables to be non-negative have the same in the primal and the dual.

Example 6: Suppose the primal problem is

Maximise $z=c_lx_l+c_2x_2$

subject to $a_{11}\, x_l. + a_{12}\, x_2 \le b_l$

$a_{21}\, x_l. + a_{22}\, x_2 \le b_2$

$a_{31}\, x_l. + a_{32}\, x_2 \le b_3$

$x_1\geq 0$, $x_2\geq 0$

We have to form the dual of this problem by applying the 6 rules described above. Since the original problem has 3 constraints the dual problem will have 3 variables. Let y_l, y_2 and y_3 be the dual variables. Again, since the original problem has two variables, the dual problem will have 2 constraints. Since the original problem is a maximization problem, the constants b_l, b_2 and b_3 will appear as the coefficients in the objective function of the dual and the constants c_l and c_2 will appear as the constant terms in the righthand side of the constraints of the dual.

Further, the first column of coefficients $\begin{bmatrix} a11 \\ a21 \\ a31 \end{bmatrix}$ in the constraints will be the first row of coefficients in the constraints of the dual.Similarly, the second column of coefficients $\begin{bmatrix} a12 \\ a22 \\ a32 \end{bmatrix}$ will be the second row of coefficients in the constraints of the dual. Again, since the constraints in the original problem all the "less than equal to" type, the constraints in the dual problem will be "greater than equal to" type.

The dual problem will then be as follows:

maximize $w = b_1y_1 + b_2y_2 + b_3y_3$

subject to $a_{11}y_1 + a_{21}y_2 \geq c_1$

$a_{21}y_1 + a_{22}y_2 \geq c_2$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$

Example: Original problem:

Maximise $z=4x+6y$

subject to $1/2\ x + y \leq 4$

$2x+y \leq 8$

$4X-2y \leq 2$

$x \geq 0, y \geq 0.$

Dual problem:

Minimise $w = 4u + 8v + 2w.$

Subject to $1/2\ u + 2v + 4w \geq 4$

$u+v-2w \geq 6$

$u \geq 0,\ v \geq 0, w \geq 0.$

Last Table for the solution of the original problem

			4	6	0	0	0
C_j	Basic variable	Values of the basic variable	x	y	s_1	s_2	s_3
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
6	y	3	0	1	4/5	0	-1/10
0	s_2	1	0	0	-8/5	1	-3/10
4	x	2	1	0	2/5	0	1/5
	z_j	26	4	6	32/5	0	1/5
	C_j-z_j	-	0	0	-32/5	0	-1/5

Last Table for the solution of the dual problem

			4	8	2	0	0	M	M
C_j	Basic variable	Values of the basic variable	u	v	w	s_1	s_2	A_1	A_2
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
2	w	1/5	0	3/10	1	-1/5	1/10	1/5	-1/10
4	u	32/5	1	16/10	0	-2/5	-4/5	2/5	4/5
	z_j	26	4	7	2	-2	-3	2	3
	C_j-z_j	-	0	1	0	2	3	M-2	M-3

Here also it is seen that the value of the objective function is the same in both the problems. In the original problem the elements in the net evaluation row under columns s_1, s_2 and s_3 are -32/5, 0 and -1/5. Hence, the solution of the three dual variables will be u = 32/5, v = 0, and w =-115.

The knowledge of the dual is important for two main reasons.

1) The dual variables have economic interpretations. The values of the dual variables may be useful in taking managerial decisions.

2) The solution of a L.P. problem may be easier through the dual than through the primal problem.

Summary

- A problem wherein the objective is to allot the limited available resources to the jobs in such a way as to optimise the overall effectiveness, i.e., minimise the total cost or maximise the total profit, is called mathematical programming. Mathematical programming wherein constraints are expressed as linear equalities/inequalities is called linear programming. Linear programming deals with the optimisation of the total effectiveness expressed as a linear function of decision variables, known as the objective function, subject to a set of linear equalities/inequalities known as constraints.
- Steps involved in the mathematical formulation of a linear programming problem (LPP) are: i) Identification of the decision variables of the problem; ii) expressing the objective function as a linear function of the decision variables; iii) identifying the limited available resources to write the constraints as linear inequalities or equalities in terms of decision variables; iv) introducing the nonnegative restrictions.
- Graphical method is used to solve linear programming problems having two decision variables. The graphical method of solving linear programming programs comprises the following steps: i) The graphs are plotted for the equations corresponding to the given inequalities for constraints as well as restrictions; ii) The region corresponding to each inequality is shaded; iii) After shading the regions for each inequality, the most common

shaded portion, i.e., the region obtained on superimposing all the shaded regions is determined. This is the region where all the given inequalities, including non-negative restrictions are satisfied. This common region is known as the feasible region or the solution set or the polygonal convex set; iv) Each of the corner points (vertices) of the polygon is then determined; v) The objective function at each corner point is evaluated.

- How to formulate the dual of an LPP, weak and strong duality theorems which relate the primal and dual basic feasible solutions. The Dual simplex method to solve LPP for which a basic optimal, but infeasible, solution is known.

Keywords

- Basic Feasible Solutions: These solutions are basic as well as feasible.
- Basic Solution: Any set of values of the variables in which the number of non-zero valued variables is equal to the number of constraints is called a Basic Solution.
- Constraints: The linear inequalities or the side condition.
- Dual Problem: Associated with every linear programming there is a linear programming problem. Which is called its dual problem.
- Feasible Solution: A set of values of decision variables, which satisfies the set of constraints and the non-negativity restrictions.
- Linear Programming: Linear programming is the analysis of problems in which linear function of a number of variables is to be maximized (or minimized) when those variables are subject to a number of restraints in the form of linear inequalities.
- Maximini or Minimax Principle: If a player lists the worst possible outcomes of all his potential strategies then he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes.
- Mixed Strategy: Decision making rule in which a player decides in advance, to choose his courses of action with some definite probability distribution.
- Primal: The original L.P.P. is called the primal problem
- Objective Function: The function to be maximized or minimized.

Self Assessment

1. A constraint in an LP model restricts

A. value of the objective function

B. value of the decision variable

C. use of the available resources

D. all of the above

2. In graphical method of linear programming problem if the iso-cost line coincide with a side of region of basic feasible solutions we get

A. Unique optimum solution

B. unbounded optimum solution

C. no feasible solution

D. Infinite number of optimum solutions

3. The linear function of the variables which is to be maximize or minimize is called

A. Constraints
B. Objective function
C. Decision variable
D. None of the above

4. The first step in formulating a linear programming problem is
A. Identify any upper or lower bound on the decision variables
B. State the constraints as linear combinations of the decision variables
C. Understand the problem
D. Identify the decision variables

5. A basic solution is called non-degenerate, if
A. All the basic variables are zero
B. None of the basic variables is zero
C. At least one of the basic variables is zero
D. None of these

6. The graph of $x \leq 2$ and $y \geq 2$ will be situated in the
A. First and second quadrant
B. Second and third quadrant
C. First and third quadrant
D. Third and fourth quadrant

7. Which of the terms is not used in a linear programming problem
A. Slack variables
B. Objective function
C. Concave region
D. Feasible solution

8. In. L.P.P----
A. objective function is linear
B. constraints are linear
C. Both objective function and constraints are linear
D. None of the above

9. Constraints means----
A. limitations are expressed in mathematical equalities (or inequalities)
B. Assumption
C. goal is to be achieved
D. None of the above.

10. Dual of the dual is the
A. Primal
B. Dual
C. Either primal or dual

D. None of these

11. If the 'i' constraint of a primal (maximization) is equality, then the dual (minimization) variable 'yi' is:

A. ≤
B. ≥
C. Unrestricted in sign
D. None of the above

12. In linear programming, dual price represents

A. Mean and maximum price
B. Unit worth of a resource
C. Minimum and mean price
D. Minimum and maximum price

13. In primal-dual solutions, the dual problem solution can be obtained by solving other problems classified as

A. Unrestricted problem
B. Original problem
C. Double problem
D. Restricted problem

14. When using a graphical solution procedure, the region bounded by the set of constraints is called the

A. solution.
B. feasible region.
C. infeasible region.
D. maximum profit region.

15. The graphic method of LP uses

A. objective equations.
B. constraint equations.
C. linear equations.
D. all of the above.

Answers for Self Assessment

1.	A	2.	D	3.	B	4.	D	5.	B
6.	B	7.	C	8.	C	9.	A	10.	A
11.	C	12.	B	13.	B	14.	B	15.	D

Review Questions

1. What is the difference between formulating and solving a linear programming problem?

2. What are the basic steps in formulating a linear program?
3. What is the feasible region of an LP?
4. Solve the following LPP by two phase simplex method.

   Minimise: $Z= 15/2x_1-3x_2$

   subject to $3x_1 - X_2 - X_3 \geq 3$

   $x_1 - x_2 + x_3 \geq 2$

   $(x_1, x_2, x_3) \geq (0,0,0)$
5. Maximise $-2x_1 + 3x_2 - x_4 + x_5$ subject to $x_1 + x_2 - x_3 + x_4 \leq 5$

   $3x_1 - 5x_2 + X_4 \leq 7$

   $4x_1 + 2x_2 - x_3 - 6x_5 \leq 10$

   $4x_1 - 2x_2 + x_3 + 6x_5 \leq -10$

   $(x_1, x_2, x_3, x_4, x_5) \geq (0,0,0,0,0)$
6. Minimise $Z = 10x_1 + 4x_2$ subject to the constraints

   $4x_1 + x_2 \geq 80$

   $2x_1 + x_2 \geq 60$

   $x_1 \geq 0,\ x_2 \geq 0$

Further Readings

- . S. Hillier / G. J. Liebennan (2001), Introduction to Operations Research, McGraw-Hill ,7th Edition.
- M. Glicksman (2001). An Introduction to Linear Programming and the Theory of Games, Dover Publ., Mineola, NY.
- Essential Mathematics for Economists- Nutt Sedester, peter Hawmond, Prentice Hall Publication

Web Links

https://www.brainkart.com/article/Solution-of-LPP-by-graphical-method_37041/

http://www.math.wsu.edu/students/odykhovychnyi/M201-04/Ch06_1-2_Simplex_Method.pdf

Unit 05: Maxima and Minima

CONTENTS

Objectives

- define critical points, stationary points, saddle points, local maxima and local miniina;
- state a necessary condition for functions to have local extrema and apply it;
- state and prove the theorem known as "second derivative test" which gives a sufficient condition for finding local maxima and minima;
- use Hessian for classifying local maxima and local minima; and
- apply Lagrange's multiplier method for finding the stationary points when the variables are subject to some constraints.

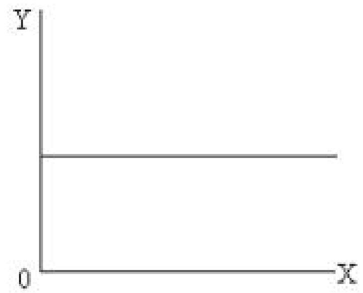
Introduction

The primary objective of this unit is to characterize Optimal points, which are also called 'extremal points' and lay down the conditions that need to be satisfied to classify a point as an 'extremum', either a maximum or a minimum. Optimum (plural: optima) is the generic term for maximum and minimum. The process of finding an optimum is called optimisation. The basic idea is that a decision maker (for example a consumer or a firm) has an objective function that the decision-maker is attempting to optimise (that is, maximise or minimise).

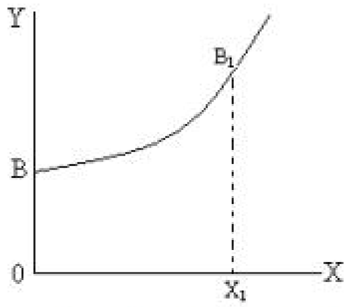
The question that needs to be answered at the outset is as follows: What is the importance of 'locating an extrema' in the context of Economics? The process is important because an economic unit (consumer, producer, etc.) is often faced with various different alternatives. For instance, a consumer has to choose from different commodity bundles, or a producer has to choose amongst various combinations of factor inputs, viz. Labour, capital, etc. The economic agent has to choose one particular alternative, which very often either maximise something (e.g. a producer will maximise profit, a consumer maximises her utility) or minimise something (e.g. cost of producing a given output). Economically, this process of maximisation or minimisation is characterised as a 'process of optimisation' or 'the quest for the best'. However, from the standpoint of a mathematician, the location of a 'maximum' or a 'minimum' does not carry forth any notion of optimality. To solve an optimisation problem the first task of the economic agent is to construct an 'objective function'. The dependent variable of this function is the so-called 'object', which has to be either maximised or minimised. The independent variable(s) of the function are the choice/decision/policy variables that can be manipulated by the agent to achieve the desired goal. The optimisation process involves choosing a value of the independent variable that will yield an extreme value ('minimum' or 'maximum' as the case might be) for the requisite dependent variable of the objective function. The process of locating an extremum value (either a maximum or a minimum) is discussed in the following sections. Note, only classical technique for locating extreme positions, using differential calculus, will be discussed. For simplicity it will be assumed that the objective function consists of a single independent variable.

5.1 Maxima and Minima of Functions

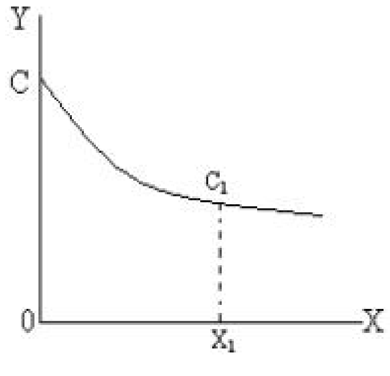
To start with, we assume the function y = f(x) is differentiable (and hence continuous) throughout its range. The function may be represented by one of the following types of graphs:



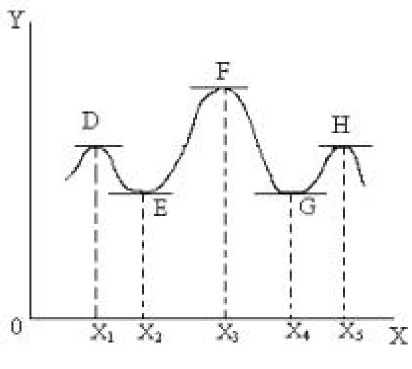
i) Constant



ii) Monotonically increasing



iii) Monotonically decreasing



iv) Fluctuating

In the first case, the function is the same for all values of x. We shall consider the values of y as both maximum and minimum value.

In the second case, the value of the function is monotonically increasing. Let us restrict the domain of x to the interval $0 \le x \le x$. Then in this closed interval, y can assume a maximum of B_1x_1 and a minimum of OB. These maximum and minimum values of the function are called absolute maximum (global) and absolute (global) minimum respectively. Such maximum and minimum values of a function are called extreme values or extrema. Accordingly, in the third case, OC becomes the global maxima and X_1C_1 the global minima if we restrict in the interval of x to $0 \le x \le x$.

Let us consider the fourth case. Here y oscillates with changes in x. The function generates three peaks, D, F and H and two bottoms E and G. The value of the function at D is the highest in comparison to the values at other points in its the immediate neighborhood. Symbolically,

$f(x - \in) < f(x_1)$ and $f(x_1 + \in) < f(x_1)$ as $\in \to 0$.

Thus, we can say that the value of the function at D is maxima, at least in some small interval $x_1 - \in < x < x + \in$. Similarly, in another small interval $x_3 - \in < x < x + \in$, the value of the function is maximum at $x = x_3$ and in the neighborhood of F,

$f(x_3 - \in) < f(x_3)$ and $f(x_3 + \in) < f(x_3)$

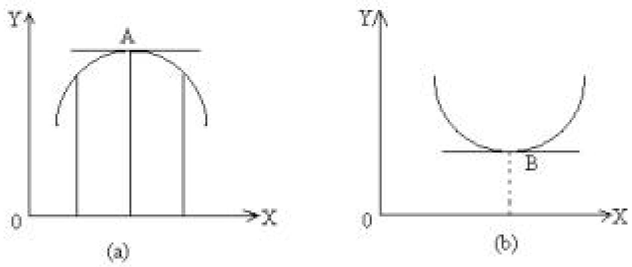
The function attains another maximum in the neighborhood of point F. These maximum values are called local (relative) maxima. A relative maximum gives us the information that the value of the function is maximum in the neighborhood of the maximum point only, and the function can assume higher values elsewhere. For example, in the fourth figure, points D and H describe relative maximum, though the function assumes a higher value at F. The relative maxima are unlike global maxima. The global maximum is the highest value of the function with reference to its entire domain and hence unique.

A function can have several local maxima and unique global maxima. The concept of local minima and global minima can be explained in the same manner. For a continuous function, the global maximum must always be greater than the global minimum.

The local extrema always appear as point interior to an interval, however small. These local extrema are thus strictly interior extrema. In the following discussion, we shall consider only these types of extrema.

5.2 Identification of Maxima and Minima

Consider the following figure:



The extremum described by (a) is a maximum, while that of (b) is a minimum. In Figure (a) the curve $y = f(x)$ takes a turn at point A. To the left of A, the curve is increasing i.e., $f'(x)>0$ while to the its right f(x) is decreasing i.e., $f'(x)<0$. Thus, at the turning point A, $f'(x)$ must be zero i.e., $f'(x_0) = 0$. Similarly, at point B in Figure (b), $f'(x_0)=0$. Therefore, we can conclude that an extremum can take place only at the stationary points where $f'(x)=0$. The extreme values are always stationary values.

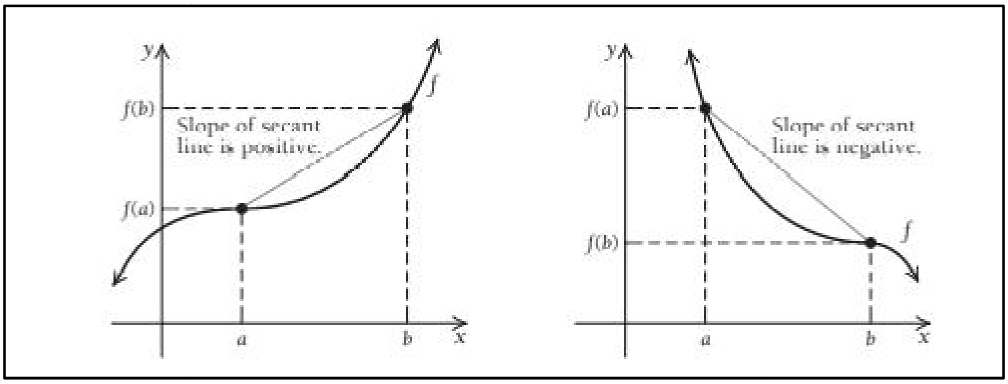
Definition:

Let f be defined on an interval I containing 'c'

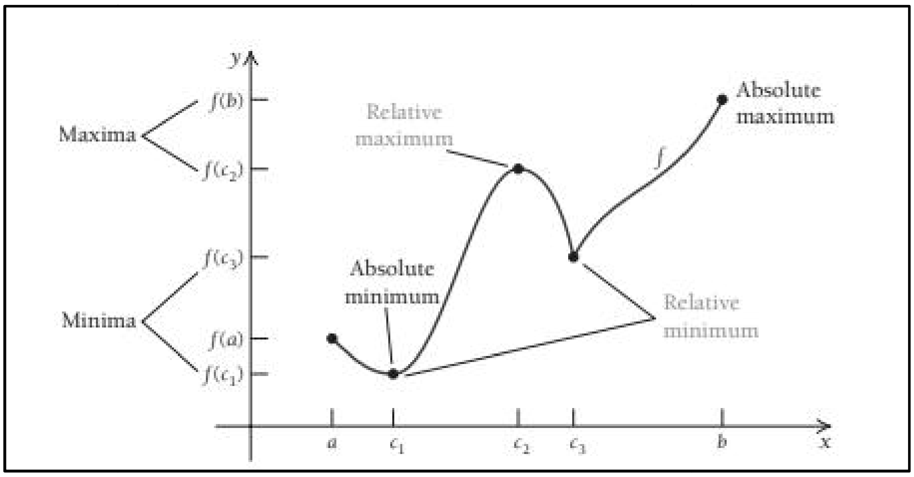
1. f (c) is the (absolute) minimum of f on I if $f(c) \le f(x)$ for all x in I.

2. f (c) is the (absolute) maximum of f on I if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are called the extreme values or extreme, of the function on the interval.



2. If f is a continuous function defined on a closed and bounded interval [a,b], then f has both a minimum and a maximum value on the interval [a,b]. This is called the extreme value theorem and its proof is beyond the scope of our course.



Note that at $x = x_0$, the point A on graph is not an absolute maximum because $f(x_2) > f(x_0)$. But if we consider the interval (a,b), then f has a maximum value at $x = x_0$ in the interval (a,b). Point A is a point of local maximum of f. Similarly, f has a local minimum at point B.

5.3 First Derivative Test and Relative Optima

If the first derivative of a function f (x) at $x = x_0$i.e., if $f'(x_0) = 0$, then the value of the function at x0 i.e., f (x_0) will be

i) A relative maximum if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x0to its immediate right.

ii) A relative minimum if the derivative $f'(x)$ changes its sign from negative to positive from the immediate left of the point $x_0$to its immediate right.

iii) Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of the point x_0.

The value of the dependent variable, at which, the first derivative of the function is equal to zero i.e. at x_0 is referred to as the critical value of x. The value of the function at its critical point i.e.f (x_0) is known as the stationary value. The point with the coordinates equal to x_0 and f (x_0) is accordingly called the stationary point.

Point A is a relative maximum because for all values of x in the immediate left of x_2, the function is rising (the first derivative of f (x) is positive) and for all values of x in the immediate right of x_2, the function is falling (the first derivative of f (x) is negative). It is only at x_2, the critical point, the first derivative of the function is zero and f (x_2) is the corresponding stationary value. Note, the slope of the tangent to the function at A i.e. AT is parallel to the x-axis and is equal to zero. Analogously, one can see that point B is a point of relative minimum.

Example 1 $y = 50 + 90x - 5x^2$

Solution:

Step 1: Find the first derivative of the function.

$F'(x) = 90-10x$ or $10(9-x)$

Step 2: Equate the first derivative of the function to zero

$10(9-x) = 0$

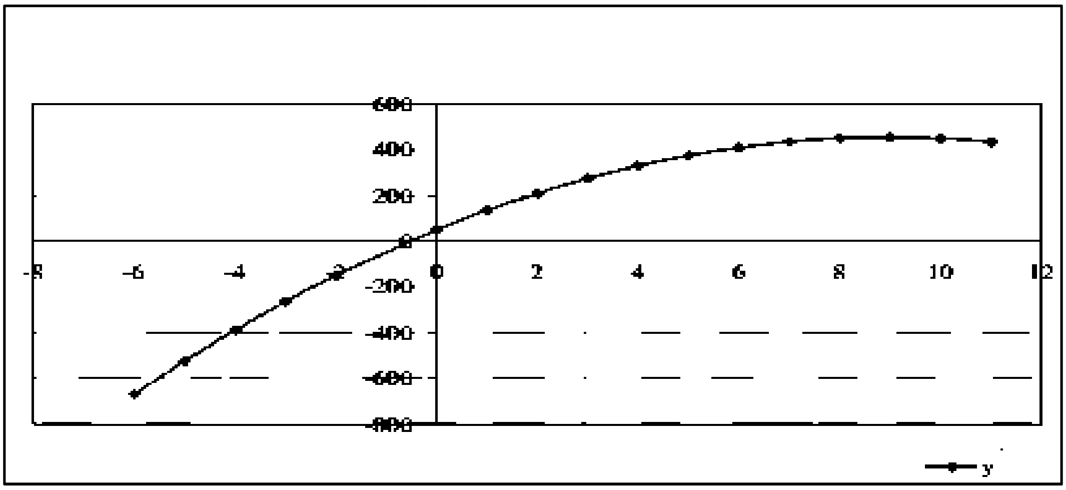
Step 3: Solve for x to obtain the critical value

The critical value of the function is $x = 9$ and the corresponding stationary value is

$y = f(9) = 455$.

x	**-6**	**-3**	**0**	**3**	**6**	**9**	**10**	**11**
y	**-670**	**-265**	**50**	**275**	**410**	**455**	**450**	**435**

The graph of this function is shown: $y = 50 + 90x - 5x^2$



It is easily verified that for all points in the neighbourhood left of $x = 9$, the function is increasing, implying that its first derivative is positive. Similarly, for all points in the immediate neighbourhood right of $x = 9$, the function is decreasing, implying its first derivative is negative. This satisfies condition (i) of the first derivative test and establishes, the critical value of $x = 9$ (located in the peak of the hill!) as a relative maximum. The corresponding stationary value of the function is $y = 455$.

Example 2: Consider the following function whose domain is assumed to be the interval [0, α).

$Y = (1/3)x^3 - x^2 + x + 10$

Differentiating with respect to x we get the first derivative as

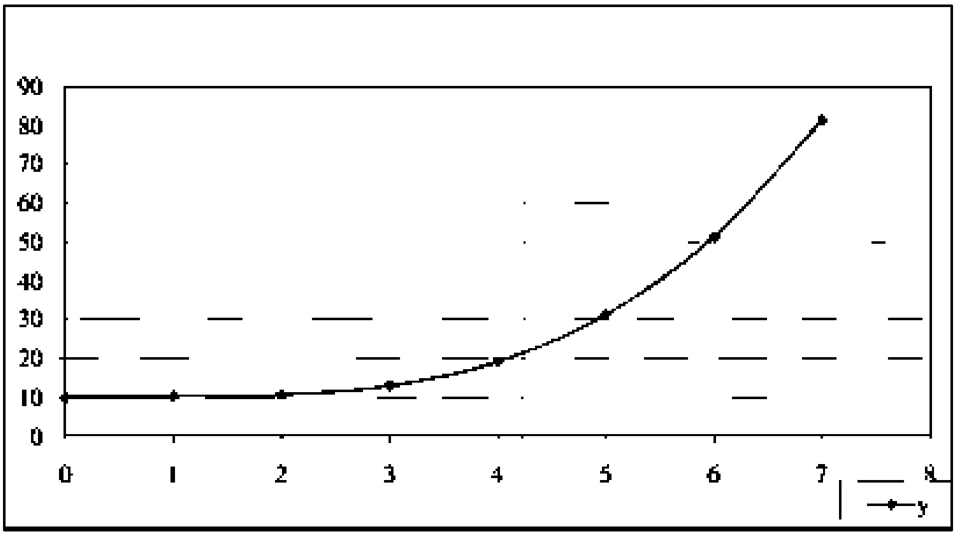
$x^2 - 2x + 1$ or $(x-1)^2$

Setting the first derivative of the function equal to zero yields $x = 1$ as a critical value of x. The corresponding stationary value of y is 10.33. To ascertain whether the stationary value is also a relative extremum we have to perform the first derivative test. The graphic representation of the function in Example 3 is as follows:

x	0	1	2	3	4	5	6
y	10	10.33	10.66	13	19.33	31.66	52

The graph of the function is presented $y = (1/3)\,x^3 - x^2 + x + 10$

The function attains a zero slope at the point where x = 1. Even though f ′ (1) is zero -which implies f (1) is a stationary value— the derivative does not change its sign from the left -hand side neighbourhood of x = 1 to the other. In fact, as confirmed by the graph, the function is more or less flat in the immediate region of x = 1. On the basis of the first derivative test mentioned earlier it can be asserted that the stationary value f (1) = 10.33 is neither a relative maximum nor a relative minimum.



In sum, a relative extremum must be a stationary value (although the reverse is not necessarily true). To find the relative maximum or minimum of a given function, the first step would be to find the critical value of the dependent variable at which the first derivative of the function is equal to zero. This will enable us to find the stationary values of the function. To ascertain whether the stationary value is also a relative maximum or relative minimum one needs to apply the first derivative test.

Note :

First Derivative Test for Local Maxima and Minima Let c be a critical number of f i.e., $f'(c) = 0$

If $f'(x)$ changes sign from positive to negative at c then f (c) is a local maximum.

If $f'(x)$ changes sign from negative to positive at c then f(c) is a local minimum

5.4 The Second Order Derivative And Second-Order Condition For Optimum

Assuming that the first derivative f'(x) is itself a function of x, the second derivative of the function is obtained by differentiating this function again with respect to x. Symbolically, the second derivative is represented as f''(x). The double prime indicates that the function y = f (x) has been differentiated twice with respect to x. The expression (x) following the double prime indicates that the second derivative is also a function of x. If the second derivative f''(x) exists for all values in the domain, the function f (x) is said to be twice differentiable; if, in addition, f''(x) is continuous, the function f (x) is said to be twice continuously differentiable.

Example 3: Find the second derivative of the following function

$Y = f(x) = 4x^3 + 5x^2 - 3x + 10$

Step 1: Differentiate the equation in Example 4 with respect to x to find the first derivative. We obtain the following equation:

$f'(x) = 12x^2 +10x-3$

Step 2: Now differentiate this equation with respect to x to obtain the second derivative of the original function:

$F''(x)= 24x+10$

5.5 Interpretation of The Second Order Derivative

The first derivative of the function i.e., $f'(x)$ measures the slope of the function or the rate of change of the function. If the first derivative is positive, i.e., if $f'(x)> 0$, then the function is increasing; and if the derivative is negative i.e., if $f'(x)<0$, then the function is decreasing. Analogously the second derivative i.e., $f''(x)$ measures the rate of change of the first derivative $f'(x)$. If the second derivative is positive i.e., if $f''(x)>0$, then there is an increasing rate of change; and when $f''(x)<0$, then the rate of change is decreasing. In other words, the second derivative measures the rate of change of the rate of change of the original function f (x). Note that if $f'(x)> 0$and $f''(x)> 0$, then this means that the function has a positive slope which is changing at an increasing rate. In other words, the function is said to be increasing at an increasing rate. Conversely,if $f'(x)< 0$and $f''(x)<0$, then this means that the function has a negative slope which is changing at a decreasing rate. In other words, the function is said to be decreasing at a decreasing rate.

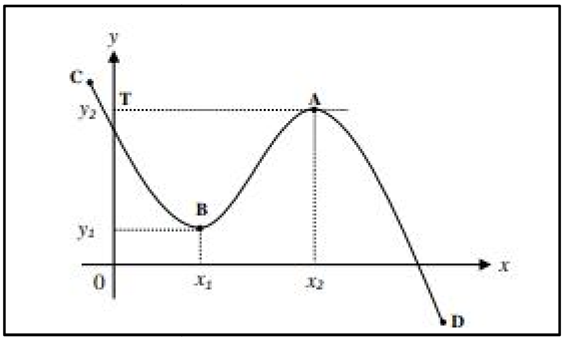
The Second Order Derivative Test

This test uses the second derivative of the function in question, hence the name.

Assume that $f(x_0) = 0$, (so x0 is a critical point!), then

1) If $f''(x_0) > 0$ then $f(x_0)$ is a relative minimum value.

2) If $f''(x_0)< 0$ then $f(x_0)$ is a relative maximum value.



As mentioned earlier, the zero slope condition, in other words $f'(x) = 0$at $x = x_0$, was deemed to be a 'necessary condition' for $f(x_0)$ to be a relative extremum. Since this is based on the first derivative of the function, it is also known as the first-order-condition. Once we verify that the first order condition is satisfied, the negative (positive) sign of $f''(x)$ at $x = x_0$ is sufficient to ensure that x_0 corresponds to a relative maximum (minimum). Since the sufficiency condition is based on the second derivative of the function, it is also referred to as the second-order-condition.

To get a clearer understanding of how the second derivative enables us to determine whether the stationary value is a 'relative maximum' or a 'relative minimum'. Recall, the extreme values of this function lies in the level stretch, either in the bottom of the hill (point B) or at the peak of the hill (point A). Around the point on the graph corresponding to the relative maximum value (point A), the graph is concave down. This makes sense, since A (x_2, y_2) is the highest point on the graph in an interval around x_2, so the graph must 'bend down' away from the peak, making it concave down. If we pick any two points in this region of the graph, then the straight line joining these two points will lie entirely below the graph except for the two end points on the curve.

This ensures that at x equal to x_2, $f''(x_2) < 0$ is satisfied. Similarly, around the point B (x_1, y_1) the graph is convex up. Once again, this makes sense. This point is the lowest point on the graph in an interval around x_1, so the graph has to 'bend up' away from the point (x_1, y_1), making it convex up. If we pick any two points in this region of the graph, then the straight line joining these two points will lie entirely above the graph except for the two end points on the curve. This ensures that at x equal to x_1, $f''(x_1) > 0$ is satisfied.

For a Relative Maximum		For a Relative Minimum	
$dy/dx = 0$	(First-order Condition)	$dy/dx = 0$	(First-order Condition)
$d^2y/dx^2 < 0$	(Second-order Condition)	$d^2y/dx^2 > 0$	(Second-order Condition)

Example 4: Find the maxima and minima for the following function:

$y = 3x^4 – 10x^3 + 6x^2 + 5$

Solution: First order condition (F.O.C.)

$12x^3 – 30x^2 + 12x = 0$ or,

$3x (4x – 2) (x – 2) = 0$.

Either, $x = 0$ or, $x = 2$ or $x = ½$

Second, order condition (S.O.C.):

At $x = 0$, $f''(x) = 12>0$.

At $x = 2$, $f''(x) = - 9 < 0$

Hence the function attains maximum at $x = ½$ and minimum at $x=0$ and $x = 2$.

Guidelines to find Local Maxima and Local Minima

The function f is assumed to posses the second derivative on the interval I.

Step 1 : Find f ' (x) and set it equal to 0.

Step 2 : Solve f ' (x) = 0 to obtain the critical numbers of f.

Let the solution of this equation be α, ……….

We shall consider only those values of x which lie in I and which are not

end points of I.

Step 3 : Evaluate f ´ (α)

If f (α) < 0, f (x) has a local maximum at x = α and its value if f (α)

If f (α) > 0, f (x) has a local minimum at x = α and its value if f (α)

If f (α) = 0, apply the first derivative test.

Step 4 : If the list of values in Step 2 is not exhausted, repeat step 3, with that

value

Example 5: Find the points of local maxima and minima, if any, of each of the following functions. Find also the local maximum values and local minimum values.

a. $f (x) = x^3 -6x^2+9x+1$

Solution: $f (x) = x^3 -6x^2+9x+1$

Thus, $f'(x)=3x^2 -12x +9= 3(x^2 -4x+3) =x(x-1)(x-3)$

To obtain critical number of f, we set $f'(x) = 0$ this yields $x = 1, 3$.

Therefore, the critical number of f are $x = 1, 3$.

Now $f'(x) = 6x – 12 = 6(x – 2)$

We have $f'(1) = 6(1 – 2) = – 6 < 0$ and $f (3) = 6(3 – 2) = 6 > 0$

Using the second derivative test, we see that f (x) has a local maximum at

x = 1 and a local minimum at x = 3. The value of local maximum at x = 1 is

f (1) = 1– 6+ 9 +1 = 5 and the value of local minimum at x = 3 is

f (3) = 3^3– 6(3^2) + 9(3) + 1 = 27 – 54 + 27 + 1 = 1.

5.6 First-Order Condition for Objective Function with Two Variable

Assume that

Z= f(x,y)

The first-order necessary condition for an extremum (either maximum or minimum) again involves dz = 0. However, now since there are two choice variables, the first-order condition is modified as follows:

dz = 0 for arbitrary non-zero values of dx and dy

The rationale behind this is similar to the explanation of the condition dz = 0 for the one variable case: an extremum point must necessarily be a stationary point; at a stationary point dz = 0 ever for infinitesimal change in the two variables x and y. Totally differentiating equation (3), we get

$D_z = f_x dx + f_y dy$

where f x = df/dx = partial derivative with respect to x and fy = df/dy =partial derivative with respect to y.

Now dx ≠ 0; dy ≠ 0

And at the stationary point dz = 0

(A) and (B) can hold simultaneously only if fy= f x=0

Hence the first-order condition for optimisation (location of extremum points) for an objective function with two variables is:

$$f_x = f_y = 0, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

the first-order condition is necessary but not sufficient. To develop the sufficiency condition, we must look into the second-order to the differential, which is related to second-order partial derivatives.

Example 6: Assuming that the second-order condition is satisfied, find out the profit maximising values of quantity (Q) and advertising expenditure (A) for a producer with the following profit function (Π):

Π =400-3Q^2-4Q+2QA-5A^2+48A

The first-order condition for profit-maximisation requires that the partial derivative

$$\frac{\partial \Pi}{\partial Q} = \frac{\partial \Pi}{\partial A} = 0$$

Partially differentiating equation (4) with respect to A keeping Q as constant gives us:

$$\frac{\partial \Pi}{\partial A} = 2Q - 10A + 48$$

Partially differentiating equation (4) with respect to Q keeping A as constant gives us:

$$\frac{\partial \Pi}{\partial A} = -6\mathrm{Q} - 4 + 2\mathrm{A}$$

Since $\frac{\partial \pi}{\partial Q} = \frac{\partial \pi}{\partial A} = 0$ as the first order condition we get:

$$2Q - 10A + 48 = 0$$

And

$$-6Q - 4 + 2A$$

Equations (5) and (6) can be written as:

Q-5A=-24

-3Q+A=2

From equation (8), we get A=3Q+2 . Substituting for A in equation (7) and we then solve for Q as follows:

Q-5(2+3Q)=-24

Q-10-15Q=-24

-14Q=14

Q=1

Hence A=2+3*1=5

So, we obtain that Q =1 and A = 5 are the quantity and advertising expenditure level which maximises profit for the firm using the first-order necessary condition.

5.7 Second-Order Condition for Objective Function with Two Variables

Let us now examine the sufficiency condition for optimisation where the objective function has two decision variables. Using the concept of d2 z, we can state the second-order sufficient condition for a maximum of z = f(x,y) as follows:

$d^2z < 0$ for arbitary non-zero values of dx and dy.

The rationale behind this is similar to that of the d2 z condition explained in the case where the objective function has one variable. Analogously, the second-order sufficiency condition for a minimum of z = f(x,y) is the following:

$d^2z > 0$ for arbitary non-zero values of dx and dy.

Note that d2 z is a function of the second order partial derivatives fxx, fxy and fyy. Intuitively, it is clear that the second-order sufficiency condition can be translated in terms of these derivatives. However, the actual translation would require knowledge of quadratic form – the discussion of which is given in section 8.5. Hence, we will state the main results here:

For any values of dx and dy, not both zero:

To sum up, the first-order and the second-order condition for optimization in case of an objective function z = f(x,y) is depicted in the following:

Table 1:First-Order and Second-Order Condition for Optimisation

	Maximum	**Minimum**
First-Order Necessary Condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-Order Sufficiency Condition	$f_{xx}, f_{yy} < 0$ *and* $f_{xx}f_{yy} > f^2_{xy}$	$f_{xx}, f_{yy} > 0$ *and* $f_{xx}f_{yy} > f^2_{xy}$

The second order condition is applicable only after the first-order condition has been fulfilled.

we had solved for the optimal product (Q) and the advertising expenditure (A), assuming that second-order condition is satisfied at the optimal point. Now let us examine whether it is actually satisfied or not.

Recall that

$\Pi = 400 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A$

$\Pi_A = \frac{\partial \pi}{\partial A} = 2Q - 10A + 48$

$\Pi_Q = \frac{\partial \Pi}{\partial A} = -6Q + 2A - 4$

Setting them equal to zero and solving for Q and A yields Q* = 1 and A* = 5, where * denotes the optimal level.

For the second-order condition we need to derive the following partial derivatives:

$$\frac{\partial \Pi^2}{\partial A^2}, \frac{\partial \Pi^2}{\partial Q^2} \text{ and } \frac{\partial \Pi^2}{\partial A \partial Q}$$

Partial differentiation of equation with respect to A gives us:

$\Pi_{AA} = \frac{\partial \Pi^2}{\partial A^2} = -10 < 0$

Partial differentiation of equation with respect to Q gives us:

$\Pi_{QQ} = \frac{\partial \Pi^2}{\partial Q^2} = -6 < 0$

Partial differentiation of equation with respect to A gives us:

$\Pi_{AQ} = \frac{\partial \Pi^2}{\partial A \partial Q} = 2$

Now

$\Pi_{AA} * \Pi_{QQ} = -10 * -6 = 60$

and $(\Pi AQ)^2 = (2)^2 = 4$

Hence $\Pi_{AA} * \Pi_{QQ} > (\Pi_{AQ})^2$

Second-order condition is satisfied for the output level (Q) equal to 1 and the advertising expenditure (A) equal to 5.

5.8 Constrained Optimization Problems

In economic applications, unconstrained optimisation is a relatively rare case. More often, we find instances of constrained optimisation, that is optimisation subject to a constraint. What this means is that there is a side condition on the optimisation exercise. The domain of the function that is sought to be optimised is restricted by one or more side relations. Consider the case of utility maximization by a consumer. An individual as consumer has unlimited wants. But she is constrained by her budget set. So, she maximizes her utility subject to her budget constraint. Similarly, a producer would want to minimize her cost subject to a given level of output.

The unit begins with a discussion of how to find stationary values of the objective function. It is shown that in simple cases we can use the method of substitution. But in more complex cases, different methods have to be employed. We will see the importance of the Lagrangian multiplier and how it is used in static constrained optimization. After this the unit discusses second-order conditions for constrained optimization.

The Method of Substitution:

With the help of an example from co-ordinate geometry, we shall learn to maximize/minimize a function subject to a constraint which restricts the domain of the function. Consider a simple example. Find the smallest circle centered on (0,0) which has a point common with the straight-line

$x+y = 10$.

The equation for the circle is

$x^2 + y^2 = r^2$.

The smallest circle will be one with the smallest radius. The restriction is that the circle must have a point in common with a given straight line. Without this restriction, the smallest circle can easily be seen to be a circle with radius zero, i.e. a point.

Thus, our problem is the following:

Minimise

$x^2 + y^2$ subject to x + y = 10

Here the unconstrained solution x=0, y=0 will not be available. The constraint prohibits this solution. What we have to do is to consider as the domain only those values of x and y for which x + y = 10. So, we see that the constraint has diminished the domain. How to find the solution? From the constraint we find y x = − 10.

Now if we substitute this into the minim and $x^2 + y^2$ we get $(X^2) + (10-X)^2$, which incorporates the constraint.

Let us now minimize this expression

$$\frac{d^2}{dx^2}[x^2 + (10 - x)^2] = 2x + 2(10 - x)(-1) = 4x - 20$$

For a stationary value 4x-20 =0 or x=5

To check whether the stationary value is truly a minimum we differentiate the function once again with respect to x. Thus,

$$\frac{d^2}{dx^2}[x^2 + (10 - x)^2]$$

= d/dx=(4x-20)

=4>0

Which proves that we have minimized the expression $x^2 + y^2$

We know that the constraint makes y =10-x= 5, thus giving us a solution x=5,y = 5 for which $x^2 + y^2$ =5

In solving the problem our method involved several steps:

Step 1 : Solve the constraint equation to get one variable in terms of the other. Above we found y in terms of x. Thus, y x = − 10

Step 2 : Substitute this solution in the objective function to ensure that the domain of the function is a set of pairs of values of x and y which satisfy the constraint. Note that the objective function thus modified is a function of x only.

Step 3 : Differentiate this modified function to get the first derivative. Find the value of x for which this derivative equal zero.

Step 4 : Put this value of x in the constraint to find the value of y.

Step 5 : Calculate the value of the objective function for this pair of values of x and y.

Step 6 : Check the second order condition to find whether the stationary point is an extremum

In general, it may not be easy or indeed possible to solve the constraint equation. In such cases it would appear that we are stuck at step (1) of the procedure described above. For instance, we may have a constraint equation like $x^3 + 2x^2 y + 9y^3 - 2y - 117 = 0$. To find y in terms of x, or x in terms of y from this complicated equation is extremely difficult. Suppose now that the problem has been referred to a mathematician and while we wait for the solution, we prepare the ground for our computation.

The Lagrange Multiplier Method

Consider once more the problem:

Maximise f(x, y) subject to g(x, y) = 0

If gy(x, y) ≠ 0, then y = h(x), so that the problem is transformed into:

Maximise f(x,h(x))

1st order condition gives us $f_x + f_y h'(x) = 0$ ……..(i)

g(x,h(x)) = 0 is an identity so that we have

$g_x + g_y h'(x) = 0$

Now define $\lambda = \frac{f_y}{g_y}$Multiplying (ii) by $\lambda = \frac{f_y}{g_y}$, we get

$$\lambda g_x + f y^{h'(x)} = 0$$

From (i) and (iii), fx- λ gx=0

From, $2 = \frac{f_y}{g_y}, f_y - h g_y = 0$

Also the constraint is g(x,y) = 0

(1), (2) and (3) are three equations in three variables, x, y and λ. When we solve these three simultaneous equations, we get the value of x and y which solve our problem; we also get the value of λ which is Lagrangian multiplier, something which does not seem to have any relevance to our problem.

Example 12: Maximise $5x^2 + 6y^2 - xy$, subject to constraint x+2y=24

Solution: Given OF: $5x^2 + 6y^2 - xy$

CF : x+2y = 24

24-x-2y= 0

Let v=OF+ λCF =$5x^2 + 6y^2 - xy + 24\lambda - \lambda x - 2\lambda y$

$$v_x = 10x + 0 - y + 0 - 2 \sim 0 = 0, 10x - y = \lambda$$

$v_y = 0 + 12y - x + 1 - 0 - 2\lambda = 0 \; or \; 12y - x = 2\lambda$

From (13) and (14) we get 20x-2y=12y-x, x=2/3y

Since we do not get clear values of x and y,

Therefore, we find $v\lambda$ =2u-x-2y=0

24-2/3y-2y=0

(2/3+2)y=24

Y=9

X=6

Hence constrained maximization takes place when x= 6 and y=9

A consumers utility function is given as u=(y+1)(x+2) . If his budget constraint is 2x+5y=51, how of x and y he should consumer to maximize his satisfaction.

5.9 Economics Applications of Maximum-Minimum

Example 7: A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit, in dollars, must be

P(x) = 1000 – x

The manufacturer also determines that the total cost of producing x units is given by

C(x) = 3000 + 20x.

a) Find the total revenue.

b) Find the total profit.

c) How many units must the company produce and sell in order to maximize profit?

d) What is the maximum profit?

e) What price per unit must be charged in order to make this maximum profit?

Solution:

a. R(x) = Total revenue

= (Number of units)*(Price per unit)

=x*p

=x(1000-x)=1000x-x^2

b. P(x)= Total revenue - Total cost

= R(x)-C(x)

=(1000-x^2)-(3000+20x)

=-x^2 +980x-3000

c. To find the maximum value of , we first find P'(x)

P'(x) = - 2x + 980.

This is defined for all real numbers, so the only critical values will come from solving P'(x)

- 2x + 980=0

X=490

There is only one critical value. We can therefore try to use the second derivative to determine whether we have an absolute maximum.

P''(x) = - 2

Thus,P''(490) is negative, and so profit is maximized when 490 units are produced and sold.

d) The maximum profit is given by

P(490)=-$(490)^2$ +980*490-3000

=237100

Thus, the stereo manufacturer makes a maximum profit of $237,100 by producing and selling 490 stereos.

e) The price per unit needed to make the maximum profit is

4 p = 1000 - 490 = 510.

Example 8: A monopolist sells two products x and y for which the demand functions are

x =25- 0.5Px

y =30 – Py

and the combined cost function is

c = x^2 + 2xy + y^2 + 20

Find (a) the profit-maximizing level of output for each product, (b) the profit-maximizing price for each product, and (c) the maximum profit.

a) Since Π = TRx+ TRy -TC, in this case

$\Pi = P_X x + P_y y - c$

Px =50 -2x

Py =30-2y

Substituting in

Π=(50 -2x)x + (30 - y)y -(x^2 +2xy + y^2 + 20)

=50x– 3x^2 +30y-2y^2 -2xy-20

The first-order condition for maximizing is:

Π x = 50 - 6x -2y = 0

Π y = 30 – 4y -2x= 0

Solving simultaneously, $\bar{x} = 7$ and $\bar{y} = 4$. Testing the second-order conditon, $\Pi_{xx} = -6$, $\Pi_{yy} = -4$, and $\Pi_{xy} = -2$. With both direct partials negative and $\Pi_{xx}\Pi_{yy} > (\Pi_{xy})^2$, Π is maximized

b) Substituting $\bar{x} = 7$, $\bar{y} = 4$

Px=50 -2(7) = 36 Py=30-4= 26

c) Substituting $\bar{x} = 7$, $\bar{y} = 4$ in $\Pi = 215$.

(xy) 2 , is maximized.

Example 8:Let the inverse demand function and the cost function be given by

P = 50 − 2Q and C = 10 + 2q

respectively, where Q is total industry output and q is the firm's output.

First consider first the case of uniform-pricing monopoly, as a benchmark. Then in this case Q = q and the profit function is

$\pi(Q) = (50 - 2Q)Q - 10 - 2Q = 48Q - 2Q^2 - 10$

solving $d\pi/dQ = 0$ we get Q = 12, P = 26, π = 278, CS = 12(50−26) /2 = 144, TS = 278 + 144 = 422

Monopoly:

Q	P	π	CS	TS
12	26	278	144	422

Now let us consider the case of two firms, or duopoly. Let q1 be the output of firm 1 and q2 the output of firm 2. Then Q = q1 + q2 and the profit functions are:

$\pi_1 (q_1, q_2) = q_1 [50 - 2 (q_1 + q_2)] - 10 - 2q_1$

$\pi_2 (q_1, q_2) = q_2 [50 - 2 (q_1 + q_2)] - 10 - 2q_2$

A Nash equilibrium is a pair of output levels (q_1, q_2) such that:

$$\pi 1(q_1^*, q_2^*) \geq \pi_1(q_1, q_2^*) q_1 \geq 0$$

And

$$\pi_1(q_1^*, q_2^*) \geq \pi_1(q_1^* 9 q_2) \geq 0$$

This means that, fixing q_2 at the value q_2 and considering π * 1 as a function of q_1 alone, this function is maximized at q_1 = q . But a necessary condition for this to be true is that $\frac{\partial \Pi_1}{\partial q_1}(q_1^*, q_2^*) = 0$

Similarly, fixing q_1 at the value q_1 and considering π * 2 as a function of q_2 alone, this function is maximized at q_2 = q . But a necessary condition for this to be true is that $\frac{\partial \Pi_2}{\partial q_2}(q_1^*, q_2^*) = 0$

Thus the Nash equilibrium is found by solving the following system of two equations in the two unknowns q_1 and q_2:

$$\frac{\partial \Pi_1}{\partial q_1}(q_1^*, q_2^*) = 50 - 4q1 - 2q2 - 2 = 0$$

$$\frac{\partial \Pi_2}{\partial q_2}(q_1^*, q_2^*) = 50 - 2q1 - 4q2 - 2 = 0$$

The solution is q1*=q2*= 8 , Q = 16, P = 18, π * * = = 1 = π2 = 118, CS = 16(50−18) 2 = 256, TS = 118 + 118 + 256 = 492.

5.10 Economic Order Quantity

Inventory control is an important consideration in business. In particular, for each shipment of raw materials, a manufacturer must pay an ordering fee to cover handling and transportation. When the raw materials arrive, they must be stored until needed, and storage costs result. If each shipment of raw materials is large, few shipments will be needed, so ordering costs will be low, while storage costs will be high. On the other hand, if each shipment is small, ordering costs will be high because many shipments will be needed, but storage costs will be low. Example 8 shows how the methods of calculus can be used to determine the shipment size that minimizes total cost.

Example 9: A bicycle manufacturer buys 6,000 tires a year from a distributor. The ordering fee is $20 per shipment, the storage cost is 96 cents per tire per year, and each tire costs rupees 21. Suppose that the tires are used at a constant rate throughout the year and that each shipment arrives just as the preceding shipment is being used up. How many tires should the manufacturer order each time to minimize cost?

Solution: The goal is to minimize the total cost, which can be written as

Total cost =storage cost +ordering cost +purchase cost

Let x denote the number of tires in each shipment and C(x) the corresponding total cost in dollars. Then,

Ordering cost =(ordering cost per shipment)(number of shipments)

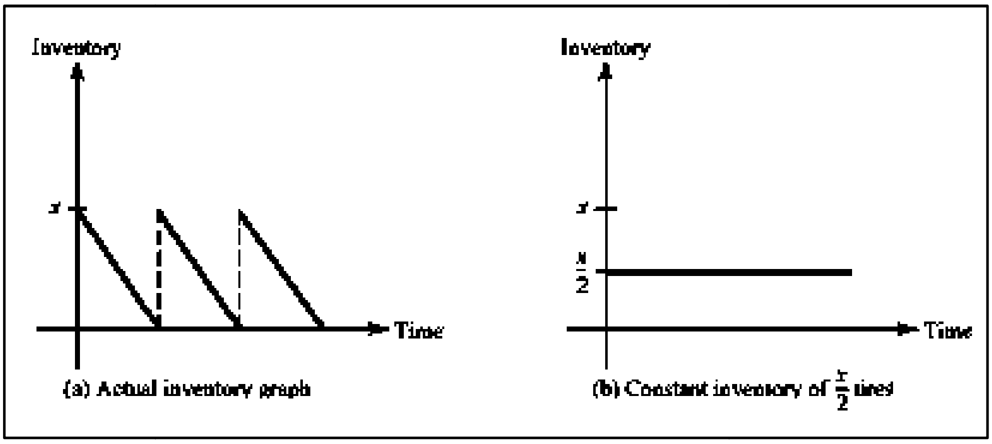
Since 6,000 tires are ordered during the year and each shipment contains x tires, the number of shipments is 6000/x and so

Ordering cost=20*(6000/x)

=120000/x

Purchase cost= (total number of tires ordered)(cost per tire) = 6,000(21)=126,000

The storage cost is slightly more complicated. When a shipment arrives, all x tires are placed in storage and then withdrawn for use at a constant rate. The inventory decreases linearly until there are no tires left, at which time the next shipment arrives. The situation is illustrated in Figure This is sometimes called just-in-time inventory management.



The average number of tires in storage during the year is x/2, and the total yearly storage cost is the same as if x/2 tires were kept in storage for the entire year. This assertion, although reasonable, is not really obvious, and you have every right to be unconvinced. It follows that

Storage cost= (average number of tires stored)(storage cost per tire)

=x /2 (0.96) =0.48

Putting it all together, the total cost is

C(x)= 0.48x+120000/x+126000

and the goal is to find the absolute minimum of C(x) on the interval

$0 \le x \le 6{,}000$

The derivative of C(x) is

$C'(x) = 0.48 - 120,000 / x$

which is zero when

$x^2 = 120000/0.48 = 250000$ or $x = \pm 500$.

Since x = 500 is the only critical number in the relevant interval $0 \le x \le 6,000$, you can apply the second derivative test for absolute extrema. You find that the second derivative of the cost function is

$C''(x) = 240000/x^3$

which is positive when x > 0. Hence, the absolute minimum of the total cost C(x) on the interval $0 \le x \le 6,000$ occurs when x = 500; that is, when the manufacturer orders the tires in lots of 500.

5.11 Bilateral Monopoly

Example 10: As can he computed from the previous models, the profit under the joint profit maximizing model (Rs. 21,870) is greater than the sum of the profits under either monopoly 5400 + 12,150 = 17,550) or monopsony (18,225 + 3037.5 = 21,262.5) market. Hence, there is an incentive for the single buyer to move as close as possible to the profit level under the monopsony market (ap = 8,225). Similarly, there is an incentive for the single seller to move as close as possible to the profit level under monopoly market (xp= Rs. 12,150). Goals cannot be achieved simultaneously. In this case some hind of compromises are needed for the bilateral model to have a unique solution of the intermediate product price. Within this framework, we reformulate the bilateral monopoly problem into the following goal programming model.

Minimize $\alpha \Delta\pi_{zz} + \beta \Delta\Pi yy$

Subject to $\pi_z + \Delta\pi_{zz} = \pi_{zz}$

$$\Pi y + \Delta\Pi yy = \Pi yy$$

$$\pi_z + \Pi y = \Pi$$

π_{zz} *and* $\pi_{yy} \ge 0$

Where

$\pi_z =$ *actual profit level of the single buyer in the bilateral monopoly model*

$\Pi y =$ *actual profit level of the single seller in the bilateral monopoly model*

$\Delta\pi_{zz} =$ *the deviational variable (underachievement) that represents the difference between the actual profit level of the single buyer (or a,) and the profit level under the monopsony*

$\Delta\Pi yy$= the deviational variable (underachievement) that represents the difference between the actual profit level of the single seller (ore,) and the profit level under the monopoly market ('^»'d

$\Pi *=$ totalprofitofthejointprofitmaximizationprobl‹

On one end, we could assume $\beta = 0$, i.e., the objective function is reduced to minimizing $\Delta\pi_{zz}$. As a result, the deviational variable for the single buyer would be zero, that is, $\Delta\pi_{zz} = 0$ and $\pi_z = \pi *_{zz} =$ Rs. 18,225, Πy - 3645, and $\Delta\Pi yy$ = 8505 (or 12,150 - 3,645) with the same value ofx, y and z. This is equivalent to the revised solution with P_y = 148.5. On the other end, we assume that α = 0, i.e., the objective function is reduced to minimizing$\Delta\Pi yy$. As a result, the deviational variable for the single seller would be zero, that is, $\Pi yy = 0$ and $\pi_y = \pi *_{yy}$ = 12,150, π_z = 9720, $\Delta\pi_{zz}$ = 8505 (or 18,225 - 9,720). This is equivalent to the revised solution *with* P_y = 306. The value of the deviational variables (Rs. 8505) represents the amount of revenue either the single seller or single buyer falls short of the goal if the market were monopoly cr monopsony, respectively. If the market were a monopoly, the maximum revenue goal would $\pi *_{yy}$= 12,150 for the single seller; and $\pi *_{zz}$ = 18,225 for the single buyer if the market were a monopsony. The sum of both a and a*p exceeds the maximum joint profit under the collusion (Rs.21,870) by Rs.8,505. The solution bounds on the revised model are a special case of the goal programming model or equationsthroughin which either single seller or single buyer takes up all the amount of the underachievement from the goal.

However, it is not likely that a single party would shoulder all the burden. An alternative solution to the bilateral monopoly model would be to assume that both parties stiare equal burden, i.e., $\Delta\pi_{zz}$ = $\Delta\Pi yy$.

5.12 Law of Equi-marginal utility

Let us assume that a consumer is consuming only two goods X_1 and X_2 . The utility which she receives from consuming X_1 and X_2 is given by the utility function U=f (X_1 , X_2) and satisfies the property of eventual diminishing marginal utility. The consumer has a given money income to be spent on these two goods during the period we are analysing her behaviour. She cannot influence the prices, P_1 and P_2,, of the goods, through her own action. Prices are given as parameters in decision-making (consumption) as this consumer is one of the numerous consumers demanding X_1 and X_2 . Thus, she has no market power. Since she is required to spend her entire income on X1 and X2 , the budget equation is given as,

$M = P_1 X_1 + P_2 X_2$

Where M is her nominal income. Since the consumer is a utility maximiser, her consumption problem can be formulated as follows:

Maximise

$U = f (X_1, X_2)$

subject to the budget constraint

$M = P_1 X_1 + P_2 X_2$.

By using the Marshallian Equi-Marginal Principle (which is based on the Lagrange Multiplier Technique), we get the equilibrium condition,

$MU_1 / P_1 = MU_2 / P_2 = \lambda$

This is the first order condition (necessary) for achieving equilibrium. The second order condition (sufficiency) of equilibrium is given by the law of eventual diminishing utility. The second order condition will be automatically fulfilled so long as the marginal utility schedules for each good MU1 and MU2 are both downward sloping. It must be noted that whenever a consumer maximises utility, equilibrium is said to be attained. What the equilibrium condition says is that to maximise total utility [U = f (X1 , X2)] an individual consumer must equalise the ratio of marginal utility to price for each and every good and which in turn must be equal to constant marginal utility of money. In other words, to obtain maximum total utility a rational consumer must equalise the marginal utility per rupee of expenditure on each and every line of expenditure (that is, on each and every good).

This relationship represents the consumer's equilibrium condition. A consumer attains equilibrium when she maximises total utility from consuming X_1 and X_2 . The equilibrium condition can also be stated in an alternative form:

$MU_I / MU_2 = P_1 / P_2$...(1)

The ratio of marginal utilities of goods X1 and X2 must equal the price ratio of the same two goods. This in turn, must equal marginal utility of money, which is constant by assumption.

Condition (1) above is the famous Marshallian law of the equi-marginal utility. It can be shown that if the ratio of marginal utility of the two goods is not equal to the price ratio, then without spending more in the aggregate, just by re-allocating the given amount of money income as between the two goods X1 and X2, the consumer can increase her total utility from consumption.

Example 11: The following table gives an individual's marginal utility schedules for goods X1 and X2 . If the prices of X1 and X2 are Rs. 2.00 each and that the individual has Rs. 20.00 of Income, which she spends on X1 and X2 , what is the individual's equilibrium purchase of X1 and X2 ?

Q 1 2 3 4 5 6 7 8 9 10 11
MU_1 16 14 11 10 9 8 7 6 5 3 1
MU_2 15 13 12 8 6 5 4 3 2 1 0

$MU_1/P_1 = MU_2/P_2$

and the budget constraint must be fully satisfied. From the above table we derive the following.

Q	1 2 3 4 5 6 7 8 9 10 11
MU_1 / P_1	8 7 5.5 5 4.5 4 3.5 3 2.5 1.5 0.5
MU_2 / P_2	7.5 6.5 6 4 3 2.5 2 1.5 1 0.5 0

At $X_1 = 6$ MU_1 / P_1 is 4 At $X_2 = 4$ units MU_2 / P_2 is 4 Hence $MU_1/ P_1 = MU_2 / P_2 = 4$.

The amount spent is $P_1 X_1 + P_2 X_2$ which is Rs.20.00 (2*6+2*4 = 12 + 8). Money income is also Rs. 20.00. Hence, the budget constraint is satisfied. The equilibrium purchase is $X_1 = 6$ units and $X_2 = 4$ units.

Since MU_1 falls from 16 to 1 as X_1 increases from 1 to 11 and MU_2 declines from 15 to zero as X_2 increases from 1 to 11, the second order condition is also fulfilled.

When there are more than one combination of two goods (X_1 , X_2) at which the equimarginal principle holds, one has to take recourse to the budget constraint to obtain the equilibrium combination and all other combinations violating the budget constraint have been rejected.

It should be noted that when the consumer consumes n goods, the law of equi- marginal utility would then read as:

$MU_1 /P_1 = MU_2 /P_2 = MU_3 /P_3 =..... =..MUn /Pn = \lambda$ (the marginal utility of money)

with the second-order conditions (the law of eventual diminishing marginal utility must hold for each of the n goods).

Summary

- Stationary Points: The points, at which first order derivatives are zero, are called stationary points. The values of the function at those points are called stationary values.
- Local (relative) Maxima and Global (absolute) Maxima: A function attains local maximum at any particular point in the domain of definition implies that the value of the function is maximum in the neighborhood of that point only, and the function can assume higher values elsewhere. This can be non-unique. The global maximum is the highest value of the function with reference to its entire domain and hence unique.
- Local (relative) Minima and Global (absolute) Minima: A function attains local minimum at any particular point in the domain of definition implies that the value of the function is minimum in the neighborhood of that point only, and the function can assume lower values elsewhere. This can be non-unique. The global minimum is the lowest value of the function with reference to its entire domain and hence unique.
- Points of Inflextion: The point of inflextion is defined as a point at which a curve changes its curvature. The sufficient condition for a point of inflextion is f x and f x ''( ) 0 '''( ) 0 = ≠ . Thus, if a function has '( ) 0, ''( ) 0 '''( ) 0 f x f x and f x == ≠ at the point x, the point is said to be stationary and inflextional.

Keywords

Minima: minimum point

Maxima: Maximum Point

Absolute minima: exactly minimum of all point sin consideration

Relative minima: comparatively minimum point

Saddle point: stationary point

Self Assessment

1. Select the correct necessary condition, in case of Maxima and Minima in Multi-variable function.

A. ∂u/ ∂x=∂u/ ∂y=0
B. ∂u/ ∂x=∂u/ ∂y≠0
C. ∂u/ ∂x=∂u/ ∂y=1
D. ∂u/ ∂x=∂u/ ∂y≠1

2. What is the saddle point?

A. Point where function has maximum value
B. Point where function has minimum value
C. Point where function has zero value
D. Point where function neither have maximum value nor minimum value

3. For function f(x,y) to have minimum value at (a,b) value is?

A. fxx*fyy– (fxy)2>0 and fxx, fyy<0
B. fxx*fyy– (fxy)2<0 and fxx, fyy<0
C. fxx*fyy– (fxy)2>0 and fxx, fyy>0
D. fxx*fyy– (fxy)2=0 and fxx, fyy =0

4. For function f(x,y) to have maximum value at (a,b) value is?

A. fxx*fyy– (fxy)2>0 and fxx, fyy>0
B. fxx*fyy– (fxy)2>0 and fxx, fyy<0
C. fxx*fyy– (fxy)2<0 and fxx, fyy>0
D. fxx*fyy– (fxy)2=0 and fxx, fyy =0

5. For homogeneous function with no saddle points we must have the minimum value as ____________

A. 90
B. 1
C. equal to degree
D. 0

6. The drawback of Lagrange's Method of Maxima and minima is?

A. Maxima or Minima is not fixed
B. Nature of stationary point is cannot be known
C. Accuracy is not good
D. Nature of stationary point is known but cannot give maxima or minima

7. Maximize the function x + y – z = 1 with respect to the constraint xy=36.

A. 0

B. -8
C. 8
D. No Maxima exists

8. What does the first order partial derivative shows in cost maximization?

A. Slope of isoquant curve= slope of Iso-cost lines
B. Rate of change in isoquant curve= slope of Iso-cost lines
C. Slope of isoquant curve= Rate of change in Iso-cost lines
D. Slope of isoquant curve ≠ slope of Iso-cost lines

9. What will be the slope under first order condition for maximization of utility?

A. $dx_2/dx_1=fx_1/fx_2$
B. $dx_2/dx_1=-fx_1/fx_2$
C. $dx_1/dx_2=fx_1/fx_2$
D. $dx_2/dx_1=fx_2/fx_1$

10. What is the second order condition for maximization of utility?

A. $d_2x_1/dx_{22}>0$
B. $d_2x_1/dx_{22}<0$
C. $d_2x_2/dx_{21}>0$
D. $d_2x_1/dx_{22}=0$

11. What does the first order condition show about maximization of output?

A. Slope of isoquant curve be greater than slope of iso-cost line
B. Slope of isoquant curve different from slope of iso-cost line
C. Slope of isoquant curve must be same as slope of iso-cost line
D. None of the above

12. What is the second order condition for maximization of utility?

A. $p_2/p_1=fx_1/fx_2$
B. $p_1/p_2=fx_1/fx_2$
C. $p_2/p_1=fx_2/fx_1$
D. $p_2/p_1=1$

13. What is Nash Equilibrium?

A. A pair of output levels at which the firms are being maximized.
B. Nash equilibrium in game theory is a situation in which a player will continue with their chosen strategy, having no incentive to deviate from it.
C. Only a
D. Both a and b

14. What is duopoly?

A. Where a number of firms are more than 2
B. Where number of firms are equal to 2
C. Where only single firms exist
D. None of the above

15. What are two type of costs on which Economic order quantity depends?

A. Inventory carrying costs
B. Procurement or set up costs.
C. Both a and b
D. None of the above

Answers for Self Assessment

1.	A	2.	D	3.	C	4.	B	5.	D
6.	A	7.	D	8.	A	9.	B	10.	A
11.	C	12.	B	13.	D	14.	B	15.	C

Review Questions

1. Find the maximum profit and the number of units that must be produced and sold in order to yield the maximum profit. Assume that revenue, R(x), and cost, C(x):
   a. $R(x)= 2x$, $C(x) = 0.01x^2 + 0.6x + 30$
   b. $R(x) = 50x - 0.5x^2$, $C(x)= 10x + 3$
2. Raggs, Ltd., a clothing firm, determines that in order to sell x suits, the price per suit must be $P=150-0.5x$

   It also determines that the total cost of producing x suits is given by

   $C(x)= 4000+0.25x^2$

   a) Find the total revenue, .

   b) Find the total profit, .

   c) How many suits must the company produce and sell in order to maximize profit? d) What is the maximum profit?

   d) What price per suit must be charged in order to maximize profit

3. A firm faces the production function $Q = 20K^{0.4}L^{0.6}$. It can buy inputs K and L for Rs. 400 a unit and Rs.200 a unit respectively. What combination of L and K should be used to maximize output if its input budget is constrained to Rs. 6,000?

4. A consumer spends all her income of Rs. 120 on the two goods A and B. Good A costs Rs.10 a unit and good B costs Rs.15. What combination of A and B will she purchase if her utility function is $U = 4A^{0.5} B^{0.5}$?

5. A store expects to sell 800 bottles of perfume this year. The perfume costs Rs.20 per bottle, the ordering fee is Rs.10 per shipment, and the cost of storing the perfume is 40 cents per bottle per year. The perfume is consumed at a constant rate throughout the year, and each shipment arrives just as the preceding shipment is being used up. a. How many bottles should the store order in each shipment to minimize total cost? b. How often should the store order the perfume?

6. Use this information to classify each critical number of f (x) as a relative maximum, a relative minimum, or neither:

a. $f'(x) = x^3 (2x- 3)^2 (x+1)^5 (x+7)$

b. $f^1(x) = \frac{x(x-2)^2}{x^4+1}$

c. $f'(x) = \sqrt[3]{x}(3-x)(x+1)^2$

7. The first derivative of a certain function is $f'(x) = x(x - 1)^2$.

a. On what intervals is f increasing? Decreasing?

b. On what intervals is the graph of f concave up? Concave down?

c. Find the x coordinates of the relative extrema and inflection points of f.

d. Sketch a possible graph of f(x).

8. Suppose that $q = 500 - 2p$ units of a certain commodity are demanded when p dollars per unit are charged, for $0 \le p \le 250$.

a. Determine where the demand is elastic, inelastic, and of unit elasticity with respect to price.

b. Use the results of part (a) to determine the intervals of increase and decrease of the revenue function and the price at which revenue is maximized.

c. Find the total revenue function explicitly and use its first derivative to determine its intervals of increase and decrease and the price at which revenue is maximized.

d. Graph the demand and revenue functions

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Unit 06: Matrices

CONTENTS

Objectives

After studying this unit, you will be able to,

- define and give examples of various types of matrices;
- evaluate the sum, difference, product and scalar multiples of matrices;
- evaluate determinants find minors and cofactors of square matrices of different orders; and apply properties of determinants.

Introduction

The knowledge of matrices has become necessary for the individuals working in different branches of science, technology, commerce, management and social sciences. In many economic analyses, variables are assumed to be related by sets of linear equations. Matrix algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in conventional algebraic notation. The concept of determinant and is based on that of matrix. In this unit, we introduce the concept of matrices and its elementary properties. Further, discusses the determinant, and a number associated with a square matrix and its properties.

Definition

A rectangular array of numbers is called a matrix.

The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS. A matrix having mrows and n columns is said to have the order m *n.

Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}.$$

Then ,

$$a_{11} = 1,\ a_{12} = 3,\ a_{13} = 7,\ a_{21} = 4,\ a_{22} = 5,\ a_{23} = 6.$$

6.1 Matrices

A set of mn numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called m×n matrix (read "m by n matrix") .

An m×n matrix is expressed as

$$A=\begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------ \\ --- & ---- & ------ \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix}$$

(i) A matrix is denoted by capital letters A, B, C, etc. of the English alphabets.

(ii) First suffix of an element of the matrix indicates the position of row and second suffix of the element of the matrix indicates position of column. e.g. 23 a means it is an element in the second row and the third column.

(iii) The order of a matrix is written as "number of rows ×number of columns".

6.2 Order of a Matrix

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. If there are 3 rows and 2 columns in a matrix, then its order is written as (3, 2) or (3 x 2) read as three by two. In general, if m are rows and n are columns of a matrix, then its order is (m x n).

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

are matrices of orders (2 x 3), (3 x 1) and (4 x 4) respectively.

Example 1: Write the order of the matrix

$$A = \begin{bmatrix} 9 & 7 & 8 & -3 & -8 \\ 4 & 3 & 6 & 1 & -10 \\ 10 & 12 & 15 & 2 & 5 \end{bmatrix}$$

Also write the elements a_{23}, a_{14}, a_{35}, a_{22}, a_{31}, a_{32}.

Solution: Order of the matrix A is 3×5 and the desired elements are: Matrices and Determinants

$a_{23}=6$, $a_{22}=3$,

$a_{14}= -3$, $a_{31}=10$,

$a_{35}= 5$, $a_{32}=12$

Example 2: Write all the possible orders of the matrix having following elements.

(i) 8

(ii) 13

Solution: (i) All the 8 elements can be arranged in single row, i.e. 1 row and 8 columns.

Or

They can be arranged in two rows with 4 elements in each row, i.e. 2 rows and 4 columns.

Or

in four rows with 2 elements in each row, i.e. 4 rows and 2 columns.

Or

in eight rows with 1 element in each row, i.e. 8 rows and 1 column.

∴ the possible orders are 1 *8, 2* 4, 4 *2, 8 *1.

(iii) All the 13 elements can be arranged in single row, i.e. 1 row and 13 columns.

Or

in 13 rows with 1 element in each row, i.e. 13 rows and 1 column.

∴ the possible orders are1* 13, 13 *1.

6.3 Types of Matrices

On the basis of number of rows and number of columns and depending on the values of elements, the type of a matrix gets changed. Various types of matrix are explained as below:

Row Matrix

A matrix having only one row is called a row matrix.

For example, [2 5 7], [8 9], [1 0 3 2] all are row matrices

Column Matrix

A matrix having only one column is called a column matrix. For example,

$$\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \end{bmatrix}$$

all are column matrices.

Null or Zero Matrixes:

A matrix in which each element is „0" is called a Null or Zero matrix. Zero matrices are generally denoted by the symbol O. This distinguishes zero matrix from the real number 0.

For example $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 4.

The matrix O_{mxn} has the property that for every matrix A_{mxn},

A + O = O + A = A

Square matrix:

A matrix A having same numbers of rows and columns is called a square matrix. A matrix A of order m x n can be written as A_{mxn}. If m = n, then the matrix is said to be a square matrix. A square matrix of order n x n, is simply written as A_n.

$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

are square matrix of order 2 and 3.

Main or Principal (leading)Diagonal:

The principal diagonal of a square matrix is the ordered set of elements a_{ij}, where i = j, extending from the upper left-hand corner to the lower right-hand corner of the matrix. Thus, the principal diagonal contains elements a_{11}, a_{22}, a_{33} etc.

For example, the principal diagonal of

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

consists of elements 1, 2 and 0, in that order.

Particular cases of a square matrix:

(a)Diagonal matrix:

A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

A square matrix A = [aij $]_{n\times n}$ is said to be diagonal matrix if aij=0, ∀i≠j

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are diagonal matrices.

Notes:

(i) For a diagonal matrix all non-diagonal elements must be zero.

(ii) (ii) In a diagonal matrix some or all the diagonal elements may be zero.

(b) Scalar Matrix:

A diagonal matrix in which all the diagonal elements are same, is called a scalar matrix i.e. Thus

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

(c) Identity Matrix or Unit matrix:

A scalar matrix in which each diagonal element is 1(unity) is called a unit matrix. An identity matrix of order n is denoted by I_n.

Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the identity matrices of order 2 and 3 .

is an identity matrix if and only if aij = 0 for i ≠ j and aij = 1 for i = j

Notes: If a matrix A and identity matrix I are conformable for multiplication, then I has the property that AI = IA = A i.e.,

I is the identity matrix for multiplication.

d. Equal Matrices:

Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B i.e for each i, j, $a_{ij} = b_{ij}$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{4}{2} & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$$

hen A = B because the order of matrices A and B is same and aij = bij for every i , j.

e. Upper Triangular Matrix

A square matrix A = [a_{ij}] n×n is said to be upper triangular matrix if all the elements below the principal diagonal are zero.

$$\begin{matrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 4 \end{matrix}$$

f.Lower Triangular Matrix

A square matrix A = [a_{ij}] n×n is said to be lower triangular matrix if all the elements above the principal diagonal are zero.

$$\begin{matrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 1 \end{matrix}$$

2. Write orders and types of the following matrices:

a. $\begin{bmatrix} 2 & 9 \\ 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

c. $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

d. $\begin{matrix} 2 & 5 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{matrix}$

e. $\begin{matrix} 2 \\ 9 \\ 6 \end{matrix}$

Solution: Order Type

(i) 2×2 Square matrix [rows and columns are equal in number.]

(ii) 2×2 Diagonal matrix [Θall the non-diagonal elements are zero.]

(iii) 3×3 Identify matrix [all the diagonal elements are unity and nondiagonal element are zero.]

(iv) 3×3 Lower triangular matrix [all the elements above the principal diagonal are zero]

(v) 3×1 Column matrix [it has only one column.]

6.4 Operations on Matrices

In school times, a child first learns the natural numbers and then learns how these numbers are added, subtracted, multiplied and divided. Similarly, here also we now see as to how such operations (except division) are applied on matrices. These operations are explained by first giving a general formula and then examples followed by some exercises.

a. Addition and subtraction of Matrices

Addition of two matrices A and B make sense only if they are of the same order and obtained by adding their corresponding elements. It is denoted by A + B.

That is, if $A= [a_{ij}]\ m\times n$, $B = [\ b_{ij}\ ]\ m\times n$ then $A + B = [aij+bij\ ]_{m\times n}$

Subtraction of two matrices A and B make sense only if they are of the same order, and is given by

$A - B = A + (- B) = A+ (-1) B$, i.e. A – B means addition of two matrices A and – B. So, if $A = [aij\ ]_{m\times n}$, $B = [\ b_{ij}\ ]\ m\times n$, then $A - B = [aij+ (-1)bij\ ]_{m\times n} = [aij-bij]\ m\times n$

$$A = \begin{bmatrix} -5 & 1 & -3 \\ 6 & 0 & 2 \\ 2 & 6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 5 \\ -8 & 10 & 3 \\ -2 & -3 & -9 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -3 & 5 & 2 \\ -2 & 10 & 5 \\ 0 & 3 & -8 \end{bmatrix} \qquad A - B = \begin{bmatrix} -7 & -3 & -8 \\ 14 & -10 & -1 \\ 4 & 9 & 10 \end{bmatrix}$$

Properties of Addition of Matrices

If A, B, C are of the same orders over R, (i.e. elements of A, B, C are real numbers) then

(i) A + B = B + A (commutative law).

(ii) (A + B) + C = A + (B + C) (associative law).

(iii) A + O = O + A = A, where O is a null matrix. (existence of additive identity) .

(iv) For a given matrix A, there exists a matrix B of the same order such that A + B = O = B + A. Here B is called additive inverse of A. (existence of additive inverse).

b. Product of Matrices:

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B. Then the product matrix AB has the same number of rows as A and the same number of columns as B.

Thus, the product of the matrices A_{mxp} and B_{pxn} is the matrix $(AB)_{mxn}$. The elements of AB are determined as follows:

The element Cij in the ith row and jth column of (AB)mxn is found by cij = $a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots\ldots\ldots + a_{in}b_{nj}$

for example, consider the matrices

$$A_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B_{2x2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Since the number of columns of A is equal to the number of rows of B, the product AB is defined and is given as

Notes:

1. Multiplication of matrices is not commutative i.e., AB ≠ BA in general.

2. For matrices A and B if AB = BA then A and B commute to each other

3. A matrix A can be multiplied by itself if and only if it is a square matrix. The product A.A in such cases is written as A2. Similarly, we may define higher powers of a square matrix i.e., $A * A^2 = A^3$, $A^2 * A^2 = A4$

4. In the product AB, A is said to be pre multiple of B and B is said to be post multiple of A.

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Thus, c_{11} is obtained by multiplying the elements of the first row of A i.e., a_{11}, a_{12} by the corresponding elements of the first column of B i.e., b_{11}, b_{21} and adding the product.

Similarly, c_{12} is obtained by multiplying the elements of the first row of A i.e., a_{11}, a_{12} by the corresponding elements of the second column of B i.e., b_{12}, b_{22} and adding the product. Similarly, for c_{21}, c_{22}.

$$\begin{bmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{bmatrix} \text{X} \begin{bmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} \text{ , } & a_{11} \times b_{12} + a_{12} \times b_{22} \\ a_{21} \times b_{11} + a_{22} \times b_{21} \text{ , } & a_{21} \times b_{12} + a_{22} \times b_{22} \end{bmatrix}$$

3. Solve

$A=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}$

$AB=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}*\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}$

$=\begin{bmatrix} 5 & 4 \\ 5 & 9 \end{bmatrix}$

4. Solve:

$A=\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

Solution:

Since A is a (2 x 3) matrix and B is a (3 x 2) matrix, they are conformable for multiplication. We have

$AB=\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}*\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

$=\begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix}$

$=\begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}$

Properties of Matrix Multiplication

If A, B, C are three matrices such that corresponding multiplications hold then

(1) A(BC) = (AB)C (associative law)

(2) (i) A(B + C) = AB + AC (left distributive law)

(ii) (A + B)C = AC +BC (right distributive law)

(3) If A is a square matrix of order n, then $I_nA = AI_n = A$, where I_n is the identity matrix of order n.

Remark

Commutative law does not hold, in general, i.e. AB ≠ BA, in general. But for some cases AB may be equal to BA. This has been explained below:

(i) AB may be defined but BA may not be defined and hence AB ≠ BA in this case.

For example, let A be a matrix of order 3×2 and B be a matrix of order 2×4 .

Here AB is defined and is of order 3×4

But BA is not defined (Θnumber of columns of B ≠ number of rows of A).

(ii) AB and BA both may defined but may not be of same order and hence AB BA. ≠

For example, let A be a matrix of order be 3×2 and B be a matrix of order 2×3 .

Here as number of columns of A = number of rows of B.

∴ AB is defined and is of order 3×3.

Also, number of columns of B = number of rows of A.

Hence BA is defined but of order 2×2 .∴ AB ≠BA .

(iii) AB and BA both may be defined and of same order but even then, they may not be equal.

6.5 Determinant of Square Matrices

Determinant is a number associated with each square matrix. In this section, we will deal with determinant of square matrices of order 1, 2, 3 and 4. Determinants of square matrices of order greater than 4 can be evaluated in a similar fashion.

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. The determinants are defined only for square matrices. It is denoted by det A or |A| for a square matrix A.

The determinant of the (2 x 2) matrix

A = $\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$

is given by det A = |A| = $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix}$

= a11 a22 – a12 a21

Example 5: If A= $\begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$

Solution: |A| = $\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix}$

= 9 – (–2) = 9 + 2 = 11

The determinant of the (3 x 3) matrix

A= $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$, denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is given as, det A = |A|

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Each determinant in the sum (In the R.H.S) is the determinant of a submatrix of A obtained by deleting a particular row and column of A.

These determinants are called minors. We take the sign + or – , according to (–1)i+jaij Where i and j represent row and column.

Minor and Cofactor of Element:

The minor Mij of the element aij in a given determinant is the determinant of order (n – 1 x n – 1) obtained by deleting the ith row and jth column of Anxn.

For example in the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The minor of the element a11 is M11= $\begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix}$

The minor of the element a11 is M12= $\begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix}$

The minor of the element a11 is M13= $\begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$ and so on

The scalars Cij = $(-1)^{i+j}M_{ij}$are called the cofactor of the element $a_{ij}$of the matrix A.

The value of the determinant in equation (1) can also be found by its minor elements or cofactors, as $a_{11}M_{11} – a_{12}M_{12} + a_{13}M_{13}$

Or

$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Hence the det A is the sum of the elements of any row or column multiplied by their corresponding cofactors. The value of the determinant can be found by expanding it from any row or column.

6.6 Properties of Determinants

Expanding a determinant in which the elements are large number can be a very tedious task. It is possible, however, by knowing something of the properties of determinants, to simplify the working. So here are some of the main properties.

(i) The value of a determinant remains unchanged if rows are changed to column and column to rows.

$\begin{vmatrix} a1 & a2 \\ b1 & b2 \end{vmatrix} = \begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$

(ii) If two rows (or two columns) are interchanged, the sign of the determinant is changed

$\begin{vmatrix} a2 & b2 \\ a1 & b1 \end{vmatrix} = - \begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$

Notes:

(i) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

(ii) If value of determinant 'Δ' becomes zero by substituting x = α, then x – α is a factor of 'Δ'.

(iii) If all the elements of a determinant above or below the main diagonal consist of zeros, then the value of the determinant is equal to the product of diagonal elements.

(iii) If two rows (or two columns) are identical, the value of the determinant is zero.

$$\begin{vmatrix} a1 & a2 \\ b1 & b2 \end{vmatrix}=0$$

(iv) If the elements of any one row (or column) are all multiplied by a common factor, the determinant is multiplied by that factors.

$$\begin{vmatrix} ka1 & kb1 \\ a2 & b2 \end{vmatrix}=k\begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$$

(v) If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.

$$\begin{vmatrix} a1+kb1 & b1 \\ a2+kb2 & b2 \end{vmatrix}=\begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$$

1. Find the minor of each element of the following matrices:

   a. $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$

Solution:

a. Let A= $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$

Let Mij denotes the minor of (i, j)th element of the matrix A, i, j = 1, 2

∴ $M_{11}=|-7|=-7$

Determinant obtained after deleting first row and first column of matrix A= |– 7|

Similarly, $M_{12}=|4|=4$, $M_{21}=|5|=5$, $M_{22}=|2|=2$

2. Find the cofactor of each element of the following matrices:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Solution: let Mij be the minor of every element

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48 = -3$$

$$M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42 = -6$$

$$M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3$$

$$M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 18 - 24 = -6$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 27 = -12$$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 -14 = -16$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6 - 12 = -6$$

$$M_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3$$

The cofactor matrix A is

$$A = \begin{vmatrix} +(-3) & -(-6) & +(-3)) \\ -(-6) & +(-12) & -(-6) \\ +(-3) & -(6) & +(-3) \end{vmatrix}$$

$$A = \begin{vmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{vmatrix}$$

6.7 Adjoint and Inverse of a Matrix

(i) The adjoint of a square matrix A = [aij] n×n is defined as the transpose of the matrix. [aij] n×n , where Aij is the co-factor of the element aij. It is denoted by adj A

$$\text{If } A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then } adj\ A=\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}, \text{ where } A_{ij} \text{ is co-factor of } a_{ij}.$$

(ii) A (adj A) = (adj A) A = |A| I, where A is square matrix of order n.

(iii) A square matrix A is said to be singular or non-singular according as |A| = 0 or |A| ≠ 0, respectively.

(iv) If A is a square matrix of order n, then |adj A| = $|A|^{n-1}$.

(v) If A and B are non-singular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

(vi) The determinant of the product of matrices is equal to product of their respective determinants, that is, |AB| = |A| |B|.

(vii) If AB = BA = I, where A and B are square matrices, then B is called inverse of A and is written as B = A–1. Also $B^{-1} = (A^{-1})^{-1} = A$.

(viii) A square matrix A is invertible if and only if A is non-singular matrix. (ix) If A is an invertible matrix, then A^{-1} = (1 / | A |)* (adj A)

3. Find out the inverse of

$$\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution: let A= $\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & -6 \\ 0 & 1 & -1 \end{bmatrix}$

A^{-1}=adj(A)/|A|

To find out the adj(A), first we have to find out cofactor(A).

a11 = -6, a12 = 4, a13 = 4

a21 = 1, a22 = -1, a23 = -1

a13 = -6, a32 = 2, a33 = 4

So, cofactor(A) = $\begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}$

adj(A) = $[\text{cofactor}(A)]^T$

adj(A) = $[\text{cofactor}(A)]^T$= $\begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}^T$

adj(A) = $\begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}$

Then, | A | = 1(0-6)+1(-4-0)+2(4-0) = -6-4+8 = -2

A^{-1}=adj(A)/|A| = $\dfrac{\begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}}{-2}$

A^{-1}= $\begin{bmatrix} 3 & -1/2 & 3 \\ 2 & 1/2 & -1 \\ -2 & 1/2 & -2 \end{bmatrix}$

6.8 Solution of Equations using Matrices

One of the most important applications of matrices is to the solution of linear simultaneous equations. On this leaflet we explain how this can be done.

Writing simultaneous equations in matrix form

Consider the simultaneous equations

$$x + 2y = 4$$

$$3x - 5y = 1$$

Provided you understand how matrices are multiplied together you will realize that these can be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing

A= $\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$, X= $\begin{pmatrix} x \\ y \end{pmatrix}$ and B= $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

we have AX = B

This is the matrix form of the simultaneous equations. Here the only unknown is the matrix X, since A and B are already known. A is called the matrix of coefficients.

Solving the simultaneous equations

Given

$$AX = B$$

we can multiply both sides by the inverse of A, provided this exists, to give

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$, the identity matrix. Furthermore, IX = X, because multiplying any matrix by an identity matrix of the appropriate size leaves the matrix unaltered.

So

$X = A^{-1} B$

4. Example. Solve the simultaneous equations x + 2y = 4

$$3x - 5y = 1$$

Solution. We have already seen these equations in matrix form: $\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

We need to calculate the inverse of A= $\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$

$$A^{-1} = \frac{1}{(1)(-5) - (2)(3)} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}$$

Then X is given by

$X = A^{-1}B$

$= -\frac{1}{11}\begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$= -\frac{1}{11}\begin{pmatrix} -22 \\ -11 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

5. Solve the simultaneous equations

   2x + 4y = 2

   -3x + y = 11

   Solution. In matrix form:

$$\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$$

We need to calculate the inverse of $A= \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$

$$A^{-1}= \frac{1}{(2)(1)-(4)(-3)}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$$

Then X is given by

$X = A^{-1}B$

$= \frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 11 \end{pmatrix}$

$= -\frac{1}{14}\begin{pmatrix} -42 \\ 28 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

Hence x = −3, y = 2 is the solution of the simultaneous equations. You should check the solution by substituting x = −3 and y = 2 into both given equations, and verifying in each case that the left-hand side is equal to the right-hand side.

Summary

1. A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n matrix : when m = n, it is called a square matrix. To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes, which respectively specify the rows and the columns. This aij is the element in the ith row and j th column of the matrix A. In the notation, the matrix A is denoted by [aij].
2. Two matrices can be multiplied only when the number of column in the first is equal to the number of row in the second. Thus matrices $[a_{ij}]$ = A and B= $[b_{jk}]$ are compatible for multiplication and $[c_{ik}]$ =C is their multiplication is not commutative, that is AB ≠ BA in general.
3. If A be any matrix, then a matrix B, if it exists, such that AB = BA = I is called the inverse of A.
4. Inverse of a square matrix A^{-1} =Adj A/ | A |
5. Product of a square matrix and its inverse $AA^{-1} = A^{-1}A = I$
6. Solution of a set of linear equations

   A.X=B is $X=A^{-1}.B$

Keywords

- Matrix: A rectangular array of numbers (elements).
- Row Matrix: One row only.
- Column Matrix: One column only.
- Equal Matrices: Corresponding elements equal.
- Diagonal Matrix: All elements zero except those on the leading diagonal.
- Null Matrix: All elements zero.

Self Assessment

1. A matrix having one row and many columns is known as?
A. Row matrix
B. Column matrix
C. Diagonal matrix
D. None of the mentioned

2. A square matrix A = [aij]nxn, if aij = 0 for i > j then that matrix is known as _______
A. Upper triangular matrix
B. Lower triangular matrix
C. Unit matrix
D. Null matrix

3. Two matrixes can be added if _______
A. rows of both the matrices are same
B. columns of both the matrices are same
C. both rows and columns of both the matrices are same
D. number of rows of first matrix should be equal to number of columns of second

4. If the order of matrix A is m*p. And the order of B is p*n. Then the order of matrix AB is?
A. m × n
B. n × m
C. n × p
D. m × p

5. A square matrix in which all elements except at least one element in diagonal are zeros is said to be a
A. Identical matrix
B. Null/zero matrix
C. Column matrix
D. Diagonal matrix

6. Two matrices A and B are multiplied to get AB if
A. both are rectangular
B. both have same order
C. no of columns of A is equal to columns of B
D. both are square matrices

7. Which of the following matrix multiplications would not be possible?
A. a 5×6 matrix with a 6×3
B. a 4×3 matrix with a 3×3
C. a 2×2 matrix with a 3×3
D. a 5×6 matrix with a 3×3 matrix

8. A diagonal matrix having equal elements is called a
A. square matrix
B. identical matrix
C. scalar matrix
D. rectangular matrix

9. Generally, the matrices are denoted by
A. capital letters
B. numbers
C. small letters
D. operational signs

10. According to determinant properties, when two rows are interchanged then the signs of determinant
A. Must changes
B. Remains same
C. Multiplied
D. Divided

11. In matrices, the determinant of a matric is denoted by
A. Vertical lines around matrix
B. Horizontal lines around matrix
C. Bracket around matrix
D. None of the above

12. According to determinant properties, the determinant equals to zero if row is
A. Multiplied to row
B. Multiplied to column
C. Divided to row
D. Divided to column

13. According to determinant properties. The determinant of resulting matrix equals to k delta if elements of rows are
A. Multiplied to constant k
B. Added to constant k
C. Multiplied to constant k
D. Divided to constant

14. The matrix which does not have an inverse by solving it, is classified as
A. Unidentified matrix
B. Linear matrix

C. Non-singular matrix
D. Singular matrix

15. Which of the following is not a property of determinant?
A. The value of determinant changes if all of its rows and columns are interchanged
B. The value of determinant changes if any two rows or columns are interchanged
C. The value of determinant is zero if any two rows and columns are identical
D. The value of determinant gets multiplied by k, if each element of row or column is multiplied by k

Answers for Self Assessment

1.	A	2.	A	3.	C	4.	A	5.	D
6.	C	7.	A	8.	C	9.	A	10.	A
11.	D	12.	A	13.	C	14.	D	15.	A

Review Questions

1. Write each sum as a single matrix:

a. A= $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \end{bmatrix}$ and B= $\begin{bmatrix} 6 & 3 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

b. A= $|1 \quad 3 \quad 5|$ and B= $|0 \quad -2 \quad 1|$

2. Write each product as a single matrix:

A. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

B. $|3 \quad -2 \quad 5|\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

3. Solution by matrix inverse method

i) A= $\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

ii) A= $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

4. Solve equations by using matrix

I. -2x + 3y = 8

3x - y = -5

II. 2 x - y + 3 z = - 3

- x - y + 3 z = - 6

x - 2y - z = - 2

Further Readings

- Hoffman, K. H, and Kunze Ray, Linear Algebra, Prentice Hall of India.
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Unit 07: Input-Output Analysis

CONTENTS

Objectives

After studying this unit, you will be able to,

- Apply the theory of matrices and linear equation in a market demand and supply Model;
- Analysis of Input-Output Model;
- Work on how solution is attained in a framework of several variables production problems related to input-output;

Introduction

The concept of Input-Output (I-0) analysis originated by the eighteenth-century French economists Quesnay in the name of 'Tableau Economies'. In its modem form, the 1-0 analysis has been developed by the American economist Wassily W. Leontief in his famous work 'Structure of the American Economy' in the year 1951.

I-0 analysis is a method of analyzing how an industry undertakes production by using the output of other industries in the economy and how the output of the given industries used up in other industries or sectors. Since various industries are interdependent, i.e., the output of one industry is an input for the others, their mutual relationship ultimately must lead to equilibrium between Input-Output Analysis supply and demand in the economy consisting of 'n' industries. 1-0 analysis is also known as the inter-industry analysis as it explains the interdependence and interrelationship among various industries. Since the analysis is concerned with the relations among inputs, use of a commodity in the production by an industry, it is known as the input-output analysis. In other words, the I-0 analysis explains the interdependence of inputs and outputs of various industries in the economy.

7.1 Input-Output Analysis

Input- Output analysis explains the interdependence and interrelationship of inputs and outputs of various industries in the economy. It is a method of analyzing how an industry undertakes production by using the output of other industries in the economy, and how the output of the given industry is used up in other industries of the same economy.

In its "static" version, the Input- Output analysis of Professor W. Leontief, a Nobel Prize winner, deals with this particular version: What level of output should each of the n industries in an economy produce, in order that it is just be sufficient to satisfy the total demand for the product? I n view of the inter industry dependence, any set of "correct" output levels for the n industries must be one that is consistent with all the input requirements in the economy, so that no bottlenecks will arise anywhere.

I-0 analysis was an attempt made by Prof. Leontief to take account of 'general equilibrium' phenomena in the 'empirical' analysis of 'production'. These three italicized elements are the main features of I-0 analysis. First, the I-0 analysis deals almost exclusively with production. The problem is essentially technological. Given the quantities of available resources and the state of technology, the analysis is concerned with the use of various inputs by the industries and outputs derived from them.

The second distinctive feature of I-0 analysis is its devotion to empirical investigation. This is primarily what distinguishes it from the work of Walras and later general equilibrium theorists. I-0 employs a model, which is more simplified and also narrower in the sense that it seeks to encompass fewer phenomena than does the usual general equilibrium theory. Its narrowness lies in its exclusive emphasis on the production side of the economy.

The third distinctive feature is its emphasis of general equilibrium phenomena where everything depends on everything else. Thus, in the two-industry model coal is an input for steel industry and steel is an input for coal industry, though both are the output of respective industries. According to I-0 analysis, it is not possible to find some industries as being in the 'earlier' stages of production and some other industries as being in the 'later' stages. For the production of coal, steel is needed; whereas, for the production of steel, coal is required. No one can say whether the coal industry or the steel industry is earlier or later in the hierarchy of production.

The basic problem, then, is to see what can be left over for final consumption and how much of each output will be used up in the course of the productive activities which must be undertaken to obtain these net outputs. Solution of this problem can be used in predicting future production requirements if usable demand estimates can somehow be obtained. Particularly, it can be used for economic planning including problems of economic development in 'backward areas' as well as problem of military mobilization. A more modest purpose that it has already successfully begun to serve is the provision of a very illuminating detailed structure for national income accounting.

7.2 Assumptions

As it was stated earlier, the intransigence of empirical materials and the computational problems have forced on 1-0 analysis to adopt a number of simplifying assumptions even more extreme than those usually employed in our theoretical models.

The economy can be meaningfully divided into finite number of sectors (industries) on the basis of the following assumptions:

i) Each industry produces only one homogeneous output. NO two products Application are produced jointly; but if at all there is such a case, then it is assumed that they (products) are produced in fixed proportions.

ii) Each producing sector satisfies the properties of linear homogeneous production function. In other words, production of each sector is subject to constant returns to scale so that k-fold change in every input will result in an exactly k-fold change in the output.

iii) A far stronger assumption is that each industry uses a fixed input ration for the production of its output; in other words, input requirement per unit of output in each sector remains fixed and constant. The level of output in each sector (industry) uniquely determines the quantity of each input, which is purchased.

iv) The final demand for the commodities is given from outside the system. The total amount of the primary factor is also given. These two are called open end of the system and for this, the model is called 'open model'. In contrast in the 'closed model' all the variables are determined within the system.

v) The model is static in the sense that all variables in it refer to the same period of time. The static model is also known as flow model.

7.3 Structure

For the purpose of simplification, let us consider an economy with only two producing sectors: agriculture and industry. If we concentrate on agriculture, it will produce a given output of goods in a given time period, say, one year. These goods will be used in various ways. In other words, agricultural products will have several destinations. These destinations are broadly classified into two groups: (a) output going as intermediate input to the agricultural sector itself and to the industrial sector and (b) output going to the final demand sector for final consumption. For example, wheat is an output of agricultural sector and it may be demanded (i) as input in the agricultural sector itself (in the form of seed) for further production of wheat; and in industry for the manufacture of bread and (ii) for final consumption by the final demand sector. In the same manner, the output of the industry may be demanded by agricultural or industrial sectors as intermediates input and by final demand sector for final consumption.

In addition to the intermediate (also known as secondary) inputs, each sector requires primary inputs. This primary input is in the form of services of factors like land, labour, capital and entrepreneurship and is supplied by the final demand sector, also known as household sector.

Let us consider an input-output table involving agriculture and industry. Such a table is often called an input-output transactions matrix.

Table 1: Input-Output Transactions Matrix

Producing Sector	**Purchasing Sector**		**Final Demand (Consumption)**	**Total Output**
	1 (Agriculture)	2 (Industry)		
1 (Agriculture) 2 (Industry)	X_{11} X_{21}	X_{12} X_{22}	d_1 d_2	X_1 X_2
Primary Input (labour)	L_1	L_2		

In the above input-output transaction matrix, Xij (i = 1, 2; j = 1, 2) denote the output of ith producing sector that is being used as intermediate input in j th producing sector. In the above table, agricultural sector is denoted by 1 and industrial sector by 2. Thus, we can interpret X11 as the output of agricultural sector that is being used as an intermediate input in agriculture. Similarly, X21 can be interpreted as the output of industrial sector that is being used as input in agriculture etc. In addition, the output for each producing sector goes for the final use also. Thus, d1 is the output of agriculture and d2 is the output of industry going for the final use (consumption). Further, X1 and X12 denote the total output of agriculture and industry respectively. In view of the above introduction, we can write

$X_{11} + X_{12} + d_1 = X_1$ (1)

$X_{21} + X_{22} + d_2 = X_2$ (2)

Finally, the elements in the last row of the above table i.e. , L1 and L2 denote the requirement of primary input (say, labour) by the respective sector. These primary inputs, as mentioned earlier, are presumed t be supplied by the household sector. It is clear that our input-output table is an oversimplified representation. However, the table can readily be made more realistic by introducing more producing sectors, additional kinds of final uses (say, investment, government expenditure and net exports) and other categories of primary inputs (that is, land, capital and entrepreneurship).

Let us denote this amount by a_{ij}, where $a_{ij}=\frac{xij}{xj}$. If xij and xj denote value of outputs, then aij can be interpreted in value terms. Thus a_{ij}, = 0.04 can mean that forty paise worth of the ith commodity is required to produce onerupee worth of the jth commodity. This a_{ij}, is called an input coefficient.

We can arrange the input-coefficients of an economy with a given number of producing sectors in the form of a matrix. This matrix is called an input coefficient matrix.

We should note that the sum of elements in each column gives the requirement of secondary input to produce a rupee worth of output in that producing sector. As a result, the sum of the elements in each column of the input coefficient matrix should be less than 1, since; it does not include the cost of the primary inputs per rupee worth of the output. Assuming that there is pure competition with free entry, the primary input cost per one rupee worth output for a producing sector should be one minus the relevant column sum of the elements of the input coefficient matrix. For the economy considered above, if the cost of primary inputs per rupee worth of output for the two producing sectors are l_1 and l_2 respectively; then

$l_1 = 1- (a_{11} + a_{21})$

and

$l_2 = 1- (a_{12} + a_{22})$

The total output for each producing sector can be expressed in terms of the input coefficients, by replacing $xj\ a_{ij}= xij$ in equations (1) and (2). Thus, for out two producing sector economy

$X_1=a_{11}X_1+a_{12}X_2+a_{12}X_2+d_1$

or

$X_1-a_{11}X_1-a_{12}X_2=d_1$

or

$(1-a_{11})X_1-a_{12}X_2=d_1$ (3)

and similarly

$X_2=a_{21}X_1+a_{22}X_2 +d_2$

or

$X_2-a_{21}X_1-a_{22}X_2 =d_2$

or

$-a_{21}X_1+(1-a_{22})X_2-a_{12}X_2- a=d_1$ (4)

Writing equations (3) and (4) in the matrix form

$$\begin{pmatrix} 1-a11 & -a12 \\ -a21 & 1-a22 \end{pmatrix}\begin{pmatrix} X1 \\ X2 \end{pmatrix}=\begin{pmatrix} d1 \\ d2 \end{pmatrix}$$

Suppose

$A=\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}, x=\begin{pmatrix} X1 \\ X2 \end{pmatrix}$ and $d=\begin{pmatrix} d1 \\ d2 \end{pmatrix}$ (5)

(I - A) X = d (6)

Where I is a (2×2) identity matrix. The matrix (I-A) is called the technology matrix. If (I -A) is non-singular (that is, $|I - A| \neq 0$, equation (6) can be solved for X. Thus

$X= (I - A)^{-1}\ d$ (7)

It is important to note that for a given technology, as embodied in the technology matrix, equation (7) can be used to determine the total output that is needed to be produced by different producing sectors to satisfy a given final demand for the commodities.

7.4 Closed Model

In the closed model, all the goods are intermediate in nature, for everything that is produced, is produced for the sake of satisfying the input requirements of the other industries of the model.

Let,

aij = Required minimum input of commodity i, per unit of output of commodity j (the first subscript refers to the input, and the second to the output),)

aij = In order to produce each unit of jth commodity, the input need for the ith commodity)

aij = How much of the ith commodity be used for the production of each unit of the jth commodity)

aij = In order to produce each unit of the jth commodity, the input need for the ith commodity must be a fixed amount)

aij = "Input Coefficient" which is assumed to be fixed.

For an n industry economy, the input coefficients can be arranged into a matrix, A = [aij]; as shown in the table 2, in which each column specifies the input requirements for the production of one unit of the output of a particular industry.

It is to note that if no industry uses its own product as an input, then the elements in the principal diagonal of matrix A will all be zero.

Table 2: : Input Coefficient Matrix (closed model)

	Output				
Input ↓	I	II	III	...	N
I	a_{11}	a_{12}	a_{13}	...	a_{1n}
II	a_{21}	a_{22}	a_{23}	...	a_{2n}
III	a_{31}	a_{32}	a_{33}	...	a_{3n}
...	...	...	...	...	...
N	a_{n1}	a_{n2}	a_{n3}	...	a_{nn}

Here, in a static model, it is to observe that each column sum represents the final input cost incurred in producing a Rupee's worth of some commodity. Symbolically,

$$\sum_{i=1}^{n} aij = 1; (j = 1,2,3, \dots \dots . . n)$$

where the summation is over i, that is, over the elements appearing in the various rows of a specific column j. If industry I is to produce an output just sufficient to meet the input requirements of the n industries of the closed sector, its output level x_1 must satisfy the following equation:

$$x_1 = a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots\dots + a_1nxn$$

That is in general,

$$x_1=a_{11}x_1+a_{12}x_2+\dots\dots\dots\dots\dots+a_1nxn=\sum_{i=1}^{n} aij.xj; = 1; (j = 1,2,3, \dots \dots . . n)$$

where $a_{ij}x_j$ represents the input demand from the jth industry; and $\sum_{i=1}^{n} aij.xj$ represents the total amount of xi needed as input for the n industries.

7.5 Coefficient Matrix and Open Model

Our open model in matrix notation is given by, X = AX + F, where A is the input coefficient matrix, F is the final demand vector and X is the total output matrix. The input coefficient matrix or 'technology matrix' represented by [a,,] is of great importance. Each element must be non-negative, i.e., we rule out the possibility of negative inputs. But to maintain complete interdependence among the industries each element of [a,,] matrix must be positive and no element can exceed unity, i.e., we rule out the possibility of negative outputs. Each column of this matrix specifies the input requirements for the production of 1 unit of a particular commodity; thus, sum of the elements in each column must be less than unity. Symbolically, this fact may be stated as: $\sum_{i=1}^{n} aij < 1; (j = 1,2,3, \dots \dots . . n)$and each a,, is non-negative, i.e., either zero or 1=1 greater than zero. The cost of the primary inputs (which is also termed as 'value added') needed in producing a unit of jth commodity would be (1-$\sum_{i=1}^{n} aij$)

If this wen not true, it would mean that the total value of intermediate products used by an industry exceeded the value of its input. This in turn would mean that the value added by that industry was negative. Now, this is not impossible, but, if we assume that the wage bill cannot be negative, it means that the industry must be making losses (indeed, losses greater in absolute value than its wage bill). An industry in which value added is negative is not covering variable costs (intermediate inputs plus the wage bill), and we know from the elementary micro theory that in such a case, losses will be reduced by closing down. Thus, we do not want to describe such an industry in our technology matrix at all. Given the assumption of CRS, we've described the technology by a constant coefficient matrix. We should notice that we have also built in the assumption that there are no externalities. An externality in production would exists if, for example, a factory discharged waste into a river so that a factory further downstream had to use the resources to clean the water before it could use it. In this case, the resource requirement of the later factory would not depend solely on its output but would, also depend on the activity of the former.

7.6 Solution to Open Model

If the n industries in table 1 constitute the entirety of the economy, then all their products would be for the sole purpose of meeting the input demand of the same n industries (to be used in further production) as against the final demand (such as consumer demand, not for further production). At the same time, all the inputs used in the economy would be in the nature of intermediate inputs (those supplied by the n industries) as against primary inputs (such as labour, not an industrial product).

To allow for the presence of final demand and primary inputs, we must include in the model an open sector outside the n industry network.

Here, in a open model, it is to observe that each column sum represents the partial input cost (not including the cost of primary inputs) incurred in producing a Rupee's worth of some commodity. Symbolically,

$$\sum_{i=1}^{n} aij < 1; (j = 1,2,3, \dots \dots . . n)$$

where the summation is over i, that is, over the elements appearing in the various rows of a specific column j. Thus, the value of primary inputs needed in producing a unit of the jth commodity should be (1-$\sum_{i=1}^{n} aij > 0$)

	Output					Final
Input ↓	**I**	**II**	**III**	...	**N**	**Demand**
I	a_{11}	a_{12}	a_{13}	...	a_{1n}	d_1
II	a_{21}	a_{22}	a_{23}	...	a_{2n}	d_2
III	a_{31}	a_{32}	a_{33}	...	a_{3n}	d_3
...	...	...	...	...	...	...
N	a_{n1}	a_{n2}	a_{n3}	...	a_{nn}	d_n
Primary Input	a_{01}	a_{02}	a_{03}	...	a_{0n}	–

If industry I is to produce an output just su¢ cient to meet the input requirements of the n industries as well as the Önal demand of the open sector, its output level x1 must satisfy the following equation:

$x_1 = a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots. + a_{1n}x_n + d_1)$

$=d_1 = x_1 - (a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots. + a_{1n}x_n)$

$=(1- a_{11})\ x1\ -a_{12}x_2 - a_{1n}x_n = d_1$

By the same token, the output levels of all the n industries should satisfy the following system of linear equations:

$-a_{21}x_1 – (1- a_{22})x_2 - \ldots\ldots\ldots\ldots..- a_{2n}x_n = d_2$

$-a_{31}x_1 – a_{32}x_2 - (1- a_{33})x_3 - \ldots\ldots\ldots\ldots..- a_{2n}x_n = d_2$

$-a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots\ldots\ldots.(1- a_{nn})xn = d_n$

In matrix form, this may be written as:

In the matrix notation this may be written as:

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}$$

(I - A) X = F,

i.e X= $(I - A)^{-1}$ F

Here A is the given matrix of input coefficients, while X and F are the vectors of output and final demand of each producing sector. If II-Al* 0, then [I-A]-' exists, we can then estimate for either of the two matrices X and F by assuming one of them to be given exogenously. It is to be observed that, the assumptions made in 1-0 analysis go a long way in making the problem simplified. For example, with the assumption of linear homogenous function, it is possible to write a linear equation of each producing sector, which then can be easily transformed into matrix notation. On the other hand, as long as the input coefficients remain fixed (as assumed), the matrix A will not change or [I-A] will not change. Therefore, in finding the solution of X = [LA]-' F, only one matrix inverse needs to be performed even if we are to consider thousands of different finals, demand vectors according to alternative development targets. Hence, such an assumption of fixed technical coefficient has meant considerable savings in computational effort.

7.7 Hawkin Simon conditions

Many a times I-0 solution may give output expressed by negative numbers. If our solution gives negative outputs, it means that more than one unit of a product is used up in the production of every one unit of that product; it is definitely an unrealistic situation. Such a system is not a viable. Hawkins Simon condition guards against such eventualities. Our basic equation is X = $[I-A]^{-1}$ F, in order that this does not give negative numbers as a solution, the matrix [I-A], which in fact is

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix}$$ hould be such that

i) the determinant of the matrix must always be positive, and

ii) the diagonal elements: ($I-a_{11}$), ($1-a_{22}$),. ... ($I-a_{nn}$) should all be positive or, in other words, elements, a1 1, a22,...,a_{nn} should all be less than one. Thus, one unit of output of any sector should use not more than one unit of its own output. These are called Hawkins-Simon conditions. Further, the first condition, that implies D > 0, implies that (for 2 industry case) $\begin{pmatrix} 1-a11 & -a12 \\ -a21 & 1-a22 \end{pmatrix} > 0$, or,

($1-a_{11}$)($1-a_{22}$)-$a_{12}a_{21}$>0. This condition implies that the direct and indirect requirement of any commodity to produce one unit of that commodity must also be less than one. On the other hand, the interpretation is always that all subgroups of commodities should be 'self-sustaining', directly and indirectly.

Example 1: Suppose [A]= $\begin{pmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{pmatrix}$

Then [I-A]=$\begin{pmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{pmatrix}$

and the value of the determinant [I-A] = (-) 8.12, which is less than zero. As the Hawkins-Simon conditions are not satisfied, no solution will be possible in this case.

7.8 General Equilibrium: 3-Industry case

- Simplest case: Suppose we divide the economy into 3 sectors:

1. Agriculture
2. Manufacturing
3. Services

- The three industries each use inputs from two sources:

1. Domestically produced commodities form the three industries
2. Other inputs, such as imports, labour, and capital.

- The outputs of the industries have two broad uses or destinations:

1. Inputs to production of the three industries (intermediate inputs)
2. Final demand (Consumption, Investment, Government expenditure, Exports)

	Output					
		Agriculture	Manufactures	Services	Final demand	Total
Input	Agriculture	30	40	0	30	100
	Manufactures	10	200	50	140	400

	Services	20	80	200	200	500
	Final demand	40	80	250	230	600
	Total	100	400	500	600	1600

Take for example manufacturing:

- Its output is worth 400 crore, which is allocated as follows:
- ❖ 10crore is used by the agricultural sector
- ❖ 200 crore as intermediate goods for the manufacturing sector
- ❖ 50 crore is used by the services sector.
- ❖ 140 crore is the final demand (consumption, investment, government expenditure & exports

- In order to produce, the manufacturing sector uses inputs worth of e400, of which
- ❖ 40crore comes from the agricultural sector,
- ❖ 200crore from the manufacturing sector (intermediate inputs),
- ❖ 80 crore from the services sector,
- ❖ 80 crore from other sources, including imports, labour and capital

Define the following 2 vectors

b= $\begin{bmatrix} 30 \\ 140 \\ 200 \end{bmatrix}$ is the vector of final demands for output of the industry sectors

x= $\begin{bmatrix} 100 \\ 400 \\ 500 \end{bmatrix}$ is the vector of total ouput of the industry sectors

As said above, one (critical) assumption is that each sector produces according to fixed proportion technological coefficients (also called input-output coefficients).

7.9 Application of Leontief Inverse

Example 1: Agriculture uses 20 crore from the services sector. Given that the value of its total inputs is 100 crore, then services represent 20/100 =0:20 of its total inputs.

	Output					
Input		Agriculture	Manufactures	Services	Final demand	Total
	Agriculture	3/10	1/10	0	1/20	1/16
	Manufactures	1/10	1/2	1/10	7/30	1/4
	Services	1/5	1/5	2/5	1/3	5/16
	Final demand	2/5	1/5	1/2	23/60	3/8
	Total	1	1	1	1	1

In matrix form:

$$A=\begin{bmatrix} 3/10 & 1/10 & 0 \\ 1/10 & 1/2 & 1/10 \\ 1/5 & 1/5 & 2/5 \end{bmatrix}$$ is the matrix of inter-industry coefficients

One important consequence of the input-output analysis is that we can express the vector of total demand (x) as a function of the final demand (b) and the matrix of inter-industry Coefficients (A):

$$x = Ax + b$$

Then:

$$x - Ax = b$$

$$(I - A)\, x = b$$

If (I - A) has an inverse:

$(I - A)^{-1}(I - A)\, x = (I\ \ A)^{-1}b$

$x = (I - A)^{-1}b$

Thus, The matrix $(I - A)^{-1}$ is known as the input-output inverse, or the Leontieff Inverse

$$(I - A)^{-1} = \begin{bmatrix} 0.7 & -.1 & 0 \\ -\ 0.1 & 0.5 & -0.1 \\ -0.2 & -0.2 & 0.6 \end{bmatrix}$$

Thus, $x = (I\ \ A)^{-1}b$

$$X=\begin{bmatrix} 1.49 & 0.32 & 0.05 \\ 0.42 & 2.23 & 0.37 \\ 0.64 & 0.85 & 1.81 \end{bmatrix}\begin{bmatrix} 1/20 \\ 7/30 \\ 1/3 \end{bmatrix}$$

$$X=\begin{bmatrix} 1.49 * \frac{1}{20} + 0.32 * \frac{7}{30} + 0.05 * 1/3 \\ 0.42 * \frac{1}{20} + 2.23 * \frac{7}{30} + 0.37 * 1/3 \\ 0.64 * \frac{1}{20} + 0.85 * \frac{7}{30} + 1.81 * 1/3 \end{bmatrix}$$

$$X=\begin{bmatrix} 0.165 \\ 0.664 \\ 0.833 \end{bmatrix}$$

7.10 Determination of Equilibrium Prices

Let the prices of commodities 1, 2, 3,. .. be pl, p2, p3 ,... respectively, and the price of the primary factor inputs be w (here, primary factor is labor; so w represents wage rate), then the technology matrix or transaction matrix in quantity may be converted into that in value terms. The problem can be posed as follows:

Sector	1	2	Final demand
1	$a_{11}X_1p_1$	$a_{12}X_2p_1$	F_1p_1
2	$a_{21}X_1p_2$	$a_{22}X_2p_2$	F_2p_2
Primary Input	$l_1X_1\,w$	$l_2X_2\,w$	
Total Costs	$a_{11}X_1p_1 + a_{21}X_1p_2 + l_1X_1\,w$	$a_{12}X_2p_1 + a_{22}X_2p_2 + l_2X_2\,w$	

With pure competition and free entry, profit in each industry must be zero, i.e., revenues equal costs. Hence, for the first industry receipts are (output x I price) and cost is $a_{11}X_1p_1 + a_{21}X_1p_2 + I_1x_1$ w. Same is true for the second industry.

Hence, for equilibrium; $p_1X_1 = a_{11}X_1p_1 + a_{21}X_1p_2 + I_1X_1w$; and $p_2X_2 = a_{12}X_2p_1 + a_{22}X_2p_2 + {}_{12}X_2w$, which simplify to $p_1 = a l_1 p_1 + a_{21}p_2 + l_1w$; and $p_2X_2 = a_{12}p_1$ I + $a_{22}p_2 + 1_2W$, which can be put in matrix form as under.

$$\begin{pmatrix} 1 - a11 & -a12 \\ -a21 & 1 - a22 \end{pmatrix}\begin{pmatrix} p1 \\ p2 \end{pmatrix} = \begin{pmatrix} I1W \\ I2W \end{pmatrix}$$

Notice that the set of coefficients here are transposed, this matrix is transposed of [I-A].

$$\begin{pmatrix} p1 \\ p2 \end{pmatrix} = \begin{pmatrix} I1W \\ I2W \end{pmatrix}\begin{pmatrix} 1 - a11 & -a12 \\ -a21 & 1 - a22 \end{pmatrix}^{-1}$$

Therefore, $p_1 = 1/D(A_{11}I_1 + A_{12}I_2)w$

$P_2 = 1/D(A_{21}I_1 + A_{22}I_2)w$

where A1 I, f D D A12, etc., are the cofactors of the matrix [I-A] as in the preceding cases.

7.11 Uses of Input–Output Analysis

Input–output analysis has a wide range of applications:

(1) Input–output analysis is used to obtain projections of demand, output, employmentand investment for a country or region.

(2) Input–Output analysis is helpful in providing necessary information for formulationof economic policies. It is used in economic development planning, studies of inter-regional and international economic relationships.

(3) Input–Output analysis is useful for national income accounting as it providesdetailed breakdown of the macro aggregates and the money flows.

(4) The inter relations between various sectors, as revealed in the input–output table, provide indication regarding prospective trends in which they are likely to combine with each other.

(5) Given a certain final output target, it can show the production requirements of various sectors.

7.12 Limitations of Input–Output Analysis

Despite its many useful applications, the input–output analysis has many shortcomings

which are given below:

(i) The input–output analysis is based on the assumption of fixed input co-efficients or proportions. Over time, technology and input prices change, and these are likely to greatly affect the proportions in which inputs are combined in the production of many commodities. This necessitates frequent and continuous updatings of input–output tables, which are very costly and time consuming.

(2) Input–output analysis is based on linear equations relating outputs of one industry to inputs of the others. This appears unrealistic

(3) In the input–output analysis labour is the only input which is considered scarce. This is not true in practice.

(4) Final demand (i.e, the purchases by consumers and the government) in the input–output analysis is taken as given and treated as independent of the production sector.

Though input–Output analysis has its short–comings, yet it is considered an important tool for the analysis of economic decisions of the government and in development planning.

Summary

1. The fundamental feature of any economic system is the interdependence of its constituent parts. The partial equilibrium theory is an inadequate tool for analyzing this interdependence between

the basic micro-economic units- the consumers and firms. General equilibrium theory, on the other hand, is basically designed to deal with the problem of interdependence. It seeks to analyze the interaction of the consumer and the firms in the determination of prices and quantities of goods and factor inputs.

2. Any disturbance in one market not only changes the equilibrium condition of the concerned market but also creates impacts on the equilibrium conditions of other markets in the economy. All the markets in the economy are interdependent of each other, General equilibrium exists in an economy when all the markets in the economy are in equilibrium simultaneously.

3. Walrasian system asserts that the excess demand is inversely related to price, i.e, the increase in the excess demand occurs due to decrease in prices.

4. The input-output analysis has been pioneered by Leontief. It is an application of general equilibrium theory. It is concerned with an empirical study of interdependence among various parts or sectors of the economy.

5. The input-output analysis divides the economy into a number of sectors or industries, including households and the government as “industries” of final demand: Each industry is viewed as selling its output to other industries; these outputs become inputs for the purchasing industries is viewed as a purchaser of the outputs of other industries. Thus, the interdependence of each industry on the other is established.

6. It basically states that more than on unit of a product cannot be used up in the production of every unit of that product. If A is the technological coefficient matrix then, according to Hawkins-Simon condition, determinant of [I-A] must be positive and all principal minors of [IA] must also be positive.

7.The matrix [aij], which basically represents input requirement from ith industry to produce one unit output of jth industry, is known as technological coefficient matrix.

8.The I-0 model that considers 'final demand bill' as exogenous factor is said to be as open I-0 model and in closed I-0 model 'final demand bill' is considered as endogenous factor.

Keywords

Condition: constraints

Input: Receipts

Output: Issuance Final Product

Self Assessment

1. In Leontief’s model own input coefficients are less than----

A. Four
B. Three
C. Two
D. One

2. Hawkins Simon condition explains the----- condition in Input output model

A. Instability
B. Stability
C. Uniqueness
D. None of the above

3. In Input Output model only----product is produced

A. One
B. Two
C. Three
D. Four

4. Leontief's model explains a----- equilibrium.

A. static
B. dynamic
C. both static and dynamic
D. none of the above

5. Factors of production in Input Output model is----

A. one
B. two
C. three
D. four

6. Leontief's model is open due to the fact that final demand is given from----- the system

A. within
B. outside
C. both (a) and (b)
D. none of the above.

7. Isoquants in Leontief's input- output model is

A. Variable coefficient
B. Fixed coefficient
C. Sometimes fixed sometimes variable
D. None of the above.

8. Leontief's input output model is based on the concept of-----

A. Consumption function
B. Partial Equilibrium
C. General Equilibrium.
D. All of the above.

9. The isoquants in Input Output model is

A. convex to the origin
B. concave to the origin
C. L shaped.
D. straight line.

10. Input-Output Technique was invented by

A. Gunnar Myrdal

B. Wassily Leontief
C. Hollis B. Chenery
D. Robert Solow

11. In an open economy input-output model, the viability is examined by

A. Hawkins-Simon conditions
B. Kuhn-Tucker conditions
C. Dickey-Fuller test
D. Goldfeld-Quandt test

12. Consumption Possibility curve (for one factor) in Leontief's model is a straight line due to the assumption of—

A. increasing returns to scale
B. decreasing returns to scale
C. constant returns to scale
D. none of the above.

13. Which one is efficient economy according to Leontief Input-Output Model?

A. More inner demand
B. More external demand
C. Inner demand = external demand
D. None of the above

14. Choose the correct formula to calculate Total production in Input-Output closed model.

A. X= AX
B. X=X/1-A
C. X=A
D. X= A/X

15. What does row and column represent in Input-Output Model?

A. Row- Output distribution/Factor payment, Column- Input Distribution
B. Row- Input Distribution, Column- Output distribution/Factor payment
C. Both a and b
D. None of the above

Answers for Self Assessment

1.	D	2.	B	3.	A	4.	A	5.	A
6.	B	7.	B	8.	C	9.	C	10.	B
11.	A	12.	C	13.	B	14.	A	15.	A

Review Questions

1. Suppose that the economy of a certain region depends on three industries: service, electricity and oil production. Monitoring the operations of these three industries over a period of one year, we were able to come up with the following observations:

a. To produce 1 unit worth of service, the service industry must consume 0.3 units of its own production, 0.3 units of electricity and 0.3 units of oil to run its operations.

b. To produce 1 unit of electricity, the power-generating plant must buy 0.4 units of service, 0.1 units of its own production, and 0.5 units of oil.

c. Finally, the oil production company requires 0.3 units of service, 0.6 units of electricity and 0.2 units of its own production to produce 1 unit of oil.

2. Consider an open economy with three industries: coal-mining operation, electricity-generating plant and an auto-manufacturing plant. To produce 1 of coal, the mining operation must purchase 0.1 of its own production, 0.30 of electricity and 0.1 worth of automobile for its transportation. To produce 1 of electricity, it takes 0.25 of coal, 0.4 of electricity and 0.15 of automobile. Finally, to produce 1 worth of automobile, the auto-manufacturing plant must purchase 0.2 of coal, 0.5 of electricity and consume 0.1 of automobile. Assume also that during a period of one week, the economy has an exterior demand of 50,000 worth of coal, 75,000 worth of electricity, and 125,000 worth of autos. Find the production level of each of the three industries in that period of one week in order to exactly satisfy both the internal and the external demands.

3. What is an open input-output model?

4. Discuss the importance of Hawkins-Simon Conditions in an input-output model.

5. The following is a input-output transactions matrix for a two-producing sector economy for the year: 2019-2020.

	Output				
		Agriculture	Manufactures	Final demand	Total
Input	Agriculture	17.70	3.27	20	38.77
	Manufactures	7.77	9.77	17	32.70
	Final demand	17.70	19.70		
	Total	38.77	32.70		

The Planning Commission of the economy feels that the final consumption demand for the products of the two sectors will rise by Rs. 10 Crore and Rs. 7 Crore respectively in the year: 2020-2021. Assuming unchanged technology and fixed prices, what will be the changes in the output of the two sectors? Also present the estimated input-output transactions matrix for the year 2020- 2021.

6. A three sector input-output matrix [I-A] is given as:

$$\begin{bmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{bmatrix}$$

with labor coefficients (per unit of output) as 0.4, 0.7, 1.2, if the household demand for the outputs of the 3 sectors is 1000, 5000 and 4000, determine the level of output and employment.

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Unit 08: Differential Calculus

CONTENTS

Objectives

After studying this unit, you will be able to,

- Calculate differentiation
- Calculate differentiation of sum, difference, multiplication and division
- Use of differentiation in economics

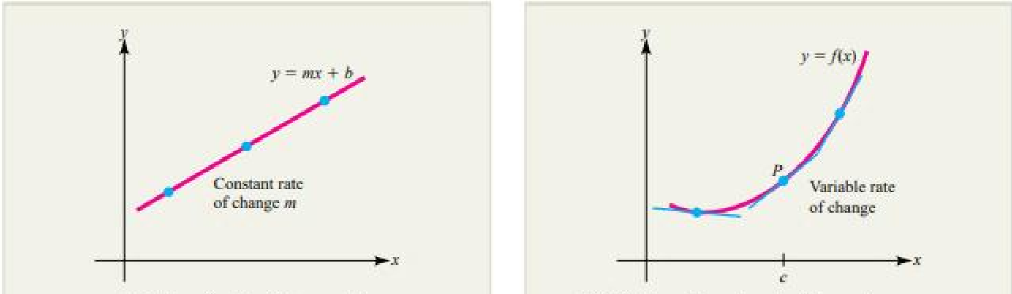
Introduction

Calculus is the mathematics of change, and the primary tool for studying change is a procedure called differentiation. In this section, we shall introduce this procedure and examine some of its uses, especially in computing rates of change. Here, and later in this chapter, we shall encounter rates such as velocity, acceleration, production rates with respect to labor level or capital expenditure, the rate of growth of a population, the infection rate of a susceptible population during an epidemic, and many others.

Calculus was developed in the seventeenth century by Isaac Newton (1642–1727) and G. W. Leibniz (1646–1716) and others at least in part in an attempt to deal with two geometric problems: Tangent problem: Find a tangent line at a particular point on a given curve. Area problem: Find the area of the region under a given curve. The area problem involves a procedure called integration in which quantities such as area, average value, present value of an income stream, and blood flow rate are computed as a special kind of limit of a sum. This initiated differential calculus, which has been the foundation for the development of modern science. It has also been of central importance to the theoretical development of modern economics.

8.1 Slope and Rates of Change

a linear function $L(x) = mx + b$ changes at the constant rate m with respect to the independent variable x. That is, the rate of change of L(x) is given by the slope or steepness of its graph, the line $y = mx + b$ (Figure 1a). For a function f(x) that is not linear, the rate of change is not constant but varies with x. In particular, when x=c, the rate is given by the steepness of the graph of f(x) at the point P(c, f(c)), which can be measured by the slope of the tangent line to the graph at P (Figure 1b). The relationship between rate of change and slope is illustrated in Fig. 1.



(a) A linear function $L(x) = mx + b$ changes at the constant rate m.

(b) If $f(x)$ is nonlinear, the rate of change at $x = c$ is given by the slope of the tangent line at $P(c, f(c))$.

Fig. 1.1 Rate of change is measured by slope.

8.2 Meaning of Derivative

Differentiation is a method used to find the slope of a function at any point. Although this is a useful tool in itself, it also forms the basis for some very powerful techniques for solving optimization problems, which are explained in this and the following chapters. The basic technique of differentiation is quite straightforward and easy to apply. Consider the simple function

$y = 6x^2$

To derive an expression for the slope of this function for any value of x the basic rules of differentiation require you to:

(a) multiply the whole term by the value of the power of x, and

(b) deduct 1 from the power of x.

In this example there is a term in x^2 and so the power of x is reduced from 2 to 1. Using the above rule the expression for the slope of this function therefore becomes

$2 \times 6x^{2-1} = 12x$.

This is known as the derivative of y with respect to x, and is usually written as dy/dx, which is read as 'dy by dx'.

Tangents and Derivatives

The tangent to a curve at a point is a straight line which just touches the curve at that point. We now give a more formal definition of the same concept.

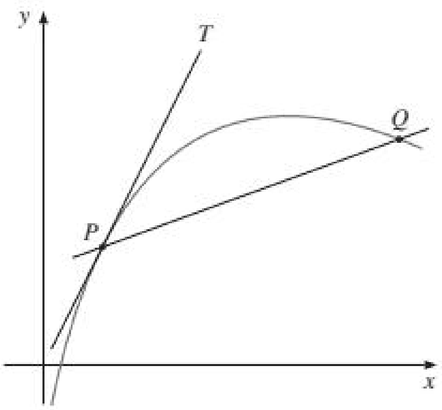


Figure 1

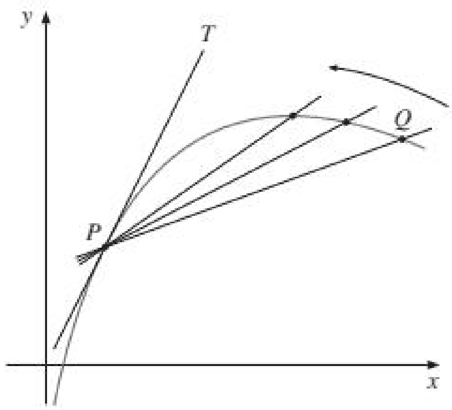
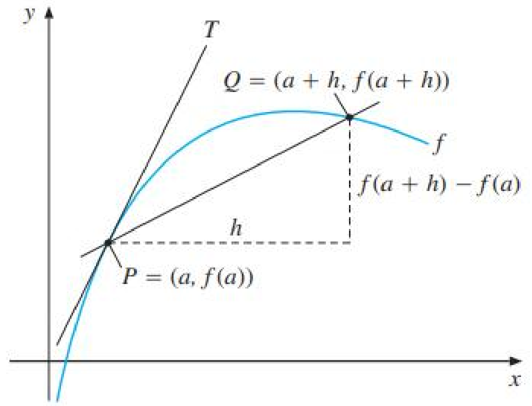


Figure 2

The geometrical idea behind the definition is easy to understand. Consider a point P on a curve in the xy-plane (see Fig. 1). Take another point Q on the curve. The entire straight line through P and

Q is called a secant. If we keep P fixed, but let Q move along the curve toward P, then the secant will rotate around P, as indicated in Fig. 2. The limiting straight line P T toward which the secant tends is called the tangent (line) to the curve at P. Suppose that the curve in Figs. 1 and 2 is the graph of a function f. The approach illustrated in Fig. 2 allows us to find the slope of the tangent P T to the graph of f at the point P.



This fraction is often called a Newton quotient of f . Note that when h = 0, the fraction becomes 0/0 and so is undefined. But choosing h = 0 corresponds to letting Q = P. When Q moves toward P along the graph of f, the x-coordinate of Q, which is a + h, must tend to a, and so h tends to 0. Simultaneously, the secant P Q tends to the tangent to the graph at P. This suggests that we ought to define the slope of the tangent at P as the number that mPQ approaches as h tends to 0. In the previous section we called this number f′(a). So, we propose the following definition of f′(a):

f′(a)= {the limit as h tends to 0 of} = $\frac{f(a+h)-f(a)}{h}$

It is common to use the abbreviated notation $\lim_{h\to0}$, or $\lim_{h\to0}$, for "the limit as h tends to zero" of an expression involving h.

Definition of the Derivative

The derivative of the function f at point a, denoted by f ′(a), is given by the formula

$$f'(a)= \lim_{h\to0} f(a+h) - f(a)/h$$

The number f′(a) gives the slope of the tangent to the curve y = f (x) at the point (a, f (a)). The equation for a straight line passing through (x1, y1) and having a slope b is given by y − y1 = b (x − x1).

Definition of the Tangent

The equation for the tangent to the graph of y = f (x) at the point (a, f (a)) is

$$y - f(a) = f'(a)(x - a)$$

Example 1: Find the derivative of the function $f(x) = 16x^2$

Solution: The difference quotient for f(x) is

$$\frac{f(a+h)-f(a)}{h} = \frac{16(x+h)^2-16x^2}{h}$$

$$= \frac{16(x^2+2hx+h^2)-16x^2}{h}$$

$$= (32hx+16h^2)/h$$

$$= 32x+16h$$

Thus, the derivative of $f(x)=16x^2$ is the function

$$f'(x)= \lim_{h\to0} f(x+h) - f(x)/h = \lim_{h\to0} f(32x+16h)$$

$$= 32x$$

8.3 Increasing and Decreasing Functions

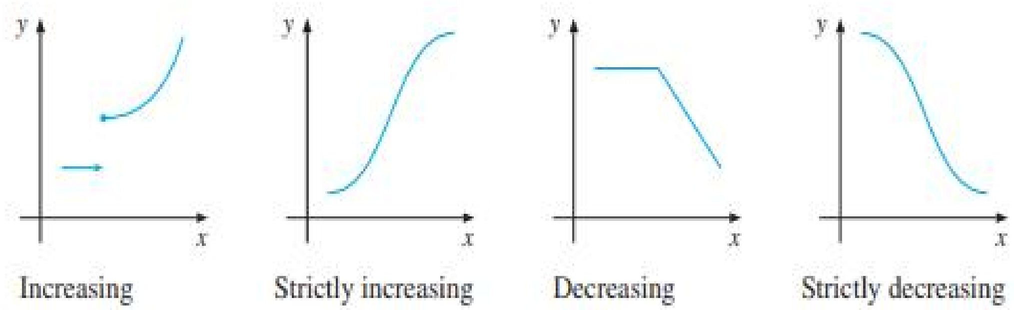
The terms increasing and decreasing functions have been used previously to describe the behavior of a function as we travel from left to right along its graph. In order to establish a definite terminology, we introduce the following definitions.

If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is **increasing** in I

If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is **strictly increasing** in I

If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is **decreasing** in I

If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is **strictly decreasing** in I



Note that we allow an increasing (or decreasing) function to have sections where the graph is horizontal. This does not quite agree with common language. Few people would say that their salary increases when it stays constant!

To find out on which intervals a function is (strictly) increasing or (strictly) decreasing using the definitions, we have to consider the sign of f (x_2)–f (x_1) whenever $x_2 > x_1$. This is usually quite difficult to do directly by checking the values of f (x) at different points x. In fact, we already know a good test of whether a function is increasing or decreasing, in terms of the sign of its derivative:

$f'(x) \geq 0$ for all x in the interval I $\Longleftrightarrow$ f is increasing in I (1)

$f'(x) \leq 0$ for all x in the interval I $\Longleftrightarrow$ f is decreasing in I (2)

Using the fact that the derivative of a function is the slope of the tangent to its graph, the equivalences in (1) and (2) seem almost obvious. An observation which is equally correct is the following:

$f'(x) = 0$ for all x in the interval I $\Longleftrightarrow$ f is constant in I

(3)

8.4 Rates of Change

The derivative of a function at a particular point was defined as the slope of the tangent to its graph at that point. Economists interpret the derivative in many important ways, starting with the rate of change of an economic variable.

Suppose that a quantity y is related to a quantity x by y = f (x). If x has the value a, then the value of the function is f (a). Suppose that a is changed to a + h. The new value of y is f (a + h), and the change in the value of the function when x is changed from a to a + h is f (a + h) − f (a). The change in y per unit change in x has a particular name, the average rate of change of f over the interval from a to a + h. It is equal to

$$\frac{f(a+h)-f(a)}{h}$$

The instantaneous rate of change of f at a is $f'(a)$

This very important concept appears whenever we study quantities that change. When time is the independent variable, we often use the "dot notation" for differentiation with respect to time. For example, if x(t) = t^2, we write x(t)= 2t.

Sometimes we are interested in studying the proportion f′(a)/f (a), interpreted as follows:

The relative rate of change of f at a is f′(a)/f (a).

In economics, relative rates of change are often seen. Sometimes they are called proportional rates of change. They are usually quoted in percentages per unit of time—for example, percentages per year (or per annum, for those who think Latin is still a useful language).

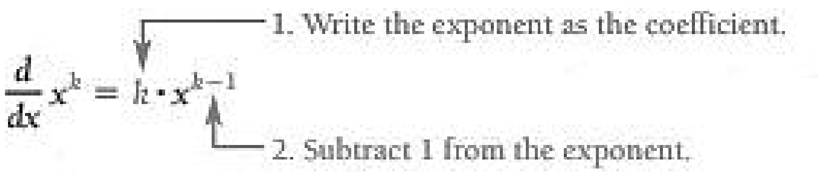
Rules for Limits:

Since limits cannot really be determined merely by numerical computations, we use simple rules instead. Their validity can be shown later once we have a precise definition of the limit concept.

Suppose that f and g are defined as functions of x in a neighborhood of a (but not necessarily at a).

The Power Rule

To find the derivative of a power function, bring the exponent to the front of the variable as a coefficient and reduce the exponent by 1.



THEOREM 1 The Power Rule

For any real number k, if $y = x^k$, then

$$\frac{d}{dx}x^k = k \cdot x^{k-1}.$$

Example 2: Use Theorem 1 to compute the derivative of:

(a) y = x^5

(b) y = $3x^8$

(c) y = x^{100}/100

Solution:

a) y = x^5

= $4x^{5-1}$

= $4x^4$

b) y= $3x^8$

=$3x^{8-1}$

=$3x^7$

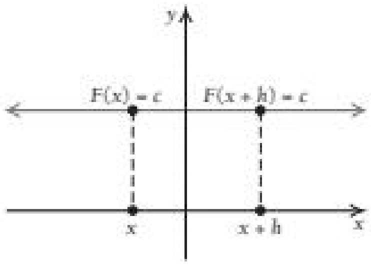
c) y = x^{100}/100

= ($100x^{100-1}$)/100

=x^{99}

The Derivative of a Constant Function

Consider the constant function given by Note that the slope at each point on its graph is 0.



Theorem 2

The derivative of a constant function is 0. That is, d/dx* c = 0.

Example 3: Find each of the following derivatives:

a) $7x^4 = 7\ d/dx(x^4) = 7.4.x^{4-1} = 28x^3$

b) -9x= d/dx(-9x) =-9

c) $(1/5x^2) = 1/5.d/dx\ .x^{-2}$

$= 1/5(-2)x^{-2-1}$

$= -2/5.\ x^{-3}$

$= -2/5x^3$

The Derivative of a Sum or a Difference

THEOREM 4 The Sum–Difference Rule

Sum. The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Difference. The derivative of a difference is the difference of the derivatives:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Example 4: Find each of the following derivatives:

a. $\frac{dy}{dx}(5x^3 - 7) = d/dx\ (5x^3) - d/dx\ (7)$

$= 5\ d/dx\ (x^3) - 0$

$= 5.3x^2$

$= 15x^2$

b. $d/dx\ (24x - x^{1/2} + 5/x) = d/dx\ (24x) - d/dx(x^{1/2}) + d/dx\ (5/x)$

$= 24\ d/dx\ (x) - d/dx\ (x^{1/2}) + 5\ d/dx\ x^{-1}$

$= 24.1 - 1/2\ (x)^{1/2-1} + 5(-1)\ x^{-1-1}$

$= 24 - ½(x)^{-1/2} – 5/x^2$

The Product Rule

A function can be written as the product of two other functions. For example, the function $F(x) = x^3.x^4$ can be viewed as the product of the two functions $f(x) = x^3$ and $g(x) = x^4$ yielding F(x)= f(x).g(x): Is the derivative of the product of the derivatives of its factors, f(x) and g(x) ? The answer is no. To see this, note that the product of and is and the derivative of this product is 7x. However, the derivatives of the two functions are$3x^2$ and $4x^3$ and the product of these derivatives is $12x^5$ This example shows that, in general, the derivative of a product is not the product of the derivatives. The following is a rule for finding the derivative of a product.

THEOREM 4 The Sum–Difference Rule

Sum. The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Difference. The derivative of a difference is the difference of the derivatives:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Let's check the Product Rule for

There are five steps:

1. Write down the first factor.

2. Multiply it by the derivative of the second factor.

3. Write down the second factor.

4. Multiply it by the derivative of the first factor.

5. Add the result of steps (1) and (2) to the result of steps (3) and (4)

Example 5: Find derivative of $(x^4 -2x^3-7)$ $(3x^2-5x)$.

Solution: $f(x)= (x^4 -2x^3-7)$, $g(x)= (3x^2-5x)$

$f'(x)= 4x^3 -6x^2$ and $g(x)= 6x-5$.

$Fx.gx= f(x).g'(x)+ g(x).f'(x)$

$= (x^4 -2x^3-7).6x-5 + (3x^2-5x)(4x^3 -6x^2)$

Example 6: $F(x)= (x^2 +4x-11)$ $(7x^3 – x^{1/2})$, find $F'(x)$. Do not simplify.

Solution: $F(x)= (x^2 +4x-11)(7x^3 -x^{1/2})$

$F'(x)= (x^2 +4x-11)$ $(21x^2 -1/2x^{-1/2})+$ $(7x^3 – x^{1/2})$ $(2x+4)$

The Quotient Rule

The derivative of a quotient is not the quotient of the derivatives. To see why, consider x^3 and x^2. The quotient is x^5/x^2 and the derivative of this quotient is $3x^2$. The individual derivatives are $5x^4$ and 2x the quotient of these derivatives, $5x4/2x$, is $(5/2)x^3$, which is not $3x^3$.

The rule for differentiating quotients is as follows.

THEOREM 6 The Quotient Rule

If $Q(x) = \frac{N(x)}{D(x)}$, then $Q'(x) = \frac{D(x) \cdot N'(x) - N(x) \cdot D'(x)}{[D(x)]^2}$.

The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(If we think of the function in the numerator as the first function and the function in the denominator as the second function, then we can reword the Quotient Rule as "the derivative of a quotient is the second function times the derivative of the first function minus the first function times the derivative of the second function, all divided by the square of the second function.")

So,

$$\frac{d}{dx}\left[\frac{N(x)}{D(x)}\right] = \frac{D(x) \cdot N'(x) - N(x) \cdot D'(x)}{[D(x)]^2}$$

There are six steps:

1. Write the denominator.

2. Multiply the denominator by the derivative of the numerator.

3. Write a minus sign.

4. Write the numerator.

5. Multiply it by the derivative of the denominator.

6. Divide by the square of the denominator.

Example 7: For $Q(x)= x^5/x^2$, Find $Q'(x)$.

Solution: $Q'(x)= (x^2.5x^4 - x^5 \cdot 2x)/(x^2)^2$

$=(5x^6 -2x^6)/x^4$

$= 3x6/x^4$

$= 3x^2$

Example 8: Differentiate $F(x)= (1+x^2)/x^3$

Solution: $f'(x)= (x^3 \cdot 2x -(1+x^2).3x^2)/(x^3)^2$

$= (2x^4-3x^2-3x^4)/x^6$

$= (-x^4 -3x^2)/x^6$

$= (-x^2 -3)/x^4$

8.5 Implicit Differentiation

The functions you have worked with so far have all been given by equations of the form y = f(x) in which the dependent variable y on the left is given explicitly by an expression on the right involving the independent variable x. A function in this form is said to be in explicit form. For example, the functions are all functions in explicit form.

$Y= x^2 +3x+1$, $y= (x^3 +1)/(2x-3)$ and $y= \sqrt[1/2]{1 - x^{\wedge}2}$

Sometimes practical problems will lead to equations in which the function y is not written explicitly in terms of the independent variable x; for example, equations such as

$x^2y^3 -6 =5y^3 +x$ and $x^2y +2y^3= 3x+2y$

Since it has not been solved for y, such an equation is said to define y implicitly as a function of x and the function y is said to be in implicit form.

Differentiation of Functions in Implicit Form

Suppose you have an equation that defines y implicitly as a function of x and you want to find the derivative. For instance, you may be interested in the slope of a line that is tangent to the graph of the equation at a particular point. One approach might be to solve the equation for y explicitly and then differentiate using the techniques you already know. Unfortunately, it is not always possible to find y explicitly. For example, there is no obvious way to solve for y in the equation $x^2 y +2y^3 = 3x +2y$

Moreover, even when you can solve for y explicitly, the resulting formula is often complicated and unpleasant to differentiate. For example, the equation $x^2 y^3 -6 =x+5y^3$ can be solved for y to give

$x^2 y^3 -5y^3 =x+6$

$y^3 (x^2-5)= x+6$

$y= \frac{x+6}{x^2-5}{}^{\wedge}1/3$

The computation of $\frac{dy}{dx}$for this function in explicit form would be tedious, involving both the chain rule and the quotient rule. Fortunately, there is a simple technique based on the chain rule that you can use to find $\frac{dy}{dx}$without first solving for y explicitly. This technique, known as implicit differentiation, consists of differentiating both sides of the given (defining) equation with respect to x and then solving algebraically for $\frac{dy}{dx}$Here is an example illustrating the technique.

Example 9: Find if $x^2y + y^2 = x^3$

Solution: $\frac{d}{dx}[x^2 f(x)+ (f(x) +(f(x))^2] = \frac{d}{dx}(x^3)$

$=[x^2\frac{df}{dx}+f(x)d/dx(x^2)] +2 f(x)\frac{df}{dx} = 3x^2$

Thus, we have

$X^2\frac{df}{dx}+f(x)(2x)+2f(x)df/dx=3x^2$

$X^2\frac{df}{dx} +2 f(x)df/dx= 3x^2 -2x f(x)$

$[x^2 +2 f(x)]df/dx= 3x^2 -2x f(x)$

$df/dx= \frac{3x^2-2x\,f(x)}{x^2+2y}$

Implicit Differentiation
■ Suppose an equation defines y implicitly as a differentiable function of x.
To find $\frac{dy}{dx}$:
1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x and use the chain rule when differentiating terms containing y.
2. Solve the differentiated equation algebraically for $\frac{dy}{dx}$in terms of x and y.

8.6 Logarithmic Differentiation

The idea of a logarithm arose as a device for simplifying computations. For instance, since log ab = log a+log b (log means log10), one could find the product ab of two numbers a and b by looking up their logarithms in a table, adding those logarithms, and then looking up the antilogarithm of the result using the table again (this last step amounting to raising 10 to the power log a + log b). Thus, the product (considered difficult) is replaced by a sum (considered easy). Similarly, logarithms replace quotients by differences and powers by products.

Given an equation y = y(x) expressing y explicitly as a function of x, the derivative y 0 is found using logarithmic differentiation as follows:

- Apply the natural logarithm ln to both sides of the equation and use laws of logarithms to simplify the right-hand side.
- Find y′ using implicit differentiation.
- Replace y with y(x).

Example 10: Differentiate loge $(x^2 + 3x + 1)$

We solve this by using the chain rule and our knowledge of the derivative of $\log_e x$.

$\frac{d}{dx}$loge $(x^2 + 3x + 1) = \frac{d}{dx}$ (loge u) (where $u = x^2 + 3x + 1$)

$= \frac{d}{du}$ (loge u) × $\frac{dy}{dx}$ (by the chain rule)

$= 1/u \times \frac{dy}{dx}$

$= 1/x^2 + 3x + 1 \times \frac{d}{dx}(x^2 + 3x + 1)$

$= 1/(x^2 + 3x + 1) \times (2x + 3)$

$=(2x + 3)/x^2 + 3x + 1$

Example 11: Differentiate $\ln(2x^3 + 5x^2 - 3)$.

Solution: $\frac{d}{dx}\ln(2x^3 + 5x^2 - 3) = \frac{d}{dx}\ln u$ (where $u = (2x^3 + 5x^2 - 3)$

$= d/du \ln u \times du/dx$ (by the chain rule)

$= 1/u \times du/dx$

$=1/(2x^3 + 5x^2 - 3) \times d/dx(2x^3 + 5x^2 - 3)$

8.7 Application of Differentiation

In economics the differential calculus has had many prolific applications. It is convenient at this stage to list some of the functional relationships which recur most frequently in the work of the economists: In a market the quantity demanded, q, depends on several factors, such as the price of the good (p), average consumer income (m), the price of a complement (pc), the price of a substitute (ps) and population (n). This relationship can be expressed as the demand function.

$q = f(p, m, pc, ps, n)$

In introductory economics courses, price elasticity of demand is usually defined as

e = percentage change in quantity demanded /percentage change in price

This definition implicitly assumes ceteris paribus, even though there may be no mention of other factors that influence demand. The same implicit assumption is made in the more precise measure of point elasticity of demand with respect to price:

$E= \frac{p}{q} * \frac{1}{\frac{dq}{dx}} = p/q\left(\frac{dq}{dx}\right)$

Example 12: For the demand function

$q = 35 - 0.4p + 0.15m - 0.25pc + 0.12ps + 0.003n$

where the terms are as defined at the start of this section, what is price elasticity of demand when p = 24?

Solution: We know the value of p and we can easily derive the partial derivative $\partial q/\partial p = -0.4$. Substituting these values into the elasticity formula

$E = p/q\left(\frac{dq}{dx}\right)$

$= (24)(-0.4)/(35-0.4(24)+0.15m-0.25pc+0.12ps+0.003n$

The actual value of elasticity cannot be calculated until specific values for m, pc, ps and n are given. Thus, this example shows that the value of point elasticity of demand with respect to price will depend on the values of other factors that affect demand and which consequently determine the position on the demand schedule.

Example 13: Economics: Demand for DVD Rentals. Klix Video has found that demand for rentals of its DVDs is given by

$Q= d(x)=120-20x$

where q is the number of DVDs rented per day at x dollars per rental. Find each of the following.

a) The quantity demanded when the price is rupees 2 per rental

b) The elasticity as a function of x

c) The elasticity at x=2 and x=4. Interpret the meaning of these values of the elasticity.

d) The value of x for which E(x)=1. Interpret the meaning of this price.

e) The total-revenue function, R(X)= x.D(x)

f) The price x at which total revenue is a maximum.

Solution:

a) For x=2, we have D(2)= 120-20(2)=80. Thus, 80 DVDs per day will be rented at a price of rupees 2 per rental.

b) To find the elasticity, we first find the derivative D'(x):

D'(x)=-20

Then we substitute -20 for D'(x) and 120-2x for D(x) in the expression for elasticity:

$E(x)=-\frac{x*D\prime(X)}{D(x)}= -\frac{x*(-20)}{120-20x}=\frac{20x}{120-20x}=\frac{x}{6-x}$

c) $E(2)=\frac{2}{6-2}=1/2$

At the elasticity is which is less than 1. Thus, the ratio of the percent change in quantity to the percent change in price is less than 1. A small percentage increase in price will cause an even smaller percentage decrease in the quantity sold.

E(4)= 4/6-4=2

At x=4, the elasticity is 2, which is greater than 1. Thus, the ratio of the percent change in quantity to the percent change in price is greater than 1. A small percentage increase in price will cause a larger percentage decrease in the quantity sold.

d) We set E(x)=1 and solve for p

x/6-x =1

x=6-x

x=3

thus, when the price is rupees 3 per rental, the ratio of the percent change in quantity to the percent change in price is 1.

e) Total revenue R(x) is given by x.D(x). Then,

R(x)= x* D(x)= x(120-20x)=120x-20x^2

f) To find the price x that maximizes total revenue, we find R'(x):

R'(x)= 120-40x

R'(x) exists for all x in the interval [0,∞). Thus, we solve:

R'(x)= 120-40x=0

-40x=-120

X=3

Since there is only one critical value, we can try to use the second derivative to see iff we have a maximum.

R''(x)=-40<0.

Thus, R''(3) is negative, so R(3) is a maximum. That is, total revenue is a maximum at rupees 3 per rental.

THEOREM 15

Total revenue is increasing at those x-values for which $E(x) < 1$.

Total revenue is decreasing at those x-values for which $E(x) > 1$.

Total revenue is maximized at the value(s) of x for which $E(x) = 1$.

P

le 13: The population density x miles from the center of a city is given by a function of the form Q(x) = Ae^{-kx}. Find this function if it is known that the population density at the center of the city is 15,000 people per square mile and the density 10 miles from the center is 9,000 people per square mile.

Solution: For simplicity, express the density in units of 1,000 people per square mile. The fact that Q(0) =15 tells you that A =15. The fact that Q(10) = 9 means that

$9= 15e^{-10k}$ or $3/5=e^{-10k}$

Taking the logarithm of each side of this equation, you get

Ln 3/5 — 10k or k= (-ln3/5)/10≈0.051

Hence the exponential function for the population density is $Q(x) =15e^{-0.051x}$

Summary

- For any function y = f(x)
  - $\lim_{h\to 0} f(a+h) - f(a)/h$ (if it exists) is called the derivative off at x, denoted by f′(x).
  - The function f′ is the derived function. The derivative f′(x) is the slope of the tangent to the curve y = f(x) at the point (x, y). The derivative also gives the rate of change of the function with respect to the independent variable.
- The derivative of a constant function is 0.
- The function y = [x} is derivable at every point except at x = 0.
- Every derivable function is continuous. The converse is not true, that is, there exist functions which are continuous but not differentiable.
- (Cf)′= cf′, c is a constant
- (f+g)′= f′-g′
- (fg)′= fg′+gf′
- $(f/g)' = (g'f-f'g)/g^2$
- Differ enation is used in economics to determine elasticity

Keywords

- Differential coefficient: Differentiation
- Growth: Increment
- Function: Work
- Infinite: Which does not have an end
- Elasticity: Resilience as like spring

Self Assessment

1. If a function's FIRST derivative is negative at a certain point, what does that tell you?

A. The function is increasing at that point
B. The function is decreasing at that point
C. The concavity of the function is up at that point
D. The concavity of the function is down at that point

2. In which of the following differ enation can be used for estimating rate of change

A. Manufacturing
B. Education

C. Advertising result

D. All of the above

3. Given $y=x^3\ln x$, dy/dx is

A. $3x^2\ln x$

B. $3x^2\ln x+x3.1/x$

C. x^2

D. 3x

1. Find the derivative of ex^2.

A. ex^2

B. 2x

C. $2ex^2$

D. $2xex^2$

5. Implicit functions are those functions ____________

A. Which can be solved for a single variable

B. Which cannot be solved for a single variable

C. Which can be eliminated to give zero

D. Which are rational in nature

6. How the elasticity of demand will be differentiated?

A. dq/dp.p/q

B. dp/dq. q/p

C. dq/dp. q/p

D. None of the above.

7. Consider price elasticity of demand inelastic, what happen to revenue if price increases

A. Will revenue increase

B. Will revenue decrease

C. Will revenue be constant

D. Will revenue be 0

8. If the first derivative is less than 0 on first interval (a,c) and first derivative is greater than 0 on second interval (c,b) what will be extreme?

A. Relative maxima

B. Relative minima

C. Relative maxima and minima(extreme)

D. Neither extreme nor minima

9. If f’’x>0, while f’x<0, the what will be shape of curve?

A. Concave up, decreasing

B. Concave down, increasing

C. Concave up, increasing

D. Concave down, decreasing

10. If $f'(c)=0$, at c1,c2,c3,c4. what does it show?

A. Tangent line is horizontal to all the points
B. Tangent line is vertical to these points
C. Tangent line will not exist
D. Tangent line will not be maximized

11. The definition of the first derivative of a function f(x) is

A. $f'x= f(x+\Delta x)+f(x)/\Delta x$
B. $f'x= f(x+\Delta x)-f(x)/\Delta x$
C. $f'x= \lim \Delta x \rightarrow 0 :f(x+\Delta x)+f(x)/\Delta x$
D. $f'x= \lim \Delta x \rightarrow 0 :f(x+\Delta x)-f(x)/\Delta x$

12. Find the differentiation of $x^3 + y^3 - 3xy + y^2 = 0$?

A. $(x^2-y)/(x-y^2-2y)$
B. $(3x^2-3y)/(3x-3y^2-2y)$
C. $(3x^3-3y)/(3x-3y^2-2y)$
D. $(3x^2-y)/(3x-3y^2-y)$

13. Find the differentiation of $x^4 + y^4 = 0$.

A. x^3/y^4
B. x^4/y^3
C. x^3/y^3
D. x^3/y^3

14. Implicit functions are those functions ____________

A. Which can be solved for a single variable
B. Which cannot be solved for a single variable
C. Which can be eliminated to give zero
D. Which are rational in nature.

15. Which one out of the following is an example of an implicit function?

A. Complementary goods
B. Substitute goods
C. Both a and b
D. None of the above

Answers for Self assessment

1.	B	2.	D	3.	B	4.	D	5.	B
6.	A	7.	A	8.	B	9.	A	10.	A
11.	D	12.	B	13.	C	14.	B	15.	C

Review Questions

1. Find the value of
   a. $^{d}/_{dx}(-6x^2)$
   b. $^{d}/_{dx}$ $(5x^6+2x)$
   c. Logxy $=x^2+y^2$ wrt x.
   d. $Ax^2+2hxy+by^2=0$
2. Find the derivative:
   a. $F(x)= x^2-4x+5$
   b. $F(x)= 1/3x^3-3x-4$
   c. $F(x)= 3x^5-5x^3$
3. Suppose the demand q and price p for a certain commodity are related by the linearequation q =240 - 2p (for 0< p<120).
   a. Express the elasticity of demand as a function of p.
   b. Calculate the elasticity of demand when the price is p =100. Interpret your answer.
   c. Calculate the elasticity of demand when the price is p= 50. Interpret your answer.
   d. At what price is the elasticity of demand equal to =1? What is the economic significance of this price?
4. Given the total cost function C = 5q +q^2/50 and the demand function q = 400 -20p.
   (a) Find the total revenue function
   (b) Maximise the total revenue function
   (c) Maximise profit function
5. Given find each of the following.
   (x)=62x^2+27500 and
   R(x)=x^3-12x^2+40x+10,
   Find each of the following:
   a) Total profit, P(X).
   b) Total cost, revenue, and profit from the production and sale of 50 units of the product
   c) The marginal cost, revenue, and profit when 50 units are produced and sold.
6. a. What is the slope of the function y = 6x4 when x = 2?
   b. What is the slope of the function $y = 0.2x^4$, when x = 3?

Further Readings

- Mathematics for Economics-Council for economic education
- Essential Mathematics for Economists- Nutt Sedester, peter Hawmond, Prentice Hall Publication
- Mathematics for Economists- Carl P Simone, Lawrence Bloom.
- Mathematics for Economist- Simone and Bloom, Viva Publication.

Web Links

http://ebooks.lpude.in/arts/ma_economics/year_1/DECO403_MATHEMATICS_FOR_ECONOMISTS_ENGLISH.pdf

https://ocw.mit.edu/ans7870/resources/Strang/Edited/Calculus/Calculus.pdf

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Unit 09: Differential Calculus: functions of two or more Variable

CONTENTS

Objectives

After studying this unit, you will be able to,

- Calculate Partial differentiation
- Calculate maxima and minima
- Use of differentiation in economics

Introduction

In last chapter, we discussed how here is a function f, the variables move a little bit, and f moves. The question is how much f moves and how fast. Previous chapter has shown f(x), a function of one variable. Now we have f(x, y) or f(x, y, z)-with two or three or more variables that move independently. As x and y change, f changes. The fundamental problem of differential calculus is to connect Δx and Δy to Δf.

Calculus solves that problem in the limit. It connects dx and dy to df. In using this language, I am building on the work already done. You know that df/dx is the limit of Δf/Δx. Calculus computes the rate of change-which is the slope of the tangent line. The goal is to extend those ideas to

fix, y) $=x^2 - y^2$ or f(x, y) $=\sqrt{x^\wedge 2 + y^\wedge 2}$ or f(x, y, z)=2x +3y +4z

These functions have graphs, they have derivatives, and they must have tangent.

The heart of this chapter is summarized in six lines. The subject is differential calculus-small changes in a short time. Still to come is integral calculus-adding up those small changes. We give the words and symbols for f(x, y), matched with the words and symbols for f(x).

Curve y =f(x) vs. Surface z =f(x, y)

df becomes two partial derivatives $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$

The graph of z =f(x, y) is a surface in xyz space. There are three variables-x and y are independent, z is dependent. Above (x, y) in the base plane is the point (x, y, z) on the surface. Since the printed page remains two-dimensional, we shade or color or project the surface. The eyes are extremely good at converting two dimensional images into three-dimensional understanding--they get a lot of practice. The mathematical part of our brain also has something new to work on-two partial derivatives.

9.1 Partial Derivative Defined

Partial derivative is referred to the derivative of a multivariate function when only one of the independent variables is allowed to change, other variables remaining constant.

Partial differentiation is a technique for deriving the rate of change of a function with respect to increases in one independent variable when all other independent variables in the function are held constant. Therefore, if the production function Q = f(K, L) is differentiated with respect to L, with K held constant, we get the rate of change of total product with respect to L, in other words MPL .

The basic rule for partial differentiation is that all independent variables, other than the one that the function is being differentiated with respect to, are treated as constants.

Partial Derivatives ■ Suppose $z = f(x, y)$. The partial derivative of f with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad f_x(x, y)$$

and is the function obtained by differentiating f with respect to x, treating y as a constant. The partial derivative of f with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad f_y(x, y)$$

and is the function obtained by differentiating f with respect to y, treating x as a constant.

Example 1: If $y = 14x + 3z^2$, find the partial derivatives of this function with respect to x and z.

Solution: The partial derivative of function y with respect to x is

$$\frac{\delta y}{\delta x}=14$$

The $3z^2$ disappears as it is treated as a constant. One then just differentiates the term 14x with respect to x

Similarly, the partial derivative of y with respect to z is

$$\frac{\delta y}{\delta z}=6z$$

(The 14x is treated as a constant and disappears. One then just differentiates the term $3z^2$ with respect to z.)

Example 2: Find the partial derivatives of the function $y = 6x^2z$.

Solution: In this function the variable held constant does not disappear as it is multiplied by the other variable. Therefore, treating z as a constant

$\partial y / \partial x = 12xz$

And, treating x (and therefore x^2) as a constant

$\partial y / \partial z = 6x^2$

Example 3: Find the partial derivatives fx and fy if $f(x, y) = xe^{-2xy}$

Solution: From the product rule,

$f_x(x, y) = x(-2ye^{-2xy}) + e^{-2xy} = (-2xy +1)e^{-2xy}$

and from the constant multiple rule,

$f_y(x, y) = x(-2xe^{-2xy}) = 2x^2e^{-2xy}$

9.2 Geometric Interpretation of Partial Derivatives

Functions of two variables can be represented graphically as surfaces drawn on three-dimensional coordinate systems. In particular, if z= f(x, y), an ordered pair (x, y) in the domain of f can be identified with a point in the xy plane and the corresponding function value z=f(x, y) can be thought of as assigning a "height" to this point. The graph of f is the surface consisting of all points (x, y, z) in three-dimensional space whose height z is equal to f(x, y).

The partial derivatives of a function of two variables can be interpreted geometrically as follows. For each fixed number y0, the points (x, y_0, z) form a vertical plane whose equation is $y=y_0$. If z= f(x, y) and if y is kept fixed at $y = y_0$, then the corresponding points $(x, y_0, f(x, y_0))$ form a curve in a three-dimensional space that is the intersection of the surface z = f(x, y) with the plane $y = y_0$. At each point on this curve, the partial derivative is simply the slope of the line in the plane $y = y_0$ that is tangent to the curve at the point in question. That is, is the slope of the tangent line "in the x direction."As shown in fig.1

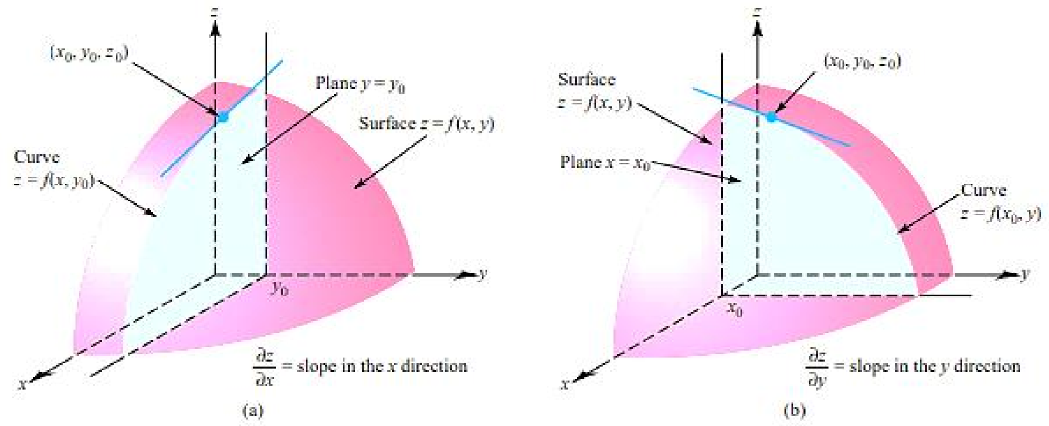


Fig.1: Geometric interpretation of partial derivatives.

Similarly, if x is kept fixed at $x = x_0$, the corresponding points $(x_0, y, f(x_0, y))$ form a curve that is the intersection of the surface z = f(x, y) with the vertical plane $x = x_0$. At each point on this curve, the partial derivative is the slope of the tangent line in the plane $x = x_0$. That is, is the slope of the tangent line "in the y direction." As shown in fig.1

9.3 Euler's theorem

A function is said to be homogeneous if multiplication of all its arguments by some arbitrary constant, say k, multiplies the function by k raised to some power.

Thus, the function $f(x_1, x_2)$ is said to be homogeneous of degree n if

$f(kx_1, kx_2) = k^n f(x_1, x_2)$.

The power of k is called the degree of homogeneity. For a homogeneous function, the sum of indices for each term of the function is the same. Thus, the function f(x1, x2) = $x_1^2 + x_1x_2 + x_2^2$ is a homogeneous function of degree two. This is because $f(kx_1, kx_2) = k^2 f(x_1, x_2)$.

Similarly, $z = x^3y + x^2y^2 + y^4$ is a homogeneous function of degree four.

Property-1: The homogeneous function z = f(x, y) of degree n can be written

as $z = x^n y\, \varphi\left(y/x\right)$ or, $z = y^n \psi\left(x/y\right)$

Property-2: The first order partial derivative for the homogeneous function z = f(x, y) of degree n are homogeneous function of degree (n-1).

Property-3: Let z = f(x, y) be a homogeneous function of degree n. Then we have

$Xf_x + yf_y = nf(xy,)$

Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted. If every means of production is credited equal to its marginal productivity and total production is liquidated completely. In mathematical formula Euler's Theorem can be indicated. If production, P = f (L, K) is Linear Homogeneous Function:

Example 3: For the production function $Q = 20K^{0.5} L^{0.5}$

(i) derive an expression for MPL, and

(ii) show that MPL decreases as one moves along an isoquant by using more L.

Solution 3: (i) MPL is found by partially differentiating the production function $Q = 20K^{0.5} L^{0.5}$ with respect to L. Thus

$MPL = \partial Q/\partial L = 10k^{0.5}L^{-0.5}$

MPL function will continuously slope downward,

ii) If the function for MPL above is multiplied top and bottom by $2L^{0.5}$, then we get

$MPL = (2L^{0.5}/ L^{0.5}) *(10K^{0.5}/L^{0.5})=Q/2L$

An isoquant joins combinations of K and L that yield the same output level. Thus if Q is held constant and L is increased then the function (1) shows us that MPL will decrease. (Note that moving along an isoquant entails using more L and less K to keep.

Example 4: A firm faces the production function

$Q = 12K^{0.4} L^{0.4}$

and can buy the inputs K and L at prices per unit of £40 and £5 respectively. If it has a budget of £800 what combination of K and L should it use in order to produce the maximum possible output?

Solution: The problem is to maximize the function $Q = 12K^{0.4} L^{0.4}$ subject to the budget constraint

$40K + 5L = 800$

The theory of the firm tells us that a firm is optimally allocating a fixed budget if the last 1 spent on each input adds the same amount to output, i.e. marginal product over price should be equal for all inputs. This optimization condition can be written as

$$\frac{MPk}{Pk} = \frac{MPL}{Pl}$$

The marginal products can be determined by partial differentiation:

$MPK = = -\partial Q/\partial K = 4.8K^{-0.6} L^{0.4}$ . . . (3)

$MPL = = -\partial Q/\partial L = 4.8K\text{-}0.4 L^{0.6}$ . . . (4)

Substituting (3) and (4) and the given prices for PK and PL into (2)

$$\frac{4.8K^{\wedge}-0.6L^{\wedge}0.4}{\substack{40\\40}} = \frac{4.8K^{\wedge}0.4L^{\wedge}-0.6}{5}$$

Dividing both sides by 4.8 and multiplying by 40 gives

$K^{-0.6} L^{0.4} = 8K^{0.4}L^{-0.6}$

Multiplying both sides by $K^{0.6}L^{0.6}$ gives

L = 8K

Substituting (5) for L into the budget constraint (1) gives

40K + 5(8K) = 800

40K + 40K = 800

80K = 800

Thus, the optimal value of K is

K = 10

and, from (5), the optimal value of L is

L = 80.

Note that although this method allows us to derive optimum values of K and L that satisfy condition (2) above, it does not provide a check on whether this is a unique solution, i.e., there is no second-order condition check. However, it may be assumed that in all the problems in this section the objective function is maximized (or minimized depending on the question) when the basic economic rules for an optimum are satisfied.

9.4 Maxima and Minima: Stationary points

Intuitively, we regard a function f(x) as increasing where the graph off is rising, and decreasing where the graph is falling. For instance, the graph in Figure 3.1 shows the military spending by the former Soviet bloc countries during the crucial period 1990–1995 following the dissolution of the Soviet Union as a percentage of GDP (gross domestic product). The shape of the graph suggests that the spending decreased dramatically from 1990 until 1992, before increasing slightly from 1992 to 1994, after which it decreased still more.

Fig. 2: Military expenditure of former Soviet bloc countries as a percentage of GDP.

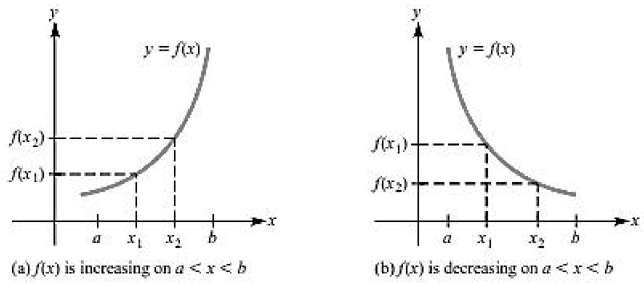


Fig 3: Intervals of increase and decrease

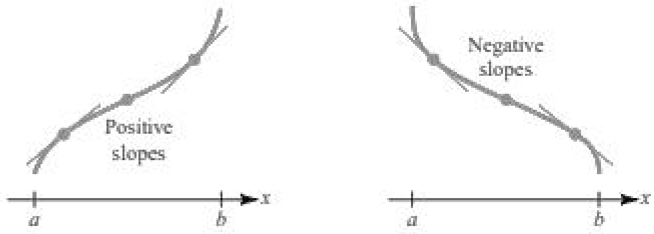
Increasing and Decreasing Functions

■ Let f(x) be a function defined on the interval a< x < b, and let x_1 and x_2 be two numbers in the interval. Then

f(x) is increasing on the interval if $f(x_2)>f(x_1)$ whenever $x_2> x_1$.

f(x) is decreasing on the interval if $f(x_2) >f(x_1)$ whenever $x_2> x_1$

As demonstrated in Figure 4a, if the graph of a function f(x) has tangent lines with only positive slopes on the interval a <x < b, then the graph will be rising and f(x) will be increasing on the interval. Since the slope of each such tangent line is given by the derivative f′(x), it follows that f(x) is increasing (graph rising) on intervals where f′(x) >0. Similarly, f(x) is decreasing (graph falling) on intervals where f′(x) < 0 (Figure 4b).



(a) $f'(x) > 0$ on $a < x < b$, so $f(x)$ is increasing.

(b) $f'(x) < 0$ on $a < x < b$, so $f(x)$ is decreasing.

Fig. 4: Derivative criteria for increasing and decreasing functions.

Procedure for Using the Derivative to Determine Intervals of Increase and Decrease for a Function f. Step 1. Find all values of x for which f′(x)=0 or f′(x) is not continuous, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number c from each interval a <x < b determined in step 1 and evaluate f′(c). Then,

If f′(c)>0, the function f(x) is increasing (graph rising) on a <x < b.

If f′(c) < 0, the function f(x) is decreasing (graph falling) on a <x < b.

Example 5: Find the intervals of increase and decrease for the function

$f(x) =2x^3 +3x^2-12x-7$

Solution: The derivative of f(x) is

$f'(x)=6x^2 + 6x -12 =6(x+ 2)(x - 1)$

which is continuous everywhere, with f ′(x) =0 where x =1 and x =-2. The numbers -2 and 1 divide the x axis into three open intervals; namely, x <-2, -2 <x < 1, and x >1. Choose a test number c from each of these intervals; say, c=-3 from x <-2, c = 0 from -2 < x < 1, and c = 2 from x > 1. Then evaluate f ′(c) for each test number:

f′(-3) = 24>0 f′(0)=-12<0 f′(2) =24>0

We conclude that f′(x)>0 for x <-2 and for x> 1, so f(x) is increasing (graph rising) on these intervals. Similarly, f ′(x)<0 on -2 <x < 1, so f(x) is decreasing (graph falling) on this interval. These results are summarized in Table 3.1. The graph of f(x) is shown in Figure 3.4.

Interval	Test Number c	f′(c)	Conclusion	Direction of Graph
$x < -2$	-3	$f'(-3) > 0$	f is increasing	Rising
$-2 < x < 1$	0	$f'(0) < 0$	f is decreasing	Falling
$x > 1$	2	$f'(2) > 0$	f is increasing	Rising

Example 7: Find the intervals of increase and decrease for the function $f(x) = x^2/(x – 2)$

Solution: The function is defined for $x \neq 2$, and its derivative is

$F'(x) = \frac{(x-2)(2x)-x^2(1)}{(x-2)^2} = x(x-4)/(x-2)^2$

which is discontinuous at x =2 and has f′(x)=0 at x=0 and x =4. Thus, there are four intervals on which the sign of f ′(x) does not change: namely, x <0, 0 < x<2, 2<x <4, and x > 4. Choosing test numbers in these intervals (say's - 2, 1, 3, and 5, respectively), we find that

f′(-2) = 3/4>0,

f ′(3) =-3 <0

f′(3)=-3<0

f ′(5) =5/9> 0

We conclude that f(x) is increasing (graph rising) for x< 0 and for x > 4 and that it is decreasing (graph falling) for 0< x < 2 and for 2 <x< 4. These results are summarized in the arrow diagram displayed next [the dashed vertical line indicates that f(x) is not defined at x= 2].

This behavior identifies x = 2 as a vertical asymptote of the graph of f(x).

Relative Extrema ■ The graph of the function f(x) is said to have a relative maximum at x=c if f(c) ≥ f(x) for all x in an interval a < x < b containing c. Similarly, the graph has a relative minimum at x = c if f(c)≤ f(x) on such an interval. Collectively, the relative maxima and minima of f are called its relative extrema.

Critical Numbers and Critical Points ■ A number c in the domain of f(x) is called a critical number if either f′(c)=0 or f′(c) does not exist. The corresponding point (c, f(c)) on the graph of f(x) is called a critical point for f(x). Relative extrema can only occur at critical point

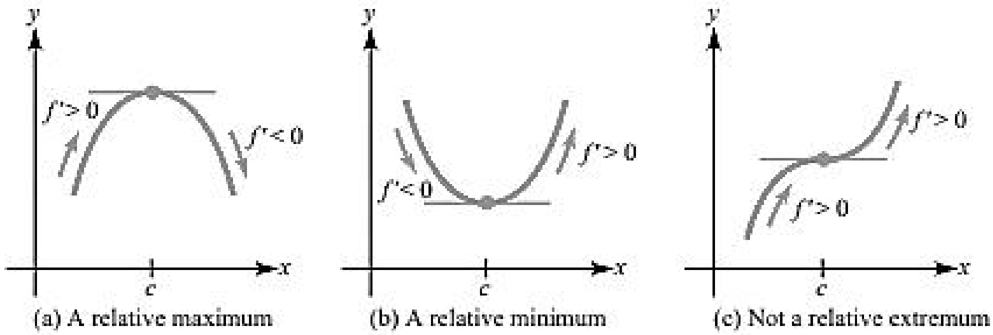


Fig. 5: Three critical points (c, f(c)) where f ′(c) = 0.

Example 8: Find all critical numbers of the function

f(x)= $2x^4 - 4x^2 + 3$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Solution: The polynomial f(x) is defined for all x, and its derivative is f′(x) = $8x^3 - 8x = 8x(x^2=1)$ = 8x(x- 1)(x+ 1)

Since the derivative exists for all x, the only critical numbers are where f′(x) = 0; that is, x =0, x = 1, and x =-1. These numbers divide the x axis into four intervals, on each of which the sign of the derivative does not change; namely, x <-1, -1 < x < 0, 0 < x < 1, and x <- 1. Choose a test number c in each of these intervals (say, -5,-1/2,1/4 and 2, respectively) and evaluate f′(c) in each case:

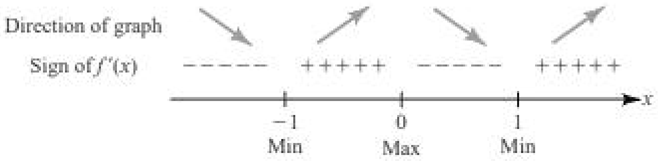
F′(-5) =-960 < 0

f(-1/2)= 3 <0

f′(1/4) =15/ 8<0

f(2) = 48 > 0

Thus, the graph of f falls for x <1 and for 0 < x <1, and rises for -1 <x<0 and for x >1, so there must be a relative maximum at x = 0 and relative minima at x =-1 and x = 1, as indicated in this arrow diagram



Once you determine the intervals of increase and decrease of a function f and find its relative extrema, you can obtain a rough sketch of the graph of the function. Here is a step-by-step description of the procedure for sketching the graph of a continuous function f(x) using the derivative f′(x).

9.5 Application of derivatives in two or more than two functions

Pure competition:

Example 9: A firm sells its output in a perfectly competitive market at a fixed price of £200 per unit. It buys the two inputs K and L at prices of rupees 42 per unit and rupees 5 per unit, respectively, and faces the production function

$q = 3.1K^{0.3}L^{0.25}$

What combination of K and L should it use to maximize profit?

Solution: $TR = pq = 200(3.1K^{0.3} L^{0.25}) = 620K^{0.3}L^{0.25}$

$TC = 42K + 5L$

Therefore the profit function the firm wishes to maximize is

$\pi = TR - TC = 620K^{0.3}L^{0.25} - 42K - 5L$

First-order conditions for a maximum require

$\partial\pi / \partial K = -186K^{-0.7}L^{0.25} - 42 = 0$

$\partial\pi / \partial L = -186K^{0.3}L^{-0.75} - 5 = 0$

Giving

$-186K^{-0.7}L^{0.25} - 42 = 0$

$186L^{0.25} = 42K^{0.7}$ and $155K^{0.3} = 5L^{0.75}$

$L^{0\cdot25}=K^{0.7}*42/186$ . . = (1)

$31K^{0.3} = L^{0.75}$ (2)

Taking (1) to the power of (3)

$L^{0.75}=(42/186)\text{^}3 *K^{2.1}$

Setting (3) equal to (2)

$42^3/186^3 *K^{2.1} = 31K^{0.3}$

$K^{1.8}= \frac{31(186)\text{^}3}{42\text{^}3} = 2692.481$

K= 80.471179 (4)

Substituting (4), the value for K, into (1)

$L^{0.25} = 42/186\ (80.471179)^{0.7} = 4.8717455$

$L = (L^{0.25})^4 = (4.8717455)^4 = 563.29822$

Therefore, first-order conditions suggest that the optimum values are (to 2 decimal places) L = 563.3 and K = 80.47

Checking second-order conditions:

$\partial^2\pi/\ \partial K^2 = -\ (0.7)\ 186K^{-1.7}L^{0.25}$

$=-130.2(80.47)^{-1.2}\,(563.3)0.25 = -0.3653576 <0$

$\partial^2\pi/\ \partial L^2 = -\ (0.75)\ 155K^{0.3}L^{-1.75}$

$=-116.25(80.47)^{-1.7}(563.3)^{0.75}$

$= -0.0066572<0$

$\partial^2\pi/\ \partial L\partial K = (0.25)186K^{-0.7}L^{-1.75}$

$(\partial^2\pi/\ \partial K^2)(\partial^2\pi/\ \partial L^2)= (-0.3653576)(-0.0066572) =0.0024323$

$(\partial^2\pi/\ \partial L\partial K)^2 = (0.0186404)^2=0.0003475$

$=46.5(80.47)^{-0.7}(563.3)^{-0.75}=0.0186404$

Therefore, as

$(\partial^2\pi/\ \partial K^2)(\partial^2\pi/\ \partial L^2)= (\partial^2\pi/\ \partial L\partial K)^2$

all second-order conditions for maximum profit are satisfied when K = 80.47 and L = 563.3

The actual profit will be

$\pi = 620K^{0.3}L^{0.25} - 42K - 5L$

$= 620(80.47)^{0.3}(563.3)^{0.25} - 42(80.47) - 5(563.3)$

$= 11{,}265.924 - 3{,}379.74 - 2{,}816.5$

$= 5{,}069.68$

9.6 Monopolist Producing Two Commodities

Example 10: A multiplant monopoly operates two plants whose total cost schedules are

$TC_1 = 36 + 0.003q^3{}_1$

$TC_2 = 45 + 0.005q^3{}_2$

If its total output is sold in a market where the demand schedule is p = 320 − 0. lq, where $q = q_1 + q_2$, how much should it produce in each plant to maximize total profits?

Solution: The total revenue is $TR = pq = (320 - 0.1q)q = 320q - 0.1q^2$

Substituting $q_1 + q_2 = q$ gives

$TR = 320(q_1 + q_2) - 0.1(q_1 + q_2)^2$

$= 320q_1 + 320q_2 - 0.1(q2_1 + 2q_1q_2 + q^2{}_2)$

$= 320q_1 + 320q_2 - 0.1q2_1 - 0.2q_1q_2 - 0.1q^2{}_2$

Thus profit will be

$\pi = TR - TC = TR - TC1 - TC2$

$= (320q_1 + 320q_2 - 0.1q^2{}_1 - 0.2q_1q_2 - 0.1q^2{}_2) - (36 + 0.003q^3{}_1) - (45 + 0.005q^3{}_2)$

$= 320q_1 + 320q_2 - 0.1q^2{}_1 - 0.2q_1q_2 - 0.1q^2{}_2 - 36 - 0.003q^3{}_1 - 45 - 0.005q^3{}_2$

First-order conditions for a maximum require

$\partial\pi\ /\partial\ q_1= 320-0.2q_1-0.2q_2-0.009q_1{}^2=0$

And $\partial\pi\ /\partial\ q_2=320-0.2q_1-0.2q_2-0.015q_2{}^2=0$

Subtracting (2) from (1)

$-0.009q_1{}^2 +0.015q_2{}^2=0$

q_2^2= (0.009/0.015)q_1^2 =0.6q_1^2

q_2=√0.6q_1^2= 0.7746q_1

Substituting (3) for q2 in (1)

320 − 0.2q_1 − 0.2(0.7746q_1) − 0.009$q^2{}_1$

= 0 320 − 0.2q_1 − 0.15492q_1 − 0.009$q^2{}_1$ = 0

0 = 0.009$q^2{}_1$ + 0.35492q_1 − 320

Using the quadratic formula to solve (4)

By disregarding the negative solution, this gives plant 1 output

Q_1 = 169.87

Substituting this value for q_1 into (3) q_2 = 0.7746(169.87216) = 131.58 (to 2 dp)

Checking second-order conditions:

$\partial^2\pi/\ \partial q_1^2$= -0.2-0.018$q_1$(169.87) =-3.25766<0

$\partial^2\pi/\ \partial q_2^2$= -0.2-0.03$q_2$(131.58) =-4.1474<0

$\partial^2\pi/\ \partial q1\partial q2$ = -0.2

Thus, using the shorthand notation for the above second-order derivatives,

$(\pi_{11})^*(\pi_{22})$ = (−3.25766)(−4.1474) = 13.51 > 0.04 = (−0.2)2 = $(\pi_{12})^2$

Therefore, all second-order conditions for a maximum value of profit are satisfied when q_1 = 169.87 and q_2 = 131.58.

9.7 Discriminating Monopolist

A monopolist sells in two markets. The inverse demand curve in market 1 is

p_1 = 200 q_1

while the inverse demand curve in market 2 is

p_2 = 300-q_2

The firm's total cost function is

c (q_1 + q_2) = (q_1 + q_2) ^{2}

The firm is able to price discriminate between the two markets.

(a) What price will it charge in each market

(b) What quantities will the monopolist sell in the two markets?

Solution:

This is a straightforward problem which entails setting marginal revenue equal to marginal cost in each market. The only complication is that the total cost function is nonlinear implying, an increasing marginal cost. This implies that we have to consider both markets at the same time since e.g. an increase in the output sold in one market increases the common marginal cost relevant to solving the optimal output in the other market. Hence solving the problem will entail solving both market outputs simultaneously; in other words, we will have to solve an equation system.

(a) Note first that the marginal cost is

MC = c′ (q_1 + q_2) = 2 (q_1 + q_2)

Next compute the revenue and the marginal revenue from each market. For market 1 we obtain R1 (q_1) = p_1q_1 = (200 q_1) q_1 = 200q_1 -q_1^2

and hence MR_1 = R'_1 (q_1) = 200-2q_1

For market 2 we obtain

n $R_2(q_2) = p_2q_2 = (300\ q_2)\ q_2 = 300q_2 - q_2^2$

and hence

$MR_2 = R'_2(q_2) = 300\text{-}2q$

2q2: The monopolist will set marginal revenue in each market equal to the (common) marginal cost. Hence, in equilibrium

$MR_1 = 200\text{-}q^*_1 = 2\ (q^*1 + q^*2) = MC$

$MR_2 = 300\text{-}\ q^*_2 = 2\ (q^*1 + q^*2) = MC$

This is an equation system with two equations and two unknown. From the First equation we obtain $200\text{-}2q_1 = 2(q^*_1 + q^*_2)$

which, solving for q^*_1 in terms of q_{*2}

yields $q^*1 = 50\text{-}q^*_2/2$

Using this to replace q^*_1

in the second equation then yields the following equation in q^*_2

$300\text{-}\ 2q^*_2 = 2\ (50\text{-}q^*_2/2+ q^*_2)$

or

$300\text{-}\ 2q^*2 = 100\text{-}q^*_2 + 2q^*_2$

or

$300=100+3q_2$

Solving for q^*_2 thus yields

$q^*2 = 200/3 \approx 66:67$

Using this equilibrium value to replace q2 in the equation for market 1 we then obtain

$q^*_1 = 50\ \text{-}q^*_2/2 = 50\text{-}\ (200/3)/2 = 50/\ 3 \approx 16:67$

Hence, the quantities sold by the monopolist will be $q^*1 = 50/\ 3$ and $q^*2 = 200/3$

b) The equilibrium prices are found simply by plugging the equilibrium quantities into the inverse demand functions. For market 1

$p^*_1 = 200\text{-}\ q^*_1 = 200\text{-}\ 50/3 = 550/3 \approx 183.33$

while for market 2

$p^*2 = 300\ \text{-}q^*_2 = 300\text{-}\ 200/\ 3 = 700/\ 3 \approx 233.33$

Summary

- Differentiation is used in economics to determine elasticity.
- Demand product shows that demand of any commodity is the product of price of that commodity. But demand of any commodity is also related to price of other related commodity. Cross demand tells that if the price of related commodity changes, in that case demand of that commodity also changes.
- Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted.
- Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means.
- Suppose $y_1 = f_1(x)$ and $y_2 = f_2(x)$ i.e y1 and y2 are the two functions of x. On differentiating both with respect to x.

  $Dy_1/dx = f'_1(x)dy2/dx= f'_2(x)$

Keywords

- Partial: Unfair, Divided
- Homogenous: Undifferentiated, similar
- Theorem: Practically which can be proved
- Elasticity: Resilience as like spring
- Profit: Profits are defined as the excess of total revenue over total costs. Symbolically it can be expressed as, P (profit) = TR – TC. i.e., (Total Revenue – Total Cost)

Self Assessment

1. Who has been credited to devise the Cobb-Douglas Production Function?

A. C W Cobb and D H Douglas

B. Cobb and Marshal

C. Douglas and Arastu

D. Above all

2. Production function is

A. P = ALβKαu

B. P = ALαKβu

C. P= LαKβ

D. P= AKβu

3. If the first derivative is less than 0 on first interval (a,c) and first derivative is greater than 0 on second interval (c,b) what will be extreme?

A. Relative maxima

B. Relative minima

C. Relative maxima and minima(extreme)

D. Neither extreme nor minima

4. When will be the curve concave up in second order derivative?

A. If first derivative =0

B. If first derivative>0

C. If second derivative=0

D. If second derivative >0

5. When will be the inflection point achieved?

A. f′′x=0

B. F′′x>0

C. F′′x<0

D. F′′x not equal to 0

6. If f′x is decreasing and concave down at 'a' point, while at another critical point it is f′x is decreasing and concave down at 'b' point, on the other-side, if f′x is increasing and concave up on 'c' then where inflection exist?

A. Inflection at (a,b) and inflection at (b,c)

B. No Inflection at (a,b) and no inflection at (b,c)

C. Inflection at (a,b) and no inflection at (b,c)

D. No inflection at (a,b) and inflection at (b,c)

7. Stationary point is a point where, function f(x,y) have?

A. $\partial f/\partial x = 0$

B. $\partial f/\partial y = 0$

C. $\partial f/\partial x = 0$ & $\partial f/\partial y = 0$

D. $\partial f/\partial x < 0$ and $\partial f/\partial y > 0$

8. Necessary condition of euler's theorem is _________

A. z should be homogeneous and of order n

B. z should not be homogeneous but of order n

C. z should be implicit

D. z should be the function of x and y only

9. At the maximum or minimum point of function the sign of dy/dx changes.

A. True

B. False

10. With the help of differentiation, level of equilibrium in monopoly is assessed.

A. True

B. False

11. To calculate elasticity, …………………….. is used in economics.

A. Differentiation

B. Differential equations

C. Integration

D. None of the above

12. If the first derivative is less than 0 on first interval (a,c) and first derivative is greater than 0 on second interval (c,b) what will be extreme?

A. Relative maxima

B. Relative minima

C. Relative maxima and minima(extreme)

D. Neither extreme nor minima

13. If f''x>0, while f'x<0, the what will be shape of curve?

A. Concave up, decreasing

B. Concave down, increasing

C. Concave up, increasing

D. Concave down, decreasing

14. Under what condition, a cost will be minimum?

A. Mc''>0

B. Mc"<0

C. Mc''=0

D. Mc=0

15. Under, perfect competition, equilibrium level be attained
A. P= AVC
B. P=MC
C. P=AC+AVC
D. None of the above

Answers for Self Assessment

1.	A	2.	B	3.	B	4.	D	5.	D
6.	D	7.	C	8.	A	9.	B	10.	A
11.	A	12.	B	13.	A	14.	A	15.	B

Review Questions

1. Derive the total differentials of the following production functions.

(a) $Q = 20K^{0.6} L^{0.4}$

(b) $Q = 48K^{0.3} L^{0.2} R^{0.4}$

(c) $Q = 6K^{0.8} + 5L^{0.7} + 0.8K^2L^2$

2. If $y = 40x^{0.4} z^{0.3}$ and $x = 5z^{0.25}$, find the total effect of a change in z on y

3. A monopolist has a cost function given by c (q) = q^2 and faces an inverse demand curve given by p (q) = 120 -q.

(a) What is his profit-maximizing output level? What price will the monopolist charge?

(b) If a lump-sum tax of rupees 100 were put on this monopolist, what would be its profit maximizing output level?

(c) If you wanted to choose a price ceiling for this monopolist so as to maximize consumer plus producer surplus, what price ceiling should you choose?

(d) How much output will the monopolist produce at this price ceiling?

(e) Suppose that you put a specific tax on the monopolist of rupees 20 per unit of output. What would its profit-maximizing level of output be?

4. A firm is a monopoly seller of good q and faces the demand schedule p = 200 − 2q, where p is the price in pounds, and the short-run production function q = 4L 0.5. If it can buy labour at a fixed wage of rupees 8, how much L should be employed to maximize profit, assuming other inputs are fixed?

5. A firm faces the non-linear demand schedule $p = (650 - 0.25q)^{1.5}$. What output should it sell to maximize total revenue?

6. In a perfectly competitive market the demand schedule is $p = 120 - 0.5q^2$ and the supply schedule is $p = 20 + 2q^2$. If the government imposes a per-unit tax t on the good sold in this market, what level of t will maximize the government's tax yield?

Further Readings

- Mathematics for Economics-Council for economic education
- Essential Mathematics for Economists- Nutt Sedester, peter Hawmond, Prentice Hall Publication
- Mathematics for Economists- Carl P Simone, Lawrence Bloom.
- Mathematics for Economist- Simone and Bloom, Viva Publication

Web Links

http://ebooks.lpude.in/arts/ma_economics/year_1/DECO403_MATHEMATICS_FOR_ECONOMISTS_ENGLISH.pdf

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Unit 10: Integration

CONTENTS

Objectives

- define the indefinite integral of a function
- evaluate certain standard integrals by finding the anti-derivatives of the integrands
- use the rules of the algebra of integrals to evaluate some integrals

Introduction

How can a known rate of inflation be used to determine future prices? What is the velocity of an object moving along a straight line with known acceleration? How can knowing the rate at which a population is changing be used to predict future population levels? In all these situations, the derivative (rate of change) of a quantity is known and the quantity itself is required. Here is the terminology we will use in connection with obtaining a function from its derivative.

Antidifferentiation ■ A function $F(x)$ is said to be an *antiderivative* of $f(x)$ if

$$F'(x) = f(x)$$

for every x in the domain of $f(x)$. The process of finding antiderivatives is called *antidifferentiation* or *indefinite integration*.

Integrating a function means finding another function which, when it is differentiated, gives the first function. It is basically differentiation in reverse and is sometimes referred to as anti-differentiation. The rules for integration are the reverse of those for differentiation. Economists like to express their ideas using graphs which show relationships between various variables, but sometimes it is important to put more precise values on analyzed functions. This is where the mathematical technique of integration comes in. It is commonly used by economists to calculate areas in graphs and to analyse probability distributions, of which you will learn more in a statistics course. The remainder of this chapter will look at some of the basic rules and methods of integration and show their applications to problems in economics and finance.

Anti-differentiation is the process of differentiation in reverse. Given a function we determine another function such that the derivative of is that is,

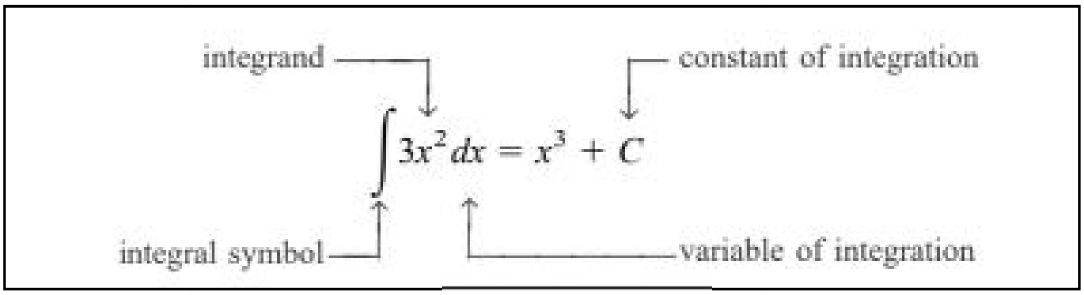
For example, let $f(x)=2x$. The function $f(x)=x^2$ is an anti-derivative $f(x)$ since $(d/dx)x^2 =2x$. However, other functions also have a derivative of $2x$. Therefore, an anti-derivative of $f(x)=2x$ is any function that can be written in the form $F(x)=x^2 +C$, where is a constant. This leads us to the following theorem.

If is an anti-derivative of a function we write

$$\int f(x)\,dx = F(x) + C$$

This equation is read as "the anti-derivative of with respect to is the set of functions." The expression on the left side is called an indefinite integral. The symbol $\int$ is the integral sign and is a command for antidifferentiation. The function is called the integrand, and the meaning of will be made clear when we develop the geometry of integration.

In the context of the indefinite integral$\int$ f(x) dx= F(x) + C, the integral symbol is $\int$, the function f(x) is called the integrand, C is the constant of integration, and dx is a differential that specifies x as the variable of integration. These features are displayed in this diagram for the indefinite integral off (x) $\int 3x^2$.



For any differentiable function F, we have since by definition,

$$\int {'f}(x)\,dx = F(x) + C$$

F(x) is an anti-derivative of F'(x). Equivalently,

$$\int \frac{dF}{dx}\,dx = F(x) + C$$

This property of indefinite integrals is especially useful in applied problems where a rate of change F'(x) is given and we wish to find F(x).

Rules for Integrating Common Functions

The **constant rule:** $\int k\,dx = kx + C$ for constant k

The **power rule:** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for all $n \neq -1$

The **logarithmic rule:** $\int \frac{1}{x}\,dx = \ln|x| + C$ for all $x \neq 0$

The **exponential rule:** $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ for constant $k \neq 0$

Example 1: Find these integrals:

a. $\int 3\,dx$

Use the constant rule with k=3: $\int 3\,dx = 3x + c$

b. $\int x^{17}\,dx$

Use the power rule with n=17: $\int x^{17}\,dx = \frac{1}{18}x^{18} + C$

c. $\int \frac{1}{\sqrt{x}}dx$

Use the power rule with n=-1/2: $\int \frac{dx}{(x)^{\frac{1}{2}}} = \frac{1}{1/2}x^{1/2} + C = 2\sqrt{x} + C$

d. $\int e^{-3x}\,dx$

Use the exponential rule with k=-2:

$$\int e^{-3x}\,dx = 1/3e^{\wedge} - 3x + C$$

Algebraic Rules for Indefinite Integration

The **constant multiple rule:** $\int kf(x)\,dx = k\int f(x)\,dx$ for constant k

The **sum rule:** $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

The **difference rule:** $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

Example 2: Find the following integrals:

a. $\int (2x^5 + 8x^3 - 3x^2 + 5)dx$

By using the power rule in conjunction with the sum and difference rules and the multiple rule, you get

$\int (2x^5 + 8x^3 - 3x^2 + 5)dx = 2\int x^5\,dx + 8\int x^3\,dk - 3\int x^2\,dx + \int 5\,dx$

$$= 2\left(\frac{x^{\sigma}}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C$$

$= 1/3x^6 + 2x^4 - x^3 + 5x + C$

b. There is no "quotient rule" for integration, but at least in this case, you can still divide the denominator into the numerator and then integrate using the method in part (a):

$$\int \left(\frac{x^3 + 2x - 7}{x}\right) dx$$

$$= \int \left(x^2 + 2 - \frac{7}{x}\right) dx$$

$$= \frac{1}{3}x^3 + 2x - 7\,ln|x| + c$$

c. $\int \left(3e^{-5t} + \sqrt{t}\right) dt$

$$= \int \left(3e^{-5t} + t^{1/2}\right) dt$$

$$= 3\left(\frac{1}{5}e^{-5t}\right) + \frac{1}{3/2}t^{3/2} + c$$

$$= -\frac{3}{5}e^{-5t} + \frac{2}{3}t^3 / 2 + c$$

10.1 Integration by Substitution

The majority of functions that occur in practical situations can be differentiated by applying rules and formulas. Integration, however, is at least as much an art as a science, and many integrals that appear deceptively simple may actually require a special technique or clever insight.

For example, we easily find that

$$\int x^7\,dx = \frac{1}{8}x^8 + C$$

by applying the power rule, but suppose we wish to compute

$$\int (3x + 5)^7\,dx$$

We could proceed by expanding the integrand $(3x+5)^7$ and then integrating term by term, but the algebra involved in this approach is daunting. Instead, we make the change of variable

$$u = 3x + 5 \quad so\ that$$

$$du = 3dx \quad \text{or}$$

$$dx = \frac{1}{3}du$$

Then, by substituting these quantities into the given integral, we get

$$\int (3x+5)^7\,dx = \int u^7 \left(\frac{1}{3}du\right)$$

$$= \frac{1}{3}\left(\frac{1}{8}u^8\right) + C$$

$$= \frac{1}{24}u^8 + C \qquad \text{power rule}$$

$$\frac{1}{-24}(3x+5)^8 + c \qquad \text{since } u=3x+5$$

We can check this computation by differentiating using the chain rule

$$\frac{d}{dx}\left[\frac{1}{24}(3x+5)^8\right] = \frac{1}{24}[8(3x+5)^7(3)] = (3x+5)^7$$

which verifies that $(3x + 5)^8$ is indeed an anti-derivative of $(3x + 5)^7$.

The change of variable procedure we have just demonstrated is called integration by substitution, and it amounts to reversing the chain rule for differentiation. To see why, consider an integral that can be written as

$$\int f(x)\,dx = \int g(u(x))u'(x)\,dx$$

Suppose G is an anti-derivative of g, so that G′= g. Then, according to the chain rule

$$\frac{d}{dx}[G(u(x))]$$

$$= G'(u(x))u'(x)$$

$= g(u(x))u'(x)$ Since G′=g

Therefore, by integrating both sides of this equation with respect to x, we find that

$$\int f(x)\,dx = \int g(u(x))u'(x)\,dx$$

$$= \int \left(\frac{d}{dx}[G(u(x))]\right)dx$$

$$= G(u(x)) + C \qquad since \int G' = G$$

In other words, once we have an anti-derivative for g(u), we also have one for f(x).

A useful device for remembering the substitution procedure is to think of u=u(x) as a change of variable whose differential du= u′(x) dx can be manipulated algebraically. Then

$$\int f(x)\,dx$$

$= \int g(u(x))u'(x)\,dx$

$= \int g(u)\,du$ substitute du for u′(x) dx

$= G(u) + c$ where G is an antiderivative of g

$= G(u(x)) + c$ substitute u(x) for u

Here is a step-by-step procedure for integrating by substitution.

Using Substitution to Integrate $\int f(x)\,dx$

Step 1. Choose a substitution $u = u(x)$ that "simplifies" the integrand $f(x)$.

Step 2. Express the entire integral in terms of u and $du = u'(x)\,dx$. This means that *all* terms involving x and dx must be transformed to terms involving u and du.

Step 3. When step 2 is complete, the given integral should have the form

$$\int f(x)\,dx = \int g(u)\,du$$

If possible, evaluate this transformed integral by finding an antiderivative $G(u)$ for $g(u)$.

Step 4. Replace u by $u(x)$ in $G(u)$ to obtain an antiderivative $G(u(x))$ for $f(x)$, so that

$$\int f(x)\,dx = G(u(x)) + C$$

Example 3: $\int \sqrt{2x} + 7\,dx$

Solution: We choose u =2x + 7 and obtain

du=dx so that dx=1/2du

Then the integral becomes

$$\int \sqrt{2x+7}\,dx = \int \sqrt{u^1/2\left(\frac{1}{2}du\right)}$$

$$= \frac{1}{2}\int u^{1/2}\,dU \text{ since } \sqrt{u} = u\text{^}1/2$$

$$= \frac{1}{2}\frac{u^3/2}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C \text{ power rule}$$

$$= \frac{1}{3}(2x+7)^{3/2} + C \text{ substitute 2x+7 for u}$$

10.2 Integration by Parts

Product Rule for differentiation:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x, we get

$$uv = \int u\frac{dv}{dx}dx + \int V\frac{du}{dx}dx$$

$$= \int u\,dv + \int v\,du$$

Solving for we get the following theorem.

Integrations by parts formula
$\int u\,dv = uv - \int v\,du$

This equation can be used as a formula for integrating in certain situations— that is, situations in which an integrand is a product of two functions, and one of the functions can be integrated using the techniques we have already developed. For example,

$$\int xe^x\,dx$$

Can be considered as

$$\left(\int xe^x\, dx\right) = \int u\, dv$$

where we let

u=x and dv=$e^x\, dx$

In this case, differentiating u gives

du=dx

and integrating dv gives

v=e^x , We select C= 0 to obtain the simplest anti-derivative.

Then the Integration-by-Parts Formula gives us,

$$\int (x)(e^x\, dx) = (x)(e^x) - \int (e^x)(dx)$$
$$= xe^x - e^x + C$$

This method of integrating is called integration by parts. As always, to check, we can simply differentiate.

Using Integration by Parts
1. If you have had no success using substitution, try integration by parts.
2. Use integration by parts when an integral is of the form $\int f(x)g(x)\, dx$
Match it with an integral of the form $\int u\, dv$ by choosing a function to be u=f(x), where f(x) can be differentiated, and the remaining factor to be, dv=g(x)dx, where g(x) can be integrated.
3. Find du by differentiating and v by integrating.
4. If the resulting integral is more complicated than the original, make some other choice for and.
5. To check your result, differentiate

Example 4: *Evaluate*: $\int x\sqrt{5x+1}\, dx$

Solution: We let

u=x and dv= $(5x+1)^{1/2}\, dx$

Then du=dx and v= $\frac{2}{15}(5x+1)^{3/2}$

Note that we have to use substitution in order to integrate dv:

$$\int (5x+1)^{1/2}\, dx$$
$$= \frac{1}{5}\int (5x+1)^{1/2} 5\, dx$$

= $\frac{1}{5}\int w^{1/2}\, dw$ Substitution (w=5x+1, dw=5dx)

$$v = \frac{1}{5}\left(\frac{w^{1/2+1}}{\frac{1}{2}+1}\right) = \frac{2}{15}w^{3/2} = \frac{2}{15}(5x+1)^{3/2}$$

Using the Integration-by-Parts Formula gives us

$$\int x\left(\sqrt{5x+1}\right) dx = x\cdot\frac{2}{15}(5x+1)^{3/2} - \int \frac{2}{15}(5x+1)^{3/2}\, dx$$
$$= \frac{2}{15}x(5x+1)^{\frac{3}{2}} - \frac{2}{15} * \frac{2}{25}(5x+1)^{5/2} + C$$

$$= \frac{-2}{15} x(5x+1)^{3/2} - \frac{4}{375}(5x+1)^{5/2} + C$$

10.3 Integration by Partial Fractions

The integrand may be in the form that it can be integrated only after resolving it into partial fractions. Here, in this section, we are going to deal with integration of such functions: First of all we discuss the process of resolving such functions into partial fractions:

Important steps for resolving into partial fractions are:

1. Check degree of numerator, if it is less than that of denominator, go to step 2 and if it is greater than or equal to that of denominator, then first divide the numerator by the denominator and then go to step 2.
2. We may have one of the following main types of functions which we will dealt as discussed below:

Type 01: Denominator involve all linear factors with exponent as unity:

$$\frac{X+5}{(X-1)(X-2)(X-3)}$$

StepLet

$$\frac{X+5}{(X-1)(X-2)(X-3)} = \frac{A}{X-1} + \frac{B}{X-2} + \frac{C}{X-3} \text{ ....(1)}$$

Step II: Equate each of the factors of denominator to zero.

i.e. $x - 1 = 0 \Rightarrow x = 1$, $x - 2 = 0 \Rightarrow x = 2$, $x - 3 = 0 \Rightarrow x = 3$

Step III Put x = 1, 2, 3 everywhere (in the given expression) but not in the factor from which it has come out,

$$\frac{A}{X-1} = A = \frac{1+5}{(1-2)(1-3)} = \frac{6}{2} = 3$$

Step IV B=(2+5)/(2-1)(2-3)= 7/-1=-7

Step V: C= (3+5)(3-1)(3-2)=8/(2*1)=4

Thus, we may write,

$$\frac{X+5}{(X-1)(X-2)(X-3)} = \frac{3}{X-1} + \frac{-7}{X-2} + \frac{4}{X-3}$$

R.H.S. is nothing but the partial fractions of the given expression. Here we note that integration of R.H.S. is directly available.

Type 02: Denominator involves all linear factors but some have 2, 3, etc. as exponents

$$\frac{x^2+x+5}{(x+5)(x+2)^3}$$

Step 1:

$$\frac{x^2+x+5}{(x+5)(x+2)^3} = \frac{A}{X+5} + \frac{B}{X+2} + \frac{C}{(X+2)^\wedge 2} + \frac{D}{(X+2)^\wedge 3} \text{..........................(1)}$$

Multiply on both sides by denominator of L.H.S. in this case by

$$(x+5)(x+2)^3 = x^2 + x + 5$$

$= A(x+2)^3 + B(x+5)(x+2)^2 + C(x+5)(x+2) + D(x+5)$

Step II Equate each of the factors to zero. i.e.

$x + 5 = 0 \Rightarrow x = -5$, $x + 2 = 0 \Rightarrow x = -2$

Step III Put x =–5 in (1) we get value of A, as given below

$(-5)^2 +(-5)+(5)= A(-5+2)^3 +B(0) +C(0) +D(0)$

$\Rightarrow 25 =-27A \Rightarrow A=-25/27$

Step IV Put x =–2 in (1) we get value of D, as given below

$(-2)^2 +(-2)+ 5= A(0)+ B(0) +C(0)+ D(-2 +5)$

$\Rightarrow 7=3D \Rightarrow D=7/3$

Step V In order to find the values of B, C we have to equate the coefficients of different powers of x on both sides of (1).

In present case equating coefficients of 3 x and constant terms, we get

$0 = A + B \dots (2)$

$5 = 8A + 20B +10C + 5D \dots (3)$

By putting value of A from Step III and value of D from step IV in equations (2) and (3), we get

0 = 27 25 – + B = 0⇒ B= 25/27

$5=8(-25/27) +20B +10C+5(7/3)$

$\Rightarrow 10C=5+200/27-20(25/27)-35/3= C=-48$

Thus, we may write,

$$\frac{x^2+x+5}{(x+5)(x+2)^3}=\frac{-25/27}{X+5}+\frac{25/27}{X+2}+\frac{-48}{(X+2)^\wedge 2}+\frac{7/3}{(X+2)^\wedge 3}$$

Type 03: Denominator involves quadratic expressions. We will not discuss the problems based on this type, because it will involve the integral formulae which are beyond our contents.

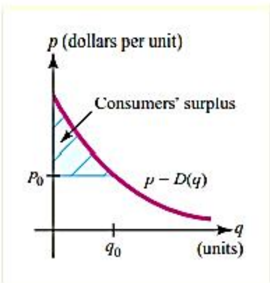
Type 4: Denominator involves higher powers of quadratic expressions. This type is also not discussed here because it will involve the integral formulae which are beyond our contents

10.4 Economic Applications of Integration

a. Consumer and Producer Surplus

Consumer Surplus: The consumer surplus is the total area under the curve minus the total expenditure. This surplus is the total utility minus the total cost and is given by

$$\int_0^0 D(x)\,dx - QP$$

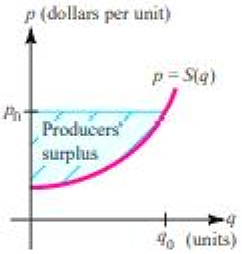


Consumers' Surplus ■ If q_0 units of a commodity are sold at a price of p_0 per unit and if $p = D(q)$ is the consumers' demand function for the commodity, then

$$\begin{bmatrix}\text{Consumers'}\\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{total amount consumers}\\ \text{are willing to spend}\\ \text{for } q_0 \text{ units}\end{bmatrix} - \begin{bmatrix}\text{actual consumer}\\ \text{expenditure}\\ \text{for } q_0 \text{ units}\end{bmatrix}$$

$$CS = \int_0^{q_0} D(q)dq - p_0 q_0$$

Producer Surplus: Producers' surplus is the other side of the coin from consumers' surplus. Recall that the supply function p = S(q) gives the price per unit that producers are willing to accept in order to supply q units to the marketplace. However, any producer who is willing to accept less than $p0=S(q_0)$ dollars for q_0 units gains from the fact that the price is p_0. Then producers' surplus is the difference between what producers would be willing to accept for supplying q_0 units and the price they actually receive. Reasoning as we did with consumers' surplus, we obtain the following formula for producers' surplus.



Producers' Surplus ■ If q_0 units of a commodity are sold at a price of p_0 dollars per unit and $p = S(q)$ is the producers' supply function for the commodity, then

$$\begin{bmatrix}\text{Producers'}\\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{actual consumer}\\ \text{expenditure}\\ \text{for } q_0 \text{ units}\end{bmatrix} - \begin{bmatrix}\text{total amount producers}\\ \text{receive when } q_0\\ \text{units are supplied}\end{bmatrix}$$

$$\text{PS} = p_0 q_0 - \int_0^{q_0} S(q)\, dq$$

Example 1: A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is

$p = D(q) = 0.1q^2 + 90$

dollars per tire, and the same number of tires will be supplied when the price is

$p = S(q) = 0.2q^2 + q + 50$

dollars per tire.

a. Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.

b. Determine the consumers' and producers' surplus at the equilibrium price.

Solution:

a. The supply and demand curves are shown in Figure. Supply equals demand when

$-0.1q^2 + 90 = 0.2q^2 + q + 50$

$0.3q^2 + q - 40 = 0$

$Q = 10$ (reject $q \approx -13.33$)

and $p = -0.1(10)^2 + 90 = 80$ dollars per tire. Thus, equilibrium occurs at a price of rupees 80 per tire, and then 10,000 tires are supplied and demanded.

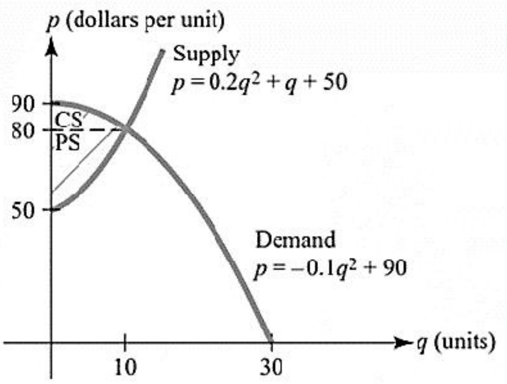
b. Using $p_0 = 80$ and $q_0 = 10$, we find that the consumers surplus is

$\text{CS} = \int_0^{10} (-0.1q^2 + 90)\, dq - 80(10)$

$= \left[-0.1\left(\frac{q^3}{3}\right) + 90q\right]\Big|_0^{10} - 80(10)$

$\approx 866.67 - 800 = 66.67$

or \$66,670 (since $q_0 = 10$ is really 10,000). The consumers' surplus is the area of the shaded region labeled CS)



The producers' surplus is

$$PS= 80(10) - \int_0^{10}(0.2q^2 + q + 50)\,dq$$

$$=80(10) - \left[0 \cdot 2\left(\frac{q^3}{3}\right) + \left(\frac{q^2}{2}\right) + 50q\right]\Bigg|_0^{10}$$

$$\approx 800\text{-}616.67\text{=}183.33$$

or \$183,330. The producers' surplus is the area of the shaded region labeled PS.

b) Investment and the Stock of Capital

Let net investment I is the rate of change of the stock of capital K. If time is treated as a continuous variable, we can express this as

$$I(t) = \frac{dk(t)}{dt}$$

Thus, if the rate of investment I(t) is known, the capital stock K(t) can be estimated through the formula,

$$k(t) = \int I(t)\,dt$$

Example 5: The rate of net investment is given by I(t)= $12t^{1/3}$ and the initial stock of capital at t = 0 is 25 units. Find the equation for the stock of capital.

Solution: $k(t) = \int 12t^{1/3}\,dt$

$$= 12\left(\frac{3}{4}\right)t^{4/3} + C$$

$=9t^{4/3} +c$

As k(0)=c=25 given,

K(t)= $9t^{4/3} +25$

c) **Consumption Over Time**

The integral is a natural tool to calculate net change and total accumulation of more quantities than just distance and velocity. Integrals can be used to calculate growth, decay, and, as in the next example, consumption. Whenever you want to find the cumulative effect of a varying rate of change, integrate it.

Example 6: From 1970 to 1980, the rate of potato consumption in a particular country was C(t)= 2.2 1.1^t millions of bushels per year, with t being years since the beginning of 1970. How many bushels were consumed from the beginning of 1972 to the end of 1973?

Solution: We seek the cumulative effect of the consumption rate for $2 \le t \le 4$.

Step 1: Riemann sum. We partition [2, 4] into subintervals of length Δt and let tk be a time in the kth subinterval. The amount consumed during this interval is approximately

CtkΔt million bushels.

The consumption for $2 \leq t \leq 4$.is approximately

ΣCtkΔt million bushels.

Step 2: Definite integral. The amount consumed from t 2 to t 4 is the limit of these sums as the norms of the partitions go to zero.

$$\int_2^4 C(t)\,dt = \int_2^4 (2\cdot 2 + 1 \square 1^t)\,dt$$

Step 3: Evaluate. Evaluating numerically, we obtain

NINT($2.2 + 1.1^t$, t, 2, 4)≈ 7.066 million bushels

d) Learning curve

From Experience, we learn that the direct labour input per unit of product steadily decline. In the direct labour requirement, the learning curve is the graphic technique, which is employed to describe the rate of decline. The general form of the learning curve is

$$f(x) = \alpha x^{\beta}, -1 \leq \beta < 0, \alpha < 0$$

Where f(x) denotes the number of hours of direct labour required to produce x units of product. The graph of the learning curve is shown in fig. below, which indicates that as x increases, f(x) decreases. Thus we can use this curve to predict the number of production hours needed for further work, once it has been obtained for a gross production process and is given as $\int_a^b f(x)\,dx$, where a, b denotes no. of units produced.

Example 6: Because of learning experience, there is a reduction in labour requirement of a firm. The number of hours of labour required to produce the xth unit as $f(x)=Ax^{\alpha}$, $A>0, -1 \leq \alpha \leq 0$. After producing 36 units, a firm has a learning curve $f(x)=200x^{-1/2}$. Find the labour hours required to produce the next 13 units.

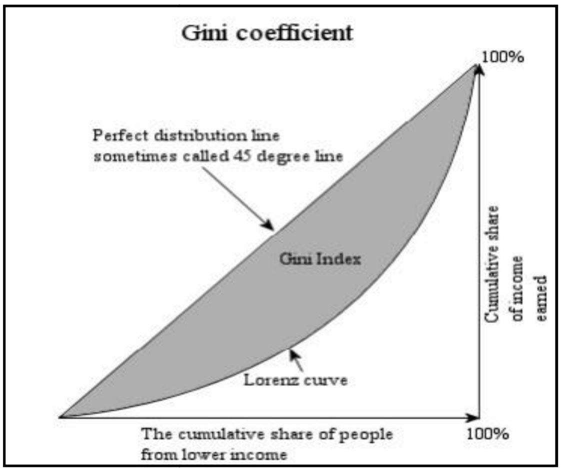
Solution: Required labour unit= $\displaystyle\int_{36}^{49} 200x^{\frac{-1}{2}}\,dx = 200\left(\frac{x^{\frac{1}{2}}}{1/2}\right)_{36}^{49}$

$$400\left(49^{1/2} - 36^{1/2}\right) = 400(7-6) = 400hrs$$

e) Ginni Coefficient or Lorenz Curve

We use Lorenz curve to find the inequality of distribution of income, wealth etc. We plot the cumulative percentage of a variable like income of individuals, against cumulative percentage of no. of individuals, against cumulative percentage of no. of individuals. When these two percentages are same everywhere, then graph will be a st. line inclined at an angle 45 degree with x-axis. In other situations, we have Lorenz curve OAR, which indicates the inequality of income distribution. We determine the degree of inequality by Ginni Coefficient:

Where p denotes shaded area covered by this curve and q denotes the other area on right of OR.



If the total area of the square is unity, then p+q is half of total area i.ep+q= ½

Therefore, G= p/(p+q)= (p+q-q)/(p+q)

= 1-q/(p+q)= 1-2q

If the cumulative % of individuals be x, we can use the Lorenz function f(x)to determine Gini coefficient G by using definite integrals.

We can also write area of square as 100*100=10000, so we have shaded area

Therefore, p= $5000 - \int_0^{1000} f(x)\, dx$

Example 7: Lovely Autos made an analysis of production which shows that the present equipment and workers, the production is 20,000 units per day. It is estimated that the rates of production P w.r.to change in the number of additional worker.

X is dp/dx=100-3√x

What is the production with 36 additional workers.

b) Equation of Lorenz curve drawn in a unit is given by f(x)= ax^2 +bx, 0≤x≤1, a, b<1. Find the Gini Coefficient.

Solution: a: Give $\frac{dP}{dx} = 100 - 3\sqrt{x}, Integrating$

P= $\int \left(100 - 3\sqrt{x}\right) dx = 1007c - 2x^{3/2} + k$

When x=0, P= 20,000, 20,000=K

Therefore, P=100-2$X^{3/2}$ +20000

When x=36, P= 100*36-2$(36)^{3/2}$ +20,000

= 3600-2*216+20,000

= 23168

b. Area under the curve between (0,1) is = $\int_0^{p}(ax^2 + bx)\, dx$

= $\left(\frac{ax^2}{3} + \frac{bx^2}{2}\right) := \frac{a}{3} + \frac{b}{2}$

As half area on one side of diagonal is ½, the area between the diagonal and the curve is p= ½-(a/3+b/2)

Therefore, Gini Coefficient = (shaded area/half area)*p =(½-a/3-b/2)/1/2

Summary

- Consumer's Surplus: This notion was introduced by Alfred Marshall to measure the net benefit that a consumer enjoys from his act of purchasing a particular commodity in the market. It is defined in terms of the excess of the consumer's total willingness to pay in units of money over his actual expenditure.
- Definite Integral: The definite integral of the function f(x) over the interval (a, b) is expressed symbolically as ∫ b a f (x)dx , read as "integral of f with respect to x from a to b. The smaller number a is termed the lower limit and b, the upper limit of integration. Geometrically, this definite integral denotes the area under the curve representing f (x) between the points x = 0 and x = b. Note that the definite integral ∫ b a f (x) is a number
- Indefinite Integral: The indefinite integral is basically reverse differentiation. To differentiate means to find the rate of change (derivative) of a given function. Indefinite integration reverses the process and finds the unknown function whose rate of change (derivative) is given.
- There are two main methods to find the integration of any function:
  i) Integration by Substitution

ii) Integration By parts

- In this operation integration is done by changing the given integrand in standard formula.
- If integrand is a form of Ø[f(x)]*f/x i.e. it is a function of any amount f(x) and product of integral coefficient f(x) of this same amount f Ø (x) or may be written in this form, then we do integration considering this amount equal to t.

Keywords

- Substitution: Replacement
- Methods: Manner, process
- Integration: It is basically differentiation in reverse and is sometimes referred to as anti-differentiation
- Consumer Surplus: The consumer surplus is the total area under the curve minus the total expenditure.
- Ginni Coefficient: inequality index

Self Assessment

1. Define Integration?

A. How can a known rate of inflation be used to determine future prices?
B. What is the velocity of an object moving along a straight line with known acceleration?
C. How can knowing the rate at which a population is changing be used to predict future population levels?
D. All of the above

2.If G′(x)= f (x), then the integration ∫f (x) dx =G(x) + C is correct, but if G(x) is anything other than f (x), will integration is correct or not?

A. True
B. False
C. None of the above
D. Undefined.

3. What is an integration of substitution?

A. Integration by substitution is the reverse of applying the Chain Rule of Differentiation.
B. Integration by substitution is the similar to the Chain Rule of Differentiation.
C. Integration by substitution is the reverse of applying the Implicit differentiation.
D. Integration by substitution is the reverse of applying the Chain Rule of Differentiation.

4. Integration of function y = f(x) from limit x1 < x < x2 , y1 < y < y2, gives ___________

A. Area of f(x) within x1 < x < x2
B. Volume of f(x) within x1 < x < x2
C. Slope of f(x) within x1 < x < x2
D. Maximum value of f(x) within x1 < x < x2

5. Integration of function is same as the ___________

A. Joining many small entities to create a large entity
B. Indefinitely small difference of a function

C. Multiplication of two function with very small change in value

D. Point where function neither have maximum value nor minimum value

6. The mean value of a function f (x) from a to b is given by

A. f(a)+f(b)/2

B. f(a)+2f(a+b)/2+fb

C. (area under the curve from a to b)/(width of the interval from a to b)

D. none of the above.

7. Calculate the integration of e^-3x

A. 1 /3 .e^-3x

B. -1 /3 .e^-3x + C

C. -3 .e^-3x + C

D. /3 .e^-3x + C

8.What will be the formula to do Integration-by-Parts?

A. √u.dv = uv - √v.du

B. √v.dv = uv - √v.dx

C. √u.dv = udu - √v.du

D. √u.dv = dxv - √v.du

9.Partial Fractions of 1/(x^2-1) are equivalent to

A. Ax + B/(x2-1)

B. A/(x+1) + (B/(x-1)

C. A/(x+1)

D. B/(x+1)

10.For a repeated factor $(x-a)^3$ in denominator which contains linear factors, how to make partial fractions.

A. A/(x-a) +B (x+a)

B. A/(x-a), B $(x+a)^2$

C. A/(x-a), B $(x+a)^2$, C/$(x-a)^3$

D. A/(x-a) -B (x+a)

11. To resolve a combined fraction into its parts is called

A. Rational fraction

B. Partial fraction

C. Combined fraction

D. None of the above

12. What is a definite integral?

A. The definite integral gives the net area between the graph of a continuous function f and the x-axis over an interval (a,b)

B. The definite integral gives the gross area between the graph of a continuous function f and the x-axis over an interval (a,b)

C. The definite integral gives the definite interval area between the graph of a continuous function f and the x-axis over an interval (a,b)

D. none of the above

13. if improper integral is existing with limit and without limit, what will be the nature of it?

A. diverge, diverge

B. diverge, converge

C. converge, diverge

D. converge, converge

14. Consumer saving can be calculated by which method

A. Differentiation

B. Integration

C. Differential

D. Cramer Rule

15. Choose the correct option of Gini Coefficient

A. GI = $\int_0^1 (x - L(x)$

B. GI = $2\int_0^1 (x - L(x)$

C. GI = $\int_0^2 (x - L(x)$

D. GI = $\int_{-1}^1 (x - L(x)$

Answers for Self Assessment

1.	D	2.	B	3.	A	4.	A	5.	A
6.	C	7.	B	8.	A	9.	B	10.	B
11.	C	12.	A	13.	C	14.	A	15.	B

Review Questions

1. Find: $\int xe^{2x}\,dx$
2. Find: $\int x\sqrt{x+5}dx$
3. A manufacturer has determined that when q thousand units of a particular commodity are produced, the price at which all the units can be sold is p = D(q) dollars per unit, where D is the demand function D(q)= 10 -q$e^{0.02q}$

   a. At what price are 5,000 (q0 = 5) units demanded?

   b. Find the consumers' surplus when 5,000 units are demanded.

4. Find the Gini index for an income distribution whose Lorentz curve is the graph of the function L(x) = xe^{x-1} for 0 ≤x ≤1.

5. In a certain state, the Lorentz curves for the distributions of income for lawyers and engineers are y = L_1(x) and y = L_2(x), respectively, where L1(x) = 0.6x ^{2} + 0.4x and L2(x) = x ^{2} e^{x-1} Find the Gini index for each curve. Which profession has the more equitable distribution of income?

6. Use the method of integration by parts to solve the integrals:

a) $\int x^{5}(x^{4}+2x)dx$

b) $\int x^{-4}(x^{2}+3)dx$

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Unit 11: Differential and Difference Equations

CONTENTS

Objectives

- distinguish between ordinary and partial differential equations;
- define homogeneous, non-homogeneous linear and non-linear ordinary differential equations;
- distinguish between the order and degree of a differential equation

Introduction

The subject of differential equations constitutes a part of mathematics that plays an important role in understanding the physical sciences. ln fact, it is the source of most of, represent any the ideas and theories which constitute higher analysis. In physics, engineering, chemistry and many other disciplines it has become necessary to build a mathematical model to represent certain problems. These mathematical models often involve the search for an unknown function that satisfies an equation in which derivatives of the unknown function play an important role. Such equations are called differential equations. The primary purpose of differential equations is to serve as a tool for studying change in the physical world.

You may recall that if y = f(x) is a given function then its derivative 3 can be dx interpreted as the rate of change of y with respect to x. Sir Isaac Newton observed that certain important laws of natural sciences can be phrased in terms of equations involving rates of change. The most famous example of such a natural law is Newton's second law of motion. Newton was able to model the motion of a particle by an equation involving an unknown function and one or more of its derivatives.

As early as the 1690s, scientists such as Isaac Newton, Gottfried Leibniz, Jacques Bernoulli, Jean Bernoulli and Christian Huygens were engaged in solving differential equations. Many of the methods which they developed are in use till today. In the eighteenth century the mathematicians Leonhard Euler, Daniel Bernoulli, Joseph Lagrange and others contributed generously to the development of the subject. The pioneering work that led to the development of ordinary differential equations as a branch of modern mathematics is due to Cauchy, Riemann, Picard, Poincare, Lyapunov, Birkhoff and others.

Not only are differential equations applied by physicists and engineers, but they are being used more and more in certain biological problems such as the study of animal populations and the study of epidemics. Differential equations have also proved useful in economics and other social sciences. Besides its uses, the theory of differential equations involving the interplay of functions and their derivatives, is interesting in itself.

11.1 Differential Equations

Differential equations deal with continuous system, while the difference equations are meant for discrete process. Generally, a difference equation is obtained in an attempt to solve an ordinary differential equation by finite difference method. Thus, a difference equation is a relation between the differences of unknown function at one or more general values of the independent variable.

Further, suppose we are given a relation of the type $f(x,t_1, t_2, ..., t_n) = 0$ involving (n+ 1) variables x and $t_1, t_2, ..., t_n$, where the value of x depends on the values of the variables $t_1, t_2, ... t_n$. In this case $t_1, t_2, ..., t_n$ are called independent variables and x is called the dependent variable. For instance, consider an equation $y = x^2+2x+3$. Here x is an independent variable and y is a dependent variable. Similarly, if $z = x^2+y^2+2xy$ then x and y are independent variables and z is a dependent variable.

Any equation which gives the relation between the independent and dependent variables and the derivatives of dependent variables is called a differential equation.

Definition:

An equation involving one (or more) dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

For example,

$\frac{dy}{dx}=\cos x$

$y= x\frac{dy}{dx}+a/\frac{dy}{dx}$

$y^2\frac{dy}{dx}+xy\frac{dy}{dx}=nzx$

are all differential equations.

In Eqn. (3), $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are partial derivatives of z w.r.t. x and y respectively. The ax av partial derivatives of a function of two variables z = f(x,y) w.r.t. one of the independent variables, can be defined as

$$\frac{\partial y}{\partial x} = \frac{\partial z}{\partial \varkappa} = \frac{\partial f}{\partial x} = f_\varkappa(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta \varkappa, y) - f(x, y)}{\Delta \varkappa}$$

When the limit exists and is independent of the path of approach. is the first order ax partial derivative of z w.r.t'. x and is obtained by differentiating z w.r.t x treating y as a -constant. Iris read as 'del z by del x'. Similarly, first order partial derivative of z w.r.t. y is denoted by $\frac{\partial z}{\partial \varkappa}$ (or$\frac{\partial f}{\partial x}$ - or f_y(x, y)), so that

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y + y) - f(x, y)}{\Delta y}$$

Note that equations of the type

$$\frac{d}{dx}(xy) = y + xy'$$

are not differential equations. In this equation if you expand the lefthand side then you will find the lefthand side is the same as the right-hand side. Such equations are called identities. Moreover, a differential equation may have more than one dependent variable. For instance,

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} + x = y$$

is a differential equation with dependent variables x and y and the independent variable t.

Differential equations are classified into various types. The most obvious classification' of differential equations is based on the nature of the dependent variable and its derivative (or derivatives) in the equation. Accordingly, we divide differential equations into three classes: ordinary, partial and total. The following definitions give these three types of equations.

Definition: A differential equation involving only ordinary derivatives (that is, derivatives with respect to a single independent variable) is called an ordinary differential equation (abbreviated as ODE).

Equations

$$\frac{d^2y}{dx^2} + y = x^2$$

$$\left(\frac{dy}{dx}\right)^2 = [\sin xy + 2]^2$$

$$y = x\frac{d\dot{y}}{d^x} + r\sqrt{1 + (y')^2}$$

are all ordinary differential equations.

Definition: A differential equation containing partial derivatives, that is, the derivative of one (or more) dependent variable with respect to two or more independent variables is called a partial differential equation (abbreviated as PDE).

The examples of partial differential equations are

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 v}{\partial y}\frac{+\partial_v^2}{20 - z^2} = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 0$$

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial u \partial^2 u}{\partial x \partial t^2} + xtu = 0$$

11.2 Order of a Differential Equation

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Consider the following differential equations:

$$\frac{dy}{dx} = e^x$$

$$\frac{d_y^2}{dx^2} + y = 0$$

$$\frac{d^3y}{dx^3} + x\left(\frac{d^2y}{dx^2}\right)^3 = 0$$

The equations involve the highest derivative of first, second and third order respectively. Therefore, the orders of these equations are 1, 2 and 3 respectively.

Degree of a differential equation

To study the degree of a differential equation, the key point is that the differential equation must be a polynomial equation in derivatives, i.e., y′, y″, y‴ etc. Consider the following differential equations:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0 \qquad 1$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0 \qquad 2$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0 \qquad 3$$

We observe that all equation 1 is a polynomial equation either in y‴, y″ and y′, equation 2 is a polynomial equation in y′ (not a polynomial in y though). Degree of such differential equations can be defined. But equation 3 is not a polynomial equation in y′ and degree of such a differential equation cannot be defined.

By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

Order and degree (if defined) of a differential equation are always positive integers.

11.3 Order and Degree (if defined) of a Differential Equation are always Positive Integers

i) Differential equations with variables separable

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \qquad (1)$$

If F (x, y) can be expressed as a product g (x) h(y), where, g(x) is a function of x and h(y) is a function of y, then the differential equation (1) is said to be of variable separable type. The differential equation (1) then has the form

$$\frac{dy}{dx} = h(y) \cdot g(x) \qquad (2)$$

If h(y) ≠ 0, separating the variables, (2) can be rewritten as

$$\frac{1}{h(y)}\,dy = g(x)\,dx \qquad (3)$$

Integrating both sides of (3), we get

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx \qquad (4)$$

Thus, (4) provides the solutions of given differential equation in the form

H(y)=G(x)+C

Here, H (y) and G (x) are the anti-derivatives of 1/h (y) and g(x) respectively and C is the arbitrary constant.

Example 1: Find the general solution of the differential equation $\frac{dy}{d\gamma} = \frac{\varkappa+1}{2-y}, (y+2)$

Solution: We have $\frac{dy}{d\gamma} = \frac{\varkappa+1}{2-y}$

Separating the variables in equation (1), we get

$$(2-y)\,dy = \int (x+1)\,dx$$

Integrating both sides of equation (2), we get

$$\int (2-y)\,dy = \int (\varkappa+1)\,dx$$

$$2y - y^2/2 = x^3/2 + x + C_1$$

Or $$x^2 + y^2 + 2x - 4y + 2C_1 = 0$$

Or $$x^2 + y^2 + 2x - 4y + C = 0, \text{ Where } C = 2C_1$$

which is the general solution of equation (1).

Example 2: Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that y = 1, when x = 0

Solution: If $y \neq 0$, the given differential equation can be written as

$$\frac{dy}{y^2} = -4\varkappa\,dx$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{y^2} = -4\int \varkappa\,dx$$

Or $$\frac{-1}{y} = -2x^2 + c$$

Or $$y = \frac{1}{2x^2} - C$$

Substituting y = 1 and x = 0 in equation (2), we get, C = – 1.

Now substituting the value of C in equation (2), we get the particular solution of the given differential equation as $Y=1/2x^2+1$

11.4 Homogeneous Differential Equations

Consider the following functions in x and y

$F_1(x, y) = (y^2 + 2xy)$,

$F_2(x, y) = 2x - 3y$,

$F_3(x, y) = \cos y/x$

$F_4(x, y) = \sin x + \cos y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$

$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$

$F_3(\lambda x, \lambda y) = \cos \cos y\, yx x\lambda = \lambda 0\, F_3(x, y)$

$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda n\, F_4(x, y)$, for any $n \in N$

Here, we observe that the functions F_1, F_2, F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda n\, F(x, y)$ but F_4 cannot be written in this form. This leads to the following definition:

A function F(x, y) is said to be homogeneous function of degree n if

F(λx, λy) = λn F(x, y) for any nonzero constant λ.

We note that in the above examples, F_1, F_2, F_3 are homogeneous functions of

degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

We also observe that

$F_1(x, y) = x^2 \left(\frac{y^2}{x^2} \frac{+2y}{x}\right) = x^2 h_1 \left(\frac{y}{n}\right)$

$F_2(x, y) = y^2 \left(1 + \frac{2x}{y}\right) = y^2 h_2 \left(\frac{x}{y}\right)$

$F_2(x, y) = x^1 \left(2 - \frac{3y}{x}\right) = x^1 h_3 \left(\frac{y}{x}\right)$

$F_2(x, y) = y^1 \left(2\frac{x}{y} - 3\right) = y^1 h_4 \left(\frac{x}{y}\right)$

$F_3(x, y) = x^0 \cos\left(\frac{y}{x}\right) = \gamma^0 h_5 \left(\frac{y}{x}\right)$

$F_4(x, y) \neq x^n h_6 \left(\frac{y}{x}\right)', n\varepsilon N$

Therefore, a function F (x, y) is a homogeneous function of degree n if

$F(x, y) = x^n g \left(\frac{y}{x}\right)^a y^n h \left(\frac{x}{y}\right)$

A differential equation of the form dy dx = F (x, y) is said to be homogenous if F(x, y) is a homogenous function of degree zero.

Steps to solve Homogenous differential equation:

$$\boldsymbol{\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)} \qquad \textbf{1)}$$

We make the substitution y = v.x 2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + \frac{x\,dv}{dx} \qquad 3)$$

Substituting the value of dy/ dx from equation (3) in equation (1), we get

$$v + x\frac{dv}{dx} = g(v)$$

$$x\frac{dv}{dx} = y(\dot{v}) - v \qquad 4)$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v)-v} = \frac{dx}{x} \qquad 5)$$

Integrating both sides of equation (5), we get

$$\int \frac{dV}{g(V)-V} = \int \frac{1}{x}dx + C \qquad 6)$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by y/x .

If the homogeneous differential equation is in the form dx/dy = F (x, y) where F(x,y) is homogenous function of degree zero, then we make substitution x / y = v i.e., x = vy and we proceed further to find the general solution as discussed above by writing F(x , y)=dx/dy= h(x/y).

Example 3: Show that the differential equation (x – y) dy/dx = x + 2y is homogeneous and solve it

Solution: The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x+2y}{x-y} \qquad 1)$$

Let $F(x, y) = \frac{x+2y}{x-y}$

Now $F(\lambda x, \lambda y) = \frac{(x+2y)}{x-y} = \lambda\text{^}0F(x, y)$

Therefore, F(x, y)is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

11.5 Linear Differential Equations

A differential equation of the from

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^{x}$$

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$$

Another form of first order linear differential equation is

$\frac{dy}{d^{x}} + P_y = Q$ 1)

Multiply both sides of the equation by a function of x say g (x) to get

$g(x)\frac{dy}{dx} + p(g(x))y = Qg(x)$ 2)

Choose g (x) in such a way that R.H.S. becomes a derivative of y.g (x).

i.e. g(x) dy/dx + P. g(x) y = d/dx [y . g (x)]

or g(x) dy/ dx + P. g(x) y = g(x) dy/ dx + y g′ (x)

⇒ P. g (x) = g′ (x)

or P = (g′ (x)/g (x)

Integrating both sides with respect to x, we get

$$\int p\,dx = \int \frac{g_x'}{gx}dx$$

Or ∫P.dx = log (g (x))

Or $g(x) = e^{\int P\,dx}$

On multiplying the equation (1) by $g(x) = e^{\int P\,dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int P\,dx}$ is called Integrating Factor (I.F.) of the given differential equation

Substituting the value of g (x) in equation (2), we get

$$e^{p\,dx}\frac{dy}{dx} + Pe^{\rho\,dx}y = Qe^{\rho\,dx}$$

Or $$\frac{d}{dy}ye^{P}\,dx = Qe^{\rho\,dx}$$

Integrating both sides with respect to x, we get

$ye^{\rho\,dx}$ = Q $e^{\rho\,dx}$ dx

or y = Q $e^{\rho\,dx}$ dx/ $e^{\rho\,dx}$ *+C

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form dy/dx + Py= Q where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = P$e^{\rho\,dx}$.

(iii) Write the solution of the given differential equation as

y (I.F) = Q × I.F dx +C

In case, the first order linear differential equation is in the form dx/dy +P_1x = Q_1, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{P1\ dy}$ and the solution of the differential equation is given by

x . (I.F) = (Q × I.F) dy +C

Example 4: Find the general solution of the differential equation dy/dx -y= cosx

Solution: Given differential equation is of the form

$\frac{dy}{dx} + P_y = Q$, where P=-1 and Q=cos x

I.F= $e^{dx} e^{x}$

Multiplying both sides of equation by I.F, We get

$$e^{-x}\frac{dy}{dx} - e^{x}y = e^{x} \, cos \, x$$

or dy/dx ye^{-x} = e^{-x} cos x

On integrating both sides with respect to x, we get

y^{e-x} = ∫ e^{-x} cos x dx +C

Let I= ∫ e^{-x} cos x dx

= cosx(e^{-x}/-1)- $(-sinx)(e^{-x})$ dx

= cosx(e^{-x}/-1)- ∫ $sinxe^{-x}$ dx

= cosx(e^{-x}/-1)- [$sinxe^{-x}$ - ∫ $cosx(-e^{-x})$dx]

= $cosxe^{-x+}$ sinx e^{-x} - ∫ cos xe^{-x} dx

Or I= -e^{-x} cos x + sin x e^{-x} –I

2I = (sin x – cos x) e^{x}

I = (sinx- cosx) e^{-x}/2

Substituting the value of I in equation (1), we get

ye^{-x} =(sinx- cosx) e^{-x}/2 +C

y = (sinx- cosx)/2+ Ce^{x}

which is the general solution of the given differential equation

11.6 Difference Equation

Differential equations deal with continuous system, while the difference equations are meant for discrete process. Generally, a difference equation is obtained in an attempt to solve an ordinary differential equation by finite difference method. Thus, a difference equation is a relation between the differences of unknown function at one or more general values of the independent variable.

The study of dynamics in economics is important because it allows to drop out the (static) assumption that the process of economic adjustment inevitable leads to an equilibrium. In a dynamic context, this stability property has to be checked, rather than assumed away. Let time be a discrete denoted t = 0, 1, ...

A function X : N → R^n that depends on this variable is simply a sequence of vectors of n dimensions X0, X1, X2, . . .

If each vector is connected with the previous vector by means of a mapping f : $R^n \rightarrow R^n$ as

$X_{t+1} = f(X_t)$, t = 0, 1, . . . ,

then we have a system of first–order difference equations. In the following definition, we generalize the concept to systems with longer time lags and that can include t explicitly.

$$\Delta^2 y_n + \Delta_{yn} + y_n = n^2$$

Where $\Delta yn = y_{n+1} - y_\eta$

$$\Delta 2y_m = y_{n+2} - 2yn + 1 + yn$$

On simplification, it reduces to

$Y_{n+2} - y_{n+1} + y_n = n^2$

It may be further expressed as

$(E^2 - E + 1)y_n = n^2$ since we define $Ey_n = y_{n+1}$ for n>0

Order of a difference equation is the difference between the largest and the smallest argument occurring in the difference equation divided by the unit of increment.

e.g. The equation $y_{n+2} - 2yn + 1 + 2y_n = 2^n$ is of order 2.

As = $\frac{\textit{the largest argument} - \textit{the smallest argument}}{\textit{Unit of increment}} = \frac{(n+2)-n}{1} = 2$

Solution of a difference equation is the expression for the unknown function (say, yn) which satisfies the given difference equation. The general solution of a difference equation contains as many arbitrary constants as the order of the difference equation. Such an expression on substitution in the equation makes the right hand and left-hand members identically equal. However, particular solutions are obtained by assigning particular values to the arbitrary constants in the general solution.

For example, if c1 and c2 are the arbitrary constants of period 1, $y_\gamma = C_1 3^n + C_2(-1)^n$ is the general solution of the equation $y_{n+2} - 2y_{n+1} - 3y_n = 0$ and $y_n = 2(3)^n + 5(-1)^n$ is a particular solution with $c_1 = 2$ and $c_2 = 5$.

Example: Form the difference equation from the relation $y_n = C_1 3^n + C_2(-1)^n$

Solution: Take n as (n + 1) and (n + 2) in the relation $y_n = C_1 3^n + C_2(-1)^n$, we have

$$y_{n+1} = C_1^{3^{n+1}} + C_2(-1)^{n+1} = 3C_1 3n - C_2(-1)^n$$

$$y_{n+2} = C_1 3^{n+2} + C_2(-1)^{n+2} = 9C_1^{3^n} + C_2(-1)^n$$

On eliminating c1 and c2 from above two relations, we get the desired difference equation of order 2 as $\begin{bmatrix} yn & 1 & 1 \\ yn+1 & 3 & -1 \\ yn+2 & 9 & 1 \end{bmatrix}$=0 or $y_{n+1}(-1-3) - y_{n+1}(1-9) + y_n(3+9) = 0$

$$y_{n+2} - y_{n+1} - 3y_n = 0$$

Homogeneous Linear Difference Equations With Constant Coefficients

A difference equation is said to be linear if the variables involved in it (say, y_{n+1}, y_{n+2} etc.) occur to the first degree only and are not multiplied together.

Hence, a linear difference equation is of the form

$a_0 y_{n+r} + a_1 y_{n+r-1} \dots\dots\dots\dots\dots + a_r y_n = f(n) \dots$(1)

Where a_1, a_2, a3, ,… are constants and f(n) is a function of n or a constant, is called a linear difference equation with constant coefficients. Further, if f(n) = 0, the equation is said to be homogenous. Hence, equation (1), in homogenous form, may be written as

$$(E^r + a_1 E^{r-1} + a_r)y_n = 0$$

where E is the shift operator such that $E'y_n = y_{\eta+r}$

In general, the properties of linear difference equations with constant coefficients are analogous to those of linear differential equations with constant coefficients.

If $y_1(n)$, $y_2(n)$, yr(n) are r independent solutions of the equation (2), then its general solution will be

$U_n = C_1 y_1(n) + C_2 y_n(n) + \cdots \dots\dots\dots\dots\dots\dots\dots . . C_r y_r(n)$ (3)

where c_1, c_2, c_3 are the r arbitrary constants

If v_n is a particular solution of (1), then the complete solution $y_n = u_n + vn$…(4) where u_n is called as Complementary function and v_n is particular integral.

Case 1: Linear homogenous difference equation with constant Coefficient of the first order

$$\boldsymbol{y_{x+1} - A_1 y_x = 0}$$

Let $y_x = \beta x$ then $\beta^{x+1} - A_1\beta^x = o_9\beta^x(\beta - A_1) = 0$

Therefore, $\beta = A_1$

Example 2: Solve: $y_{x+1} - 2y_x = 0$

Sol. Let yx= βx

Then by substitution, we obtain the characteristic equation

$$\beta^{x+1} - 2\beta^x = 0, \beta(\beta - 2) = 0$$

Therefore, $\beta = 2$

Thus, the solution is $y_x = C\beta^x = C2^x$

Case 2: Linear homogenous difference equation with constant coefficients of order 2 or higher. Then

$$\boldsymbol{y_{x+2} + A_1 y_{x+1} + A_2 y_x = 0}$$

Let $y_x = \beta x$, Then the auxiliary equation is

$$\beta^2 + A_1\beta + A_2 = 0$$

Subcase 1: when the two roots are $\boldsymbol{\beta 1}$ *and* $\boldsymbol{\beta 2}$ are real and distinct the solution is

$$y_x = C_1\beta_1^x + C_2\beta_2^x$$

Subcase 2: When the two roots are equal, the solution is where, $\boldsymbol{\beta 1 = \beta 2 = \beta}$

$$y_x = C_1\beta^x + C_2 x\beta^x$$

Subcase 3: When the roots are conjugate complex numbers

Let the roots be $\beta 1$ & $\beta 2$

Let $\beta 1 = a + ib, \beta 2 = a - ib$

The required solution is

$$y_x = p^x(C_1 \cos\theta x + C_2 \sin\theta x)\ where\ p = \sqrt{a^2 + b^2}, \tan\theta = \frac{b}{a}$$

Example 3: Solve the difference equation $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

Solution: The given equation in symbolic form is written as

$(E^3 - 2E^2 - 5E + 6)U_n = 0$

Corresponding auxiliary equation becomes

$E^3 - 2E^2 - 5E + 6 = 0$

Or $(E-1)(E+2)(E-3) = 0$ i.e. $E = 1, -2, 3$.

Hence the solution is $U_n = C_1(1)^n + c_2(-2) + c_3(3)^n$

11.7 Second Order Difference Equations

A difference equation not involving successive values of y_t greater than y_{t+2} is said to be of order 2. Some examples of second-order difference equations are:

i) $u_{t+2} - 3u_{t+1} + 5u_t = 10_t$

ii) $y_{t+2} - 4y_t = 0$

iii) $y_{t+2} - 3y_{t+1} + 4y_t = 6 + t$

In general, any second order linear, non-homogenous difference equation with constant coefficient takes the form,

$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$

wherea_1 and a_2 are constants and φ(t) is a function of t only. If φ(t) = 0, we get

$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$

which is a second order linear, homogeneous difference equation with constant coefficients.

Example 4: Solve $y_{t+2} - 4y_t = 0$

Solution: The characteristic equation (with $y_t = 1$ and $y_{t+2} = b^2$) of the difference equation is

$b^2 - 4 = 0 \Rightarrow b^2 = 4 \Rightarrow b = \pm 2$

Thus, solutions are $b_1 = 2$ and $b_2 = -2$. These are real and distinct.

Therefore, the general solution to the given equation is:

$y = A_1 2^t + A_2 (-2)^t$

11.8 Non-Homogeneous Linear Difference Equation of Order 2

As discussed earlier, the general solution to equation will consist of two parts, y_c and y_p. Here, y_C, the complementary function, is the general solution of the corresponding homogeneous difference equation, and we have already learned how to find it. Now we will learn to find y_p, the particular integral for the difference equation. In fact, p y depends on the function φ(t) on the righthand side of the equation. We will discuss methods for obtaining the particular integral for some special types of φ (t) only. To be specific, we will be discussing only the following cases:

Case 1: When φ (t) is a constant function, say a, we choose

$Y_t = k$

as a possible solution of equation, where k is a constant to be determined by substituting $Y_t = k$

Example 5: Solve $y_{t+2} - 4y_t = 5$

Solution: Note that we are being given a non-homogenous equation, and the homogenous equation corresponding to this equation is

$y_{t+2} - 4y_t = 0$

which is the equation that has been solved in example 1. We already know the characteristic roots of the homogenous equation, giving us the complementary function as the general solution.

$\therefore y_c = A_1 2^t + A_2(-2)^t$

Now we proceed to find the particular integral. Since φ(t) = 5, in this case we choose

$Y_t = k$

as a trial solution for finding the particular solution of the given equation.

y =k implies $y_{t+2} = k$ [y being a constant function]

Now substituting values of y_t and y_{t+2} in our non-homogeneous equation, we get

$k - 4k = 5$

$k = -5/3$

$\therefore y_p = -5/3$

is a particular integral p y of the given equation. Thus, the complete solution of equation = is given by $y_{t+2} - 4y_t = 5$

$y_t = y_c + y_p$

$= A_1 2^t + A_2 (-2)^t - 5/3$

Case 2: When φ(t) is a polynomial of degree n in t, we choose the general polynomial

$y_t = p_0 + p_1 t + \ldots\ldots p_n t^n$

of degree n as a possible candidate for the particular integral and find the constants p_0, $p_1, \ldots\ldots\ldots\ldots p_n$ by substituting in the given difference equation.

Example 6: Solve $y_{t+2} + 10y_{t+1} + 25y_t = 1+t^2$

Solution: To get the particular integral, we choose the general polynomial

$y_t = p_0 + p_1 t + p_2 t^2$

since we have φ(1) =1 + t ^{2} a polynomial of degree 2. The difference equation in the our example contains, besides yt , the terms y (t+1) & y $_{t+2}$. We thus find these values from our general polynomial:

$y_{t+1} = p_0 + p_1(t + 1) + p_2(t + 1)^2$

$= (p_0 + p_1 + p_2) + (p_1 + 2\, p_2)t + p_2\, t$

$y_{t+2} = p_0 + p_1(t + 2) + p_2(t + 2)^2$

$= (p_0 + 2\, p_1 + 4p_2) + (p_1 + 4\, p_2)t + p_2\, t^2$

Now, on substituting the values of yt , yt+1 and yt+2 in our difference equation, we get

$[(p_0 + 2\, p_1 + 4p_2) + (p_1 + 4\, p_2)t + p_2\, t^2] + 10\, [(p_0 + p_1 + p_2) + (p_1 + 2\, p_2)t + p_2\, t^2] + 25\, [p_0 + p_1 t + p_2 t^2] = 1 + t^2$

Collecting the coefficients of like terms on L.H.S, we get

$(36p_0 + 12\, p_1 + 14\, p_2) + (36\, p_1 + 24\, p_2)t + (36\, p_2)t^2 = 1 + t^2$

Now, equating the coefficients of both the sides,

$36p_0 + 12p_1 + 14p_2 = 1$

$36p_1 + 24p_2 = 0$

$36p_2 = 1$

Solving these equations, we get $p_2 = 1/36$, $p_1 = -1/39$, po= -19/8424

Thus we get $y_p = -19/8424 + (-1/39)t + (1/36)t^2$

as a particular integral for the given difference equation. Thus, a complete solution to the given difference equation

$y_{t+2} + 10y_{t+1} + 25y_t = 1+t^2$

$y_t = y_c + y_p$

$y_t = (A_1 + tA_2)(-5) + (-19/8424 - 1/39*t + 1/36*t2)$

Case 3: When (t) φ = a ^{t} for some constant a

In this case the choice for the trial functions depends upon whether a is a root of the characteristic equation of the given difference equation or not. If a is not a root of the characteristic equation, then , $y_t = ca^t$ where c is a constant to be determined, works as our trial function.

Whereas, if a is a root of the characteristic equation, then we check if it is repeating or not. If it is repeating m times, we choose, $y_t = ca^t t^m$ as our trial equation. If a is a root of the characteristic equation and is not repeated, then m = 1, and we take , $y_t = ca^t t$ as our trial equation. If it is a root repeated twice, then m = 2, and we take $y_t = ct^2 a^t$ as our trial equation, and so on.

Example 7: Solve $y_{t+2} - 5y_{t+1} + 6y_t = 7^t$

Solution: It is easy to see that in this case, the characteristics equation to the given difference equation will have b1 = 3 and b2 = 2 as the real and distinct characteristic roots. Thus, the complimentary function (C.F) will be given by

$y_c = A_1 3^t + A_2 2^t$

To find y_p, note that (t) φ = 7^t and 7 is not a root of the characteristic equation, therefore we choose the following as our trial function:

$Y_t = c7^{t+1}$

Now, we find y_{t+1} and y_{t+2} from the trial function.

$Y_{t+1} = c.7^{t+1}$

$Y_{t+2} = c.7^{t+2}$

Now, on substituting the values of y_t, y_{t+1} and y_{t+2} in our difference equation, we get

$c.7_{t+2} - 5.c.7_{\ t+1} + 6.\ c.7_t = 7$

Dividing both sides by 7^t we get

$49\ c - 35\ c + 6\ c = 1 \Rightarrow c = 1/20$

$\therefore$yp = $1/20 * 7^t$ is the required particular integral (PI)

Now, the complete solution of the difference equation $y_{t+2-}5y_{t+1}+6y_t=7_t$ turns out to be:

Y= CF+PI

$= A_1 3^t + A_2 2^t + 1/20*7^t$

Case 4: When $\emptyset(t) = (a_0 + a_1 t + \ldots + a_n t^n)\ a^t$, where a, $a_0, \ldots a_n$ are constants. In this case $\emptyset(t)$ is a product of functions considered in case 2 and case 3. The trial solution to be considered for y_p is also a product of the solutions considered in case 2 and case 3, respectively.

Thus if $\emptyset(t) = (a_0 + a_1 t + \ldots + a_n t^n)\ a^t$, we choose $y_{t=}$ $(p_\circ + p_1 t + \ldots\ldots\ldots\ldots\ldots\ p_n t^n)$ as our trial equation, if a is not a root of the characteristic equation of the associated linear homogeneous difference equation and $y_{t=}\ t^m (p_\circ + p_1 t + \ldots\ldots\ldots\ldots\ldots\ p_n t^n) a^t$, if a is a root of the characteristic equation repeated m times. For example, if the difference equation is

$Y_{t+2} - 5\ y_{\ t+1} + 6\ y_{t\,=}\ 3t^2\, 7^t$

then we shall choose, $y_t = (p_0 + p_1 t + p_2\ t^2)\ 7^t$

as the trial function, because in this case, the corresponding characteristic equation $b^2 - 14b + 49 = 0$ has 7 as a root repeated twice. This necessitated the induction of the factor 2 t in the trial equation. The values of the constantsp0, p1,…,p_n in the trial functions above can be obtained by substituting the value of t y of the trial function, in the respective difference equations. The calculation of the constants, of course, becomes more and more complicated with higher values of m and n in our trial equations. Even the difference equations of the type mentioned above need a lot of calculation. But the learner shall seldom encounter problem with higher values of n and m. We advise the learner to solve the above level of difference equations, along with some simple problems of this type given in the exercise that follows this unit. The techniques for solving difference equations of order 2, discussed here, are general in nature and can be applied to linear difference equations of any order. In particular, linear difference equations of order 1 can be easily solved using these techniques

Example 8: Solve $y_{t+1} + 5y_t = 6^t$

Solution: In this case the characteristic equation of the associated homogeneous difference equation has −5 as the only root and $\emptyset(t) = 6^{\ t}$ is a function of the type discussed in case 3. Therefore, we choose $y_t = c.6^t$ as our trial equation because 6 is not a root of the characteristic equation. This gives $y_{t+1} = c.6^{t+1}$. On substituting the values of $y_t$and y_{t+1} is the given difference equation we get,

$c.6^{\ t+1} + 5.\ c.6^t = 6$

Dividing both sides by 6^t, we get

6c+5c=1

C=1/11

$\therefore$ The particular integral (P.I), $y_p = 1/11*6^t$

Also, C.F., $Y_c = A(-5)^T$

$\therefore$The complete solution is y_t= C.F+P.I

$= A(-5)^t + 1/11*6^t$

11.9 Economic Applications

Difference equations have wide ranging applications in economics. In dynamic analysis, when the time variable t is allowed to take only discrete values and we are interested in finding a time path, given the pattern of change of a variable y over time variable t, the difference equations come out to be quite handy. We take up some situations in economics where the problem can be converted into a difference equation whose solution provides a solution to the problem in hand.

The Cobweb Model

Example 9: Use the cobweb difference equation solution to answer the question what happens in the market where

$Q_t^d = 400 - 20P_t$ and $Q_t^s = -50 + 10P_{t-1}$

if there is a sudden one-off change in Q_t^s to 160?

Solution: Substituting the values for this market a = 400, b = −20, c = −50 and d = 10 into the general cobweb difference equation solution

$$P_t = \frac{a-C}{d-b} + A\left(\frac{d}{b}\right)^t$$

gives

$P_t = \frac{400-(-50)}{10-(-20)} + A\left(\frac{10}{-20}\right)^t = 450/30 + A(-0.5)^t = 15 + A(-0.5)^t$

To find the value of A we then substitute in the known value of P0.

The question tells us that the initial 'shock' output level Q_0 is 160 and so, as price adjusts until all output is sold, P_0 can be calculated by substituting this quantity into the demand schedule. Thus,

$Q_0^d = 160 = 400 - 20P$

$20P_0 = 240$

$P_0 = 12$

Substituting this value into the general difference equation solution (2) above gives, for time period 0,

$P0 = 12 = 15 + A(-0.5)^0$

$12 = 15 + A$ since $(-0.5)^0 = 1$

$A = -3$

Thus the complete solution to the difference equation in this example is

$P_t = 15 - 3(-0.5)^t$

and so we can see that, as t gets larger, the value of $(-0.5)^t$ approaches zero. This is because

$$\left|\frac{d}{b}\right| = |-0.5| = 0.5 < 1$$

and so the stability condition outlined above is satisfied.

Note:

Stability

From this solution we can see that the stability of the model depends on the value of d/b. If A is a non-zero constant, then there are three possibilities

(i) If $\left|\frac{d}{b}\right| < 1$ then $\left(\frac{d}{b}\right)^t \to 0$ as $t \to \infty$

This occurs in a stable market. Whatever value the constant A takes the value of the complementary function gets smaller over time. Therefore the divergence of price from its equilibrium also approaches zero. (Note that it is the absolute value of $|d/b|$ that we consider because b will usually be a negative number.)

(ii) If $\left|\frac{d}{b}\right| > 1$ then $\left(\frac{d}{b}\right)^t \to \infty$ as $t \to \infty$

This occurs in an unstable market. After an initial disturbance, as t increases, price will diverge from its equilibrium level by greater and greater amounts.

(iii) If $\left|\frac{d}{b}\right| = 1$ then $\left|\left(\frac{d}{b}\right)^t\right| = 1$ as $t \to \infty$

11.10 Growth models and Lagged market equilibrium models

Example 10: There is initially an equilibrium in the basic Keynesian model

$Y_t = C_t + I_t$

$C_t = 650 + 0.5Y_{t-1}$

with I_t remaining at 300. Then It suddenly increases to 420 and remains there. What will be the actual level of Y six time periods after this change?

Solution: The initial equilibrium in period 'minus 1' before the change is

$$Y* = \frac{a + I_t}{1 - b} = \frac{650 + 300}{1 - 0.5} = \frac{950}{0.5} = 1900$$

Therefore the value of C in time period 0 when the increase in I takes place is

$C_0 = 650 + 0.5(1{,}900) = 650 + 950 = 1{,}600$

and so the value of Y_t immediately after this shock is

$Y_0 = C_0 + I_0 = 1{,}600 + 420 = 2{,}020$

The new equilibrium level of Y is

$$Y* = \frac{a + I_t}{1 - b} = \frac{650 + 420}{1 - 0.5} = \frac{1070}{0.5} = 2140$$

Substituting these values into the general solution for the lagged Keynesian macro-economic model difference equation, we get the general solution for this example, which is

$Yt = Y* + (Y0 - Y*)b^t$

$= 2{,}140 + (2{,}020 - 2{,}140)0.5^t$

$= 2{,}140 - 120(0.5)^t$

Therefore, six time periods after the increase in investment

$Y^6 = 2{,}140 - 120(0.5)^6$

$= 2{,}140 - 1.875$

$= 2{,}138.125$

Example 11: Use a spreadsheet to estimate Y_t for the twelve time periods after It is increased to 140, assuming that $Y_t$is determined by the distributed lag Keynesian model

$Y_t = C_t + I_t$

$Ct = 320 + 0.5Y_{t-1} + 0.3Y_{t-2}$

and that the system had previously been in equilibrium with I at 90.

Solution 11: The initial equilibrium level Y* satisfies the equations

$Y^* = C_t + 90$ (1)

$C_t = 320 + 0.5Y^* + 0.3Y^*$

$= 320 + 0.8Y^*$

By substitution of (2) into (1)

$Y^* = 320 + 0.8Y^* + 90$

$0.2Y^* = 410$

$Y^* = 2{,}050 = Y_{t-1} = Y_{t-2}$

Thus, when I increase to 140

$C_0 = 320 + 0.5(2{,}050) + 0.3(2{,}050) = 1{,}960$

$Y_0 = C_0 + I_0 = 1{,}960 + 140 = 2{,}100$

$C_1 = 320 + 0.5(2{,}100) + 0.3(2{,}050) = 1{,}985$

$Y_1 = C_1 + I_1 = 1{,}985 + 140 = 2{,}125$ etc.

Harrod-Domar One Sector Model

An economy produces one good Q with capital K through a production function Qt = bKt, where b = constant productivity of capital. Accumulation of capital between t and t+1 is given by

It = Kt+1 – Kt, where It = investment in t.

Saving St = sQt

Equilibrium level of income is determined at the equality of savings and investment. So,

St = It or,

sQt = Kt+1 – Kt.

Since Qt = bKt, we have sbKt = Kt–1 – Kt

or, Kt+1 = (1 + sb)Kt, a homogeneous first order linear difference equation. Therefore, solution to this equation is given by

Kt = (1 + sb)t K0.

Since b is productivity of capital in the model, we write b 1 = capital output ratio = v (say).

Now $k_t = \left(1 + \frac{s}{v}\right)^t k_0$

And

$$Q_t = \left(1 + \frac{s}{v}\right)^t Q_0$$

Remember that s/v = warranted rate of growth and constituted by two basic parameters s and v. We can find out the output growth rate given s and v.

Samuelson multiplier acceleration model

This model assumes that the national income yt consists of three components i) Consumption (Ct) ii) Investment (It) iii) Government Expenditure (Gt)

The Consumption (Ct) in the time period t is assumed to be proportional to the income of the previous period, y_{t-1}. Investment (It) in period t is assumed to be proportional to the increase in the consumption in the time period t over the consumption in the time period t –1, i.e., It is assumed to be proportional to $C_t - C_{t-1}$. These set of assumptions give rise to the following set of equations:

Income Function $y_t = C_t + I_t + G_0$

Consumption FunctionCt = γy_{t-1} ($0 < \gamma < 1$)

Investment Function It = $\alpha(C_t - C_{t-1})$ ($\alpha > 0$)

The government expenditure is assumed to be constant equal toG0 .The constant γ represents the marginal propensity to consume and α represents the acceleration coefficient.

Substituting the value Ct from consumption function into investment function, we get

Now substituting the value of consumption function and the value of I_t obtained above in the income function (yt), we get

$y_t = \gamma y_{t-1} + \alpha\gamma(y_{t-1} - y_{t-2}) + G0$

 $y_t - \gamma y_{t-1} - \alpha\gamma(y_{t-1} - y_{t-2}) = G0$

$y_t - \gamma(1 + \alpha)y_{t-1} + \alpha\gamma y_{t-2} = G0$

By replacing t by t + 2 in the above equation, this equation can be rewritten as

 $y_{t+2} - \gamma(1 + \alpha) y_{t+1} + \alpha\gamma y_t = G0$

Now we have a linear non-homogeneous difference equation of order 2 that can be easily solved.

Since 0 $\varphi(t) = G_0$, a constant, the particular integral (P.I) is easy to obtain. The calculation of complimentary function (C.F) may pose some problems because in this case the characteristic equation of the associated homogeneous difference equation (with $y_{t+2} = x^2$, $y_{t+1} = x$, and $y_t = 1$) is

$x^2 - \gamma(1 + \alpha) x + \alpha\gamma = 0$

This equation has roots

$$X = \frac{\gamma(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha Y}}{2} = 0$$

The nature of roots depends on the constants α and γ,and a number of cases may arise. We have no intention of going into the complicated details as we have already established our claim that there are situations in economic analysis which can be tackled using difference equations.

Summary

1. Cobweb Model: A model where production or supply responds to price with a one-period lag. This model is often used to analyse the demand-supply mechanism for markets of agricultural commodities.
2. Constant Coefficient Difference Equation: A difference equation has constant coefficient if the coefficients ai's associated with the y values are constant and do not change over time.

3. Difference Equation: A difference equation is an equation involving the values of an unknown function y(x) for different values of x. The independent variable – time in problems of economic dynamics – takes only discrete values. The form of the equation is, $y_t = a_1y_{t-1} + a_2y_{t-2} + \ldots + {}_{anyn-1} + b$, where a1, a2, …., an and b are constants, is an example of an n-th order linear, constant coefficient, difference equation.
4. Homogeneous Difference Equation: A difference equation is homogeneous if the constant term b is zero.
5. Linear Difference Equation: A difference equation is linear if (i) the dependent variable y is not raised to any power and (ii) there are no product terms.
6. Non-homogeneous Difference Equation: A difference equation is nonhomogeneous if the constant term, b, is non-zero.
7. Order of a Difference Equation: It is determined by the maximum number of periods lagged.
8. An equation involving one (or more) dependent variables and its derivatives w.r. to one or more independent variables is called a differential equation.
9. A differential equation involving only ordinary derivatives is called an ODE. c) A differential equation involving partial derivatives is called a partial differential equation (PDE).
10. The order of a differential equation is the order of the highest order derivative appearing in the equation.
11. The degree of a differential equation is the highest exponent of the highest order derivative appearing in it after the equation has been expressed in the form free from radicals and fractions of the derivatives.
12. In a differential equation, when the dependent variable and its derivatives occur in the first degree only, and not as higher powers or products, the equation is said to be linear.
13. If an ordinary differential equation is not linear, it is said to be non-linear.

Keywords

1. Cobweb Model: A model of quantity demanded and supplied where demand depends on current price while quantity supplied depends on price prevailing one period earlier.
2. Homogeneous difference equation: A difference equation is homogeneous if its constant term (the term containing no y) is zero, otherwise it is non-homogenous.
3. Order of a difference equation: The order of a difference equation is the order of the higher difference it contains
4. Difference equation: a difference equation is a relation between the differences of unknown function at one or more general values of the independent variable.
5. Differential equation: An equation involving one (or more) dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

Self Assessment

1. The degree of the differential equation $(d^2y/dx^2)^3 + (dy/dx)^2+1=0$

A. 3
B. 2
C. 1
D. not defined

1. The order of the differential equation $2x^2(d^2y/dx^2)^3$ -3(dy/dx)+y=0

A. 2
B. 1
C. 0
D. not defined

3. Which of the following equations is an exact Differential Equation?
A. $(x^2 + 1)$ dx – xydy = 0
B. xdy + (3x – 2y) dx = 0
C. 2xy dx + $(2 + x^2)$ dy = 0
D. x^2y dy – ydx = 0

4. A differential equation is considered to be ordinary if it has
A. one dependent variable
B. more than one dependent variable
C. one independent variable
D. more than one independent variable

5. Classify the following differential equation: exdydx + 3y = x2y.
Exactly one option must be correct)
A. Separable and not linear.
B. Linear and not separable.
C. Both separable and linear.
D. Neither separable nor linear.

6. How many arbitrary constants will be there in the general solution of a second order differential equation?
A. 3
B. 4
C. 2
D. 1

7.Which of the following functions is a solution for the differential equation dydx-14x=0
A. y=7x2
B. y=7x
C. $y=x^7$
D. y=14x

8. Difference equation is used in:
A. Discrete time analysis
B. Continuous time analysis
C. Digital analysis
D. None of the mentioned

9. Difference equation in discrete systems is similar to the ____________ in continuous systems.
A. Difference equation
B. Differential equation
C. Quadratic equation
D, None of the mentioned

10. What is the homogenous solution of the system described by the first order difference equation y(n)+ay(n-1)=x(n)?
A. $c(a)^n$(where 'c' is a constant)
B. $c(a)^{-n}$
C. $c(-a)^n$
D. $c(-a)^{-n}$

11. The solution obtained by assuming the input x(n) of the system is zero is ____________

A. General solution
B. Particular solution
C. Complete solution
D. Homogenous solution/Complementary Solution

12. In case of Differential equation, set y=ex to obtain the solution. For difference equation, y_x is set equal to β^x

A. True
B. False

13. To get the general solution of homogenous function of first order in case of roots are equal, the solution will be of

A. $y= e^{mx}(C1+C2x)$
B. $y= C_1e^{m_1x} + C_2e^{m_2x}$
C. $y= -e^{mx}(C1+C_2x)$
D. None of the above

14. Choose the correct examples of First Order Difference Equation:

A. Harrod-Domar model of growth
B. Cobweb Model
C. Lagged Income Determination
D. All of the above

15. In case of Samuelson Multiplier Acceleration Model, if roots are real and unequal as $\alpha(1+\beta^2)>4\beta$

A. Fluctuations and non-stability
B. No Fluctuations and non-stability
C. Fluctuations and stability
D. No Fluctuations and stability

Answers for Self Assessment

1.	B	2.	A	3.	B	4.	C	5.	C
6.	C	7.	A	8.	A	9.	B	10.	C
11.	D	12.	A	13.	A	14.	D	15.	D

Review Questions

1. Investigate the behaviour of price in a market, i.e., the stability of a system with demand and supply functions: a) Dt = 86 – 0.8 Pt

St = –10 + 0.8 Pt –1

b) Dt = 86 – 0.8 Pt

2. Establish the stability condition of Samuelson's multiplier-accelerator interaction model.

3. Find the solution of the equation $y_{t+1} + \frac{1}{4} y_t = 5 \; for \; y_0 = 2.$

4. A Keynesian macroeconomic model with a single-time-period lag on the consumption function, as described below, is initially in equilibrium and the level of It is given at 500.

Yt = Ct + It

Ct = 750 + $0.5Y_{t-1}$ It is then increased to 650. Use difference equation analysis to find the value of Y_t in the fourth time period after this disturbance to the system. Will it then be within 1% of its new equilibrium level?

6. Solve the following second-order linear difference equations:

a. $Y_{t+2} - 5Y_{t+1} + 6Y_t = 0$

b. $Y_{t+2} + 4Y_{t+1} + 4Y_t = 6$

c. $Y_{t+2} - 3Y_{t+1} - 40Y_t = 1 + t$

d. $Y_{t+2} + 6Y_{t+1} + 9Y_t = t + t^2$

7. Solve the following differential equations:

a. $dy + ydx = 2xy^2e^X dx$

b. $dx + 2/yxdy = 2x^2y^2dy$

Further Readings

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Web Links

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https://nitkkr.ac.in/docs/14-Difference%20Equations%20and%20their%20Applications.pdf

Unit 12: Linear Algebra

CONTENTS

Objectives

After studying this unit, you will be able to,

- basic concepts of a matrix
- methods of representing large quantities of data in matrix form
- explain the ideas of linear combination and linear dependence ofvectors;
- discuss the effect that the change of basis has on the matrix of a linear transformation.

Introduction

The knowledge of matrices has become necessary for the individuals working in different branches of science, technology, commerce, management and social sciences. In many economic analyses, variables are assumed to be related by sets of linear equations. Matrix algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in conventional algebraic notation. The concept of determinant and is based on that of matrix. In this unit, we introduce the concept of matrices and its elementary properties. Further, discusses the determinant, and a number associated with a square matrix and its properties.

Definition

A rectangular array of numbers is called a matrix.

The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS. A matrix having mrows and n columns is said to have the order m*n.

Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}.$$

Then,

$$a_{11} = 1,\ a_{12} = 3,\ a_{13} = 7,\ a_{21} = 4,\ a_{22} = 5,\ a_{23} = 6.$$

12.1 Matrices

A set of mn numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called m×n matrix (read "m by n matrix").

An m×n matrix is expressed as

$$A=\begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------ \\ --- & ---- & ------ \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix}$$

(i) A matrix is denoted by capital letters A, B, C, etc. of the English alphabets.

(ii) First suffix of an element of the matrix indicates the position of row and second suffix of the element of the matrix indicates position of column. e.g. 23 a means it is an element in the second row and the third column.

(iii) The order of a matrix is written as "number of rows ×number of columns".

12.2 Order of a Matrix

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. If there are 3 rows and 2 columns in a matrix, then its order is written as (3, 2) or (3 x 2) read as three by two. In general, if m are rows and n are columns of a matrix, then its order is (m x n).

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

are matrices of orders (2 x 3), (3 x 1) and (4 x 4) respectively.

Example 1: Write the order of the matrix

$$A = \begin{bmatrix} 9 & 7 & 8 & -3 & -8 \\ 4 & 3 & 6 & 1 & -10 \\ 10 & 12 & 15 & 2 & 5 \end{bmatrix}$$

Also write the elements a_{23}, a_{14}, a_{35}, a_{22}, a_{31}, a_{32}.

Solution: Order of the matrix A is 3×5 and the desired elements are: Matrices and Determinants

a_{23}=6, a_{22}=3,

a_{14}= -3, a_{31}=10,

a_{35}=5, a_{32}=12

Example 2: Write all the possible orders of the matrix having following elements.

(i) 8

(ii) 13

Solution: (i) All the 8 elements can be arranged in single row, i.e. 1 row and 8 columns.

Or
They can be arranged in two rows with 4 elements in each row, i.e. 2 rows and 4 columns.
Or
in four rows with 2 elements in each row, i.e. 4 rows and 2 columns.
Or
in eight rows with 1 element in each row, i.e. 8 rows and 1 column.
∴ the possible orders are 1 *8, 2* 4, 4 *2, 8 *1.

(iii) All the 13 elements can be arranged in single row, i.e. 1 row and 13 columns.

Or

in 13 rows with 1 element in each row, i.e. 13 rows and 1 column.

∴ the possible orders are1* 13, 13 *1.

12.3 Types of Matrices

On the basis of number of rows and number of columns and depending on the values of elements, the type of a matrix gets changed. Various types of matrix are explained as below:

Row Matrix

A matrix having only one row is called a row matrix.
For example, [2 5 7], [8 9], [1 0 3 2] all are row matrices

Column Matrix

A matrix having only one column is called a column matrix. For example,

$$\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \end{bmatrix}$$

all are column matrices.

Null or Zero Matrix:

A matrix in which each element is „0" is called a Null or Zero matrix. Zero matrices are generally denoted by the symbol O. This distinguishes zero matrix from the real number 0.

For example $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 4.

The matrix O_{mxn} has the property that for every matrix A_{mxn},
A + O = O + A = A

Square matrix:

A matrix A having same numbers of rows and columns is called a square matrix. A matrix A of order m x n can be written as A_{mxn}. If m = n, then the matrix is said to be a square matrix. A square matrix of order n x n, is simply written as A_n.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

are square matrix of order 2 and 3.

Main or Principal (leading)Diagonal

The principal diagonal of a square matrix is the ordered set of elements a_{ij}, where i = j, extending from the upper left-hand corner to the lower right-hand corner of the matrix. Thus, the principal diagonal contains elements a_{11}, a_{22}, a_{33} etc.
For example, the principal diagonal of

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

consists of elements 1, 2 and 0, in that order.

Particular cases of a square matrix

(a)Diagonal matrix:

A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

A square matrix A = [aij]$_{n\times n}$ is said to be diagonal matrix if aij=0, ∀i≠j

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are diagonal matrices.

Notes:

(i) For a diagonal matrix all non-diagonal elements must be zero.

(ii) In a diagonal matrix some or all the diagonal elements may be zero.

(b) Scalar Matrix:

A diagonal matrix in which all the diagonal elements are same, is called a scalar matrix i.e. Thus

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

(c) Identity Matrix or Unit matrix:

A scalar matrix in which each diagonal element is 1(unity) is called a unit matrix. An identity matrix of order n is denoted by I_n.

Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the identity matrices of order 2 and 3 .

is an identity matrix if and only if aij = 0 for i ≠ j and aij = 1 for i = j

Notes: If a matrix A and identity matrix I are conformable for multiplication, then I has the property that AI = IA = A i.e., I is the identity matrix for multiplication.

d. Equal Matrices:

Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B i.e for each i, j, $a_{ij} = b_{ij}$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{4}{2} & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$$

hen A = B because the order of matrices A and B is same and aij = bij for every i , j.

e. Upper Triangular Matrix

A square matrix A = [a_{ij}] n×n is said to be upper triangular matrix if all the elements below the principal diagonal are zero.

$$\begin{matrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 4 \end{matrix}$$

f.Lower Triangular Matrix

A square matrix A = [a_{ij}] n×n is said to be lower triangular matrix if all the elements above the principal diagonal are zero.

$$\begin{matrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 1 \end{matrix}$$

Example2: Write orders and types of the following matrices:

a. $\begin{bmatrix} 2 & 9 \\ 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

c. $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

d. $\begin{matrix} 2 & 5 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{matrix}$

e. $\begin{matrix} 2 \\ 9 \\ 6 \end{matrix}$

Solution: Order Type

(i) 2× 2 Square matrix [rows and columns are equal in number.]

(ii)2× 2 Diagonal matrix [Θall the non-diagonal elements are zero.]

(iii) 3× 3 Identify matrix [all the diagonal elements are unity and nondiagonal element are zero.]

(iv) 3×3 Lower triangular matrix [all the elements above the principal diagonal are zero]

(v) 3×1 Column matrix [it has only one column.]

12.4 Adjoint and inverse of a matrix

(i) The adjoint of a square matrix A = [aij] n×n is defined as the transpose of the matrix. [aij] n×n , where Aij is the co-factor of the element aij. It is denoted by adj A

If $A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then *adj* $A=\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$, where A_{ij} is co-factor of a_{ij}.

(ii) A (adj A) = (adj A) A = |A| I, where A is square matrix of order n.

(iii) A square matrix A is said to be singular or non-singular according as |A| = 0 or |A| ≠ 0, respectively.

(iv) If A is a square matrix of order n, then |adj A| = $|A|^{n-1}$.

(v) If A and B are non-singular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

(vi) The determinant of the product of matrices is equal to product of their respective determinants, that is, |AB| = |A| |B|.

(vii) If AB = BA = I, where A and B are square matrices, then B is called inverse of A and is written as B = A–1. Also $B^{-1} = (A^{-1})^{-1} = A$.

(viii) A square matrix A is invertible if and only if A is non-singular matrix. (ix) If A is an invertible matrix, then A^{-1} = (1 / | A |)* (adj A)

Example 03: Find out the inverse of

$$\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution: let A= $\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & -6 \\ 0 & 1 & -1 \end{bmatrix}$

A^{-1}=adj(A)/|A|

To find out the adj(A), first we have to find out cofactor(A).

a11 = -6, a12 = 4, a13 = 4
a21 = 1, a22 = -1, a23 = -1
a13 = -6, a32 = 2, a33 = 4

So, cofactor(A) = $\begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}$

adj(A) = $[\text{cofactor}(A)]^T$

adj(A) = $[\text{cofactor}(A)]^T = \begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}^T$

adj(A) = $\begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}$

Then, | A | = 1(0-6)+1(-4-0)+2(4-0) = -6-4+8 = -2

A^{-1}=adj(A)/|A| = $\dfrac{\begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}}{-2}$

A^{-1}= $\begin{bmatrix} 3 & -1/2 & 3 \\ 2 & 1/2 & -1 \\ -2 & 1/2 & -2 \end{bmatrix}$

12.5 Linear independence and dependence of vectors

A vector is a list of numbers. There are (at least) two ways to interpret what this list of numbers mean: One way to think of the vector as being a point in a space. Then this list of numbers is a way of identifying that point in space, where each number represents the vector's component that dimension. Another way to think of a vector is a magnitude and a direction, e.g. a quantity like velocity ("the fighter jet's velocity is 250 mph north-by-northwest"). In this way of think of it, a vector is a directed arrow pointing from the origin to the end point given by the list of numbers.

A homogenous system such as

$$\begin{bmatrix}1 & 2 & -3\\3 & 5 & 9\\5 & 9 & 3\end{bmatrix}\begin{bmatrix}x1\\x2\\x3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Can be viewed as a vector equation

$$x1\begin{bmatrix}1\\3\\5\end{bmatrix}+x2\begin{bmatrix}2\\5\\9\end{bmatrix}+x3\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

the vector equation has the trivial solution ($x_1=0, x_2=0, x_3=0$)

Linear Dependent

A set of vectors $\{v_1, v_2, \ldots\ldots\ldots v_p\}$ in R^n is said to be linearly independent if the vector equation

$$xv_1+xv_2+\ldots\ldots x_pv_p=0$$

has only the trivial solution. The set $\{v_1, v_2, \ldots\ldots\ldots v_p\}$ is said to be linearly dependent if there exists weights $c_1, \ldots\ldots\ldots c_p$, not all 0, such that

$$c_1v_1+c_2v_2+\ldots\ldots c_pv_p=0$$

↑

linear dependence relation (when weights are not all zero)

Example 4. Let $v_1=\begin{bmatrix}1\\3\\5\end{bmatrix}$, $v_2=\begin{bmatrix}2\\5\\9\end{bmatrix}$, $v_3=\begin{bmatrix}-3\\9\\3\end{bmatrix}$

a. Determine if $\{v_1, v_2, v_3\}$ is linearly independent

b. If possible, find a linear dependence relation among v_1, v_2, v_3.

Solution: $x1\begin{bmatrix}1\\3\\5\end{bmatrix}+x2\begin{bmatrix}2\\5\\9\end{bmatrix}+x3\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Augmented matrix:

$$\begin{bmatrix}1 & 2 & -3 & 0\\3 & 5 & 9 & 0\\5 & 9 & 3 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 2 & -3 & 0\\0 & -1 & 18 & 0\\0 & -1 & 18 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 2 & -3 & 0\\0 & -1 & 18 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

x_3 is a free variable→there are non-trivial solutions.

$\{v_1, v_2, v_3\}$ is linearly independent

b) Reduced Echelon form: $\begin{bmatrix}1 & 2 & -3 & 0\\0 & -1 & 18 & 0\\0 & 0 & 0 & 0\end{bmatrix}\rightarrow\begin{bmatrix}x1\\x2\\x3\end{bmatrix}$

let x_3=………………(any non-zero number).

Then x_1=…………..and x_2=…………………..

$$\ldots\begin{bmatrix}1\\3\\5\end{bmatrix}+\ldots\begin{bmatrix}2\\5\\9\end{bmatrix}+\ldots\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Or

$\ldots.v_1+\ldots v_2+\ldots v_3=0$

(one possible linear dependence relation)

A linear dependence relation such as

$$-33\begin{bmatrix}1\\3\\5\end{bmatrix}+18\begin{bmatrix}2\\5\\9\end{bmatrix}+1\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Can be written as the matrix equation:

$$\begin{bmatrix}1 & 2 & -3\\3 & 5 & 9\\5 & 9 & 3\end{bmatrix}\begin{bmatrix}-33\\18\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Each linear dependence relation among the columns of A corresponds to a non-trivial solution to AX=0.

The columns of matrix A are linearly independent if and only if the equation Ax=0 has only the trivial solution.

Linear independence

A finite, nonempty set of vectors {v1, v2,..., vk} in a vector space V is said to be linearly independent if the only values of the scalars c1, c2,...,ck for which

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

are $c_1 = c_2 = \cdots = c_k = 0$.

Example 5: Let us check whether the set S = {(I, 0, 0), (1, 1, 0), (1, 1, l)}is linearly independent or linearly dependent. Consider the equation

α (l, 0, 0) + β (1, 1, 0) + γ (1, 1, 1) = (0,0, 0),

i.e., (α+ β+ γ, β+γ,γ)= (0,0, 0)

The equality of two vectors requires:

α+ β+ γ =0

β+ γ =o

γ = 0

This is a triangular system with the unique solution α= β= γ= 0. Hence S is a linearly independent subset of R^3.

12.6 Characteristic Roots and Vectors

Eigenvalues are associated with eigenvectors in Linear algebra. Both terms are used in the analysis of linear transformations. Eigenvalues are the special set of scalar values that is associated with the set of linear equations most probably in the matrix equations. The eigenvectors are also termed as characteristic roots. It is a non-zero vector that can be changed at most by its scalar factor after the application of linear transformations. And the corresponding factor which scales the eigenvectors is called an eigenvalue.

Eigenvalue

Eigenvalues are the special set of scalars associated with the system of linear equations. It is mostly used in matrix equations. 'Eigen' is a German word that means 'proper' or 'characteristic'. Therefore, the term eigenvalue can be termed as characteristic value, characteristic root, proper values or latent roots as well. In simple words, the eigenvalue is a scalar that is used to transform the eigenvector. The basic equation is

$Ax = \lambda x$

The number or scalar value "λ" is an eigenvalue of A.

Eigenvectors

Eigenvectors are the vectors (non-zero) that do not change the direction when any linear transformation is applied. It changes by only a scalar factor. In a brief, we can say, if A is a linear transformation from a vector space V and x is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of x.

An Eigenspace of vector x consists of a set of all eigenvectors with the equivalent eigenvalue collectively with the zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "n × n" matrix and λ is an eigenvalue of matrix A, then x, a non-zero vector, is called as eigenvector if it satisfies the given below expression;

$Ax = \lambda x$

x is an eigenvector of A corresponding to eigenvalue, λ.

Notes:

- There could be infinitely many Eigenvectors, corresponding to one eigenvalue.
- For distinct eigenvalues, the eigenvectors are linearly dependent.
- Steps to find the eigenvalues and eigenvectors of a 2x2 matrix

Properties of Eigenvalues

a. Eigenvectors with Distinct Eigenvalues are Linearly Independent

b. Singular Matrices have Zero Eigenvalues

c. If A is a square matrix, then λ = 0 is not an eigenvalue of A

d. For a scalar multiple of a matrix: If A is a square matrix and λ is an eigenvalue of A. Then, $a\lambda$ is an eigenvalue of aA.

e. For Matrix powers: If A is square matrix and λ is an eigenvalue of A and n≥0 is an integer, then λ^n is an eigenvalue of A^n.

f. For polynomials of matrix: If A is a square matrix, λ is an eigenvalue of A and p(x) is a polynomial in variable x, then p(λ) is the eigenvalue of matrix p(A).

g. Inverse Matrix: If A is a square matrix, λ is an eigenvalue of A, then λ-1 is an eigenvalue of A^{-1}

h. Transpose matrix: If A is a square matrix, λ is an eigenvalue of A, then λ is an eigenvalue of A^t

Steps to find the eigenvalues and eigenvectors of a 2x2 matrix

1. Set up the characteristic equation, using $|A - \lambda I| = 0$
2. Solve the characteristic equation, giving us the eigenvalues (2 eigenvalues for a 2x2 system)
3. Substitute the eigenvalues into the two equations given by $A - \lambda I$
4. Choose a convenient value for x_1, then find x_2
5. The resulting values form the corresponding eigenvectors of A (2 eigenvectors for a 2x2 system)

Example 6: We start with a system of two equations, as follows:

$y_1 = -5x_1 + 2x_2$

$y_2 = -9x_1 + 6x_2$

Write those equations in matrix form as:

$\begin{bmatrix} y1 \\ y2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

In general we can write the above matrices as:

y = Av,

where,

$y = \begin{bmatrix} y1 \\ y2 \end{bmatrix}$, $A = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ and $v = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

Step 1. Set up the characteristic equation, using $|A - \lambda I| = 0$

Our task is to find the eigenvalues λ, and eigenvectors v, such that:

y = λv

We are looking for scalar values λ (numbers, not matrices) that can replace the matrix A in the expression y = Av.

So to find λ such that :

$$-5x_1 + 2x_2 = \lambda x_1$$

$$-9x_1 + 6x_2 = \lambda x_2$$

Rearranging gives:

$$-(5 - \lambda)x_1 + 2x_2 = 0$$

$$-9x_1 + (6 - \lambda)x_2 = 0 \qquad (1)$$

This can be written using matrix notation with the identity matrix I as:

$|A - \lambda I|v = 0$

$(A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})V = 0$

$(A - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix})V = 0$

Clearly, there is a trivial solution $v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The resulting equation, using determinants, $|A-\lambda I|=0$ is called the characteristic equation.

Step 2. Solve the characteristic equation, giving us the eigenvalues (2 eigenvalues for a 2x2 system) the coefficient determinant from equations (1)

$|A-\lambda I|=\begin{vmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{vmatrix}$

$= (-5-\lambda)(6-\lambda)-(-9)(2)$

$= -30-\lambda+\lambda^2+18$

$=-\lambda+\lambda^2-12$

$=(\lambda+3)(\lambda-4)$

Now this equals 0 when:

$(\lambda+3)(\lambda-4)=0$

That is, when:

$\lambda=-3 \text{or} 4$.

These two values are the eigenvalues for this particular matrix A.

Step 3. Substitute the eigenvalues into the two equations given by A –λI

Case 1: λ_1=-3

When $\lambda=\lambda_1=-3$, equations (1) become:

$$[-5-(-3)]x_1+2x_2=0$$
$$-9x_1+[6-(-3)]x_2=0$$

That is:

$$-2x_1+2x_2=0$$
$$(2)$$
$$-9x_1+9x_2=0$$

Dividing the first line of Equations (2) by −2 and the second line by -9 (not really necessary, but helps us see what is happening) gives us the identical equations:

$$x_1-x_2=0$$
$$x_1-x_2=0$$

Step 4. Choose a convenient value for x_1, then find x_2

There are infinite solutions of course, where $x_1=x_2$. We choose a convenient value for $x_1$of, say $x_2=1$.

Step 5. The resulting values form the corresponding eigenvectors of A (2 eigenvectors for a 2x2 system)

So the corresponding eigenvector is:

$$V_1=\begin{bmatrix}1\\1\end{bmatrix}$$

Case 1: $\lambda_2=4$

When $\lambda=\lambda_2=4$, equations (1) become:

$$[-5-(-4)]x_1+2x_2=0$$
$$-9x_1+[6-(-4)]x_2=0$$

That is:

$$-9x_1+2x_2=0$$
$$(2)$$
$$-9x_1+2x_2=0$$

We choose a convenient value for x_1 of 2, giving x_2=9. So,the corresponding eigenvector is:

$$V_1=\begin{bmatrix}2\\9\end{bmatrix}$$

Notes:

1. In the above example, we were dealing with 2×2 system, and we found 2 eigenvalues and 2 corresponding eigenvectors.

2. If we had a 3×3 system, we would have found 3 eigenvalues and 3 corresponding eigenvectors.

3. In general, $n\times n$ system will produce {n}n eigenvalues and n corresponding eigenvectors.

Example 7: Find the eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution: To find λ such that:

$$x_2 = \lambda x_1$$
$$x_1 - x_2 + x_3 = \lambda x_2$$
$$x_2 = \lambda x_3$$

That is:

$$x_2 - \lambda x_1 = 0$$
$$x_1 + x_3 - (x_2 + \lambda x_2) = 0$$
$$x_2 - \lambda x_3 = 0$$

The characteristic equation, using $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & 1 & 0 \\ 1 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$= -\lambda^3 - \lambda^2 + 2\lambda$
$= -\lambda(\lambda^2 + \lambda - 2)$
$= -\lambda(\lambda+2)(\lambda-1)$
$= 0$

This occurs when $\lambda_1 = 0, \lambda_2 = -2$ or $\lambda_3 = 1$

Case 1: λ=0

$$x_2 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$x_2 = 0$$

Clearly, $x_2=0$, $x_1=1$, giving $x_3=-1$

So for the eigenvalue $\lambda_1 = 0$, the corresponding eigen vector is v1= $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Case 1: λ=-2

$$2x_1 + x_2 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$

Clearly, $x_2=-2$, $x_1=1$, giving $x_3=1$

So,for the eigenvalue $\lambda_1 = -2$, the corresponding eigen vector is v1= $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Case 1: λ=1

$$-x_1 + x_2 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$

Clearly, choosing $x_2=1$, $x_1=1$, giving $x_3=1$

So,for the eigenvalue $\lambda_1 = -2$, the corresponding eigen vector is v1= $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Summary:
To solve the eigenvalue problem for an n by n matrix, follow these steps:
1. Compute the determinant of A-λI. Withλ subtracted along the diagonal, thisdeterminant starts with λ^nor-λ^n. It is a polynomial in λ of degree n.
2. Find the roots of this polynomial, by solving det.(A -λ I)=0. The n roots arethe n eigenvalues of A. They make A-λI singular.
3. For each eigenvalue, solve (A-λI)x=0 to find an eigenvector x.

12.7 Quadratic forms

Consider the expression $a_{11}x_l^2 + 2a_{12}x_lx_2 + a_{22}x_2^2$where a_{ll}, a_{12} and a_{22} aregiven constants and x_l and x_2 are real variables. Expressions such as this,being homogeneous of degree 2 in x_l and x_2, is called quadratic forms.

Generally, the sign of the expression would depend not on a's but also on the values of x_l and x_2. But sometimes given the values of a's, it is possible to determine the sign of the expression, whatever x_l and x_2 may be.

For example, $Q(x_1, x_2) = x_1^2 - 4x_1x_2 + 4x_2^2 = (x_1-2x_2)^2$
So, the sign of this is positive for all x_1 and x_2.
Similarly, $Q(x_1x_2) = -10x_1^2 + 6x_1x_2 - x_2^2 = -x_1^2 - (3x_1 - x_2)^2$

Therefore, the sign is negative for all (x_1, x_2). Such quadratic forms, whichhave a definite sign for all (x_1, x_2) are said to be either positive definite ornegative definite depending on the sign.
Others,which could vary in sign, aresaid to be indefinite.

In general, with variables $x_1, x_2, \ldots, x_n$, we may have a quadratic form as:

$Q(x_1, x_2, \ldots\ldots x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} aijxixj$

Thus, Q(.) is positive definite if$Q(x) > 0 \;\forall\; X \in R_n$,
Or, positive semi-definite if$Q(x) > 0 \;\forall\; X \in R_n$,
Identical definitions hold for negative definite and negative semi-definitequadratic forms.

Consider the equation:

$Q(x, ,x,) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

$= a_{11}[\ (x_1 + \frac{a12}{a11}x_2)^2 + (a_{22}/a_{11} - a_{12}^2/a_{11}^2)^* x_2^2$

$= a_{11}[\ (x_1 + \frac{a12}{a11}x_2)^2 + (a_{11}a_{22} - a_{12}^2)/a_{11}).x_2^2$

Clearly, $Q > 0$ for all(x_1, x_2)if $a_{11} > 0$ and $(a_{11}a_{22} - a_{12}^2)/a_{11})>0$

Thus, $a_{ll} > 0$, $a_{11}a_{22} - a_{12}^{2*} > 0$ are sufficient conditions for $Q > 0$. These are necessary also, as for $a_{ll} < 0$, Q can be less than zero with high x_l and low x_2 or as for $(a_{11}a_{22} - a^2_{12}) < 0$. Q can be less than zero if $a_{ll} < 0$.

Summary

- A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n matrix : when m = n, it is called a square matrix. To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes, which respectively specify the rows and the columns. This aij is the element in the ith row and j th column of the matrix A. In the notation, the matrix A is denoted by [aij].
- If A be any matrix, then a matrix B, if it exists, such that AB = BA = I is called the inverse of A.Inverse of a square matrix A-1 =Adj A/ | A |
- Ax=λx says that eigenvectors x keep the same direction when multiplied by A.
- Ax=λx also says that det.(A-λI)= 0. This determines n eigenvalues.
- The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.
- The sum of theλ 's equals the sum down the main diagonal of A (the trace).The product of the 's equals the determinant.

Keywords

- Matrix: A rectangular array of numbers (elements).
- Row Matrix: One row only.
- Column Matrix: One column only.
- Equal Matrices: Corresponding elements equal.
- Diagonal Matrix: All elements zero except those on the leading diagonal.
- Null Matrix: All elements zero

- Characteristic Roots and vector: eigen value and eigen vector

Self Assessment

1. Let A be a square matrix of order 3 × 3, then | kA| is equal to

A. k| A |
B. k2 | A |
C. k3| A |
D. 3k | A |

2. Which of the following is correct?

A. Determinant is a square matrix.
B. Determinant is a number associated to a matrix.
C. Determinant is a number associated to a square matrix.
D. None of these

3. Determinant is a number associated to a matrix.

A. True
B. False

4. A matrix with only one column is called:

A. A Null matrix.
B. A row matrix.
C. Homogeneous matrix.
D. None of the above

5. If Ax = b is a system of n linear equations in n unknowns such that det (A)$\neq 0$, then the system has:

A. Infinitely many solutions.
B. Unique solution.
C. Both (a) and (b).
D. None of the above.

6. Transpose of a rectangular matrix is a:

A. Rectangular matrix.
B. Diagonal matrix.
C. Square matrix.
D. Scalar matrix

7. If |A|=0, then A is:

A. Zero matrix.
B. Singular matrix.
C. Non-singular matrix.

D. None of the above.

8. For a non-trivial solution |A| is:

A. |A| > 0
B. |A| < 0
C. |A| = 0
D. None of the above.

9. Two vectors are linearly dependent if and only if they lie:

A. On a line parallel to x-axis.
B. On the same line through origin
C. On a line parallel to y-axis.
D. None of the above.

10. A basis is a linearly independent set that is as large as possible:

A. True.
B. False.
C. May be.
D. None of the above.

11. The scalar λ is characteristic root of the matrix A if:

A. λI) is singular
B. λI is non -singular
C. A is singular.
D. None of the above.

12.If eigenvalue of matrix A is λ, then eigenvalue of A2 is:

A. 1
B. 1/λ
C. λ2
D. None of the above.

13. If A is invertible matrix and eigenvalue of A is λ, then eigenvalue of A−1 is:

a) 1
b) 1/ λ
c) λ2
d) None of the above.

14.In matrices, the inter-industry demand is summarized as

A. Input-Output Matrix
B. Output-input matrix
C. Linear buying matrix

D. Linear selling matrix

15. In input-output analysis, if the exogenous sectors of the open input output model is absorbed in to the system as just another sector _____

A. The transaction matrix

B. A technology coefficient

C. Leontief closed model

D. None of the above

Answers for Self Assessment

1.	A	2.	A	3.	B	4.	D	5.	B
6.	A	7.	B	8.	C	9.	B	10.	A
11.	A	12.	C	13.	B	14.	A	15.	C

Review Questions

1. Write each sum as a single matrix:

a. $A=\begin{bmatrix}5 & 1 & 4\\ 3 & -1 & 9\\ & & \end{bmatrix}$ *and* $B=\begin{bmatrix}1 & -1 & 0\\ -2 & 1 & 0\end{bmatrix}$

b. $A=|1 \quad 3 \quad 0|$ and $B=|4 \quad -3 \quad 1|$

2. Write each product as a single matrix:

A. $\begin{bmatrix}2 & 5 & 4\\ 3 & 6 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 8 & 4\\ 3 & 1\end{bmatrix}$

B. $|3 \quad -2 \quad 5|\begin{pmatrix}1\\ 2\\ -2\end{pmatrix}$

3.Solution by matrix inverse method

i) $A=\begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix}$

4. Determine whether the given set of vectors is linearly independent or linearly dependent in Rn. In the case of linear dependence, find a dependency relationship.

a.{(1, −1), (1, 1)}.

b. {(2, −1), (3, 2), (0, 1)}.

c. {(1, −1, 0), (0, 1, −1), (1, 1, 1)}

5. Evaluate the eigenvalues and eigenvectors of the following matrices:

a. $A=\begin{bmatrix}3 & 1 & 1\\ 1 & -2 & 1\\ 0 & 0 & 2\end{bmatrix}$

b. $B=\begin{bmatrix}5 & -6 & -6\\ -1 & 4 & 2\\ 3 & -6 & 4\end{bmatrix}$

6. A 3 by 3 matrix B is known to have eigenvalues 0, 1,2. This information is enough

to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of B^TB

(c) the eigenvalues of B^TB

(d) the eigenvalues of $(B^2+ I)^{-1}$.

Further Readings

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Unit 13: Optimization

CONTENTS

Objectives

After studying this unit, you will be able to,

- Understand the concave and quasi-concave functions
- Explain the methods of optimization in equality and inequality function.

Introduction

Concavity and Convexity are to do with the shapes that certain functions have. We shall see that concavity and convexity of functions are features decided by the second order derivatives of functions. We shall be relating the concept of convexity also to the related but rather important concept of quasi-concavity. Throughout, we shall also be describing some important economic applications of convexity. We shall take functions, mainly from Microeconomics since you have a course on Principles of Microeconomics in this semester, and see which of the important functions that you come across in that course are convex and which are concave, and what are the implications of these.

13.1 Concave Function

A convex function is a function with the property that the set of points which are on or above its graph is a convex function. In terms of the definition of a convex set that we just saw, a function is a convex function if it has the property that the chord joining any two points on its graph lie on or above the graph.

A function f is concave if and only if any pair of distinct point p and R in the domain of f and $0 < \theta < 1$

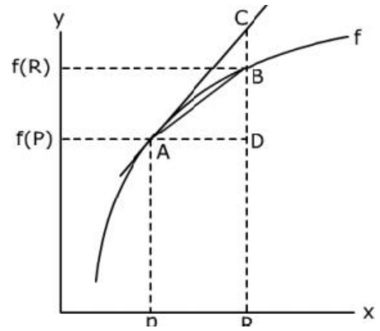
$f(p + (1-\theta)R) \geq f(p) + (1-\theta) f(R)$

The definition can be extended to strict concavity by changing the weak inequality ≥ to the strict inequality >.

A function f is convex if and only if any pair of distinct points p and R in domain of f and for $0<\theta<1$

$f(p\theta + (1-\theta) R) \leq \theta f(p) + R((1-\theta) f(R)$

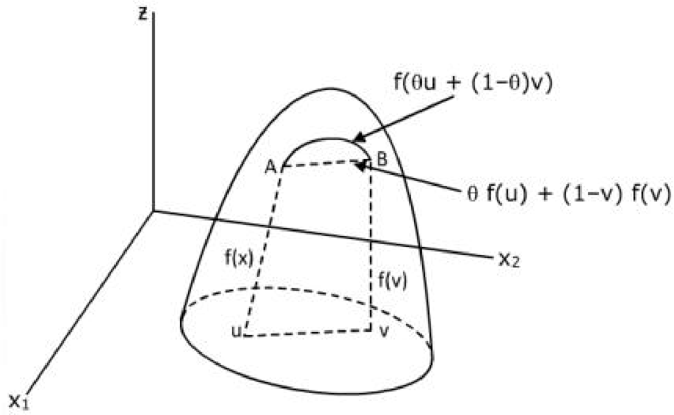
The right-hand side is the height of line segment and the left-hand side is the height of the arc AB.



Till now we have been discussing concavity and convexity of functions of one variable only. The conditions for concavity and convexity, strict and non-strict can be defined for functions of many variables. We shall discuss the concept of concavity and convexity for a two-variable function.

$z = f(x_1, x_2)$

The function f(x,y) is concave (convex) if and only if for any pair of distinct points A and B on its graph (a-surface) the line segment lies either on or below (above) the surface except at point A and B. Strict concavity requires the line segment AB lies below the arc AB. Imagine a dome-shaped surface. The surface of convex function typically be bowl shaped. For non-strictly concave and convex function the line segment AB is allowed to lie on the surface itself, some portion of the surface, or even the entire surface may be flat rather than curved



A function is concave if the set of all points which are on or below the graph of the function is a convex set. We may say that a function f is concave if and only if –f is convex. Hence from this, we see that the properties of a concave function can be derived from the corresponding properties of convex functions. Thus,a function f is concave if and only if

$f(\alpha x_1 + (1-\alpha) x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2)$

If f is a differentiable concave function, then

$f(a+h) \leq + f(a)+hf'(a)$

Also, we say that a function f is strictly concave if –f is strictly convex.

13.2 Characteristics of Concave Function

Let us first understand how the sign of the first-derivative determines whether a function is increasing or decreasing:

$f'(x) \geq 0$ on (a,b) $\Leftrightarrow$ f(x) is increasing on (a,b).

Similarly, $f'(x) \leq 0$ on (a,b) $\Leftrightarrow$ f(x) is decreasing is ,on (a,b).

Let us recall that the first derivative measures the slope of the tangent at that point on the curve f. Hence, first derivative increasing means that the slope of the tangent line is increasing. This means that as x increases, the slope at the tangent to the curve f(x) gets progressively steeper. The change in the steepness of the tangent line leads us towards the idea of convexity using second-order derivatives.

Now consider the function f (x) = 10 – x^2. We see that the first derivative is – 2x and the second derivative is – 2. Thus, this function is decreasing in x for all x> 0 and is increasing in x for all x< 0. The slope, however, is falling for all values of x. This means that when f′(x) > 0, it is becoming less steep, while when f(x) < 0 the function is becoming steeper in absolute value, but the slope is becoming more negative. Since the function f (x) = 10 – x 2 has a negative second derivative, it has properties that are opposite of those of a convex function. It is a concave function. A twice differentiable function f (x) is concave if f″(x) ≤ 0 on all points of its domain. Also, a twice differentiable function f (x) is strictly concave if f″(x) < 0 on all points of its domain except possibly at a single point.

Two points can be made here. First, a linear function, since it has a second derivative equal to zero, satisfies the condition for both a convex function as well as a concave function. Secondly, since multiplying by – 1 reverses an inequality, we could say that f (x) is concave if – f (x) is convex, and that f (x) is strictly concave if – f (x) is strictly convex.

A function f is concave on I ⇔ f″ x≤0 for all I^0

Till now we have considered convex and concave functions over an interval. It would be interesting to see what can be said about convexity and concavity at a particular point. The sign of the second derivative at a point x = a provides some useful information. If f ″(a)is positive, then f (x) is changing at an increasing rate as x increases through a, and the slope of the tangent to the curve y = f (x) increases as we pass through the point x = a. The tangent to the curve turns in an anticlockwise direction and the curve is convex from below when viewed at this point. On the other hand, if f″(a) is negative, then f (x) changes at a decreasing rate, the slope turns in the clockwise direction, and the curve is concave from below at the point where x = a. These results concerning the second derivative are independent of the value of f′(a) and whether the tangent to the curve slopes upwards, downwards or is horizontal at the point x = a. Therefore

i)f ″(a) ≥ 0 implies that the function changes at an increasing rate as the function passes through point a and the function is convex from below at the point x = a.

ii) f″(a) ≤ 0 implies that the function changes at a decreasing rate as the function passes through point a and the function is concave from below at the point x = a.

The numerical value of f″(a)shows how quickly the change in the value of f(x) changes and how great is the curvature of the curve y = f(x)at the point x = a.

13.3 Quasi-Concavity

Let x and y be two distinct points in the domain of a function f, and let the segment xy in the domain of the function give rise to the arc CD on the graph of the function. Suppose point D is higher or equal in height to point C. Then the function is said to be quasi-concave if all points on the arc CD (other than C and D) are higher than or equal in height at point C. The function f would be a quasi-convex one if all points on the arc CD are equal to or lower in height at point D. The function f would be strictly quasi-concave (quasi-convex) if all points on the arc are strictly higher than point C (lower than point D). We may state here that any strictly quasi-concave (strictly quasi-convex) is quasi concave (quasi-convex) but the converse is not true. Usually, a quasi-concave function that is not concave has a shape like a bell or like a portion of a bell, and a quasi-convex has a shape like an inverted bell. A concave function is a little like a dome and a convex function like an inverted dome.

Now let us convert these geometric characterizations into an algebraic definition of quasi-concavity and quasi-convexity. A function f is quasi-concave if and only if, for any distinct points x and y in the convex set domain of f, and for 0 < λ < 1,

$f(y) \geq f(x) \Rightarrow f[\lambda x + (1-\lambda)y] \geq f(x)$

From these definitions, we can state three results

Result 1: If f(x) quasi-concave (strictly quasi-concave), then –f(x) is quasi-convex (strictly quasi-convex).

Result 2: Any concave (convex) function is quasi-concave (quasi-convex) but the converse is not true. Similarly any strictly concave (strictly convex) function is strictly quasi-concave (strictly quasi-convex) but the converse is not true.

Result 3: If a function f(x) is linear, then it is quasi-concave as well as quasi-convex.

Example1: Consider the example in lecture notes 1 for a function of two variables. Locate the stationary points of $f(x)= 12x^5 - 45x^4 + 40x^3 +5$ and find out if the function is convex, concave or neither at the points of optima based on the testing rules discussed above.

Solution: $f'x= 60x^4 -180x^3 +120x^2=0$

$= x^4-3x^3+2x^2=0$

Or x= 0,1,2.

Consider the point $x =x^* = 0$

$f''(x^*) = 240(x^*)^3-540(x^*)^2+240x^*= 0$ at $x^*=0$

$f'''(x^*) = 720(x^*)^2-1080x^*+240=240$ at $x^*=0$

Since the third derivative is non-zero $x = x^* = 0$ is neither a point of maximum or minimum but it is a point of inflection. Hence the function is neither convex nor concave at this point.

Consider $x = x^* = 1$

$f''(x^*) = 240(x^*)^3-540(x^*)^2+240x^*= -60$ at $x^*=1$

Since the second derivative is negative, the point $x = x^* = 1$ is a point of local maxima with a maximum value of $f(x) = 12 - 45 + 40 + 5 = 12$. At this point the function is concave since $\partial^2f/\partial x^2< 0$.

Consider $x = x^* = 2$

$f''(x^*) = 240(x^*)^3-540(x^*)^2+240x^*= 240$ at $x^*=2$

Since the second derivative is positive, the point $x = x^* = 2$ is a point of local minima with a minimum value of $f(x) = -11$. At this point the function is convex since $\partial^2f/\partial x^2> 0$.

13.4 Characterization of Interior Optima

i) Lagrange Multiplier (Equality Constraints)
ii) Karush-John-Kuhn-Tucker (Inequality Constraints)

The Lagrange multiplier method can be used for most types of constrained optimization problems. The best way to explain how to use the Lagrange multiplier is with an example and so we shall work through the problem in Examples from the previous section using the Lagrange multiplier method. The firm is trying to maximize output $Q = 12K^{0.4}L^{0.4}$ subject to the budget constraint $40K + 5L = 800$. The first step is to rearrange the budget constraint so that zero appears on one side of the equality sign. Therefore

$0 = 800 - 40K - 5L$ (1)

We then write the 'Lagrange equation' or 'Lagrangian' in the form

G = (function to be optimized) + λ(constraint)

where G is just the value of the Lagrangian function and the Greek letter λ (lambda) is known as the 'Lagrange multiplier'. (Do not worry about where these terms come from or what their actual values are. They are just introduced to help the analysis. Note also that in some other texts a 'curly' L is often used to represent the Lagrange function. This can confuse students because economics problems frequently involve labour, represented by L, as one of the variables in the function to be optimized. This text therefore uses the notation 'G' to avoid this confusion.)

In this problem the Lagrange function is thus

$$G = 12K^{0.4} L^{0.4} + \lambda(800 - 40K - 5L) \quad (2)$$

Next, we derive the partial derivatives of G with respect to K, L and λ and set them equal to zero, i.e. find the stationary points of G that satisfy the first-order conditions for a maximum.

$$\partial G/\partial K = -4.8K^{-0.6} L^{0.4}- 40\lambda=0 \quad (3)$$

$$\partial G/\partial L = -4.8K^{0.4L-0.6}- 5\lambda=0 \quad (4)$$

$$\partial G/\partial \lambda = 800-40K-5L=0 \quad (5)$$

You will note that (5) is the same as the budget constraint (1). We now have a set of three linear simultaneous equations in three unknowns to solve for K and L. The Lagrange multiplier λ can be eliminated as, from (3),

$$0.12K^{-0.6} L^{0.4} = \lambda$$

and from (4)

$$0.96K^{0.4} L^{-0.6} = \lambda$$

Therefore,

$$0.12K^{-0.6} L^{0.4} = 0.96K^{0.4} L^{-0.6}$$

Multiplying both sides by $K^{0.6} L^{0.6}$

$$0.12L = 0.96K$$

$$L = 8K$$

Substituting (6) into (5)

$$800 - 40K - 5(8K) = 0$$

$$800 = 80K$$

$$10 = K$$

Substituting back into (5)

$$800 - 40(10) - 5L = 0$$

$$400 = 5L$$

$$80 = L$$

Example2: A firm can buy two inputs K and L at 18 per unit and 8 per unit respectively and faces the production function $Q = 24K^{0.6} L^{0.3}$. What is the maximum output it can produce for a budget of 50,000? (Work to nearest whole units of K, L and Q.

Solution: The budget constraint is

$$50{,}000 - 18K - 8L = 0$$

and the function to be maximized is $Q = 24K^{0.6} L^{0.3}$

The Lagrangian for this problem is therefore

$G = 24K^{0.6} L^{0.3} + \lambda(50{,}000 - 18K - 8L)$

Partially differentiating to find the stationary points of G gives

$\partial G/\partial K = - 14.4K^{-0.4} L^{0.3} - 18 \lambda = 0$

$$14.4L^{0.3}/18K^{0.4} = \lambda \qquad (1)$$

$\partial G/\partial L = 7.2K^{0.6} L^{-0.7} - 8 \lambda = 0$

$$7.2K^{0.6}/8L^{0.7} = \lambda \qquad (2)$$

$\partial G/\partial \lambda = 500000 - 18K - 8L = 0 \qquad (3)$

Setting (1) equal to (2) to eliminate λ

$$14.4L^{0.3}/18K^{0.4} = 7.2K^{0.6}/8L^{0.7}$$

$$115.2L = 129.6K$$

$$L = 1.125K \qquad (4)$$

Substituting (4) into (3)

$$50{,}000 - 18K - 8(1.125K) = 0$$

$$50{,}000 - 18K - 9K = 0$$

$$50{,}000 = 27K$$

$$1{,}851.8519 = K \qquad (5)$$

Substituting (5) into (4)

L = 1.125(1,851.8519) = 2,083.3334

Thus, to the nearest whole unit, optimum values of K and L are 1,852 and 2,083 respectively.

We can check that when these whole values of K and L are used the total cost will be

TC = 18K + 8L = 18(1,852) + 8(2,083) = 33,336 + 16,664 = 50,000

and so the budget constraint is satisfied. The actual maximum output level will be

$Q = 24K^{0.6} L^{0.3} = 24(1,852)^{0.6} (2,083)^{0.3} = 21,697$ units

Although the same mathematical method can be used for various economic applications, you still need to use your knowledge of economics to set up the mathematical problem in the first place.

iii) Karush-John-Kuhn-Tucker (Inequality Constraints)

It was previously established that for both an unconstrained optimization problem and an optimization problem with an equality constraint the first-order conditions are sufficient for a global optimum when the objective and constraint functions satisfy appropriate concavity/convexity conditions. The same is true for an optimization problem with inequality constraints.

The Kuhn-Tucker conditions are both necessary and sufficient if the objective function is concave and each constraint is linear or each constraint function is concave, i.e. the problems belong to a class called the convex programming problems.

Consider the following optimization problem:

Minimize f(X) subject to gj(X) ≤ 0 for j = 1,2,…,p ; where $X = [x_1, x_2 \ldots x_n]$

Then the Kuhn-Tucker conditions for X* = [x1 * x2 * . . . xn*] to be a local minimum are

$\partial f/\partial x_1 + \sum_{j=1}^{m} \lambda_j \partial g/\partial x_1 = 0$ i=1,2,………………….n

$\lambda_j gj = 0$ j=1,2,………………….n

$gj \le 0$ j=1,2,………………….m

$\lambda_j \ge 0$ j=1,2,………………….m

In case of minimization problems, if the constraints are of the form gj(X) ≥ 0, then λ j have to be non-positive in (1). On the other hand, if the problem is one of maximization with the constraints in the form gj(X) ≥ 0, then λ j have to be nonnegative.

Example3: Minimize$f=x_1^2+2x_2^2+3x_3^2$ subjecttotheconstraints

$g_1=x_1-x_2-2x_3\le 12$

$g_2=x_1+2x_2-3x_3\le 8$

usingKuhn-Tuckerconditions.

Solution:

The Kuhn-Tucker conditions are given by

A) $\partial f/\partial x_1 + \lambda_1 \partial f/\partial x_1 + \lambda_2 \partial f/\partial x_1 = 0$

i.e.

$$2x_1+\lambda_1+\lambda_2=0 \quad (2)$$

$$4x_2-\lambda_1+2\lambda_2=0 \quad (3)$$

$$6x_3-2\lambda_1-3\lambda_2=0 \quad (4)$$

B) $\lambda_j g_j=0$

i.e.

$$\lambda_1(x_1-x_2-2x_3-12)=0 \quad (5)$$

$$\lambda_2(x_1+2x_2-3x_3-8)=0 \quad (6)$$

C) $g_j \le 0$

i.e.,

$$x_1-x_2-2x_3-12\le 0 \quad (7)$$

$$x_1+2x_2-3x_3-8\le 0 \quad (8)$$

d) $\lambda_j \ge 0$

i.e.,

$$\lambda_1\ge 0 \quad (9)$$

$$\lambda_2\ge 0 \quad (10)$$

From (5) either $\lambda_1=0$ or, $x_1-x_2-2x_3-12=0$

<u>*Case1*</u>: $\lambda_1=0$

From *(2), (3) and (4)* we have $x_1=x_2=-\lambda_2/2$ and $x_3=\lambda_2/2$.

Using these in *(6)* we get $\lambda_2^2+8\lambda_2=0, \therefore \lambda_2 = 0$ or -8

From *(10)*, $\lambda_2\ge 0$, therefore, $\lambda_2=0$, $\mathbf{X}^*=[0,0,0]$, this solution set satisfies all of *(6)* to *(9)*

<u>*Case2*</u>: $x_1-x_2-2x_3-12=0$

Using *(2), (3) and (4)*, we have

$$-\frac{\lambda_1-\lambda_2}{2}-\frac{\lambda_1-2\lambda_2}{4}-\frac{2\lambda_1+3\lambda_2}{3}-12=0 \text{ or,}$$

$17\lambda_1+12\lambda_2=-144$. But conditions

(9) and *(10)* give us $\lambda_1\ge 0$ and $\lambda_2\ge 0$ simultaneously, which cannot be possible with $17\lambda_1+12\lambda_2=-144$.

Hence the solution set for this optimization problem is $\mathbf{X}^*=[0\ 0\ 0]$

Example 4: Minimize $f=x_1^2+x_2^2+60x_1$ subject to the constraints

$g_1=x_1-80\ge 0$

$g_2=x_1+x_2-120\ge 0$ using Kuhn-Tucker conditions.

Solution:

The Kuhn-Tucker conditions are given by

a) $\partial f/\partial x_i + \lambda_1\partial g_1/\partial x_i + \lambda_2\partial g_2/\partial x_i + \lambda_3\partial g_3/\partial x_i=0$

i.e.

$$2x_1+60+\lambda_1+\lambda_2=0 \quad (11)$$

$$2x_2+\lambda_2=0 \quad (12)$$

B) $\lambda_j g_j = 0$

i.e.

$$\lambda_1(x_1 - 80) = 0 \qquad (13)$$

$$\lambda_2(x_1 + x_2 - 120) = 0 \qquad (14)$$

C) $g_j \leq 0$

i.e.,

$$x_1 - 80 \geq 0 \qquad (15)$$

$$x_1 + x_2 + 120 \geq 0 \qquad (16)$$

d) $\lambda_j \leq 0$

i.e.,

$$\lambda_1 \leq 0 \qquad (17)$$

$$\lambda_2 \leq 0 \qquad (18)$$

From (13) either $\lambda_1 = 0$ or, $(x_1 - 80) = 0$

Case 01: $\lambda_1 = 0$

From (11) and (12) we have $x_1 = -\frac{\lambda_2}{2} - 30$ and $x_2 = -\frac{\lambda_2}{2}$

Using these in *(14)* we get $\lambda_2(\lambda_2 - 150) = 0$; $\therefore \lambda_2 = 0$ *or* -150

Considering $\lambda_2 = 0$

$$, \mathbf{X}^* = [30, 0].$$

But this solution set violates *(15)* and *(16)*

For $\lambda_2 = -150$, $\mathbf{X}^* = [45, 75]$.

But this solution set violates *(15)*.

Case 02: $(x_1 - 80) = 0$

Using $x_1 = 80$ in *(11)* and *(12)*, we have

$$\lambda_2 = -2x_2$$

$$\lambda_1 = 2x_2 - 220 \qquad (19)$$

Substitute (19) in (14), we have

$$-2x_2(x_2 - 40) = 0.$$

For this to be true, either

$$x_2 = 0 \text{ or } x_2 - 40 = 0$$

For x2 0, 1 220. This solution set violates (15) and (16)

For x_2–40=0,λ_1=–140and $\lambda_2 = -80$. This solution set is satisfying all equations from (15) to (19) and hence the desired. Therefore, the solution set for this optimization problem is X* = [80 40].

Summary

- A convex function is a function with the property that the set of points which are on or above its graph is a convex function. In terms of the definition of a convex set that we just saw, a function is a convex function if it has the property that the chord joining any two points on its graph lie on or above the graph.
- f ″(a) ≥ 0 implies that the function changes at an increasing rate as the function passes through point a and the function is convex from below at the point x = a.
- f″(a) ≤ 0 implies that the function changes at a decreasing rate as the function passes through point a and the function is concave from below at the point x = a.
- The function f(x,y) is concave (convex) if and only if for any pair of distinct points A and B on its graph (a-surface) the line segment lies either on or below (above) the surface except at point A and B. Strict concavity requires the line segment AB lies below the arc AB. Imagine a dome-shaped surface.
- Any concave (cKarush–Kuhn–Tucker onvex) function is quasi-concave (quq
- quasi-convex) but the converse is not true. Similarly, any strictly concave (strictly convex) function is strictly quasi-concave (strictly quasi-convex) but the converse is not true.
- If a function f(x) is linear, then it is quasi-concave as well as quasiconvex.

Keywords

- Convex: A convex function is a function with the property that the set of points which are on or above its graph is a convex function.
- Concave: a concave function is the negative of a convex function.
- Quasiconcave: f(xj) ≥ f(xi) ⇒ f'(xi) (xj – xi) ≥ 0
- Lagrange: the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints
- Karush–Kuhn–Tucker: The Karush–Kuhn–Tucker (KKT) conditions, also known as the Kuhn–Tucker conditions, are first derivative tests (sometimes called first-order necessary conditions) for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied.

SelfAssessment

1. In a simple one-constraint Lagrange multiplier setup, the constraint has to be always one dimension lesser than the objective function.

A. True
B. False

2. If function changes at a decreasing rate as the function passes through point a and the function is concave from below at the point x= a.

A. f’’(a)>0
B. f’’(a)<0
C. f’(a)=0
D. none of the above

3. If f is a differentiable concave function, then, which of the following equations shows strictly concave?

A. f (a+h) ≤ f (a) + hf ′(a)
B. f (a+h) ≥ f (a) + hf ′(a)
C. f (a+h) = f (a) + hf ′(a)

D. f (a+h) ≠ f (a) + hf '(a)

4. Any concave function is quasi-concave but quasi-concave is also concave
A. True
B. False

5. f (y)≥f (x) ⇒ f [λx + (1- λ)y] ≤ f(y) shows:
A. Quasi-concave Function
B. Quasi-convex Function
C. Concave Function
D. Convex Function

6. When will be the inflection point achieved?
A. f′′x=0
B. F′′x>0
C. F′′x<0
D. F′′x not equal to 0

7. If f′x is decreasing and concave down at 'a' point, while at another crii7. If f′x is decreasing and concave down at ‘a’ point, while at another critical point it is f′x is decreasing and concave down at 'b' point, on the other-side, if f′x is increasing and concave up on 'c' then where inflection exist?
A. Inflection at (a,b) and inflection at (b,c)
B. No Inflection at (a,b) and no inflection at (b,c)
C. Inflection at (a,b) and no inflection at (b,c)
D. No inflection at (a,b) and inflection at (b,c)

8. If f′′x>0, while f′x<0, the what will be shape of curve?
A. Concave up, decreasing
B. Concave down, increasing
C. Concave up, increasing
D. Concave down, decreasing

9. What is Lagrange Multiplier?
A. The Lagrange multiplier technique finds the maximum or minimum of a multivariable function when there is some constraint on the input values.
B. Lagrange multipliers are used in multivariable calculus to find maxima and minima of a function subject to constraints.
C. The Lagrangian function is a technique that combines the function being optimized with functions describing the constraint or constraints into a single equation.
D. All of the above

10. In optimization problems with equality constraints:
A. the number of constraints equals the number of choice variables.
B. the number of constraints may equal the number of choice variables.
C. the number of constraints must exceed the number of choice variables.
D. the number of constraints may exceed the number of choice variables.

11. In optimization problems with inequality constraints:
A. the number of constraints equals the number of choice variables.
B. the number of constraints may equal the number of choice variables.
C. the number of constraints must exceed the number of choice variables.
D. the number of constraints must be smaller than the number of choice variables.

12. In optimization problems with inequality constraints, the Kuhn-Tucker conditions are:
A. sufficient conditions for (x0, ..., xN) to solve the optimization problem.
B. necessary conditions for (x0, ..., xN) to solve the optimization problem.
C. sufficient but not necessary conditions for (x0, ..., xN) to solve the optimization

D. problem.

13. The drawback of Lagrange's Method of Maxima and minima is?
A. Maxima or Minima is not fixed
B. Nature of stationary point is cannot be known
C. Accuracy is not good
D. Nature of stationary point is known but cannot give maxima or minima

14. In case of Minimization problem, in Kuhn Tucker Conditions, What should be the form of constraints?
A. $gj(X) \geq 0$ then $\lambda \leq 0$
B. $gj(X) \leq 0$ then $\lambda \geq 0$
C. $gj(X) = 0$ then $\lambda \leq 0$
D. $gj(X) \geq 0$ then $\lambda = 0$

15. In case of Lagrange Equality constraints, when the function will be minimum?
A. If all eigen values are positive
B. If all eigen values are negative
C. Both a and b
D. None of the above

Answers for SelfAssessment

1.	A	2.	B	3.	A	4.	B	5.	A
6.	A	7.	D	8.	A	9.	D	10.	A
11.	D	12.	D	13.	B	14.	A	15.	A

Review Questions

1. Comment upon the Convexity/concavity of the following functionsover the set of non-negative real numbers.
   a. $f(x)= - (2/5)\, x^2 + 5x - 10$
   b. $f(x)= 5x^2 - 7x$
   c. $f(x)= x^{1/2}$
2. Describe the concavity and/or convexity of $f(x) = 7x^3 - 42x^2 + 12x + 97$ over its domain?
3. The notion of Quasi-concavity is weaker than the notion of Concavity, do you agree?
4. A firm has a budget of £570 to spend on the three inputs x, y and z whose prices per unit are respectively 4, 6 and 3. What combination of x, y and z will maximize output given the production function Q =
   $2x^{0.2}y^{0.3}z^{0.45}$?
5. Make up your own constrained optimization problem for an objective function with three variables and solve it.

Further Readings

- Allen, R.G,D, Mathematical Analysis for Economists, London: Macmillan and Co. Ltd
- Knut Sydsaeter and Peter J. Hammond, Mathematics for Economic Analysis, Prentice Hall
- Carl P. Simon and Lawrence Blume, Mathematics for Economists, London: W .W. Norton & Co.

Unit 14: Trigonometric Functions

CONTENTS

Objectives

After studying this unit, you will be able to,

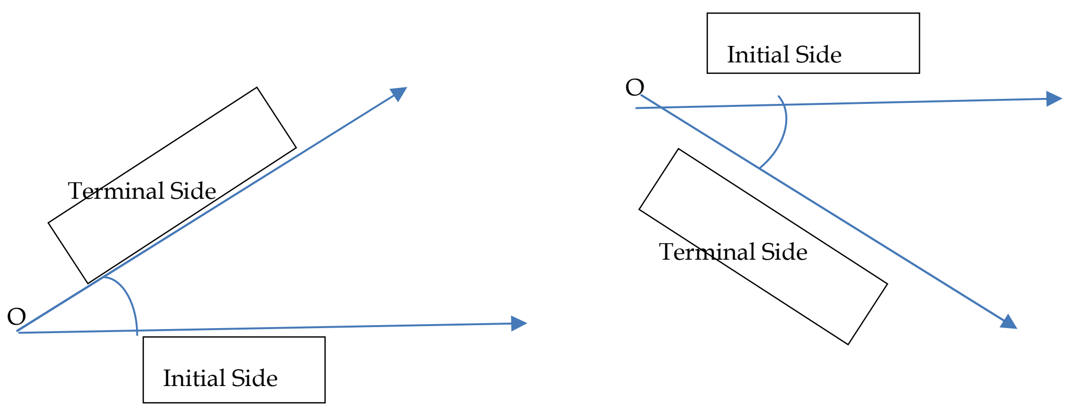
- discuss the importance and use of angles
- understand the use of trigonometry functions

Introduction

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

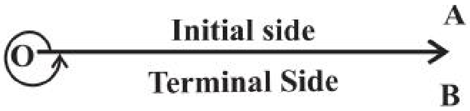
14.1 Angles

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative (Fig. 1)



i) Positive Angle ii) Negative Angle

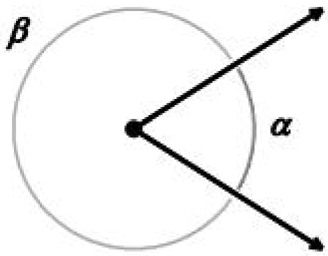
The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. one complete revolution from the position of the initial side as indicated in Fig.2.



This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure.

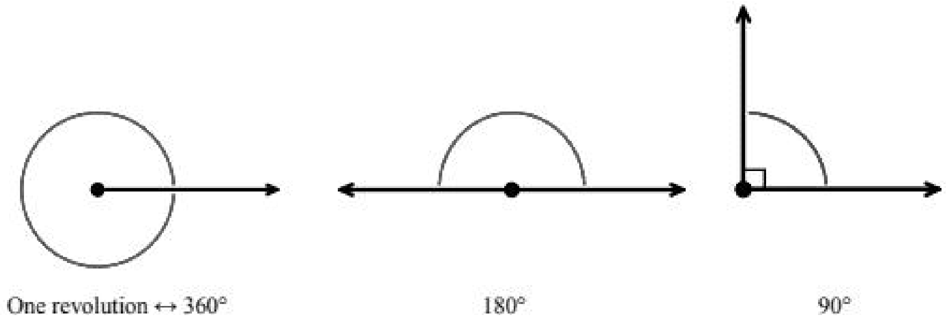
The measure of an angle is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as the following pictures indicate

Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.1 clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book, we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles. So, for instance, we have



Degree Measure

One commonly used system to measure angles is degree measure. Quantities measured in degrees are denoted by a small circle displayed as a superscript. One complete revolution is 360°, and parts of a revolution are measured proportionately. Thus, half of a revolution (a straight angle) measures (½ (360°) =180 °; a quarter of a revolution measures (1/4) (360°) =90, and so on.



Note that in the previous figure we have used the small square to denote a right angle, as is commonplace in geometry. If an angle measures strictly between 0° and 90° it is called an acute angle and if it measures strictly between 90° and 180° it is called an obtuse angle.

Using our definition of degree measure, 1° represents the measure of an angle which constitutes 1 360 of a revolution. Even though it may be hard to draw, it is nonetheless not difficult to imagine an angle with measure smaller than 1°.

There are two ways to subdivide degrees.

1. The first, and most familiar, is decimal degrees. For example, an angle with a measure of 30.5° would represent a rotation halfway between 30° and 31° or, equivalently 30.5/360=61/720 of a full rotation.

2. The second way to divide degrees is the Degree-Minute-Second (DMS) system, used in surveying, global positioning and other applications requiring measurements of longitude and latitude. In the DMS system, one degree is divided equally into sixty minutes, and in turn each minute is divided equally into sixty seconds. In symbols, we write 1°= 60' = and 1' =60", from which it follows that 1 3600".

Example 1: Convert 42.125° to the DMS system.

Solution. To convert 42.125° to the DMS system, we first note that 42.125°=42+0.125.

Converting the partial amount of degrees to minutes, we find

0.125(60'/1°)

= 7.5'

=7'+ 0.5'.

Next, converting the partial amount of minutes to seconds gives

0.5' (60''/1')

=30".

The result is

42.125 =42°+ 0.125°

=42 + 7.5'

=42 +7'+ 0.5'

=42 +7' +30"

=42°7'30".

Example 2: Convert 117°15'45" to decimal degrees.

Solution. To convert 117°15'45" to decimal degrees, we first compute

15'(1°/60')

=1/ 4

=0.25

and

45"(1°/3600")

=1/ 80

=0.0125.

Then we find

=117 15'45"

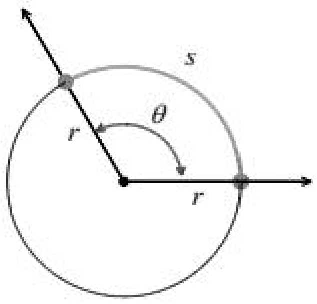
=117+ 15' +45"

=117 +0.25 +0.0125

=117.2625

Radian Measure

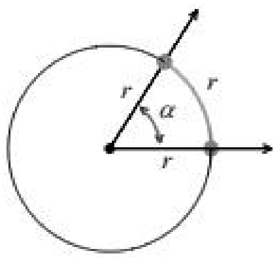
Suppose now we take a portion of the circle, so instead of comparing the entire circumference C to the radius r, we compare some arc measuring s units in length to the radius r, as depicted below. Let θ be the central angle subtended by this arc; that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc. Using proportionality arguments, it stands to reason that the ratio s r should also be a constant among all circles with the same central angle θ. This ratio, s/r , defines the radian measure of an angle.



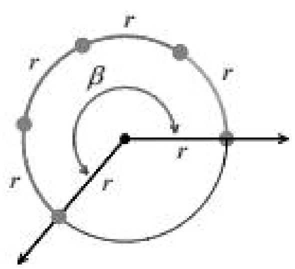
To get a better feel for radian measure, we note that

1. An angle with radian measure 1 means the corresponding arc length s equals the radius r of the circle. Hence, s= r .
2. When the radian measure is 2, we have s = 2r .
3. When the radian measure is 3, s =3r, and so forth.

Thus, the radian measure of an angle θ tells us how many radius lengths we need to sweep out along the circle to subtend the angle θ.



α has radian measure 1



β has radian measure 4

Since one revolution sweeps out the entire $2\pi r$, one revolution has radian measure $2\pi r/r = 2\pi$. From this we can find the radian measure of other central angles using proportions, just like we did with degrees. For instance, half of a revolution has radian measure (1/ 2) 2π =π; a quarter revolution has radian measure 1/4)(2π)=π/2 , and so forth.

Relation between Radian and Degree

For converting between degrees and radians, since one revolution counter-clockwise measures 360° and the same angle measures 2π radians, we can use the proportion 2πradians /360°, or its reduced equivalent radiansπ/180°, as the conversion factor between the two systems.

Degree – Radian Conversion:

- To convert degree measure to radian measure, multiply by $\frac{\pi \text{ radians}}{180^\circ}$.
- To convert radian measure to degree measure, multiply by $\frac{180^\circ}{\pi \text{ radians}}$.

Example 3: Convert the following measures.

1. 60° to radians

2.-5π/6 radians to degrees

3. 1 radian to degrees

Solution.

1. 60° = 60(πradians/180)

= πradians/3

2. -5π/6 radians= (-5π/6 radians) (180°/πradians

=-150°[12]

3. 1 radian= (1 radian) (180°/πradians)

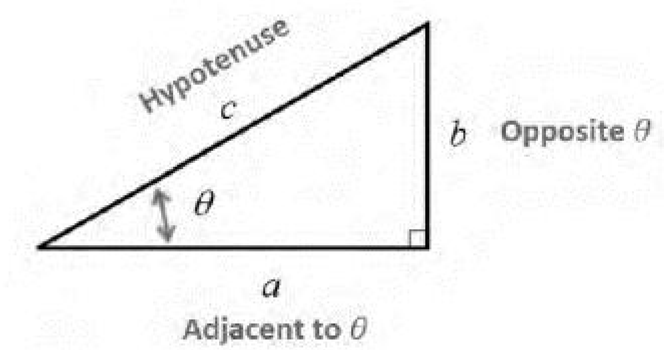
=180/π

≈ 57.2958

14.2 Trigonometric Functions

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

We consider the generic right triangle below with acute angle θ. The side with length a is called the side of the triangle adjacent to θ; the side with length b is called the side of the triangle opposite θ; and the remaining side c (the side opposite the right angle) is called the hypotenuse.



The six commonly used trigonometric functions are defined below:

The Trigonometric Functions: Suppose θ is an acute angle residing in a right triangle. If the length of the side adjacent to θ is a, the length of the side opposite θ is b, and the length of the hypotenuse is c, then

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$.

The following are important properties of the trigonometric functions.

1. For all right triangles with the same acute angle θ, because they are similar, the values of the resulting trigonometric functions of θ will be identical. This is a result of the property of equivalent proportions of corresponding sides within similar triangles.

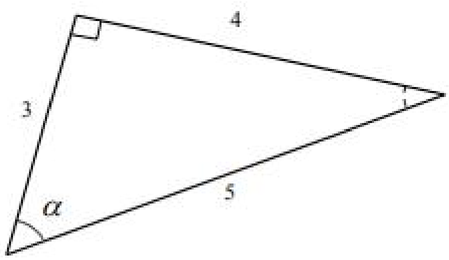
2. Cosecant, secant and cotangent are reciprocal functions of sine, cosine and tangent, respectively. Thus, if we know the sine, cosine and tangent values for an angle, we can easily determine the remaining three trigonometric functions. In particular,

$$\csc(\theta) = 1/\sin(\theta)$$

$$\sec(\theta) = 1/\cos(\theta)$$

$$\cot(\theta) = 1/\tan(\theta)$$

Example 4: Use the following triangle to evaluate sin (α) , cos(α) , tan(α) , csc(α), sec(α) and cot(α).



Solution. From the definitions of trigonometric functions,

$\sin(\alpha) = \frac{oppoiste}{hypotenuse} = 4/5$

$\cos(\alpha) = \frac{adjacent}{hypotenuse} = 3/5$

$\tan(\alpha) = \frac{oppoiste}{adjacent} = 4/5$

The reciprocals of these three function values result in the remaining three trigonometric function values:

$$\csc(\theta) = 1/\sin(\theta) = 5/4$$

$$\sec(\theta) = 1/\cos(\theta) = 5/3$$

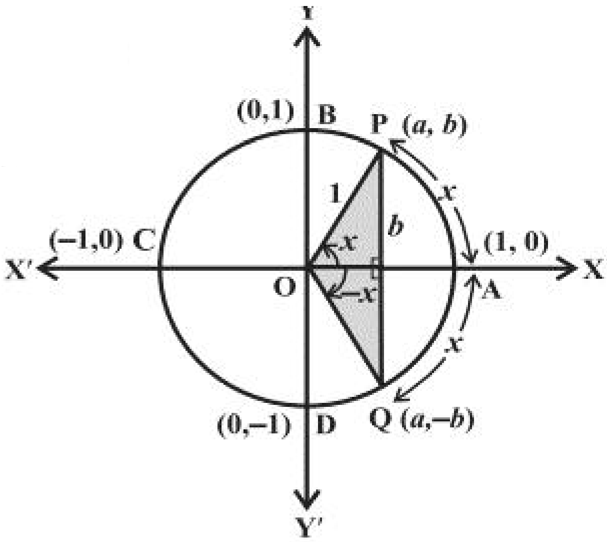
$$\cot(\theta) = 1/\tan(\theta) = 3/4$$

Sign of Trigonometric Functions

Let P (a, b) be a point on the unit circle with center at the origin such that ∠AOP = x. If ∠AOQ = – x, then the coordinates of the point Q will be (a, –b). Therefore,

cos (– x) = cos x

and sin (– x) = – sin x

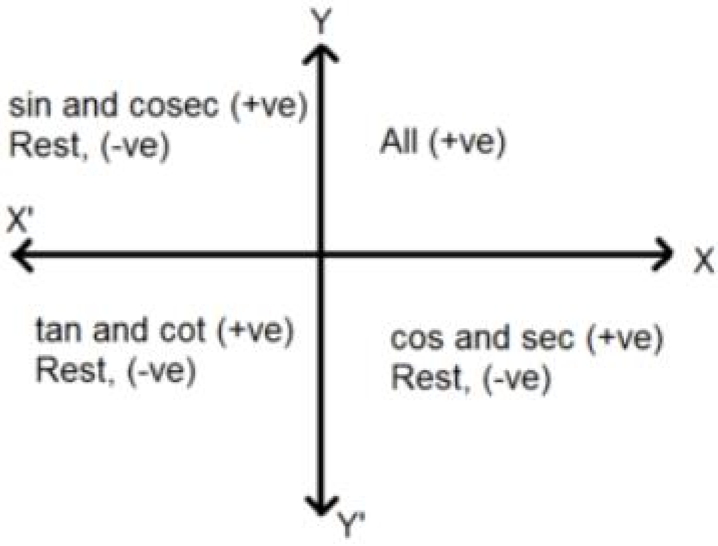


Since for every point P (a, b) on the unit circle, – 1 ≤ a ≤ 1 and – 1 ≤ b ≤ 1, we have – 1 ≤ cos x ≤ 1 and –1 ≤ sin x ≤ 1 for all x. Therefore, sin x is positive for 0 < x < π, and negative for π < x < 2π. Similarly, cos x is positive for 0 < x < π 2, negative for π 2 < x < 3π 2 and also positive for 3π 2 < x < 2π.Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

Signs of Trigonometric Ratios in Different Quadrants

θ lies in Quadrant →	I	II	III	IV
Trigonometric Functions ↓				
sin θ	+ ve	+ ve	– ve	– ve
cos θ	+ ve	– ve	– ve	+ ve
tan θ	+ ve	– ve	+ ve	– ve
cosec θ	+ ve	+ ve	– ve	– ve
sec θ	+ ve	– ve	– ve	+ ve
cot θ	+ ve	– ve	+ ve	– ve

Diagram Showing Signs of Trigonometric Ratios



Example 6: If cos x = – 3/5, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since cos x =− 3/5, we have sec x = −5/ 3

Now $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - 9/25 = 16/25$

Hence sin x = ± 4/ 5.

Since x lies in third quadrant, sin x is negative. Therefore

sin x = – 4 /5

which also gives

cosec x = – 5/ 4

Further, we have tan x = sin/cos x = 4/ 3 and cot x = cos/ sin x = 3/ 4.

Example 7: If cot x = – 5/12, x lies in second quadrant, find the values of other five trigonometric functions.

Solution: Since cot x = – 5 /12, we have tan x = – 12/ 5

Now $\sec^2 x = 1 + \tan^2 x = 1 + 144/25 = 169/25$

Hence sec x = ± 13/5

Since x lies in second quadrant, sec x will be negative. Therefore

sec x = – 13/5,

which also gives

cos x = −5/13

Further, we have

sin x = tan x cos x = (– 12/ 5) × (– 5 /13) = 12/ 13

and cosec x = 1/ sin x = 13/12.

Trigonometric Functions of Sum and Difference of Two Angles

1. sin (– x) = – sin x

2. cos (– x) = cos x

3. cos (x + y) = cos x cos y – sin x sin y

4 . cos (x – y) = cos x cos y + sin x sin y

5. cos (π/2 – x) = sin x

6. sin (π/2 – x) = cos x

7. sin (x + y) = sin x cos y + cos x sin y

8. sin (x – y) = sin x cos y – cos x sin y

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

cos (π/2 +x) = – sin x

sin (π/2 +x) = cos x

cos (π – x) = – cos x

sin (π – x) = sin x

cos (π + x) = – cos x

sin (π + x) = – sin x

cos (2π – x) = cos x

sin (2π – x) = – sin x

10. If none of the angles x, y and (x + y) is an odd multiple of π/2,

Then

tan (x + y) = (tanx+tany)/(1-tanxtany)

11. tan (x - y) = (tanx-tany)/(1+tanxtany)

12. cot (x + y) = (cotx+coty)/(1-cotxcoty)

13.cot (x - y) = (1+cotxcoty)/(coty-cotx)

14.cos 2x= (1-tan^{2}x)/(1+tan^{2}x)

15.sin 2x= (2tanx)/(1+tan^{2}x)

Example 8: cos 75°

Solution: cos 75°= cos(30°+ 45°)

=cos 30°cos 45°-sin30°sin45°

= $\sqrt{3}/2 * 1/\sqrt{2} - 1/2\ (1/\sqrt{2})$

= $(\sqrt{2}-1)2\sqrt{2}$

Example 9: cos(30° +x)=$\sqrt{3}$/2 cosx-1/2 sinx

Solution: Cos 30° cos x- sin30°sinx

LHS= $\sqrt{3}$/2 cosx-1/2 sinx

LHS=RHS

Summary

- Radian measure = π/ 180 × Degree measure
- Degree measure = 180 /π × Radian measure
- Measurement of an Angle.
  English System: 1 right angle = 90°, 1° = 60 minutes = 60′, and 1′ = 60 second =60 ″
- Circular System: 2 right angles = 180°
- Trigonometrical Ratios (circular functions)
  $\sin^2 x+\cos^2 x=1$,
  $1+\tan^2 x=\sec^2 x$,
  $1+\cot^2 x=\mathrm{cosec}^2 x$
  $\tan x= \sin x/\cos x$
  $\sec x= 1/\cos x$
  $\mathrm{cosec}\, x= 1/\sin x$

Keywords

- Angle: Angle is a measure of rotation of a given ray about its initial point.
- Radian: 1 revolution measured in radians is 2π, where π is the constant approximately.
- Degree:A circle is comprised of 360°, which is called one revolution.
- Radian Measure: the radian measure of an angle θ tells us how many radius lengths we need to sweep out along the circle to subtend the angle θ.
- Trigonometry: It is derived from the Greek word's 'trigon' and 'metron' which means 'measuring the sides of a triangle'.

Self Assessment

1. Angles between 0 ° and 90 ° lies in

A. 2nd quadrant
B. 3rd quadrant
C. 1st quadrant
D. 4th quadrant

2. The value of 240 ° into radians should be

A. 4π⁄3
B. 3π⁄4
C. π⁄4
D. π⁄6

3. The union of two non-collinear rays with some common end point is called

A. vertex
B. angle
C. degree
D. radius

4. π radians are equal to

A. 180 °
B. 360 °
C. 90 °
D. 270 °

5. Angles between 270 ° and 360 ° lies in

A. 1st quadrant
B. 2nd quadrant
C. 3rd quadrant
D. 4th quadrant

6. The value of $5\pi/6$ into degrees should be

A. 135 °
B. 90 °
C. 120 °
D. 150 °

7. In which quadrant are Cotangent and Tangent positive?

A. Quadrant 1
B. Quadrant 3
C. Quadrant 2
D. Quadrant 4

8. In which quadrant are Cosine and Tangent negative?

A. Quadrant 4
B. Quadrant 2
C. Quadrant 1
D. Quadrant 3

9. 1. Tanθ equals to

A. $1/\text{cosec}\theta$
B. $1/\sec\theta$
C. $1/\cos\theta$
D. $1/\cot\theta$

10. $1 + \tan^2\theta$ equals to

A. $\text{cosec}^2\theta$
B. $\sec^2\theta$
C. $\sin^2\theta$
D. $\cos^2\theta$

11. $\text{Sin}^2\theta + \cos^2\theta$ equals to

A. 1

B. 0
C. 0.5
D. 2

12. The fundamental trigonometric ratios are

A. 4
B. 5
C. 6
D. 3

13. Which of the following is correct?

A. sin(-x) = -sin(x)
B. sec(-t) = -sec(t)
C. sin(π + x) = sin(x)
D. cos(x) = -cos(x)

14. cot x + tan x =

(A) cot 2x
(B) 2 cot2x
(C) sec x cosec x
(D) cot22x

15. Is the following equation correct or not: Sin(α- β)= SinαCosβ- CosαSinβ

(A) Yes
(B) No

Answers for Self Assessment

1.	C	2.	A	3.	B	4.	A	5.	D
6.	D	7.	B	8.	B	9.	D	10.	B
11.	A	12.	C	13.	A	14.	C	15.	A

Review Questions

1.Find the general solution for each of the following equations:

a. cos 4 x = cos 2 x
b. cos 3x + cos x – cos 2x = 0
c. sin 2x + cos x = 0
d. sec2 2x = 1– tan 2x
e. sin x + sin 3x + sin 5x = 0

2. Find the value of tan π /8.

3.Prove that

a. $\cos^2 x + \cos^2(x+\pi/3) + \cos^2(x-\pi/3) = 3/2$

b. (sin 3x + sin x) sin x + (cos 3x – cos x) cos x = 0

c.(cos x + cos y) ^{2} + (sin x – sin y) ^{2} = 4 cos^2(x +y)/2

4. Find sin x/2, cos x /2 and tan x /2 in each of the following:

a.tan x = − 4/3, x in quadrant II

b. cos x = − 1/ 3, x in quadrant III

c. sin x = 1/4, x in quadrant II

5. Convert the following radian measures into degrees.

a. 7π/3

b. π/6

c. π/18

d. 3π/4

6. Convert the following degree measures into radians

a. 45°

b. 510°

c. -240°

d. 0°

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