Basic Mathematics II

DMTH202

Edited by:
Richa Nandra

LOVELY PROFESSIONAL UNIVERSITY

BASIC MATHEMATICS-II

Edited By
Richa Nandra

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SYLLABUS

Basic Mathematics-II

Objectives: This course is the second course of the basic mathematics series and is designed to give the basic and necessary knowledge of integration and their application in daily life, formation of basic differential equations and elementary concept of Probability. After completion of this course student will be able to find the area of region enclosed by the curves by the method of integration, solve any first order differential equation and solve the basic problems of probabilities.

Sr. No.	Description
1.	Integration as inverse process of differentiation, Integration by substitution
2.	Integration by partial fraction and by parts
3.	Definite integral, evaluation of definite integrals by substitution
4.	Simple properties of definite integral
5.	Application in finding the area under simple curve and within two curves
6.	Formation of differential equation
7.	Solution of differential equation of first order and first degree by separation of variables
8.	Homogeneous equation and linear equation
9.	Permutation, Combinations
10.	Random Experiments, Event, Axiomatic Approach to Probability

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Unit 1: Integration

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Objectives

After studying this unit, you will be able to:

- Understand the Integration as inverse process of differentiation
- Illustrate the process of integration by Substitution

Introduction

Differential Calculus is centered on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions. If a function f is differentiable in an interval I, i.e., its derivative f 2 exists at each point of I, then a natural question arises that given f 2 at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the indefinite integral of the function and such process of finding anti derivatives is called integration. Such type of problems arises in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types: (a) the problem of finding a function whenever its derivative is given, (b) the problem of finding the area bounded by the graph of a function under certain conditions. These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the Integral Calculus. There is a connection, known

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as the Fundamental Theorem of Calculus, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering.

1.1 Integration as an Inverse Process of Differentiation

Integration is defined as the inverse process of differentiation. Rather than differentiating a function, we are provided the derivative of a function and requested to locate its primitive, i.e., the original function. This type of process is known as Integration or anti differentiation.

1.1.1 The Fundamental Theorem of Calculus

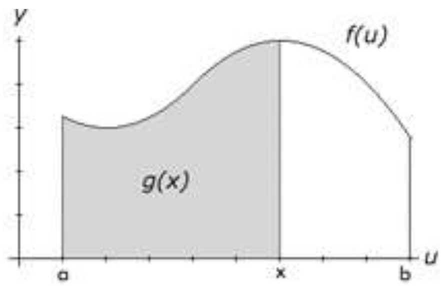
The Fundamental Theorem of Calculus identifies the relationship among the processes of differentiation and integration. That relationship says that differentiation and integration are inverse processes.

The Fundamental Theorem of Calculus: Part 1

If f is a continuous function on [a,b], then the function indicated by

$$g(x) = \int_a^x f(u)du \text{ for } x \in [a,b]$$

is continuous on [a,b], differentiable on (a,b) and g′(x) = f(x).



If f(t) is continuous on [a,b], the function g(x) which is equal to the area enclosed by the u-axis and the function f(u) and the lines u=a and u=x will be continuous on [a,b] and differentiable on (a,b). Most prominently, while we differentiate the function g(x), we will discover that it is equal to f(x). The above graph demonstrates the function f(u) and the area g(x).

The Fundamental Theorem of Calculus : Part 2

If f is a continuous function on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f.

If f is continuous on [a,b], the definite integral with integrand f(x) and limits a and b is just equal to the value of the **antiderivative** F(x) at b minus the value of F at a. This property permits us to simply resolve definite integrals, if we can locate the antiderivative function of the integrand.

Part one and part two of the Fundamental Theorem of Calculus can be combined as below.

Combining the Fundamental Theorem of Calculus Part 1 and Part 2

Let f be a continuous function on [a,b].

Notes

1. $If\ g(x) = \int_a^x f(u)du, \text{then } g'(x) = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a), \text{where } F \text{ is any antiderivative of } f$

As integration is the inverse process to differentiation. So rather than multiplying by the index and dropping the index by one, we enhance the index by one and divide by the new index. The + *C* occurs since the derivative of any constant term is zero.

C is known as the (arbitrary) **constant of integration**.

Did u know? The value of C can be instituted when suitable additional information is given, and this provides a specific integral.

The rule for integration is

$\int ax^n dx = \dfrac{ax^{n+1}}{n+1} + C$ provided $n \neq -1$.

Generally, $\int f'(x)dx = f(x) + C$ or $\int \left(\dfrac{dy}{dx}\right) dx = y + C$

The inverse relationship among differentiation and integration means that, for each statement regarding differentiation, we can write down an equivalent statement concerning integration.

Example: Find the equation of the curve for which $\dfrac{dy}{dx} = 4x^3 + 6x^2$ passes via the point *(1, 3)*.

Integrating provides

$y = \int (4x^3 + 6x^2)dx = x^4 + 2x^3 + C$

Substituting $x = 1$ and $y = 3$ offers $3 = 1 + 2 + C$, thus $C = 0$

$y = x^4 + 2x^3$

The subsequent step is, when we are specified a function to integrate, to execute rapidly via all the typical differentiation formulae, until we come to one which is suitable to our problem.

Alternatively, we have to learn to identify a specified function as the derivative of another function.

Task Given $\dfrac{d}{dx}(x^4) = 4x^3$, then evaluate $\int 4x^3 dx$.

Self Assessment

Fill in the blanks:

1. .... is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.

2. The functions that could possibly have given function as a derivative are called .... of the function.

Notes

3. .... is defined as the inverse process of differentiation.
4. The Fundamental Theorem of Calculus identifies the .... among the processes of differentiation and integration.
5. The + *C* occurs since the derivative of any constant term is ....
6. .... is known as the constant of integration.
7. The value of C can be instituted when suitable additional information is given, and this provides a specific ....
8. The .... relationship among differentiation and integration means that, for each statement regarding differentiation, we can write down an equivalent statement concerning integration.
9. $\frac{d}{dx}(\sin x) = \cos x$, so $\int \cos x\, dx =$.................... .

1.2 Integration by Substitution

Integration can be simply performed by means of integration formulas, If the integral is in the typical form where we can simply pertain formulas. But if the function which is to be integrated is not in the typical form then it is either tough or impracticable to utilize integration formulas to integrate. In that case we are required to use Integration by Substitution method to integrate a specified function.

In the process of Integration by substitution, we reduce an integral in non-standard form into a integral in standard form by altering the variable into a new variable with appropriate substitution.

1.2.1 The Guess-and-Check Method

The Guess-and-Check Method, a high-quality strategy for locating simple ant derivatives is to *guess* an answer (by means of knowledge of differentiation rules) and then *check* the answer by means of differentiating it.

Did u know? If we get the predictable result, then we're completed; or else, we rework the guess and check again.

The method of guess-and-check is functional in reversing the chain rule. The chain rule shows:

$$\frac{d}{dx}(f(g(x))) = \underbrace{f'}_{\text{Derivative of outside}}\overbrace{(g(x))}^{\text{inside}}.\ \underbrace{g'(x)}_{\text{Derivative of inside}}$$

Therefore, any function which is the consequence of applying the chain rule is the product of two factors: the "derivative of the outside" and the "derivative of the inside." If a function contains this form, its anti derivative is f (g(x)).

Example: Find $\int 3x^2 \cos(x^3)dx.$

Solution:

The function $3x^2\cos(x^3)$ appears as the result of applying the chain rule: there is an "inside" function x^3 and its derivative $3x^2$ occurs as a factor. As the outside function is a cosine which has a sine (x^3) as an ant derivative, we guess for the anti derivative. Differentiating to check provides.

Notes

$$\frac{d}{dx}(\sin(x^3)) = \cos(x^3).(3x^2)$$

As this is what we started with, we know that

$$\int 3x^2 \cos(x^3)\,dx - \sin(x^3) + C.$$

The fundamental thought of this method is to attempt to locate an inside function whose derivative occurs as a factor.

This functions even when the derivative is missing a constant factor, as in the subsequent example.

Example: Find $\int t\, e^{(t^2+1)} dt$.

Solution:

It appears like t^2 +1 is an inside function. So we guess $e^{(t^2+1)}$ for the anti derivative, as taking the derivative of exponential outcomes in the recurrence of the exponential jointly with other terms from the chain rule.

Now we check:

$$\frac{d}{dt}(e^{(t^2+1)}) - (e^{(t^2+1)})\, 2t.$$

The original guess was excessively large by a factor of 2. We modify the guess to $\frac{1}{2}e^{(t^2+1)}$ and check yet again:

$$\frac{d}{dt}\left(\frac{1}{2}e^{(t^2+1)}\right) - \frac{1}{2}e^{(t^2+1)}2t - e^{(t^2+1)}\,t$$

Therefore, we know that

$$\int t\, e^{(t^2+1)} dt = \frac{1}{2}e^{(t^2+1)} - C$$

Example: Find $\int x^3 \sqrt{x^4+5}\, dx$

Solution:

At this point the inside function is $x^4 + 5$, and its derivative occurs as a factor, with the exemption of a missing 4. Therefore, the integrand we have is more or less of the form

$$g'(x)\sqrt{g(x)}$$

with $g(x) = x^4 + 5$. As $x^{3/2}/(3/2)$ is an anti derivative of the outside function $\sqrt{x}$, we might guess that an anti derivative is

$$\frac{(g(x))^{3/2}}{3/2} = \frac{(x^4+5)^{3/2}}{3/2}$$

Let's check:

$$\frac{d}{dx}\left(\frac{(x^4+5)^{3/2}}{3/2}\right) = \frac{3}{2}\frac{(x^4+5)^{1/2}}{3/2}.4x^3 = 4x^3(x^4+5)^{1/2}$$

Notes

Thus $\frac{(x^4+5)^{3/2}}{3/2}$ is very big by a factor of 4. The exact anti derivative is

$$\frac{1}{4}\frac{(x^4+5)^{1/2}}{3/2}=\frac{1}{6}(x^4+5)^{3/2}$$

Thus

$$\int x^3\sqrt{x^4+5}\,dx=\frac{1}{6}(x^4+5)^{3/2}-C$$

As a concluding check:

$$\frac{d}{dx}\left(\frac{1}{6}(x^4+5)^{3/2}\right)-\frac{1}{6}.\frac{3}{2}(x^4+5)^{1/2}4x^4-x^3(x^4+5)^{1/2}$$

> *Notes* As we have observed in the previous examples, anti differentiating a function frequently includes "correcting for" constant factors: if differentiation generates an extra factor of 4, anti differentiation will need a factor of $\frac{1}{4}$.

1.2.2 The Method of Substitution

When the integrand is intricated, it assists to formalize this guess-and-check method as below:

To Make a Substitution

Let w be the "inside function" and $dw=w'(x)\,dx=\frac{dw}{dx}dx$.

Let's do again the first example by means of a substitution.

Example: Find $\int 3x^2\cos(x^3)\,dx$.

Solution:

As before, we gaze for an inside function whose derivative occurs—in this case x^3. Let $w = x^3$. Then $dw=w'(x)dx=3x^2dx$. The original integrand can now be entirely rewritten in terms of the new variable w:

$$\int 3x^2\cos(x^3)dx=\int cos\frac{(x^3)}{w}.\frac{3x^2dx}{dw}=\int\cos w\,dw=\sin w+C=\sin(x^3)-C$$

By altering the variable to w, we can shorten the integrand. We now have cos w, which can be anti differentiated more simply. The concluding step, after anti differentiating, is to convert back to the original variable, x.

1.2.3 Trigonometric Substitutions

As we know Substitutions permit us to solve complicated-looking integrals by translating them into something more manageable. Now, we shall scrutinize a different type of substitution: a trigonometric substitution, which will permit us to integrate more functions.

Notes

Before starting, however, let us remind some helpful trigonometric identities.

The first is the Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$. If we divide all terms by $\cos^2\theta$, then we have:

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2} \Leftrightarrow \tan^2\theta+1=\sec^2\theta .$$

The well-known trigonometric identities

$$a^2-a^2\sin^2(t) = a^2\cos^2(t) \qquad (1)$$
$$a^2-a^2\tan^2(t) = a^2\sec^2(t) \qquad (2)$$
$$a^2\sec^2(t) = a^2\tan^2(t) \qquad (3)$$

may be used to eradicate radicals from integrals. Particularly when these integrals entail $\sqrt{a^2 \pm x^2}$ and $\sqrt{x^2 \pm a^2}$.

1. For $\sqrt{a^2-x^2}$ set $x = a\sin(t)$. In this case we converse regarding sine-substitution.

2. For $\sqrt{a^2+x^2}$ set $x = a\tan(t)$. In this case we converse regarding tangent-substitution.

3. For $\sqrt{x^2+a^2}$ set $x = a\sec(t)$. In this case we converse regarding secant-substitution.

The expressions $a^2 \pm x^2$ and $x^2 \pm a^2$ should be observed as a constant plus-minus a square of a function. Here, x displays a function and a a constant. For example, x^2-2x+3 can be observed as one of the two previous expressions. Certainly, if we complete the square we obtain

$$x^2-2x+3=(x-1)^2+2$$

where $a^2 = 2$. Thus from the above substitutions, we will set $x-1=\sqrt{2}\tan(t)$.

The following examples exemplify how to apply trigonometric substitutions:

Example: Find $\int x^3\sqrt{4-x^2}\,dx$

Solution:

It is simple to observe that sine-substitution is the one to use. Set $x = 2\sin(t)$ or equivalently $t=\sin^{-1}(x/2)$. Then $dx = 2\cos(t)dt$ which provides us

$$\int x^3\sqrt{4-x^2}\,dx=\int 8\sin^2(t)\sqrt{4-4\sin^2(t)2}\cos(t)dt.$$

Easy calculations provide

$$\int x^3\sqrt{4-x^2}\,dx=32\int \sin^2(t)\cos^2(t)dt.$$

Technique of integration of powers of trigonometric functions give

$$\int \sin^3(t)\cos^2(t)dt=\int\left(1-\cos^2(t)\right)\cos^2(t)\sin(t)dt$$

which recommends the substitution $v=\cos(t)$. Hence $dv=-\sin(t)dt$ which implies

$$\int\left(1-\cos^2(t)\right)\cos^2(t)\sin(t)dt=-\int\left(1-v^2\right)v^2dv=-\frac{v^3}{3}+\frac{v^5}{5}+C.$$

Notes

Thus, we have

$$\int x^3\sqrt{4-x^2}\,dx = -32\frac{v^3}{3} + 32\frac{v^5}{5} + C.$$

This will not answer completely the problem since the answer should be given as a function of x. As $v = \cos(t) = \sqrt{1-\sin^2(t)} = \sqrt{1-(x/2)^2}$, we obtain after easy simplifications.

Example: Evaluate $\int_0^3 \sqrt{x^2+6xdx}$

Solution:

First let us complete the square for $x^2+6x.$ We obtain

$$x^2+6x = (x+3)^2 - 9$$

which recommends the secant-substitution $x + 3 = \sec(t)$. So we have $dx = 3\sec(t)\tan(t)$ and $x^2 + 6x = 9(\sec^2(t) - 1) = 9\tan^2(t)$. Observe that for $x = 0$, we have $\sec(t) = 1$ which provides $t = 0$ and for $x = 3$, we have $\sec(t) = 2$ which provides $t = \pi/3$. So, we have

$$\int_0^3 \sqrt{x^2+6x}\,dx = \int_0^{\pi/2} 3\tan(t)3\sec(t)\tan(t)\,dt = 9\int_0^{\pi/3}\tan^2(t)\sec(t)\,dt.$$

By means of the trigonometric identities, we obtain

$$\int_0^{\pi/3}\tan^2(t)\sec(t)\,dt = \int_0^{\pi/3}\left(\sec^2(t) - \sec(t)\right)dt.$$

The technique of integration connected to the powers of the secant-function provides $\int \sec(t)\,dt = \text{In}\left|\sec(t)+\tan(t)\right| + C$ and

$$\int \sec^3(t)\,dt = \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\text{In}\left|\sec(t)+\tan(t)\right| + C$$

which entails

$$\int_0^{\pi/3}\left(\sec^3(t) - \sec(t)\right)dt = \left[\frac{1}{2}\sec(t)\tan(t) - \frac{1}{2}\text{In}\left|\sec(t)+\tan(t)\right|\right]_0^{\pi/3}.$$

One would ensure easily that

$$\int_0^3 \sqrt{x^2+6x}\,dx = 9\int_0^{\pi/3}\tan^2(t)\sec(t)\,dt = \sqrt{3} - \frac{1}{2}\text{In}\left(2+\sqrt{3}\right).$$

Useful trigonometric identities:

$$\begin{aligned} 1-\sin^2(t) &= \cos^2(t) \\ 1+\tan^2(t) &= \sec^2(t) \\ \sec^2(t)-1 &= \tan^2(t) \end{aligned}$$

Example: Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx.$

Solution:

Observe that we cannot use a u-substitution.

Notes

If we let $u = 1 - x^2$, then $du = -2xdx$, but we don't have $-2xdx$.

Also, integration by parts will not function.

If we let $u = (1 - x^2)^{-1/2}$, then $du = x(1 - x^2)^{-3/2}dx$. $dv = dx$, and so $v = x$.

Putting it together, we have: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{x}{\sqrt{1-x^2}} - \int \frac{x^2}{\left(1-x^2\right)^{3/2}}\,dx$, which has only made the integral worse.

But what happens if we let $x = \sin\theta$. Observe that the quantity inside of the square root turns out t be $1 - \sin^2\theta = \cos^2\theta$, by the Pythagorean Identity above.

Also, observe that $dx = \cos\theta d\theta$. Unlike before, here we do not have to solve for dx; it is already given to use.

So, now we just plug everything in and solve.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \Rightarrow \frac{x}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta \Rightarrow \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta \Rightarrow \int \frac{\cos\theta}{\cos\theta}\,d\theta \Rightarrow \int d\theta \Rightarrow \theta + C$$

The question remains, what is θ? We need to solve for θ in terms of x.

If we gaze above, we observe such a relationship. Recall, we had let $x = \sin\theta$. That means that $\theta = \sin^{-1}(x)$ or $\arcsin(x)$.

So, we have that $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C.$

Example: Evaluate $\int \frac{1}{x^2+4x+13}\,dx.$

First we complete the square in the denominator by observing that $13 = 4 + 9$ and then that provides us $x^2 + 4x + 4 + 9 = (x + 2)^2$.

$$\int \frac{1}{x^2+4x+13}\,dx = \int \frac{1}{x^2+4x+4+9}\,dx = \int \frac{1}{(x+2)^2\,3^2}\,dx.$$

Then we let $x + 2 = 3\tan\theta$. Observe that $dx = 3\sec^2\theta d\theta$.

After substituting, we have

$$\int \frac{1}{(x-2)^2+3^2}\,dx \int \frac{3\sec^2\theta d\theta}{(3\tan\theta)^2+9} = \int \frac{3\sec^2\theta d\theta}{9\tan^2\theta+9} = \int \frac{3\sec^2\theta d\theta}{9\sec^2\theta} = \int \frac{1}{3}\,d\theta = \frac{1}{3}\theta + C.$$

Solving for θ, we have $\theta = \arctan\left(\frac{x+2}{3}\right).$

Thus, $\int \frac{1}{x^2 4x+13}\,dx = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C.$

Example: Evaluate $\int \frac{dx}{d^2\sqrt{4-x^2}}$

Notes

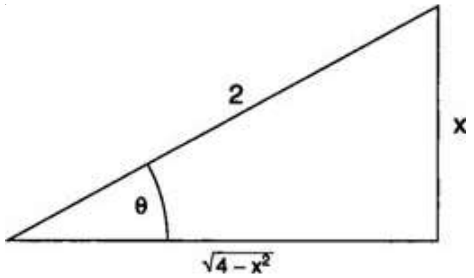
Solution:

Since the radical has the form $\sqrt{a^2 - x^2}$

let $x = a \sin\theta = 2 \sin\theta$

$dx = 2 \cos\theta d\theta$

and $\sqrt{4 - x^2} = 2\cos\theta$



hence,
$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta d\theta}{(4\sin^2\theta)(2\cos\theta)}$$
$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$
$$= \frac{1}{4}\int \sec^2\theta d\theta$$
$$= \frac{1}{4}\cot\theta + C$$
$$= \frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$
$$= \frac{\sqrt{4-x^2}}{4x} + C$$

Example: Evaluate $\int \frac{dx}{\sqrt{25+x^2}}$

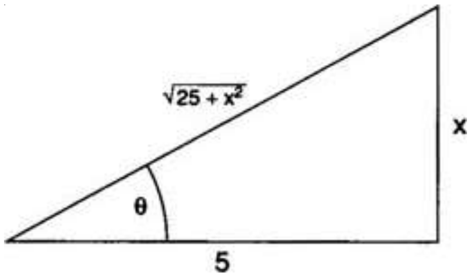
Solution:

Since the radical has the form $\sqrt{a^2 + x^2}$

let $x = a \tan\theta = 5 \sin\theta$

$dx = 5 \sec^2\theta d\theta$

and $\sqrt{25 - x^2} = 5\cos\theta$



hence,
$$\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{5\sec^2\theta d\theta}{5\sec\theta}$$
$$= \int \sec\theta d\theta$$
$$= \text{In}\left|\sec\theta + \tan\theta\right| + C$$
$$= \text{In}\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C$$

1.2.4 Why does Substitution Work?

The substitution technique makes it appear as if we can treat *dw* and *dx* as disconnected entities, even canceling them in the equation *dw* = (*dw/dx*)*dx*. Now we illustrate how this works. Let us have an integral of the form $\int f(g(x))g'(x)dx$, where g(x) is the inside function and *f*(x) is the outside function. If *F* is an antiderivative of *f*, then *F'* = *f*, and by means of chain rule $\frac{d}{dx}(F(g(x))) = f(g(x))g'(x)$.

Thus,

$$\int f(g(x))g'(x)dx - F(g(x)) + C$$

Now write *w*=*g(x)* and *dw/dx*=*g'(x)* on both sides of this equation:

$$\int f(w)\frac{dw}{dx}dx - F(w) + C$$

Conversely, knowing that *F'* = *f* illustrates that

$$\int f(w)dx - F(w) + C$$

So, the following two integrals are equal:

$$\int f(w)\frac{dw}{dx}dx - \int F(w)\,dw$$

Substituting *w* for the inside function and writing *dw* = *w'(x)dx* leaves the indefinite integral unchanged.

Let's return to the second example that we did by guess-and-check.

Example: Find $\int t\,e^{(t^2+1)}dt$.

Solution:

Here the inside function is t^2 +1, with derivative 2*t*. As there is a factor of *t* in the integrand, we try

$$w = t^2 + 1$$

So

dw=*w'(t)dt* = *2t dt*

Observe, though, the original integrand has only *t dt*, not *2t dt*. We then write $\frac{1}{2}aw = t\,dt$ and then substitute:

Notes

$$\int t\,e^{(t^2+1)}dt - \int e^{\overbrace{(t^2+1)}^{w}}\underbrace{t\,dt}_{\frac{1}{2}dw} - \int e^{w}\frac{1}{2}dw - \frac{1}{2}\int e^{w}dw - \frac{1}{2}e^{w} + C - \frac{1}{2}e^{(t^2+1)} + C$$

This provides the similar answer as we found by means of guess-and-check.

Why didn't we put $x\frac{1}{2}\int e^{w}dw - \frac{1}{2}e^{w} + \frac{1}{2}C$ in the preceding example? As the constant C is arbitrary, it doesn't actually matter whether we add C or $\frac{1}{2}C$. The convention is always to add C to whatever antiderivative we have computed.

Now let's do again the third example that we solved formerly by guess-and-check.

Example: Find $\int x^3\sqrt{x^4+5}\,dx$.

Solution:

The inside function is $x^4 + 5$, with derivative $4x^3$. The integrand has a factor of x^3, and as the only thing missing is a constant factor, we attempt

$w = x^4 + 5$

Then

$$dw = w'(x)dx = 4x^3dx$$

Giving

$$\frac{1}{4}aw = x^3dx$$

So,

$$\int x^3\sqrt{x^4+5}\,dx - \int\sqrt{w}\,\frac{1}{4}dw - \frac{1}{4}\int w^{1/2}dw - \frac{1}{4}.\frac{w^{3/2}}{3/2} + C - \frac{1}{6}(x^4+5)^{3/2} - C$$

Again, we get the similar result as with guess-and-check.

Caution We can apply the substitution method when a **constant** factor is absent from the derivative of the inside function. Though, we may not be able to utilize substitution if anything other than a constant factor is missing. For example, setting $w = x^4 - 5$ to find

$$\int x^2\sqrt{x^4+5}\,dx$$

does us no good since $x^2\,dx$ is not a constant multiple of $dw=4x^3\,dx$. Substitution functions if the integrand encloses the derivative of the inside function, *to within a constant factor*.

Some people favor the substitution method over guess-and-check as it is more systematic, but both methods attain the same result. For uncomplicated problems, guess-and-check can be quicker.

Example: Find $\int e^{\cos\theta}\sin\theta\,d\theta$.

Solution:

Consider $w = \cos\theta$ as its derivative is $-\sin\theta$ and there is a factor of $\sin\theta$ in the integrand. This provides

$dw = w'(\theta)\, d\theta = -\sin\theta\, d\theta$,

So

$-dw = \sin\theta\, d\theta$

Therefore

$$\int e^{\cos\theta} \sin\theta\, d\theta - \int e^{w}(-dw) - (-1)\int e^{w}\, dw - -e^{w} + C - -^{\theta} e^{\cos} \;\; + C$$

Example: Find $\int \frac{e^t}{1+e^t} dt$.

Solution:

Observing that the derivative of $1 + e^t$ is e^t, we notice $w = 1 + e^t$ is a good quality choice. Then $dw = e^t\, dt$, so that

$$\int \frac{e^t}{1+e^t} dt = \int \frac{1}{1+e^t} e^t\, dt = \int \frac{1}{w} dw = 1n|w| + C$$

$$= 1n\left|1+e^t\right| + C$$

As the numerator is e^t dt, we might also have attempted $w = e^t$. This substitution leads to the integral $\int (1/(1+w))dw$, which is better than the original integral but needs another substitution, $u = 1 + w$, to terminate. There are frequently several different methods of doing an integral by substitution.

> *Notes* Observe the pattern in the preceding example: having a function in the denominator and its derivative in the numerator shows a natural logarithm. The next example follows the similar pattern.

Example: Find $\int \tan\theta\, d\theta$.

Solution:

Remember that $\tan\theta = (\sin\theta)/(\cos\theta)$. If $w = \cos\theta$, then $dw = -\sin\theta\, d\theta$, so

$$\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta = \int \frac{-dw}{w} = -1n|\cos\theta| - C$$

Example: Compute $\int_0^2 xe^{x^2} dx$

Solution:

To assess this definite integral by means of the Fundamental Theorem of Calculus, we first want to find an antiderivative of $f(x) = xe^{x^2}$. The inside function is x^2, so we consider $w = x^2$. Then $dx = 2x\, dx$, so

$$\frac{1}{2} dx = x dx.$$

Notes

Thus,

$\int \sec^2 x\, dx$

Now we locate the definite integral

$$\int xe^{x^2} dx = \frac{1}{2}e^{x^2}\Big|_0^2 - \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1).$$

There is another method observe at the same problem. After we recognized that

$$\int xe^{x^2} dx = \frac{1}{2}e^w + C,$$

our next two steps were to substitute w by x^2, and then x by 2 and 0. We could have directly substituted the original limits of integration x = 0, and x = 2, by the analogous w limits. As $w = x^2$, the w limits are $w = 0^2 = 0$ (when x = 2) and $w = 2^2 = 4$ (when x = 2), so we get

$$\int_{x=0}^{x=2} xe^{x^2} dx - \frac{1}{2}\int_{w=0}^{w=4} e^w dx - \frac{1}{2}e^w\Big|_0^4 - \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1).$$

As we would guess, both methods provide the similar answer.

Task Evaluate $\int x^2\sqrt{1+x^3}dx$ by using substitution.

1.2.5 More Complex Substitutions

In the examples of substitution illustrated until now, we guessed an expression for w and expected to find (or some constant multiple of it) in the integrand. What if we are not so fortunate? It turns out that it frequently works to let w be some muddled expression contained inside, say, a cosine or under a root, even if we cannot see instantly how such a substitution assists.

Example: Find $\int\sqrt{1+\sqrt{x}}\,dx$.

Solution:

Here, the derivative of the inside function is nowhere to be observed. Yet, we try $w = 1+\sqrt{x}$.

Then

$w = 1+\sqrt{x}$, so$(w-1)^2 = x$, so.

Thus 2(w – 1 dw = dx).

We have

$$\begin{aligned}\int\sqrt{1+\sqrt{x}}\ dx &= \int\sqrt{w2}(w-2)dw = 2\int w^{1/2}(w-1)\\ &= 2\int(w^{3/2} - w^{1/2})dx = 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + C\\ &= 2\left(\frac{2}{5}\left(1+\sqrt{x}\right)^{5/2} - \frac{2}{3}\left(1+\sqrt{x}\right)^{3/2}\right) + C\end{aligned}$$

Notes

Observe that the substitution in the previous example again converts the inside of the messiest function into something easy.

Caution As the derivative of the inside function is not waiting for us, we have to solve for x so that we can obtain dx completely in terms of w and dw.

Self Assessment

Fill in the blanks:

10. In the process of Integration by............................., we reduce an integral in non-standard form into a integral in standard form by altering the variable into a new variable with appropriate substitution.
11. The .... Method, a high-quality strategy for locating simple ant derivatives is to *guess* an answer (by means of knowledge of differentiation rules) and then *check* the answer by means of differentiating it.
12. The method of guess-and-check is functional in .... the chain rule.
13. Any function which is the consequence of applying the chain rule is the .... of two factors: the "derivative of the outside" and the "derivative of the inside."
14. .... a function frequently includes "correcting for" constant factors.
15. We can apply the substitution method when a constant factor is absent from the derivative of the .... function.

1.3 Summary

- Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.
- The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function.
- The formula that gives all the anti derivatives is called the indefinite integral of the function and such process of finding anti derivatives is called integration.
- There is a connection, known as the Fundamental Theorem of Calculus, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering.
- The Fundamental Theorem of Calculus defines the relationship among the processes of differentiation and integration.
- The inverse relationship among differentiation and integration means that, for each statement about differentiation, we can write down a matching statement regarding integration.
- In the method of Integration by substitution, we reduce a integral in non-standard form into a integral in standard form by altering the variable into a new variable with appropriate substitution.
- The Guess-and-Check Method is a high-quality strategy for locating simple ant derivatives is to *guess* an answer (by means of knowledge of differentiation rules) and then *check* the answer by differentiating it.

Notes

- Some people favor the substitution method over guess-and-check as it is more methodical, but both methods attain the similar result.

1.4 Keywords

Anti-derivatives: The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function.

Guess and Check method: The Guess-and-Check Method is a high-quality strategy for locating simple ant derivatives is to *guess* an answer (by means of knowledge of differentiation rules) and then *check* the answer by differentiating it.

Integration: The formula that gives all the anti derivatives is called the indefinite integral of the function and such process of finding anti derivatives is called integration.

1.5 Review Questions

1. Illustrate how integration can be defined as an Inverse Process of Differentiation.
2. Depict the concept of the Fundamental Theorem of Calculus with examples.
3. What is Guess-and-Check Method? Illustrate with examples.
4. Use substitution to state each of the following integrals as a multiple of $\int_a^b (1/w)dw$ for some a and b. Then evaluate the integrals.

   (a) $\int_0^1 \frac{x}{1+x^2} dw$ (b) $\int_0^{x/4} \frac{\sin x}{\cos x} dw$
5. Find the antiderivatives of:

   (a) $x\sin(x^2+1)$ (b) $x^2\sin(x^3+1)$
6. If appropriate, assess the integral $\int x\sin(x^2)dx$ by substitution. If substitution is not appropriate, mention it, and do not assess.
7. If suitable, assess the integral $\int \frac{x^2}{1+x^2}dx$ by substitution. If substitution is not suitable, mention it, and do not assess.
8. Find $\int 4x(x^2+1)dx$ by means of two methods:

   (a) Do the multiplication first, and then anti-differentiate.

   (b) Use the substitution $w = x^2$.

   (c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?
9. Evaluate $\int(-\sin x)dx$ by using the method of substitution.
10. Evaluate $\int \sec^2 x\, dx$ by using the method of substitution.

Answers: Self Assessment

Notes

1. Integral Calculus
2. anti derivatives
3. Integration
4. relationship
5. zero
6. C
7. integral
8. inverse
9. sin $x + c$.
10. substitution
11. Guess and Check
12. reversing
13. product
14. Anti differentiating
15. inside

1.6 Further Readings

Books

Douglas S. Kurtz, Jaroslav Kurzweil, Charles Swartz, *Theories of Integration*, World Scientific.

G. H. Hardy, T. W. Körner, *A Course of Pure Mathematics*, Cambridge University Press.

Morris Kline, *Calculus: An Intuitive and Physical Approach*, Courier Dover Publications.

Ron Larson, David C. Falvo, *Calculus: An Applied Approach*, Cengage Learning

Online link

http://mathsci.ucd.ie/modules/math1200/calculus/notes-20.pdf

Unit 2: Integration by Partial Fraction

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of integration by partial fraction
- Discuss the partial fraction theorem
- Identify how to find the coefficients in the partial fraction expansion

Introduction

We recognize the process to integrate polynomials and negative power of x–a. By the method of "partial fractions" we can translate any rational function into a polynomial and fractions each one with negative powers of just one factor (x–a); this permits us to integrate any rational function, when we identify the process to factor its denominator entirely.

2.1 Integration by Partial Fractions

A rational function is defined as the proportion of two polynomials in the form of $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If the degree of $P(x)$ is less than the degree of $Q(x)$,

Notes

then the rational function is known as proper, or else, it is known as improper. The improper rational functions can be abridged to the proper rational functions by long division procedure.

Caution The degree of the numerator ought to be less than the degree of the denominator.

2.1.1 Idea of Method of Partial Fractions

Here we show general rational function P(x)/Q(x) as a sum of integrable terms.

Assume that the degree of P(x) is p and that of Q(x) is q. If p is at least q, we can hide P by Q (through synthetic, i.e., long division) to attain a polynomial, D(x), and a remainder, R(x)/Q(x), along with the degree, r, of R less than q.

The fraction R(x)/Q(x) then moves toward 0 as YxY augments in every path in the multifaceted plane.

Each polynomial can be factored into linear factors (if complex numbers can occur in the factors). As a result we get:

$$Q(x) = (x - q_1)^{m_1}(x - q_2)^{m_2} \cdots (x - q_k)^{m_3}$$

where m_k is the multiplicity of the k^{th} root of Q(x); the sum's of the m's being q.

Caution If the rational function is improper, make use of "long division" of polynomials to carve it as the sum of a polynomial and a proper rational function "remainder."

2.1.2 The Partial Fraction Theorem

You can consider R(x) / Q(x) as a sum over the roots q_j of the terms of the form

$$\frac{a_{j0} + a_{j1}(x - q_j) + \cdots a_{j(m_j-1)}(x - q_j)^{m_j-1}}{(x - q_j)^{m_j}}$$

for suitable constants a_{jk}.

Notes Observe that this theorem permits us to integrate P(X) / Q(x); we are required only to integrate the polynomial D(x) and the different inverse powers appearing in this sum, (presuming we can calculate the a_{jk} here.)

Proof of the Partial Fraction Theorem

Let Q(a) = 0 and the root a of this equation contains multiplicity k.

Next, we have Q(x) = $(x - a)^k$Z(x) and Z(a) = c for non-zero c.

Let consider further that R(a) = d for non-zero d. (or else we could factor (x – a) out of both R and Q and lessen their degrees.)

Then $\frac{R(x)}{Q(x)} - \frac{d}{c}(x-a)^{-k}$ acts at worst like $u(x-a)^{-(k-1)}$ at x = a, since the rational function $\frac{R(x)}{Z(x)} - \frac{d}{c}$ disappears at x = a, and must consequently have a factor (x – a) in it.

Notes

Did u know? By substraction of a suitable multiple of an suitable inverse power, we can attain a rational function that is less singular than R(x) / Q(x) at an random root of Q

Notes If we persist such substraction until we have detached all the singularities of R(x) / Q(x) at all the roots, we will be left with a rational function that still disappears at infinity, and now contains no finite singularities. The only such function is 0; so that R(x) / Q(x) must equal the sum of the substractions; which statement is the theorem.

Self Assessment

Fill in the blanks:

1. By the method of "................................" we can translate any rational function into a polynomial and fractions each one with negative powers of just one factor (x–a).
2. A .... function is defined as the proportion of two polynomials in the form of P(x)/Q(x) , where P (x) and Q(x) are polynomials in x and Q(x) $\neq 0$.
3. If the degree of P(x) is less than the degree of Q(x), then the rational function is known as ....
4. The improper rational functions can be abridged to the proper rational functions by .... procedure.
5. Each polynomial can be factored into .... factors.
6. The degree of the numerator must be .... than the degree of the denominator.
7. The partial fraction theorem permits us to integrate P(X) / Q(x); we are required only to integrate the polynomial D(x) and the different .... powers appearing in this sum.

2.2 Finding the Coefficients in the Partial Fraction Expansion

Here, we will find the a_{jk} in the expression:

$$(*)\frac{R(x)}{Q(x)} = \sum_j \frac{a_{j0} + a_{ji}(x-q_j) + \cdots a_{j(m_j-1)}(x-q_j)^{m_j-1}}{(x-q_j)^{m_j}}$$

There are four different methods to do this.

2.2.1 Method 1: Expansion

Consider $Z_k(x) = \dfrac{Q(x)}{(x-q_j)^{m_j}}$

Use the constant, linear, quadratic or higher approximation to $\dfrac{R(x)}{Z_k(x)} at\ x = q_j$, to attain:

$a_{jk} = \dfrac{1}{k!}\dfrac{d^k}{dx^k}\dfrac{R(x)}{Z_j(x)}$ at $x = q_j$.

constant, linear, quadratic or higher approximation

Notes

Example: of method 1

Articulate $\frac{5x^2 2x - 7}{(x-2)^2(x-3)}$ in integrable form:

$$\frac{a_{10} + a_{11}(x-2)}{(x-2)^2} + \frac{a_{20}}{x-3}$$

Deduce $Z_1(x) = x - 3, Z_2(x) = (x-2)^2$

Use formulae to obtain

$$a_{10} = \frac{R(2)}{Z_1(2)} = \frac{5*4+2*2-7}{-1} = -17$$
$$a_{11} = \frac{R'(2)Z_1(2) - Z_1'(2)R(2)}{Z_1^2(2)} = \frac{(20+2)(-1)-1(17)}{(-1)^2} = -39$$
$$a_{20} = \frac{R(3)}{Z_2(3)} = \frac{5*9+2*3-7}{1} = 44$$

Task Evaluate the integral $\int \frac{x}{x^2+4x+4}dx$ using partial fraction expansions.

2.2.2 Method 2: Cover Up

Mimic proof of theorem:

Here, set $R_{j0} = R, k = 0$.

1. Deduce: $a_{jk} = \frac{R_{jk}(q_j)}{Z_j(q_j)}$

2. Set $R_{j(k+1)}x = \frac{R_{jk}(x) - a_{jk}Z_j(x)}{(x-q_j)}$

3. Set $k = k + 1$, go to step 1.

Example: of method 2

In the preceding example, you figured out a_{10} and a_{20} as before but get a_{11} by replacing R(x) by R_{11}(x) provided by:

$$R_{11}(x) = \frac{R(x) - a_{10}Z_1(x)}{x - q_1}$$
$$= \frac{5x^2 + 2x - 7 - (-17)(x-3)}{x-2}$$
$$= 5x + 29$$
$$a_{11} = \frac{R_{11}(2)}{Z_1(2)} = -39$$

Notes

2.2.3 Method 3: Evaluate and Solve Equations

Assess both sides of equation (*) at r points where r is the number of unidentified coefficients.

Setting the sides equal at these points provide k linear equations for these unknowns.

Solve them. (Suitable points to select are usually 0,1,–1, or near infinity.)

Example: of method 3

Considering that $a_{10} = -17$; $a_{20} = 44$, assess both sides near infinity. The left side appears as $\frac{5}{x}$, the right side appears as $\frac{44 + a_{12}}{x}$; conclude $a_{12} = -39$.

To verify, assess at some other point and ensure the sides are equivalent there.

set $x = 0$;

$$\begin{aligned} \text{LHS} &= \frac{-7}{-12} \\ &= \frac{7}{12} \\ \text{RHS} &= \frac{-17}{4} - \frac{39}{-2} + \frac{44}{-3} \\ &= \frac{-51 + 234 - 176}{12} = \frac{7}{12} \end{aligned}$$

2.2.4 Method 4: Common Denominator

Write both sides of (*) as polynomials aided by Q(x).

The coefficients in these polynomials of every power of x must consent; these provide linear equations for the unknown. Solve them.

Example: of method 4

Find

$$5x^2 + 2x - 7 = a_{10}(x-3) + a_{11}(x-2)(x-3) + a_{20}(x-2)^2$$

Deduce

$$\begin{aligned} 5 &= a_{11} + a_{20} \\ 2 &= a_{10} - 5a_{11} - 4a_{20} \\ -7 &= -3a_{10} + 6a_{11} + 4a_{20} \end{aligned}$$

Solve these equations.

Examples of the methods discussed above

Notes

Example:

$$P(x) = 7x^2 + 6x + 2,\ Q(x) = x(x+1)(x+2)$$

By first method, you obtain

$$q_1 = 0,\ q_2 = -1,\ q_3 = -2$$
$$R(x) = P(x)$$
$$Z_1(x) = (x+1)(x+2),\ Z_2(x) = x(x+2),\ Z_3(x) = x(x+1)$$
$$a_{10} = \frac{R(0)}{Z_1(0)} = \frac{2}{2} = 1$$
$$a_{20} = \frac{R(-1)}{Z_2(-1)} = \frac{3}{-1} = -3$$
$$a_{30} = \frac{R(-2)}{Z_3(-2)} = \frac{28-12+2}{2} = 9$$
$$\frac{P(x)}{Q(x)} = \frac{1}{x} - \frac{3}{x+1} + \frac{9}{x+2}$$

Integrating, you obtain

$$\int \frac{P(x)}{Q(x)}\,dx = \ln\left(\frac{x(x+2)^9}{(x+1)^3}\right) + C$$

Example:

$$P(x) = 2x + 1,\ Q(x) = x(x-1)^2$$
$$q_1 = 0,\ q_2 = 1$$
$$R(x) = P(x)$$
$$Z_1(x) = (x-1)^2,\ Z_2(x) = x$$
$$a_{10} = \frac{R(0)}{Z_1(0)} = 1$$
$$a_{20} = \frac{R(1)}{Z_2(1)} = \frac{3}{1} = 3$$

Method 1:

$$a_{21} = \left(\frac{R(x)}{Z_2(x)}\right)' \text{ at } x = 1$$
$$= -\frac{1}{1^2} = -1$$

Method 2:

$$R_{21}(x) = \frac{R(x) - a_{20}Z_2(x)}{x-1} = \frac{2x+1-3x}{x-1}$$
$$\frac{R_{21}(1)}{Z_2(1)} = -1$$

Notes

Method 3:

Verify at x near infinity

LHS is near $\frac{2}{x^2}$

RHS is near $\frac{a_{10}}{x} + \frac{a_{20}}{x^2} + \frac{a_{21}}{x}$

conclusion: $a_{10} + a_{21} = 0$

you get: $a_{21} = -a_{10} = -1$

Self Assessment

Fill in the blanks:

8. After having determined the right outline for the partial fraction decomposition of a rational function, we are required to calculate the .... coefficient.

9. The method that uses the constant, linear, quadratic or higher approximation is known as ....

10. Assess both sides of equation (*) at r points where r is the number of unidentified coefficients is included in .... method.

2.3 Partial Fractions

Every proper rational fraction cab be expressed as a sum of simple fractions whose denominators are of the form $(9x+b)^n$ and $(9x^2+bx+c)^n$, n being a positive integer.

2.3.1 Distinct Linear Factors

To each linear factor ax+b occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to the determined.

2.3.2 Repeated Linear Factors

To each linear factors ax+b occurring n times in the denominator of a proper rational fraction. These corresponds a sum of n partial fractions of the form.

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ----+ \frac{A_n}{(ax+b)^n}$$

where the A's are constants to be obtained of course $A_n \neq 0$

2.3.3 Distinct Quadratic Factors

To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a simple partial fraction of the form $\frac{Ax+B}{9x^2+bx+c}$ where A and b are constants to be determined.

2.3.4 Repeated Quadratic Factors

Notes

To each irreducible quadratic factor, occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form.

$$\frac{A_1x+B_1}{9x^2+bx+c}+\frac{A^2x+b}{(9x^2+bx+c)^2}+----------+\frac{A^2x+b}{(9x^2+bx+c)^2}$$

Task Make distinction between Distinct Linear Factors and Repeated Linear Factors.

Example: Evaluate $\int \frac{x-5}{x^2-5x+6}dx$

Solution:

Let $\frac{x-5}{(x-2)(x-3)}=\frac{A}{(x-2)}+\frac{B}{(x-3)}$

$\therefore x-5=A(x-2)+B(x-3)$ ....(1)

Putting $x=3$ in (1)

$$-3=-B \Rightarrow B=3$$

Now, $\int \frac{x-5}{x^2-5x+6}dx=\int\left(\frac{-2}{(x-2)}+\frac{3}{(x-3)}\right)dx$

$$=-2\log(x-2)+3\log(x-3)$$
$$=-\log(x-2)^2+\log(x-3)^3$$
$$=\log\frac{(x-3)^3}{(x-2)^2}$$

Example: Evaluate $\int \frac{1}{x(x^2-1)}dx$

Solution:

Let $\int \frac{1}{x(x^2-1)}dx=\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{x+1}$

$\therefore 1=A(x-1)(x+1)+Bx(x+1)+cx(x-1)$

Putting x = 0, then

$$1=A(-1)(1)\Rightarrow A=-1$$

Putting x=a, then

$$1=B.1.(2)\Rightarrow B=\frac{1}{2}$$

Notes

Hence $\int \frac{1}{x(x^2-1)}dx = \int \left(\frac{1}{x}+\frac{1}{2}.\frac{1}{(x-1)}+\frac{1}{2}.\frac{1}{(x-1)}\right)dx$

$$= \log x + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1)$$
$$= \log x\sqrt{x-1}\sqrt{x+1}$$
$$= \log(x\sqrt{x^2-1})$$

Example: Evaluate $\int \frac{x}{(x+1)(x-1)}dx$

Solution:

Let $\int \frac{x}{(x+1)(x-1)}dx = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$\therefore x = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$

$x = A(x^2+4x+4) + B(x^2+3x+2) - 1(x+1)$

comparing the coefficients of x both the sides

$A + B = 0$ ....(1)

$4A + 3B + 2C = 1$ ....(2)

$4A + 2B + C = 0$ ....(3)

on solving equation (1), (2), (3), we get

A = – 1, B =1, C = 1

Hence $\int \frac{dx}{(x+1)(x+2)^2} = -\int \frac{1}{x+1}dx + \int \frac{1}{(x+2)^2}dx$

$$= \log(x+1) + \log(x+2) - \frac{1}{(x+2)}$$
$$= \log\frac{(x+2)}{(x+1)} - \frac{1}{(x+2)} Ans.$$

Example: Evaluate $\int \frac{x}{(x-1)(x^2+4)}dx$

Solution:

Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

Or $x = A(x^2+4) + (Bx+C)(x-1)$

$x = A(x^4+4) + B(x^2+x)C(x-1)$

comparing the coefficients of x both side

Notes

$A + B = 0$ ....(1)

$-B + C = 1$ ....(2)

$4A - C = 0$ ....(3)

adding (1) and (2) we get

$A + C = 1$ ...(4)

Adding (3) and (4) we get

5A=1 $\Rightarrow$ a=1/5

$$\therefore B = -\frac{1}{5}$$

$$0rC = \frac{6}{5}$$

Hence $\int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\int \frac{1}{(x-1)}dx - \frac{1}{5}\int \frac{x-6}{x^2+4}$

$$= \frac{1}{5}\int \frac{1}{x-1}dx - \frac{1}{5}\int \frac{x}{x^2+4}dx + \frac{6}{5}\int \frac{1}{x^2+4}dx$$
$$= \frac{1}{5}\log(x-1) - \frac{1}{5}.\frac{1}{2}\log(x^2+4) + \frac{6}{5x^2}\tan^{-1}\frac{x}{2}$$
$$= \frac{1}{5}\log(x-1) - \frac{1}{10}\log(x^2+4) + \frac{3}{5}\tan^{-1}x/2$$

Example: Evaluate $\int \frac{x^4 - x^2 - x - 1}{x^3 - x^2}dx$

Solution:

Here the integrand is an improper fraction therefore we divide the numerator by the denominator and obtain

$$\frac{x^4 - x^2 - x - 1}{x^3 - x^2} = x - \frac{x+1}{x_3 - x^2} = x - \frac{x+1}{x^2(x-1)}$$

How, let

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\therefore x + 1 = Ax(x-1) + B(x-1) + Cx^2$$

on comparing the coefficients of x, we get

$A + C = 0$ ....(1)

$B = 1$

$-A-B = 1$

$-A = 2 \rightarrow A = -2$

Notes

$$\frac{x^4 - x^2 - x - 1}{x^3 - x^2}dx = \int\left\{x - \left(\frac{-2}{x} + \frac{1}{x^2} + \frac{2}{x-1}\right)\right\}dx$$

$$= \int x dx + 2\int\frac{1}{x}dx - \int x^{-2}\, dx - 2\int\frac{dx}{x-1}$$

$$= \frac{x^2}{2} + 2\log x + \frac{1}{x} - 2\log(x-1)$$

$$= \frac{x^2}{2} - \frac{1}{x} + 2\log\frac{x}{x-1}$$

Did u know? The integrand is an improper rational function. By "long division" of polynomials, we can rephrase the integrand as the sum of a polynomial and a proper rational function "remainder":

Example: Evaluate $\int\frac{x^2+2}{(x-1)(x-2)^3}dx$

Solution:

Let $y = x - 2 \Rightarrow x = y \pm 2$

And $dx = dy$

$$\therefore \int\frac{(y+2)^2+2}{(y+2-1)y^3}dy$$

$$= \int\frac{y^2+4y+6}{y^5(y+1)}dy$$

How let $\dfrac{y^2+4y+6}{y^3(y+1)} = \dfrac{A}{y} + \dfrac{B}{y^2} + \dfrac{C}{y^3} + \dfrac{D}{y+1}$

$$y^2 + 4y + 6 = Ay^2(y+1) + By(y+1) + C(y+1) + Dy^3$$

Putting $y = 0$

$6 = C \Rightarrow C = 6$

Again putting $y = -1$

$1 - 4 + 6 = -D \Rightarrow D = -3$

$$y^2 + 4y + 6 = A(y^3 + y^2) + B(y^2 + y) + 6(y+1) - 3y^3$$

Comparing the coefficients of y for $A - 3 = 0$ $A = 3$

$A + B = 1 \Rightarrow B = -2$

Here $\displaystyle\int\frac{x^2+2}{(x-1)(x-2)^3}dx = \int\left\{\frac{3}{(x-2)} - \frac{2}{(x-2)^2} + \frac{6}{(x-2)} - \frac{3}{x-1}\right\}dx$

Notes

$$= 3\int\frac{dx}{x-2} - 2\int(x-2)^{-2}dx + 6\int(x-2)^{-3}dx - 3\int\frac{dx}{x-1}$$

$$= 3\log(x-2) + \frac{2}{(x-2)} - \frac{3}{(x-2)^2}$$

$$= 3\log\frac{x-2}{x-1} + \frac{2x+7}{(x-2)^2}$$

Self Assessment

Fill in the blanks:

11. Every proper rational fraction can be expressed as a sum of .... fractions.
12. To each linear factor ax+b occurring once in the denominator of a proper rational fraction, there corresponds a .... partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to the determined.
13. In case of .... Linear Factors, to each linear factors ax+b occurring n times in the denominator of a proper rational fraction.
14. In case of Distinct Quadratic Factors, to each .... quadratic factor occurring once in the denominator of a proper rational fraction, there corresponds a simple partial fraction of the form $\frac{Ax+B}{9x^2+bx+c}$ where A and b are constants to be determined.
15. In case of Repeated Quadratic Factors, to each irreducible quadratic factor, occurring n times in the denominator of a proper rational fraction, there corresponds a .... of n partial fractions.

2.4 Summary

- By the method of "partial fractions" we can translate any rational function into a polynomial and fractions each one with negative powers of just one factor (x–a).
- Each polynomial can be factored into linear factors (if complex numbers can occur in the factors).
- The partial fraction theorem permits us to integrate P(X) / Q(x); we are required only to integrate the polynomial D(x) and the different inverse powers appearing in this sum.
- In partial fraction theorem, by substraction of a suitable multiple of an suitable inverse power, we can attain a rational function that is less singular than R(x) / Q(x) at an random root of Q.
- Finding the Coefficients in the Partial Fraction Expansion includes four methods such as expansion, cover up, evaluate and solve equations, and common denominator.
- Every proper rational fraction cab be expressed as a sum of simple fractions whose denominators are of the form $(9x+b)^n$ and $(9x^2+bx+c)^n$, n being a positive integer.
- To each linear factor ax+b occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to the determined.

Notes

- To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a simple partial fraction of the form $\frac{Ax+B}{9x^2+bx+c}$ where A and b are constants to be determined.

2.5 Keywords

Distinct Quadratic Factor: To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a simple partial fraction of the form $\frac{Ax+B}{9x^2+bx+c}$ where A and b are constants to be determined.

Partial Fraction: By the method of "partial fractions" we can translate any rational function into a polynomial and fractions each one with negative powers of just one factor (x–a).

2.6 Review Questions

1. What is partial fraction theorem? Illustrate the proof of partial fraction theorem.
2. Depict various methods used in finding the Coefficients of the Partial Fraction Expansion.
3. Explicate the working of Method 3: evaluate and solve equations. Give example.
4. Make distinction between Linear Quadratic Factors and Repeated Quadratic Factors.
5. Write the fractions $\frac{3x}{(x-2)x-4)}$ as sum of partial fractions and then integrate with respect to x.
6. Write the fractions $\frac{1}{x^2-4x+4}$ as sum of partial fractions and then integrate with respect to x.
7. Write the fractions $\frac{x^2}{(x-1)(x-2)(x-3)}$ as sum of partial fractions and then integrate with respect to x.
8. Write the fractions $\frac{x}{(x-1)(3x-2)(x+3)}$ as sum of partial fractions and then integrate with respect to x.
9. Write the fractions $\frac{1}{x(x-2)}$ as sum of partial fractions and then integrate with respect to x.
10. Write the fractions $\frac{1}{x^3+x}$ as sum of partial fractions and then integrate with respect to x.

Answers: Self Assessment

1.	partial fractions	2.	rational
3.	proper	4.	long division

Notes

5. linear
6. less
7. inverse
8. unknown
9. expansion
10. evaluate and solve equations
11. simple
12. single
13. Repeated
14. irreducible
15. sum

2.7 Further Readings

Books

Douglas S. Kurtz, Jaroslav Kurzweil, Charles Swartz, *Theories of Integration*, World Scientific.

G. H. Hardy, T. W. Körner, *A Course of Pure Mathematics*, Cambridge University Press.

Morris Kline, *Calculus: An Intuitive and Physical Approach*, Courier Dover Publications.

Ron Larson, David C. Falvo, *Calculus: An Applied Approach*, Cengage Learning.

Online links

www.intmath.com

calc101.com/partial_fractions.html

Notes

Unit 3: Integration by Parts

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of integration by parts
- Recognize the usage of integration by parts
- Discuss the substitution with respect to integration by parts

Introduction

As we know, there is a power rule intended for derivatives; there is a power rule intended for integrals. There is a chain rule intended for derivatives; there is a chain rule intended for integrals. There is a product rule intended for derivatives; and now what do you think will be there for integrals? There is integration by parts for integrals. In this unit, you will understand the concept of integration by parts with their examples as well.

3.1 Integration by Parts

This is a method depending on the product rule for differentiation, for articulating one integral in provisions of another. It is mostly functional for integrating functions that are products of two types of functions: like power times an exponent, and functions including logarithms.

If $f(x)$ and $g(x)$ be two given functions of x we know that $\frac{d}{dx}\{f(x).g(x)\} = f(x).g(x) + g(x).f(x)$

Hence, by definition

$$f(x)\phi(x) = \int f(x).g^1(x)dx + \int f^1(x).g(x)d$$

$$or \int f(x.g1(x)dx = f(x).g(x) - \int f1(x).g(x)dx$$

Notes To apply this formula, the integrand should be expressible as the product of two functions such that one of them can be easily integrated. This is taken as the second function.

To write the result in a more symmetrical form, replace $f(x)$ by $f_1(x)$ and write $f_2(x)$ for $g(x)$. Then for $g(x)$ we shall have to write $\int f_2(x)dx$ the above equation then becomes.

$$\int f_1(x)f_2(x)dx = f_1(x)\int f_2(x) - \int\left\{f_1^1(x)\int f_2(x)dx\right\}dx$$

i.e. the integral of the product of two functions = first function $\times$ integration of second – Integral of diff. {Coeft. Of first integral of second}

Integration with the help of this rule is called integration by parts. The success of the method depends upon choosing the first function in such a way that the second term on the right hand side may be easy to the product is regarded as the first function.

Caution Case must be taken in choosing the first function.

Integration by parts permits us to integrate numerous products of functions of x. We consider one aspect in this product to be f (this also occurs on the right-hand-side, together with df/dx). The other aspect is considered to be dg/dx (on the right-hand-side only g occurs – i.e. the other factor integrated concerning x).

Notes It is important to note that

1. Unity may be taken in certain cases as one of the functions.
2. The formula of integration by parts can be applied more than once if necessary.

Did u know? If the integral on the right-hand side reverts to the original form, the value of the integral can be immediately inferred by transposing the forms to the left-hand side.

3.1.1 Usage

Integration by parts is used when we observe two dissimilar functions that don't appear to be associated to each other via a substitution.

Example: $\int 2xe^{x^2}dx$ does not need integration by parts as $2x$ is the derivative of x^2.

Usually, one will observe a function that contains two dissimilar functions. We emphasize here four dissimilar types of products for which integration by parts can be accessed (in addition to which factor to tag f and which one to tag dg/dx). These are:

(i)
$$\int \underset{\substack{\uparrow \\ u}}{x^n} \cdot \underset{\substack{\uparrow \\ \frac{dv}{dx}}}{\begin{Bmatrix} \sin bx \\ or \\ \cos bx \end{Bmatrix}} dx$$

(ii)
$$\int \underset{\substack{\uparrow \\ u}}{x^n} \cdot \underset{\substack{\uparrow \\ \frac{dv}{dx}}}{e^{ax}} dx$$

Notes

(iii) $$\int \underset{\dfrac{dv}{dx}}{\underset{\uparrow}{x^r}} \cdot \underset{u}{\underset{\uparrow}{\text{In}(ax)}}\,dx$$

(iv) $$\int \underset{u}{\underset{\uparrow}{e^{ax}}} \cdot \underset{\dfrac{dv}{dx}}{\underset{\uparrow}{\begin{Bmatrix} \sin bx \\ or \\ \cos bx \end{Bmatrix}}}\,dx$$

Here you can consider u as 'f'and v as 'g', as we have used f and g variables in the unit.

A general fault of those accessing integration by parts is abandoned to put a dx with the term in both df and dg. This is a negligible point, but, for the want of comprehensiveness, it requires to be incorporated on transitional steps.

Caution Keep in mind whenever we utilize integration by parts, we make use of everything inside of the integral for f and dg that comprises the dx.

Example:

$$\int x\frac{\log xdx}{I} = \log\int xdx - \int\left\{dx \log x \int xdx\right\}dx$$

$$= \log x\frac{x^2}{2} - \int\frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{1}{2}x^2\log x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\log x - \frac{1}{4}x^2 Ans.$$

Example:

$$\int x\cos xdx = x\int\cos x - \int(a.\sin n)dx$$

$$= x\sin x + \cos x Ans.$$

Example:

$$\int\log xdx = \log x\int 1dx - \int\frac{1}{x}(x)dx$$

$$= x\log x - \int dx = x\log x - x$$

$$= x(\log - 1)$$

Example:

$$\int e^x\cos xdx$$

Let I = $\int e^x\cos xdx$

Notes

$$= e^n \sin x - \int e^n \sin x dx$$
$$= e^n \sin x - \left\{ e^n(-\cos x) - \int e^n(-\cos x) dx \right\}$$
$$= e^n \sin x + e^n \cos x - \int e^n \cos x dx$$
$$= e^n(\sin n + \cos n) - I$$
$$\therefore = \frac{1}{2} e^n (\sin n + \cos x) Ans.$$

Example:

$$\int 1.\sin^{-1} x \, dx = \sin^{-1} x \times x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$x \sin^{-1} x - \frac{1}{2} \int \frac{2x}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x + \text{In}\left(\sqrt{1\text{-}x^2}\right) + C$$

Example:

$$\int x^2 \cos 2x dx$$
$$= x^2 \frac{\sin 2x}{2} - \int 2x.\frac{\sin 2x}{2} dx$$
$$= \frac{x^2}{2} \sin 2x - \left\{ x \frac{(-\cos 2x)}{2} - \int \frac{(-\cos 2x)}{2} \right\} dx$$
$$= \frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{\sin 2x}{4}$$

Example:

$$\int \text{In } x \, dx$$

Set

$f = \ln x$; $dg = dx$

Deduce

$$df = \frac{dx}{x}; g = x + C$$

Any value of C can be used here.

Here and in the other examples, we select $C = 0$.

Get

$$\int \text{In } x \, dx = x \text{ In } x - \int dx$$
$$= x (\text{In } x - 1)$$

Notes

Example:

$\int \tan^{-1} x \, dx$

Set

$f = \tan^{-1}x$; $dg = dx$

Deduce

$$df = \frac{dx}{1+x^2}; g = x$$

Get

$$\therefore \int \tan^{-1} x \, dx = \int 1.\tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2}$$
$$= x \tan^{-1} x - \frac{1}{2}\int \frac{d(x^2)}{1+x^2}$$
$$= x \tan^{-1} x - \frac{1}{2} \text{In } (1+x^2)$$

Example:

$\int xe^{2x} dx$

Set

$f = x$; dg $= e^{2x}dx$

Deduce

$$df = dx; g = \frac{e^{2x}}{2}$$

Get

$$\int xe^{2x}dx \int x \,.\, e^{2n} \, dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}$$
$$= e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right)$$

Example:

$\int \sec^n x \, dx$

Let $G(n) = \int \sec^n x \, dx$

Set

$$f = \sec^{n-2} x = \cos^{2-n} x; dg = \sec^2 x \, dx$$

Deduce

Notes

$$df = (n-2)\tan x \sec^{n-2} x\, dx; g = \tan x \text{ (except for } n = 2\text{, when } df = 0).$$

Use

$$\tan^2 x = \sec^2 x - 1$$

to rewrite

$$g\, df = (n-2)\left(\sec^n x - \sec^{n-2} x\right)^{n-2}$$

Get

$$G(n) = \sec^{n-2} x \tan x \big] - (n-2)\left(G(n) - G(n-2)\right)$$

Reorganize

$$(n-1)G(n) = \sec^{n-2} x \tan x \big] + (n-2)G(n-2)$$

Iterate this to get

$$G(n) = \left(\sec^{n-2} x \tan x\right)\left(\frac{1}{n-1} + \frac{n-2}{(n-1)(n-3)}\cos^2 x + \frac{(n-2)(n-4)}{(n-1)(n-3)(n-5)}\cos^4 x + \cdots\right)\Bigg]$$

When n is even this discontinues automatically; when n is odd, the outcome is in our table of simple trigonometric integrals for n = 1:

$G(1)$ In (sec x + tan x)]

Example:

$sqrt(1+x^2)$

First method

To evaluate: $\int \sqrt{1+x^2}\, dx$

Substituting

$x = \tan y$

You obtain

$$\int \sqrt{1+x^2}\, dx = \int \sec^3 y\, dy$$
$$= G(3)$$

Using the above example you obtain

$$G(3) = \frac{\sec y \tan y + \text{In } (\sec y + \tan y)}{2}$$

$$G(3) = \frac{x\sqrt{1+x^2} + \text{In}\left(x + \sqrt{1+x^2}\right)}{2}$$

Alternate method by substitution

To evaluate: $\int \sqrt{1+x^2}\, dx$

Notes

Set

$x = \sinh y$

Get

$$\sqrt{1+x^2} = \cosh y,\ dx = \cosh y\ dy$$

$$\int \sqrt{1+x^2}\,dx = \int \cosh^2 y\ dy$$

$$= \int \frac{e^{2y} + 2 + e^{-2y}}{4} dy$$

$$= \left(\frac{e^{2y} - e^{-2y}}{8} + \frac{y}{2}\right)$$

$$= \left(\frac{\sinh 2y}{4} + \frac{y}{2}\right)$$

$$\sinh 2y = 2\sinh y \cosh y = 2x\sqrt{1+x^2}$$

$$\int \sqrt{1+x^2}\,dx = \frac{x\sqrt{1+x^2} + \text{arc} \sinh x}{2}\Bigg]$$

Task Evaluate the following integral: $\int 2x^2 e^x dx$

Self Assessment

Fill in the blanks:

1. .... is a method depending on the product rule for differentiation, for articulating one integral in provisions of another.

2. If $f(x)$ and $g(x)$ be two given functions of x we know that $\frac{d}{dx}\{f(x).g(x)\} =$ ....

3. The integral of the product of two functions = first function × integration of second – Integral of {..}

4. The success of the integration by parts method depends upon choosing the first function in such a way that the second term on the right hand side may be easy to the product is regarded as the .... function.

5. If the integral on the right-hand side reverts to the .... form, the value of the integral can be immediately inferred by transposing the forms to the left-hand side.

6. Integration by parts is used when we observe two .... functions that don't appear to be associated to each other via a substitution.

7. Whenever we utilize integration by parts, we make use of everything inside of the integral for f and dg that comprises the....................................... .

8. Integration by parts is mostly functional for integrating functions that are .... of two types of functions.

State whether the following statements are true or false:

9. Integration by parts is a method depending on the power rule for differentiation, for articulating one integral in provisions of another.

10. Integration by parts permits us to integrate numerous products of functions of x.
11. A general mistake of those accessing integration by parts is neglected to put a dx with the term in both df and dg.
12. Integration by parts is used when we observe two similar functions that don't appear to be associated to each other via a substitution.

3.2 The Substitution $z = \tan(x/2)$

The magnificent substitution $z = \tan(x/2)$ permits alteration of any trigonometric integrand into a rational one.

Let us consider our integrand as a rational function of $\sin(x)$ and $\cos(x)$.

After the substitution $z = \tan(x/2)$ we get an integrand that is a rational function of z, which can then be assessed by partial fractions.

Theorem:

If $z = \tan(x / 2)$, then

$$dx = \frac{2dz}{1+z^2},$$

$$\cos x = \frac{1-z^2}{1+z^2},$$

and

$$\sin x = \frac{2z}{1+z^2}$$

and any rational function of xdx turns out to be a rational function of zdz.

Proof.

By the rules for differentiation we contain, for $z = \tan(x/2)$,

$$dz = \frac{\sec^2(x/2)}{2}dx = \frac{1+z^2}{2}dx$$

The angle addition formulae provide us:

$$\begin{aligned}\sin x &= 2\sin(x/2)\cos(x/2)\\ &= 2\tan(x/2)\cos^2(x/2)\\ &= \frac{2z}{\sec^2(x/2)}\\ &= \frac{2z}{1+z^2}\end{aligned}$$

$$\begin{aligned}\cos x &= \cos^2(x/2) - \sin^2(x/2)\\ &= (1-z^2)\cos^2(x/2)\\ &= \frac{(1-z^2)}{\sec^2(x/2)}\\ &= \frac{(1-z^2)}{(1+z^2)}\end{aligned}$$

Notes

Example:

Integrate $I = \int \frac{dx}{3+4\cos x}$

With $z = \tan(x/2)$ we obtain

$$I = \int \frac{2dx}{(1+z^2)(3+4(1-z^2)/(1+z^2))}$$
$$= 2\int \frac{dz}{7-z^2}$$
$$= \frac{\text{In}\left|(z+\sqrt{7})/(z-\sqrt{7})\right|}{\sqrt{7}} + C$$

where we have used

$$\frac{1}{a^2-z^2} = \frac{1}{2a}\left(\frac{1}{a-z} + \frac{1}{a+z}\right)$$

Task Evaluate the following integral: $\int_0^{\pi/4} \frac{dx}{1-\sin x}$.

Self Assessment

Fill in the blanks:

13. The magnificent substitution $z = \tan(x/2)$ permits alteration of any trigonometric integrand into a .... one.
14. If $z = \tan(x/2)$, then dx = ...
15. After the substitution $z = \tan(x/2)$ we get an .... that is a rational function of z.

3.3 Summary

- Integration by parts is a method depending on the product rule for differentiation, for articulating one integral in provisions of another.
- By Integration by parts, we mean $f(x)\phi(x) = \int f(x).g^1(x)dx + \int f^1(x).g(x)d$.
- To write the result in a symmetrical form, replace $f(x)$ by $f_1(x)$ and write $f_2(x)$ for $g(x)$. Then for $g(x)$ we shall have to write $\int f_2(x)\,dx$ the above equation then becomes $\int f_1(x)f_2(x)dx = f_1(x)\int f_2(x) - \int\left\{f_1^1(x)\int f_2(x)dx\right\}dx$ i.e. the integral of the product of two functions.
- The success of the method depends upon choosing the first function in such a way that the second term on the right hand side may be easy to the product is regarded as the first function.
- If the integral on the right-hand side reverts to the original form, the value of the integral can be immediately inferred by transposing the forms to the left-hand side.

Notes

- Integration by parts is used when we observe two dissimilar functions that don't appear to be associated to each other via a substitution.
- Whenever we utilize integration by parts, we make use of everything inside of the integral for f and dg that comprises the dx.
- The magnificent substitution $z = \tan(x/2)$ permits alteration of any trigonometric integrand into a rational one.

3.4 Keywords

Integration by Parts: It is a method depending on the product rule for differentiation, for articulating one integral in provisions of another.

Substitution z=tan(x/2): The magnificent substitution $z = \tan(x/2)$ permits alteration of any trigonometric integrand into a rational one.

3.5 Review Questions

1. Integrate $\int x\sqrt{x+1}\,dx$
2. Integrate $\int x \text{ In } x\,dx$
3. Integrate $\int x^2 \text{ In } 4x\,dx$
4. Integrate $\int xe^2\,dx$
5. Integrate $\int x\sec^2 x\,dx$
6. Integrate $\int x^2e^{3x}\,dx$
7. Integrate $\int \arcsin x\,dx$
8. Integrate $\int x\cos 3x\,dx$
9. Integrate $\int \arcsin 3x\,dx$
10. Integrate $\int 2x \arctan x\,dx$

Answers Self Assessment

1.	Integration by parts	2.	$f(x).g(x) + g(x).f(x)$
3.	diff. Coeft. of first integral of second	4.	first
5.	original	6.	dissimilar
7.	dx	8.	products
9.	False	10.	True

Notes

11. True
12. False
13. rational
14. $\dfrac{2dz}{1+z^2}$
15. integrand

3.6 Further Readings

Books

Edward Thomas Dowling, *Schaum's Outline of Theory and Problems of Mathematical Methods for Business and Economics,* McGraw-Hill Professional, 1993.

R.S. Bhardwaj, *Business Statistics,* Excel Books, 1999.

R.S. Bhardwaj, *Mathematics for Economics and Business,* Excel Books, E 2.

Online link

http://math.arizona.edu

Unit 4: Definite Integral

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand definite integral as the limit of a sum
- Discuss the fundamental theorem of integral calculus

Introduction

The Definite Integral comprises extensive number of applications in mathematics, the physical sciences and engineering. The speculation and application of statistics, for instance, is based greatly on the definite integral; via statistics, many conventionally non-mathematical regulations have turn out to be greatly reliant on mathematical thoughts. Economics, sociology, psychology, political science, geology, and many others specialized fields make use of calculus notions.

4.1 Definite Integral as the Limit of a Sum

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$. Divide the interval $[a,b]$ into n equal parts each of width h by points

a+h, a+2h, a + 3h, …, a+(n – 1) h.

Then, $\underbrace{h+h+\ldots+h}_{n\ times}=b-a$

$$\Rightarrow nh=b-a \Rightarrow h=\frac{b-a}{n}.$$

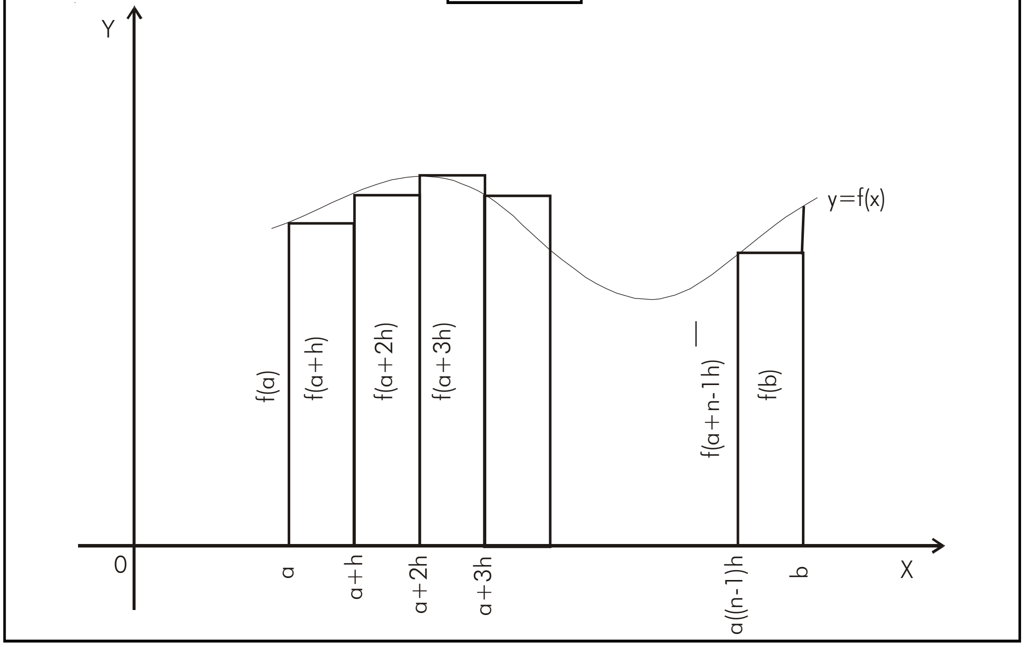
Now the areas of inner rectangles are:

$$hf(a), hf(a+h), hf(a+2h), hf(a+3h), \ldots, hf(a+\overline{n-1}\,h).$$

[$\therefore$ The breadth of inner rectangles is h and their heights are $f(a), f(a+h), \ldots, f(a+\overline{n-1}h)$ respectively and area of rectangle is breadth $\times$ height]

Notes

Figure 4.1



The sum of these areas:

$$A = hf(a) + hf(a+h) + hf(a+2h) + hf(a+3h) + \ldots + hf(a+\overline{n-1}h)$$
$$= h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + hf(a+\overline{n-1}h)]$$

This area is close to the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$.

If n increases, the number of rectangles will increases and the width of rectangles will decrease.

Did u know? A will give closer approximation of the area enclosed by the curve $y = f(x)$, x-axis and the coordinates $x = a$, $x = b$.

Thus the area of region bounded by curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$ is

$$\lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h]$$

where $h = \dfrac{b-a}{n}$

$$= \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h]$$

where $nh = b - a [\because as\ n \to \infty, h \to 0]$

Notes This area is also the limiting value of any area which is among that of the rectangles beneath the curve and that of the rectangles over the curve.

Caution For the sake of ease, we shall consider rectangles with height identical to that of the curve at the left hand edge of every subinterval.

We take this expression as the definition of a definite integral and denote it by $\int_a^b f(x)dx.$

Hence,

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h]$$

where $nh = b - a$.

The process of evaluating a definite integral by using the above definition is called integration from first principles or integration by ab-initio method or integration as the limit of a sum.

The following results are useful in evaluating definite integrals as limit of sums:

1. $1 + 2 + 3 + \ldots + (n-1) = \dfrac{n(n-1)}{2}$

2. $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$

3. $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \left[\dfrac{n(n-1)}{2}\right]^{-2}$

4. $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$

5. $\sin a + \sin(a+h) + \sin(a+2h) + +\sin[a+n–1]$

$$= \frac{\sin\left[a + \left(\frac{n-1}{2}\right)h\right]\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$$

6. $\cos a + \cos(a+h) + \cos(a+2h) + \ldots + cis[a + (n-1)h]$

$$= \frac{\cos\left[a + \left(\frac{n-1}{2}\right)h\right]\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$$

Caution The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable.

The variable of integration is known as a dummy variable.

Example: Evaluate the following definite integrals as limit of sums:

1. $\int_1^2 (2x+3)dx$

2. $\int_{-1}^1 (x+3)dx$

Notes

Solution:

1. $\int_1^2 (2x+3)dx$

$f(x) = 2x+3, a = 1, b = 2 \text{ and } nh = b - a = 2 - 1 = 1$
$f(a) = f(1) = 5$
$f(a+h) = f(1+h) = 2(1+h)+3 = 5+2h$
$f(a+2h) = f(1+2h) = 2(1+2h)+3 = 5+2.2h$
$f(a+3h) = f(1=3h) = 2(1+3h)+3 = 5+3.2h$
:
:
$f(a+\overline{n-1}h) = f(1+\overline{n-1}h) = 2(1+\overline{n-1}h)+3 = 5+\overline{n-1}.2h$

$$Now, \int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$$

$$\Rightarrow \int_1^2 (2x+3)\,dx = \lim_{h\to 0} h[f(1)+f(1+h)+f(a+2h)+f(1+3h)+\ldots+f(1+\overline{n-1}h)$$

$= \lim_{h\to 0} h[5+5+2h.h+5+2.2h+5+3.2h+\ldots+5+\overline{n-1}2h]$
$= \lim_{h\to 0} h[5n+2h(1+2+3+\ldots+(n-1)]$
$= \lim_{h\to 0} h\left[5n+2h\dfrac{n(n-1)}{2}\right]$
$= \lim_{h\to 0} h[5n+nh(n-1)] = \lim_{h\to 0}[5nh+nh(nh-h)]$
$= \lim_{h\to 0} h[5+(1-h)] \qquad [\because nh = 1]$
$= 5+1 = 6$

2. $\int_{-1}^1 (x+3)\,dx$

$f(x) = x+3, a = 1, b = 1 \text{ and } nh = b - a = 1 - 1 = 2$
$f(a) = f(-1) = 2$
$f(a+h) = f(-1+h) = -1+h+3 = 2+h$
$f(a+2h) = f(-1+2h) = -1+2h+3 = 2+2h$
$f(a+3h) = f(-1=3h) = -1+3h+3 = 2+3h$
:
:
$f(a+\overline{n-1}h) = f(-1+\overline{n-1}h) = -1+\overline{n-1}h+3 = 2+(n-1)h$

$$Now, \int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$$

$$\int_{-1}^1 (x+3)\,dx = \lim_{h\to 0} h[f(-1)+f(-1+h)+f(-1+2h)+(-1+3h)+\ldots+f(-1+\overline{n-1}h)]$$

$= \lim_{h\to 0} h[2+2+h+2+2h+2+3h+\ldots+2+(n-1)h]$
$= \lim_{h\to 0} h[2n+h+2h+3h+\ldots+(n-1)h]$

Notes

$$= \lim_{h\to 0} h[2n + h(1+2+3+\ldots+(n-1)]$$
$$= \lim_{h\to 0} h\left[2n + h\frac{n(n-1)}{2}\right] = \lim_{h\to 0} h\left[2nh + \frac{nh(nh-h)}{2}\right]$$
$$= \lim_{h\to 0} h[4 + (2-h)] \qquad [\because nh = 2]$$
$$= 4 + 2 = 6.$$

Example:

Evaluate the following definite integrals as limit of sums:

1. $\int_1^4 (x^2 - x)\,dx$ 2. $\int_1^3 (2x^2 + 5x)\,dx$ 3. $\int_2^4 (2x^2 + 3x + 1)\,dx$

Solution:

1. $\int_1^4 (x^2 - x)\,dx$

$f(x) = x^2 - x, a = 1, b = 4\, and\, nh = b - a = 4 - 1 = 3$

$f(a) = f(1) = 0$

$f(a+h) = f(1+h) = (1+h)^2 - (1+h) = 1 + h2 + 2h - 1 - h = h^2 + h$

$f(a+2h) = f(1+2h) = (1+2h)^2 - (1+2h) = 1 + 2^2h^2 + 4h - 1 - 2h = 2^2h^2 + 2h$

$f(a+3h) = f91 = 3h) = (1+3h)^2 - (1+3h) = 1 + 3^2h^2 + 6h - 1 - 3h = 3^2h^2 + 3h$

:

:

$f(a + \overline{n-1}h) = f(1 + \overline{n-1}h) = (1 + \overline{n-1}h)^2 - (1 + \overline{n-1}h)$

$= 1 + (n-1)^2h^2 + 2(n-1)h - 1 - (n-1)h = (n-1)^2h^2 + (n-1)h$

Now, $\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a + \overline{n-1}h)]$

$$\Rightarrow \int_1^4 (x^2 - x)dx = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + f(1+3h) + \ldots + f(1 + \overline{n-1}h)]$$

i. $= \lim_{h\to 0} h[0 + h^2 + h + 2^2h^2 + 2^2h^2 + 3h + \ldots + (n-1)^2h^2 + (n-1)h]$

$$= \lim_{h\to 0} h[h^2(1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + h(1+2+3+\ldots+(n-1))]$$
$$= \lim_{h\to 0} h\left[h^2\frac{n(n-1)(2n-1)}{6} + h\frac{n(n-1)}{2}\right]$$
$$= \lim_{h\to 0} \left[\frac{nh(nh-h)2nh-h}{6} + \frac{nh(nh-h)}{2}\right]$$
$$= \lim_{h\to 0} \left[\frac{3(3-h)(6-h)}{6} + \frac{3(3-h)}{2}\right] \qquad [\because nh = 3]$$
$$= \frac{3(3)(6)}{6} + \frac{3(3)}{2} = 9 + \frac{9}{2} = \frac{27}{2}$$

Notes

2. $\int_1^3 (2x^2 + 5x)\,dx$

$f(x) = 2x^2 + 5x, a = 1, b = 3 \text{ and } nh = b - a = 3 - 1 = 2$

$f(a) = f(1) = 7$

$f(a+h) = f(1+h) = 2(1+h)^2 + 5(1+h)$

$= 2(1 + 2h + h^2) + + 5h = 7 + 9h + 2h^2$

$f(a+2h) = f(1+2h) = 2(1+2h)^2 + 5(1+2h)$

$= 2(1 + 4h + 2^2h^2) + 5 + 10h = 7 + 2.9h + 2^2.2h^2$

:

:

$f(a+\overline{n-1}h) = f(1+\overline{n-1}h) = 2(1+\overline{n-1}h)^2 + 5(1+\overline{n-1}h)$

$= 2(1 + 2(n-1)h + (n-1)^2h^2) + 5 + 5(n-1)h$

$= 7 + (n-1).9h + (n-1)^2.2h^2$

Now $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+h) + f(a+2f(a+3h)$

$+ \ldots + f(a+\overline{n-1}h)]$

$\Rightarrow \int_1^3 (2x^2 + 5x) = \lim_{h\to 0} h\,[f(1) + f(1+h) + f(1+2h) + f(1+3h)$

$+ \ldots + f(a + \overline{n-1}h)]$

$= \lim_{h\to 0} h[7 + 7 + 9h + 2h^2 + 7 + 2.9h + 2^2.2h^2 + 7 + 3.9h$

$+3^2.2h^2 + \ldots + 7 + (n-1).9h + (n-1)^2.2h^2]$

$= \lim_{h\to 0} h[7n + 9h(1 + 2 + 3 + \ldots + (n-1))$

$= \lim_{h\to 0} h\left[7n + 9h\frac{n(n-1)}{2} + 2h^2\frac{n(n-1)(2n-1)}{6}\right]$

$= \lim_{h\to 0} h\left[14 + 9(2-h) + \frac{2}{3}(2-h)(4-h)\right]$ $[\because nh = 2]$

$= 14 + 18 + \frac{16}{3} = \frac{112}{3}$

3. $\int_2^4 (2x^2 + 3x + 1)\,dx$

$f(x) = 2x^2 + 3x + 1, a = 2, b = 4 \text{ and } nh = b - a = 4 - 2 = 2$

$f(a) = f(2) = 15$

$f(a+h) = f(2+h) = 2(2+h)^2 + 3(2+h) + 1$

$= 2(4 + 4h + h^2) + 6 + 3h + 1 = 15 + 11h + 2h^2$

$f(a+2h) = f(2+2h) = 2(2+2h)^2 + 3(2+2h) + 1$

$= 2(1 + 8h + 2^2h^2) + 6 + 6h + 1$

$= 15 + 2.11h + 2^2.2h^2$

$f(a+3h) = f(2+3h) = 2(2+3h)^2 + 3(2+3h) + 1$

$= 2(4 + 12h + 3^2h^2) + 6 + 9h + 1 = 15 + 3.11h + 3^2.2h^2$

:

:

$$f(a+\overline{n-1}h) = f(2+\overline{n-1}h) = 2(2+\overline{n-1}h)^2 + 3(2+\overline{n-1}h) + 1$$
$$= 2(4+2(n-1)h+(n-1)^2h^2)+6+3(n-1)h+1$$
$$= 15+(n-1).11h+(n-1)^2.2h^2$$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h) + \ldots + f(a+\overline{n-1}h)]$

$$\Rightarrow \int_2^4 (2x^2+3x+1)\,dx$$
$$= \lim_{h\to 0} h\,[f(2)+f(2+h)+f(2+2h)+f(2+3h)+\ldots+f(2+\overline{n-1}h)]$$
$$= \lim_{h\to 0} h[15+15+11h+2h^2+15+2.11h+2^2.2h^2+15+3.11h+3^2.2h^2 + \ldots + 15+(n-1)11h+(n-1)^2.2h^2]$$
$$= \lim_{h\to 0} h[15n+11h(1+2+3+\ldots+(n-1))+2h^2(1^2+2^2+3^2+\ldots+(n-1)^2)]$$
$$= \lim_{h\to 0} h\left[15n+11h\frac{n(n-1)}{2}+2h^2\frac{n(n-1)(2n-1)}{6}\right]$$
$$= \lim_{h\to 0} h\left[15nh+\frac{11}{2}nh(nh-h)+\frac{nh(nh-h)(2nh-h)}{3}\right]$$
$$= \lim_{h\to 0}\left[30+11(2-h)+\frac{2(2-h)(4-h)}{3}\right] = 30+22+\frac{16}{3} = \frac{172}{3}$$

Example:

Evaluate as limit of sums $\int_1^2 x^3+1dx$

Solution:

$$\int_1^2 x^3+1dx$$

$f(x)=x^3+1, a=1, b=2$ and $nh=b-a=2-1=1$

$f(a)=f(1)=2$

$f(a+h) = f(1+h) = (1+h)^3+1 = 1+3h+3h^2+h^3+1 = 2+3h+3h^2+h^3$

$f(a+2h) = f(1+2h) = (1+2h)^3+1 = 1+2.3h+2^2.3h^2+2^3h^3+1$

$= 2+2.3h+2^2.3h^2+2^3h^3$

$f(a+3h) = f(1+3h) = (1+3h)^2+1 = 1+3.3h+3^2.3h^2+3^3.h^3+1$

$= 2+3.3h+3^2.3h^2+3^3h^3$

:

:

$f(a+\overline{n-1}h) = f(1+\overline{n-1}h) = (1+\overline{n-1}h)^3+1 = 1+(\overline{n-1})3h+(n-1)^2.3h^2+(n-1)^3h^3+1$

$= 2+(n-1).3h+(n-1)^2.3h^2+(n-1)^3.h^3$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$

Notes

$$\Rightarrow \int_1^2 (x^3+1)\,dx = \lim_{h\to 0} h\,[f(1)+f(1+h)+f(1+2h)+f(1+3h)+\ldots+f(1+\overline{n-1}h)]$$

$$= \lim_{h\to 0} h[2+2+3h+3h^2+h^3+2+2.3h+2^2.3h^2+2^3.h^3+2+3.3h+3^2.3h^2+3^3h^3$$
$$+\ldots+2+(n-1)3h+(n-1)^2-3h^2+(n-1)^3h^3]$$

$$= \lim_{h\to 0} h[2n+3h(1+2+3+\ldots+(n-1))+3h^2(1^2+2^2+3^2+\ldots+(n-1)^2$$
$$+h^3(1^3+2^3+3^3).+\ldots+(n-1)^3]$$

$$= \lim_{h\to 0} h\left\{2n+3h\frac{n(n-1)}{2}+3h^2\frac{n(n-1)(2n-1)}{6}+h^3\frac{n(n-1)}{2}\right\}$$

$$= \lim_{h\to 0} h\left\{2nh+\frac{3}{2}nh(nh-h)(2nh-h)+\frac{1}{4}[nh(nh-h)]^2\right\}$$

$$= \lim_{h\to 0}\left[2+\frac{3}{2}(1-h)+\frac{1}{2}(1-h)(2-h)+\frac{1}{4}(1-h)^2\right]$$

$$= 2+\frac{2}{3}+1+\frac{1}{4} = 3+\frac{2}{3}+\frac{1}{4} = \frac{47}{12}$$

Example: Evaluate the following definite integrals as limit of sums:

1. $\int_2^5 e^x dx$ 2. $\int_{-1}^1 e^{2x} dx$

3. $\int_1^3 e^{-x} dx$ 4. $\int_0^1 e^{2-3x}$

Solution:

1. $\int_2^5 e^x dx$

$$f(x) = ex, a = 2, b = 5 and\, nh = b-a = 2 = 3$$
$$f(a) = f(2) = e^2$$
$$f(a+h) = f(2+h) = e^{2+h} = e^2.e^{2h}$$
$$f(a+2h) = f(2+2h) = e^{2+2h} = e^2.e^{2h}$$
$$f(a+3h) = f(2+3h) = e^{2+3h} = e^2.e^{3h}$$
$$\vdots$$
$$\vdots$$
$$f(a+\overline{n-1}h) = f(2=\overline{n-1}h) = e^{2+(n-1)^h} = e^2.e^{(n-1)h}$$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$

$$\int_2^3 e^{2x} dx = \lim_{h\to 0} h[e^2+e^2.e^h+e^2.e^{2h}+e^2.e^{3h}+\ldots+e^2.e^{(n-1)h}]$$

$$= \lim_{h\to 0} h\frac{e^2(e^{nh}-1)}{e^h-1} \qquad \left[\because a+ar+ar^2+\ldots+ar^{n-1} = \frac{a(r^n-1)}{r-1}\right]$$

$$= \lim_{h\to 0}\frac{h}{e^h-1}.e^2(e^3-1) \qquad [\because nh = 3]$$

$$= e^2(e^3-1) = e^5-e^2 \qquad \left[\text{Using } \lim_{x\to 0}\frac{e^x-1}{x} = 1\right]$$

Notes

2. $\int_{-1}^{1} e^{2x} dx$

$f(x) = ex^{2x}, a = -1, b = 1 \text{ and } nh = b - a = 1 - (-1) = 2$

$f(a) = f(-1) = e^{-2}$

$f(a+h) = f(-1+h) = e^{2(-1+h)} = e^{-2}.e^{2h}$

$f(a+2h) = f(-1+2h) = e^{2(-1+2h)} = e^{-2+4h} = e^{-2}.e^{4h}$

$f(a+3h) = f(-1+3h) = e^{2(-1+3h)} = e^{-2+6h} = e^{-2}.e^{6h}$

:

:

$f(a+\overline{n-1}h) = f(-1+\overline{n-1}h) = e^{2(-1+\overline{n-1}h)} = e^{-2+2(n-1)^h} = e^{-2}.e^{2(n-1)h}$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h)]$

$\Rightarrow \int_{-1}^{1} e^{2x} dx = \lim_{h\to 0} h[f(-1) + f(-1=h) + f(-1+2h) + f(-1+3h) + \ldots + f(-1+\overline{n-1}h)]$

$= \lim_{h\to 0} h[e^{-2} + e^{-2}.e^{2h} + e - 2.e^{4h} + e - 2e^{6h} + \ldots + e^{-2}e^{2(n-1)h}]$

$= \lim_{h\to 0} h\left[e - 2\frac{(e^{2nh}-1)}{e^{2h}-1}\right] = \lim_{h\to 0} \frac{h}{3^{2h}-1}.e - 2(e^4 - 1) \qquad [\because nh = 2]$

$= \lim_{h\to 0} \frac{2h}{e^{2h}-1}.\frac{1}{2}e^{-2}(e^4-1) = \frac{1}{2}e^{-2}(e^4-1) = \frac{1}{2}(e^2 - e^{-2})$

3. $\int_1^3 e^{-x}\, dx$

$f(x) = e^{-x}, a = -1, b = 3 \text{ and } nh = b - a = 3 - 1 = 2$

$f(a) = f(1) = e^{-1}$

$f(a+h) = f(1+h) = e^{-(1+h)} = e^{-1}.e^{-h}$

$f(a+2h) = f(1+2h) = e^{-(1+2h)} = e^{-1-2h} = e^{-1}.e^{-2h}$

$f(a+3h) = f(1+3h) = e^{-1(1+3h)} = e^{-1}.e^{-3h}$

:

:

$f(a+\overline{n-1}h) = f(1+\overline{n-1}h) = e^{1(1+(n-1)h)} = e^{-1-(n-1)^h} = e^{-1}.e^{-(n-1)h}$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h)]$

$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + (1+3h) + \ldots + f(1+(n-1)h)]$

$= \lim_{h\to 0} h[e^{-1} + e^{-1}e^{-h} + e^{-1}e^{-2h} = e^{-1}e^{-3h} + \ldots + e^{-1}e^{-(n-1)h}]$

$= \lim_{h\to 0} h\frac{e^{-1}(e^{-nh}-1)}{e^{-h}-1} == \lim_{h\to 0} \frac{h}{e^{-h}-1}e^{-1}(e^{-nh}-1)$

$= \lim_{h\to 0} -\left(\frac{-h}{e^{-h}-1}\right)e^{-1}(e^{-2}-1)$

$= -e - 1(e^{-2}-1) = -e^{-3} + e^{-1} = e^{-1} - .e^{-3}$

Notes

4. $\int_0^1 e^{2-3x}\,dx$

$f(x) = e^{2-3x}, a = 0, b = 1$ and $nh = b - a = 1 - 0 = 1$

$f(a) = f(0) = e^2$

$f(a+h) = f(h) = e^{2-3h} = e^2.e^{-3h}$

$f(a+2h) = f(2h) = e^{2-3h} = e^2.e^{-3h}$

$f(a+3h) = f(3h) = e^{2-9h} = e^2.e^{-9h}$

:

:

$f(a+\overline{n-1}h) = f(1+\overline{n-1}h) = e^{2-3(n-1)h} = e^2.e^{-3(n-1)h}$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$

$$\Rightarrow \int_0^1 e^{2-3x}dx = \lim_{h\to 0} h[f(0)+f(h)+f(2h)+f(3h)+\ldots+f(a+\overline{n-1}h)]$$

$$= \lim_{h\to 0} h[e^2 + e^2.e^{-3h} + e^2.e^{-6h} + e^2.e^{-9h} + \ldots + e^2 \cdot e^{-3(n-1)h}]$$

$$= \lim_{h\to 0} h\frac{e^2(e^{-3nh}-1)}{e^{-3nh}-1} = \lim_{h\to 0} \frac{h}{e^{-3h}-1}\cdot e^2(e^{-3nh}-1)$$

$$= \lim_{h\to 0} h\frac{-3h}{e^{-3h}-1}\cdot\left(\frac{1}{-3}\right)e2(e^{-3}-1) \qquad [\because nh = 1]$$

$$= -\frac{1}{3}e^2(e^{-3}-1) = -\frac{1}{3}(e^{-1}e^2) = \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$$

Example:

Evaluate the following definite integrals as limit of sums:

1. $\int_2^4 2^x x$ 2. $\int_{-1}^2 5^x x$

Solution:

1. $\int_2^4 2\,x$

$f(x) = 2^x, a = 2, b = 4$ and $nh = b - a = 4 - 2 = 2$

$f(a) = f(2) = e^2 = 4$

$f(a+h) = f(2+h) = e^{2+3h} = 2^2.2^h = 4.2^h$

$f(a+2h) = f(2+2h) = 2^{2+2h} = 2^2.2^{2h} = 4.2^{2h}$

$f(a+3h) = f(2+3h) = 2^{2+3h} = 22.2^{3h} = 4.2^{3h}$

:

:

$f(a+\overline{n-1}h) = f(2+\overline{n-1}h) = e^{2+(n-1)h} = 2^2.2^{(n-1)h} = 4.2^{(n-1)h}$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$

$$\Rightarrow \int_2^4 2^x\,dx = \lim_{h\to 0} h[f(2)+f(2+h)+f(2+2h)+f(2+3h)+\ldots+f(2+\overline{n-1}h)]$$

$$= \lim_{h\to 0} h[4+4.2^h+4.2^{2h}+4.2^{3h}+\ldots+4.2^{(n-1)h}]$$

$$= \lim_{h\to 0} h\,\frac{4(2^{nh}-1)}{2^h-1} = \lim_{h\to 0} \frac{h}{2^h-1}\,4(2^{nh}-1)$$

$$= \lim_{h\to 0} h\frac{h}{2^h-1}4\,(2^2-1)$$

$$= \frac{1}{\log 2}4(3) \qquad \left[\because \lim_{h\to 0}\frac{a^x-1}{x} = \log a\right]$$

$$= \frac{12}{\log 2}$$

2. $\int_{-1}^{2} 5^x x$

$f(x) = 5^x, a = -1, b = 2$ and $nh = b - a = 2-(-1) = 3$

$$f(a) = f(-1) = 5^{-1}$$
$$f(a+h) = f(-1+h) = e^{-1+h} = 5^{-1}.5^h$$
$$f(a+2h) = f(-1+2h) = 5^{-1+2h} = 5^{-1}.5^{2h}$$
$$f(a+3h) = f(-1+3h) = 5^{-1+3h} = 5^{-1}5^{3h}$$
$$\vdots$$
$$\vdots$$
$$f(a+\overline{n-1}h) = f(-1+\overline{n-1}h) = 5^{-1+(n-1)h} = 5^{-1}5^{(n-1)h}$$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$

$$\Rightarrow \int_{-1}^{2} 5^x\,dx = \lim_{h\to 0} h[f(-1)+f(-1+h)+f(-1+2h)+f(-1+3h)+\ldots+f(-1+\overline{n-1}h)]$$

$$= \lim_{h\to 0} h[5^{-1}+5^{-1}5^h+5^{-1}.5^{2h}+5^1.5^{3h}+\ldots+5^{-1}5^{(n-1)h}]$$

$$= \lim_{h\to 0} h\,\frac{5^{-1}(5^{nh}-1)}{5^h-1} = \lim_{h\to 0} \frac{h}{5^h-1}\,5^{-1}(5^3-1)$$

$$= \lim_{h\to 0} h\frac{h}{2^h-1}4\,(2^2-1)$$

$$= \frac{1}{\log 5}\left(5^2-\frac{1}{5}\right) = \frac{124}{5\log 5}$$

Example:

Evaluate the following definite integrals as limit of sums:

1. $\int_a^b \sin x\,dx$ 2. $\int_0^{\pi/4} \cos x\,dx$ 3. $\int_{\pi/6}^{\pi/2} \sin^2 x\,dx$

Solution:

1. $\int_a^b \sin x dx$

$f(x) = \sin x$ and $nh = b - a$

$f(a) = \sin a$

Notes

$f(a+h) = \sin(a+h)$

$f(a+h) = \sin(a+2h)$

$f(a+3h) = \sin)\ a+3h)$

$f(a+\overline{n-1}h) = \sin(a+\overline{n-1}h)$

Now, $\int_a^b f(x) = dxlinh\,|\,f(0) + f(a+h) + f(a+2h) + f(a+3h) + ... + f(a+\overline{n-1}h)\,|$

$$\Rightarrow \int_a^b \sin x dx = \lim h[\sin a + \sin(a+h) + \sin(a+2h) + \sin(a+3h) + ... + \sin(a+(n-a)h)$$

$$= \lim_{h\to 0} h \frac{\sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$$

$$= \lim_{h\to 0} h \frac{h}{\sin\left(\frac{h}{2}\right)} \sin\left\{a+\frac{nh-h}{2}\right\}\sin\left(\frac{b-a}{2}\right)$$

$$= \lim_{h\to 0} h \frac{2}{\sin\left(\frac{h}{2}\right)} 2\sin\left\{a+\frac{b-a-h}{2}\right\}\sin\left(\frac{b-a}{2}\right)$$

$$= 2\sin\left(\frac{2a+b-a}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= 2\sin\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right) = \cos\alpha - \cos b$$

2. $\int_0^{\pi/4} \cos x dx$

i. $f(x) = \cos x$, $\alpha = 0$, $b = \pi/4$, $nh = b - a\ \pi/4$

ii. $f(\alpha) = f(0) = \cos 0$

iii. $f(a+h) = f(h) = \cos h$

iv. $f(a+2h) = f(2h) = \cos = 2h$

v. $f(a+3h) = f(3h) = \cos 3\ h$

$f(a+\overline{n-1}h) = f(\overline{n-a}h) = \cos(n-a)h$

Now, $\int_a^b f(x) = \lim f(a) + f(a+h) + f(a+2h) + f(a+3h) + ... + f(a+\overline{n-1}h)\,|$

$$\Rightarrow \int_a^b \cos x dx = \lim h[f(0) + f(h) + f(3h) + ... + f(\overline{n-1})h]$$

$$= \lim_{h\to 0} h \frac{\cos\left\{0+\left(\frac{n-1}{2}\right)\right\}\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$$

$$= \lim_{h\to 0} h \frac{h}{\sin\left(\frac{h}{2}\right)} \cos\left\{\frac{nh-h}{2}\right\}\sin\frac{nh}{2}$$

Notes

$$= \lim_{\to 0} h \frac{\frac{h}{2}}{\sin\frac{h}{2}} 2\cos\frac{\frac{\pi}{4}-h}{2}\sin\frac{\pi}{8}$$

$$= 2\cos\frac{\pi}{8}\sin\frac{\pi}{8} = \sin 2\left(\frac{\pi}{8}\right)\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

3. $\int_{\pi/6}^{\pi/2} \sin^2 x dx$

$$f(x)\sin^2 x, \alpha = \frac{\pi}{6}, b = \frac{\pi}{2} and\ nh = b - a = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

$$f(a) f\left(\frac{\pi}{6}\right) = \sin^2\frac{\pi}{2} = \frac{1-\cos\frac{\pi}{3}}{2}$$

$$f(a+h) = f\left(\frac{\pi}{6}+h\right)\sin^2\left(\frac{\pi}{6}+h\right) = \frac{1-\cos\left(\frac{\pi}{3}+2h\right)}{2}$$

$$f(a+2h) = f\left(\frac{\pi}{6}+2\right) = \sin^2\left(\frac{\pi}{6}+2h\right)$$

$$= \frac{1-\cos\left(\frac{\pi}{3}+4h\right)}{2} = \frac{1-\cos\left(\frac{\pi}{3}+2.2h\right)}{2}$$

$$f(a+3h) = f\left(\frac{\pi}{6}+3h\right) = \sin^2\left(\frac{\pi}{6}+3h\right)$$

$$= \frac{1-\cos\left(\frac{\pi}{3}+6h\right)}{2} = \frac{1-\cos\left(\frac{\pi}{3}+3.2h\right)}{2}$$

$$f(a+\overline{n-1}h) = f\left(\frac{\pi}{6}+\overline{n-1}h\right) = \sin^2\left(\frac{\pi}{6}+\overline{n-1}h\right)$$

$$= \frac{1-\cos\left(\frac{\pi}{3}+2\overline{n-1}h\right)}{2}$$

Now, $\int_a^b f(x)dx = \lim_{h\to 0} h \,|\, f(a) + f(a+h) + f(a+2h) + f(a+3h) + ... + f(a+\overline{n-1}h)$

$$\Rightarrow \int_{\pi/6}^{\pi/2} \sin^2 x dx = \lim h\left[f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{6}+h\right) + f\left(\frac{\pi}{6}+2h\right) + f\left(\frac{\pi}{6}+3h\right) + f\left(\frac{\pi}{6}+\overline{n-1}h\right)\right]$$

$$= \lim h\left[\frac{1-\cos\frac{\pi}{3}}{2} + \frac{1-\cos\left(\frac{\pi}{3}+2h\right)}{2}\frac{1-\cos\left(\frac{\pi}{3}-2.2h\right)}{2} + \frac{1-\cos\left(\frac{\pi}{3}-3.2h\right)}{2} + \frac{1-\cos\left(\frac{\pi}{3}\right]+(n-1)2h\right)}{2}\right.$$

$$= \lim\frac{h}{2}\left\{n - \left[\cos\frac{\pi}{3} + \cos\left(\frac{\pi}{3}+2h\right) + \cos\left(\frac{\pi}{3}+2.2h\right) + \cos\left(\frac{\pi}{3}+3.2h\right) + .. + \cos\left(\frac{\pi}{3}+(n-1)2h\right)\right]\right\}$$

$$\underset{h\to 0}{=\lim}\frac{h}{2}\left\{n - n - \frac{\left\{\cos\frac{\pi}{3} + \cos\left(\frac{n-1}{2}\right)2h\right\}\sin\left(n\frac{2h}{2}\right)}{\sin\left(\frac{2h}{2}\right)}\right\}$$

Notes

$$\lim_{h \to 0} \left[\frac{nh}{2} - \frac{\frac{h}{2}}{\mathrm{sinh}} \cos\left[\frac{\pi}{3} + \frac{2nh-2h}{2}\right] \sin nh \right]$$

$$= \lim \left[\frac{\pi}{6} - \frac{h}{\mathrm{sinh}} \frac{1}{2} \cos\left[\frac{\pi}{3} + \frac{2\frac{\pi}{3} - 2h}{2} \right] \sin\frac{\pi}{3} \right]$$

$$= \frac{\pi}{6} - \frac{1}{2}\cos\frac{2\pi}{3}\sin\frac{\pi}{3} = \frac{\pi}{6} - \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

Task Evaluate the following definite integrals as limit of sums:

1. $\int_0^{\pi/2} (x + \cos x)dx$
2. $\int_0^{\pi/6} (\cos x + \sin x)dx$

Self Assessment

Fill in the blanks:

1. Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$. Divide the interval $[a,b]$ into n equal parts each of width h by points $a + h$, $a+2h$, $a + 3h$, …, $a+(n-1)$ h. Then, $\underbrace{h+h+\ldots+h}_{n\ times} =$ ....
2. If n increases, the number of rectangles will increases and the ....of rectangles will decrease.
3. The area of region bounded by curve y =................................, x-axis and the ordinates $x = a$, $x = b$ is $\lim\limits_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h)]$ where $h = \frac{b-a}{n}$
4. The area in the case of limit as a sum is also the ....value of any area which is among that of the rectangles beneath the curve and that of the rectangles over the curve.
5. $\int_a^b f(x)dx = \lim\limits_{h\to 0} h$ .... where $nh = b - a$.
6. The process of evaluating a definite integral by using the definition $\int_a^b f(x)dx = \lim\limits_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+\overline{n-1}h)]$ is called integration from first principles or integration by .... method or integration as the limit of a sum.
7. $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \left[\frac{n(n-1)}{2}\right]^{-2}$ = ...
8. $\cos a + \cos(a+h) + \cos(a+2h) + \ldots + cis[a+(n-1)h] =$ ....

Notes

9. $1^2+2^2+3^2+\ldots+(n-1)^2=\dfrac{n(n-1)(2n-1)}{6}$ ....

10. $1^3+2^3+3^3+\ldots+(n-1)^3=\left[\dfrac{n(n-1)}{2}\right]^{-2}$ ....

4.2 Fundamental Theorem of Integral Calculus

Fundamental theorem of integral calculus states that if $f(x)$ is a continuous function defined on closed interval $[a,b]$ and $F(x)$ is integral of $f(x)$ i.e., $\int f(x)dx = F(x), then \int f(x)dx = F(x),$ then a is called lower limit, b is called upper limit and $F(b) - F(a)$ is called the value of the definite integral and is always unique.

Notes This theorem is very useful as it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

Did u know? The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand.

This toughens the relationship among differentiation and integration.

Example: Evaluate the following integrals:

(i) $\int_1^3 (2x+1)^3 dx$

(ii) $\int_{-1}^2 \frac{1}{3x-2}dx$

(iii) $\int_0^{\pi/2} \sin^2 x dx$

(iv) $\int_0^{\pi/4} \tan x dx$

(v) $\int_0^{\pi/4} \sec x dx$

(vi) $\int_0^1 \frac{1}{\sqrt{1+x^2}}dx$

(vii) $\int_0^1 \frac{dx}{1+x^2}$

(viii) $\int_2^3 \frac{x}{x^2+1}dx$

Solution:

(i) $I=\int_1^3 (2x+1)^3 dx = \left.\frac{(2x+1)^4}{4\times 2}\right]_1^3 = \left.\frac{1}{8}(2x+1)^4\right]_1^3 = \frac{1}{8}\left[(7)^4-(3)^4\frac{1}{8}(2320)\right] = 290$

Notes

(ii) $I = \int_{-1}^{2} \frac{1}{3x-2} dx = \frac{\log|3x-2|}{3}\Bigg]_{-1}^{2} = \frac{1}{3}[\log|4| - \log|-5|] = \frac{1}{3}\log\frac{4}{5}$

(iii) $I = \int_{0}^{\pi/2} \sin 2x dx = \frac{\cos 2x}{3}\Bigg]_{-1}^{\pi/2} = -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{2}\right) - \cos 2(0)\right]$

$= -\frac{1}{2}[\cos\pi - \cos 0] = -\frac{1}{2}[-1-1] = -\frac{1}{2}(-2) = 1$

(iv) $I = \int_{0}^{\pi/4} \tan x dx = \log|\sec x|\Bigg]_{0}^{\pi/4} = \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0|$

$= \log\sqrt{2} - \log 1 = \log\sqrt{2} = \frac{1}{2}\log 2$

(v) $I = \int_{0}^{\pi/4} \sec x dx = \log|\sec x + \tan x|\Bigg]_{0}^{\pi/4} = \log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log|\sec 0 + \tan 0|$

$= \log|\sqrt{2} - +1| - \log|1+0| = \log|\sqrt{2}+1|$

(vi) $I = \int_{0}^{1} \frac{1}{\sqrt{1+x^2}} dx = \sin^{-1} x\Bigg]_{0}^{1} = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{4}$

(vii) $I = \int_{0}^{1} \frac{x}{1+x^2} dx = \tan^{-1} x\Bigg]_{0}^{1} = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$

(viii) $I = \int_{2}^{3} \frac{x}{x^2+1} dx = \frac{1}{2}\int_{2}^{3} \frac{2x}{x^2+1} dx = \frac{1}{2}\log|x^2+1|\Bigg]_{2}^{3}$

$\frac{1}{2}[\log|10| - \log|5|] = \frac{1}{2}\log\left[\frac{10}{5}\right] = \frac{1}{2}\log 2$

Example:

Evaluate the following integrals:

(i) $\int_0^4 \frac{dx}{x^2-4}$

(ii) $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$

(iii) $\int_0^2 \frac{dx}{4+x-x^2}$

(iv) $\int_0^1 \sqrt{x-x^2}\, dx$

Solution:

(i) $\int_0^4 \frac{dx}{x^2-4} = \int_0^4 \frac{dx}{x^2-(2)^2}$

$= \frac{1}{4}\log\left|\frac{x-2}{x+2}\right|\Bigg]_3^4 = \frac{1}{4}\left[\log\left|\frac{2}{6}\right| - \log\left|\frac{1}{5}\right|\right] = \frac{1}{4}\log\left(\frac{5}{3}\right)$

(ii) $\int_0^4 \frac{dx}{4+x-x^2} = -\int_0^4 \frac{dx}{\sqrt{(x+1)^2+2}} = \log|x+1+\sqrt{x^2|2x+3}\Bigg]_0^4$

$= \log|5+\sqrt{27}| - \log|1+\sqrt{3}| = \log\left|\frac{5+\sqrt[3]{3}}{1+\sqrt{3}}\right| = \log|7+4\sqrt{3}|$

Notes

(iii) $\int_0^2 \frac{dx}{4+x-x^2} = -\int_0^2 \frac{dx}{x^2-x-4}$

$$\int_0^2 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{17}}{2}\right)^2} = -\int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\frac{\sqrt{17}}{2}}\log\left|\frac{\frac{\sqrt{17}}{2}+x-\frac{1}{2}}{\frac{\sqrt{17}}{2}-x+\frac{1}{2}}\right]_0^2\Bigg|$$

$$= \frac{1}{\sqrt{17}}\left\{\log\left|\frac{\frac{\sqrt{17}}{2}+x-\frac{1}{2}}{\frac{\sqrt{17}}{2}-x+\frac{1}{2}}\right| - \log\left|\frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}}\right|\right\}$$

$$= \frac{1}{\sqrt{17}}\left\{\log\left|\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}\times\frac{\frac{\sqrt{17}}{2}+\frac{1}{2}}{\frac{\sqrt{17}}{2}-\frac{1}{2}}\right|\right\}$$

$$= \frac{1}{\sqrt{17}}\log\left|\frac{\left(\sqrt{17}+3\right)\left(\sqrt{17}+1\right)}{\left(\sqrt{17}-3\right)\left(\sqrt{17}-1\right)}\right| = \frac{1}{\sqrt{17}}\log\left|\frac{12+5\sqrt{17}}{4}\right|$$

(iv) $\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{-[x^2-x]}\,dx$

$$= \int_0^1 \sqrt{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}\,dx = \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$= \left|\frac{x-\frac{1}{2}}{2}\sqrt{x-x^2}\right|_0^1 + \frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}}\Bigg]_0^1$$

$$= \left|\frac{2x-1}{4}\sqrt{x-x^2}\right|_0^1 + \frac{1}{8}\sin^{-1}(2x-1)\Big]_0^1$$

$$= \frac{1}{4}\sqrt{0} + \frac{1}{8}\sin^{-1}(1) - \frac{-1}{4}\sqrt{0} - \frac{1}{8}\left|\sin^{-1}(-1) - \sin^{-1}(0)\right|$$

$$= \frac{2}{8}\sin^{-1}(1) = \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$$

Example: Evaluate the following integrals:

(i) $\int_1^2 \frac{dx}{(x+1)(x+2)}$

(ii) $\int_0^3 \frac{dx}{x^2(x+1)}$

Solution:

(i) $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Notes

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\Rightarrow 1 = A(x+2) + B(x+1)$$

Putting $x = -1$, we get $A = 1$

Putting $x = -2$, we get $B = -1$

$$I = \int_1^2 \frac{dx}{(x+1)(x+2)} = \int_1^2 \left[\frac{1}{x+1} + \frac{1}{x+2}\right] dx$$

$$= \log|x+1| - \log|x+2| \;\Big]_1^2$$

$$= (\log 3 - \log 4) - (\log 2 - \log 3) = \log\frac{3}{4} - \log\frac{2}{3} = \log\frac{9}{8}$$

(ii) $\quad I = \int_0^3 \frac{dx}{x^2(x+1)}$

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(a+1) + B(x+1) + Cx^2$$

Putting $x = -1$, we get $C = 1$

Putting $x = 1$, we get $A = -1$

$$I = \int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \left[-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right] dx$$

$$= -\log|x| - \frac{1}{x} + \log|x+1| \Big]_0^3$$

$$= \left(-\log 3 - \frac{1}{3} + \log 4\right) - \left(-\log|1| - \frac{1}{1} + \log|2|\right)$$

$$= -\log 3 - \frac{1}{3} + \log 4 + 1 - \log 2 = \frac{2}{3} + \log\frac{2}{3}$$

Example: Evaluate the following integrals:

(i) $\int_0^{\pi/4} \sin 2x \cos 3x dx$

(ii) $\int_0^{\pi/4} \cos^3 x dx$

(iii) $\int_0^{\pi/4} \sqrt{1 + \sin 2x\, dx}$

(iv) $\int_0^{\pi/2} \sqrt{1 + \cos x dx}$

(v) $\int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx$

Solution:

(i) $\quad I = \int_0^{\pi/4} \sin 2x \cos 3x dx$

Notes

$$= \frac{1}{2}\int_0^{\pi/4} 2\sin 2x\cos 3x dx = \frac{1}{2}\int_0^{\pi/4}[\sin 5x + \sin(-x)]dx$$

$$= \frac{1}{2}\int_0^{\pi/4}[\sin 5x \sin x]\,dx = \frac{1}{2}\left[\frac{-\cos 5x}{5} + \cos x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{-1}{5}\cos\frac{5\pi}{4} + \cos\frac{\pi}{4}\right) - \left(-\frac{1}{5}\cos 0 + \cos 0\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} + \frac{1}{5} - 1\right]$$

$$= \frac{1}{2}\left[\frac{6-4\sqrt{2}}{5\sqrt{2}}\right] = \frac{1}{\sqrt[5]{2}}\left(3 - \sqrt[2]{2}\right) = \frac{1}{10}\left(3\sqrt{2} - 4\right)$$

(ii) $I = \int_0^{\pi/4} \cos^3 x dx = \frac{1}{4}\int_0^{\pi/4}(3\cos x + \cos 3x)dx$

$$= \frac{1}{4}\left[3\sin c + \frac{\sin 3x}{3}\right]_4^{\pi}$$

$$= \frac{1}{4}\left\{\left(3\sin\frac{\pi}{4} + \frac{1}{3}\sin\frac{3\pi}{4}\right) - \left(3\sin 0 - \frac{1}{3}\sin 0\right)\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{\sqrt{2}} + \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)\right\} = \frac{1}{\sqrt[4]{2}}.\frac{10}{3} = \frac{5}{\sqrt[6]{2}}$$

(iii) $I = \int_0^{\pi/4} \sqrt{1 + \sin 2x dx}$

$$= \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x dx}$$

$$= \int_0^{\pi/4} \sqrt{(\sin x + \cos)^2}\,dx = \int_0^{\pi/4} \sin x + \cos x dx$$

$$= -\cos x + si\,nx]_0^{\pi/4}$$

$$= \left(-\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) - (-\cos 0 + \sin 0) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 - 0 = 1$$

(iv) $I = \int_0^{\pi/2} \sqrt{1 + \cos x dx} = \int_0^{\pi/2} \sqrt{2\cos^2\frac{x}{2}dx}$

$$= \sqrt{2}\int_0^{\pi/2} \cos\frac{x}{2}dx$$

$$= \sqrt{2}\left.\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} = 2\sqrt{2}\sin\frac{x}{2}\Big]_1^{\pi/2}$$

$$= 2\sqrt{2}\left[\sin\frac{\pi}{4} - \sin 0\right] = 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right] = 2$$

(v) $I = \int_0^{\pi} \frac{\sin x}{\sin x + \cos x}dx$

Let sin x = A (sin x + cos x) + B (– cos x + sin x)

$\Rightarrow a = A + B$

$0 = A - B$

$\Rightarrow A = \frac{1}{2}B$

Notes

$$\therefore I = \int_0^{\pi} \frac{\frac{1}{2}(\sin x + \cos x) + \frac{1}{2}(-\cos x + \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int_0^{\pi} 1 dx + \frac{1}{2}\int_0^{\pi} \frac{-\cos x + \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} x \Big]_0^{\pi} + \frac{1}{2} \log|\sin x + \cos x| \Big]_0^{\pi}$$

$$= \frac{1}{2}|\pi - 0| + \frac{1}{2}\{\log|\sin\pi + \cos\pi| - \log|\sin 0 + \cos 0|\}$$

$$= \frac{\pi}{2} + \frac{1}{2}\{\log|-1| - \log|1|\} = \frac{\pi}{2}$$

Example: Evaluate the following integrals:

1. $\int_0^1 xe^x dx$

2. $\int_1^2 \log x dx$

3. $\int_0^{\pi/2} x \cos x dx = \sin x \Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx$

4. $\int_0^{\pi} \cos 2x \log \sin x dx$

5. $\int_0^{\pi/6} (2 + 3x^2) \cos 3x dx$

6. $\int_2^4 \frac{x^2 + x}{\sqrt{2x} + 1} dx$

7. $\int_1^e e^x \left(\frac{1 + x \log x}{x}\right) dx$

Solution:

1. $\int_0^1 xe^x dx$

$$= xe^x \Big]_0^1 - \int_0^1 e^x dx$$

$$= [1.e^1 - 0] - e^x \Big]_0^1 = e - [e - e]^0 = e - e + 1 = 1$$

2. $\int_1^2 \log x dx$

$$= \log x . x]_1^2 - \int_1^2 \frac{1}{x} . x dx$$

Notes

$$= (\log 2)(2) - (\log 1)(1) - \left\{ \int_1^2 1dx \right\}$$

$$= 2\log 2 - \left\{ x]_1^2 \right\}$$

$$= 2\log 2 - \{2-1\} = 2\log 2 - 1$$

3. $$\int_0^{\pi/2} x\cos x dx = \sin x \Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx$$

$$= \left[\frac{\pi}{2}\sin\frac{\pi}{2} - 0 \right] - \cos x]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1$$

4. $$\int_0^{\pi} \cos 2x \log \sin x dx$$

$$= \log \sin x \left(\frac{\sin 2x}{2} \right) \Bigg]_0^{\pi} - \int_0^{\pi} \frac{\cos x}{\sin x} . \frac{\sin 2x}{2} dx$$

$$= 0 - \int_0^{\pi} \frac{\cos x}{\sin x} . \frac{2\sin x \cos x}{2} dx = -\int_0^{\pi} \cos^2 x dx = -\int_0^{\pi} \frac{1+\cos 2x}{2} dx$$

$$= -\frac{1}{2}\int_0^{\pi} 1 + \cos 2x dx = -\frac{1}{2}\left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= -\frac{1}{2}\left[\left(\pi + \frac{\sin 2x}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right] = -\frac{\pi}{2}$$

5. $$\int_0^{\pi/6} (2+3x^2)\cos 3x dx$$

$$= (2+3x^2)\frac{\sin 3x}{3} \Bigg]_6^{\pi} - \int_6^{\pi} 6x \frac{\sin 3x}{3} dx$$

$$= \left\{ \left(2 + 3\frac{\pi^2}{36} \right) \frac{1}{3} \sin 3\left(\frac{\pi}{6} \right) - 0 \right\} - 2\int_0^{\pi/6} \frac{-\cos 3x}{3} dx$$

$$= \left(2 + \frac{\pi^2}{12} \right) \frac{1}{3} - 2\left\{ -x\frac{\cos 3x}{3} \right]_0^{\pi/6} - \int_0^{\pi/6} \frac{-\cos 3x}{3} dx$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - 2\left\{ (0+0) + \frac{1}{9}\sin 3x \right]_0^{\pi/6} \Bigg\}$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - \left\{ \frac{2}{9}[1-0] \right\} = \frac{2}{3} + \frac{\pi^2}{36} - \frac{2}{9} = \frac{\pi^2}{36} + \frac{4}{9}$$

6. $$I = \int_2^4 \frac{x^2+x}{\sqrt{2x}+1} dx$$

$$= \int_2^4 (x2+x) - \frac{1}{\sqrt{2x}+1} dx$$

$$= (x2+x)\frac{(2x+1)^{1/2}}{\frac{1}{2}.2} \Bigg]_2^4 - \int_2^4 \{(2x+1)\sqrt{2x+1}\} dx$$

$$= \{(20)(9)^{1/2} - (6)(5)^{1/2} - \int_2^4 (2x+1)^{3/2} dx$$

Notes

$$= \{60 - \sqrt[6]{5}\} - \left.\frac{(2x+1)^{5/2}}{\frac{5}{2}.2}\right]_2^4$$

$$= \{60 - \sqrt[6]{5}\} - \left\{\frac{9^{5/2}}{5} - \frac{5^{5/2}}{5}\right\} = 60 - \sqrt[6]{5} - \frac{243}{5} + \sqrt[5]{5} = \frac{57}{5} - \sqrt{5}$$

7. $\int_1^e e^x \left(\frac{1 + x\log x}{x}\right) dx$

$$= \int_2^{e^2} \frac{1}{\log x}.1dx - \int_2^{e^2} \frac{1}{(\log x)^2} dx$$

$$= \left.\frac{1}{\log x}.x\right]_e^{e^2} - \int_2^{e^2} \frac{1}{(\log x)^2}.\frac{1}{x}.xdx - \int_2^{e^2} \frac{1}{(\log x)^2} dx$$

$$= \left\{\frac{e^2}{\log e^2} - \frac{e}{\log e}\right\} + \int_e^{e^2} \frac{1}{(\log x)} dx - \int_e^{e^2} \frac{1}{(\log x)^2} dx$$

$$= \frac{e^2}{2\log e} - e = \frac{e^2}{2} - e.$$

Example: If $\int_a^b x^3 dx = 0$

$$\Rightarrow \left.\frac{x^4}{4}\right]_a^b = 0 \Rightarrow \frac{b^4}{4} - \frac{a^4}{4} = 0$$

$$\Rightarrow b^4 = a^4 \Rightarrow a = \pm b$$

Also $\int_a^b x^2 dx = \frac{2}{3}$

$$\Rightarrow \left.\frac{x^3}{3}\right]_a^b = \frac{2}{3} \Rightarrow \frac{b^3}{3} - \frac{a^3}{3} = \frac{2}{3} \Rightarrow b^3 - a^3 = 2$$

When $a = b$, then

$a^3 - a^3 = 2$ which is absurd $\therefore a = -b$

$$\Rightarrow -a^3 - a^3 = 2 \Rightarrow -2a^3 = 2 \Rightarrow a^3 = -1 \Rightarrow a = -1$$

thus $b = 1$

Hence, $a = -1$ and $b = 1$

Example: If $\int_0^k \frac{dx}{2 + 8x^2} = \frac{\pi}{16}$, Find the value of k.

$$\Rightarrow \frac{1}{8}\int_0^k \frac{dx}{\frac{1}{4} + x^2} = \frac{\pi}{16}$$

$$\Rightarrow \frac{1}{8}\int_0^k \frac{dx}{\left(\frac{1}{2}\right)^2 + x^2} = \frac{\pi}{2}$$

Notes

$$\Rightarrow \frac{1}{\frac{1}{2}}\tan^{-1}\frac{x}{\frac{1}{2}}\Bigg]_0^k = \frac{\pi}{2}$$

$$\Rightarrow 2\tan^{-1}2x\Big]_0^k = \frac{\pi}{2}$$

$$\Rightarrow 2\tan^{-1}2k - 0 = \frac{\pi}{2}$$

$$\Rightarrow 2\tan^{-1}2k = \frac{\pi}{4} \Rightarrow 2k = \tan\frac{\pi}{4}$$

$$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

Task Evaluate the following integrals:

1. $\int_2^3 \frac{dx}{x^2-1}$
2. $\int_0^1 \frac{1-x^2}{1+x^2}dx$

Self Assessment

Fill in the blanks:

11. .... states that if $f(x)$ is a continuous function defined on closed interval $[a, b]$ and $F(x)$ is integral of $f(x)$.
12. $\int f(x)dx = F(x)$, then $\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$ ....
13. In Fundamental theorem of integral calculus, a is called lower limit, b is called upper limit and $F(b) - F(a)$ is called the.................................of the definite integral.
14. The Fundamental theorem of integral calculus is very useful as it gives us a method of calculating the definite integral more easily without calculating the ....
15. The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the ....

4.3 Summary

- The Definite Integral comprises extensive number of applications in mathematics, the physical sciences and engineering.
- Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$. Divide the interval $[a, b]$ into n equal parts each of width h by points $a + h$, $a + 2h$, $a + 3h$, …, $a + (n - 1) h$.

Notes

- In case of definite integral as limit of sum, the areas of inner rectangles are $hf(a), hf(a+h), hf(a+2h), hf(a+3h), \ldots, hf(a+\overline{n-1}h)$.
- The area In case of definite integral as limit of sum, is close to the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$.
- If n increases, the number of rectangles will increases and the width of rectangles will decrease.
- The process of evaluating a definite integral by using the above definition is called integration from first principles or integration by ab-initio method or integration as the limit of a sum.
- Fundamental theorem of integral calculus states that if $f(x)$ is a continuous function defined on closed interval $[a, b]$ and $F(x)$ is integral of $f(x)$ i.e., $\int f(x)dx = F(x)$, then

  $$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

- Fundamental theorem of integral calculus is very useful as it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

4.4 Keywords

Definite Integral: The Definite Integral comprises extensive number of applications in mathematics, the physical sciences and engineering.

Fundamental Theorem of Integral Calculus: It states that if $f(x)$ is a continuous function defined on closed interval $[a, b]$ and $F(x)$ is integral of $f(x)$.

4.5 Review Questions

1. Elucidate the concept of Definite Integral as the Limit of a Sum. Give examples.
2. Evaluate the definite integral $\int_0^3 x + 3dx$ as limit of sums.
3. Evaluate the definite integral $\int_1^5 1 - xdx$ as limit of sums.
4. Evaluate the definite integral $\int_2^4 3x + 1dx$ as limit of sums.
5. Evaluate the definite integral $\int_0^2 3x^2 + 2dx$ as limit of sums.
6. Evaluate the definite integral $\int_1^4 2x - x^2dx$ as limit of sums.
7. Evaluate the definite integral $\int_1^3 x^2 + 2xdx$ as limit of sums.

Notes

8. Evaluate the integral $\int_1^4 \sqrt{3x+7}\,dx$

9. Evaluate the integral $\int_0^5 \frac{1}{\sqrt{1+2x}}\,dx$

10. Evaluate the integral $\int_{-1}^1 \frac{1}{2x-3}\,dx$

11. Evaluate the integral $\int_1^3 (x^2-3x)\,dx$

12. Evaluate the integral $\int_0^{1/2} \frac{dx}{\sqrt{1-x}}\,dx$

Answers: Self Assessment

1. $b - a$
2. width
3. $f(x)$
4. limiting
5. $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+\overline{n-1}h)]$
6. ab-initio
7. $=\left[\frac{n(n-1)}{2}\right]^{-2}$
8. $=\frac{\cos\left[a+\left(\frac{n-1}{2}\right)h\right]\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$
9. $=\frac{n(n-1)(2n-1)}{6}$
10. $=\left[\frac{n(n-1)}{2}\right]^{-2}$
11. Fundamental theorem of integral calculus
12. $F(x)\big]_a^b = F(b) - F(a)$
13. value
14. limit of a sum
15. integrand

Notes

4.6 Further Readings

Books

Douglas S. Kurtz, Jaroslav Kurzweil, Charles Swartz, *Theories of Integration*, World Scientific

G. H. Hardy, T. W. Körner, *A Course of Pure Mathematics*, Cambridge University Press

Morris Kline, *Calculus: An Intuitive and Physical Approach*, Courier Dover Publications

Ron Larson, David C. Falvo, *Calculus: an Applied Approach*, Cengage Learning.

Online link

www.intmath.com

Unit 5: Definite Integrals by Substitution

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the Substitution rule for definite integrals
- Discuss the concept of definite integrals by substitution

Introduction

As we know, the first step in performing a definite integral is to calculate the indefinite integral and that hasn't tainted. We will still calculate the indefinite integral initially. This signifies that we by now know how to perform these. Now, in this unit, we make use of the substitution rule to locate the indefinite integral and then perform the evaluation.

5.1 Substitution Rule for Definite Integrals

There are though, two methods to treat with the assessment step.

The steps for performing integration by substitution for definite integrals are the similar as the steps for integration by substitution for indefinite integrals apart from we must alter the bounds of integration and we do not require subbing back in for u.

1. Let $u = g(x)$.
2. Find $du/dx = g'(x)$
3. Let $du = g'(x)\, dx$. Now, confirm that this is included in the unique integral. If not, then you cannot utilize this method.
4. Substitute u in for $g(x)$ and du in for $g'(x)\, dx$.
5. Locate the new bounds of integration by plugging in the lower bound into u. That consequence will be the new lower bound. Then plug in the upper bound into u. This will be the new upper bound.
6. Integrate the new integral.
7. Plug in the new bounds and calculate.

Notes

So, we can say, the Substitution Rule for Definite Integrals state: If f is continuous on the range of $u = g(x)$ and $g'(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Did u know? One of the methods of performing the assessment is perhaps the most understandable at this point, but also has a point in the procedure where we can dig up in problem if we aren't paying awareness.

Self Assessment

Fill in the blanks:

1. The first step in performing a definite integral is to calculate the .... integral.
2. The steps for performing integration by substitution for definite integrals are .... as the steps for integration by substitution for indefinite integrals.
3. If f is continuous on the range of $u = g(x)$ and $g'(x)$ is continuous on $[a, b]$, then ....

State whether the following statements are true or false:

4. Performing integration by substitution for definite integrals is different from performing integration by substitution for indefinite integrals.
5. The new bounds of integration are located by plugging in the lower bound.

5.2 Use Substitution to Find Definite Integrals

To Use Substitution to find Definite Integrals, you are required to perform either:

- Calculate the indefinite integral, articulating an antiderivative in terms of the original variable, and then assess the consequence at the original limits, or
- Translate the original limits to new limits in provisions of the new variable and do not translate the antiderivative back to the original variable.

  Let us now describe both methods of performing the evaluation step.

  Consider the following definite integral.

  We will illustrate here, the evaluation of the given integral by means of two different methods.

  $$\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\,dt$$

  Let's begin off from first method of dealing with the assessment step.

Caution We are required to be cautious with this method as there is a point in the procedure where if we aren't paying awareness we'll obtain the wrong solution.

Solution 1:

Notes

Firstly, we calculate the indefinite integral by means of the substitution rule.

Here, the substitution is,

$$u = 1 - 4t^3 \qquad du = -12t^2 dt \qquad \Rightarrow \qquad t^2 dt = -\frac{1}{12} du$$

Plugging this into the integral provides,

$$\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt = -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\, du$$
$$= -\frac{1}{9} u^{\frac{3}{2}} \Big|_{-2}^{0}$$

Observe that we didn't perform the evaluation yet. This is where the latent problem occurs with this solution method. The limits specified here are from the original integral and thus are values of *t*. We have *u*'s in our solution. We can't plug values of *t* in for *u*.

Thus, we will have to get back to *t*'s before we perform the substitution.

Did u know? Getting back to t's before performing the substitution is the standard step in the substitution procedure, but it is frequently forgotten when performing definite integrals.

Note also that here, if we don't move back to *t*'s we will have a little problem in that one of the evaluations will finish up providing us a complex number.

So, concluding this problem provides,

$$\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt = -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}} \Big|_{-2}^{0}$$
$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right)$$
$$= \frac{1}{9}\left(33\sqrt{33} - 1\right)$$

Thus, that was the first solution method. Now let us observe the second method.

Solution 2:

Notes This solution method isn't actually all that dissimilar from the first method. In this method we are going to keep in mind that when performing a substitution we want to remove all the *t*'s in the integral and write all in terms of *u*.

When we say all here we actually mean all.

Caution Keep in mind that the limits on the integral are also values of *t* and we're going to translate the limits into *u* values.

Converting the limits is quite simple as our substitution will tell us how to relate *t* and *u* therefore all we require to do is plug in the original *t* limits into the substitution and we'll obtain the new *u* limits.

Notes

Here is the substitution (it's similar as the first method) in addition to the limit conversions.

$$u = 1 - 4t^3 \qquad du = -12t^2 dt \qquad \Rightarrow \qquad t^2 dt = -\frac{1}{12} du$$

$$t = -2 \qquad \Rightarrow \qquad u = 1 - 4(-2)^3 = 33$$
$$t = 0 \qquad \Rightarrow \qquad u = 1 - 4(0)^3 = 1$$

The integral is now,

$$\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt = -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}} du$$
$$= -\frac{1}{9} u^{\frac{3}{2}} \Big|_{33}^{1}$$

As with the first method let us gap here a second to remind us what we're performing. Here, we've converted the limits to u's and we've also obtained our integral in terms of u's and so here we can just plug the limits straightforwardly into our integral.

Did u know? Here we would not plug our substitution back in. Performing this here would cause troubles as we would have t's in the integral and our limits would be u's.

We have given below the rest of this problem.

$$\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt = -\frac{1}{9} u^{\frac{3}{2}} \Big|_{33}^{1}$$
$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33} - 1\right)$$

We got precisely the similar answer and this time didn't have to concern regarding going back to t's in our answer.

Thus, we've observed two solution methods for computing definite integrals that need the substitution rule. Both are valid solution methods and each include their uses. We will be using the second completely though because it makes the evaluation step a little simpler.

Notes Observe that, though we will persistently remind ourselves that this is a definite integral by putting the limits on the integral at every step. Without the limits it's easy to overlook that we had a definite integral when we've gotten the indefinite integral calculated.

Task Illustrate the two methods used to evaluate definite integral by substitution.

Let us perform some examples.

Example: Evaluate each of the following:

(a) $\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$

Notes

(b) $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx$

(c) $\int_{0}^{\frac{1}{2}} e^y + 2\cos(\pi y)\,dy$

(d) $\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$

Solution:

Because we've done performed a few substitution rule integrals, here we aren't going to put a lot of exertion into elucidating the substitution part of things here.

(a) $\int_{-1}^{5} (1+w)(2w+w^2)^5\,dw$

The substitution and converted limits are,

$$u = 2w + w^2 \qquad du = (2+2w)\,dw \qquad \Rightarrow (1+w)\,dw = \frac{1}{2}dw$$

$$w = -1 \quad \Rightarrow \quad u = -1 \qquad\qquad w = 5 \quad \Rightarrow \quad u = 35$$

At times a limit will stay similar after the substitution. Don't get thrilled when it takes place and don't imagine it to occur all the time.

Here is the integral,

$$\int_{-1}^{5} (1+w)(2w+w^2)^5\,dw = \frac{1}{2}\int_{-1}^{35} u^5\,du$$

$$= \frac{1}{2}u^6\Big|_{-1}^{35} = 153188802$$

(b) $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx$

Here is the substitution and converted limits for this problem,

$$u = 1 + 2x \qquad du = 2dx \qquad \Rightarrow dx = \frac{1}{2}du$$

$$x = -2 \quad \Rightarrow \quad u = -3 \qquad\qquad x = -6 \quad \Rightarrow \quad u = -11$$

The integral is then,

$$\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx = \frac{1}{2}\int_{-3}^{-11} 4u^{-3} - \frac{5}{u}\,du$$

$$= \frac{1}{2}\left(-2u^{-2} - 5\,\text{In}\,|u|\right)\Big[_{3}^{11}$$

$$= \frac{1}{2}\left(-\frac{2}{121} - 5\,\text{In}\,11\right) - \frac{1}{2}\left(-\frac{2}{9} - 5\,\text{In}3\right)$$

$$= \frac{112}{1089} - \frac{5}{2}\,\text{In}\,11 + \frac{5}{2}\,\text{In}\,3$$

(c) $\int_{0}^{\frac{1}{2}} e^y + 2\cos(\pi y)\,dy$

This integral requires to be divided into two integrals as the first term doesn't need a substitution and the second does.

Notes

$$\int_0^{\frac{1}{2}} e^y + 2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}} e^y dy + \int_0^{\frac{1}{2}} 2\cos(\pi y)\,dy$$

Here is the substitution and converted limits for the second term.

$$u = \pi y \qquad du = \pi dy \qquad \Rightarrow \qquad dy = \frac{1}{\pi}du$$

$$y = 0 \qquad \Rightarrow \qquad u = 0 \qquad\qquad y = \frac{1}{2} \qquad \Rightarrow \qquad u = \frac{\pi}{2}$$

Here is the integral.

$$\int_0^{\frac{1}{2}} e^y + 2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}} e^y dy + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du$$

$$= e^y\Big|_0^{\frac{1}{2}} + \frac{2}{\pi}\sin u\Big|_0^{\frac{\pi}{2}}$$

$$= e^{\frac{1}{2}} - e^0 + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\sin 0$$

$$= e^{\frac{1}{2}} - 1 + \frac{2}{\pi}$$

(d) $\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$

This integral will need two substitutions. So first divide the integral so we can accomplish a substitution on each term.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = \int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right)dz - \int_{\frac{\pi}{3}}^{0} 5\cos(\pi - z)\,dz$$

There are the two substitutions for these integrals.

$$u = \frac{z}{2} \qquad du = \frac{1}{2}dz \qquad \Rightarrow \qquad dz = 2du$$

$$z = \frac{\pi}{3} \qquad \Rightarrow u = \frac{\pi}{6} \qquad\qquad z = 0 \qquad \Rightarrow \qquad u = 0$$

$$v = \pi - z \qquad dv = -dz \qquad \Rightarrow \qquad dz = -dv$$

$$z = \frac{\pi}{3} \qquad \Rightarrow \qquad v = \frac{2\pi}{3} \qquad\qquad z = 0 \qquad \Rightarrow \qquad v = \pi$$

Here is the integral for this problem.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = 6\int_{\frac{\pi}{6}}^{0} \sin(u)\,du + 5\int_{\frac{2\pi}{3}}^{\pi} \cos(v)\,dv$$

$$= -6\cos(u)\Big|_{\frac{\pi}{6}}^{0} 5\sin(v)\Big|_{\frac{2x}{3}}^{x}$$

$$= 3\sqrt{3} - 6 + \left(\frac{5\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} - 6$$

Example: Evaluate each of the following.

(a) $\int_{-5}^{5} \frac{4t}{2 - 8t^2}\,dt$

Notes

(b) $\int_{3}^{5} \frac{4t}{2-8t^2} dt$

Solution:

(a) $\int_{-5}^{5} \frac{4t}{2-8t^2} dt$

Be cautious with this integral. The denominator is zero at and both of these are in the gap of integration. Thus, this integrand is not constant in the interval and so the integral can't be completed.

Be cautious with definite integrals and be on the lookout for division by zero problems. In the preceding section they were simple to spot as all the division by zero problems that we had there were at zero. Once we go into substitution problems though they will not always be so easy to spot so confirm that you first take a rapid look at the integrand and observe if there are any continuity problems with the integrand and if they take place in the interval of integration.

(b) $\int_{3}^{5} \frac{4t}{2-8t^2} dt$

Now, here the integral can be completed since the two points of discontinuity, $x = \pm\frac{1}{2}$, are both outside of the interval of integration. The substitution and converted limits here are,

$u = 2 - 8t^2 \qquad du = -16t\, dt \qquad \Rightarrow \qquad t\, dt = -\frac{1}{16} dt$

$t = 3 \qquad \Rightarrow \qquad u = -70 \qquad t = 5 \qquad \Rightarrow \qquad u = -198$

The integral is then,

$$\begin{aligned}\int_{3}^{5} \frac{4t}{2-8t^2} dt &= -\frac{4}{16}\int_{-70}^{-198} \frac{1}{u} du \\ &= -\frac{1}{4}\text{In}|u|\Big|_{-70}^{-198} \\ &= -\frac{1}{4}\left(\text{In}(198) - \text{In}(70)\right)\end{aligned}$$

Example: Evaluate each of the following:

(a) $\int_{0}^{\text{In}(1+\pi)} e^x \cos\left(1-e^x\right) dx$

(b) $\int_{e^2}^{e^6} \frac{\left[\text{In } t\right]^4}{t} dt$

(c) $\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}} dP$

(d) $\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x)) dx$

(e) $\int_{\frac{1}{50}}^{2} \frac{e^{\frac{2}{w}}}{w^2} dw$

Solution:

(a) $\int_{0}^{\text{In}(1+\pi)} e^x \cos\left(1-e^x\right) dx$

Notes

The limits are a little strange in this case, but that will occur sometimes so don't get too energized regarding it. Here is the substitution.

$u - 1 - e^2 \qquad du = -e^x dx$

$x = 0 \qquad \Rightarrow \qquad u = 1 - e^0 = 1 - 1 = 0$

$x = \text{In}(1+\pi) \qquad \Rightarrow \qquad u = 1 - e^{\text{In}(1-\pi)} = 1 - (1-\pi) = -\pi$

The integral is then,

$$\int_0^{\text{In}(1+\pi)} e^x \cos\left(1 - e^2\right) dx = -\int_0^{-\pi} \cos u \, du$$

$$= -\sin(u)\Big|_0^{-\pi}$$

$$= -\left(\sin(-\pi) - \sin 0\right) = 0$$

(b) $\displaystyle\int_{e^2}^{e^6} \frac{[\text{In } t]^4}{t} dt$

Here is the substitution and converted limits for this problem.

$u = In\ t \qquad du = \frac{1}{t} dt$

$t = e^2 \qquad \Rightarrow \qquad u = \text{In } e^2 = 2 \qquad\qquad t = e^6 \qquad \Rightarrow \qquad u = \text{In } e^6 = 6$

The integral is,

$$\int^{e^6} \frac{[In\ t]^4}{t} dt = \int_2^6 u^4 du$$

$$= \frac{1}{5} u^5 \Big|_2^6$$

$$= \frac{7744}{5}$$

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2 + \sec(3P)}} dP$

Here is the substitution and converted limits and don't get too thrilled about the substitution. It's a little chaotic in the case, but that can take place on occasion.

$u = 2 + \sec(3P) \qquad du = 3 \sec(3P) \tan(3P) dP \qquad \Rightarrow \qquad \sec(3P)\tan(3P)dP = \frac{1}{3} du$

$P = \frac{\pi}{12} \qquad \Rightarrow \qquad u = 2 + \sec\left(\frac{\pi}{4}\right) = 2 + \sqrt{2}$

$P = \frac{\pi}{9} \qquad \Rightarrow \qquad u = 2 + \sec\left(\frac{\pi}{3}\right) = 4$

Here is the integral,

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2 + \sec(3P)}} dP = \frac{1}{3} \int_{2+\sqrt{2}}^{4} u^{-\frac{1}{3}} du$$

$$= \frac{1}{2} u^{\frac{2}{3}} \Big|_{2+\sqrt{2}}^{4}$$

$$= \frac{1}{2}\left(4^{\frac{2}{3}} - \left(2 + \sqrt{2}\right)^{\frac{2}{3}}\right)$$

So, not only was the substitution chaotic, but we also a untidy answer, but again that's life on occasion.

Notes

(d) $\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

This problem not as awful as it appears. Here is the substitution and converted limits.

$$u = \sin x \qquad du = \cos x\, dx$$

$$x = \frac{\pi}{2} \quad \Rightarrow \quad u = \sin\frac{\pi}{2} = 1 \qquad x = -\pi \quad \Rightarrow \quad u = \sin(-\pi) = 0$$

The cosine in the very front of the integrand will get substituted away in the differential and so this integrand in fact simplifies down considerably. Here is the integral.

$$\begin{aligned}\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx &= \int_0^1 \cos u\, du \\ &= \sin(u)\Big|_0^1 \\ &= \sin(1) - \sin(0) \\ &= \sin(1)\end{aligned}$$

Don't get energized about these types of answers. On occasion we will finish up with trig function assessments like this.

(e) $\int_{\frac{1}{50}}^{2} \frac{e^{\frac{2}{w}}}{w^2}\,dw$

This is also a complicated substitution (at least until you see it). Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}\,dw \qquad \Rightarrow \qquad \frac{1}{w^2}\,dw = -\frac{1}{2}\,du$$

$$w = 2 \quad \Rightarrow \quad u = 1 \qquad w = \frac{1}{50} \quad \Rightarrow \quad u = 100$$

Here is the integral.

$$\begin{aligned}\int_{\frac{1}{50}}^{2} \frac{e^{\frac{2}{w}}}{w^2}\,dw &= -\frac{1}{2}\int_{100}^{1} e^u\, du \\ &= -\frac{1}{2}e^u\Big|_{100}^{1} \\ &= \frac{1}{2}\left(e^1 - e^{100}\right)\end{aligned}$$

Task Evaluate $\int_1^2 \frac{x\,dx}{\left(x^2+2\right)^3}$ by using substitution method.

Self Assessment

Fill in the blanks:

6. The first method used to evaluate definite integral by substitution is to calculate the indefinite integral, articulating an antiderivative in terms of the original variable, and then assess the consequence at the .... limits.
7. The second method used to evaluate definite integral by substitution is to .... the original limits to new limits in provisions of the new variable and do not translate the antiderivative back to the original variable.
8. Sometimes a .... will stay similar after the substitution.

Notes

9. There is a point in the procedure where if we aren't paying awareness we'll obtain the .... solution.
10. The second solution method isn't actually all that dissimilar from the .... method.
11. Getting back to t's before performing the substitution is the standard step in the .... procedure, but it is frequently forgotten when performing definite integrals.
12. If we don't move back to *t*'s we will have a little problem in that one of the evaluations will finish up providing us a ....
13. Be cautious with definite integrals and be on the lookout for .... problems.

State whether the following statements are true or false:

14. In first method, when performing a substitution we want to remove all the *t*'s in the integral and write all in terms of *u*.
15. We are required to be alert with the first method as there is a point in the process where if we aren't paying awareness we'll obtain the wrong solution.

5.3 Summary

- The first step in performing a definite integral is to calculate the indefinite integral and that hasn't tainted.
- The steps for performing integration by substitution for definite integrals are the similar as the steps for integration by substitution for indefinite integrals apart from we must alter the bounds of integration and we do not require subbing back in for u.
- The Substitution Rule for Definite Integrals state: If f is continuous on the range of $u = g(x)$ and $g'(x)$ is continuous on $[a, b]$, then $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
- To Use Substitution to find Definite Integrals, you are required to perform either, calculate the indefinite integral, articulating an antiderivative in terms of the original variable, and then assess the consequence at the original limits.
- Also, another method is to translate the original limits to new limits in provisions of the new variable and do not translate the antiderivative back to the original variable.
- We are required to be cautious with this method as there is a point in the procedure where if we aren't paying awareness we'll obtain the wrong solution.
- The second solution method isn't actually all that dissimilar from the first method.
- Both are valid solution methods and each include their uses.

5.4 Keyword

Substitution Rule: The Substitution Rule for Definite Integrals state: If f is continuous on the range of $u = g(x)$ and $g'(x)$ is continuous on $[a, b]$, then $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

5.5 Review Questions

1. Illustrate the steps for performing integration by substitution for definite integrals with example.

Notes

2. Out of the two different methods used to evaluate definite integral by substitution, which one is better to use? Illustrate.

3. Evaluate $\int_{\pi}^{3\pi/2} \sqrt{\sin x + 1} \cos x \, dx$. by using substitution.

4. Evaluate $\int_{1}^{2} \frac{3}{t^2 + 4} \, dt$ by using substitution.

5. Evaluate $\int_{1}^{3} (9 + x)^2 \, dx$ by using substitution.

6. Evaluate $\int_{0}^{1} (x + 5)^4 \, dx$ by using substitution.

7. Evaluate $\int_{-1}^{1} (1 + x)^3 \, dx$ by using substitution.

8. Evaluate $\int_{0}^{\pi/2} \cos(1 + x) \, dx$ by using substitution.

9. Evaluate $\int_{-1}^{0} x^3 \left(1 + 2x^4\right)^3 \, dx$ by using substitution.

10. Evaluate $\int_{1}^{4} \sqrt{2x + 1} \, dx$ by using substitution.

Answers: Self Assessment

1.	indefinite	2.	similar
3.	$\int\limits_{g(a)}^{g(b)} f(u) \, du$	4.	False
5.	True	6.	original
7.	translate	8.	limit
9.	wrong	10.	first
11.	substitution	12.	complex number
13.	division by zero	14.	False
15.	True		

5.6 Further Readings

Books

Douglas S. Kurtz, Jaroslav Kurzweil, Charles Swartz, *Theories of Integration*, World Scientific.

G. H. Hardy, T. W. Körner, *A Course of Pure Mathematics*, Cambridge University Press.

Morris Kline, *Calculus: an Intuitive and Physical Approach*, Courier Dover Publications.

Ron Larson, David C. Falvo, *Calculus: an Applied Approach*, Cengage Learning.

Online link

tutorial.math.lamar.edu/Classes/CalcI/SubstitutionRuleDefinite.aspx

Notes

Unit 6: Properties of Definite Integral

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of properties of definite integral
- Explain the various definite integral properties

Introduction

As we know, integration in basic word is the inverse of differentiation. There are two types of integrals definite and indefinite integral. Definite integral is where the integration is performed in a specific interval. Usually the integration is performed between a to b, a is known as lower limit and b is known as upper limit. For integration of definite integral we integrate the specified function first then apply the upper and lower limit (we subtract upper limit from lower limit). There are certain properties of definite integral which are discussed in this unit.

Why do we necessitate to study these properties of the definite integral? These properties will approach to the fundamental theorem of calculus. This theorem is perhaps one of the most functional in all of mathematics. It converts the integral from a mathematical inquisitiveness to a prevailing tool that is accessed in science, engineering, economics and many other areas.

6.1 Properties of the Integral

The fundamental properties of integrals are simply attained for us since the integral is defined straightforwardly by differentiation. Therefore we can pertain all the rules we know regarding derivatives to attain analogous facts with reference to integrals.

Notes

Did u know? The properties of definite integral make the integration simpler.

6.1.1 Integrability on all Subintervals

When a function involves a calculus integral on an interval it must also contain a calculus integral on all subintervals.

Theorem (integrability on subintervals): If f is integrable on a closed, bounded interval $[a,b]$ then f is integrable on any subinterval $[c,d] \subset [a,b]$.

6.1.2 Additivity of the Integral

When a function involves a calculus integral on two contiguous intervals it must also contain a calculus integral on the amalgamation of the two intervals.

Did u know? The integral on the large interval is the sum of the other two integrals.

Theorem (additivity of the integral): Suppose f and g be integrable functions on the interval $[a; b]$. Subsequently $f+g$ is also integrable on $[a; b]$ and we have

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof:

Let f and g are integrable on [a; b]. Select an arbitrary n $\in$ N. Then we observe that there exists a step function $s_n \le f$ so that $\int_a^b s_n(x)\,dx \ge \underline{I}(f) - \frac{1}{4n} = \int_a^b f(x)\,dx - \frac{1}{4n}$, or else $\underline{I}(f) - \frac{1}{4n}$ would be an upper bound to $S = \{\int_a^b s(x)\,dx \mid s \le f$; s is a step function} less than $\underline{I}(f)$, in destruction of the definition $\underline{I}(f) = \sup(S)$

Similarly we locate $t_n, \tilde{s}_n$ and with $\tilde{t}_n$, $\int_a^b t_n(x)\,dx \le \int_a^b f(x)\,dx + \frac{1}{4n}, \int_a^b \tilde{s}_n(x)\,dx \ge \int_a^b g(x)\,dx - \frac{1}{4n}$.

and $\int_a^b \tilde{t}_n(x)\,dx \le \int_a^b g(x)\,dx + \frac{1}{4x}$.

Merging these equations, and by means of the additivity for step functions provides us

That

$$\int_a^b (s_n + \tilde{s}_n)(x)\,dx \ge \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \frac{1}{2n},$$

$$\int_a^b (t_n + \tilde{t}_n)(x)\,dx \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \frac{1}{2n}.$$

Furthermore we know that $s_n + \tilde{s}_n \le f + g \le t_n + \tilde{t}_n$ (for all $n \in N$), so $s_n + \tilde{s}_n$ is a step function bounding $f + g$ from below, and $t_n + \tilde{t}_n$ is a step function bounding $f + g$ from above. Therefore we locate that (still, for all $n \in N$) we have

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \frac{1}{2n} \ge \overline{I}(f+g) \ge \underline{I}(f+g) \ge \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \frac{1}{2n}$$

for all $n \in N$.

Notes

As the lower integral and the upper integral are now appeared to be equivalent we discover that $f + g$ is integrable, and its integral equals the value of the lower integral, that is

$$\int_a^b (f+g)(x)\,dx = \overline{I}(f+g) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

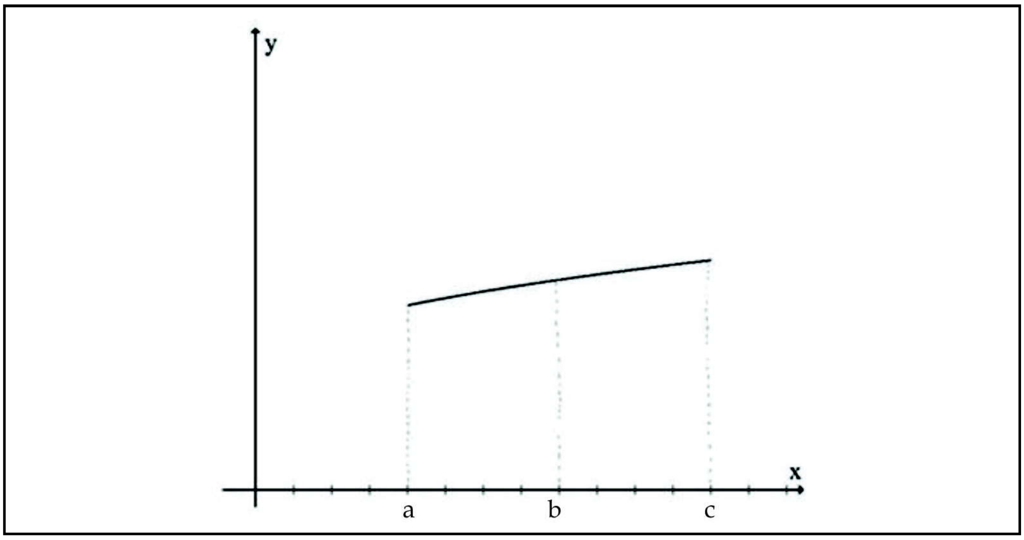
as preferred.

Alternate Theorem

This theorem can also be established as

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

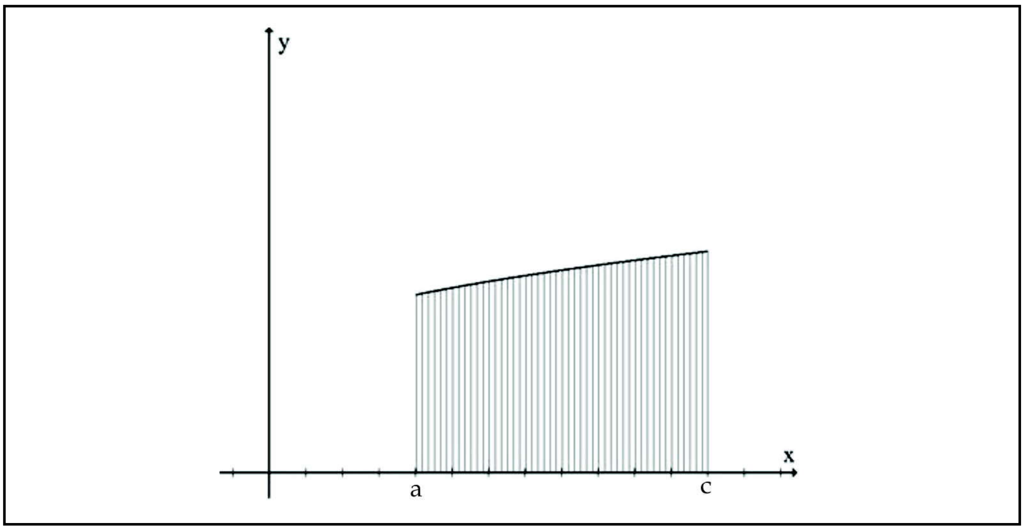
Let us suppose we have a function $f(x)$, and three points on the x axis: a, b and c:



We have that $a < b < c$. The definite integral

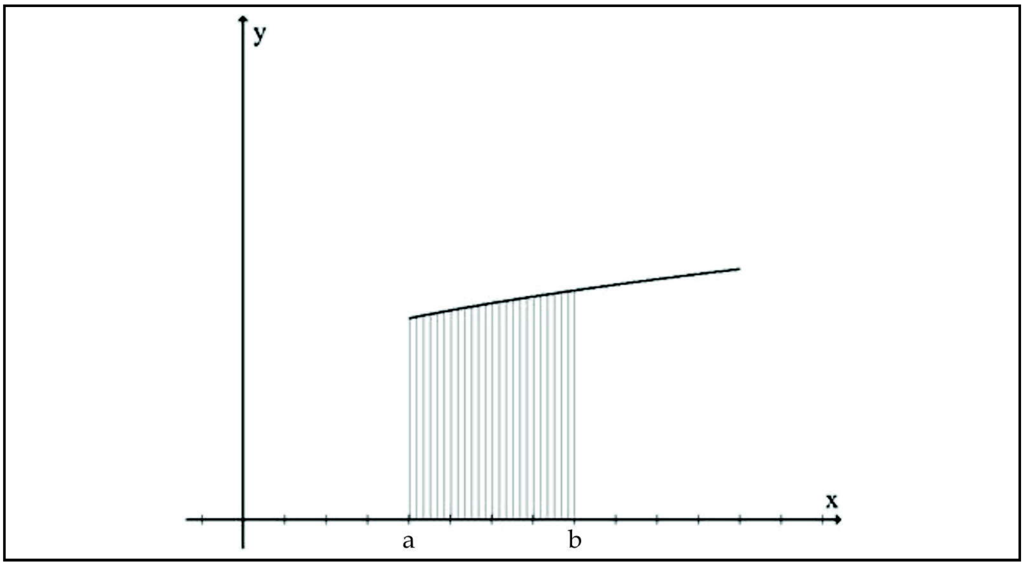
$\int_a^c f(x)\,dx$

Provides us the area under the curve from a to c:

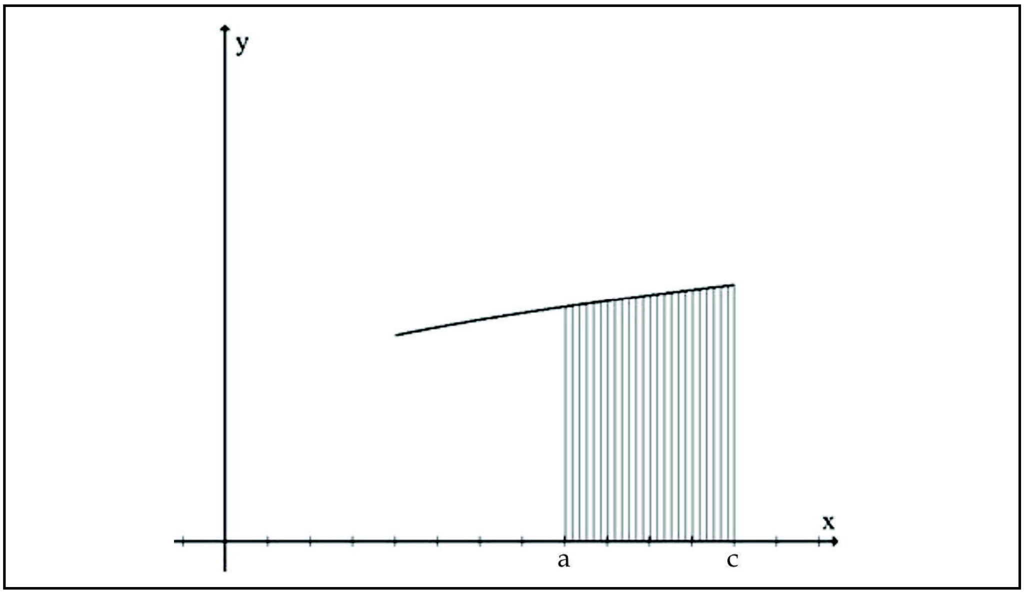


Notes

The integral from a to b equals this area:



And the integral from b to c equals this area:



We can observe evidently from the graphs that if we sum these two last areas we'll obtain the area from a to c. That is:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Notes These numbers don't require to be in the order I. That is, it is not essential for them to seize the relationship: a < b < c.

Notes

6.1.3 Inequalities for Integrals

Larger functions contain larger integrals. The formula for *inequalities*:

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

if $f(x) \le g(x)$ for all but finitely several points x in (a,b).

Theorem (integral inequalities): Assume that the two functions f, g are both integrable on a closed, bounded interval $[a, b]$ and that $f(x) \le g(x)$ for all $x \in [a,b]$ with probably finitely many exceptions. Then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

The proof is a simple exercise in derivatives. We recognize that if H is consistently continuous on $[a, b]$ and if $\frac{d}{dx}H(x) \ge 0$ for all but finitely several points x in (a,b) then $H(x)$ must be non-decreasing on $[a,b]$.

Task Accomplish the details needed to prove the inequality formula of Theorem..

6.1.4 Linear Combinations

Formula for linear combination is as below:

$$\int_a^b \left[rf(x) + sg(x)\right] dx = r\int_a^b f(x)\,dx + s\int_a^b g(x)\,dx \quad (r, s \in \mathbb{R}).$$

This is a specific statement of what we mean by this formula: If both functions $f(x)$ and $g(x)$ contain a calculus integral on the interval $[a,b]$ then any linear combination $r\,f(x)+s\,g(x)$ $(r, s \in \mathrm{R})$ also encloses a calculus integral on the interval $[a,b]$ and, furthermore, the identity is ought to be hold. We know that

$$\frac{d}{dx}\left(rF(x) + sG(x)\right) = rF'(x) + sG'(x)$$

at any point x at which both F and G are differentiable.

Example: We have

$$\int_0^1 \left(x^2 - 2x\right)dx = \int_0^1 x^2 dx - 2\int_0^1 x\,dx.$$

$$\int_0^1 x^2\,dx = \frac{1}{3} \text{ and } \int_0^1 x\,dx = \frac{1}{2}.$$

Thus

$$\int_0^1 (x^2 - 2x)dx = \frac{1}{3} - 2\,\frac{1}{2} = -\frac{2}{3}.$$

6.1.5 Integration by Parts

Integration by parts formula:

$$\int_a^b F(x)G'(x)\,dx = F(x)G(x) - \int_a^b F'(x)G(x)\,dx$$

Notes

The purpose of the formula is enclosed in the product rule for derivatives:

$$\frac{d}{dx}\big(F(x)G(x)\big) = F(x)G'(x) + F'(x)G(x)$$

which holds at any point where both functions are differentiable.

Caution One must provide strong enough hypotheses that the function $F(x)G(x)$ is an indefinite integral for the functioning the sense required for our integral.

6.1.6 Change of Variable

The *change of variable* formula (i.e., integration by substitution):

$$\int_a^b f\big(g(t)\big)g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx.$$

The meaning of the formula is enclosed in the following statement which includes a sufficient condition that permits this formula to be proved: Let I be an interval and $g:[a, b] \to I$ a continuously differentiable function. Let that $F: I \to \mathrm{R}$ is an integrable function. Then the function $F\big(g(t)\big)g'(t)$ is integrable on $[a, b]$ and the function f is integrable on the interval $[g(a),g(b)]$ (or rather on $[g(a),g(b)]$ if $g(b) < g(a)$) and the identity holds. There are diverse assumptions under which this might be applicable.

The proof is an application of the chain rule for the derivative of a composite function:

$$\frac{d}{dx}F\big(G(x)\big) = F'\big(G(x)\big)G'(x).$$

Task Show that the integral $\int_0^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ exists and use a change of variable to determine the exact value.

6.1.7 Derivative of the Definite Integral

What is $\frac{d}{dx}\int_a^x ff(t)\,dt$?

We know that $\int_a^x f(t)\,dt$ is an indefinite integral of f and so, by definition, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ at all but finitely many points in the interval (a, b) if f is integrable on $[a, b]$.

If we require to know more than that then there is the following edition which we have already proved:

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ at all points $a < x < b$ at which f is continuous.

Caution We should keep in mind, however, that there may also be various points where f is discontinuous and so far the derivative formula holds.

If we go ahead of the calculus interval, then the same formula is valid $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ but there may be several more than finitely many exceptions probable. For "most" values of t this is true but there may even be infinitely many exceptions probable. It will still be true at points

Notes

of continuity but it must also be true at most points when an integrable function is poorly discontinuous.

Examples of Definite Integral Properties

1. The integral of $y = 4x + 3$ from 2 to 2 is 0 since the integral of any function in which both bounds are the exact similar number is zero.
2. If the integral from 1 to 4 of the function $y = x^2$ is 21, then the integral from 4 to 1 of x^2 is -21 since when the bounds of an integral are toggled, the integral is similar, but has the opposite sign (positive transforms to negative, or negative transforms to positive).

Self Assessment

Fill in the blanks:

1. .... integral is where the integration is performed in a specific interval.
2. If the integration is performed between a to b, then a is known as lower limit and b is known as .... limit.
3. For integration of definite integral we integrate the specified .... first then apply the upper and lower limit.
4. The fundamental properties of integrals are simply attained for us since the integral is defined straightforwardly by ....
5. When a function involves a calculus integral on an interval it must also contain a calculus integral on all ....
6. When a function involves a calculus integral on two .... intervals it must also contain a calculus integral on the amalgamation of the two intervals.
7. The integral on the large interval is the .... of the other two integrals.
8. Suppose f and g be integrable functions on the interval $[a; b]$. Subsequently $f + g$ is also integrable on $[a; b]$ and we have .... = $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
9. The formula for .... is $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ if $f(x) \le g(x)$ for all but finitely several points x in (a,b).
10. If both functions $f(x)$ and $g(x)$ contain a calculus integral on the interval $[a,b]$ then any .... $r f(x)+s g(x)$ ($r, s \in$ R) also encloses a calculus integral on the interval $[a,b]$.
11. The integration by .... formula is defined by

    $$\int_a^b F(x)G'(x)dx = F(x)G(x) - \int_a^b F'(x)G(x)dx.$$
12. The .... formula is defined by $\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx$.
13. $\int_a^x f(t)\,dt$ is an .... integral of f and so, by definition, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
14. Complete the following: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx +$ ....

Notes

15. Complete the following:

...............................= $= r\int_a^b f(x)dx + s + \int_a^b g(x)dx(r, s \in \mathbb{R})$.

6.2 Summary

- There are certain properties of definite integral which makes the integration simpler.
- The fundamental properties of integrals are simply attained for us since the integral is defined straightforwardly by differentiation.
- When a function involves a calculus integral on an interval it must also contain a calculus integral on all subintervals.
- When a function involves a calculus integral on two contiguous intervals it must also contain a calculus integral on the amalgamation of the two intervals.
- The formula for *inequalities* is $\int_a^b f(x)dx \le \int_a^b g(x)dx$ if $f(x) \le g(x)$ for all but finitely several points x in (a,b).
- If both functions $f(x)$ and $g(x)$ contain a calculus integral on the interval $[a,b]$ then any linear combination $r f(x)+s g(x)$ ($r, s \in$ R) also encloses a calculus integral on the interval $[a,b]$ and, furthermore, the identity is ought to be hold.
- *Integration by parts* formula is the purpose of the formula is $\int_a^b F(x)G'(x)dx = F(x)G(x) - \int_a^b F'(x)G(x)dx$ enclosed in the product rule for derivatives: $\frac{d}{dx}(F(x)G(x)) = F(x)G'(x) + F'(x)G(x)$ which holds at any point where both functions are differentiable.
- The first two are similar to the properties of limits. The other two are very perceptive and relate to the notion of area.

6.3 Keywords

Change of Variable: The *change of variable* formula (i.e., integration by substitution):

$$\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(x)dx.$$

Fundamental Theorem of Calculus: This theorem converts the integral from a mathematical inquisitiveness to a prevailing tool that is accessed in science, engineering, economics and many other areas.

6.4 Review Questions

1. Supply the details needed to prove the integration by parts formula in the special case where F and G are continuously differentiable everywhere.
2. Supply the details needed to prove the change of variable formula in the special case where F and G are differentiable everywhere.
3. Let $F(x) = |x|$ and $G(x)= x^2 \sin^{-1}$, $G(0) = 0$. Does

   $\int_0^1 F'(G(x))G'(x)dx = F(G(1)) - F(G(0)) = |\sin 1|$?

Notes

4. Supply the details needed to prove the change of variable formula in the special case where G is strictly increasing and differentiable everywhere
5. Show that the function $f(x) = x^2$ is integrable on [–1,2] and compute its definite integral there.
6. Show that each of the following functions is not integrable on the interval stated:
   (a) $f(x) = 1$ for all x irrational and f $(x) = 0$ if x is rational, on any interval $[a,b]$.
   (b) $f(x) = 1$ for all x irrational and f (x) is undefined if x is rational, on any interval $[a,b]$.
   (c) $f(x) = 1$ for all x 6= 1,1/2,1/3,1/4, . . . and $f(1/n) = cn$ for some sequence of positive numbers $c_1, c_2, c_3, \ldots$, on the interval [0,1].
7. Determine all values of p for which the integrals

   $$\int_0^1 x^p dx \; or \int_1^\infty x^p dx$$

8. Are the following additivity formulas for infinite integrals valid:
   (a) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^b f(x)dx + \int_b^\infty f(x)dx?$
   (b) $\int_{-\infty}^{\infty} f(x)dx = \sum_{n-1}^{\infty} \int_{n-1}^{n} f(x)dx?$
   (c) $\int_{-\infty}^{\infty} f(x)dx = \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} f(x)dx?$
9. Evaluate the following integral using properties of definite integrals and interpreting integrals as areas:

   $$\int_{-1}^{6} (4x-2)dx$$

10. Evaluate the following integral using properties of definite integrals and interpreting integrals as areas.

   $$\int_{-2}^{2} \left(5u^3 - 5u^9 + \frac{\pi}{2}\right)du$$

Answers: Self Assessment

1.	Definite	2.	upper
3.	function	4.	differentiation
5.	subintervals	6.	contiguous
7.	sum	8.	$\int_a^b (f+g)(x)dx$
9.	inequalities	10.	linear combination
11.	parts	12.	change of variable
13.	indefinite	14.	$\int_b^c f(x)dx$
15.	$\int_a^b [rf(x)+sg(x)]\,dx$		

6.5 Further Readings

Notes

Books

Douglas S. Kurtz, Jaroslav Kurzweil, Charles Swartz, *Theories of integration*, World Scientific

G. H. Hardy, T. W. Körner, *A course of pure mathematics*, Cambridge University Press

Morris Kline, *Calculus: an intuitive and physical approach*, Courier Dover Publications

Ron Larson, David C. Falvo, *Calculus: an applied approach*, Cengage Learning

Online link

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Notes

Unit 7: Definite Integral Applications

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Objectives

After studying this unit, you will be able to:

- Understand the application in finding the area under simple curve
- Discuss the application in finding the area within two curves

Introduction

Applications of the definite integral include finding the area under simple curve and finding the area within two curves. In this unit, you will understand definite integral applications in detail with explained examples.

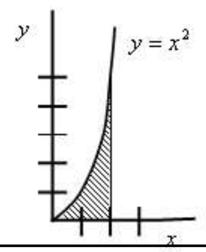
7.1 Area under Simple Curve

We begin with the area under a simple curve, a straight line. A straight line, y = const, above a distance in x is a rectangle and the area of a rectangle is the height multiplied by the width. For some more anticipation appearance at the area under, or inside, a triangle created by the coordinate axes and $y = kx$. These two areas, the rectangle and the triangle, include straight sides, but what in relation to an area surrounded by a quadratic, $y = x^2$. This is a whole new problem. As a first approach to this problem, gaze at succeeding estimations to the area.

Graph the function between $x = 0$ and $x = 2$.

The area under this curve is less than the area within a triangle created with base with the x axis from 0 to 2, height from $y = 0$ to 4 and the inclined height from the point (0,0) to (2,4). Such a triangle has area $(1/2)2 \times 4 = 4$.

This is a first estimation.



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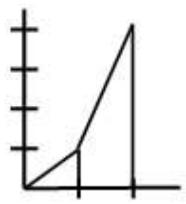
The curve $y = x^2$ g traverse throughout the points (0.0), (1,1) and (2,4) so persist this estimation approach by finding the area of this triangle and trapezoid grouping. The area of the triangle is (1/2)1 × 2 = 1/2. The area of a trapezoid is (1/2)(sum of the opposite faces) (height) which for this trapezoid is (1/2)(1 + 4)(1) = 2.5. The sum of these areas is 3.

This is an improved estimation.



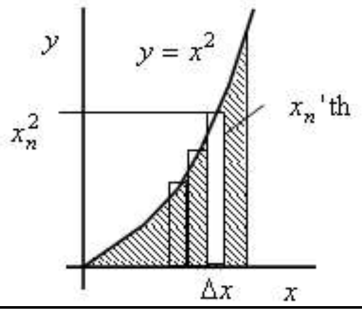
Caution This estimation approach can be carried to superior correctness by making slighter and slighter trapezoids.

Make the trapezoids minute enough and they get somewhat close to rectangles.

Notes Observe that this estimation strategy can be applied to any function; power, exponential, trigonometric or any combination theorem. Therefore we have an estimation method that can be executed out to any extent of correctness as long as we are enthusiastic to make the comprehensive computations.

With this foreword to areas and an estimation approach to areas surrounded by functions gaze now at what is known as the fundamental theorem of integral calculus.

Make use of the similar curve $y = x^2$ as an example, although any curve would function as well, and consider estimating the area not with trapezoids, but with a compilation of slender rectangles. The rectangles can be created in a numerous methods, inside the curve, outside the curve or by means of a mid-value. It actually doesn't make any dissimilarity how they are created since we are going to take the limit by having their width to zero. The ones displayed here are an average height. See the x_n'th rectangle of width Dx that contains height $x_n^{\,2}$.



The area under this curve can be represented as a sum of similar rectangles. With this vision, the area under the curve is

$$A \approx \sum_n \left(x_n^2\right) \Delta x$$

with the area getting closer and closer to the definite area since the width of the rectangles reduces and their number augments.

Notes

Did u know? By means of a limit approach, and the knowledge that this outlines, or integral, over a particular range in x is the area under the curve, A is the limit of the sum as Dx leads to zero.

$$A = \lim_{\Delta x \to 0} \sum_{n} \left(x_n^2 \right) \Delta x = \int_0^x x^2 dx$$

The integral is scrutinized as the area produced by summing an infinite number of rectangles of infinitely minute width. The antiderivatives of these integrals can be revealed by the estimation method sketched to be equal to the area under the curve. In definite practice the integrals are frequently mentioned not as 0 to x but as from a lower limit to a higher limit corresponding to the area preferred.

The next three examples demonstrate the utilization of integrals in locating the area of a rectangle, triangle and area under a quadratic. These integrals exhibit that the area under the curve as displayed by the integral is in fact the antiderivative with the specified limits. This is easily confirmed by performing estimations as summarized above.

Example: Find the area under the curve $y = 5$, enclosed by the lines $x = 0$ and $x = 5$.

Graph the function.

It is a straight line at $y = 5$, analogous to the x-axis. To locate the area, integrate $5dx$ from $x = 0$ to $x = 5$.

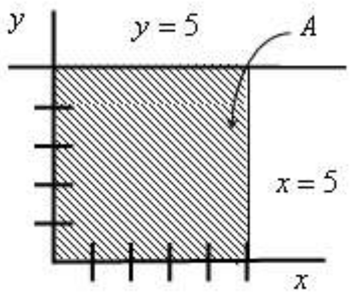
This area integral is represented as

$$A = \int_0^5 5\,dx$$

The 0 and 5 mean, assess the integral at 5 and then subtract the value for 0.

The operations are

$$A = \int_0^5 5\,dx = 5x\Big|_0^5 = 5(5) - 5(0) = 25$$



Example: Find the area under the curve $y = 2x$ among $x = 0$ and $x = 3$.

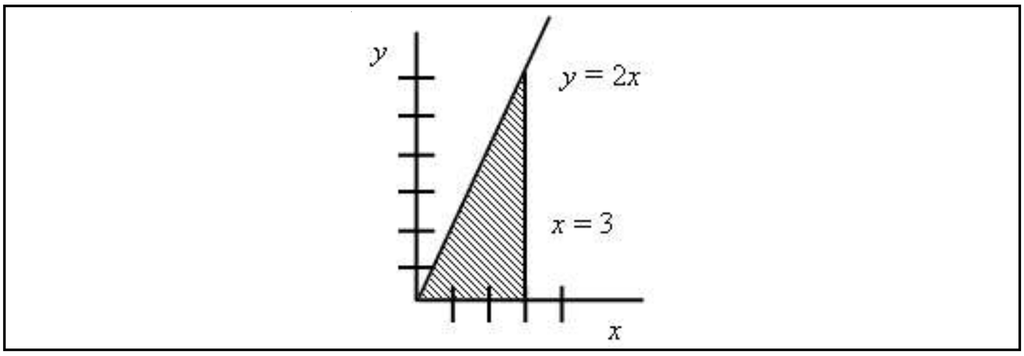
Initially, graph the curve.

The area, by integration is

$$A = \int_0^3 x\,dx = \frac{x^2}{2}\Bigg|_0^3 = 9 - 0 = 9$$

Notes

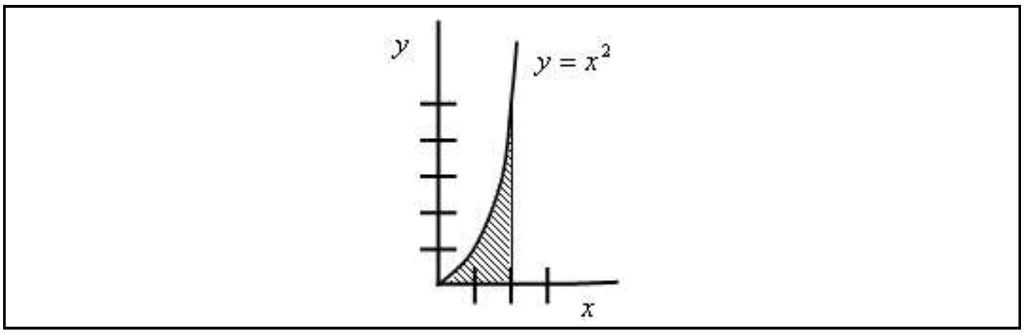
This curve $y = 2x$ generates a triangle with the x-axis and the line $x = 3$. The area of this triangle is one-half the base times the height (1/2)3·6 = 9, the same value as attained via integration.



Example: Find the area under the curve $y = x^2$ connecting $x = 0$ and $x = 2$.

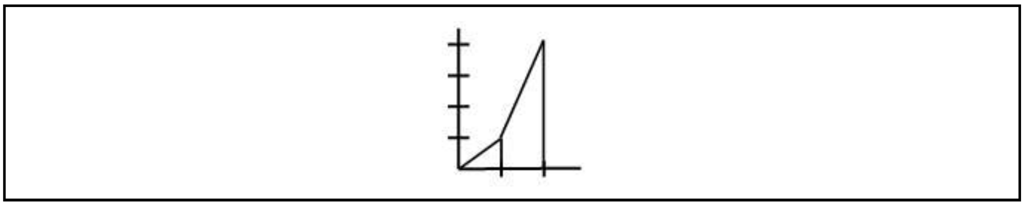
First, graph the curve. Using the integral method, the area is

$$A = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3}{3} - 0 = \frac{8}{3} \approx 2.7$$



The area under this curve is less than the area within a triangle generated with base along the x axis from 0 to 2, height from $y = 0$ to 4 and the inclined height from the point (0,0) to (2,4). Such a triangle contains area (1/2)2·4 = 4, and as predictable is more than the area of ≈2.7 calculated with the integral.

The curve $y = x^2$ traverses throughout the points (0.0), (1,1) and (2,4) so carry on this estimation approach by locating the area of this triangle and trapezoid. The area of the triangle is (1/2)1 ·2 = 1/2. The area of a trapezoid is (1/2)(sum of the opposite faces)(height) which here is (1/2)(1 + 4)(1) = 2.5. The sum of these areas is 3, nearer to the 2.7 attained via the integral.



If this course were sustained with more narrower and narrower trapezoids the area would lead to 2.7 attained via the integral.

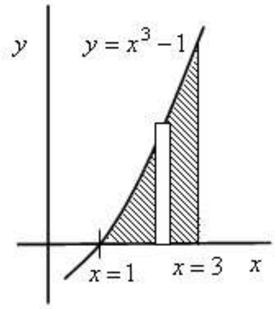
Example: Find the area under the curve $y = x^{\tilde{3}}$ 1 from $x = 1$ to $x = 3$.

Graph the curve over the series of interest.

Notes

This is a cubic. It increases abruptly, and it crosses the y-axis at and the x-axis at 1 and passes through the point (3,8). The rectangle displays one of the rectangles that is being summed in the integration course.

The shaded area is the integral



$$A = \int_1^3 \left(x^3 - 1\right) dx = \left[\frac{x^4}{4} - x\right]_1^3$$

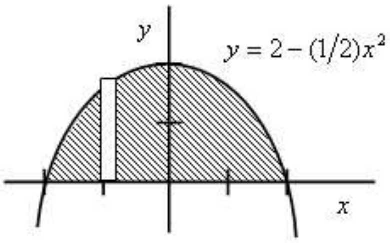
Now assess the integral

$$A = \left(\frac{81}{4} - \frac{12}{4}\right) - \left(\frac{1}{4} - \frac{4}{4}\right) = \frac{69}{4} + \frac{3}{4} = \frac{72}{4} = 18$$

Example: Find the area enclosed by $y = \tilde{2}\ (1/2)x^2$ and the x-axis. Graph the function.

This function is a parabola. It opens down and traverses the y-axis at y = 2.

The limits on the integral have to be from where the curve crosses the x-axis on the negative side to where it crosses on the positive part. To locate these points set *and solve for x.*



Recognizing how to graph this curve permits you to focus on the calculus part of the problem.

The shaded area is the area preferred so the integral is

$$A = \int_{-2}^{2}\left[2-(1/2)^{x^2}\right]dx = \left[2x - \frac{x^3}{6}\right]_{-2}^{2} = \left[2(-2) - \frac{(-2)^3}{6}\right] - \left[2(2) - \frac{(-2)^3}{6}\right]$$

$$A = \left[4 - \frac{4}{3}\right] - \left[-4 - \frac{-8}{6}\right] = \left[\frac{12}{3} - \frac{4}{3}\right] - \left[\frac{12}{3} + \frac{4}{3}\right] = \left[\frac{8}{3}\right] - \left[-\frac{8}{3}\right] = \frac{16}{3}$$

Second Solution:

There is a slight faster, a little simpler, and a little less prone to fault method of performing this problem. Keep in mind the symmetry that was so cooperative in graphing parabolas. Not just

Notes

is there a symmetry in the graph of the curve between 0 and 2 and 0 and – 2, but the area under the curve from 0 to 2 is identical as the area under the curve from 0 to – 2. Thus, the complete area among this curve and the x-axis is twice the area computation between x = 0 and x = 2. Observe the lack of negative numbers in this solution.

$$A = 2\int_0^2 \left[2-(1/2)x^2\right]dx = 2\left[2x-\frac{x^3}{6}\right]_0^2 = 2\left\{\left[2(2)-\frac{2^3}{6}\right]-[0]\right\} = 2\left\{4-\frac{4}{3}\right\} = 2\left\{\frac{12}{3}-\frac{4}{3}\right\} = \frac{16}{3}$$

In performing area problems gaze for symmetry that will make the problem simpler and cut down on the quantity of numbers you have to influence.

Example: An investment in a definite mining process is returning an total per month that goes with the following formula

$A = 0.25x^2 - 14x + 300$

where x is the month and A is the amount for that month. This formula is applicable for 40 months.

What is the overall return for the first 12 months?

Solution:

The curve begins out at \$300 for the zeroth month and drops off with time. In this problem the height of a rectangle beneath the curve for every value of x *displays* the money obtained that month and the sum of these rectangles is the total quantity for the period summed. For 12 months this amount is the area under the curve from the zeroth to the eleventh month, or the integral from 0 to 11.

$$T = \int_0^{11}\left(0.25x^2 - 14x + 300\right)dx$$

$$T = \left[0.25\frac{x^3}{3} - 14\frac{x^3}{2} + 300x\right]_0^{11}$$

$$T = \left[0.25\frac{11^3}{3} - 14\frac{11^2}{2} + 300(11)\right]$$

$$T = \$2564$$

Self Assessment

Fill in the blanks:

1. A straight line, y = const, above a distance in x is a ....
2. The area of a rectangle is the height multiplied by the ....
3. The area of a ....is (1/2)(sum of the opposite faces) (height).
4. The rectangles can be created in a numerous methods, inside the curve, outside the curve or by using a ....
5. It actually doesn't make any variation how the rectangles are created since we are going to take the limit by having their width to ....
6. By means of a .... approach, and the knowledge that this outlines, or integral, over a particular range in x is the area under the curve, A is the limit of the sum as Dx leads to zero.

Notes

7. In definite practice the integrals are frequently mentioned not as 0 to x but as from a lower limit to a .... limit corresponding to the area preferred.

8. The .... of the integrals can be revealed by the estimation method sketched to be equal to the area under the curve.

9. The integrals exhibit that the area under the curve as displayed by the integral is in fact the antiderivative with the .... limits.

10. Make the trapezoids minute enough and they get somewhat .... to rectangles.

State whether the following statements are true or false:

11. This estimation approach can be carried to superior correctness by making slighter and slighter trapezoids.

12. The integral is scrutinized as the area produced by summing a finite number of rectangles of infinitely minute width.

7.2 Area within Two or more Curves

The technique for identifying the area within two or more curves is an imperative application of integral calculus. There are three major steps to this procedure. They are:

1. Graph the two or more equations.
2. Find out the points of intersection.
3. Set up and assess the definite integral.

In the following examples, this process is demonstrated.

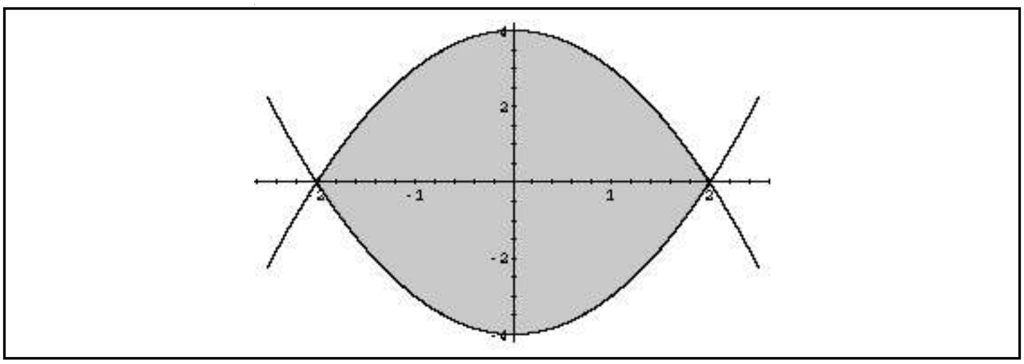
Did u know? The technique for identifying the area within two or more curves allows us identify the area of irregular shapes by assessing the definite integral.

Example: Find the area between the curves $f(x) = 4 - x\ 2$ and $g(x) = x\ 2 - 4$.

Solution:

Step 1: Graph the functions. (See figure)

The motive for graphing the two equations is to be capable to find out which function is on top and which one is on the bottom. At times, you can also find out the points of intersection. From this graph, it is apparent that $f(x)$ is the upper function, $g(x)$ is the lower function, and that the points of intersection are $x = -2$ and $x = 2$.



Step 2: Find out the points of intersection.

If you did not recognize the points of intersection from the graph, solve for them algebraically or by means of your calculator. To locate them algebraically, set each equation equivalent to each other.

$4 - x^2 = x^2 - 4 \rightarrow -2x^2 = -8 \rightarrow x^2 = 4 \rightarrow x = -2$ or $x = 2$

Step 3: Set up and estimate the integral.

Recollect from previous notes, when we were locating the area between the curve and the x-axis, we had to find out the upper and the lower curve. Then the area was defined to be the next integral.

$$Area = \int_a^b (upper\ curve - lower\ curve)\, dx$$

So the definite integral would be as below.

$$Area = \int_{-2}^{2}\left[\left(4-x^2\right)-\left(x^2-4\right)\right]dx = \int_{-2}^{2}\left(8-2x^2\right)dx$$

Now, let us assess the integral.

$$\begin{aligned} Area &= \int_{-2}^{2}\left[\left(4-x^2\right)-\left(x^2-4\right)\right]dx = \int_{-2}^{2}\left(8-2x^2\right)dx \\ &= \left[8x-\frac{2}{3}x^3\right]\Bigg|_{-2}^{2} = \left(8(2)-\frac{2}{3}(2)^3\right)-\left(8(-2)-\frac{2}{3}(-2)^3\right) \\ &= \left(16-\frac{16}{3}\right)-\left(-16+\frac{16}{3}\right)=\frac{64}{3} \end{aligned}$$

If you observe the graph of the two functions cautiously, you should have observed that we could have used some balance when setting up the integral. The region is symmetric regarding both the x- and the y-axis. If we had utilized the y-axis symmetry, the consequent integral would have had bounds of 0 and 2, and we would have had to take 2 times the area to discover the total area. Here is this integral.

$$\begin{aligned} Area &= 2\int_{0}^{2}\left[\left(4-x^2\right)-\left(x^2-4\right)\right]dx = \int_{-2}^{2}\left(8-2x^2\right)dx \\ &= 2\left[8x-\frac{2}{3}x^3\right]\Bigg|_{-2}^{2} = \left[\left(8(2)-\frac{2}{3}(2)^3\right)-\left(8(0)-\frac{2}{3}(0)^3\right)\right] \\ &= 2\left[16-\frac{16}{3}\right]=2\left(\frac{32}{3}\right)=\frac{64}{3} \end{aligned}$$

If we had used both symmetries, the consequential integral would still have bounds of 0 and 2, but the upper function would have been f (x) and the lower function would be y = 0 (the x-axis). To locate the total area, we would have to obtain this area times 4. Here is this integral.

$$\begin{aligned} Area &= 4\int_{0}^{2}\left[\left(4-x^2\right)-0\right]dx = 4\int_{0}^{2}\left(4-x^2\right)dx \\ &= 4\left[4x-\frac{1}{3}x^3\right]\Bigg|_{0}^{2} = 4\left[\left(4(2)-\frac{1}{3}(2)^3\right)-\left(4(0)-\frac{1}{3}(0)^3\right)\right] \\ &= 4\left[8-\frac{8}{3}\right]=4\left(\frac{16}{3}\right)=\frac{64}{3} \end{aligned}$$

Example: Find the area between the curves $f(x) = 0.5\sec^2 x$ and $g(x) = -4\sin^2 x$ over the interval $[-\pi/3, \pi/3]$.

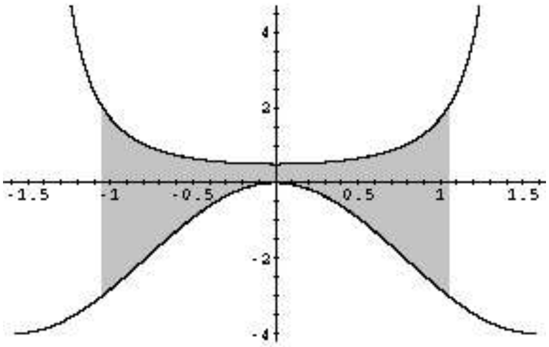
Notes

Solution:

Step 1: Graph the functions.

Observe that $f(x)$ is the upper function, and $g(x)$ is the lower function. As the interval that we will be integrating over is specified, skip step 2.

Step 3: Set up and assess the integral.



$$Area = \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\left(\frac{1}{2} \sec^2 x \right) - \left(-4 \sin^2 x \right) \right] dx$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{1}{2} \sec^2 x + \frac{4}{2} \left(1 - \cos^2 x \right) \right] dx$$

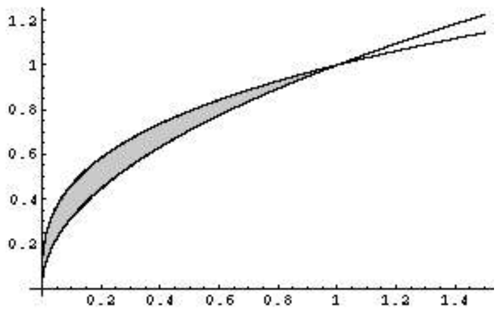
$$= \frac{1}{2} \tan x + 2x - \sin^2 x \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{4\pi}{3}$$

Now, let us undertake a problem in which we integrate relating to y.

Example: Find the area between the curves $x = y^3$ and $x = y^2$ that is contained in the first quadrant.

Solution:

Step 1: Graph the functions. (See figure)



Since both equations are x in terms of y, we will integrate with respect to y. When integrate with respect to x, we have to determine the upper function and the lower function. Now that we are integrating with respect to y, we must determine what function is the farthest from the y-axis. The function that is the farthest from the y-axis is $x = y^2$. So that will be our upper curve. The lower curve will be the curve that is nearest to the y-axis. In this case, it is the function $x = y^3$.

Notes

Step 2: Find the points of intersection.

Set the two equations equal to each other.

$y^2 = y^3 \rightarrow y^2 - y^3 = 0 \rightarrow y^2(1 - y) = 0 \rightarrow y = 0$ or $y = 1$

Step 3: Set up and evaluate the integral.

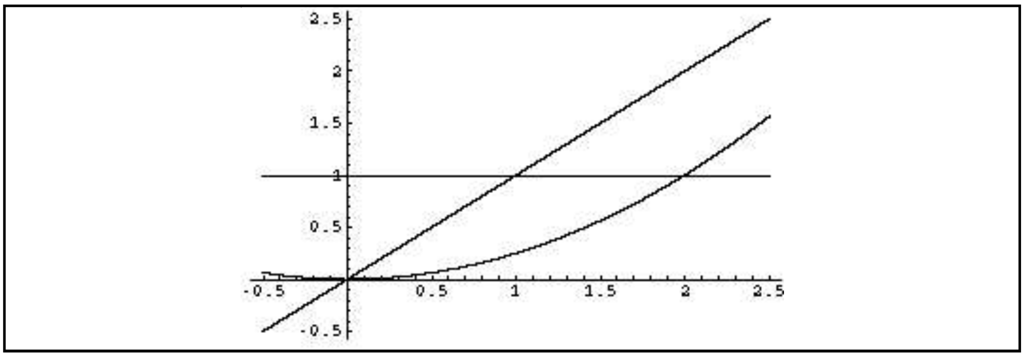
$$Area = \int_0^1 \left(y^2 - y^3\right) dy = \frac{1}{3}y^3 - \frac{1}{4}y^4 \Big|_0^1$$

Example: Find the area of the region enclosed by the curves $y = x$, $y = 1$, and $y = x^2/4$ that lies in the first quadrant.

Solution:

Step 1: Graph the functions. (See figure)

Here, you will observe that if we choose to integrate with respect to x, then we will have to split the area into two pieces. One will be on the interval [0, 1], and the other on the interval [1, 2]. If we choose to integrate with respect to y, then we will only have one area to compete with. So let us do it regarding y first. The upper function will be $y = x^2/4$, and the lower function will be $y = x$.



Step 2: Find out the points of intersection.

Observe that the one general intersection point of the two functions is the point (0, 0). So we will integrate from $y = 0$ to $y = 1$.

Step 3: Set up and assess the integral.

To integrate with respect to y, we must solve $y = x^2/4$ for x in terms of y.

$$y = \frac{x^2}{4} \rightarrow 4y = x^2 \rightarrow x = 2y^{\frac{1}{2}}$$

$$Area = \int_0^1 \left(2y^{\frac{1}{2}} - y\right) dy = \frac{4}{3}y^{\frac{3}{2}} - \frac{1}{2}y^2 \Big|_0^1$$

$$= \left(\frac{4}{3}(x)^{\frac{3}{2}} - \frac{1}{2}(1)^2\right) - \left(\frac{4}{3}(0)^{\frac{3}{2}} - \frac{1}{2}(0)^2\right) = \frac{5}{6}$$

Now, let us integrate with respect to x. Keep in mind that we will have to have two integrals. For the interval [0, 1] the upper function is $y = x$ and the lower function is $y = x^2/4$. On the interval [1, 2] the upper function is $y = 1$ and the lower function is $y = x^2/4$.

Notes

$$Area = \int_0^1 \left(x - \frac{1}{4}x^2\right)dx + \int_1^2 \left(1 + \frac{1}{4}x^2\right)dx$$

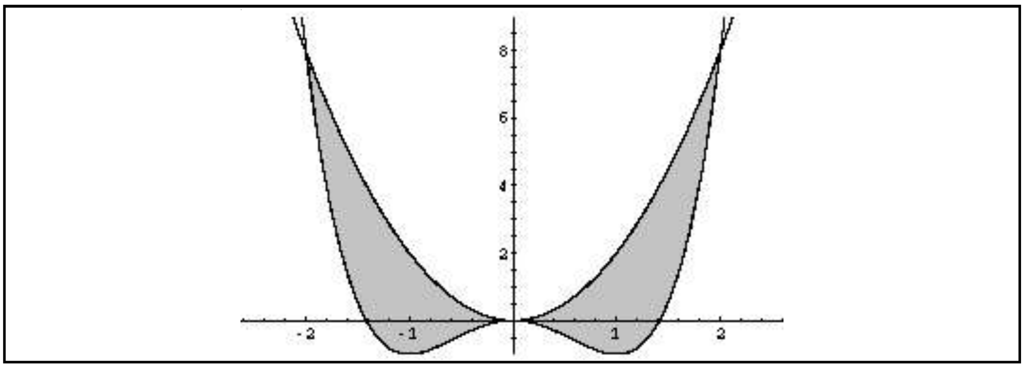
Example: Find the area bounded by the curves $y = 2x^2$ and $y = x^4 - 2x^2$.

Solution:

Step 1: Graph the functions. (See figure)

Observe that this area is symmetric, so we can utilize symmetry to abridge the process of locating the area among the curves. Notice that $y = 2x^2$ is the upper curve.

Step 2: Determine the intersection points.



We must find out the points of intersection. To perform this, set the two equations equal to each other and solve for x. You should find out that $x = -2$, $x = 0$, and $x = 2$ are the points of intersection. As we are using symmetry, we will be integrating over the interval [0, 2] and considering that area 2 times for the total area.

Step 3: Set up and evaluate the integral.

$$Area = 2\int_0^2 \left[2x^2 - \left(x^2 - 2x^2\right)\right]dx = 2\int_0^2 \left(4x^2 - x^4\right)dx$$

$$= 2\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]\Bigg|_0^2 = \frac{128}{15}$$

Notes Observe that for all of the examples, the curves are graphed to recognize which curve was the upper curve, and which one was the lower curve.

Caution You can utilize your graphing calculator for graphing curves this, or if you need a printed copy, go to the computer lab and make use of Maple to plot one that you can print out.

Self Assessment

Fill in the blanks:

13. The technique for identifying the area within two or more curves is an imperative application of ....

Notes

14. The three major steps to technique for identifying the area within two or more curves are graphing the two or more equations, finding out the points of.............................., and Setting up and assessing the definite integral.

15. The curves are graphed to recognize which curve was the .... curve, and which one was the lower curve.

7.3 Summary

- Applications of the definite integral include finding the area under simple curve and finding the area within two curves.
- A straight line, y = const, above a distance in x is a rectangle and the area of a rectangle is the height multiplied by the width.
- We have an estimation method that can be executed out to any extent of correctness as long as we are enthusiastic to make the comprehensive computations.
- The rectangles can be created in a numerous methods, inside the curve, outside the curve or by means of a mid-value.
- In definite practice the integrals are frequently mentioned not as 0 to x but as from a lower limit to a higher limit corresponding to the area preferred.
- The technique for identifying the area within two or more curves is an imperative application of integral calculus.
- Identifying the area within two or more curves allows us identify the area of irregular shapes by assessing the definite integral.
- There are three major steps to this procedure: Graph the two or more equations, Find out the points of intersection, and Set up and assess the definite integral.

7.4 Keyword

Estimation Approach: We have an estimation approach that can be executed out to any extent of correctness as long as we are enthusiastic to make the comprehensive computations.

7.5 Review Questions

1. Illustrate the application of finding the area Under Simple Curve with example.
2. Exemplify the application of finding the area within two curves with example.
3. Find the area of the region bounded by $y = 2x$, $y = 0$, $x = 0$ and $x = 2$.
4. Find the area bounded by $y = x^3$, $x = 0$ and $y = 3$.
5. Find the area bounded by the curves $y = x^2 + 5x$ and $y = 3 - x^2$.
6. Find the area bounded by the curves $y = x^2$, $y = 2 - x$ and $y = 1$.
7. Find the area of the region enclosed between the curves $y = x^2 - 2x + 2$ and $-x^2 + 6$.
8. Find the area of the region enclosed by $y = (x-1)\ 2 + 3$ and $y = 7$.
9. Find the area of the region bounded by $x = 0$ on the left, $x = 2$ on the right, $y = x^3$ above and $y = -1$ below.

Notes

Answers: Self Assessment

1.	rectangle	2.	width
3.	trapezoid	4.	mid-value
5.	zero	6.	limit
7.	higher	8.	antiderivatives
9.	specified	10.	close
11.	True	12.	False
13.	integral calculus	14.	intersection
15.	upper		

7.6 Further Readings

Books

Douglas S. Kurtz, Jaroslav Kurzweil, Charles Swartz, *Theories of Integration*, World Scientific

G. H. Hardy, T. W. Körner, *A Course of Pure Mathematics*, Cambridge University Press

Morris Kline, *Calculus: an Intuitive and Physical Approach*, Courier Dover Publications

Ron Larson, David C. Falvo, *Calculus: an Applied Approach*, Cengage Learning

Online link

http://www.analyzemath.com/calculus/Integrals/area_under_curve.html

Unit 8: Formation of Differential Equation

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define differentiation equation
- Understand the classification of differential equation
- Discuss the formation of differential equation

Introduction

From times immemorial, man has made progress through observing the nature, questioning and improvising himself. Scientists observe the positions of various celestial bodies, noting their movements and finally discovering the paths traversed by these bodies and later on representing them in the form of equations known as laws. This process of mathematical formulations of natural phenomena lead to a differential equation -which is a natural choice as phenomena change in time or change in other constants. Thus, in the fields of physics, engineering, economics, finance, management, etc., differential equations play a very important role. They arise in many practical problems where variation of distance of a moving particle, variation of current in an electric circuit, in mechanical systems, agriculture, etc. A differential equation involves independent variables, dependent variables, their derivatives and constants.

8.1 Definition

A differential equation is an equation, which involves differentials or differential coefficient.

Ordinary differential equations are those, which involve only one independent variable. Thus

(i) $\frac{dy}{dx} + y\cos x = \sin x$

(ii) $\frac{d^2y}{dx^2} + a^2y = \tan ax$

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(iii) $x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) + a = 0$

(iv) $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2} = \frac{d^2y}{dx^2} \, or \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

are examples of ordinary differential equations.

Did u know? The differential equations are solved and the solutions thus obtained are interpreted in the context of the problem.

Self Assessment

Fill in the blanks:

1. A .... involves independent variables, dependent variables, their derivatives and constants.
2. Ordinary differential equations are those, which involve only one .... variable.
3. The process of .... formulations of natural phenomena leads to a differential equation.

State whether the following statements are true or false:

4. Ordinary differential equations involve more than one independent variable.

8.2 Classification of Differential Equation

A differential equation can be classified as ordinary or partial differential equation. A differential equation involving a single independent variable and the derivatives with respect to it, is called an **ordinary differential equation**.

Example: The equations (i), (ii), (iii) and (iv) given above are ordinary differential equations.

Did u know? An ordinary differential equation contains total derivatives or total differentials.

A differential equation involving two or more independent variables and the partial derivatives with respect to them, is called a **partial differential equation**.

Example: the equations:

(i) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Notes

(ii) $$\frac{\partial^2 v}{\partial t^2} = C\left(\frac{\partial^2 v}{\partial x^2}\right)^2$$

(iii) $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nz$$

are partial differential equations.

Self Assessment

Fill in the blanks:

5. A differential equation can be classified as an ordinary or .... differential equation.
6. A differential equation involving a single independent variable and the derivatives with respect to it, is called an .... differential equation.
7. A differential equation involving .... independent variables and the partial derivatives with respect to them, is called a partial differential equation.

8.3 Formation of a Differential Equation

At times a family of curves can be displayed by a single equation. In this case the equation includes an arbitrary constant c. By allocating different values for c, we obtain a family of curves. Here c is known as the parameter or arbitrary constant of the family.

Differential equations are formed by elimination of arbitrary constants. To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order. Elimination of n arbitrary constants leads us to n^{th} order derivatives and hence a differential equation of the n^{th} order.

Notes By eliminating the arbitrary constants from the specified equation and the equations attained by the differentiation, we obtain the requisite differential equations.

Example: From the differential equation of all circles of radius r.

Solution:

The equation of any circle of radius r is

$$(x - h)^2 + (y - k)^2 = r^2, \qquad \ldots\ldots\ldots\ldots(1)$$

where (h, k) the coordinates of the centre.

Differentiating (1) w.r.t. x, we get

$$2(x - h) + 2(y - k)\frac{dy}{dx} = 0$$

Notes

or $(x - h) + (y - h)\dfrac{dy}{dx} = 0$ ....(2)

Differentiating again, we have

$$1 + (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{..........(3)}$$

From (3), $$y - k = -\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}$$

and from (2), $$x - h = -(y-k)\frac{dy}{dx} = \frac{\dfrac{dy}{dx}\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}$$

Substituting the values of (x – h) and (y – k) in (1), we get

$$\frac{\left(\dfrac{dy}{dx}\right)^2\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^2}{\left(\dfrac{d^2y}{dx^2}\right)^1} + \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^2}{\left(\dfrac{d^2y}{d^2x}\right)^1} = r^2$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right] = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Task Find the differential equations of all straight lines in a plane.

Example: Eliminate the constants from the equation

$y = e\,(C_1 \cos x + c_2 \sin x)$ ....(1)

and obtain the differential equation.

Solution:

There are two arbitrary constants c_1 and c_2 in equation (1).

Notes

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x\left(c_1 \cos x + c_2 \sin x\right) + e^x\left(-c_1 \sin x + c_2 \cos x\right)$$

or $$\frac{dy}{dx} = y + e^x\left(-c_1 \sin x + c_2 \cos x\right) \qquad \text{.........(2)}$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} = e^x\left(-c_1 \sin x + c_2 \cos x\right) + e^x\left(-c_1 \cos x - c_2 \sin x\right)$$
$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y \qquad [\text{by (1) and (2)}]$$

or $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation.

Notes The differential equation of two arbitrary constants family is attained by differentiating the equation of the family twice and by eliminating the arbitrary constants.

Example: Form a differential equation to represent the family of curves y = A cos x + B sin x

Solution:

Since, y = A cos x + B sin x

$$\frac{dy}{dx} \cong \text{ A} \sin x + \text{B} \cos x$$

$$\frac{d^2y}{dx^2} \cong \text{ A} \cos x + \text{B} \sin x \cong (\text{A} \cos x + \text{B} \sin x)$$

$$\cong y$$

i.e., $$\frac{d^2y}{dx^2} + y = 0.$$

Hence, the required differential equation is

$$\frac{d^2y}{dx^2} + y = 0.$$

Example: Form a differential equation of the family of curves $y = \text{A}ex + \text{B}\tilde{e}^{2x}$ for different values of A and B.

Solution:

Given, $y = \text{A}ex + \text{B}\tilde{e}^{2x}$ ...(1)

Notes

Differentiation (1) w.r.t. x, we get

$$\frac{dy}{dx} = \mathrm{A}\,e^{x} - 2\mathrm{B}e^{-2x} \qquad ...(2)$$

Differentiating (2) again, we get

$$\frac{d^2y}{dx^2} = \mathrm{A}\,e^{x} + 4\mathrm{B}e^{-2x}$$

$$\cong \left(Ae^{\tilde{x}} 2B\tilde{e}^{2x}\right) + 2\left(Ae^{x} + B\tilde{e}^{2x}\right)$$

$$= -\frac{dy}{dx} + 2y \qquad \text{from (1) and (2).}$$

Thus, the required differential equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0.$$

Example: Form the differential equation of simple harmonic motion given by x = A cos(nt + B)

Solution:

The given equation is x = A cos(nt + B) ...(1)

Differentiating (1) with respect to 't', we get

$$\frac{dx}{dt} = -\mathrm{A}\sin(nt + \mathrm{B}).n \qquad ...(2)$$

Differentiating (2) again with respect to 't', we have

$$\frac{d^2x}{dt^2} = -\mathrm{A}\cos(nt + \mathrm{B}).n^2 = -\mathrm{A}n^2\cos(nt + \mathrm{B})$$

$\cong n2x$, from (1)

or, $\dfrac{d^2x}{dt^2} + n^2x = 0$ is the required differential equation.

Example: Form the differential equation from the relation $\sin^{-1}x + \sin^{-1}y = C$

Solution:

Given $\sin^{-1}x + \sin^{-1}y = C$

Differentiating (1) w.r.t. 'x', we have

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+y^2}}\frac{dy}{dx} = 0$$

Notes

or, $\dfrac{dy}{dx} = -\dfrac{\sqrt{1-y^2}}{\sqrt{1+x^2}}$ is the required differential equation.

Example: Eliminate C from the equation $y = Ce^{\sin^{-1}x}$

Solution:

$$y = Ce^{\sin^{-1}x} \qquad ...(1)$$

Differentiating (1) w.r.t. 'x', we get

$$\frac{dy}{dx} = Ce^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{y}{\sqrt{1-x^2}}$$

i.e., $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{1-x^2}}$ is the required differential equation.

Caution The specified equation is differentiated as many times as there are arbitrary constants.

> *Task* Eliminate the arbitrary constants and obtain the differential equation:
>
> $y = A \cos 2x + B \sin 2x$

Self Assessment

Fill in the blanks:

8. By allocating different values for c, we obtain a family of curves where c is known as the .... of the family.
9. Differential equations are formed by .... of arbitrary constants.
10. To eliminate .... arbitrary constants, we require two more equations besides the given relation.
11. The elimination of two arbitrary constants lead us to .... order derivatives.
12. Elimination of n arbitrary constants leads us to n^{th} order derivatives and hence a differential equation of the .... order.
13. By eliminating the arbitrary constants from the specified equation and the equations attained by the ...., we obtain the requisite differential equations.

State whether the following statements are true or false:

14. The specified equation is differentiated as many times as there are arbitrary constants.
15. Elimination of n arbitrary constants leads to a differential equation of the $(n+1)^{th}$ order.

Notes

8.4 Summary

- A differential equation involves independent variables, dependent variables, their derivatives and constants.
- A differential equation involving a single independent variable and the derivatives with respect to it, is called an ordinary differential equation.
- An ordinary differential equation contains total derivatives or total differentials.
- A differential equation involving two or more independent variables and the partial derivatives with respect to them, is called a partial differential equation.
- By allocating different values for c, we obtain a family of curves where c is known as the parameter or arbitrary constant of the family.
- Differential equations are formed by elimination of arbitrary constants.
- Elimination of n arbitrary constants leads us to n^{th} order derivatives and hence a differential equation of the n^{th} order.
- By eliminating the arbitrary constants from a particular equation and the equations achieved by the differentiation, we obtain the requisite differential equations.

8.5 Keywords

Arbitrary Constant: By allocating different values for c, we obtain a family of curves where c is known as the parameter or arbitrary constant of the family.

Differential Equation: A differential equation involves independent variables, dependent variables, their derivatives and constants.

***Ordinary Differential Equation**:* A differential equation involving a single independent variable and the derivatives with respect to it, is called an ordinary differential equation.

***Partial Differential Equation**:* A differential equation involving two or more independent variables and the partial derivatives with respect to them, is called a partial differential equation.

8.6 Review Questions

1. Form the differential equation for $y = a\cos^3 x + b\sin 3x$ where a and b are arbitrary constants.
2. Form the differential equation of $y = ae^{bx}$ where a and b are the arbitrary constants.
3. Find the differential equation for the family of concentric circles $x^2 + y^2 = a^2$, a is the arbitrary constant.
4. Obtain the differential equation of the family of circles with fixed radius r and center on the y-axis.
5. Form the differential equation of all circles with their centers on the line $y = 2x$.
6. Form the differential equation of simple harmonic motion given by $x = A\cos(n\tilde{t}\,\alpha)$.
7. Obtain the differential equation from the relation

$$i.\ y = C_1e^{3x} + C_2e^{2x} + C_3e^{x}$$

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8. Form the differential equation from the relation $xy = Ae^{x} + B\tilde{e}^{x}$, where A and B are arbitrary constants.

9. Form the differential equation of all ellipses with centers at the origin.

10. Form the differential equation of the family of curves $y = e^x(A \cos x + B \sin x)$, where A and B are arbitrary constants.

Answers: Self Assessment

1.	differential equation	2.	independent
3.	mathematical	4.	False
5.	partial	6.	ordinary
7.	two or more	8.	parameter or arbitrary constant
9.	elimination	10.	two
11.	second	12.	n^{th}
13.	differentiation	14.	True
15.	False		

8.7 Further Readings

Books

D.A. Murray, *Introductory Course in Differential Equations*, Orient Blackswan.

Dr. Sanat Kumar Adhikari, Sanat Adhikari, *Basics of Professional Mathematics*, Firewall Media.

Zafar Ahsan, *Differential Equations and their Applications*, PHI Learning Pvt. Ltd.

Online link

www.purecoder.net

Notes

Unit 9: Solution of Differential Equation

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the order & degree of differential equation
- Illustrate the concept of solution of differential equation
- Understand the geometric meaning of a differential equation of the first order and first degree
- Discuss the equations in which the variables are separable

Introduction

From times immemorial, man has made progress through observing the nature, questioning and improvising himself. Scientists observe the positions of various celestial bodies, noting their movements and finally discovering the paths traversed by these bodies and later on representing them in the form of equations known as laws.

This process of mathematical formulations of natural phenomena lead to a differential equation – which is a natural choice as phenomena change in time or change in other constants.

Thus, in the fields of physics, engineering, economics, finance, management, etc., differential equations play a very important role. They arise in many practical problems where variation of

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distance of a moving particle, variation of current in an electric circuit, in mechanical systems, agriculture, etc. A differential equation involves independent variables, dependent variables, their derivatives and constants. These equations are solved and the solutions thus obtained are interpreted in the context of the problem.

9.1 Order & Degree of Differential Equation

The order of a differential equation is the order of highest derivative appearing in the equation.

The degree of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radials and fractions so far as the derivatives are concerned.

Thus from the above differential equations:

(i) is of the first order and first degree;

(ii) is of the second order and first degree;

(iii) is of the first order and second degree;

(iv) is of the second order and second degree;

Did u know? Equation of degree higher than one is also called non-linear.

Self Assessment

Fill in the blanks:

1. A .... involves independent variables, dependent variables, their derivatives and constants.

2. The .... of a differential equation is the order of highest derivative appearing in the equation.

3. The .... of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radials and fractions so far as the derivatives are concerned.

4. Equation of degree higher than one is also called ....

9.2 Solution of a Differential Equation

Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution or integral of the differential equation.

For example, the equation y = A cos$x$ + B sin$x$ is the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$. Because, we have, y = A cos$x$ + B sin$x$.

Upon differentiating it, $\frac{dy}{dx} \cong$ A sin$x$ + B cos$x$.

Again differentiation gives,

$$\frac{d^2y}{dx^2} \cong (\text{A} \cos x + \text{B} \sin x).$$

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Substituting these values in the given equation,

$$\frac{d^2y}{dx^2} + y \cong (\text{A} \cos x + \text{B} \sin x) + \text{A} \cos x + \text{B} \sin x = 0.$$

Notes

1. The solution of a differential equation is also called a primitive.
2. The order of the differential equation determines the number of arbitrary constants in the solution.

9.2.1 General Solution

The solution of a differential equation, in which the number of arbitrary constants is equal to the order of the differential equation is called the **general solution** or **complete solution** or the **complete primitive**.

For example, $y = C_1 e^x + C_2 \tilde{e}^x$ involving two arbitrary constants C_1 and C_2 is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$ of second order.

Similarly, the solution $y = \text{A} \cos x + \text{B} \sin x$ obtained for the differential equation $\frac{d^2y}{dx^2} + y = 0$ is the general solution.

9.2.2 Particular Solution

The solution obtained from the general solution by giving particular values to the arbitrary constants, is called a **particular solution** of the differential equation.

For example, $y = \cos x$ (taking A = 1 and B = 0) and $y = \sin x$ (by taking A = 0 and B = 1) are some particular solutions of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Caution Different techniques can be used to find the solutions of differential equations of first order and first degree depending on type of given differential equation.

Self Assessment

Fill in the blanks:

5. Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution or .... of the differential equation.
6. The solution of a differential equation, in which the number of arbitrary constants is equal to the order of the differential equation is called the ....

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7. The solution obtained from the general solution by giving particular values to the arbitrary constants, is called a .... of the differential equation.

9.3 Solution and Constant of Integration

A solution or integral of a differential equation is a relation between the variables, by means of which and the derivatives obtained therefore, the equation is satisfied.

Finding the unknown function is called solving or integrating the differential equation. The solution or integral of a differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the general solution (or complete integral or complete primitive). A solution obtainable from the general solution by giving particular values to the constants is called a particular solution.

E.g. If $y = A\cos x + B\sin x$ ....(1)

Then $\frac{dy}{dx} = -A\sin x - B\cos x$

and $\frac{d^2y}{dx^2} = -A\cos x - B\sin x$

or $\frac{d^2y}{dx^2} = -y$

$\Rightarrow \frac{d^2y}{dx^2} + y = 0$ ....(2)

Thus (1) is a solution of (2)

Note: Here y = A cos x + B sin x (involving two arbitrary constants A and B) is the general solution of $\frac{d^2y}{dx^2} + y = 0$ of second order equation.

Caution The solution of differential equation of nth order is its particular solution if it contains less than n arbitrary constants.

Self Assessment

Fill in the blanks:

8. A solution or integral of a differential equation is a .... between the variables, by means of which and the derivatives obtained therefore, the equation is satisfied.

9. Finding the .... function is called solving or integrating the differential equation.

10. The solution or integral of a differential equation is also called ...., because the differential equation can be regarded as a relation derived from it.

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9.4 Differential Equations of the First Order and First Degree

All differential equations of the first order and first degree cannot be solved; they can be solved, however, if they belong to one or the other of the following standard forms (categories) by the standard methods:

1. Equations in which the variables are separable,
2. Homogeneous equations,
3. Linear Equations,
4. Exact Equations.

9.4.1 Geometric Meaning of a Differential Equation of the First Order and First Degree

Let $f\left(x, y, \frac{dy}{dx}\right) = 0$ ....(1)

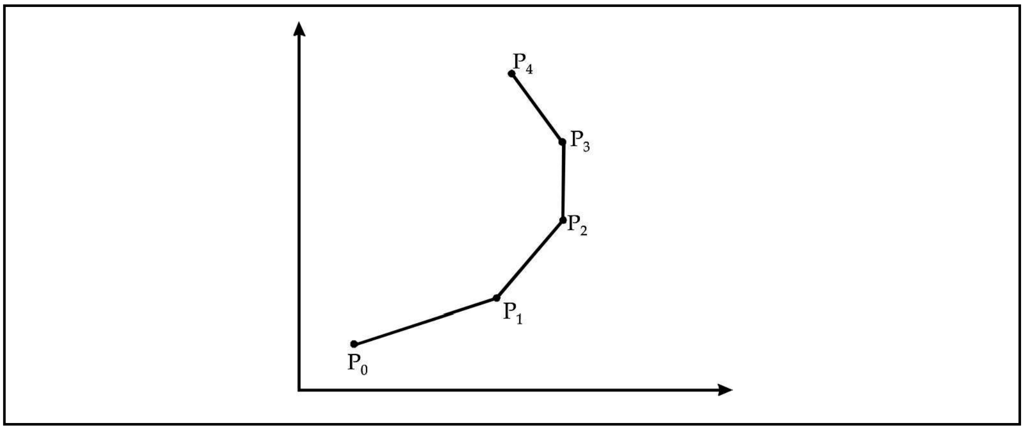
be a differential equation of the first order and first degree since the direction of a curve at a particular point is determined by drawing a tangent line at that point, i.e., its slope is given by $\frac{dy}{dx}$ at that particular point.

Let $P_0(x_0, y_0)$ be any point in the plane.

Let $m_0 = \frac{dy_0}{dx_0} = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$ be the slope of the curve at P_0 derived from (1).

Taking a neighboring point $P_1(x_1, y_1)$ such that the slope of P_0P_1 is m_0. Let $m_1 = \frac{dy_1}{dx_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ be the slope of the curve at P_1 derived from (1).

Taking a neighboring point P_2 (x_2, y_2) such that the slope P_1 P_2 is m_2.



Continuing like this, we get a succession of points. If the points are taken sufficiently close to each other, they obtain an approximate smooth curve C: y = (x) which is a solution of (1) corresponding to the initial point P_o (x_o, y_o). Any point on C and the slope of the tangent at that point satisfy (1). If the moving point starts at any other point, not on C and moves as before, it will describe another

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curve. The equation of each such curve is a particular solution of the differential equation (1). The equation of the system of all such curves is the general solution of (1).

9.4.2 Equations in Which the Variables are Separable

A differential equation of first order and first degree is of the form

$$\frac{dy}{dx} + f(x,y) = 0 \qquad ...(1)$$

which is sometimes written as

$$\text{M}dx + \text{N}dy = 0, \qquad ...(2)$$

where M and N are functions of x and y or constants and f is some known function of x and y. The following are some standard methods to solve some first order, first degree equations, which can be classified in one of the categories given below:

If a first order and first degree equation (2) can be put in the form

$$f_1(x)\, dx + f_2(y)dy = 0,$$

then it is said to be in **variable separable** form. The solution is obtained by integration, i.e.,

$$\int f_1(x)dx + \int f_2(y)dy = C,$$

where C is an arbitrary constant.

> *Notes* Many differential equations can be reduced to variable separable form by making suitable substitution. For instance, the equation of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to the form of variable separable by putting $ax + by + c = u$.

Example: Solve $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

Solution:

Here $(x^2 - yx^2)\, dx + (y^2 + xy^2)\, dx = 0$

or $$\frac{1-y}{y^2}dy + \frac{1+x}{x^2}dx = 0$$

or $$\left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \left(\frac{1}{x^2} + \frac{1}{x}\right)dx = 0$$

Integrating, we get

$$-\frac{1}{y} - \log y - \frac{1}{x} + \log x = c$$

or $$\log\left(\frac{x}{y}\right) - \left(\frac{y+x}{xy}\right) = c$$

which is the required solution.

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Example: Solve $x^2 \dfrac{dy}{dx} + y = 1$

Solution:

Here $x^2 \dfrac{dy}{dx} + y = 1$

or $\dfrac{dy}{1-y} = \dfrac{dx}{x^2}$

Integrating, we get

$$-\log(1-y) = \frac{1}{x} + c$$

or $\log(1-y) = \dfrac{1}{x} - c$

$\therefore$ $1 - y = e^{-c+\frac{1}{x}} = Ae^{\frac{1}{x}}$; where e^{-c}=A.

or $y = 1 - Ae^{\frac{1}{x}}$

which is the required solution.

Example: Solve $(xy^2 + x)dx + (yx^2 + y)dy = 0$.

Solution:

Here $x(y^2+1)dx + y\,(x^2+1)dy = 0$

Or $\dfrac{xdx}{x^2+1} + \dfrac{ydx}{y^2+1} = 0$

Integrating, we get

$$\log\frac{1}{2}\ (x^2+1) + \frac{1}{2}\log(y^2+1) = c$$

or $\log(x^2+1) + (y^2+1) = 2c$

or $(x^2+1)\ (y^2+1) = e^2e = A$, which is the required solution.

Example:

Solve $\dfrac{dy}{dx} = e^{3x-2y} + x^2e^{-2y}$

Solution:

Here $\dfrac{dy}{dx} = e^{3x-2y} + x^2e^{-2y}$

or $e^{2y}dy = \left(e^{3x} + x^2\right)dx$

Integrating, we get

$$\frac{1}{2}e^{2y} = \frac{1}{3}e^{3x} + \frac{x^3}{3} + c$$

or $\quad 3e^{2y} = 2\left(e^{3x} + x^3\right) + c_1$, where $c_1 = 6c$

which is the required solution.

Example:

Solve $(x+y+1)^2 \frac{dy}{dx} = 1$

Solution:

Putting $(x+y+1) = u$

we get $1 + \frac{dy}{dx} = \frac{du}{dx}$

or $\frac{dy}{dx} = \frac{du}{dx} - 1$

Thus the given equation reduces to

$$u^2\left(\frac{du}{dx} - 1\right) = 1$$

or $$\frac{du}{dx} = \frac{1+u^2}{u^2}$$

or $$\frac{u^2}{1+u^2}du = dx$$

or $$\left(1 - \frac{1}{1+u^2}\right)du = dx$$

Integrating, we get

$$u - \tan^{-1} u = x + c$$

or $$(x+y+1) - \tan^{-1}(x+y+1) = x + c$$

or $\quad y = \tan^{-1}(x+y+1) + c_1$, where $c_1 = c - 1$

which is the required solution.

Remark:

Different equations of the form

$$\frac{dy}{dx} = \phi(ax + by + c) \qquad \text{.....(1)}$$

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can be reduced to a form in which the variables are separable by the substitution $ax + by + c = u$

so that $a + b\dfrac{dy}{dx} = \dfrac{du}{dx}$

or $\dfrac{dy}{dx} = \dfrac{1}{b}\left(\dfrac{du}{dx} - a\right)$

Equation (1) reduces to

$$\frac{1}{b}\left(\frac{du}{dx} - a\right) = \phi(\mu)$$

or $$\frac{1}{b}\left(\frac{du}{dx} - a\right) = \phi(\mu)$$

or $$\frac{du}{a + b\phi(u)} = dx.$$

After integrating both sides, u is replaced by its values.

Example: Solve $3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$, given $y = \dfrac{\pi}{4}$ when $x = 0$

Integrating, we get

$$3\log\left(1 + e^x\right) + \log\left(\tan y\right) = \log c,$$

or $$\log\left[\left(1 + e^x\right)^3 \tan y\right] = \log c,$$

or $(1 + e^x)^3 \tan y = c.$ …(1)

which is the general solution of the given equation.

Since $y = \dfrac{\pi}{4}$ when $x = 0$, we have from (1),

$$(1 + 1)^3 \times 1 = c$$

$\Rightarrow$ $c = 8$

$\therefore$ The required particular solution is

$$(1 + e^x)^3 \tan y = 8.$$

Example:

Solve $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution:

The given equation can be re-written as

$$(\sin y + y \cos y)\, dy = x\, (2 \log x + 1)\, dx$$

Integrating both sides, we get

$$\int \sin y\, dy + \int \underset{I}{y} \underset{II}{\cos} dy = 2 \int \underset{II}{x} \log \underset{I}{x} dx + \int x dx + c$$

or $$-\cos y + y \sin y - \int 1.\sin y\, dy = 2\left[(\log x)\frac{x}{2} - \int \frac{1}{x}\frac{x^2}{2} dx\right] + \frac{x^2}{2}$$

or $$-\cos y + y \sin y + \cos y = 2\left[\frac{x^2}{2}\log x - \frac{1}{4}x^2\right] + \frac{x^2}{2} + c$$

or $$y \sin y = x^2 \log x + c.$$

which is the required general solution.

Example: Show that the curve in which the angle between the tangent and the radius vector at every point is one half of the vectorial angle is a cardioid.

Solution:

If the angle between the radius vector and the tangent at any point be j, and q the vectorial angle, then according to the given condition.

$$\varphi = \frac{\theta}{2}$$

or $$\tan\varphi = \tan\left(\frac{\theta}{2}\right)$$

$\therefore$ $$r\frac{d\theta}{dr} = \tan\frac{\theta}{2}$$

or $$\frac{dr}{r} = \cot\frac{\theta}{2} d\theta$$

Integrating, we get

$$\log r = 2\log\sin\left(\frac{\theta}{2}\right) + \log 2c$$

or $$\log r = \log 2c\sin^2\left(\frac{\theta}{2}\right)$$

$\Rightarrow$ $r = c\,(1 - \cos\theta)$,

which is the required equation of the curve. Clearly this represents a cardioid.

Example: Solve the differential equation $(1 + x)ydx + (1 + y)xdy = 0$

Solution:

The given equation can be written as

$$(1 + y)x\frac{dy}{dx} + (1 + x)y = 0.$$

Notes

Separating the variables, we get

$$\frac{1+y}{y}\,dy + \frac{1+x}{x}\,dx = 0$$

Integrating both sides, we get

$$\int \frac{1+y}{y}\,dy + \int \frac{1+x}{x}\,dx = c.$$

or, log$y$ + y + log x + x = a, a constant

or, log $xy + x + y = a$

which is the required solution.

Example: Solve $(x+y)^2\left(x\frac{dy}{dx}+y\right) = xy\left(1+\frac{dy}{dx}\right)$

Solution:

Put $x + y = u$, then differentiating w.r.t. x and y, we get respectively

$$1+\frac{dy}{dx} = \frac{du}{dx} \quad \text{and} \quad x\frac{dy}{dx}+y = \frac{dv}{dx}$$

Substituting these values in the given equation,

$$u^2\frac{dv}{dx} = v\frac{du}{dx} \quad \text{or,} \quad \frac{dv}{v} = \frac{du}{u^2}$$

Integrating both sides, we have

$$\int\frac{dv}{v} = \int\frac{du}{u^2} + C \quad \text{or,} \quad \log v = -\frac{1}{u}+C$$

$$\text{or,} \quad \log xy = -\frac{1}{x+y}+C \quad \text{or,} \quad \log xy + \frac{1}{x+y} = C.$$

Example: Solve $\frac{ydx - xdy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$

Solution:

$$\frac{-(xdy - ydx)}{\left(1-\frac{y}{x}\right)^2 x^2} = \frac{dx}{2\sqrt{1-x^2}}$$

Put $\frac{y}{x} = v$, $\frac{xdy - ydx}{x^2} = dv.$

Substituting these values in the given equation,

$$-\frac{dv}{(1-v)^2} = \frac{dx}{2\sqrt{1-x^2}},$$ Integrating we get

$$-\frac{1}{1-v} = \frac{1}{2}\sin^{-1}x + c$$

or, $$\frac{x}{y-x} = \frac{1}{2}\sin^{-1}x + C$$ is the required solution.

Notes

Task Solve the differential equation: $\dfrac{dy}{dx} + \dfrac{1+y^2}{1+x^2}$

9.4.3 Exact Differential Equations

A first order first degree equation of the form

$$M(x, y)dx + N(x, y)\, dy = 0$$

is said to be exact if its left hand side quantity is the exact differential of some function $u(x, y)$. Thus for the above differential equation to be exact,

$$du \;\; Mdx + Ndy = 0.$$

The solution of the above equation is given by

$$u(x, y) = C.$$

Did u know? Necessary and sufficient condition for a differential equation of the form $M(x, y)dx + N(x, y)dy$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Working Rule for Solving Exact Differential Equation

The solution of an exact differential equation $Mdx + Ndy = 0$ is

$$\int\limits_{y\ \text{constant}} M dx + \int (\text{terms of N that do not contain } x) dy = C$$

Example: Solve $(\tilde{x}y)\,(d\tilde{x}\;dy) = dx + dy$

Solution:

The given equation is

$$(\tilde{x}y)\,(d\tilde{x}\;dy) = dx + dy$$

or, $\quad (\tilde{x}\tilde{y}\;1)d\tilde{x}(\tilde{x}y + 1)dy = 0$

Here, $\quad M = \tilde{x}\tilde{y}\;1,\; N \cong 1(\tilde{x}y + 1).$

Now,

$\therefore \quad \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} = -1.$

Therefore, the given equation is exact. It's solution is given by

$$\int\limits_{y\ \text{constant}} M dx + \int (\text{terms of N that do not contain } x) dy = C$$

i.e., $\quad \displaystyle\int\limits_{y\ \text{constant}} (x - y - 1)dx - \int (-y + 1)dy = C$

Notes

or, $\qquad \frac{1}{2}x^2 - xy - x + \frac{1}{2}y^2 - y = C.$

or, $\qquad x^2 + y^2 - 2xy - 2x - 2y = C'.$

Example: Solve (ey + 1)cos$x$ dx + ey sin$x$ dy = 0.

Solution:

The given equation is (ey + 1) cos x dx + ey sin x dy = 0 ...(1)

This is of the form Mdx + Ndy = 0

where M = (ey + 1) cos x, N = ey sin x

$$\therefore \frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation (1) is exact.

General solution is given by

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of N independent of } x)dy = C$$

i.e., $\displaystyle\int_{y\text{ constant}} (e^y + 1)\cos x dx + \int 0 dy = C$

or, (ey + 1) sin x = C is the general solution.

Example: Solve $\left(1 + e^{x/y}\right) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

Solution:

The given equation is $\left(1 + e^{x/y}\right) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$ ...(1)

This is of the form Mdx + Ndy = 0, where

M = 1 + ex/y, N = ex/y ($\tilde{1}x/y$)

$$\therefore \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right), \quad \frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The general solution is given by

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of N independent of } x)dy = C$$

$$\Rightarrow \int_{y\text{ constant}} (1 + e^{x/y})dx + \int o.dy = C \Rightarrow x + y\,e^{x/y} = C$$

This is the required general solution.

Notes

Example: Solve $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx+\{x+\log x - x\sin y\}\,dy=0$

Solution:

This given equation is of the form Mdx + Ndy =0

where M = $y\left(1+\frac{1}{x}\right)+\cos y$, N = x + log x – sin y

$\therefore\ \frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y,\quad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$

$\therefore\ \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}$, therefore, the given equation is exact.

The solution is given by

$$\int_{y\text{ constant}} M\,dx+\int(\text{terms of N independent of }x)dy = C$$

or, $$\int_{y\text{ constant}}\left\{\left(1+\frac{1}{x}\right)+\cos y\right\}dx+\int o.dy = C$$

or, $y(x + \log x) + x\cos y = C.$

is the required general solution.

Task Solve the differential equation: $(1+e^{x/y})\,dx+\left(1-\frac{x}{y}\right)e^{x/y}\,dy=0.$

Self Assessment

Fill in the blanks:

11. A differential equation of first order and first degree is of the form $\frac{dy}{dx}+f(x,y)=0$ is sometimes written as ....
12. If a first order and first degree equation can be put in the form $f1(x)\,dx + f2(y)dy = 0$, then it is said to be in .... form.
13. Many differential equations can be reduced to variable separable form by making suitable ....
14. A first order first degree equation of the form $M(x, y)dx + N(x, y)\,dy = 0$ is said to be .... if its left hand side quantity is the exact differential of some function $u(x, y)$.
15. The .... of the exact equation is given by $u(x, y) = C$.

9.5 Summary

- A differential equation involves independent variables, dependent variables, their derivatives and constants.
- The order of a differential equation is the order of highest derivative appearing in the equation.

Notes

- The degree of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radials and fractions so far as the derivatives are concerned.
- Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution or integral of the differential equation.
- The solution of a differential equation, in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution or complete solution or the complete primitive.
- The solution obtained from the general solution by giving particular values to the arbitrary constants, is called a particular solution of the differential equation.
- A solution or integral of a differential equation is a relation between the variables, by means of which and the derivatives obtained therefore, the equation is satisfied.
- A differential equation of first order and first degree is of the form $\frac{dy}{dx} + f(x, y) = 0$ which is sometimes written as Mdx + Ndy = 0. Where M and N are functions of x and y or constants and f is some known function of x.
- A first order first degree equation of the form M(x, y)dx + N(x, y) dy = 0 is said to be exact if its left hand side quantity is the exact differential of some function $u(x, y)$.

9.6 Keywords

Complete Primitive: The solution of a differential equation, in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution or complete solution or the complete primitive.

Degree: The degree of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radials and fractions so far as the derivatives are concerned.

Differential Equation: A differential equation involves independent variables, dependent variables, their derivatives and constants.

Order: The order of a differential equation is the order of highest derivative appearing in the equation.

Particular Solution: The solution obtained from the general solution by giving particular values to the arbitrary constants, is called a particular solution of the differential equation.

9.7 Review Questions

1. Solve the differential equation $x^2(y+1)dx + y^2(x-1)dy = 0$.
2. Solve the differential equation $(1-x^2)(1-y)dx = xy(1+y)dy$.
3. Solve the differential equation $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.
4. Solve the differential equation $x\frac{dy}{dx} = +\cot y = 0$ if $y = \frac{\pi}{4}$ when $x = \sqrt{2}$.

5. Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

6. Solve the differential equation $\frac{dy}{dx} = (4x + y + 1)^2$.

7. Solve the differential equation $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$.

8. Solve the differential equation $(x + y)^2 \frac{dy}{dx} = a^2$.

9. Solve the differential equation $\frac{dy}{dx} - x \tan(y - x) = 1$.

10. Solve the differential equation $(2x - y + 1)dx + (2y - x - 1)dy = 0$.

11. Solve the differential equation $(hx + by + f)\, dy + (ax + hy + g)dx = 0$.

Answers: Self Assessment

1.	differential equation	2.	order
3.	degree	4.	non-linear
5.	integral	6.	general solution
7.	particular solution	8.	relation
9.	unknown	10.	primitive
11.	$Mdx + Ndy = 0$	12.	variable separable
13.	substitution	14.	exact
15.	exact		

9.8 Further Readings

Books

D.A. Murray, *Introductory Course in Differential Equations*, Orient Blackswan.

Dr. Sanat Kumar Adhikari, Sanat Adhikari, *Basics of Professional Mathematics*, Firewall Media.

Zafar Ahsan, *Differential Equations and their Applications*, PHI Learning Pvt. Ltd.

Online link

www.purecoder.net

Notes

Unit 10: Homogeneous Equations

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of homogeneous equations
- Discuss the equations reducible to homogeneous form

Introduction

As we know, the finest manner to resolve a new problem is to lessen it, in some manner, into the outline of a problem that you already recognize how to solve. This is what you perform with homogeneous differential equations. If you identify the truth that an equation is homogeneous you can, in some cases, carry out a substitution which will permit you to apply separation of variables to solve the equation. At this position you may be inquiring yourself, what is a homogeneous differential equation? It is just an equation where both coefficients of the differentials dx and dy are homogeneous.

10.1 Homogeneous Equations

Homogeneous functions are defined as functions where the sums of the powers of each term are the same. A homogeneous equation can be malformed into a distinguishable equation by a change of variables.

An equation of the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} \qquad \text{.....(1)}$$

is called a homogeneous function of the same degree in x and y.

If $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of degree n in x and y, then

$$f_1(x,y) = x^n \varphi_1\left(\frac{y}{x}\right) \text{ and } f_2(x,y) = x^n \varphi_2\left(\frac{y}{x}\right)$$

$\therefore$ Equation (1) reduces to

Notes

$$\frac{dy}{dx} = \frac{\varphi_1\left(\frac{y}{x}\right)}{\varphi_2\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right) \qquad \ldots\ldots\ldots..(2)$$

Putting $\frac{y}{x} = v$ or $y = vx$

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (2) becomes

$$v + x\frac{dv}{dx} = F(v)$$

or $$\frac{dv}{(v) - v} = \frac{dx}{x}$$

Integrating, we get the solution in terms of v and x. Replacing v by $\frac{y}{x}$, we get the required solution.

Did u know? Homogeneous Equations are equations that can be changed into a distinguishable equation by a variation of the dependent variable, y.

Notes The following implications are considered regarding systems of homogeneous equations, in which there are no constant terms, where all right hand sides are 0.

1. If you contain n variables, and n equations, and the determinant of the system is non-zero, in order that the corresponding matrix is non-singular, then the origin point, or 0 vector is the only solution to the equations. It is known as the trivial solution to them.

2. If there are smaller amount of linearly independent equations than there are variables, then there are other, non-trivial solutions to the homogeneous equations. These may be located by row reduction, in parametric manner, with the basis variables as parameters. Row reduction is to some extent easier in this case because there is no right hand side of the equations to manage.

Working Rule

(i) Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(ii) Separate the variables v & x, and integrate

(iii) Replace the value of v by $\frac{y}{x}$.

Notes

Example: Solve $x\,(x - y)\,dy + y^2\,dx = 0$

Solution:

Here $\dfrac{dy}{dx} = \dfrac{y^2}{x(y-x)}$

Putting $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dy}{dx}$

$$\therefore \qquad v + x\frac{dx}{dx} = \frac{v^2}{v-1}$$

$$\Rightarrow \qquad x\frac{dy}{dx} = \frac{v^2}{v-1} - v = \frac{v}{v-1}$$

or $\qquad \dfrac{v-1}{v}dv = \dfrac{dx}{x}$

or $\qquad \left(1 - \dfrac{1}{v}\right)dv = \dfrac{1}{x}dx$

Integrating, we get

$$(v - \log v) = \log x + \log c$$

or $\qquad \log e^v - \log v = \log cx$

or $\qquad \log\left(\dfrac{e^v}{v}\right) = \log cx$

$$\Rightarrow \qquad e^v = vcx$$

or $\qquad e^{y/x} = c.\dfrac{y}{x}.x$

or $\qquad y = \dfrac{1}{c}e^{y/x}$

or $\qquad y = c_1 e^{y/x}$; where $c_1 = \dfrac{1}{c}$

which is the required general solution.

Example: Solve $(x^2 - y^2)\,dx = 2xy\,dy$

Solution:

Here $\qquad \dfrac{dy}{dx} = \dfrac{x^2 - y^2}{2xy} \qquad$ ……(1)

Notes

Putting $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) reduces to $v + x\frac{dy}{dx} = \frac{1-v^2}{2v}$

or $\quad x\frac{dv}{dx} = \frac{1-v^2}{2v} - v = \frac{1-3v^2}{2v}$

or $\quad \frac{2v}{1-3v^2}dv = \frac{dx}{x}$

Integrating, we get

$$-\frac{1}{3}\log\left(1-3v^2\right) = \log x + c$$

or $\quad 3\log x + \log\left(1-3v^2\right) = -3c$

or $\quad \log x^3\left(1-3v^2\right) = -3c$

or $\quad x^3\left(1-\frac{3y^3}{x^2}\right) = e^{-3c} = c' \qquad [\because y = vx]$

or $\quad x^3\left(1-\frac{3y^2}{x^2}\right) = c'$

or $\quad x\left(x^2-3y^2\right) = c'$

which is the required general solution.

Example: Solve $y^2dx + \left(xy = x^2\right)dy = 0$

Solution:

Here $\frac{dy}{dx} = -\frac{y^2}{xy+x^2}$ ...(1)

Putting $y = vx$, so that $\frac{dv}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) reduces to $v + x\frac{dv}{dx} = \frac{v^2}{1+v}$

or $\quad x\frac{dv}{dx} = -\frac{v+2v^2}{1+v}$

or $\quad -\frac{dx}{x} = \frac{(1+v)dv}{v+2v^2}$

Notes

Integrating, we get

$$-\log x = \int \frac{(1+v)}{v(2v+1)} dv$$
$$= \int \left(\frac{1}{v} - \frac{1}{2v+1} \right) dv$$
$$= \log v - \frac{1}{2}\log(2v+1)\log c_1$$

or $$2\log x + 2\log v - \log(2v+1) = \log c, \qquad [2\log c_1 = \log c]$$

or $$\log\left(\frac{x^2v^2}{2v+1}\right) = \lg c$$

or $$\frac{x^2v^2}{2v+1} = c$$

or $$\frac{x^2\left(\frac{y^2}{x^2}\right)}{2\left(\frac{y}{x}\right)+1} = c \qquad \left[\therefore v = \frac{y}{x}\right]$$

or $$\frac{xy^2}{2y+x}$$

or $$xy^2 = c(2y+x)$$

which is the required general solution.

Example: Solve $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$.

Solution:

Here $$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \qquad \text{....(1)}$$

Putting $y = vx$, so that $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$

$\therefore$ (1) reduces to

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

or $$x\frac{dv}{dx} = \sqrt{1+v^2}$$

or $$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{Dx}.$$

Integrating, we get

$$\log\left(v+\sqrt{1+v^2}\right)=\log x+\log c$$

or $$v+\sqrt{1+v^2}=cx$$

or $$\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}=cx \qquad \left[\because v=\frac{y}{x}\right]$$

or $$y+\sqrt{x^2+y^2}=cx^2$$

which is the required solution.

Example: Solve $\left[x\tan\left(\frac{y}{x}\right)-y\sec^2\left(\frac{y}{x}\right)\right]dx+x\sec^2\left(\frac{y}{x}\right)dy=0$

Solution:

Here $$\frac{dy}{dx}=\frac{y\sec^2\left(\frac{y}{x}\right)-x\tan\left(\frac{y}{x}\right)}{x\sec^2\left(\frac{y}{x}\right)} \qquad \text{.....(1)}$$

Putting $v=\frac{y}{x}$ i.e., $y = vx$

So that $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Equation (1) reduces to

$$v+x\frac{dy}{dx}=v-\frac{\tan v}{\sec^2 v}$$

or $$\frac{\sec^2 v}{\tan v}dv+\frac{dx}{dx}=0$$

Integrating, we get

log (tan v) + log x = log c

or log (x tan v) = log c

or $$x\tan\left(\frac{y}{x}\right)=c; \qquad \left[\because v=\frac{y}{x}\right]$$

which is required solution.

Example: Solve $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$.

Notes

Solution:

Here $\frac{dx}{dy} = -\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{x/y}}$ ……….(1)

Putting $x = vy$, so that $\frac{dx}{xy} = v + y\frac{dv}{xy}$

$\therefore$ (1) reduces to

$$v + y\frac{dv}{dy} = -\frac{e^{v}(1-v)}{1+e^{v}}$$

or $$v\frac{dv}{dy} = -\frac{e^{v}(1-v)}{1+e^{v}} - v = -\frac{v+e^{v}}{1+e^{v}}$$

or $$\frac{dy}{-y} = \frac{1+e^{v}}{v+e^{v}}dv$$

Integrating, we get

$$\log c - \log y = \log(v + e^{v})$$

or $\log y + \log(v + e^{v}) = \log c$

or $y(v+e) = c$

or $x + y\, e^{e/y} = c.$

which is the required general solution.

Caution When a differential equation contain $\frac{y}{x}$ a number of times, solve it like a homogeneous equations by putting $\frac{y}{x} = v$.

Notes The function f does not rely on x & y individually but only on their proportion y/x or x/y.

Task Solve the following differential equation:

$$y - x\frac{dy}{dx} = x + y\,\frac{dy}{dx}.$$

Self Assessment

Notes

Fill in the blanks:

1. .... is just an equation where both coefficients of the differentials dx and dy are homogeneous.
2. Homogeneous functions are defined as functions where the .... of the powers of each term are the same.
3. A homogeneous equation can be malformed into a distinguishable equation by a change of ....
4. An equation of the form $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$ is called a homogeneous function of the .... degree in x and y.
5. If $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of degree n in x and y, then $f_1(x,y)=x^n\varphi_1\left(\frac{y}{x}\right)$ *and* $f_2(x,y)=x^n\varphi_2\left(\frac{y}{x}\right)$ ....
6. If you contain n variables, and n equations, and the determinant of the system is non-zero, the it is known as the .... solution to them.
7. The function f does not depend on x & y independently but only on their proportion................................... .
8. If you recognize the fact that an equation is homogeneous you can, in some cases, carry out a .... which will permit you to apply separation of variables to solve the equation.
9. If we get the solution in terms of v and x replacing v by.................................. we get the required solution.

State whether the following statements are true or false:

10. The sums of the powers of each term of homogenous function are different.
11. If there are smaller amount of linearly independent equations than there are variables, then there are other, trivial solutions to the homogeneous equations.

10.2 Equations Reducible to Homogeneous Form

A differential equation of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C} \qquad \text{.....(1)}$$

can be reduced to the homogeneous form.

Did u know? A differential equation of the form $\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$ is not homogeneous, but we can formulate it so by a transformation in the origins of x and y.

Notes

Now we consider the following cases to reduce the given differential equation to the homogeneous form.

Case I

When $\frac{a}{A} \neq \frac{b}{B}$.

Putting $x = X + h$, $y = Y + k$; (h & k are constants)

so that $dx = dX$, $dy = dY$.

Equation (1) becomes:

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+c}$$

or
$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+c)} \qquad \text{.....(2)}$$

Choosing h & k such that (2) becomes homogeneous.

Thus $ah + bk + c = 0$ and $Ah + Bk + c = 0$

so that $\frac{h}{bC-Bc} = \frac{k}{cA-Ca} = \frac{1}{aB-Ab}$

$\Rightarrow$
$$h = \frac{bC-Bc}{aB-Ab}, k = \frac{cA-Ca}{aB-Ab}$$

$$\left[\because \frac{a}{A} \neq \frac{b}{B} i.e., aB - bA \neq 0\right]$$

$\therefore$ Equation (2) becomes

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$$

which is homogeneous in X & Y and can be solved by putting $Y = vX$.

Case II

When $\frac{a}{A} = \frac{b}{B}$, i.e., $aB - bA = 0$.

The case (1) fails.

Now $\frac{a}{A} = \frac{b}{B} = \frac{1}{M}$ (say)

so that $A = ma$, $B = mb$.

$$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c} = f(ax+by)$$

$\therefore$ Equation (1) reduces to

which can be solved by putting $ax + by = t$.

Notes

Example:

Solve $$\frac{dy}{dx}=\frac{6x-2y-7}{2x+3y-6} \qquad \text{....(1)}$$

Solution:

Putting $x = X - h$, $y - Y + k$,

$$\therefore \qquad \frac{dy}{dx}=\frac{6(x+h)+2(y-k)-7}{2(x+h)+3(y+k)-6},$$

or $$\frac{dY}{dX}=\frac{6X-2Y+6h-2k-7}{2X+3Y+2h+3k-6}, \qquad \text{....(2)}$$

Choosing h and k such that (2) becomes homogeneous,

i.e., $6h - 2k - 7 = 0$ and $2h + 3k - 6 = 0$

which gives $h=\frac{3}{2}$, $k = 1$.

$\therefore$ The equation (2) becomes

$$\frac{dY}{dX}=\frac{6X-2Y}{2X+3Y}$$

Putting $Y = vX$, so that $\frac{dY}{dX}=v+X\frac{dv}{dX}$.

$$\therefore \qquad v+\frac{dv}{dx}=\frac{6-2v}{2+3v}$$

or $$\frac{dx}{x}=-\frac{1}{2}\frac{6v+4}{3v^2+4v-6}dv$$

Integrating, we have

$$\log X=-\frac{1}{2}\log\left(2v^2+4v-6\right)+\log c$$

or $$\log x+\log\left(3v^2+4v-6\right)^{1/2}=\log c$$

or $$X\sqrt{3v^2+4v-6}=c$$

or $$x\left(3v^2+4v-6\right)^{1/2}=c$$

or $$\left(x-\frac{3}{2}\right)\left[3\left(\frac{y-1}{x-\frac{3}{2}}\right)^2+4\left(\frac{y-1}{x-\frac{3}{2}}\right)-6\right]^{\frac{1}{2}}=c$$

Notes

which on simplification gives

$$3y^2 + 4xy - 6x^2 - 12y + 14x = c.$$

Example:

Solve $\dfrac{dy}{dx} = \dfrac{6x-2y-7}{3x-y+4}$ …(1)

Solution:

Here $\dfrac{a}{A} = \dfrac{b}{B}$.

So putting $3x - y = v$, so that $3 - \dfrac{dy}{dx} = \dfrac{dv}{dx}$,

$\therefore$ (1) becomes

$$\frac{dv}{dx} = 3 - \frac{2v-7}{v+4} = \frac{v+19}{v+4}$$

$$\therefore \qquad dx = \frac{v+4}{v+19}\, dv = \left(1 - \frac{15}{v+19}\right) dv$$

Integrating, we get

$$x + c = v - 15 \log(v + 19)$$

On restoring the value of v, we get

$$2x - y - 15 \log (v + 19) = c,$$

which is the required solution.

Example:

Solve $\dfrac{dy}{dx} = \dfrac{2x+3y-4}{4x+y-3}$, given that $y = 1$, when $x = 1$.

Solution:

Putting $x = X + h$ and $y = Y + k$, the given equation becomes

$$\frac{dY}{dX} = \frac{2(X+h)+3(Y+k)-4}{4(X+h)+(Y+k)-3},$$

$$= \frac{2X+3Y+2h+3k-4}{4X+Y+4h+k-3} \qquad \text{….(1)}$$

Choosing h and k such that (1) becomes homogeneous,

i.e., $2h + 3k - 4 = 0$ and $4h + k - 3 = 0$.

$\Rightarrow h = \dfrac{1}{2}, k = 1.$

Notes

Then (1) becomes $\frac{dY}{df} = \frac{2X+3Y}{4X+Y}$.

Putting $Y = vX$, so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$.

$\therefore \qquad v + X\frac{dv}{dX} = \frac{2+3v}{4+v}$

or $\qquad X\frac{dv}{dX} = \frac{2+3v}{4+v} - v = \frac{2-v-v^2}{4+v}$

or $\qquad \frac{v+4}{(v-1)(v-2)}dv = -\frac{dX}{X}$

or $\qquad \left[-\frac{2}{3(v+2)} + \frac{5}{3(v-1)}\right]dv = -\frac{1}{x}dx$

Integrating, we have

$$-\frac{2}{3}\log(v+2) + \frac{5}{3}\log(v-1) = -\log X + \log C$$

Replacing v by $\frac{Y}{X}$, we have

$$-3\log X \neq 3\log C = -2\log\left(\frac{Y+2X}{X}\right) + 5\log\left(\frac{Y-X}{X}\right)$$

On restoring the value of x & y, we get

$$5\log\left(y - x - \frac{1}{2}\right) - 2\log(y + 2x - 2) = 3\log c\,.$$

or $\qquad \frac{\left(y-x-\frac{1}{2}\right)^5}{(y+2x-2)^2} = c^3\,;$

Putting $y = 1$ at $x = 1$, we have $c^3 = -\frac{1}{25}$.

Thus $\frac{\left(y-x-\frac{1}{2}\right)^5}{(y+2x-2)^2} = -\frac{1}{2^5}$

or $\qquad (y+2x-2)^2 = -2^5\,(y-x-\frac{1}{2})^5$

$\qquad = (2x-2y+1)^5$

or $\qquad (y+2x-2)^2 = (2x-2y+1)^5$

which is the required particular solution.

Notes

Task Solve the following differential equation:

$$\frac{dy}{dx} = \frac{y-x+1}{y+x+5}.$$

Self Assessment

Fill in the blanks:

12. A differential equation of the form .... can be reduced to the homogeneous form.

13. A differential equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ is not homogeneous, but we can formulate it so by a transformation in the .... of x and y.

14. When ...., then the equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ becomes

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+c)}.$$

15. When ...., then the equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ becomes

$$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c} = f(ax+by).$$

10.3 Summary

- Homogeneous equation is just an equation where both coefficients of the differentials dx and dy are homogeneous.
- Homogeneous functions redefined as functions where the sums of the powers of each term are the same.
- A homogeneous equation can be malformed into a distinguishable equation by a change of variables.
- An equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ is called a homogeneous function of the same degree in x and y.
- If you identify the truth that an equation is homogeneous you can, in some cases, carry out a substitution which will permit you to apply separation of variables to solve the equation.
- If $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of degree n in x and y, then $f_1(x,y) = x^n \varphi_1\left(\frac{y}{x}\right)$ *and* $f_2(x,y) = x^n \varphi_2\left(\frac{y}{x}\right)$.

Notes

- A differential equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ can be reduced to the homogeneous form when $\frac{a}{A} \neq \frac{b}{B}$.

- A differential equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ can be reduced to the homogeneous form when $\frac{a}{A} = \frac{b}{B}$, i.e., $aB - bA = 0$.

10.4 Keywords

Homogeneous Equation: Homogeneous equation is just an equation where both coefficients of the differentials dx and dy are homogeneous.

Homogeneous Functions: Homogeneous functions are defined as functions where the sums of the powers of each term are the same.

10.5 Review Questions

Solve the following differential equations:

1. $(x^2 + y^2)dx = 2xydy$
2. $x^2ydy + (x^3 + x^2y - 2xy^2 - y^3)\, dx = 0$
3. $(1 + e^{x/y})\, dx + e^{x/y}\, (1 - x/y)\, dy = 0$
4. $y(8x - 9y)\, dx + 2x(x - 3y)\, dy = 0$
5. $(x^2 - 2xy + 3y^2)\, dx + (y^2 + 6xy - x^2)dy = 0$
6. $(y\, dx + x\, dy)\, x \cos(y/x) = (xdy - ydx)\, y \sin(y/x)$
7. $x^2yd - (x^3 + y^3)\, dy = 0$
8. $xdy - ydx = \sqrt{x^2 + y^2}\, dx$ given that $y = 1$ when $x = \sqrt{3}$
9. $\left[x \tan\left(\frac{y}{x}\right) - y \sec^2\left(\frac{y}{x}\right)\right] dx + x \sec^2\left(\frac{y}{x}\right) dy = 0$
10. $xdx + \sin^2\left(\frac{y}{x}\right)\left(ydx - xdy\right) = 0$
11. $\frac{dy}{dx} = \frac{y}{x + ye^{-2(x/y)}}$
12. $(x-y-2)\, dx + (x-2y-3)\, dy = 0.$
13. $(x-y+1)\, dx - (2x+2y+3)\, dy = 0.$

Notes

Answers: Self Assessment

1. Homogenous equation
2. sums
3. variables
4. same
5. $x^n \varphi_2\left(\frac{y}{x}\right)$
6. trivial
7. *y/x* or *x/y*
8. substitution
9. $\frac{y}{x}$
10. False
11. False
12. $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$
13. origins
14. $\frac{a}{A} \neq \frac{b}{B}$
15. $\frac{a}{A} = \frac{b}{B}$

10.6 Further Readings

Books

D.A. Murray, *Introductory Course in Differential Equations*, Orient Blackswan.

Dr. Sanat Kumar Adhikari, *Basics of Professional Mathematics*, Firewall Media.

Zafar Ahsan, *Differential Equations and their Applications*, PHI Learning Pvt. Ltd.

Online link

www.sosmath.com/diffeq/first/homogeneous/homogeneous.html

Richa Nandra, Lovely Professional University

Unit 11: Linear Differential Equations of First Order

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of linear equations
- Discuss the equations reducible to linear form

Introduction

An equation is basically the mathematical manner to portray a relationship among two variables. The variables may be physical quantities, possibly temperature and position for instance, in which case the equation informs us how one quantity relies on the other, so how the temperature differs with position. The easiest type of relationship that two such variables can comprise is a linear relationship. This shows that to locate one quantity from the other you multiply the first by some number, then add a different number to the outcome. In this unit, you will understand the concept of linear equations and equations reducible to linear form.

11.1 Linear Equations

An equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \ldots(1)$$

in which P & Q are functions of x alone or constant is called a linear equation of the first order.

Did u know? If you are provided a value of x, you can simply discover the value of y.

The general solution of the above equation can be found as follows:

Multiplying both sides of (1) by $e^{\int P dx}$, we have

$$\frac{dy}{dx} e^{\int P dx} + Pye^{\int P dx} = Qe^{\int P dx}$$

Notes

i.e., $\dfrac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$

Integrating, we have

$$ye^{\int P dx} = \int Qe^{\int P dx}\, dx + c, \qquad \text{.......(2)}$$

which is the required general solution.

Notes

1. $e^{\int P dx}$ is known as Integrating factor, in short, I.F.
2. Linear differential equation is commonly known as Leibnitz's linear equation.

Example:

Solve $\cos x \dfrac{dy}{dx} + y \sin x = 1$

Solution:

Given equation can be written as

$$\frac{dy}{dx} + y \tan x = \sec x \qquad \text{....(1)}$$

Here $P = \tan x$, $Q = \sec x$.

$\therefore$ $I.F. = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x.$

$\therefore$ Solution of (1) is

$y \,.\, \sec x = \int \sec x \,.\, \sec x\, dx + c$

or $y \sec x = \tan x + c.$ Ans.

Example:

Solve $\left(1 + x^2\right)\dfrac{dy}{dx} + 2xy - 4x^2 = 0.$

Solution:

Given equation is

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = \frac{4x^2}{1+x^2}. \qquad \text{....(1)}$$

Here $P = \dfrac{2x}{1+x^2}$, $Q = \dfrac{4x^2}{1+x^2}.$

$\therefore$ $I.F. = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = (1 + x^2).$

$\therefore$ The solution of (1) is

$$y\,.\left(1+x^2\right)=\int\frac{4x^2}{1+x^2}\,.\left(1+x^2\right)dx+c$$

or $$y\,.\left(1+x^2\right)=\frac{4}{3}x^3+c.$$

which is the required solution.

Caution The system is said to be consistent if it contains a solution or else the system is said to be inconsistent.

Example:

Solve $\left(1+y^2\right)dx=\left(\tan^{-1}y-x\right)dy.$

Solution:

Since the equation involves a term tan-1 dy. So it is not linear in y. However, the equation is linear in x, so that

$$\left(1+y^2\right)dx+x\,dy=\tan^{-1}y\,dy \qquad .....(1)$$

or $$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}$$

Here $$P=\frac{1}{1+y^2},\qquad Q=\frac{\tan^{-1}y}{1+y^2}.$$

$$\therefore \qquad I.F.=e^{\int P dy}=e^{\int\frac{1}{1+y^2}dy}=e^{(\tan^{-1}y)}.$$

$\therefore$ The solution of (1) is

$$xe^{(\tan^{-1}y)}=\int\frac{\tan^{-1}y}{1+y^2}\,.\,e^{\left(\tan^{-1}y\right)dy}+c \qquad .....(2)$$

Now solving

$$I=\int\frac{\tan^{-1}-y}{1+y^2}e\tan^{-1}y\,dy$$

Putting t = tan-1y

$$dt=\frac{1}{1+y^2}\,dy$$

or $$I=\int te^t\,dt$$

$$=te^t-e^t$$

Notes

Thus (2) becomes

$$x\,e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y-1) + c$$

or $$x = (\tan^{-1}y-1) + ce^{\tan^{-1}y},$$

which is the required solution.

Example:

Solve $x(1-x^2)\,dy + (2x^2y-y-ax^3)\,dx = 0.$

Solution:

The given equation is equivalent to

$$\frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}\,y=\frac{ax^2}{1-x^2}. \qquad \ldots\ldots(1)$$

Here $P = \dfrac{2x^2-1}{x(1-x^2)}$, $Q=\dfrac{ax^2}{1-x^2}$.

$$\therefore \quad I.F. = e^{\int p\,dx} = e^{\int\left(\frac{2x^2-1}{x(1-x^2)}\right)dx} = e^{\int\frac{1-2x^2}{x(x^2-1)}dx}$$

$$= e^{\int\frac{1-2x^2}{x(x-1)\,(x+1)}dx}$$

$$= e^{-\int\left(\frac{1}{x}+\frac{1}{2\,(x-1)}+\frac{1}{2\,(x+1)}\right)dx}$$

$$= e^{\log\left\{\frac{1}{x\sqrt{x^2-1}}\right\}} = \frac{1}{x\sqrt{x^2-1}}.$$

Thus the solution of (1) is

$$y.\frac{1}{x\sqrt{x^2-1}} = \int\frac{ax^2}{1-x^2}\cdot\frac{1}{x\sqrt{x^2-1}}\,dx + c$$

$$= c + a\int\frac{-x}{(1-x^2)^{3/2}}\,dx$$

$$= c - \frac{a}{2}\int\frac{dt}{t^{3/2}}, \ (t = x^2-1)$$

$$= c + \frac{a}{2}\,.\,2(t)^{-1/2}$$

Notes

$$\frac{y}{x\sqrt{x^2-1}} = c + a\,(x^2-1)^{-1/2}$$

$$\therefore \qquad y = cx\sqrt{x^2-1} + ax,$$

which is the required solution.

Example:

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

Solution:

The given equation can be written as

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Here $P = \frac{1}{\sqrt{x}}$, $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$$\therefore \qquad IF = e^{\sqrt{Pdf}} = e^{\sqrt{x^{1/2}}dx} = e^{2\sqrt{x}}.$$

Solution is

$$\therefore \qquad ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}\,dx + c$$

$$ye^{2\sqrt{x}} = \int x^{1/2}\,dx + c$$

$= 2\sqrt{x} + c$, which is the required solution.

Notes Solving a system comprising a single linear equation is simple. On the other hand, if we are concerning with two or more equations, it is enviable to have a systematic technique of identifying if the system is consistent and to discover all solutions.

Opportunely, in actual physical problems, quantities normally are associated linearly, so this equation is very generally utilized.

Task Solve the following differential equation:

$$x \log x\,.\,\frac{dy}{dx} + y = 2\log x.$$

Notes

Self Assessment

Fill in the blanks:

1. An equation is basically the mathematical manner to portray a .... among two variables.
2. An equation of the form $\frac{dy}{dx} + Py = Q$, in which P & Q are functions of x alone or constant is called a .... equation of the first order.
3. To locate one quantity from the other you .... the first by some number, then add a different number to the outcome.
4. Linear differential equation is commonly known as ....
5. On integrating a linear equation of the first order, we get .... as required general solution.
6. If we are concerning with two or more equations, it is enviable to have a systematic technique of identifying if the system is consistent and to discover all ....
7. In actual physical problems, quantities normally are associated ...., so this equation is very generally utilized.
8. The system is said to be .... if it contains a solution.

11.2 Equations Reducible to the Linear (Bernoulli's Equation)

(i) An equation of the form $\frac{dy}{dx} + Py = Qy^n$ .....(1)

where P and Q are functions of x only or constants known as Bernoulli's equation. It can be made linear.

Dividing both sides of (1) by y^n, we have

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad .....(2)$$

Putting $y^{-n} = z$

So that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$.

or $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\cdot\frac{dz}{dx}$.

Equation (2) becomes

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

or $$\frac{dz}{dx} + (1-n)\,Pz = (1-n)\,Q. \qquad .....(3)$$

which is a linear differential equation with z as the dependent variable.

If $n > 1$, then we have to add the solution y=0 to the solutions found by means of the technique illustrated above.

Notes

Caution If $n = 0$, Bernoulli's equation decreases instantaneously to the standard form first-order linear equation:

$$\frac{dy}{dx} + Py = Q$$

(ii) General equation reducible to linear form is

$$f'(y)\,\frac{dy}{dx} + Pf(y) = Q \qquad \text{.....(4)}$$

where P and Q are functions of x only or constants.

Putting $f(y) = z$, so that $f'(y)\,.\,\frac{dy}{dx} = \frac{dz}{dx}$

Equation (4) becomes

$$\frac{dz}{dx} + Pz = Q$$

which is linear.

Example:

Solve $\frac{dy}{dx} + \frac{x}{1-x^2}\,y = x\sqrt{y}$ .....(1)

Solution:

Dividing by $\sqrt{y}$, the equation (1) becomes

$$y^{-1/2}\frac{dy}{dx} + \frac{x}{1-x^2}\,y^{1/2} = x \qquad \text{.....(2)}$$

Putting $y^{1/2} = z$

So that $\frac{1}{2}\,y^{-1/2}\,\frac{dy}{dx} + \frac{dz}{dx}$

or $y^{-1/2}\,\frac{dy}{dx} = 2\,\frac{dz}{dx}$.

Equation (2) becomes

$$2\,\frac{dz}{dx} + \frac{x}{1-x^2}\;z = x$$

or $\frac{dz}{dx} + \frac{x}{2(1-x^2)}\,z = \frac{x}{2}$, which is linear in z.

Notes

$$I.F. = e^{\int \frac{x}{2(1-x^2)} dx} = e^{-\frac{1}{4} \log (1-x^2)} = (1-x^2)^{-\frac{1}{4}}.$$

$\therefore$ The solution is

$$z \cdot (I.F.) = \int \frac{x}{2} \cdot (I.F.)\, dx + c$$

or $$z \cdot (1-x^2)^{-\frac{1}{4}} = \int \frac{x}{2} (1-x^2)^{-\frac{1}{4}} dx + c$$

$$= -\frac{1}{4} \int (1-x^2)^{-\frac{1}{4}} \cdot (-2x)\, dx + c$$

$$= -\frac{1}{4} \frac{(1-x^2)^{\frac{3}{4}}}{\left(\frac{3}{4}\right)} + c$$

or $$z = -\frac{1}{3} (1-x^2) + c\, (1-x^2)^{\frac{1}{4}}$$

or $$\sqrt{y} = -\frac{1}{3} (1-x^2) + c\, (1-x^2)^{\frac{1}{4}}$$

Which is the required solution.

Example:

Solve $xy\,(1+xy^2) \frac{dy}{dx} = 1.$

Solution:

The given equation can be written as

$$\frac{dx}{dy} - y = x^2 y^3 .$$

Dividing by x^2, we have

$$x^{-2} \frac{dx}{dy} - y\, x^{-1} = y^3. \qquad \text{.....(1)}$$

Putting $x^{-1} = z$

$$-x^{-2} \frac{dx}{dy} = \frac{dz}{dy}$$

or $$x^{-2} \frac{dx}{dy} = -\frac{dz}{dy}.$$

Equation (1) becomes **Notes**

$$-\frac{dz}{dy} - yz = y^3$$

or $$\frac{dz}{dy} + yz = -y^3$$ which is linear in z.

$I.F. = e^{\int y\, dy} = e^{y^2/2}.$

$\therefore$ The solution is

$$Z.e^{y^2/2} = \int -y^3.e^{y^2/2} dy + c$$

$$= -\int y^2.e^{y^2/2}.y\ dy + c$$

$$= -\int 2t.e^t\, dt + c, \left(t = \frac{y^2}{2}\right)$$

$$= -\int 2\left[t.e^t - e^t\right] + c$$

$$Z e^{y^2/2} = -2 e^{y^2/2}\left[\frac{y^2}{2} - 1\right] + c$$

$$\therefore\ \frac{1}{x} = -2\left(\frac{y^2}{2} - 1\right) + ce^{-y^2/2}$$

which is the required solution.

Example:

Solve $\frac{dy}{dx} + x\,sin^2 y = x^3\ cos^2 y$.

Solution:

Dividing by $\cos^2 y$, we have

$$\sec^2 y.\frac{dy}{dx} + \frac{2x \sin y \cos y}{\cos^2 y} = x^3$$

or $$\sec^2 y\,\frac{dy}{dx} + 2\tan y.x = x^3 . \quad .....(1)$$

Putting $\tan y = z$

$$\therefore\quad sec^2 y\,\frac{dy}{dx} = \frac{dz}{dx}.$$

Notes

$\therefore$ (1) becomes

$$\frac{dz}{dx} + 2z. = x^3 \text{, which is linear in z.}$$

$$I.F. = 2^{\int 2x\,dx} = e^{x^2}.$$

$\therefore$ The solution is

$$ze^{x^2} = \int x^3 . e^{x^2} dx + c$$

$$= \int x^2 . e^{x^2} . x dx + c$$

$$= \frac{1}{2} \int te^t dt + c \qquad (t = x^2)$$

$$= \frac{1}{2} e^t \left(t - 1\right) + c$$

$$ze^{x^2} = \frac{1}{2} e^{x^2} \left(x^2 - 1\right) + c$$

or $$z = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

$\therefore$ $$\tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

which is the required solution.

Example:

Solve $\frac{dy}{dx} + \frac{y}{x} = y^2$.

Solution:

Dividing throughout by y^2, we have

$$\frac{1}{y^2} \neq \frac{dy}{dx} + \frac{1}{x} . \frac{1}{y} = 1 \qquad \text{.....(1)}$$

Putting $\frac{1}{y} = z$

$\therefore$ $$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}.$$

$\therefore$ Equation (1) becomes

$$\frac{dz}{dx} - \frac{1}{x} z = -1 \qquad \text{.....(2)}$$

which is linear in z.

Notes

$\therefore \qquad I.F. = e^{\int -\frac{1}{x}\,dx} = e^{-\log x} = \frac{1}{x}.$

$\therefore$ Solution of (2) is

$$z.\frac{1}{x} = \int -1\,.\,\frac{1}{x}dx + c$$

or $\qquad \frac{z}{x} = -\log x + c$

or $\qquad \frac{1}{xy} = c - \log x,$

which is the required solution.

Example:

Solve $x\frac{dy}{dx} + y = xy^3$. .....(1)

Solution:

Dividing throughout by y^3, we have

$$xy^{-3}\frac{dy}{dx} + y^{-2} = x$$

or $\qquad y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 1.$ .....(2)

Putting $y^{-2} = z$

$\therefore \qquad -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

or $\qquad y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}.$

$\therefore$ Equation (2) becomes

$$-\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$$

or $\qquad \frac{dz}{dx} - \frac{2}{x}z = -2.$ .....(3)

which is linear in z.

$\therefore$ $I.F. = e^{-\int \frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}.$

Notes

$\therefore$ Solution of (3) is

$$z.\frac{1}{x^2} = \int -2.\frac{1}{x^2}\,dx + c.$$

or $$\frac{z}{x^2} = \frac{2}{x} + c$$

or $$(2 + cx)\,xy^2 = 1,$$

which is the required solution.

Did u know? The replacements were victorious in converting the Bernoulli equation into a linear equation.

Task Solve the following differential equation:

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y.$$

Self Assessment

Fill in the blanks:

9. An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only or constants known as ....
10. If $n > 1$ in the Bernoulli's equation, then we have to .... the solution y=0 to the solutions.
11. If $n = 0$, Bernoulli's equation .... instantaneously to the standard form first-order linear equation: $\frac{dy}{dx} + Py = Q$.
12. General equation reducible to linear form is ....
13. In the general equation, P and Q are functions of x only or ....
14. Putting $f(y) = z$ in the general equation $f'(y)\frac{dy}{dx} + Pf(y) = Q$, implies ....

11.3 Summary

- The easiest type of relationship that two variables can comprise is a linear relationship.
- An equation of the form $\frac{dy}{dx} + Py = Q$ in which P & Q are functions of x alone or constant is called a linear equation of the first order.

Notes

- To locate one quantity from the other you multiply the first by some number, then add a different number to the outcome.
- On integration, we get $ye^{\int P dx} = \int Q e^{\int P dx}\, dx + c,$ as the required general solution.
- Linear differential equation is commonly known as Leibnitz's linear equation.
- The system is said to be consistent if it contains a solution. Or else the system is said to be inconsistent.
- When concerning with two or more equations, it is fortunate to have a methodical technique of determining if the system is consistent and to discover all solutions.
- An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only or constants is known as Bernoulli's equation which can be made linear.

11.4 Keywords

Bernoulli's Equation: An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only or constants is known as Bernoulli's equation.

Linear Equation: An equation of the form $\frac{dy}{dx} + Py = Q$, in which P & Q are functions of x alone or constant is called a linear equation of the first order.

11.5 Review Questions

Solve the following differential equations:

1. $(x + 2y^3)\,\dfrac{dy}{dx} = y$

2. $\dfrac{d}{dx}\left(\dfrac{dy}{dx} + \dfrac{2y}{x}\right) = 0$

3. $\dfrac{dy}{dx} + (1 - x^2)^{(-3/2)}\, y = \dfrac{x + \sqrt{1 - x^2}}{(1 - x^2)^2}$

4. $\dfrac{dy}{dx} + y = e^{e^x}$

5. $\cosh x \,\dfrac{dy}{dx} + y \sinh^2 x = 2\cosh x \sinh x$

6. $\dfrac{dy}{dx} - \left(\dfrac{2}{x}\right) y = x + \dfrac{1}{x} \sin\left(\dfrac{1}{x^2}\right)$

Notes

7. $\frac{dy}{dx} = \frac{x - y\cos x}{1 + \sin x}$

8. For $x > 0$, $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$ given that $y = 1$ when $x = \pi$

9. $2xy = y^{\tilde{2}}\, 2x^3$ where $y(1) = 2$

10. $\cos x \frac{dy}{dx} - 4y \sin x = 4\sqrt{y} \sec x$

11. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

12. $\frac{dy}{dx} = 1 - x(y - x) - x^3 (y - x)^3$

13. $2y \cos y^2 \frac{dy}{dx} - \frac{2}{x+1} \sin y^2 = (x+1)^3$

14. $e^y \left(\frac{dy}{dx} + 1 \right) = e^x$

15. $\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$

Answers: Self Assessment

1. relationship
2. linear
3. multiply
4. Leibnitz's linear equation
5. $ye^{\int P dx} = \int Q e^{\int P dx} dx + c,$
6. solutions
7. linearly
8. consistent
9. Bernoulli's equation
10. add
11. decreases
12. $f'(y) \frac{dy}{dx} + P f(y) = Q$
13. constants
14. $f'(y) \cdot \frac{dy}{dx} = \frac{dz}{dx}$

11.6 Further Readings

Notes

Books

D.A. Murray, *Introductory Course in Differential Equations*, Orient Blackswan.

Dr. Sanat Kumar Adhikari, Sanat Adhikari, *Basics of Professional Mathematics*, Firewall Media.

Zafar Ahsan, *Differential Equations and their Applications*, PHI Learning Pvt. Ltd.

Online link

www.purplemath.com/modules/solvelin.htm

Notes

Unit 12: Permutation

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of Permutation
- Understand the circular permutation
- Discuss the factorial

Introduction

Permutation is the method of arrangement of things. The expression *arrangement* is used, if the order of things *is measured*. A permutation is defined as an arrangement of a group of objects in a specific order. In this unit, you will understand various concepts of permutation such as factorials, circular permutation, etc.

12.1 Multiplication and Addition Principles

Let us start with considering the following situation: Suppose a shop sells six styles of pants. Each style is available in 8 lengths, six waist sizes, and four colours. How many different kinds of pants does the shop need to stock?

There are 6 possible types of pants; then for each type, there are 8 possible length sizes; for each of these, there are 6 possible waist sizes; and each of these is available in 4 different colours. So, if you sit down to count all the possibilities, you will find a huge number, and may even miss some out! However, if you apply the multiplication principle, you will have the answer.

So, what is the multiplication principle? There are various ways of explaining this principle. One way is the following:

Suppose that a task/procedure consists of a sequence of subtasks or steps, say, Subtask 1, Subtask 2,…, Subtask k. Furthermore, suppose that Subtask 1 can be performed in n_1, ways, Subtask 2 can be performed in n_2 ways after Subtask 1 has been performed, Subtask 3 can be performed in n_3

ways after Subtask 1 and Subtask 2 have been performed, and so on. Then the multiplication principle says that the number of ways in which the whole task can be performed is n_1, n_2 …. n_k.

Let us consider this principle in the context of boxes and objects filling them. Suppose there are m boxes. Suppose the first box can be filled up in k(1) ways. For every way of filling the first box, suppose there are k(2) ways of filling the second box. Then the two boxes can be filled up in k(1), k(2) ways. In general, if for every way of filling the first (r “ 1) boxes, the rth box can be filled up in k(r) ways, for r = 2, 3, … m, then the total number of ways of filling all the boxes is k(1), k(2)… k(m).

So let us see how the multiplication principle can be applied to the situation above (the shop selling pants). Here k(1) = 6, k(2) = 8, k(3) = 6 and k(4) = 4. So, the different kinds of pants are 6 × 8 × 6 × 4 = 1152 in number.

Self Assessment

Fill in the blank:

1. .... is the method of arrangement of things.
2. A permutation is defined as an .... of a group of objects in a specific order.

12.2 Factorial

Factorials are just products, specified by an exclamation mark. For example, "four factorial" is stated as "4!" and means 1 × 2 × 3 × 4 = 24. Generally, $n!$ means the product of all the whole numbers from 1 to n; that is, $n! = 1 \times 2 \times 3 \times ... \times n$.

Factorial identifies the number of different ORDERS in which one can assemble or place set of items

$n! = n \times (n-1) \times (n-2) \times (n-3)...3 \times 2 \times 1$

A factorial symbolizes the multiplication of consecutive natural numbers.

Did u know? For many reasons, 0! is defined to be equal to 1, not 0. Remember 0! = 1.

Example: Evaluate 6!.

1 × 2 × 3 × 4 × 5 × 6 = 720

Simplify 12!

Notes

12! = 1 × 2 × 3 × 4 ×........ × 12

12! = 479001600

When you begin performing combinations, permutations, and probability, you'll be evaluating expressions that have factorials in the numerators and the denominators.

Example: Simplify the following:

$$\frac{6!}{4!}$$

From the definition of a factorial:

$$\frac{6!}{4!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \cdot 5 \cdot 6}{\cancel{1 \cdot 2 \cdot 3 \cdot 4}} = 5 \cdot 6 = 30$$

Thus 6! ÷ 4! = 30

Simplify the following:

$$\frac{17!}{14!3!}$$

At once, you can cancel off the factors 1via 14 that will be common to both 17! and 14!. Then you can simplify what's left to obtain:

$$\frac{17!}{14!3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 14 \cdot 15 \cdot 16 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots 14 \cdot 1 \cdot 2 \cdot 3} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 14} \cdot 15 \cdot 16 \cdot 17}{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 14} \cdot 1 \cdot 2 \cdot 3} = \frac{15 \cdot 16 \cdot 17}{1 \cdot 2 \cdot 3}$$

$$= \frac{\cancel{15} \cdot \cancel{16} \cdot 17}{1 \cdot \cancel{2} \cdot \cancel{3}} = 5 \cdot 8 \cdot 17 = 680$$

Observe how we reduced what we had to write by leaving a gap (the "ellipsis", or triple-period) in the center. This gap-and-cancel process will turn out to be handy later on (such as in calculus, where you'll utilize this technique a lot), especially when you're relating with expressions that your calculator can't manage.

Example: Simplify the following:

$$\frac{(n+2)!}{(n-1)!}$$

To do this, we will write out the factorials, by means of enough of the factors to have stuff that can terminate off. The factors in the product (n + 2)! are of the form:

$$1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times (n) \times (n+1) \times (n+2)$$

Now we have created a list of factors that can cancel out:

$$\frac{(n+2)!}{(n-1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)(n)(n+1)(n+2)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)} =$$

$$= \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}\,(n+1)(n+2)}{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}} = n(n+1)(n+2) = n^3 + 3n^2 + 2n$$

Notes

Notes Make note of the manner we managed that cancellation. We expanded the factorial expressions enough that we could observe where I could cancel off duplicate factors. Although we had no idea what n might be, I could still cancel.

Example: The senior choir has rehearsed five songs for an upcoming assembly. In how many different orders can the choir perform the songs?

Solution:

There are five ways to choose the first song, four ways to choose the second, three ways to choose the third, two ways to choose the fourth, and only one way to choose the final song. Using the fundamental counting principle, the total number of different ways is

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

The choir can sing the five songs in 120 different orders.

Example:

$$\frac{9!}{3!6!}$$

$$\frac{9 \times 8 \times 7 \times 6!}{3!6!}$$

$= 84$

Example:

6! is a factor of 10!. For,

$$\begin{aligned} 10! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \\ &= 6! \cdot 7 \cdot 8 \cdot 9 \cdot 10 \end{aligned}$$

Example:

$$\frac{8!}{5!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 6 \cdot 7 \cdot 8 = 6 \cdot 56 = 336$$

Self Assessment

Fill in the blanks:

3. Factorials are just products, symbolized by an ....
4. Factorial indicates the number of different .... in which one can arrange or place set of items.
5. A factorial symolizes the multiplication of .... natural numbers.
6. n! means the .... of all the whole numbers from 1 to n; that is, $n! = 1 \times 2 \times 3 \times ... \times n$.
7. 0! is defined to be equal to .... , not 0.

Notes

12.3 Permutations

A permutation, also recognized as an "arrangement number" or "order," is a reorganization of the essentials of an ordered list S into a one-to-one correspondence with S itself. The number of permutations on a set of elements is provided by $n!$.

Example: There are $2! = 2 \cdot 1 = 2$ permutations of {1 · 2}, that is {1 · 2} and {2 · 1}, and $3! = 3 \cdot 2 = 6$ permutations of {1 · 2 · 3}, that is {1 · 2 · 3}, {1 · 3 · 2}, {2 · 1 · 3}, {2 · 3 · 1}, {3 · 1 · 2}, and {3 · 2 · 1}. The permutations of a list can be instituted in *Mathematica* by means of the command Permutations.

Caution A list of length can be verified to observe if it is a permutation of 1, ..., n by means of the command Permutation Q [*list*] in the *Mathematica* package Combinatorica' .

The number of methods of attaining an *ordered* subset of elements from a set of elements is specified by

$$nP_k \equiv \frac{n!}{(n-k)!} \tag{1}$$

where $n!$ is a factorial.

Example: There are $4!/2! = 12$ 2-subsets of {1 · 2 · 3 · 4}, that is {1 · 2}, {1 · 3}, {1 · 4}, {2 · 1}, {2 · 3}, {2 · 4}, {3 · 1}, {3 · 2}, {3 · 4}, {4 · 1}, {4 · 2}, and {4 · 3}. The *unordered* subsets comprising elements are recognized as the k-subsets of a specified set.

A demonstration of a permutation as a product of permutation cycles is exclusive.

Example: An example of a cyclic disintegration is the permutation {4 · 2 · 1 · 3} of {1 · 2 · 3 · 4}. This is indicated as (2)(143), analogous to the disjoint permutation cycles (2) and (143).

There is a huge concern of independence in choosing the illustration of a cyclic decomposition because

1. The cycles are disjoint and can so be mentioned in any order, and
2. Any rotation of a specified cycle mentions the similar cycle.

As a result, (431)(2), (314)(2), (143)(2), (2)(431), (2)(314), and (2)(143) all portray the same permutation.

A different notation that clearly recognizes the positions engaged by elements before and after application of a permutation on elements utilizes a matrix, where the first row is and the second row is the new arrangement.

Example: The permutation which toggles elements 1 and 2 and fixes 3 would be represented as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}. \qquad \text{...(2)}$$

Any permutation is also considered as a product of transpositions. Permutations are normally indicated in lexicographic or transposition order.

Notes

Did u know? There is an association between a permutation and a pair of Young tableaux called as the Scented correspondence.

A permutation of ordered objects in which no object is in its ordinary place is known as derangement (or at times, a complete permutation) and the number of such permutations is specified by the sub factorial ! n.

By means of

$$(x+y)^n \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \tag{3}$$

with $x = y = 1$ provides

$$2^n \sum_{r=0}^{n} \binom{n}{r}, \tag{4}$$

so the number of methods of selecting 0, 1, ..., or n at a time is $2n$.

The set of all permutations of a set of elements 1, ..., n can be attained by the following recursive procedure

```
   1  2
   /                  (5)
2  1

   1     2  3
         /
   1  3  2
   /
3  1     2
|
3  2     1            (6)
   \
   2  3  1
         \
   2     1  3
```

Consider permutations in which no pair of successive elements (i.e., mounting or falling successions) take place. For n = 1, 2, ... elements, the numbers of such permutations are 1, 0, 0, 2, 14, 90, 646, 5242, 47622, ...

Consider the set of integers 1, 2, ..., N be permuted and the consequent sequence be separated into rising runs. Signify the regular length of the nth run as N approaches infinity, L_n. The first few values are summed up in the following table, where e is the base of the natural logarithm.

n	L_n	approximate
1	$e - 1$	1.7182818...
2	$e^2 - 2e$	1.9524...
3	$e^2 - 3e^2 + \frac{3}{2}e$	1.9957...

Now let us define permutation.

Notes

Definition: A permutation of n objects taken r at a time is a selection of r objects from a total of n objects ($r \leq n$), where order matters.

Example: $r = n$: Suppose you have 6 (distinguishable) people to seat in 6 chairs. The first person may choose her/his chair from 6 possibilities. The second person must choose from the 5 remaining chairs, etc., until the 6[th] person has to take the only chair that's left. This arrangement can be done in 6×5×4×3×2×1, or 6! ("six factorial") ways. Although this problem is somewhat ambiguous, assume that each of the 14 objects, not each type of object, is a different color and therefore distinguishable from the others.)

Example: $r < n$: Now suppose that you have 6 (distinguishable) people but only 4 chairs; so only 4 of the 6 people will be able to sit down. How many arrangements of the 6 people can be seated in the 4 chairs? This arrangement can be done in $6 \times 5 \times 4 \times 3$ ways.

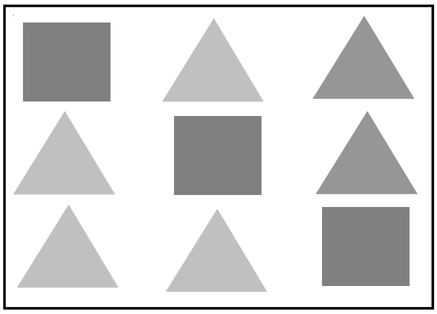
Formulas: The general formula for a permutation of n objects arranged r at a time is $P(n, r) = n!/(n - r)!$

Note that in Ex. P1, $r = n$, so $n!/(n - r)! = n!/(n - n)! = n!/0!$ Since 0! is defined to be 1, we have $n!/1 = n!$. Therefore, the answer to P1 is simply 6!.

Applying the formula to Ex. P2 yields $6!/(6 - 4)! = 6!/2! = (6 \times 5 \times 4 \times 3 \times 2 \times 1)/(2 \times 1) = 6 \times 5 \times 4 \times 3$.

Thus, the general formula applies in both cases, whether $r = n$ or $r < n$.

Example: Suppose that you have 3 objects, but they are not all distinguishable. For example, suppose that you have 1 red square and 2 green triangles. How many different arrangements of those 3 objects are possible?



If the two green triangles switched places with each other, would the arrangements be distinguishable from the ones given above? No, they would not. Therefore, the 2! ways the green triangles could be rearranged must be eliminated. So if we have 3 objects, where 2 objects are the same, then the number of possible arrangements is $P(3, 3) = 3!/2! = (3 \times 2 \times 1)/(2 \times 1) = 3$ (not 3!).

Here objects of the same type are the same color and therefore not distinguishable from each other.

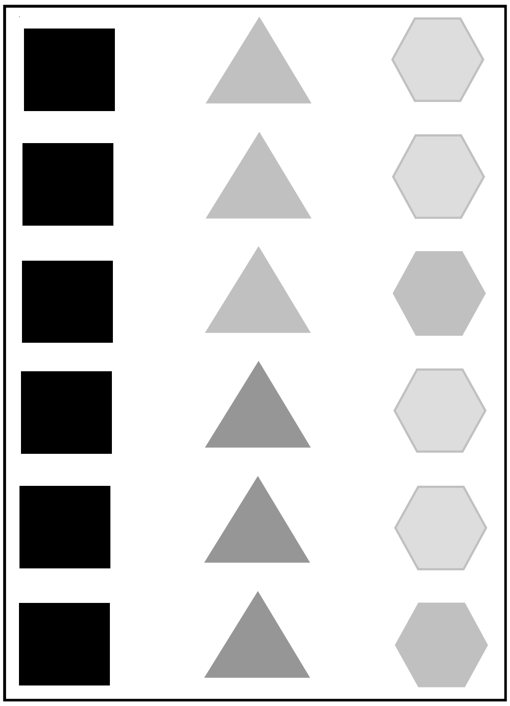
Example: Suppose that you have 6 objects—1 red square, 2 green triangles, and 3 yellow hexagons. How many different arrangements of the 6 objects are possible? In this case, you have 2 indistinguishable green triangles and 3 indistinguishable yellow hexagons. Note that you have 2! ways to interchange the 2 triangles and 3! ways to interchange the 3 hexagons. So you must take those out of the total possible permutations.

Thus,

Notes

$P(6, 6) = 6!/2!3! = (6×5×4×3×2×1)/(2×1)(3×2×1) = (6×5×4)/2 = 60$ (not 6!).

Example: Suppose you have the same set of 6 objects given in example P4 above, but this time each triangle is a different shade of green and each hexagon is a different shade of yellow



so that they are distinguishable. This time you want to select exactly one object of each type. (Assume the different types are in 3 separate bags.) How many different ways can this be done if the order in which the objects are selected doesn't matter? Here are the possibilities (i.e., the sample space):

There is 1 way to get a red square, 2 ways to get a green triangle, and 3 ways to get a yellow hexagon; so the total number of arrangements (since the order of selection doesn't matter) is $1 \cdot 2 \cdot 3 = 6$.

Example: Suppose that you have the same set of objects given in Ex. 5, but this time the order of selection makes a difference. How many ways can we order the selection?

We can choose square (S) then triangle (T) then hexagon (H), or we can choose S then H then T, or T then H then S, or T then S then H, or H-T-S or H-S-T.

Note that we have 3 choices for the first selection, 2 for the second, and 1 for the third. So the number of different ways of selecting the objects is $3 \cdot 2 \cdot 1$ or 3!. We know from example 5 that if we choose S first, there are 6 possible arrangements of the distinguishable objects when order of selection doesn't matter. If we multiply that by the number of ways the objects can be ordered, then we have $6 \cdot 3! = 36$ different arrangements of the objects when order of selection matters.

Example: Now suppose that you have 3 squares, 4 triangles, and 7 hexagons, and each of the 14 objects is a different color. The arrangement of n distinguishable objects is n!. Suppose also that the squares must be grouped together, the triangles must be grouped together, and the

Notes

hexagons must be grouped together. In how many different ways can each type of object be arranged?

The group of 3 distinguishable squares can be arranged in 3! ways, the 4 distinguishable triangles in 4! ways, and the 7 distinguishable hexagons in 7! ways. Given that we have 3 groups of distinguishable objects, we can arrange those groups in 3! ways. Thus, the number of distinguishable arrangements of the objects when they must be grouped by type is

$3! \cdot 4! \cdot 7! \cdot 3! = 4,354,560$

> *Task* A coach must select five starters from a team of 12 players. How many different ways can the coach choose the starters?

Self Assessment

Fill in the blanks:

8. A ...., also recognized as an "arrangement number" or "order," is a reorganization of the essentials of an ordered list into a one-to-one correspondence with itself.

9. A permutation of ordered objects in which no object is in its ordinary place is known as ....

10. The permutations of a list can be instituted in Mathematica by means of the command ....

11. Permutations are normally indicated in .... order.

12. A demonstration of a permutation as a product of permutation cycles is ....

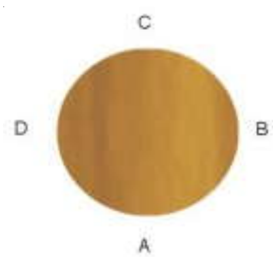
12.4 Circular Permutation

Circular-permutations consists of two phases:

(a) If clockwise and anti-clock-wise orders are dissimilar, then total number of circular-permutations is specified by (n–1)!

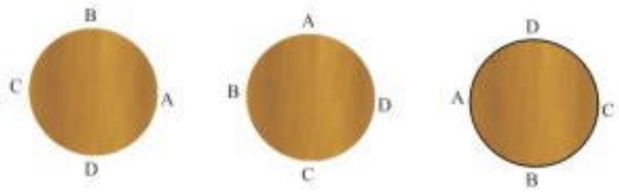
(b) If clock-wise and anti-clock-wise orders are considered as not dissimilar, then total number of circular-permutations is specified by (n–1)!/2!

Proof (a):



(a) Le us suppose that 4 persons A,B,C, and D are sitting about a round table

Moving A, B, C, D, one place in anticlock-wise direction, we obtain the following agreements:



Therefore, we use that if 4 persons are sitting at a round table, then they can be moved four times, but these four arrangements will be the similar, since the sequence of A, B, C, D, is similar. But if A, B, C, D, are sitting in a row, and they are moved, then the four linear-arrangement will be diverse.

Thus if we have '4' things, then for every circular-arrangement number of linear-arrangements =4

Likewise, if we have 'n' things, then for every circular – agreement, number of linear – arrangement = n.

Let the total circular arrangement $= p$

Total number of linear–arrangements = $n.p$

Total number of linear–arrangements = n. (number of circular-arrangements)

Or Number of circular-arrangements = 1 (number of linear arrangements)

$n = 1(n!)/n$

circular permutation = $(n–1)!$

Proof (b): When clock-wise and anti-clock wise arrangements are not dissimilar, then inspection can be completed from both sides, and this will be identical. Here two permutations will be considered as one. So total permutations will be half, thus in this case:

Circular–permutations = $(n–1)!/2$

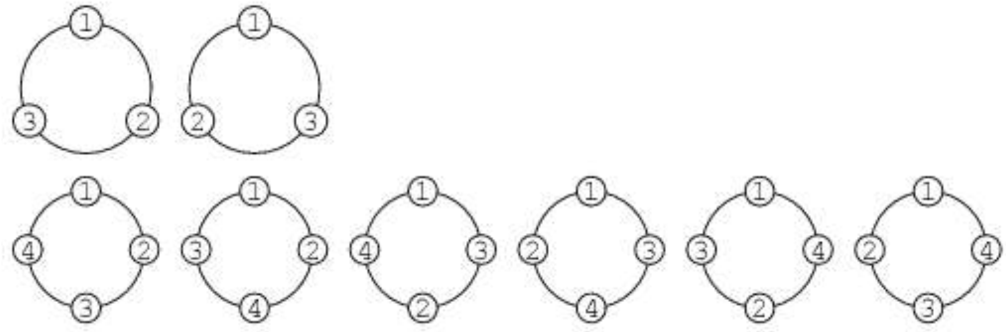
Notes Number of circular-permutations of 'n' dissimilar things taken 'r' at a time:

1. If clock-wise and anti-clockwise orders are considered as dissimilar, then total number of circular-permutations $= {}^nP_r/r$
2. If clock-wise and anti-clockwise orders are considered as not dissimilar, then total number of circular – permutation = ${}^nP_r/2r$

Therefore, we can articulate that the number of methods to assemble distinct objects along a permanent (i.e., cannot be chosen up out of the plane and twisted over) circle is $P_n = (n - 1)!$

The number is rather than the usual factorial because all cyclic permutations of objects are corresponding since the circle can be rotated.

Notes



Example: Of the 3! = 6 permutations of three objects, the (3 – 1)! = 2 different circular permutations are (1, 3, 2) and . Likewise, of the 4! = 24 permutations of four objects, the different circular permutations are (1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 2, 1), (1, 3, 2, 4), (1, 4, 2, 1), and (1, 4, 3, 2). Out of these, there are only three liberated permutations (i.e., inequivalent when flipping the circle is permitted): (1, 2, 3, 4), (1, 2, 4, 3), and (1, 3, 2, 4). The number of free circular permutations of order n is P'_n = 1 for n = 1, 2, and $P'_n = \frac{1}{2}(n-1)!$ for $n \geq 3$, offering the sequence 1, 1, 1, 3, 12, 60, 360, 2520, ...

Example: How many necklace of 12 beads each can be prepared from 18 beads of dissimilar colours?

Solution: Here clockwise and anti-clockwise arrangement s are identical.

Therefore total number of circular–permutations: ${}^{18}P_{12}/2x12$

= 18!/(6 x 24)

> *Task* Five men and three women are to be seated around a round table. Find the number of ways to seat the eight people specified that the women must be separated from one another.

Self Assessment

Fill in the blanks:

13. If clock-wise and anti-clock-wise orders are dissimilar, then total number of circular-permutations is specified by ....
14. If clock-wise and anti-clock-wise orders are considered as not dissimilar, then total number of circular-permutations is specified by ....
15. When clock-wise and anti-clock-wise arrangements are not dissimilar, then inspection can be completed from both sides, and this will be ....

12.5 Summary

- A permutation is defined as an arrangement of a group of objects in a specific order.
- Factorial determines the number of different ORDERS in which one can arrange or place set of items such as n!= $n \times (n-1) \times (n-2) \times (n-3)...3 \times 2 \times 1$.

- A permutation, also recognized as an "arrangement number" or "order," is a reorganization of the essentials of an ordered list S into a one-to-one correspondence with S itself.
- A permutation of ordered objects in which no object is in its ordinary place is known as derangement (or at times, a complete permutation) and the number of such permutations is specified by the sub factorial ! n.
- A permutation of n objects taken r at a time is a selection of r objects from a total of n objects ($r \leq n$), where order matters.
- The permutations of a list can be instituted in *Mathematica* by means of the command Permutations.
- If clockwise and anti clock-wise orders are dissimilar, then total number of circular permutations is specified by $(n - 1)!$
- ?If clock-wise and anti-clock-wise orders are considered as not dissimilar, then total number of circular-permutations is specified by (n – 1)!/2!

12.6 Keywords

Factorial: It determines the number of different ORDERS in which one can arrange or place set of items such as $n!$= n x (n–1) x (n–2) x (n–3)...3 x 2 x 1.

Permutation: A permutation of n objects taken r at a time is a selection of r objects from a total of n objects ($r \leq n$), where order matters.

12.7 Review Questions

1. In a race with 10 horses, the first, second, and third place finishers are noted. How many outcomes are there?
2. Eight persons, consisting of four married couples, are to be seated in a row of eight chairs. How many seating arrangements are there if:
   (a) There are no other restrictions
   (b) The men must sit together and the women must sit together
   (c) The men must sit together
   (d) The spouses in each married couple must sit together
3. Suppose that n people are to be seated at a round table. Show that there are $(n - 1)!$ distinct seating arrangements.
4. How many different ways can 3 red, 4 yellow and 2 blue bulbs be arranged in a string of Christmas tree lights with 9 sockets?
5. How many numbers greater than 1000 can be formed with the digits 3, 4, 6, 8, 9 if a digit cannot occur more than once in a number?
6. Expand the factorial (n + 2)! / n!
7. Evaluate (n - 1)! / (n + 1)!
8. How many permutations of 3 different digits are there, chosen from the ten digits 0 to 9 inclusive?
9. In how many ways can a committee of 5 be chosen from 10 people?

10. Given n objects of k types (where the objects within each type are indistinguishable), ri of the ith type, there are $n!r1!r2!\ldots rk!$ permutations. How many circular permutations are there of such a set?

Answers: Self Assessment

1.	Permutation	2.	arrangement
3.	exclamation mark	4.	ORDERS
5.	consecutive	6.	product
7.	1	8.	permutation
9.	derangement	10.	Permutations.
11.	lexicographic or transposition	12.	exclusive
13.	(n–1)!	14.	(n–1)!/2!
15.	identical		

12.8 Further Readings

Books

Giri & Banerjee, *Introduction to Business Mathematics*, Academic Publishers.

Jacobus Hendricks van Lint, *A Course in Combinatorics*, Cambridge University Press.

Marshall Hall, *Combinatorial Theory*, John Wiley & Sons.

Online link

http://www.wtamu.edu

Unit 13: Combinations

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of combinations
- Discuss the restricted combinations

Introduction

An arrangement of r objects, without considering order and without replication, chosen from n different objects is known as a combination of n objects taken r at a time. Combination signifies *selection* of things. The word *selection* is used when the order of things has *no significance*. In this unit, you will understand the concept of combinations and restricted combinations.

13.1 Combinations

Definition: A combination of n objects taken r at a time is a selection of r objects from a total of *n* objects ($r \leq n$), where order does not matter. The notation is usually

$$\left(\binom{n}{r}\right)$$

and read, "*n* choose *r*."

The number of such combinations is indicated by

$$_nC_r = \frac{n}{(n-r)!r!}$$

Notes

Example: Now presume that we have to create a team of 11 players out of 20 players, This is an instance of *combination*, since the order of players in the team will not consequence in a transformation in the team. Regardless of in which order we list out the players the team will continue to be the same! For a different team to be created at least one player will have to be tainted.

Notes The dissimilarity among combinations and permutations is in combinations you are counting groups and in permutations you are counting different methods to assemble items with respect to order.

The *n* and the *r* signify the similar thing in both the permutation and combinations, but the formula varies.

Caution The combination has an additional *r*! in its denominator.

Example: ***C1 — r = n:*** How many ways can a committee of 7 people be drawn from a group of 7 people?

C(7, 7) = 7!/[(7 – 7)!'"7!] = 7!/[0!'"7!] = 7!/7! = 1.

In other words, there is exactly 1 way to choose all 7 people to be on the committee since order doesn't matter.

Example: ***C2 — r < n:*** How many ways can 2 co-chairpersons be chosen from a committee of 7 people?

C(7, 2) = 7!/[(7 – 2)!×2!] = 7!/[5!×2!] = 7×6×5×4×3×2×1/(5×4×3×2×1)(2×1) = 7×6/2×1 = 7×3 = 21.

In other words, there are 21 ways to select a pair of co-chairs from a committee of 7 people.

Relating the above scenario to permutations, note that there are 7 ways to choose the first co-chair and 6 ways to choose the second co-chair. So if order mattered, we would have 42 possible choices. However, the combination of 2 co-chairs is the same, no matter which person was selected first. Thus, we must divide by the by the 2! ways the two chair persons could have been selected. Applying the actual formula, we have

P(7, 2) = 7!/(7 – 2)! = 7!/5! = 7×6 = 42.

But

C(7, 2) = 7!/[(7 – 2)!×2!] = 7!/[5!×2!] = (7×6)/(2×1) = 42/2 = 21.

Example: ***C3:*** In a club with 8 males and 11 female members, how many 5-member committees can be chosen that have 4 females?

Solution: Since order doesn't matter, we will be using our combinations formula.

How many ways can the 4 females be chosen from the 11 females in the club?

C(11, 4) = 11!/[(11 – 4)! ×4!] = 11!/(7! ×4!) = (11×10×9×8)/ (4×3×2×1) = 330 ways.

But this gives us only the 4 female members for our 5-member committee. For a 5-member committee with 4 females, how many males must be chosen? Only 1. How many ways can the 1 male be chosen from the 8 males in the club?

C(8, 1) = 8!/[(8 – 1)! × 1!] = 8!/(7! × 1!) = 8 ways.

Notes

Thus we have $8 \times 330 = 2640$ ways to select a 5-member committee with exactly 4 female members from this club.

At least 4 females?

Solution: We found that there are 2640 ways to select exactly 4 females for our 5-member committee. If we need at least 4 females, we could have 4 females or 5 females, couldn't we? How many ways can we select 5 females from the 11 females in the club, given that order doesn't matter?

$C(11, 5) = 11!/[(11 - 5)! \times 5!] = 11!/(6! \times 5!) = (11 \times 10 \times 9 \times 8 \times 7)/ (5 \times 4 \times 3 \times 2 \times 1) = 462$ ways.

So to have at least 4 females, we could have $C(11, 4) \times C(8, 1)$ or $C(11, 5)$. Since there are 330 ways to achieve the first and 462 ways to achieve the second, we have

$330 + 462 = 792$ ways to select a 5-member committee with at least 4 female members from the club.

No more than 2 males?

Solution: "No more than 2" males means we could have 1 male or 2 males on our 5-member committee. Since we have the condition of the 5-member committee, we have the following possibilities: (1) 1 male and 4 females, or (2) 2 males and 3 females. Since order doesn't matter, we have $C(8, 1) \cdot C(11, 4) \cup C(8, 2) \cdot C(11, 3)$ ways to select the committee. Applying the combinations formulas, we have $\{8!/[(8 - 1)! \cdot 1!]\} \cdot \{11!/[(11 - 4)! \cdot 4!]\} + \{8!/[(8 - 2)! \cdot 2!]\} \times \{11!/[(11 - 3)! \times 3!]\} = (8)(330) + (28)(165) = 2540 + 4620 = 7260$ ways to select a 5-member committee with no more than 2 male members from the club.

Example: Suppose we have the same club with 8 males and 11 females, but this time we want to know how many committees could be chosen with at least 1 but no more than 3 members, regardless of gender. Therefore, committees must have 1, 2, or 3 members. What are the possible committee memberships?

Solution: The possibilities are 1M or 1F or 2M or 2F or 1M & 1F or 1M & 2F or 2M & 1F or 3M or 3F. Does order matter? No, it does not. So we use our combinations formulas again for each possible committee make-up and add them to get the solution to our problem.

$C(8, 1) + C(11, 1) + C(8, 2) + C(11, 2) + C(8, 1) \cdot C(11, 1) + C(8, 1) \cdot C(11, 2) + C(8, 2) \cdot C(11, 1) + C(8, 3) + C(11, 3)$.

Example: In a discussion of 9 schools, how many intraconference football games are played throughout the period if the teams all play each other exactly once?

When the teams plays with each other, order is not an issue, we are counting match ups. For each game there is a collection of two teams playing. So we can utilize combinations to assist us out here.

Observe that if we were putting these teams in any sort of order, then we would require using permutations to explain the problem.

But here, order is not an issue, so we will use combinations.

First we are required to find n and r :

If n is the number of teams we have to select from, what do you think n here?

If you said $n = 9$ you are correct!!! There are 9 teams in the conference.

If r is the number of teams we are using at a time, what do you think r is?

If you said $r = 2$, pat yourself on the back!! 2 teams play per game.

Notes

Let's put those principles into the combination formula and see what we obtain:

$${}_nC_r = \frac{n!}{(n-r)!r} \qquad *n = 9,\ r = 2$$

$${}_9C_2 = \frac{9!}{(9-2)!2!} \qquad \text{* Eval. inside ()}$$

$$= \frac{9!}{7!2!} \qquad \text{* Expand 9! until it gets to 7! which is the larger! in the den.}$$

$$= \frac{9.8.7!}{7!2!} \qquad \text{* Cancel out 7!'s}$$

$$= \frac{9.8}{2.1}$$

$$= 36$$

If you have a factorial key, you can place it in as 9!, divided by 7!, divided by 2! and then press enter or =.

If you don't contain a factorial key, you can shorten it as revealed above and then enter it in. It is most likely best to shorten it first, since in some cases the numbers can get pretty large, and it would be awkward to multiply all those numbers one by one.

This indicated that there are 36 different games in the conference.

Example: You want to draw 4 cards from a typical deck of 52 cards. How many different 4 card hands are possible?

This would be a combination problem, since a hand would be a group of cards without considering order.

Observe that if we were putting these cards in any sort of order, then we are required to use permutations to solve the problem.

But here, order is not an issue, so we will use combinations.

First we are required to find n and r :

If n is the number of cards we have to select from, what do you think n here?

If you said n = 52 you are right!!! There are 52 cards in a deck of cards.

If r is the number of cards we are utilizing at a time, what do you think r is?

If you said r = 4, tap yourself on the back!! We desire 4 card hands.

Let us place those values into the combination formula and see what we obtain:

$${}_nC_r = \frac{n!}{(n-r)!r} \qquad *n = 52,\ r = 4$$

$${}_{12}C_4 = \frac{52!}{(52-4)!4!} \qquad \text{* Eval. inside ()}$$

Notes

$= \dfrac{52!}{48!4!}$ * Expand 52! until it obtains to 48! which is the larger! in the den.

$= \dfrac{52.51.50.49.48!}{48!4!}$ * Cancel out 48!'s

$= \dfrac{52.51.50.49}{4.3.2.1}$

= 270725

If you contain a factorial key, you can place it in as 52!, divided by 48!, divided by 4! and then press enter or =.

If you don't contain a factorial key, you can abridge it as revealed above and then enter it in. It is most likely best to simplify it first, since in some cases the numbers can get pretty large, and it would be awkward to multiply all those numbers one by one.

This signifies there are 270,725 different 4 card hands.

Example: 3 marbles are taken at random from a bag comprising 3 red and 5 white marbles.

Solve the following questions (a – d):

(a) How many different draws are there?

This would be a combination problem, since a draw would be a collection of marbles without considering order. It is like taking handful of marbles and gazing at them.

Observe that there are no particular conditions positioned on the marbles that we draw, so this is a simple combination problem.

If we were putting these marbles in any sort of order, then we would require using permutations to solve the problem.

But here, order is not an issue, so we are going to use combinations.

First we need to find *n* and *r*:

If *n* is the number of marbles we have to select from, what do you think *n* here?

If you said *n* = 8 you are right!!! There are 3 red and 5 white marbles for a total of 8 marbles.

If *r* is the number of marbles we are taking at a time, what do you think *r* is?

If you said *r* = 3, pat yourself on the back!! 3 marbles are taken at a time.

Let us place those values into the combination formula and observe what we obtain:

${}_nC_r = \dfrac{n!}{(n-r)!r!}$ *n = 8, r = 3

${}_8C_3 = \dfrac{8!}{(8-3)!3!}$ *Eval. inside ( )

$\dfrac{8!}{5!3!}$ *Expand 8! until it obtains 5! which is the larger ! in the den.

*Cancel out 5!'s

$\dfrac{8.7.6.5!}{5!3!}$

$= \dfrac{8.7.6}{3.2.1}$

$= 56$

Notes

If you contain a factorial key, you can place it in as 8!, divided by 5!, divided by 3! and then press enter or =.

If you don't have a factorial key, you can make it simpler as revealed above and then enter it in. It is most likely best to shorten it first, since in some cases the numbers can get rather large, and it would be awkward to multiply all those numbers one by one.

This indicates that there are 56 different draws.

(b) How many different draws would contain only red marbles?

This would be a **combination problem**, since a draw would be a group of marbles **without regard to order**. It is like clutching a handful of marbles and gazing at them.

In **part *a* above**, we observed all possible draws. From that list we only desire the ones that include only red.

Let us observe what the draw appears like: we would have to have 3 red marbles to fulfill this condition:

3 RED

First we are required to find n and r :

If n is the number of **RED** marbles we have to select from, what do you think n here?

If you said ***n* = 3** you are right!!! There are a total of 3 red marbles.

If r is the number of **RED** marbles we are drawing at a time, what do you think r is?

If you said ***r* = 3**, tap yourself on the back!! 3 **RED** marbles are drawn at a time.

Let us place those values into the combination formula and observe what we obtain:

$$_nC_r = \frac{n!}{(n-r)!r!} \quad *n = 3,\ r = 3$$

$$_3C_3 = \frac{3!}{(3-3)!3!} \quad \text{*Eval. inside ( )}$$

$$\frac{3!}{0!3!} \quad \text{*Cancel out 3!'s}$$

$$\frac{3!}{(1)3!}$$

$$= 1$$

If you contain a factorial key, you can place it in as 3!, divided by 0!, divided by 3! and then press enter or =.

If you don't contain a factorial key, you can abridge it as exposed above and then enter it in. It is perhaps best to shorten it first, since in some cases the numbers can get somewhat large, and it would be awkward to multiply all those numbers one by one.

This shows that there is only 1 draw out of the 56 established in part a that would enclose 3 RED marbles.

(c) How many different draws would contain 1 red and 2 white marbles?

This would be a **combination problem**, since a draw would be a group of marbles **without regard to order**. It is similar to grabbing a handful of marbles and gazing at them.

In **part *a* above**, we observed all probable draws. From that list we only want the ones that include 1 RED and 2 WHITE marbles.

Let us observe what the draw appears like: we would have to have 1 red and 2 white marbles to fulfill this condition:

1 RED 2 WHITE

First we are required to find *n* and *r*:

Jointly that would make up 1 draw. We are going to have to utilize the counting principle to assist us with this one.

Note how 1 draw is divided into two parts – red and white. We can not unite them together since we require a specific number of each one. So we will comprehend how many ways to get 1 RED and how many ways to get 2 WHITE, and by means of the counting principle, we will multiply these numbers together.

1 RED:

If *n* is the number of **RED** marbles we have to select from, what do you think *n* is in this problem?

If you said ***n* = 3** you are right!!! There are a total of 3 **RED** marbles.

If *r* is the number of **RED** marbles we are drawing at a time, what do you think *r* is?

If you said ***r* = 1**, pat yourself on the back!! **1 RED** marble is drawn at a time.

2 WHITE:

If *n* is the number of **WHITE** marbles we have to choose from, what do you think *n* is in this problem?

If you said ***n* = 5** you are right!!! There are a total of 5 **WHITE** marbles.

If *r* is the number of **WHITE** marbles we are drawing at a time, what do you think *r* is?

If you said ***r* = 2**, tap yourself on the back!! 2 **WHITE** marbles are drawn at a time.

Let us place those values into the combination formula and see what we obtain:

$$_nC_r = \frac{n!}{(n-r)!r!}$$

*RED: $n = 3$, $r = 3$
*WHITE: $n = 5$, $r = 2$

$$_3C_1 \cdot {}_5C_2 = \frac{3!}{(3-1)!1!} \cdot \frac{5!}{(5-2)!2!}$$

*Eval. inside ( )

$$= \frac{3!}{2!1!} \cdot \frac{5}{3!2!}$$

*Expand 3! until it gets to 2!
*Expand 5! until it gets to 3!

$$= \frac{3.2!}{2!1!} \frac{5.4.3!}{3!2!}$$

*Cancel out 2!'s and 3!'s

$$= \frac{3}{1} \cdot \frac{5}{2} \cdot \frac{4}{1}$$

$$= 3.10$$

$$= 30$$

If you contain a factorial key, you can put it in as 3!, times 5!, divided by 2!, divided by 1!, divided by 3!, divided by 2! and then press enter or =.

Notes

If you don't contain a factorial key, you can abridge it as exposed above and then enter it in. It is most likely best to shorten it first, because in some cases the numbers can get somewhat large, and it would be awkward to multiply all those numbers one by one.

This shows there are 30 draws that would contain 1 RED and 2 WHITE marbles.

How many different draws would contain exactly 2 red marbles?

This would be a combination problem, since a draw would be a group of marbles without considering order. It is like taking hold of a handful of marbles and gazing at them.

In part *a* above, we gazed at all possible draws. From that list we only desire the ones that enclose 2 RED and 1 WHITE marbles. Memorize that we require a total of 3 marbles in the draw. As we have to have 2 red, that makes us needing 1 white to complete the draw of 3.

Let's observe what the draw appears like: we would have to have 2 red and 1 white marbles to fulfill this condition:

2 RED 1 WHITE

First we need to find n and r:

Mutually that would make up 1 draw. We will use the counting principle to assist us with this one.

Note how 1 draw is split into two parts - red and white. We can not merge them together since we require a specific number of each one. So we will work out how many ways to get 2 RED and how many ways to get 1 WHITE, and by means of the counting principle, we will multiply these numbers together.

2 RED:

If n is the number of RED marbles we have to select from, what do you think n is in this problem?

If you said n = 3 you are correct!!! There are a total of 3 RED marbles.

If r is the number of RED marbles we are drawing at a time, what do you think r is?

If you said r = 2, pat yourself on the back!! 2 RED marble is drawn at a time.

1 WHITE:

If n is the number of WHITE marbles we have to select from, what do you think n is in this problem?

If you said n = 5 you are right!!! There are a total of 5 WHITE marbles.

If r is the number of WHITE marbles we are drawing at a time, what do you think r is?

If you said r = 1, tap yourself on the back!! 1 WHITE marble are drawn at a time.

Let us place those values into the combination formula and see what we obtain:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

*RED: n = 3, r = 2
*WHITE: n = 5, r = 1

$${}_3C_2 \cdot {}_5C_1 = \frac{3!}{(3-2)!2!} \cdot \frac{5!}{(5-1)!1!}$$

$$= \frac{3!}{2!1!} \cdot \frac{5}{4!1!}$$

*Eval. inside ( )

$$= \frac{3.2!}{2!1!} \frac{5.4!}{4!1!}$$

*Expand 3! until it gets to 2!
*Expand 5! until it gets to 4!

$$= \frac{3}{1} \cdot \frac{5}{2}$$

$$= 15$$

If you contain a factorial key, you can place it in as 3!, times 5!, divided by 1!, divided by 2!, divided by 4!, divided by 1! and then press enter or =.

If you don't contain a factorial key, you can shorten it as exposed above and then enter it in. It is most likely best to abridge it first, because in some cases the numbers can get pretty large, and it would be awkward to multiply all those numbers one by one.

This indicates that there are 15 draws that would include 2 RED and 1 WHITE marbles.

Self Assessment

Fill in the blanks:

1. A .... of n objects taken r at a time is a selection of r objects from a total of n objects ($r \le n$), where order does not matter.
2. Combination signifies .... of things.
3. The *n* and the *r* signify the similar thing in both the permutation and combinations, but the .... varies.
4. The number of combinations is indicated by ....
5. The dissimilarity among combinations and permutations is in combinations you are counting groups and in permutations you are counting different methods to assemble items with respect to ....

13.2 Restricted – Combinations

(a) Number of combinations of '*n*' dissimilar things taken '*r*' at a time, when '*p*' particular things are forever included = ${}^{n-p}C_{r-p}$.

(b) Number of combination of '*n*' dissimilar things, taken '*r*' at a time, when '*p*' particular things are forever to be excluded = ${}^{n-p}C_r$

Example: In how many manners can a cricket-eleven be selected out of 15 players? if

(i) A particular player is always selected,

(ii) A particular is never selected.

Solution:

(i) A particular player is always selected, it signifies that 10 players are chosen out of the remaining 14 players.

=. Required number of ways = ${}^{14}C_{10}$ = ${}^{14}C_4$

= 14!/4!x19! = 1365

(ii) A particular players is never selected, it means that 11 players are chosen out of 14 players.

$\Rightarrow$ Required number of ways = ${}^{14}C_{11}$

= 14!/11!x3! = 364

Notes

Theorem's

Number of ways of choosing zero or more things from 'n' different things is specified by: $2^n–1$

Proof: Number of ways of choosing one thing, out of n-things = ${}^nC_1$

Number of choosing two things, out of n-things =${}^nC_2$

Number of ways of choosing three things, out of n-things = ${}^nC_3$

Number of ways of choosing 'n' things out of 'n' things = ${}^nC_n$

$\Rightarrow$ Total number of manners of choosing one or more things out of n dissimilar things

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + ------ + {}^nC_n$

$= ({}^nC_0 + {}^nC_1 + --------- {}^nC_n) - {}^nC_0$

$= 2^n - 1$ $\quad [\ {}^nC_0=1]$

Example: John has 8 friends. In how many ways can he invite one or more of them to dinner?

Solution: John can choose one or more than one of his 8 friends.
$\Rightarrow$ Required number of ways = $2^8 – 1= 255$.

Theorem:

Number of ways of choosing zero or more things from 'n' identical things is specified by : n+1

Example: In how many ways, can zero or more letters be chosen form the letters AAAAA?

Solution: Number of ways of:

Choosing zero 'A's = 1

Choosing one 'A's = 1

Choosing two 'A's = 1

Choosing three 'A's = 1

Choosing four 'A's = 1

Choosing five 'A's = 1

$\Rightarrow$ Required number of ways = 6 [5+1]

Theorem:

Number of ways of Choosing one or more things from 'p' identical things of one type 'q' identical things of another type, 'r' identical things of the third type and 'n' different things is given by:

$(p+1)\ (q+1)\ (r+1)2^n – 1$

Example: Find the number of different options that can be prepared from 3 apples, 4 bananas and 5 mangoes, if at least one fruit is to be chosen.

Solution:

Number of ways of choosing apples = (3+1) = 4 ways.

Number of ways of choosing bananas = (4+1) = 5 ways.

Number of ways of choosing mangoes = (5+1) = 6 ways.

Total number of ways of choosing fruits = 4 x 5 x 6

But this involves, when no fruits i.e. zero fruits is chosen

Notes

⇒ Number of ways of selecting at least one fruit = (4x5x6) –1 = 119

Note: There was no fruit of a diverse type, thus here n=o

⇒ $2^n = 2^0 = 1$

Theorem:

Number of ways of choosing 'r' things from 'n' identical things is '1'.

Example: In how many ways 5 balls can be chosen from '12' identical red balls?

Solution: The balls are identical, total number of manners of choosing 5 balls = 1.

Example: How many numbers of four digits can be formed with digits 1, 2, 3, 4 and 5?

Solution:

Here $n = 5$ [Number of digits]

And r = 4 [Number of places to be filled-up]

Required number is $^5P_4 = 5!/1! = 5 \times 4 \times 3 \times 2 \times 1$

Self Assessment

Fill in the blanks:

6. Number of combinations of 'n' dissimilar things taken 'r' at a time, when 'p' particular things are forever included = ...

7. Number of combination of 'n' dissimilar things, taken 'r' at a time, when 'p' particular things are forever to be excluded = ...

13.3 Binomial Coefficients

You must be familiar with expressions like a + b, p + q, x + y, all consisting of two terms. This is why they are called binomials. You also know that a binomial expansion refers to the expansion of a positive integral power of such a binomial.

For instance, $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ is a binomial expansion. Consider coefficients 1, 5, 10, 10, 5, 1 of this expansion. In particular, let us consider the coefficient 10, of a^3b^2 in this expansion. We can get this term by selecting a from 3 of the binomials and b from the remaining 2 binomials in the product (a + b) (a + b) (a + b) (a + b) (a + b). Now, a can be chosen in C(5, 3) ways, i.e., 10 ways. This is the way each coefficient arises in the expansion.

The same argument can be extended to get the coefficients of $a^r b^{n-r}$ in the expansion of $(a + b)^n$. From the n factors in $(a + b)^n$, we have to select r for a and the remaining (n – r) for b. This can be done in C(n, r) ways. Thus, the coefficient of $a^r b^{n-r}$ in the expansion of $(a + b)^n$ is C(n, r).

Caution In view of the fact that C(n, r) = C(n, n – r), the coefficients of $a^r b^{n-r}$ and $a^{n-r} b^r$ will be the same.

r can only take the values 0, 1, 2, …, n. We also see that C(n, 0) = C(n, n) = 1 are the coefficients of a^n and b^n. Hence we have established the binomial expansion.

$$(a + b)^n = a^n + C(n, 1)\, a^{n-1} b + C(n, 2) a^{n-2} b^2 + \ldots + C(n, r)\, a^{n-r} b^r + \ldots + b^n.$$

Notes

In analogy with 'binomial', which is a sum of two symbols, we have 'multinomial' which is a sum of two or more (though finite) distinct symbols. Specifically we will consider the expansion of $(a_1 + a_2 + \ldots + a_m)^n$.

Notes For the expansion we can use the same technique as we use for the binomial expansion. We consider the nth power of the multinomial as the product of n factors, each of which is the same multinomial.

Every term in the expansion can be obtained by picking one symbol from each factor and multiplying them. Clearly, any term will be of the form where $r_1, r_2, \ldots,$ are non-negative integers such that $r_1 + r_2 + \ldots + r_m = n$. Such a term is obtained by selecting a_1 from r_1 factors, a_2 from r_2 factors from among the remaining $(n - r_1)$ factors, and so on. This can be done in:

$$C(n, r_1) \cdot C(n - r_1, r_2) \cdot C(n - r_1 - r_2, r_3) \ldots C(n - r_1 - r_2 \ldots\ldots\ldots\ldots - r_{m-1}, r_m) \text{ ways.}$$

If you simplify this expression, it will reduce to $\dfrac{n!}{r_1!r_2!\ldots r_m!}$.

So, we see that the multinomial expansion is

$$(a_1 + a_2 + \ldots + a_m)^n = \sum \frac{n!}{r_1!r_2!\ldots r_m!} a_1^{r_1}, a_2^{r_2} \ldots a_m^{r_m}$$

where the summation is over all non-negative integers $r_1, r_2, \ldots, r_m$ adding to n.

The coefficient of $a_1^{r_1}, a_2^{r_2} \ldots a_m^{r_m}$ in the expansion $(a_1 + a_2 + \ldots + a_m)^n$ is $\dfrac{n!}{r_1!r_2!\ldots r_m!}$, and is called a multinomial coefficient, in analogy with the binomial coefficient. We represent this by $C(n, r_1, r_2 \ldots, r_m)$. This is also represented by many authors as $\left[\dfrac{n}{r_1, r_2, \ldots r_m}\right]$.

For instance, the coefficient of $x^2y^2z^2t^2u^2$ in the expansion of $(x + y + z + u)^{10}$ is C(10; 2, 2, 2, 2, 2) = $10!/(2!)^5$.

Did u know? Multinomial expansion refers to the expansion of a positive integral power of a multinomial.

Self Assessment

Fill in the blanks:

8. A .... refers to the expansion of a positive integral power of such a binomial.
9. In analogy with 'binomial', which is a sum of two symbols, we have 'multinomial' which is a sum of two or more (though finite) .... symbols.
10. The coefficient of $a^r b^{n - r}$ in the expansion of $(a + b)^n$ is ....
11. .... expansion refers to the expansion of a positive integral power of a multinomial.

Notes

12. In view of the fact that $C(n, r) = C(n, n - r)$, the coefficients of $a^r b^{n-r}$ and $a^{n-r} b^r$ will be the ....,

13. The coefficient of in the expansion $(a_1 + a_2 + \ldots + a_m)^n$ is ...., and is called a multinomial coefficient, in analogy with the binomial coefficient.

13.4 Inclusion-Exclusion Principle

Let us begin with considering the following situation: In a sports club with 54 members, 34 play tennis, 22 play golf, and 10 play both. There are 11 members who play handball, of which 6 play tennis also, 4 play golf also, and 2 play both tennis and golf. How many play none of the three sports?

To answer this, let S represent the set of all members of the club. Let T represent the set of tennis playing members, G represent the set of golf playing members, and H represent the set of handball playing members. Let us represent the number of elements in A by $|A|$.

Consider the number $|S| - |T| - |G| - |H|$.

Is this the answer to the problem? No, because those who are in T as well as G have been subtracted twice. To compensate for this double subtraction, we may now consider the number $|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T|$.

Is this the answer? No, because those playing all the three games have been subtracted thrice and then added thrice. But those members have to be totally excluded also. Hence, we now consider the number

$$|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T| - |T \cap G \cap H|.$$

This is the correct answer. This reduces to 54 – 34 – 22 – 11 + 10 + 6 + 4 – 2 = 5.

To solve this problem we have made inclusions and exclusions alternately to arrive at the correct answer. This is a simple case of the principle of inclusion and exclusion.

Did u know? It is also known as the sieve principle because we subject the objects to sieves of a progressively finer mesh to arrive at a certain grading.

Let us state and prove this principle now.

13.4.1 Inclusion-Exclusion Formula

Let $A_1, A_2, \ldots, A_n$ be n sets in a universal set U consisting of N elements. Let S_k denote the sum of the sizes of all the sets formed by intersecting k of the A is at a time. Then the number of elements in none of the sets $A_1, A_2, \ldots, A_n$ is given by

$$\left|\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n\right| = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n$$

Self Assessment

Fill in the blanks:

14. Inclusion-Exclusion principle is also known as the .... principle because we subject the objects to sieves of a progressively finer mesh to arrive at a certain grading.

15. Inclusion-Exclusion formula is given by $\left|\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n\right|$ = ...

Notes

13.5 Summary

- An arrangement of r objects, without considering order and without replication, chosen from n different objects is known as a combination of n objects taken r at a time.
- The word selection is used when the order of things has no significance.
- A combination of n objects taken r at a time is a selection of r objects from a total of n objects ($r \le n$), where order does not matter.
- The notation for combination is usually $\left(\binom{n}{r}\right)$ and read, "n choose r."
- The number of combinations is indicated by ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.
- The dissimilarity among combinations and permutations is in combinations you are counting groups and in permutations you are counting different methods to assemble items with respect to order.
- Number of combinations of 'n' dissimilar things taken 'r' at a time, when 'p' particular things are forever included = ${}^{n-p}C_{r-p}$.
- Number of combination of 'n' dissimilar things, taken 'r' at a time, when 'p' particular things are forever to be excluded = ${}^{n-p}C_r$

13.6 Keywords

Combination: An arrangement of r objects, without considering order and without replication, chosen from n different objects is known as a combination of n objects taken r at a time.

Selection: The word selection is used when the order of things has no significance.

13.7 Review Questions

1. In a class of 10 students, how many ways can a club of 4 students be arranged?
2. Find the number of ways to take 4 people and place them in groups of 3 at a time where order does not matter.
3. A *poker hand* consists of 5 cards dealt without replacement and without regard to order from a deck of 52 cards.
   (a) Show that the number of poker hands is 2,598,960.
   (b) Find the probability that a random poker hand is a full house (3 cards of one kind and 2 of another kind).
   (c) Find the probability that a random poker hand has 4 of a kind.
4. Suppose that in a group of n people, each person shakes hands with every other person. Show that there are $C(n, 2)$ different handshakes.
5. Find the number of poker hands that are void in at least one suit.
6. 3 cards are drawn from a standard deck of 52 cards. How many different 3-card hands can possibly be drawn?

Notes

7. John has got 1 dollar, with which he can buy green, red and yellow candies. Each candy costs 50 cents. John will spend all the money he has on candies. How many different combinations of green, red and yellow candies can he buy?

8. The board of directors of a corporation comprises 10 members. An executive board, formed by 4 directors, needs to be elected. How many possible ways are there to form the executive board?

9. Jose has 9 friends that he wants to invite to dinner but he can only invite six of them at one time. Out of the nine friends many different groups can he invite?

10. Over the weekend, your family is going on vacation, and your mom is letting you bring your favorite video game console as well as five of your games. How many ways can you choose the five games if you have 12 games in all?

Answers: Self Assessment

1. combination
2. *selection*
3. formula
4. ${}_nC_r = \dfrac{n!}{(n-r)!r!}$
5. order
6. ${}^{n-p}C_{r-p}$
7. ${}^{n-p}C_r$
8. binomial expansion
9. distinct
10. C(n, r)
11. Multinomial
12. same
13. $\dfrac{n!}{r_1!r_2!\ldots r_m!}$
14. sieve
15. $N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n$

13.8 Further Readings

Books

Giri&Banerjee, *Introduction to Business Mathematics*, Academic Publishers.

Jacobus Hendricks van Lint, *A Course in Combinatorics*, Cambridge University Press.

Marshall Hall, *Combinatorial Theory*, John Wiley & Sons.

Online link

www.mathwarehouse.com/probability/combination.php

Notes

Unit 14: Probability

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concepts of random experiments
- Discuss the concept of event
- Recognize the axiomatic approach to probability

Introduction

We start with thinking of some event where the result is vague. Instances of such results would be the roll of a die, the quantity of rain that we obtain tomorrow, the condition of the economy in one month, etc. In every case, we don't know for certain what will occur. For instance, we don't know precisely how much rain we will obtain tomorrow.

A *probability* is a mathematical measure of the possibility of the event. It is a number that we connect to an event, say the event that we'll obtain over an inch of rain tomorrow, which imitates the possibility that we will get this much rain.

A probability can be expressed as a number from 0 to 1. If we allocate a probability of 0 to an event, this shows that this event *never* will take place. A probability of 1 connected to a particular event shows that this event *always* will happen.

In this unit, you will understand the various concepts of probability such as random experiments, sample space, events, etc.

14.1 Random Experiments

The fundamental view in probability is that of a random experiment: an experiment whose result cannot be revealed beforehand, but is however still dependent on analysis.

Instances of random experiments are:

1. Tossing a die,
2. Calculating the amount of rainfall in January,
3. Counting the number of calls coming at a telephone exchange throughout a fixed time period,
4. Choosing a random model of fifty people and scrutinizing the number of left-handers,
5. Selecting at random ten people and gauging their height.

Example: (Coin Tossing) The most basic random experiment is the experiment where a coin is tossed a number of times, say n times. Certainly, much of probability theory can be dependent on this experiment. To better recognize how these experiment performs, we can perform it out on a digital computer, for instance in Matlab.

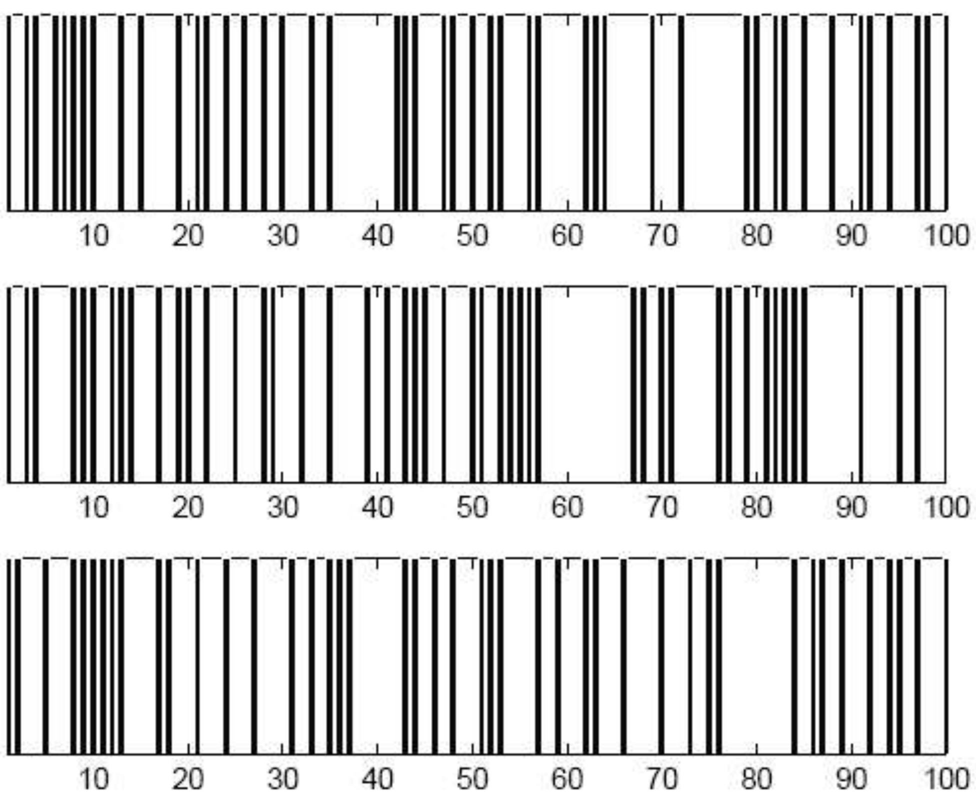
The given below simple Matlab program, creates a series of 100 tosses with a fair coin (that is, heads and tails are uniformly likely), and plots the outcomes in a bar chart.

x = (rand(1,100) < 1/2)

bar(x)

Here x is a vector with 1s and 0s, representing Heads and Tails, say. Distinctive outcomes for three such experiments are given in Figure 14.1.

Figure 14.1: Three Experiments where a Fair Coin is Tossed 100 Times



Notes

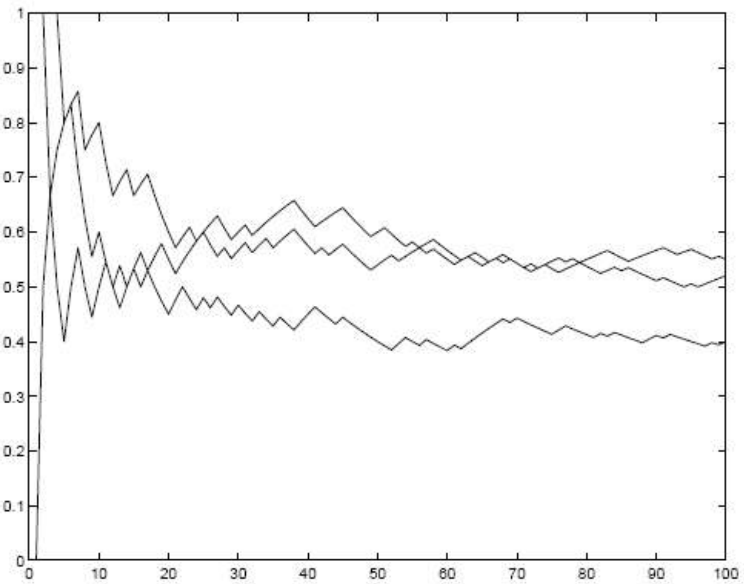
The dark bars signify when "Heads" (=1) emerges. We can also scheme the average number of "Heads" beside the number of tosses. In the same Matlab program, this is performed in two additional lines of code:

y = cumsum(x)./[1:100]

plot(y)

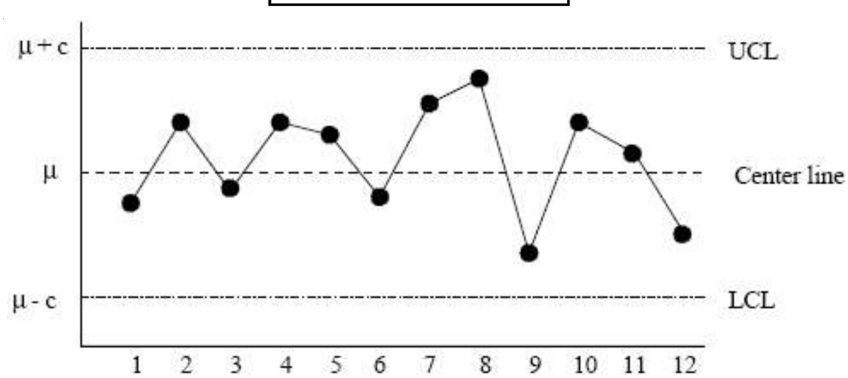
The outcome of three such experiments is portrayed in Figure 14.2. Observe that the average number of Heads appears to congregate to 1/2, but there is load of random instability.

Figure 14.2: The Average Number of Heads in n Tosses, where n = 1, . . . , 100



Example: (Control Chart) Control charts, are commonly used in manufacturing as a technique for *quality control*. Each hour the average output of the process is calculated — for instance, the average weight of 10 bags of sugar — to evaluate if the process is still "in control", for instance, if the machine still puts on average the accurate quantity of sugar in the bags. When the process > *Upper Control Limit* or < *Lower Control Limit* and an alarm is raised that the process is out of control, e.g., the machine is required to be accustomed, since it either puts too much or not sufficient sugar in the bags. The question is how to fix the control limits, as the random process obviously fluctuates around its "centre" or "target" line.

Figure 14.3: Control Chart



Notes

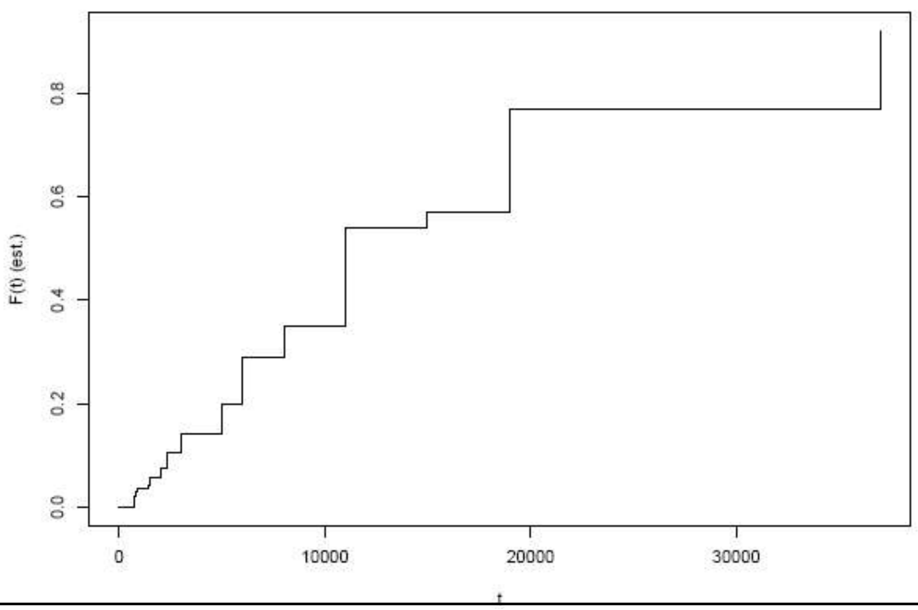
Example: (Machine Lifetime): Presume 1000 indistinguishable components are observed for failure, up to 50,000 hours. The result of such a random experiment is usually summarized through the cumulative lifetime table and plot, as specified in Table 14.1 and Figure 14.3, correspondingly. Here $\widehat{F(t)}$ signifies the proportion of components that are unsuccessful at time t. One question is how $\widehat{F(t)}$ can be modeled by means of a continuous function F, in lieu of the lifetime distribution of a usual module.

Table 14.1: The Cumulative Lifetime Table

t(h)	failed	$\widehat{F(t)}$
0	0	0.000
750	22	0.020
800	30	0.030
900	36	0.036
1400	42	0.042
1500	58	0.058
2000	74	0.074
2300	105	0.105

t(h)	failed	$\widehat{F(t)}$
3000	140	0.140
5000	200	0.200
6000	290	0.290
8000	350	0.350
11000	540	0.540
15000	570	0.570
19000	770	0.770
37000	920	0.920

Figure 14.4: The Cumulative Lifetime Table



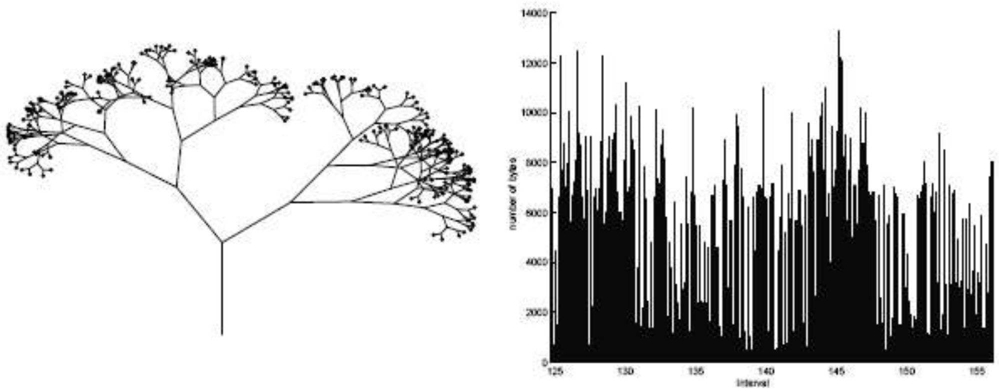
Example: A 4-engine aeroplane is able to fly on just one engine on each wing. All engines are unreliable.

Notes

Figure 14.5: An Aeroplane with Four Undependable Engines

Number the engines: 1,2 (left wing) and 3,4 (right wing). Scrutinize which engine functions correctly during a particular period of time. There are 24 = 16 probable outcomes of the experiment. Which outcomes cause "system failure"? Furthermore, if the probability of failure inside some time period is recognized for each of the engines, what is the probability of failure for the whole system? Again this can be observed as a random experiment. Below are two more pictures of randomness. The first is a computer-generated "plant", which appears astonishingly like a real plant. The second is genuine data representing the number of bytes that are broadcasted over some communications connection. An appealing trait is that the data can be displayed to exhibit "fractal" behaviour, that is, if the data is combined into smaller or larger time intervals, a comparable picture will emerge.

Figure 14.6 (a): Plant Growth (b) Telecommunications Data



We hope to depict these experiments by means of suitable mathematical models. These models is composed of three building blocks: a *sample space*, a set of *events* and a *probability*. We will now depict each of these objects.

Task Give an example of random experiments.

Self Assessment

Fill in the blanks:

1. A .... is a mathematical measure of the possibility of the event.
2. .... is an experiment whose result cannot be revealed beforehand, but is however still dependent on analysis.
3. A probability of 1 .... to a particular event shows that this event *always* will happen.

14.2 Sample Space

Notes

Even though we cannot forecast the result of a random experiment with certainty we typically can state a set of potential outcomes. This provides the first element in our model for a random experiment.

Definition

The **sample space** Ù of a random experiment is defined as the set of all achievable results of the experiment.

Example: of random experiments along with their sample spaces are:

1. Cast two dice successively,

   $\Omega = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), \ldots, (6, 6)\}$.

2. The lifetime of a machine (in days),

   Ω = R+ = {positive real numbers} .

3. The number of coming calls at an exchange throughout a specific time interval,

   $\Omega = \{0, 1, \cdots\}$ = Z+ .

4. The heights of 10 chosen people.

   $\Omega = \{(x1, \ldots, x10), xi\ e''\ 0, i = 1, \ldots, 10\} = R+^{10}$.

Here (x1, . . . , x10) displays the result that the length of the first chosen person is x1, the length of the second person is x2, etc.

Did u know? For modeling purposes it is frequently simpler to take the sample space larger than needed.

Example: The real lifetime of a machine would surely not extent the whole positive real axis. And the heights of the 10 chosen people would not surpass 3 metres.

Self Assessment

Fill in the blanks:

4. Even though we cannot forecast the result of a random experiment with certainty we typically can state a set of potential ....
5. The .... Ω of a random experiment is defined as the set of all achievable results of the experiment.
6. For modeling purposes it is frequently simpler to take the sample space .... than needed.

14.3 Events

Frequently we are not concerned in a single result but in whether or not one of a *group* of results appears. Such subsets of the sample space are known as **events**. Events will be signified by capital letters $A, B, C, \ldots$. We say that event A **appears** if the result of the experiment is one of the essentials in A.

Notes

Example: of events are:

1. The event that the sum of two dice is 10 or more,

   A = {(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)}.

2. The event that a machine lives less than 100 days,

   A = (0, 100) .

3. The event that out of fifty chosen people, five are left-handed,

   A = {5} .

Example: **(Coin Tossing)** Presume that a coin is tossed 3 times, and that we "record" every head and tail. The sample space can then be shown as

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

where, for instance, HTH means that the first toss is heads, the second tails, and the third heads. An substitute sample space is the set {0, 1}3 of binary vectors of length 3, e.g., HTH communicate to (1,0,1), and THH to (0,1,1). The event A that the third toss is heads is

$$A = \{HHH, HTH, THH, TTH\}.$$

As events are sets, we can affect the common set operations to them:

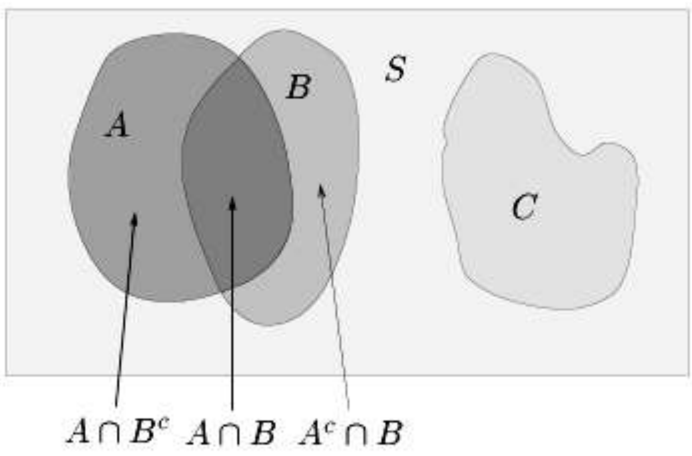
1. The set $A \cup B$ (A **union** B) is the event that A *or* B *or* both take place,
2. The set $A \cap B$ (A **intersection** B) is the event that A *and* B both take place,
3. The event Ac (A **complement**) is the event that A does *not* crop up,
4. If $A \subset B$ (A is a **subset** of B) then event A is said to *imply* event B.

Two events A and B which have no results in general, that is, $A \cap B = \phi$, are known as disjoint events.

Example: Let us cast two dice successively. The sample space is Ω = {(1, 1), (1, 2), ..., (1, 6), (2, 1),..., (6, 6)}. Let A = {(6, 1), ..., (6, 6)} be the event that the first die is 6, and let B = {(1, 6), ..., (1, 6)} be the event that the second dice is 6. Then $A \cap B$ = {(6, 1), . . . , (6, 6)} $\cap$ {(1, 6), ..., (6, 6)} = {(6, 6)} is the event that both die are 6.

It is frequently functional to portray events in a **Venn diagram**, like in Figure 14.7.

Figure 14.7: A Venn Diagram



Notes

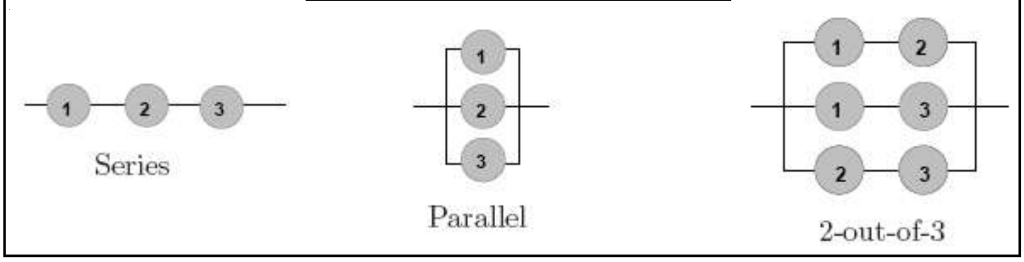
In this Venn diagram we observe

(i) $A \cap C = \phi$ and consequently events A and C are disjoint.

(ii) $(A \cap Bc) \cap (Ac \cap B) = \phi$ and therefore events $A \cap Bc$ and $Ac \cap B$ are disjoint.

Example: **(System Reliability)** In Figure 14.8 three systems are represented, each comprising of 3 undependable components. The *series* system functions if and only if (shortened as iff) *all* components work; the *parallel* system functions iff *at least one* of the components functions; and the 2-out-of-3 system functions iff at least 2 out of 3 components function.

Figure 14.8: Three Unreliable Systems



Suppose Ai be the event that the ith component is functioning, $i = 1, 2, 3$; and let Da, Db, Dc be the events that correspondingly the series, parallel and 2-out-of-3 system is working. Then,

$Da = A1 \cap A2 \cap A3$,

And

$Db = A1 \cup A2 \cup A3$.

Also,

$Dc = (A1 \cap A2 \cap A3) \cup (A1 \cap A2 \cap A3) \cup (A1 \cap A2 \cap A3) \cup (A1 \cap A2 \cap A3)$

$= (A1 \cap A2) \cup (A1 \cap A3) \cup (A2 \cap A3)$.

Two functional consequences in the theory of sets are the following, because of **De Morgan**:

If $\{Ai\}$ is a compilation of events (sets) then

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c \qquad ...(1)$$

and

$$\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c. \qquad ...(2)$$

This is simply proved by means of Venn diagrams.

Note that if we understand Ai as the event that a component functions, then the left-hand side of (1) is the event that the equivalent parallel system is not functioning. The right hand is the event that at all components are not functioning. Evidently these two events are the similar.

Notes

> *Task* Define disjoint events with example.

Self Assessment

Fill in the blanks:

7. Frequently we are not concerned in a single result but in whether or not one of a *group* of results appears. Such subsets of the sample space are known as ....
8. We say that event A **appears if** the result of the experiment is one of the .... in A.
9. Two events A and B which have no results in general, that is, $A \cap B = \phi$, are known as ....
10. If $A \subset B$ (A is a **subset** of B) then event A is said to .... event B.
11. If $\{Ai\}$ is a compilation of events (sets) then $\left(\bigcup_i A_i\right)^c$ ....

14.4 Probability

The third element in the model for a random experiment is the requirement of the probability of the events. It informs us how *likely* it is that a specific event will take place.

Definition

A probability P is a rule which allocates a positive number to every event, and which assures the following **axioms**:

Axiom 1: $P(A) \geq 0$.

Axiom 2: $P(\Omega) = 1$.

Axiom 3: For any series $A1, A2, \ldots$ of *disjoint* events we have

$$\boxed{P\left(\bigcup_i A_i\right) = \sum_i P(A_i)}. \qquad \text{...(3)}$$

Axiom 2 just specifies that the probability of the "certain" event Ω is 1. Property (1.3) is the *vital* property of a probability, and is sometimes known as the **sum rule**. It just specifies that if an event can occur in a number of different manners *that cannot occur simultaneously* then the probability of this event is just the sum of the probabilities of the composing events.

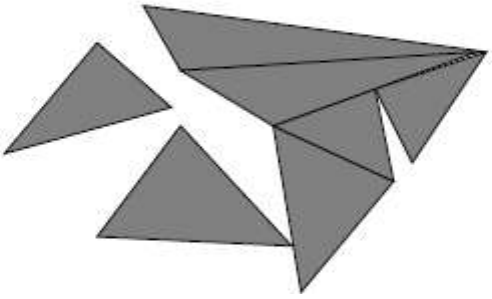
Note that a probability rule P has precisely the similar properties as the general "area measure".

Example: The whole area of the union of the triangles in Figure 14.9 is equivalent to the sum of the areas of the individual triangles. This is how you should understand property (3). But rather than gauging areas, P computes probabilities.

As a direct effect of the axioms we have the following properties for P.

Notes

Figure 14.9: The Probability Measure has the Similar Properties as the "Area"



Theorem: Let A and B be events. Then,

1. $P(\phi) = 0$.
2. $A \subset B = \Rightarrow !\ P(A) \leq P(B)$.
3. $P(A) \leq 1$.
4. $P(Ac) = 1 - P(A)$.
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof:

1. $\Omega = \Omega \cap \phi \cap \phi \cap \cdots$, so, by the sum rule, $P(\Omega) = P(\Omega) + P(\phi) + P(\phi) + \cdots$, and consequently, by the second axiom, $1 = 1 + P(\phi) + P(\phi) + \cdots$, from which it follows that $P(\phi) = 0$.
2. If $A \subset B$, then $B = A \cup (B \cap Ac)$, where A and $B \cap Ac$ are disjoint. Therefore, by the sum rule, $P(B) = P(A) + P(B \cap Ac)$, which is (by the first axiom) greater than or equal to $P(A)$.
3. This follows directly from property 2 and axiom 2, because $A \subset \Omega$.
4. $\Omega = A \cup Ac$, where A and Ac are disjoint. Therefore, by the sum rule and axiom 2: $1 = P(\Omega) = P(A) + P(Ac)$, and thus $P(Ac) = 1 - P(A)$.
5. Write $A \cup B$ as the disjoint union of A and $B \cap Ac$. Then, $P(A \cup B) = P(A) + P(B \cup Ac)$. Also, $B = (A \cap B) \cup (B \cap Ac)$, so that $P(B) = P(A \cap B) + P(B \cap Ac)$. Merging these two equations gives $P(A \cap B) = P(A) + P(B) - P(A \cap B)$.

We have now finished our model for a random experiment. It is up to the modeler to state the sample space Ω and probability gauge P which most strongly illustrates the definite experiment. This is not always as clear-cut as it appears, and sometimes it is functional to model only certain *observations* in the experiment.

Example: Consider the experiment where we toss a fair die. How should we describe Ω and P? Clearly, $\Omega = \{1, 2, \ldots, 6\}$; and some common sense displays that we should define P by

$P(A) = |A|$

$6, A \subset \Omega$,

where $|A|$ signifies the number of elements in set A. For example, the probability of receiving an even number is $P(\{2, 4, 6\}) = 3/6 = 1/2$.

In many applications the sample space is *countable*, i.e. $\Omega = \{a1, a2, \ldots, an\}$ or

$\Omega = \{a1, a2, \ldots\}$. Such a sample space is known as **discrete**.

Notes

The simplest manner to specify a probability P on a discrete sample space is to state first the probability *pi* of every **elementary event** *{ai}* and then to define

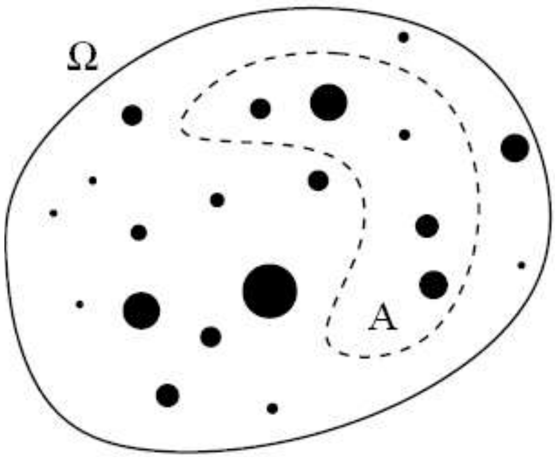
P(*A*) = _ *i*:*ai* ∈ *A*

pi , for all $A \subset \Omega$.

This thought is graphically displayed in Figure 14.10. Each element *ai* in the sample is allocated a probability weight *pi* illustrated by a black dot.

Did u know? To discover the probability of the set *A* we have to sum up the weights of all the essentials in *A*.

Figure 14.10: A Discrete Sample Space



Again, it is up to the modeller to correctly state these probabilities. Luckily, in many applications all basic events are *equally likely*, and therefore the probability of every basic event is equal to 1 divided by the total number of elements in Ω.

Since the "equally likely" principle is so significant, we invent it as a theorem.

Theorem: (Equilikely Principle): If Ω has a finite number of results, and all are uniformly likely, then the probability of each event *A* is defined as

P(*A*) = |*A*|

|Ω|.

So for such sample spaces the computation of probabilities diminishes to *counting* the number of outcomes (in *A* and Ω).

When the sample space is not countable, for example Ω = R+, it is supposed to be **continuous**.

Example: We draw at random a point in the interval [0, 1]. Every point is equally probable to be drawn. How do we state the model for this experiment?

The sample space is clearly Ω = [0, 1], which is a continuous sample space. We cannot define P by means of the simple events $\{x\}$, $x \in [0, 1]$ since each of these events must have probability 0 (!). Though we can define P as follows:

For each $0 \le a \le b \le 1$, let $P([a, b]) = b - a$.

Notes

This entirely mention P. Particularly, we can locate the probability that the point occurs into any (sufficiently nice) set *A* as the *length* of that set.

Some examples of probability are discussed below.

Example: A Poker Hand

The game of poker has various variants. General to all is the truth that players obtain - one way or another - hands of five cards each. The hands are evaluated according to a predestined ranking system. Now we shall assess probabilities of numerous hand combinations.

Poker utilizes the standard deck of 52 cards. There are C(52, 5) probable combinations of 5 cards chosen from a deck of 52: 52 cards to select the first of the five from, 51 cards to select the second one, ..., 48 to select the fifth card. The product 52 × 51 × 50 × 49 × 48 must be divided by 5!since the order in which the five cards are added to the hand is of no significance, such as 7 ♣8♣9♣10♣J♣ is the same hand as 9♣7♣10♣J♣8♣. So there are C(52, 5) = 2598960 different hands. The poker sample space comprises 2598960 uniformly probable elementary events.

The probability of either hand is obviously 1/2598960. Visualize having an urn with 52 balls, of which 5 are black and the remaining white. You are to draw 5 balls out of the urn. What is the probability that all 5 balls drawn are black?

The probability that the first ball is black is 5/52. Presuming that the first ball was black, the probability that the second is also black is 4/51. Presuming that the first two balls are black, the probability that the third is black is 3/50, ... The fifth ball is black with the probability of 1/48, given the first 4 balls were all black. The probability of drawing 5 black balls is the product:

$$\frac{3}{50}\cdot\frac{2}{49}\cdot\frac{1}{48}=\frac{1}{C(52,5)}$$

The highest ranking poker hand is a Royal Flush - a series of cards of the same suit beginning with 10, e.g., 10♣J♣Q♣K♣A♣. There are 4 of them, one for each of the four suits. So the probability of getting a royal flush is 4/2598960 = 1/649740. The probability of getting a royal flush of, say, spades ♠, is obviously 1/2598960.

Any sequence of 5 cards of the same suit is a straight flush ranked by the highest card in the sequence. A straight flush may begin with any of 2, 3, 4, 5, 6, 7, 8, 9, 10 cards and some times with an Ace where it is thought to have the rank of 1. So there are 9 (or 10) possibilities of getting a straight flush of a specified suit and 36 (or 40) possibilities of obtaining any straight flush.

Five cards of the same suit - not essentially in sequence - is a flush. There are 13 cards in a suit and C(13, 5) = 1287 combinations of 5 cards out of 13. All in all, there are 4 times as many flush combinations: 5148.

Four of a kind is a hand, such as 5♣5♠5♦5♥K♠, with four cards of the similar rank and one extra, unmatched card. There are 13 combinations of 4 equally ranked cards each of which can complete a hand with any of the remaining 48 cards. A hand with 3 cards of one rank and 2 cards of a dissimilar rank is called Full House. For a specified rank, there are C(4, 3) = 4 methods to select 3 cards of that rank; there 13 ranks to consider. There are C(4, 2) = 6 combinations of 2 cards of equal rank, but now only 12 ranks to select from. There are then 4 × 13 × 6 × 12 = 3744 full houses.

A straight hand is a straight flush without "flush", so to articulate. The card must be in series but not essentially of the same suit. If the ace is permitted to begin a hand, there are 40 ways to select the first card and then, we need to account that the remaining 4 cards could be of any of the 4 suits, providing the total of 40 × 4 × 4 × 4 × 4 = 10240 hands. Removing 40 straight flushes leaves 10200 "regular" flushes.

Notes

Three of a kind is a hand, like 5♣5♠5♦7♥K♠, where three cards have the similar rank while the remaining 2 vary in rank among themselves and the first three. There are $13 \times C(4, 3) = 52$ combinations of three cards of the similar rank. The next card could be any of 48 and the fifth any of 44 and the pair could come in any order so the products is required to be halved: $52 \times 48 \times 44 / 2 = 54912$.

Example: A spinner has 4 equal sectors colored yellow, blue, green and red. After spinning the spinner, what is the probability of landing on each color?

The possible conclusions of this experiment are yellow, blue, green, and red.

$$P(\text{yellow}) = \frac{\text{\# of ways to land on yellow}}{\text{total \# of colors}} = \frac{1}{4}$$

$$P(\text{blue}) = \frac{\text{\# of ways to land on blue}}{\text{total \# of colors}} = \frac{1}{4}$$

$$P(\text{green}) = \frac{\text{\# of ways to land on green}}{\text{total \# of colors}} = \frac{1}{4}$$

$$P(\text{red}) = \frac{\text{\# of ways to land on red}}{\text{total \# of colors}} = \frac{1}{4}$$

Example: A single 6-sided die is rolled. What is the probability of each conclusion? What is the probability of rolling an even number? of rolling an odd number?

The possible conclusions of this experiment are 1, 2, 3, 4, 5 and 6.

Probabilities:

$$P(1) = \frac{\text{\# of ways to roll a 1}}{\text{total \# of sides}} = \frac{1}{6}$$

$$P(2) = \frac{\text{\# of ways to roll a 2}}{\text{total \# of sides}} = \frac{1}{6}$$

$$P(3) = \frac{\text{\# of ways to roll a 3}}{\text{total \# of sides}} = \frac{1}{6}$$

$$P(4) = \frac{\text{\# of ways to roll a 4}}{\text{total \# of sides}} = \frac{1}{6}$$

$$P(5) = \frac{\text{\# of ways to roll a 5}}{\text{total \# of sides}} = \frac{1}{6}$$

$$P(6) = \frac{\text{\# of ways to roll a 6}}{\text{total \# of sides}} = \frac{1}{6}$$

$$P(\text{even}) = \frac{\text{\# ways to roll an even number}}{\text{total \# of sides}} = \frac{3}{6} = \frac{1}{2}$$

$$P(\text{odd}) = \frac{\text{\# ways to roll an odd number}}{\text{total \# of sides}} = \frac{3}{6} = \frac{1}{2}$$

This example shows the difference between a conclusion and an event. A single conclusion of this experiment is rolling a 1, or rolling a 2, or rolling a 3, etc. Rolling an even number (2, 4 or 6) is an event, and rolling an odd number (1, 3 or 5) is also an event.

Notes

Example: A glass jar contains 6 red, 5 green, 8 blue and 3 yellow marbles. If a single marble is selected at random from the jar, what is the probability of selecting a red marble? a green marble? a blue marble? a yellow marble?

The possible conclusions of this experiment are red, green, blue and yellow.

Probabilities:

$$P(\text{red}) = \frac{\text{\# of ways to choose red}}{\text{total \# of marbles}} = \frac{6}{22} = \frac{3}{11}$$

$$P(\text{green}) = \frac{\text{\# of ways to choose green}}{\text{total \# of marbles}} = \frac{5}{22}$$

$$P(\text{blue}) = \frac{\text{\# of ways to choose blue}}{\text{total \# of marbles}} = \frac{8}{22} = \frac{4}{11}$$

$$P(\text{yellow}) = \frac{\text{\# of ways to choose yellow}}{\text{total \# of marbles}} = \frac{3}{22}$$

The conclusions in this experiment are not equally expected to happen. You are more probable to select a blue marble than any other color. You are least likely to select a yellow marble.

14.4.1 Axiomatic Approach to Probability

The issue of what probability actually is does not have a completely acceptable answer. In some conditions it may be supportive to consider probability as displaying long-run amount or degree of belief. However these are not exact mathematical definitions. The current approach is to consider probability as a mathematical construction pleasing definite axioms.

Definition (Kolmogorov's Axioms for Probability): Probability is a function P which allocates to every event A a real number P(A) such that:

1. For every event A we have $P(A) \geq 0$
2. $P(S) = 1$
3. If A1, A2, . . . , An are events and $Ai \cap Aj = \phi$ for all i =6 j then

   $$P(A1 \cup A2 \cup \cdots \cup An) = \sum_{i=1}^{n} P(A_i)$$

   If A1, A2, . . . are events and $Ai \cap Aj$ = " for all i =6 j then

   $$P(A1 \cup A2 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

   The events fulfilling 3 are pair wise disjoint or mutually exclusive.

> *Notes* Notice that Axiom 3 has an edition for finitely numerous events and an edition for a countable infinite number of events. If S is finite then we want only worry regarding the first one of these. If S is infinite (mainly if it is not countable) then a number of intricacies creep in which we will mostly ignore in this course.

Notes

You can verify that setting P(A) = $\left|\frac{A}{S}\right|$ provides a probability. This is the case when every result in the sample space is uniformly likely.

Caution Do not assume that every outcome is equally likely without good reason.

Beginning from the axioms we can infer different properties. Optimistically, these will consent with our instinct regarding probability. The proofs of all of these are simple inferences from the axioms.

Proposition 1: If A is an event then P(Ac) = 1 – P(A).

Proof:

Let A be any event. By meaning of the complement Ac and A are disjoint events.

Since they are disjoint we can apply Axiom 3 to get

P(A ∪ Ac) = P(A) + P(Ac).

But (again by definition of the complement) A ∪ Ac = S so

P(S) = P(A) + P(Ac).

By Axiom 2 P(S) = 1 and so

1 = P(A) + P(Ac).

Rearranging this provides the result.

Notes Observe that every line of the proof is defensible by one of the axioms or a definition (or is a simple manipulation).

We can utilize the results we have proved to infer further ones like the following corollary.

Corollary 1

P(ϕ) = 0.

Proof.

By definition of complement Sc = ϕ. Hence by Proposition 1

P(ϕ) = 1 “ P(S) = 1 “ 1 = 0

where the second equality accesses Axiom 2.

Corollary 2

If A is an event then P(A) 1.

Proposition 2: If A and B are events and A B then

P(A) P(B).

Proposition 3: If A = {a1, a2, . . . , an} is an event then

P(A) = $P(A) = \sum_{i=1}^{n} P(\{a_i\})$

Self Assessment

Notes

Fill in the blanks:

12. Probability informs us how .... it is that a specific event will take place.
13. A probability rule P has precisely the .... properties as the general "area measure".
14. To discover the probability of the set A we have to sum up the .... of all the essentials in A.
15. When the sample space is not countable, it is supposed to be ....

14.5 Summary

- A probability is a mathematical measure of the possibility of the event.
- The fundamental view in probability is that of a random experiment: an experiment whose result cannot be revealed beforehand, but is however still dependent on analysis.
- Even though we cannot forecast the result of a random experiment with certainty we typically can state a set of potential outcomes.
- The sample space Ω of a random experiment is defined as the set of all achievable results of the experiment.
- Frequently we are not concerned in a single result but in whether or not one of a group of results appears. Such subsets of the sample space are known as events.
- Two events A and B which have no results in general, that is, $A \cap B = \phi$, are known as disjoint events.
- The third element in the model for a random experiment is the requirement of the probability of the events. It informs us how likely it is that a specific event will take place.
- The issue of what probability actually is does not have a completely acceptable answer. In some conditions it may be supportive to consider probability as displaying long-run amount or degree of belief.

14.6 Keywords

Disjoint Events: Two events A and B which have no results in general, that is, $A \cap B = \phi$, are known as disjoint events.

Events: Frequently we are not concerned in a single result but in whether or not one of a *group* of results appears. Such subsets of the sample space are known as events.

Probability: A probability is a mathematical measure of the possibility of the event.

Random Experiment: The fundamental view in probability is that of a random experiment: an experiment whose result cannot be revealed beforehand, but is however still dependent on analysis.

Sample Space: The sample space Ù of a random experiment is defined as the set of all achievable results of the experiment.

14.7 Review Questions

1. Illustrate the concept of random experiments with examples.
2. A die is rolled, find the probability that an even number is obtained.

Notes

3. Which of these numbers cannot be a probability?

   (a) -0.00001 (b) 0.5

   (c) 1.001 (d) 0

   (e) 1 (f) 20%

4. Two dice are rolled, find the probability that the sum is

   (a) equal to 1 (b) equal to 4

   (c) less than 13

5. A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

6. If two events 'A' and 'B' are such that $A \subset B$, then prove P(A) < =P(B).

7. Three coins are tossed. What is the probability of getting (a) all heads, (b) two heads, (c) at least one head, (d) at least two heads?

8. Illustrate the concept of Axiomatic Approach to Probability with examples.

9. A single card is chosen at random from a standard deck of 52 playing cards. What is the probability of choosing a 5 or a king?

10. A single 6-sided die is rolled. What is the probability of each outcome? What is the probability of rolling an even number?

Answers: Self Assessment

1.	probability	2.	Random experiment
3.	connected	4.	outcomes
5.	sample space	6.	larger
7.	events.	8.	essentials
9.	disjoint events	10.	imply
11.	$=\bigcap_i A_i^c$	12.	likely
13.	similar	14.	weights
15.	continuous		

14.8 Further Readings

Books

Hsu, Obability, *Random Variables*, And (So), Tata McGraw-Hill Education.

Joseph Lawson Hodges, *Basic Concepts of Probability and Statistics*, SIAM.

Yu. M. Suhov, *Probability and Statistics by Example: Basic Probability and Statistics*, Cambridge University Press.

Online link

http://www.maths.uq.edu.au/~kroese/asitp.pdf

LOVELY PROFESSIONAL UNIVERSITY
Jalandhar-Delhi G.T. Road (NH-1)
Phagwara, Punjab (India)-144411
For Enquiry: +91-1824-521360
Fax.: +91-1824-506111
Email: odl@lpu.co.in

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