SPECIAL FUNCTIONS AND INTEGRAL EQUATION

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SYLLABUS

Special functions Integral Equation

Objectives: The objective of the course is to know different methods to solve ordinary and partial differential equations and also to solve Integral equation of Fredholm and Voltera type.

Sr. No.	Content
1	Classification of second order partial differential equations, Solution of Laplace's equation, Wave and diffusion equations by separation of variable (axially symmetric cases).
2	Integral equations and algebraic system of linear equations, Volterra equation & L2_Kernels and functions.
3	Volterra equations of the first kind, Volterra integral equations and linear differential equations.
4	Fredholm equations, Solutions by the method of successive approximations.
5	Neumann's series, Fredholm's equations with Poincere Goursat Kernels, the Fredholm theorems.

CONTENT

Unit 1: Bessel's Functions

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Deduce Bessel's Differential equation from Laplace equation
- Obtain singular and non-singular points of Bessel's equations
- Obtain series solutions of Bessel's equation by Frobenius Method
- Establish recurrence relations between various Bessel's Co-efficient
- Obtain the formula for $J_n(x)$ from its generating functions
- Obtain zeroes of Bessel Functions.

Introduction

In this unit we shall be dealing with the various forms of Laplace differential equation involving Cartesian, Cylindrical and Spherical polar Co-ordinates.

Bessel's functions play a very important and central place in optical phenomical and in applied mathematical process. Just as a Fourier series, power series, Bessel's functions are quite useful in solving problems involving laplace equations in cylindrical co-ordinates. In this unit the importance is given to the following aspects of the Bessel's functions:

1. Solution of Bessel's functions $J_n(x)$, $Y_n(x)$ for various values of n as well as for different expansions involving x or $(1/x)$.

2. Recurrence relations are quite useful as they help in finding whole class of $J_n(x)$ in terms of two or three $J_n(x)$ of lower values of n i.e., $n = 0, 1, 2$.
3. Generating function for $J_n(x)$ is introduced so that certain formulas involving Bessel functions can be deduced. With the help of generating functions we can deduce recurrence relations or certain other formulas straight away.
4. Finally we also discuss the zeros of Bessel functions as they will lead us to the completeness as well as orthogonality properties of Bessel's Functions.

1.1 Bessel's Differential Equations from Laplace Equations

In dealing with the theory of potential problems in electrostatics or in gravitational field we commonly use Laplace equations

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(1)$$

Here V is a function of the Cartesian Co-ordinates. Any solution V_n of this equation, which is a homogeneous polynomial of degree n in x, y, z is called the solid spherical Harmonies.

Depending upon the symmetry of the problem we can express Laplace equation in cylindrical co-ordinates (r, θ, z) or spherical polar co-ordinates (r, θ, Φ). You must be knowing that the relations between x, y, z and r, θ, z are

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \qquad ...(2)$$

Also the relation between x, y, z and r, θ, Φ are

$$\left.\begin{aligned} x &= r\sin\theta\cos\Phi \\ y &= r\sin\theta\sin\Phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad ...(3)$$

1.2 Bessel's Differential Equations

To define Bessel functions we first of all obtain Bessel's Differential equation from Laplace's equation. To do that we write Laplace's equations (1) in cylindrical co-ordinates as

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial y}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(4)$$

We assume that V as a function of r, θ and z can be written as

$$V = R(r)\,\Theta'(\theta)\,z'(z) \qquad ...(5)$$

Where R, Θ', Z' are functions of r, θ, z alone respectively. Substituting in (4) we get

$$\Theta' Z'\frac{d^2R}{dr^2}+\frac{1}{r}\Theta' Z'\frac{dR}{dr}+\frac{RZ'}{r^2}\frac{d^2\Theta'}{d\Theta^2}+R\Theta'\frac{d^2Z}{dz^2} = 0$$

Or
$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{rR}\frac{dR}{dr}+\frac{1}{r^2}\cdot\frac{1}{\Theta'}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z'}\frac{d^2Z'}{dZ^2} = 0 \qquad ...(6)$$

Since the first three terms are independent of z, therefore the fourth term must also be independent of z. Let it be a constant c, so that

$$\frac{1}{Z'}\frac{d^2Z'}{dz^2} = c$$

Or $$\frac{d^2Z'}{dz^2} = cZ' \qquad ...(7)$$

Similarly, the third term in equation (6) must be free from θ i.e.

$$\frac{1}{\Theta'}\frac{d^2\Theta}{d\theta^2} = d$$

Or $$\frac{d^2\Theta}{d\theta^2} = d\Theta' \qquad ...(8)$$

With the help of (7) and (8) equation (6) becomes

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2}d+c = 0$$

or $$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(d+cr^2)R = 0 \qquad ...(9)$$

Let us put $kr = x$, so that

$$\frac{dR}{dr} = k\frac{dR}{dx}$$

$$\frac{d^2R}{dr^2} = k^2\frac{d^2R}{dx^2}$$

By putting these values in (9), we get

$$k^2r^2\frac{d^2R}{dx^2}+kr\frac{dr}{dx}+\left(d+\frac{cx^2}{k^2}\right)R = 0$$

Putting $c = k^2$ and $D = -n^2$, we get

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+(x^2-n^2)\,R = 0$$

Again put $R = y$ we have

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)\,y = 0$$

This is Bessel's differential equation. The solution of this equation is called cylindrical function or Bessel's function of order n, denoted as $J_n(x)$.

In this unit we shall be using certain properties of gamma function $\Gamma(x)$:

(i) $\Gamma(n)$ is defined by the integral

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \qquad n>0$$

(ii) $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}$

(iii) $\Gamma(1) = 1$

(iv) $\Gamma(1/2) = \sqrt{\pi}$

(v) $\Gamma(n + 1) = n\,\Gamma(n)$, $n > 0$

(vi) $\Gamma(n + 1) = 1.\,2.\,3.\,...n = n!$ for n a +ve integer

(vii) $\Gamma(n)\,\Gamma(1 - n) = \dfrac{\pi}{\sin n\pi}$

(viii) $\Gamma(m) = \infty$ if $m = 0$ or $-$ *ve* integer

(ix) $\Gamma(2n) = \dfrac{2^{2n-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(n+1/2)$

1.3 On Second Order Differential Equation of the Fuchs Type

Consider Bessel's equation for any n:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Or

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \qquad ...(A)$$

Let

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 1 - \frac{n^2}{x^2} \end{aligned}\right\} \qquad ...(B)$$

Thus $p(x)$ has a pole at $x = 0$ and $q(x)$ has a double pole at $x = 0$. Thus $x = 0$ is a singular point of Bessel's Differential Equation. Since

$$x\,p(x) \text{ and } x^2\,q(x), \qquad ...(C)$$

are finite at $x = 0$, the point $x = 0$ is a regular singular point of Bessel Differential equation. Also by putting $x = 1/r$ as independent variable we can show that $x = \infty$ is an irregular singular point. To see this put

$$x = \frac{1}{r}, \quad r = \frac{1}{x}$$

Then

$$\frac{dy}{dx} = \frac{dy}{dr}\frac{dr}{dx} = -\frac{dy}{dr}\left(\frac{1}{x^2}\right) = -r^2\frac{dy}{dr}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-r^2\frac{dy}{dr}\right)$$

Notes

$$= -r^2 \frac{d}{dr}\left(-r^2 \frac{dy}{dr}\right)$$

$$= r^2\left[2r\frac{dy}{dr} + r^2\frac{d^2y}{dr^2}\right]$$

$$\frac{d^2y}{dx^2} = r^4\frac{d^2y}{dr^2} + 2r^3\frac{dy}{dr}$$

So the Bessel's equation becomes

$$\frac{1}{r^2}\left(r^4\frac{d^2y}{dr^2} + 2r^3\frac{dy}{dr}\right) + \frac{1}{r}\left(-r^2\frac{dy}{dr}\right) + \left(\frac{1}{r^2} - n^2\right)y = 0$$

Or $$r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} + \left(\frac{1}{r^2} - n^2\right)y = 0$$

∴ $$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left(\frac{1}{r^4} - \frac{n^2}{r^2}\right)y = 0$$

Thus $r \to 0$ or $x = \infty$ is an irregular singular point. Since the singular points for the Bessel's equation are only 0 and ∞, therefore we can get a series solution of the Bessel's equation in powers of x which converges for $0 < x < \infty$. According to Fuchs theorem, the point is regular singular point provided $p(x)$, $q(x)$ satisfy conditions (C).

Fuchs theorem states that for $x = x_0$ to be a regular singular point, it is necessary and sufficient that $p(x)$ has at most a pole of order 1 and $q(x)$ at most a pole of order 2.

1.3.1 Series Solution of Bessel's Differential Equation

Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad ...(10)$$

Here we shall apply Frobenius method which assumes the solution to be of the form

$$y = x^k \sum_{r=0}^{\infty} x^r c_r \qquad ...(11)$$

Substituting in equation 10 we have

$$\sum_{r=0}^{\infty}\left\{C_r(k+r)(k+r-1)x^{k+r} + C_r(k+r)x^{k+r} + C_r(x^2 - n^2)x^{n+r}\right\} = 0$$

or $$\sum_{r=0}^{\infty} C_r\, x^{k+r}\left\{(k+r)(k+r-1) + (k+r) + (x^2 - n^2)\right\} \equiv 0 \qquad ...(12)$$

Equating to zero the lowest power of x i.e. x^k to zero we have

$$C_0\left\{k(k-1) + k - n^2\right\} = 0$$

or $$C_0\left\{k^2 - n^2\right\} = 0 \qquad ...(13)$$

As $C_0 \neq 0$, we have

$$K^2 - n^2 = 0 \quad \text{...(14)}$$

The equation (14) is called *indicial equation*.

So $k = n$ or $k = -n$

We first consider the case $k = n$, next equate the co-efficient of x^{k+1} to zero i.e.

$$C_1 [(k + 1)^2 - n^2] = 0$$

For $k = n$, $(k + 1)^2 - n^2 \neq 0$

So we have $C_1 = 0$ ...(15)

Putting the co-efficient of x^{k+2} to zero, we get

$$C_2 \{(k + 2)(k + 1) + k + 2 - n^2\} + C_0 = 0$$

or $C_2 [(k + 2)^2 - n^2] + C_0 = 0$

or

$$C_2 = -\frac{C_0}{(k+2)^2 - n^2}$$

$$= -\frac{C_0}{(n+2)^2 - n^2} \quad \text{for } k = n$$

or

$$= -\frac{C_0}{(2n+2)(2)} = -\frac{C_0}{(n+1)2^2} \quad \text{...(16)}$$

Putting the co-efficient of x^{k+3} to zero, we get

$$C_3 [(k + 3)^2 - n^2] + C_1 = 0$$

or

$$C_3 = -\frac{C_1}{(n+3)^2 - n^2} = 0, \text{ as } C_1 = 0$$

Putting the co-efficient of x^{k+4} to zero, we get

$$C_4 [(k + 4)^2 - n^2] + C_2 = 0$$

or

$$C_4 = -\frac{C_2}{(n+4)^2 - n^2}$$

$$= -\frac{C_2}{(2n+4)(4)}$$

$$= -\frac{C_2}{(n+2)2, 2^2} = \frac{(-1)^2 \quad C_2}{(n+1)(n+2)1.2(2)^4}$$

Proceeding in the same way we get

$$C_1 = 0 = C_3 = C_5 = C_7 = \ldots \quad \text{...(17)}$$

And

$$C_{2k} = \frac{(-1)^k \quad C_0}{(n+1)(n+2)\ldots1.2\ldots(2k) \quad 2^{2k}} \quad \text{for } 1, 2, 3 \quad \text{...(18)}$$

Notes

So $$C_6 = \frac{(-1)^3 \quad C_0}{(n+1)(n+2)(n+3)\underline{|3}\,(2)^6}$$

$$C_8 = \frac{(-1)^4 \quad C_0}{(n+1)(n+2)(n+3)(n+4)\underline{|4}\,(2)^8}$$

$$\begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

Substituting the values of k, C_0, C_1, C_2,... in equation for y we get

$$y = x^n \left\{ C_0 - \frac{k_0}{(n+1)\cdot 1}\left(\frac{x}{2}\right)^2 + \frac{k_0}{(n+1)(n+2)\underline{|2}}\left(\frac{x}{2}\right)^4 \ldots. \right\}$$

$$= C_0\, x^n \left\{ 1 - \frac{1}{(n+1)}\frac{1}{1}\left(\frac{x}{2}\right)^2 + \frac{1}{(n+1)(n+2)\underline{|2}}\left(\frac{x}{2}\right)^4 \ldots. \right\} \qquad \ldots(19)$$

If we now take C_0 to be

$$C_0 = \frac{1}{2^n\,\Gamma(n+1)} \qquad \ldots(20)$$

Where $\Gamma(n)$ is a gamma function.

As you know the properties of gamma functions $n\,\Gamma(n) = \Gamma(n+1)$, for any value of n, so we get various values of $\Gamma(n)$. The equation for y becomes

$$y = \frac{x^n}{2^n\,\Gamma(n+1)}\left\{ 1 - \frac{1}{(n+1)\cdot 1}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{(n+1)(n+2)1.2}\left(\frac{x}{2}\right)^4 + \ldots.. \right\}$$

$$= \left\{ \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n - \frac{1}{\Gamma(n+2)\cdot 1}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{\Gamma(n+3)\cdot\underline{|2}} - \left(\frac{x}{2}\right)^4 + \ldots \right\}$$

or $$y = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+1+s)\underline{|s}}\left(\frac{x}{2}\right)^{n+2s} \qquad \ldots(21)$$

Here we have used the fact that

$(n + 1)\,\Gamma(n + 1) = \Gamma(n + 2)$,

$(n + 2)\,\Gamma(n + 2) = \Gamma(n + 3)$ and so on.

Also 1, 2, 3... $s = \underline{|s} = \Gamma(s+1)$

The above solution is called Bessel's function $J_n(x)$. Thus

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+1+s)}\,\frac{1}{\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s} \qquad \ldots(22)$$

For $k = -n$ and if n is not an integer then the other solution for $k = -n$ is obtained from the equation of $J_n(x)$ by replacing $n \to -n$ i.e.

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\,\Gamma(s+1)} \left(\frac{x}{2}\right)^{2s-n} \qquad \text{...(23)}$$

Thus the general solution of Bessel's equation is

$$y = A\, J_n(x) + B\, J_{-n}(x) \qquad \text{...(24)}$$

Where A, B are arbitrary constants.

Example: Proceeding as above shown that for $n = 0$

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(\underline{|2})^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(\underline{|3})^2}\left(\frac{x}{2}\right)^6 + \ldots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \qquad \text{...(25)}$$

Prove for integer n

$$J_{-n}(x) = J_n(x)\,(-1)^n \qquad \text{...(26)}$$

To prove this consider the expression for $J_n(x)$ i.e.

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+s+1)\Gamma(s+1)} \left(\frac{x}{2}\right)^{n+2s}$$

Thus
$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)} \left(\frac{x}{2}\right)^{2s-n}$$

$$= \sum_{s=0}^{n-1} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)} \left(\frac{x}{2}\right)^{2s-n} + \sum_{s=n}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)} \left(\frac{x}{2}\right)^{2s-n} \qquad \text{...(27)}$$

In the first term we have the argument of

$$\Gamma(s + 1 - n),$$

To be negative i.e.

$$S + 1 - n$$

is $-ve$ for $s = 0$ to $n - 2$ and it is zero for $s = n - 1$. From the properties of gamma functions

$$\Gamma(s + 1 - n) \text{ is } \infty \text{ for } s + 1 - n \le 0 \qquad \text{...(28)}$$

So the first series for $J_{-n}(x)$ is zero and the expression for $J_{-n}(x)$ becomes

$$J_{-n}(x) = \sum_{s=n}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)} \left(\frac{x}{2}\right)^{2s-n}$$

Putting $S = r + n$, we have for

$s = n, n + 1, \ldots \infty$

$r = 0, 1, 2, \ldots \infty$

Thus
$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^{r+n}}{\Gamma(r+n+1-n)\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2r+n}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\Gamma(r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

or $\quad J_{-n}(x) = (-1)^n J_n(x) \qquad ...(26)$

Thus $J_{-n}(x)$ is not independent of $J_n(x)$

1.3.2 Solution of Bessel's Differential Equation when *n* is a Non-negative Integer

We had seen that when n is not an integer there are two independent solutions i.e. $J_n(x)$ and $J_{-n}(x)$.

When n is a non-negative integer

$$J_{-n}(x) = (-1)^n J_n(x) \qquad ...(26)$$

And so it is dependent on $J_n(x)$. To find a second solution we introduce Neumann Function

$$Y_v(x) = \frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v} \qquad ...(29)$$

If v is not an integer, then $Y_v(x)$ and $J_v(x)$ form a general solution of the Bessel's equation. If v is a non-negative integer, then from equation (26), equation (29) becomes an indeterminate form. To calculate the limit of (29) for $v \to n$, differentiate both the numerator and denominator with respect to v. Then setting $v \to n$, we have

$$Y_n(x) = \lim_{\upsilon\to n} Y_\upsilon(x) = \lim_{\upsilon\to n} \frac{-\pi\sin\pi_\upsilon J_\upsilon(x) + \cos\pi\upsilon J'_\upsilon(x) - J'_{v}(x)}{\pi\cos\pi\upsilon}$$

$$= \frac{1}{\pi}\frac{\partial J_\upsilon(x)}{\partial\upsilon}\bigg)_{\upsilon=n} - \frac{(-1)^n}{\pi}\frac{\partial J_{-\upsilon}(x)}{\partial\upsilon}\bigg)_{\upsilon=n} \qquad ...(29a)$$

Now from equation (21)

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)s}{\Gamma(\upsilon+s+1)\lfloor s} \left(\frac{x}{2}\right)^{\upsilon+2s}$$

$$\therefore \quad \frac{\partial J_\upsilon(x)}{\partial\upsilon} = \sum_{s=0}^{\infty} \frac{(-1)s}{\lfloor s}\left(\frac{x}{2}\right)^{2s} \left\{ \frac{\left(\frac{x}{2}\right)^\upsilon \log\frac{x}{2}}{\Gamma(\upsilon+s+1)} - \frac{\Gamma'(\upsilon+s+1)}{[\Gamma(u+s+1)]^2}\left(\frac{x}{2}\right)^\upsilon \right\}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)s}{\lfloor s}\frac{(x/2)^{\upsilon+2\upsilon}}{\Gamma(\upsilon+s+1)} \left\{\log\left(\frac{x}{2}\right) - \Psi(\upsilon+s+1)\right\}]$$

where

$$\psi(\upsilon+s+1) = \frac{\Gamma'(\upsilon+s+1)}{\Gamma(\upsilon+s+1)} \qquad ...(30)$$

thus $$\lim_{\upsilon\to n}\frac{\partial J_\upsilon(x)}{\partial\upsilon} = \sum_{s=0}^{\infty} \frac{(-1)^s\left(\frac{x}{2}\right)^{n+2s}}{\lfloor s\,\Gamma(n+s+1)} \left[\log\left(\frac{x}{2}\right) - \psi(n+s+1)\right] \qquad ...(31)$$

The expression for $J_{-n}(x)$ is from (27)

$$J_{-\upsilon}(x) = \sum_{s=0}^{n-1} \frac{(-1)^s}{\Gamma(s+1-\upsilon)\Gamma(s+1)} \left(\frac{x}{2}\right)^{2s-\upsilon} + \sum_{s=n}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(s+1-\upsilon)} \left(\frac{x}{2}\right)^{2s-\upsilon} \quad ...(27)$$

As you know from the properties of gamma functions

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad ...(32)$$

From (27) we obtain

$$\frac{\left(\frac{x}{2}\right)^{-\upsilon+2s}}{\Gamma(-\upsilon+s+1)} = \frac{\left(\frac{x}{2}\right)^{-\upsilon+2s} \Gamma(\upsilon-s)\sin(\upsilon-s)\pi}{\pi} \quad ...(33)$$

Differentiating (33), we see that for $0 \le s \le n$

$$\frac{d}{d\upsilon}\left\{\frac{(x/2)^{\upsilon+2s}\,\Gamma(\upsilon-s)\sin(\upsilon-s)\pi}{\pi}\right\}\Bigg|_{\upsilon=n} = \left[\left(\frac{1}{2}x\right)^{-\upsilon+2s} \Gamma(\upsilon-s)\left\{\pi^{-1}\psi(\upsilon-s)\sin(\upsilon-s)\pi + \right.\right.$$

$$\left.\left. + \cos(\upsilon-m)\pi - \pi^{-1}\log(x/2)\sin(\upsilon-s)\pi\right]\right._{\upsilon=n}$$

$$= \left(\frac{x}{2}\right)^{-n+2m} \Gamma(n-m)\cos(n-m)\pi$$

Therefore as $\upsilon \to n$, $\dfrac{\partial J_{-\upsilon}(x)}{\partial x}$ tends to

$$\sum_{s=0}^{n-1} \frac{(-1)^n\ \Gamma(\upsilon-s)(x/2)^{-n+2s}}{\Gamma(s+1)} + \sum_{s=n}^{\infty} \frac{(-1)^s (x/2)^{-\upsilon+2s}}{\underline{|s}\,\Gamma(-n+s)} \{-\log(x/2) + \psi(-n+s+1)\}$$

$$= (-1)^n \sum_{s=0}^{n-1} \frac{(n-s-1)}{\underline{|s}} \left(\frac{x}{2}\right)^{-n+2s} + (-1)^{n-1} \sum_{s=n}^{\infty} (-1)^m (\tfrac{1}{2}x)^{-n+2s} \left[\log\frac{x}{2} - \psi(s+1)\right] \quad ...(34)$$

Using (31) and (34) we get for Neumann Function $Y_n(x)$ with n being a non-negative integer the following

$$Y_n(x) = \frac{2}{\pi} J_n(x) \log\frac{x}{2} - \frac{1}{\pi} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \left\{\frac{\psi(s+1)+\psi(n+s+1)}{s!(n+s)!}\right\}$$

$$-\frac{1}{\pi} \sum_{s=0}^{n-1} \frac{(n-1-s)!}{s!} \left(\frac{x}{2}\right)^{-n+2s} \quad ...(35)$$

For $n = 0$, the last term does not appear. Thus $J_n(x)$ and $Y_n(x)$ form the general solution.

Thus we see that the Neumann Function $Y_n(x)$ defined by the relation

$$Y_n(x) = \frac{J_\upsilon(x)\cos\pi\upsilon - J_{-\upsilon}(x)}{\sin\pi\upsilon}$$

converges uniformly to $Y_n(x)$ given by equation (35) as $v \to n$ is any bounded closed domain in the complex x plane except for the origin. Formula (35) for $Y_n(x)$ is known as Hankel Formula.

Notes

Hankel Functions: The Hankel Function, or the Bessel Functions of the third kind are defined by

$$H_\upsilon^{(1)}(x) = J_\upsilon(x) + i\,Y_\upsilon(x)$$

$$H_\upsilon^{(2)}(x) = J_\upsilon(x) + i\,Y_\upsilon(x)$$

Prove that

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \qquad \text{...(29)}$$

Proof: $J_n(x)$ is given by

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+1+s)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s+n}$$

So $$J_{1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+3/2)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s+1/2}$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+3/2)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s}$$

Expanding

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma(3/2)\Gamma(1)}\left(\frac{x}{2}\right)^{o} - \frac{1}{\Gamma(5/2)\Gamma(2)}\left(\frac{x}{2}\right)^{2} + \frac{1}{\Gamma(7/2)\Gamma(3)}\left(\frac{x}{2}\right)^{4} \ldots\right\}$$

or $$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma(3/2)}\left\{1 - \frac{1}{3/2 \cdot \Gamma(2)}\left(\frac{x}{2}\right)^{2} + \frac{1}{3/2 \cdot 5/2 \cdot \Gamma(3)}\left(\frac{x}{2}\right)^{4} - \ldots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma(1/2)\Gamma(1/2)}\left\{1 - \frac{x^2\,(2)}{3 \cdot 2^2} + \frac{2.2}{1 \cdot 2 \cdot 3 \cdot 5}\,\frac{x^4}{(2)^4} - \ldots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{(1/2)\Gamma(1/2)}\left\{1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \ldots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{2}{\Gamma(1/2)}\left(\frac{1}{x}\right)\left\{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots\right\}$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{2}{x}\right)^{1/2} \sin x$$

Here $\Gamma(1/2) = \sqrt{\pi}$

Self Assessment

1. Prove that

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

Show that when n is any integer positive or negative

$$J_n(-x) = (-1)^n J_n(x)$$

The expression for $J_n(x)$ is given by

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s}$$

Case I let n be a positive integer. Replacing $x \to -x$ in the above equation we have

$$J_n(-x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s+1)!}\left(\frac{-x}{2}\right)^{n+2s}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s\,(-1)^{n+2s}}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s\,(-1)^{2s}}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n J_n(x) \qquad \left[as\ (-1)^{2s} = 1\right]$$

Thus $\quad J_n(-x) = (-1)^n J_n(x)$

1.4 Recurrence Formulas for $J_n(x)$

Some of the recurrence relations involving Bessel functions are as follows:

I. **$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$,**

where $J_n'(x) = \frac{d}{dx} J_n(x)$

To prove the above relation, we start from the series expansion of $J_n(x)$ as follows:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

Differentiating it w.r.t. x and multiplying by x on both sides, we have

$$x J'_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s\,(n+2s)\,x^{n+2s}}{s!(n+s)!\,2^{n+2s}}$$

$$= n\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} + x\sum_{s=1}^{\infty}\frac{(-1)^s\ (x/2)^{n+2s-1}}{(s-1)!\,(n+s)!}$$

$$= n\,J_n(x) + x\sum_{s=1}^{\infty}\frac{(-1)^s}{(s-1)!(n+s)!}\left(\frac{x}{2}\right)^{n+2s-1}$$

In the last sum, let us replace s by r as

$s = r + 1$, then

$$x\,J'_n(x) = n\,J_n(x) + x\sum_{r=0}^{\infty}\frac{(-1)^{r+1}\left(\frac{x}{2}\right)^{n+1+2r}}{r!(n+1+r)!}$$

or $$x\,J'_n(x) = n\,J_n(x) - x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+1+r)!}\left(\frac{x}{2}\right)^{n+1+2r}$$

$$= n\,J_n(x) - x\quad J_{n+1}(x)$$

As the last sum is equal to $J_{n+1}(x)$. Thus

$$\boldsymbol{x\,J'_n(x) = n\,J_n(x) - x\quad J_{n+1}(x)}$$

II. $\boldsymbol{x\,J'_n(x) + n\,J_n(x) = x\quad J_{n-1}(x)}$

Again, we have

$$x\,J'_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s\ (n+2s)}{s!(n+s)!}\frac{x^{n+2s}}{2^{n+2s}}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s\ (2n+2s-n)}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s\ (2n+2s)}{s!(n+s)!}\left(\frac{x}{2}\right)^{n-1+2s}\cdot\frac{x}{2} - n\,J_n(x)$$

$$= x\sum_{s=0}^{\infty}\frac{(-1)^s\ (n+s)}{s!(n+s)!}\left(\frac{x}{2}\right)^{n-1+2s} - n\,J_n(x)$$

$$= x\sum_{s=0}^{\infty}\frac{(-1)}{s!(n-1+s)!}\left(\frac{x}{2}\right)^{n-1+2s} - n\,J_n(x)$$

{As $(n + s)! = (n + s)(n - 1 + s)!$}

Thus identifying the sum with $J_{n-1}(x)$, we have

$$x\,J'_n(x) = x\,J_{n-1}(x) - n\,J_n(x)$$

Notes

or rearranging terms we have

$$x\,J'_n(x) + n\,J_n(x) = x\,J_{n-1}(x)$$

III. $\mathbf{2n\,J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$

To prove this we just make use of the above two recurrence relations I and II, here we have

$$x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x)$$

and $\quad x\,J'_n(x) + n\,J_n(x) = x\,J_{n-1}(x)$

Substracting we get

$$-n\,J_n(x) = n\,J_n(x) - x\,J_{n+1}(x) - x\,J_{n-1}(x)$$

or $\quad 2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)].$

Again rearranging terms we have

$$2n\,J_n(x) = x\,J_{n+1}(x) + x\,J_{n-1}(x)$$

or $\quad 2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

You can see that relation III is not independent. It depends upon I and II recurring relations.

IV. $\mathbf{2\,J'_n(x) = J_{n-1}(x) - J_{n+1}(x)}$

Hint: Add recurrence relations I and II and simplify the result.

From recurrence relation I, we can show that

$$J'_0(x) = -J_1(x)$$

V. $\dfrac{d}{dx}\left[x^{-n}\,J_n(x)\right] = -x^{-n}\,J_{n+1}(x)$

Now, the left hand side is

$$\begin{aligned}
\frac{d}{dx}\left[x^{-n}\,J_n(x)\right] &= -n\,x^{-n-1}\,J_n(x) + x^{-n}\,J'_n \\
&= x^{n-1}\,[-n\,J_n(x) + x\,J'_n(x)] \\
&= x^{n-1}\,[-n\,J_n(x) + n\,J_n(x) - x\,J_{n+1}(x)] \qquad \{\text{From recurrence relation I}\} \\
&= x^{n-1}\,[-x\,J_{n+1}(x)] \\
&= -x^{-n}\,J_{n+1}(x) = R.H.S
\end{aligned}$$

Self Assessment

2. Prove

$$\frac{d}{dx}\left[x^{n}\,J_n(x)\right] = x^{n}\,J_{n-1}(x)$$

1.5 Generating Function for $J_n(x)$

Notes

Prove that when n is positive integer $J_n(x)$ is the Co-efficient of t^n in the expansion of

$$e^{\frac{x}{2}(t-1/t)} \qquad \text{...(A)}$$

in ascending and descending powers of t. Also show that $J_n(x)$ multiplied by $(-1)^n$ is the co-efficient of t^{-n} in the expansion of the above expression.

Proof:

Expanding $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ in powers of x *i.e.*

$$e^{\frac{x}{2}(t-1/t)} = \left(e\frac{xt}{2}\right)\left(e\frac{-x}{2t}\right)$$

$$= \left\{1+\frac{xt}{2}+\frac{x^2t^2}{\lfloor 2}+\frac{x^3t^3}{3!}+.......\right\}\times\left\{1-\left(\frac{x}{2t}\right)+\left(\frac{-x}{2t}\right)^2,\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}\left(\frac{-x}{2t}\right)^3+.......\right\} \qquad \text{(B)}$$

In the above expansion, collecting the co-efficients of t^n, we have

$$\rightarrow \quad \frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n\cdot 1 - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+1}+\frac{1}{(n+2)!}\frac{1}{\lfloor 2}\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2}-...$$

$$= \frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{(n+2)!\lfloor 2}\left(\frac{x}{2}\right)^{n+4}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s}{\lfloor s\,\lfloor n+s}\left(\frac{x}{2}\right)^{n+2s} \equiv J_n(x) \qquad \text{...(C)}$$

Similarly co-efficients of t^{-n} in the above product is

$$= 1.\ \frac{(-1)^n}{\lfloor n}\left(\frac{x}{2}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}\left(\frac{x}{2}\right)+\frac{\left(\frac{x}{2}\right)(-1)^{n+2}}{\lfloor 2\,(n+2)!}\left(\frac{x}{2}\right)^{n+2}+...$$

$$= (-1)^n\left[\frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n+\frac{(-1)}{\lfloor n+1\,\lfloor 1}\left(\frac{x}{2}\right)^{n+2}+\frac{(-1)^2}{\lfloor 2\ \lfloor n+2}\left(\frac{x}{2}\right)^{n+4}+...\right]$$

$$= (-1)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{\lfloor s\,\lfloor n+s}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n J_n(x)$$

In the above product the co-efficient of t^0 is

$$= 1-\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^4\frac{1}{2^2}-\left(\frac{x}{2}\right)^6\frac{1}{2^2\cdot 3^2}+\left(\frac{x}{2}\right)^8\cdot\frac{1}{2^2\cdot 3^2\cdot 4^2}....$$

$$= 1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\frac{x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}....$$

$$= J_0(x)$$

Notes

Thus in the expansion of t,

$$e\frac{x}{2}\left(t-\frac{1}{t}\right) = J_0 + \left(t-\frac{1}{t}\right)J_1 + \left(t^2+\frac{1}{t^2}\right)J_2 + \left(t^3-\frac{1}{t^3}\right)J_3 + \ldots + \ldots.. + \left(t^n+(1)^n\frac{1}{t^n}\right)J_n + \ldots$$

$$= J_0(x) + t\left[J_1(x)+J_{-1}(x)\right] + t^2\left]\left[J_2(x)+J_{-2}(x)\right] + \ldots$$

$$= \sum_{n=-\infty}^{+\infty} t^n J_n(t)$$

Here we have used the result $J_{-n}(x) = (-1)^n J_n(x)$

(A) ***Trigonometric Expansions involving Bessel's Functions***

Show that

(a) $\cos(x \sin\phi) = J_0(x) + 2\cos 2\phi\, J_2 + 2\cos 4\phi\, J_4 + \ldots.$

(b) $\sin(x \sin\phi) = 2\sin\phi\, J_1 + 2\sin 3\phi\, J_3 + \ldots.$

(c) $\cos(x\cos\phi) = J_0 - 2\cos\phi\, J_2 + 2\cos 4\phi\, J_4 - \ldots.$

(d) $\sin(x\cos\phi) = 2\cos\phi\, J_1 - 2\cos 3\phi\, J_3 + 2\cos 5\phi\, J_5 + \ldots$

(e) $\cos x = J_0 - 2J_2 + 2J_3 - 2J_4 + \ldots$

(f) $\sin x = 2J_1 - 2J_3 + 2J_5 \ldots.$

Proof: We know from generating function that

$$e\frac{x}{2}\left(t-\frac{1}{t}\right) = \sum_{n=\infty}^{+\infty} t^n J_n(x)$$

$$= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=-1}^{\infty} t^n J_n(x)$$

$$= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=-1}^{\infty} t^{-n} J_{-n}(x)$$

$$= J_0 + \sum_{n=1}^{\infty}(t^n + (-1)^n t^{-n})J_n(x) \qquad \{\text{since } J_{-n}(x) = (-1)\, J_n(x)\}$$

Thus,

$$e^{\frac{x}{2}}\left(t-\frac{1}{t}\right) = J_0 + (t-t^{-1})J_1 + (t^2+t^{-2})J_2 + (t^3-t^{-3})J_3 + \qquad \ldots\text{(i)}$$

Let us put $t = e^{i\phi}$, $t^n = e^{in\phi}$

then (i) becomes $e^{\frac{x}{2}}(e^{i\phi}-e^{-i\phi}) = J_0 + (e^{i\phi}-e^{-i\phi})J_1 + (e^{2i\phi}+e^{2i\phi})J_2 + (e^{3i\phi}-e^{-3i\phi})J_3 + \ldots \qquad$ (ii)

Since $\quad \cos n\phi = \frac{1}{2}\left(e^{in\phi}+e^{-in\phi}\right)$

$$\sin n\phi = \frac{1}{2i}\left(e^{in\phi}-e^{-in\phi}\right)$$

So (ii) may be written as

$$e^{ix \sin\phi} = J_0 + 2^i \sin\phi\, J_1 + 2\cos 2\phi\, J_2 + 2\, i \sin 3\phi\, J_3 + \ldots \quad \text{...(iii)}$$

comparing real and imaginary part on both sides

we have

(a) $\cos(x\sin\phi) = J_0 + 2\cos 2\phi\, J_2 + 2\cos 4\phi\, J_4 + \ldots$ ...(iv)

(b) $\sin(x\sin\phi) = 2\sin\phi\, J_1 + 2\sin 3\phi\, J_3 + \ldots$ ...(v)

Replacing ϕ by $\pi/2 - \phi$ in (iv) and (v) and using $\sin\phi \to \sin(\pi/2 - \phi) = \cos\phi$, we get

(c) $\cos(x\cos\phi) = J_0 - 2\cos 2\phi\, J_2 + 2\cos 4\phi\, J_4 \ldots$ ...(vi)

(d) $\sin(x\cos\phi) = 2\cos\phi\, J_1 - 2\cos 3\phi\, J_3 + 2\cos 5\phi\, J_5 \ldots$ ...(vii)

Replacing ϕ by 0 in (iv) and (vii) we get

(e) $\cos x = J_0 - 2J_2 + 2J_4 \ldots$ ...(viiii)

and

(f) $\sin x = 2J_1 - 2J_3 + 2J_5 \ldots$ ...(ix)

1.6 On the Zeros of Bessel Functions $J_n(x)$

We know that Bessel function $J_n(x)$ satisfies the equation

$$x^2 \frac{d^2 J_n(x)}{dx^2} + x\frac{d J_n(x)}{dx} + (x^2 - n^2) J_n(x) = 0$$

Here n is a positive integer

let us put $x = \lambda\nu$,

$$\frac{dJ_n}{dx} = \frac{1}{\lambda}\frac{dJ_n}{d\nu}$$

$$\frac{d^2 J_n}{dx^2} = \frac{1}{\lambda^2}\frac{d^2 J_n}{d\nu^2}$$

So equation (1) becomes

$$\nu^2 \frac{d^2 J_n(\lambda\nu)}{d\nu^2} + \nu\frac{d J_n(\lambda\nu)}{d\nu} + (\lambda^2\nu^2 - n^2) J_n(\lambda\nu) = 0 \quad \text{...(ii)}$$

which may be written as

$$\frac{d}{d\nu}\left[\nu\frac{d J_n(\lambda\nu)}{d\nu}\right] + \left[\frac{-n^2}{\nu} + 2\lambda\nu\right] J_n(\lambda\nu) = 0 \quad \text{..(iii)}$$

let us put $R = \nu$, $P = \nu$, $Q = -\dfrac{n^2}{\nu}$

Then $$\frac{d}{d\nu}\left(R\frac{d J_n(\lambda\nu)}{d\nu}\right) + \left[Q + 2\lambda p\right] J_n(\lambda\nu) \quad \text{...(iv)}$$

Notes

Here due to $R = 0$, it can be shown that for some a i.e. $0 \le x \le a$, $J_n(\lambda\nu)$ satisfies the Boundary Condition $J_n(\lambda a) = 0$ ...(v)

And so the solutions of (iii) form an orthonormal set w.r.t. weight function $P = \nu$.

So zeros of $J_n(\lambda\nu)$ if denoted by $\propto_{in}$ $i = 1, 2,...$

Let

$$\alpha_{1a} < \alpha_{2a} < \alpha_{3a} ... \alpha_{m} ....$$

So $$\lambda a = a_{mn}$$

thus $$\lambda = \frac{\alpha_{mn}}{a} \equiv \lambda_{mn}$$

Since both J_n and $\frac{dJ_n}{d\nu}$ are continuous at $\nu = 0$, therefore for each fixed $n = 0, 1, 2...$ the Bessel function $J_n(\lambda_{mn})$ $(m = 1, 2,...)$ with $\lambda_{mn} = \frac{\alpha_m}{a}$, form an orthogonal set on the internal $0 \le x \le a$ w.r.t. weight $P = \nu$ i.e.

$$\int_0^a \nu\, J_n(\lambda_{mn}\,\nu) J_n(\lambda_{pn}\,\nu) = 0 \text{ for } p \ne m$$

So zeros of $J_n(x)$ are useful in obtaining orthogonal properties of $J_n(x)$. The details of the above discussion will be given in the later units.

Example: Prove that $J_n(x) = 0$ has no repeated roots except at $x = 0$.

Solution: If possible let $\propto$ be a repeated root of

$$J_n(x) = 0 \quad \text{at} \quad x = \alpha \qquad ...(i)$$

Thus $$J_n(\alpha) = 0 \quad \text{as well as} \quad J'_n(\alpha) = 0 \qquad ...(ii)$$

Now from recurrence formulae I and II,

$$x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x),$$

$$x\,J'_n(x) + n\,J_n(x) = x\,J_{n-1}(x),$$

We have

$$J_{n+1}(x) = \frac{n}{x}\,J_n(x) - J'_n(x) \qquad ...(iii)$$

$$J_{-n-1}(x) = \frac{n}{x}\,J_n(x) + J'_n(x) \qquad ...(iv)$$

As $J_n(x) = 0$ and $J'_n(\alpha) = 0$, we have from III and IV $J_{-n+1}(\alpha) = 0$ and $J_{n-1}(\alpha) = 0$, i.e. for the same value of $x = \alpha$, $J_n(x)$, $J_{n+1}(x)$, $J_{n-1}(x)$ are all zero x, which is absurd as we cannot have two power series having the same sum function. Then $J_n(x) = 0$ cannot have repeated roots except $x = 0$.

1.7 Illustrative Examples

Notes

Example 1: Show that

(i) $x \sin x = 2\,(2^2 J_2 - 4^2 J_4 + 6^2 J_6 - ...)$

(ii) $x \cos x = 2\,(1^2 J_1 - 3^2 J_3 + 5^2 J_5 - ...)$

Solution: (i) We know that

$$\cos(x \sin\phi) = J_0 + 2 J_2 \cos 2\phi + 2 J_4 \cos 4\phi + ... \quad ...(i)$$

Differentiating w.r.t. 'φ' we get

$$-\sin(x \sin\phi).\, x\cos\phi = 0 - 2.2\, J_2 \sin 2\phi - 2.4\, J_4 \sin 4\phi\, ... \quad ...(ii)$$

Differentiating (ii) w.r.t. 'φ', we have

$$-\cos(x \sin\phi).\,(x\cos\phi)^2 + \sin(x\sin\phi)\,(x\sin\phi)$$

$$= -2.2^2.J_2 \cos 2\phi - 2.4^2 J_4 \cos 4\phi - 2.6^2 J_0 \cos 6\phi\, ... \quad ...(iii)$$

Replacing φ by π/2 in (iii), we get

$$x \sin x = 2\,(2^2 J_2 - 4^2 J_4 + 6^2 J_6\, ...)$$

(ii) Start with

$$\sin(x\sin\phi) = 2J_1 \sin\phi + 2J_3 \sin 3\phi + ...$$

Differentiate this twice w.r.t. 'φ' as in part (i) and then replace φ by π/2. Thus we can get the required answer.

Example 2: Show that when n is integral

(a) $\pi J_n = \int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$

(b) $\pi J_0 = \int_0^\pi \cos(x\cos\phi)\,d\phi$

$$= \int_0^\pi \cos(x\sin\phi)\,d\phi$$

and hence deduce that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ...$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r.r!)^2}$$

Solution: We know that

$$\cos(x\sin\theta) = J_0 + 2J_2 \cos 2\theta + ... + 2J_{2m} \cos 2m\phi + ... \quad ...(i)$$

and $\sin(x\sin\theta) = 2\sin\theta.J_1 + 2\sin 3\theta\, J_3 + ...$

$$+2J_{2m+1} \sin(2m+1)\,\theta + ... \quad ...(ii)$$

Notes

Multiplying both sides of (i) by cos $2m\theta$ and then integrating between the limits 0 to π, we get

$$\int_0^\pi \cos(x\sin\theta)\cos 2m\theta\, d\theta$$

$$= J_0\int_0^\pi \cos 2m\theta\, d\theta + 2J_2\int_0^\pi \cos 2\theta \cos 2m\theta\, d\theta + ... + 2J_{2m}\int_0^\pi \cos^2 2m\theta\, d\theta + ...$$

$$= 0 + 0 + ... + J_{2m}\int_0^\pi (1+\cos 4m\theta)\, d\theta + ...$$

$$= \pi J_{2m}.$$

Similarly, we can prove that

$$\int_0^\pi \cos(x\sin\theta)\cos(2m+1)\theta\, d\theta = 0$$

Again multiplying both sides of (ii) by sin $(2m + 1)\theta$ and then integrating between the limits 0 to π, we get

$$\int_0^\pi \sin(x\sin\theta)\sin(2m+1)\theta\, d\theta$$

$$= 2J_1\int_0^\pi \sin\theta.\sin(2m+1)\theta\, d\theta + 2J_2\int_0^\pi \sin 3\theta \sin(2m+1)\theta\, d\theta +$$

$$+ ... + 2J_{2m+1}\int_0^\pi \sin^2(2m+1)\theta\, d\theta + ...$$

$$= 0 + 0 + ... + 2J_{2m+1}\int_0^\pi \{1-\cos 2(2m+1)\theta\}\, d\theta + ...$$

$$= J_{2m+1}\ [\theta]_0^\pi = \pi J_{2m+1}$$

Similarly,

$$\int_0^\pi \sin(x\sin\theta)\sin 2m\theta\, d\theta = 0$$

Therefore

$$\int_0^\pi \cos(2m\theta - x\sin\theta)\, d\theta = \int_0^\pi \cos 2m.\cos(x\sin\theta) d\theta$$

$$+ \int_0^\pi \sin 2m\theta.\sin(x\sin\theta)\, d\theta$$

$$= \pi J_{2m}$$

Also **Notes**

$$\int_0^{\pi} \cos\left[(2m+1)\theta - x\sin\theta\right] d\theta$$

$$= \int_0^{\pi} \cos(2m+1)\theta . \cos(x\sin\theta)\, d\theta + \int_0^{\pi} \sin(2m+1)\theta \sin(x\sin\theta)\, d\theta$$

$$= \pi J_{2m+1}$$

Hence for all positive integral n, we get

$$\int_0^{\pi} \cos(\pi\theta - x\sin\theta)\, d\theta = \pi J_n.$$

If n is negative, say $n = -m$, where m is positive, then

$$\int_0^{\pi} \cos(\pi\theta - x\sin\theta)\, d\theta$$

$$= \int_0^{\pi} \cos(-m\theta - x\sin\theta)\, d\theta$$

$$= -\int_{\pi}^{0} \cos\left\{-m(\pi-\phi) - x\sin(\pi-\phi)\right\} d\pi \qquad \text{Putting } \theta = \pi - \phi$$

$$= \int_0^{\pi} \cos\left\{-m\pi + (m\phi - x\sin\phi)\right\} d\phi$$

$$= \int_0^{\pi} \{\cos m\pi \cos(m\phi - x\sin\phi) + \sin m\pi \sin(m\phi - x\sin\phi)\, d\theta$$

$$= (-1)^m \int_0^{\pi} \cos\,(m\phi - x\sin\phi)\, d\phi$$

$$= (-1)^m \pi\, J_m(x) \qquad \text{Since } J_{-m}(x) = (-1)^m\, J_m(x)$$

$$= \pi\, J_n(x)$$

Hence for all integral values of n

$$\int_0^{\pi} \cos\,(n\theta - x\sin\theta)\, d\theta = \pi J_n$$

(b) Putting $\theta = \pi/2 + \phi$ in the value of $\cos(x \sin\theta)$ from (i), we have

$$\cos(x\cos\phi) = J_0 - 2J_2 \cos 2\phi + 2J_4 \cos 4\phi - ...$$

$$\therefore \qquad \int_0^{\pi} \cos\,(x\cos\phi)\, d\phi = J_0 \int_0^{\pi} d\theta - 2J_2 \int_0^{\pi} \cos 2\phi\, d\phi + ...$$

$$= \pi\, J_0$$

Notes

From (i) we have

$$\cos(x \sin\phi) = J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + ...$$

$$\therefore \quad \int_0^{\pi} \cos(x \sin\phi) d\phi = J_0 \int_0^{\pi} d\phi + 2J_2 \int_0^{\pi} \cos 2\phi \, d\phi + ...$$

$$= \pi J_{0.}$$

Deduction: We have to prove that

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \cos\phi) d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{x^2\cos^2\phi}{2!} + \frac{x^4\cos^4\phi}{4!} - \frac{x^6\cos^6\phi}{6!} + ...\right) d\phi \qquad ...(iii)$$

Since $\int_0^{\pi} \cos^{2r}\phi \, d\phi = \frac{1.3.5...(2r-1)}{2.4.6...(2r)} \cdot \pi$

from definite integrals.

$\therefore$ from (iii), we get

$$J_0(x) = \frac{1}{\pi}\left[\pi - \frac{x^2}{2!}\cdot\frac{1}{2}\pi + \frac{x^4}{4!}\cdot\frac{1.3}{2.4}\pi - \frac{x^6}{6!}\cdot\frac{1.3.5}{2.4.6}\pi + ...\right]$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + ...$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot (2!)^2} - \frac{x^6}{2^6 \cdot (3!)} + ...$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r.r!)}$$

Self Assessment

3. Verify directly from the representation

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\phi) d\phi$$

that $J_0(x)$ satisfies Bessel's equation in which $n = 0$

Example 3: Prove

$$\int_0^{\infty} e^{-ax} j_0(bx) dx = \frac{1}{\sqrt{\{(a^2+b^2)\}}}, a > 0.$$

Solution: From example above, we have

Notes

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\phi)dx$$

$$\therefore \int_0^{\infty} e^{-ax}j_0(bx)dx = \int_0^{\infty} e^{-ax}\left\{\frac{1}{x}\int_0^{\pi}\cos(bx\sin\phi)d\phi\right\}dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\int_0^{\infty} e^{-ax}\cos(bx\sin\phi)dx\right]d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\int_0^{\infty} e^{-ax}\frac{e^{i(bx\sin\phi)}+e^{-i(bx\sin\phi)}}{2}dx\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\int_0^{\infty}\left\{e^{-(a-ib\sin\phi)x}+e^{-(a+ib\sin\phi)x}\right\}dx\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{e^{-(a-ib\sin\phi)x}}{-(a-ib\sin\phi)}-\frac{e^{-(a+ib\sin\phi)x}}{(a+ib\sin\phi)}\right]_0^{\infty}d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{a-ib\sin\phi}+\frac{1}{a+ib\sin\phi}\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{2a\,d\phi}{a^2+b^2\sin^2\phi}$$

$$= 2.\frac{a}{\pi}\int_0^{\pi/2}\frac{\mathrm{cosec}^2\phi\,d\phi}{b^2+a^2\,\mathrm{cosec}^2\phi}$$

$$= 2.\frac{a}{\pi}\int_0^{\pi/2}\frac{\mathrm{cosec}^2\phi\,d\phi}{(a^2+b^2)+a^2\cot^2\phi}$$

$$= 2.\frac{a}{\pi}\left[\frac{1}{a\sqrt{\{(a^2+b^2)\}}}\cot^{-1}\frac{a\cot\phi}{\sqrt{\{(a^2+b^2)\}}}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi\sqrt{\{(a^2+b^2)\}}}\left[\cot^{-1}0-\cot^{-1}\infty\right]$$

$$= \frac{1}{\sqrt{\{(a^2+b^2)\}}}$$

Notes

Example 4: Using generating function or otherwise, show that

$$J_n(-x) = (-1)^n J_n(x)$$

Solution: We have

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$$

Replacing x by $-x$ in (i), we get

$$\sum_{n=-\infty}^{\infty} J_n(-x) z^n = e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{x}{2}\left(-z-\frac{1}{-z}\right)}$$

$$= \sum_{n=-\infty}^{\infty} J_n(x).(-z)^n \qquad \text{[by (i)]}$$

$$\sum_{n=-\infty}^{\infty} J_n(-x) z^n = \sum_{n=-\infty}^{\infty} J_n(x).(-1)^n z^n$$

Equating the coefficient of z^n from both sides of (ii) gives

$$J_n(-x) = (-1)^n J_n(x).$$

Example 5: If $n > -1$, show that

$$\int_0^x x^{n+1} J_n(x) dx = x^n J_{n-1}(x)$$

Solution: From recurrence formula I, we have

$$\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x) \qquad \text{...(i)}$$

Replacing n by $(n + 1)$ in (i), we get

$$\frac{d}{dx}\left\{x^{n+1} J_{n+1}(x)\right\} = x^{n+1} J_n(x) \qquad \text{...(ii)}$$

Integrating (i) w.r.t. 'x' between the limits 0 and x, we get

$$\left[x^{n+1} J_{n+1}(x)\right]_0^x = \int_0^x x^{n+1} J_n(x) dx$$

or $$\int_0^x x^{n+1} J_n(x) dx = x^{n+1} J_{n+1}(x)$$

Notes

Example 6: Show that

(a) $\int_0^x x^{-n} J_{n+1}(x)\,dx = \frac{1}{2^n \Gamma(n+1)'} - x^{-n} J_n,\ n > 1.$

(b) $\int_0^\infty x^{-n} J_{n+1}(x)dx = \frac{1}{2^n \Gamma(n+1)},\ n > -\frac{1}{2}.$

Solution:

(a) From recurrence formula II, we have

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n} J_{n+1}(x)dx \qquad ...(i)$$

Integrating (i) w.r.t. 'x' between the limits 0 and x, we get

$$\left[x^{-n} J_n(x)\right]_0^x = \int_0^x x^{-n} J_{n+1}(x)dx$$

$$\therefore\ x^{-n} J_n(x) - \lim_{x\to 0} \frac{J_n(x)}{x^n} = -\int_0^x x^{-n} j_{n+1}(x)dx \qquad ...(ii)$$

But $\lim_{x\to 0} \frac{J_n(x)}{x^n} = \lim_{x\to 0} \frac{1}{x^n} . \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2.2.(n+1)} + ...\right] = \frac{1}{2^n \Gamma(n+1)}$

Hence (ii) may be written as

$$\int_0^x x^{-n} J_{n+1}(x)dx = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x)$$

(b) Integrating (i) w.r.t. 'from 0 to ∞' we get

$$\left[x^{-n} J_n(x)\right]_0^\infty = -\int_0^\infty x^{-n} J_{n+1}(x)dx$$

$$\therefore\ \lim_{x\to 0} \frac{J_n(x)}{x^n} - \lim_{x\to 0} \frac{J_n(x)}{x^n} = -\int_0^\infty x^{-n} J_{n+1}(x)dx \qquad ...(iii)$$

As in part (a), $\lim_{x\to 0} \frac{J_{n+1}(x)}{x^n} = \frac{1}{2^n \Gamma(n+1)}$ ...(iv)

We know that for large values of x the approximate value of $J_n(x)$ is given by

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left\{x - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right\}, n > -\frac{1}{2}$$

Notes

Using (v), $\lim_{x\to\infty} \frac{J_n(x)}{x^n} = 0$

Using (iv) and (vi), (iii) reduces to

$$\int_0^\infty x^{-n} J_{n+1}(x)dx = \frac{1}{2^n \Gamma(n+1)}$$

1.8 Summary

- Bessel Differential equation is seen to have $x = 0$ as regular singular point
- $x = \infty$ is irregular singular point of the Bessel Differential.
- Bessel Differential equation is deduced from Laplace equation.
- Bessel Differential equation is of Fuchs Type and so Frobenius method of expanding solution of Bessel's equation as power series in x is valid.
- The generating function of Bessel function is given by

$$e^{\frac{x}{z}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{+\infty} t^n J_n(x)$$

- With the help of generating function we obtain recurrence relations
- It is seen that $J_n(x)$ does not have repeated zeroes except at $x = 0$.

1.9 Keywords

Ordinary point of a Differential equation is such that the solution can be expressed in terms of a power series.

Regular singular point $x = x_0$ is such that $p(x)$, $q(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

behave as

$$(x - x_0)p(x) = \text{finite} \lim_{x\to x_0},\ (x - x_0)^2 q(x) = \text{finite} \lim_{x\to x_0}$$

Recurrence relation is a relation involving a few Bessel functions i.e. it involves $J_n(x), J_{n-1}(x), J_{n+1}(x)$ and $\frac{dJ_n(x)}{dx}$.

Generating function is such a function which on expansions gives the values of $J_n(x)$.

Fuchs type differential equation satisfies the properties as given above.

Indicial equation gives the values of the parameter appearing in power series expansion of $J_n(x)$.

1.10 Review Questions

Notes

Prove that:

1. $J_2(x) = \frac{d^2 J_0(x)}{dx^2} - x^{-1}\frac{dJ_0}{dx}(x)$

2. $J_2(x) - J_0(x) = 2\frac{d^2 J_0(x)}{dx^2}$

3. $J_2(x) + 3\frac{dJ_0}{dx} + 4\frac{d^2 J_0}{dx^2}(x) = 0$

4. $2\frac{d}{dx}J_n(x) = J_{n-1}(x) - J_{n+1}(x)$

5. Solve the Differential equation

   $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{9}{4}\right)y = 0$$

   and show that

   $$y = A\ J_{\frac{3}{2}}(x) + B\,J_{\frac{-3}{2}}(x)$$

1.11 Further Readings

G. N. Watson, A Treatise on the Theory of Bessel Functions

Louis A. Pipes and L.R. Harvill, Applied Mathematics for Engineers and Physicists

K. Yosida, Lectures on Differential and Integral Equations

Jai Dev Anand, P.K. Mittal and Ajay Wadhwa, Mathematical Physics Part II

Richa Nandra, Lovely Professional University

Notes

Unit 2: Legendre's Polynomials

CONTENTS

Objectives

After studying this unit, you should be able to:

- Observe that Legendre's differential equation is obtained from the Laplace differential equation
- Obtain the Legendre's polynomial $P_n(x)$ as a power series having x^n as a maximum powerterm for $n > 0$ integer
- See recurrence relations of $P_n(x)$ help in finding all $P_n(x)$ in terms of two or three lower $P_n(x)$.
- See that a generating function is found by which various $P_n(x)$ are found.
- See that orthogonal properties of $P_n(x)$ help in expressing any function $f(x)$ in terms of various $P_n(x)$.

Introduction

The Legendre's polynomials $P_n(x)$ play an important role in potential problems i.e. in electrostatics and gravitational field. It is therefore important to study the properties of $P_n(x)$.

1. First of it is important to study the solution of Legendre's equations so that more insight to $P_n(x)$ can be seen.

Notes

2. Recurrence relations derived in this unit help us in finding unknown $P_n(x)$ in terms of two or three known Legendre polynomial

   - The Legendre's polynomials $P_n(x)$ have zeroes at some $x = x_i, i = 1, 2, ...$ i.e. $P_2(x)$ has two zeroes, $P_3(x)$ has three and so on.
   - Legendre polynomials are quite suited in numerical evaluations of certain integrals.

2.1 Legendre's Differential Equation from Laplace's Equation

Laplace's equation in spherical polar coordinates is

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0 \quad ...(A)$$

Let us put

$$V = r^n F_n(\theta, \phi) \quad ...(B)$$

Here $F_n(\theta, \phi)$ is a function of θ and ϕ. So

$$\frac{\partial V}{\partial r} = n\, r^{n-1}\, F_n$$

$$\frac{\partial V}{\partial\theta} = r^n \frac{\partial F_n}{\partial\theta}$$

$$\frac{\partial^2 V}{\partial\phi^2} = r^n \frac{\partial^2 F_n}{\partial\phi^2}$$

Substituting in Laplace equation, we get

$$\frac{\partial}{\partial r}\left(n\, r^{n+1}\, F_n\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(r^n \sin\theta\frac{\partial F_n}{\partial\theta}\right)+\frac{r^n}{\sin^2\theta}\frac{\partial^2 F_n}{\partial\phi^2} = 0$$

or

$$n(n+1)r^n\, F_n+\frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right)+\frac{r^n}{\sin^2\theta}\frac{\partial F_n}{\partial\phi^2} = 0$$

Dividing by r^n, we have

$$n(n+1)F_n+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 F_n}{\partial\phi^2} = 0 \quad ...(C)$$

Next consider the case when $F_n(\theta,\phi)$ is independent of ϕ, so

$$n(n+1)\, F_n+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right) = 0 \quad ...(D)$$

Let us put the independent variable θ in terms of x given by

$$x = \cos\theta$$

$$\frac{d}{d\theta}F_n = \frac{\partial F_n}{\partial x}\frac{\partial x}{\partial\theta}=-\sin\theta\frac{\partial F_n}{\partial x}$$

Notes

$$\sin\theta\frac{\partial}{\partial\theta}F_n = -\sin^2\theta\frac{\partial F_n}{\partial x}$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}F_n\right) = \frac{\partial}{\partial\theta}\left(-\sin^2\theta\frac{\partial F_n}{\partial x}\right)$$

$$= -2\sin\theta\cos\theta\frac{\partial F_n}{dx}+\sin^3\theta\frac{d^2F_n}{dx^2}$$

Substituting in equation (D) we have

$$n(n+1)F_n - 2\cos\theta\frac{\partial F_n}{\partial x}+\sin^2\theta\frac{d^2F}{dx^2} = 0$$

or $$n(n+1)F_n - 2x\frac{dF_x}{dx}+(1-x^2)\frac{d^2F}{dx^2} = 0$$

Rewriting it as:

$$(1-x^2)\frac{d^2F_n}{dx^2}-2x\frac{dF_n}{dx}+n(n+1)F_n = 0 \qquad \text{...(E)}$$

This equation (E) is known as Legendre's differential equations. The solution of equation (E) for positive integer values of n are known as Legendre Polynomial.

Putting Legendre equation in Fuchs form we have

$$\frac{d^2F_n}{dx^2}-\frac{2x}{(1-x^2)}\frac{dF_n}{dx}+\frac{n(n+1)}{(1-x^2)}F_n = 0 \qquad \text{...(F)}$$

Here let coefficients of $\frac{dF_n}{dx}$ and F_n be

$$\left.\begin{aligned} p(x) &= -\frac{2x}{(1-x^2)} \\ q(x) &= \frac{n(n+1)}{(1-x^2)} \end{aligned}\right\} \qquad \text{..(G)}$$

At $x=1$ and $x=-1$, both $p(x)$ and $q(x)$ have poles of the first order. So the points $x = 1$ and $x = -1$ are regular singular points of the Legendre's equations. Let us investigate the behaviour of the equation for $x=\infty$. For this purpose let us put

$$x = \frac{1}{r},\ F_n = y \qquad \text{...(H)}$$

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dr}{dx} = -r^2\frac{dy}{dr}$$

Notes

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -x^2\frac{d}{dr}\left(-r^2\frac{dy}{dr}\right)$$

$$= r^2\left(2r\frac{dy}{dx}\right) + r^4\frac{d^2y}{dr^2}$$

The equation (F) becomes

$$r^4\frac{d^2y}{dr^2} + \frac{2r^2}{r\left(1-\frac{1}{r^2}\right)}\frac{dy}{dx} - \frac{n(n+1)}{\left(1-\frac{1}{r^2}\right)}y = 0$$

$$r^4\frac{d^2y}{dr^2} + \frac{2r^3}{(x^2-1)}\frac{dy}{dx} - \frac{n(n+1)r^2}{(r^2-1)}y = 0$$

or $$\frac{d^2y}{dr^2} + \frac{2}{r(r^2-1)}\frac{dy}{dr} - \frac{n(n+1)}{r^2(r^2-1)}y = 0 \qquad \text{(I)}$$

Thus $r = 0$ or $x = \infty$ is a regular singular point of the differential equation (Legendre's). Thus we can find a solution of Legendre's equation in terms of a power series in x as well as in powers of $\frac{1}{x}$.

2.1.1 Power Series Solution of Legendre's Equation in Ascending Powers of x

$$(1-x^2)\frac{d^2F_n}{dx^2} - 2x\frac{dF_n}{dx} + n(n+1)F_n = 0 \qquad \text{...(A)}$$

As in the case of Bessel's differential equation we assume a solution of the form:

$$F_n = x^s\sum_{r=0}^{\infty}C_r x^r$$

or $$F_n = \sum_{r=0}^{\infty}C_r x^{r+s} \qquad \text{...(B)}$$

For (B) to be a solution of (A) it is necessary that when equation (B) is substituted into (A), the coefficients of every power of x vanish. So we have

$$(1-x^2)\sum_{r=0}^{\infty}(r+s)(r+s-1)C_r x^{r+s-2} - 2x\sum_{r=0}^{\infty}C_r(r+s)x^{r+s-1} + n(n+1)\sum_{r=0}^{\infty}C_r x^{r+s} \equiv 0$$

or $$\sum_{r=0}^{\infty}\left[(r+s)(r+s-1)C_r(x^{r+s-2} - x^{r+s}) - 2C_r(r+s)x^{r+s} + x^{r+s}n(n+1)C_r\right] \equiv 0$$

or $$\sum_{r=0}^{\infty}\left\{(r+s)(r+s-1)C_r x^{r+s-2} + C_r x^{r+s}[n(n+1) - 2(r+s) - (r+s)(r+s-1)]\right\} \equiv 0$$

$$\sum_{r=0}^{\infty}\left[(r+s)(r+s-1)C_r x^{r+s-2} + C_r x^{r+s}(n-r-s)(n+r+s+1)\right] \equiv 0 \qquad \text{...(C)}$$

Notes

Equating coefficients of x^{r+s-2} we get

$$C_r(r+s)(r+s-1)+(n-r-s+2)(n+r+s-1)C_{r-2} \quad = \quad 0$$

for $r=0,\ 1, ...$ ...(D)

Since the leading term is C_o so that

$C_{-1}=0, C_{-2}=0.$ Thus C_o satisfies

$$C_0(s)(s-1) \quad = \quad 0 \qquad ...(E)$$

Since $C_0 \neq 0,$ so the indicial equation is

$$s(s-1) \quad = \quad 0 \qquad ...(F)$$

giving the value $s=0$ and $s=1.$

Next putting $r=1,$ we have

$$(s+1)s\,C_1 \quad = \quad 0 \qquad ...(G)$$

So for $s=0,$ C_0 and C_1 are both arbitrary. Thus for s = 0, equation (D) becomes

$$C_r \quad = \quad \frac{(n-r+2)(n+r-1)}{r(r-1)}C_{r-2} \qquad ...(H)$$

From equation (H),

$$C_2 \quad = \quad -\frac{n(n+1)}{1.2}C_0$$

$$C_3 \quad = \quad -\frac{(n-1)(n+2)}{3.2}C_1$$

$$C_4 \quad = \quad \frac{-(n-2)(n+3)}{4.3}C_2 = \frac{n(n-2)(n+1)(n+3)}{1.2.3.4}C_0$$

$$C_5 \quad = \quad \frac{-(n-3(n+4)}{5.4}C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{1.2.3.4.5}C_1$$

....

....

....

and so

Substituting the above values of C_{r0} in equation (B) and using s = 0 value we have

$$F_n(x) \quad = \quad C_0\left[1-\frac{n(n+1)}{\underline{|2}}x^2+\frac{n(n-2)(n+1)(n+3)x^4}{1.2.3.4}...\right]+$$

$$+C_1\left[x-\frac{(n-1)(n+2)}{1.2.3}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{1.2.3.4.5}x^4...\right] \qquad ...(I)$$

By applying ratio test it may be shown that above two series converge in the interval (-1, 1)

Notes

As a problem, one can show that for $s = 1$, we can get second series by the above procedure. Since equation (I) contains two arbitrary constants so equation (I) is the general solution of Legendre's equation (A). Now if we give arbitrary coefficients C_0 and C_1 such numerical value that the polynomial (I) becomes equal to one when x is unity, we obtain for n the values 0, 1, 2, 3, ..., and obtain the following system of polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (3x^2-1)/2$$

$$P_3(x) = \frac{1}{2}(5x^3-3x);\ P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \quad ...(J)$$

The general polynomial $P_n(x)$ which satisfies Legendre's equation is given by the series

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r(2n-2r)!}{2^n(n-r)!(n-2r)!} x^{n-2x} \quad ...(K)$$

Where $N = n/2$ for even n and $N = (n-1)/2$ for n odd.

2.1.2 Solution of Legendre's Equation in Descending Powers of x

The Legendre's Equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad ...(A)$$

Let us assume

$$y = \sum_{r=0}^{\infty} C_r x^{s-r} \quad ...(B)$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} (s-r)C_r x^{s-r-1} \quad ...(C)$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (s-r)(s-r-1)\, C_r x^{s-r-2} \quad ...(D)$$

Substituting in (A), we get

$$(1-x^2)\sum_{r=0}^{\infty} C_r(s-r)(s-r-1)x^{s-r-2} - 2x\sum_{r=0}^{\infty} C_r(s-r)x^{s-r-1} + n(n+1)\sum_{r=0}^{\infty} C_r x^{s-r} = 0$$

or
$$\sum_{r=0}^{\infty} C_r \left\{(s-r)(s-r-1)x^{s-r-2} + \left[n(n+1)-(s-r)(s-r+1)\right]x^{s-r}\right\} = 0 \quad ...(E)$$

Simplifying (E) we have

$$\sum_{r=0}^{\infty} C_r \left\{(s-r)(s-r-1)x^{s-r-2} + (n-s+r)(n+s-r+1)x^{s-r}\right\} \equiv 0 \quad ...(F)$$

Equation (F) being identity, we can equate to zero the coefficients of various powers of x. Equating to zero the coefficients of highest powers of x i.e. of x^s, we have

$$C_0(n-s)(n+s+1) = 0 \quad \text{...(G)}$$

Since $C_0 \neq 0$, so the indicial equation is

$$(n-s)(n+s+1) = 0 \quad \text{...(H)}$$

The solutions of equation (H) are

$$s = n \text{ and } s = -n-1 \quad \text{...(I)}$$

Equating to zero the coefficient of the next lower power of x i.e. of x^{s-1}, we have

$$a_1(n-s+1)(n+s) = 0 \quad \text{...(J)}$$

So $a_1 = 0$, as its coefficient is not zero for both $s = n$ and $s = -n-1$.

Again equating to zero the coefficient of the general term i.e. of x^{k-r}, we have

$$C_{s-2}(s-r+2)(s-r+1)+(n-s+r)(n+s-r+1)\,C_r = 0$$

or

$$C_r = -\frac{(s-r+2)(s-r+1)}{(n-s+r)(n+s-r+1)}C_{r-2} \quad \text{...(K)}$$

Putting $r = 2$

$$C_2 = -\frac{(s)(s-1)}{(n-s+2)(n+s-1)}C_0$$

Putting $r = 3$

$$C_3 = -\frac{(s-1)(s-2)C_1}{(n-s+3)(n+s-2)} = 0, \text{ as } C_1 = 0$$

Thus

$$C_1 = C_3 = C_5 = \ldots = 0 \quad \text{(L)}$$

Now there are two values for s i.e.

$$s = n \text{ and } s = -n-1 \quad \text{(I)}$$

We first take $s = n$, then the general recurrence relation (K) becomes

$$C_r = \frac{-(n-r+2)(n-r+1)}{r(2n-r+1)}C_{r-2} \quad \text{(M)}$$

Putting $r = 2, 4, 6, \ldots$ we obtain the coefficients $C_2, C_4, C_6 \ldots$ in terms of C_o i.e.

$$C_2 = -\frac{n(n-1)}{2(2n-1)}C_0$$

Notes

$$C_4 = -\frac{(n-2)(n-3)}{n(2n-3)}C_2$$

$$= -\frac{(n-3)(n-2)(n-1)n}{1.2.4.(2n-3)(2n-1)} - C_0$$

$$C_6 = -\frac{(n-4)(n-5)}{6(2n-5)}C_4$$

$$= -\frac{(n-5)(n-4)(n-3)(n-2)(n-1)n}{2.4.6(2n-5)(2n-3)(2n-1)}C_0$$

....
....
....

Substituting these values of C's in equation (B) we have for $s = n$

$$y = C_0\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{x-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\ (2n-1)(2n-3)}x^{n-4}\ldots\right\} \quad \ldots(N)$$

$$= y_1 \text{ (say)}$$

For the second value of $s = -n-1,$ we have from equation (K)

$$C_r = -\frac{(-n-r+1)(-n-r)}{(2n+r+1)(-r)}C_{r-2}$$

or $$C_r = \frac{(n+r-1)(n+r)}{r(2n+r+1)}C_{r-2} \quad \ldots(O)$$

Putting the values of $r = 2, 4, 6, \ldots$ in equation (O)

$$C_2 = \frac{(n+1)(n+2)}{2(2n+3)}C_0$$

$$C_4 = \frac{(n+3)(n+4)}{4(2n+5)}C_2$$

$$= \frac{(n+4)(n+3)(n+2)(n+1)}{2.4(2n+3)(2n+5)} - C_0$$

....
....
....

Substituting these values of C's in equation (B) we have for $s = -n-1$

$$y = C_0x^{-n-1} + C_2x^{-n-3} + C_4x^{-n-5} + \ldots.$$

$$= C_0\left\{x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + \ldots\right\}$$

$$= y_2 \quad \ldots(P)$$

Notes

So the two solutions of Legendre's equations form the general solution

$$y = A\,y_1 + B\,y_2 \qquad \text{...(Q)}$$

In particular, if we take constant C_0 to be

$$C_0 = \frac{1.3.5...(2n-1)}{n!}$$

in equation (N), we get the solution

$$P_n(x) = \frac{1.3.5-(2n-1)}{\underline{|n}}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)(2n-3)}x^{n-4}...\right\} \qquad \text{...(R)}$$

denoted by $P_n(x)$, and is called Legendre's function of first kind.

Legendre's Functions of the Second Kind

When n is a positive integer and putting the value of C_o, as

$$C_o = \frac{n!}{1.3.5....(2n+1)} \qquad \text{...(S)}$$

in the second solution (P) we get the Legendre's function of the second kind denoted by $Q_n(x)$ i.e.

$$Q_n(x) = \frac{n!}{1.3.5...(2n+1)}\left\{x^{-n-1} + \frac{(n+1)\,(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.\,(2n+3)\,(2n+5)}x^{-n-5} + ...\right\} \qquad \text{...(T)}$$

As is seen from equation (T), $Q_n(x)$ is an infinite or non-terminating series.

Thus the general solution of Legendre's equation is

$$y = A\,P_n(x) + B\,Q_n(x) \qquad \text{...(U)}$$

2.2 Rodrigue's Formula for Legendre Polynomials

An other formula for $P_n(x)$ can be obtained from the Legendre's differential equation. Here we start with

$$u = (x^2 - 1)^n \qquad \text{...(A)}$$

Then
$$\frac{du}{dx} = 2\,nx(x^2-1)^{n-1}$$

Multiplying both sides by $(x^2 - 1)$ and transposing to left hand side, we get

$$(x^2-1)\frac{du}{dx} - 2nx(x^2-1)^n = 0$$

or
$$(x^2-1)\frac{du}{dx} - 2nx\,u = 0$$

Differentiating the above equation with respect to x, we get

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2nu + 2nx\frac{du}{dx} = 0$$

$$(1-x^2)\frac{d^2u}{dx^2} + 2(n-1)x\frac{du}{dx} + 2nu = 0 \qquad \text{...(B)}$$

Notes

We now apply Leibnitz theorem to differentiate equation r times. Here Leibnitz theorem states that the r^{th} differentiation of product of two functions is given by

$$\frac{d^r}{dx^r}(fg) = f\frac{d^r g}{dx^r} + r\left(\frac{df}{dx}\right)\frac{d^{r-1}}{dx^{r-1}}g + \frac{r(r-1)}{\underline{|2}}\frac{d^2 f}{dx^2}\frac{d^{r-2}g}{dx^{n-2}} + ... \quad ...(C)$$

So differentiating equation (B) r times we get

$$(1-x^2)\frac{d^{r+2}u}{dx^{r+2}} + r\frac{d^{r+1}u}{dx^{r+1}}.(-2x) + \frac{r(r-1)}{2}\frac{d^r u}{dx^r}(-2) + 2(n-1)\left[x\frac{d^{r+1}u}{dx^{r+1}} + r.\frac{d^r u}{dx^r}\right] + 2n\frac{d^r u}{dx^r} = 0$$

or rearranging terms

$$(1-x^2)\frac{d^{r+2}u}{dx^{r+2}} + 2x(n-1-r)\frac{d^{r+1}u}{dx^{r+1}} + \frac{d^r u}{dx^r}\{-r(r-1) + 2r(n-1) + 2n\} = 0 \quad ...(D)$$

Simplifying the above equation and putting

$$u_r = \frac{d^r u}{dx^r}, \quad ...(E)$$

We get

$$(1-x^2)\frac{d^2 u_r}{dx^2} + 2x(n-1-r)\frac{du_r}{dx^r} + (r+1)(2n-r)u_r = 0$$

We now put $r = n$ and get

$$(1-x^2)\frac{d^2 u_n}{dx^2} + 2x(-1)\frac{du_n}{dx} + (n+1)(n)u_n = 0$$

This is Legendre's equation. Hence for $r = x$, u_n satisfies Legendre's equation. Thus the Legendre's polynomial are given by

$$P_n(x) = \frac{d^n}{dx^n}(x^2-1)^n(C) \quad ...(F)$$

Where C is a constant. To evaluate C we compare the coefficients of x^n on both sides of (F) i.e.

$$\frac{(2n)!\, x^n}{2^n(n!)^2} = C\frac{d^n}{dx^n}x^{2n} = C(2n)(2n-1)...(n+1)x^{2n}$$

$$= C\frac{(2n)!}{n!}x^n$$

Thus

$$\frac{1}{(n!)2^n} = C$$

Thus

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This is Rodrigue's formula for the Legendre's polynomials. We can again find a few Legendre polynomials from this formula.

Self Assessment

1. Find

   $P_1(x), P_2(x), P_3(x)$

   from Rodrigue formula

2.3 Generating Function for Legendre Polynomials

In the following we will show that $P_n(x)$ is the coefficient of h^n in the expansion of

$$(1-2xh+h^2)^{\frac{-1}{2}}$$

for $|x|\leq|, |h|<|$

i.e.
$$(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \text{...(A)}$$

Now
$$(1-2hx+h^2)^{\frac{-1}{2}} = [1-h(2x-h)]^{\frac{-1}{2}}$$

$$= 1-\frac{1}{2}(-h)(2x-h)+\frac{1}{2}\cdot\frac{3}{2}\frac{1}{2}h^2(2x-h)^2+\ldots$$

$$+\ldots+\frac{1.3\ldots(2n-3)}{2.4.6(2n-2)}h^{n-1}(2x-h)^{n-1}+$$

$$+\frac{1.3.\ldots(2n-1)}{2.4.6\ldots.(2n)}h^n(2x-h)^n+\ldots$$

Therefore the coefficients of h^n are

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}(2x)^2+\frac{1.3.5.\ldots(2n-3)}{2.4.6\ldots(2n-2)}(2x)^{n-2}n-1_{C_1}+\frac{1.3.5\ldots(2n-5)}{2.4.6\ldots(2n-4)}n-2_{C_2}(2x)^{n-4}+\ \ldots \text{(B)}$$

$$= \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{x^n-\frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2}+\frac{(2n)(2n-2)(n-2)(n-3)}{(2n-1)(2n-3)\,\underline{|2}\,2^4}+\ldots\right\}$$

$$= \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-2)}x^{n-4}+\ldots\right\}$$

$$= P_n(x) \qquad \text{...(C)}$$

Thus

$$(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Where $P_n(x)$ is given by

$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{(n-2)(n-1)\,n\,(n-3)}{2.4.(2n-1)(2n-2)}x^{n-4}-\ldots\right\} \qquad \text{...(D)}$$

Also it can be written as

Notes

$$P_n(x) = \sum_{r=0}^{n} \frac{(-1)^r (2n-2r)!}{2^n (n-r)!(n-2r)!} x^{n-2r} \qquad \text{...(E)}$$

Example 1: From the relation

$$(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Obtain $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$.

i.e.

Prove

$$P_0(x) = 1,\ P_1(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

Example 2: Express $P(x) = x^4+2x^3+2x^2-x-3$ in terms of Legendre's polynomials.

Solution: From Example 1, we have

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{(3x^2-1)}{2},$$

$$P_3(x) = \frac{(5x^3-3x)}{2},\ P_4(x) = \frac{35x^4-30x^2+3}{8}$$

from $$P_4(x) = \frac{1}{8}(35x^4-30x^2+3),$$

$$x^4 = \frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35},$$

from $$P_3(x) = \frac{1}{2}(5x^3-3x),\ x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}x,$$

from $$P_2(x) = \frac{1}{2}(3x^2-1),\ x^2 = \frac{2}{5}P_2(x)+\frac{1}{3}$$

and $$x = P_1(x); \quad 1 = P_0(x)$$

Substituting these values, we have

$$P(x) = \frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}+2x^3+2x^2-x-3$$

$$= \frac{8}{35}P_4(x)+2x^3+\frac{20}{7}x^2-x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+2\left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]+\frac{20}{7}x^2-x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}x^2+\frac{1}{5}x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]+\frac{1}{5}x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}x-\frac{224}{105}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}P_1(x)-\frac{224}{105}P_0(x)$$

Example 3: Prove $1+\frac{1}{3}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+...$

$$+... = \log\left[\left(1+\sin\frac{\theta}{2}\right)\Big/\left(\sin\frac{\theta}{2}\right)\right]$$

Solution: From the generating function, we have

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)^{-1/2} \qquad ...(i)$$

Integrating w.r.t. h from 0 to 1, we get

$$\sum_{n=0}^{\infty}\int_0^1 h^n P_n(x)\,dh = \int_0^1 \frac{dh}{\sqrt{(1-2hx+h^2)}} \qquad ...(ii)$$

Replacing x by cos θ on both sides, (ii) gives

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\int_0^1 h^n\,dh = \int_0^1 \frac{dh}{\sqrt{(1-2h\cos\theta+h^2)}}$$

or $$\sum_{n=0}^{\infty} P_n(\cos\theta)\left[\frac{h^{n+1}}{n+1}\right]_0^1 = \int_0^1 \frac{dh}{\sqrt{\left[(h-\cos\theta)^2+\sin^2\theta\right]}}$$

or $$\sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{n+1} = \log(h-\cos\theta)+\sqrt{\left[(h-\cos\theta)^2+\sin^2\theta\right]_0}$$

$$= \log\{(1-\cos\theta)+\sqrt{[(1-\cos\theta)^2+\sin^2\theta]}\}-\log(1-\cos\theta)$$

$$= \log\{(1-\cos\theta)+\sqrt{[2(1-\cos\theta)\}]}-\log(1-\cos\theta)$$

$$= \log\frac{(1-\cos\theta)+\sqrt{2}\sqrt{(1-\cos\theta)}}{(1-\cos\theta)}$$

$$= \log\frac{\sqrt{\{(1-\cos\theta)\}}\sqrt{\{(1-\cos\theta)\}}+\sqrt{2}\sqrt{\{(1-\cos\theta)\}}}{\sqrt{\{(1-\cos\theta)\}}\sqrt{\{(1-\cos\theta)\}}}$$

$$= \log \frac{\sqrt{[(1-\cos\theta)]}+\sqrt{2}}{\sqrt{[(1-\cos\theta)]}}$$

$$= \log \frac{\sqrt{\left\{\left(2\sin^2\frac{\theta}{2}\right)\right\}}+\sqrt{2}}{\sqrt{\left\{\left(2\sin^2\frac{\theta}{2}\right)\right\}}}$$

$$= \log \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\therefore \quad \frac{P_0(\cos\theta)}{1}+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\ldots = \log \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

or
$$1+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\ldots = \log \frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} \qquad [\because P_0(\cos\theta)=1]$$

Example 4: Show that

$$(a)\ P_n(1) = 1$$

$$(b)\ P_n(-x) = (-1)^n P_n(x)$$

Hence deduce that $P_n(-1) = (-1)^n$.

Solution:

(a) We know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$

Putting $x = 1$

$$\sum_{n=0}^{\infty} h^n P_n(1) = (1-2h+h^2)^{-1/2}$$

$$= (1-h)^{-1}$$

$$= 1+h+h^2+\ldots+h^n+\ldots$$

$$= \sum_{n=0}^{\infty} h^n$$

Equating the coefficients of h^n, we get $P_n(1) = 1$.

(*b*) we have

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Now, $$(1+2xh+h^2)^{-1/2} = \{1-2x(-h)+(-h)^2\}^{-1/2}$$

$$= \sum_{n=0}^{\infty}(-h)^n P_n(x)$$

$$= \sum_{n=0}^{\infty}(-1)^n h^n P_n(x) \qquad \text{...(i)}$$

Again $$(1+2xh+h^2)^{-1/2} = \{1-2(-x)+h+h^2\}^{-1/2}$$

$$= \sum_{n=0}^{\infty} h^n P_n(-x) \qquad \text{...(ii)}$$

From (i) and (ii) we have

$$= \sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty}(-1)^n h^n P_n(x)$$

Equating the coefficients of h^n from both sides, we get

$$P_n(-x) = (-1)^n \; P_n(x).$$

Deduction: Putting $x = 1$, we have

$$P_n(-1) = (-1)^n P_n(1)$$

$$= (-1)^n [\because P_n(1) = 1].$$

Example 5: Prove that $(a)\, P_n'(1) = \frac{1}{2}n(n+1)$

$$(b)\; P_n'(-1) = (-1)^{n-1}\frac{1}{2}n(n+1)$$

Solution: $P_n(x)$ satisfies Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \text{ putting } y = P_n(x)$$

$$\because \quad (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad \text{...(i)}$$

(a) Putting $x = 1$, in (i) we have

$$-2P_n''(1) + n(n+1)P_n(1) = 0$$

$$\therefore \quad P_n'(1) = \frac{1}{2}n(n+1)P_n(1)$$

$$= \frac{1}{2}n(n+1) \qquad [\because P_n(1) = 1].$$

(b) Putting $x = -1$ in (i), we get

$$2P_n'(-1)+n(n+1)P_n(-1) = 0$$

or $$P_n'(-1) = -\frac{1}{2}n(n+1)P_n(-1)$$

$$= (-1)^{n-1}.\frac{1}{2}n(n+1) \qquad [\because P_n(-1)=(-1)^n].$$

Example 6: Prove that $P_n(0)=0,$ for n odd and

$$P_n(0) = \frac{(-1)^{n/2}n!}{2^n\{n/2!\}^2}, \text{ for } n \text{ even.}$$

Solution:

(i) We know that

$$P_n(x) = \frac{1.3.5...(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}-...\right]$$

when n is odd, say $n=(2m+1),$ then

$$P_{2m+1}(x) = \frac{1.3.5...\{2(2m+1)-1\}}{(2m+1)!}\times\left[x^{2m+1}-\frac{(2m+1)(2m+1-1)}{2.\{2(2m+1)-1\}}x^{2m+1-2}+...\right]$$

Putting $x=0,$ we get $P_{2m+1}(0)=0,$

i.e., $P_n(0) = 0$ when n is odd.

Also, we have

$$\sum_{n=0}^{\infty}h^nP_n(x) = (1-2xh+h^2)^{-1/2}$$

or $$\sum_{n=0}^{\infty}h^nP_n(0) = (1+h^2)^{-1/2}=\left\{1-(-h)^2\right\}^{-1/2}$$

$$= 1+\frac{1}{2}.(-h^2)+\frac{1.3}{2.4}(-h^2)^2+\frac{1.3.5}{2.4.6}(-h^2)^3+...+\frac{1.3.5...(2r-1)}{2.4...2r}(-h^2)^r+...$$

Hence all powers of h on the R.H.S. are even.

Equating the coefficient of h^{2m} on both sides, we have

$$P_{2m}(0) = \frac{1.3.5...(2m-1)}{2.4.6...2m}(-1)^m$$

$$= (-1)^m\frac{(2m)!}{2^{2m}(m!)^2}$$

i.e. when $n = 2\,m$, then

$$P_n(0) = \frac{(-1)^{n/2}n!}{2^n\{(n/2)!\}^2}$$

Notes

Example 7: Prove that $(1-2xz+z^2)^{\frac{-1}{2}}$ is a solution of the equation

$$z\frac{\partial^2(z\nu)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\nu}{\partial x}\right\} = 0$$

Where

$$\nu = (1-2xz+z^2)^{-1/2}$$

Solution: Let

$$\nu = (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n$$

or

$$z\nu = \sum_{n=0}^{\infty} z^{n+1} P_n$$

$\therefore$

$$z\frac{\partial^2}{\partial z^2}(z\nu) = \sum_{n=0}^{\infty}(n+1)nz^n P_n.$$

Also

$$\frac{\partial\nu}{\partial x} = \sum_{n=0}^{\infty} z^n P_n'$$

$\therefore$

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\nu}{\partial x}\right\} = \frac{\partial}{\partial x}\left[(1-x^2)\sum_{n=0}^{\infty} z^n P_n'\right]$$

$$= (1-x^2)\sum_{n=0}^{\infty} z^n P_n'' - 2x\sum_{n=0}^{\infty} z^n p_n'$$

Substituting this in the L.H.S. of the given equation, we get

$$z.\frac{\partial^2(z\nu)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\nu}{\partial x}\right\} = \sum_{n=0}^{\infty}[(n+1)nz^n P_n + (1-x^2)z^n P_n'' - 2xz^n P_n']$$

$$= \sum_{n=0}^{\infty} z^n[(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n]$$

$= 0$ since P_n is a solution of Legendre's equation.

Self Assessment

2. Show that

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)z^x.$$

Laplace's First Integral for $P_n(x)$: when n is a positive integer. Show that

$$\boldsymbol{P_n(x)} = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}\cos\phi\right]^x d\phi.$$

Proof: From integral calculus, we have

$$\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{[(a^2-b^2)]}},\text{ where } a^2 > b^2 \qquad \text{...(i)}$$

Putting $a = 1 - hx$ and $b = h\sqrt{(x^2 - 1)}$

so that $$a^2 - b^2 = (1 - hx)^2 - h^2(x^2 - 1) = 1 - 2xh + h^2$$

Thus, we have from (i)

$$\pi(1 - 2xh + h^2)^{-1/2} = \int_0^{\pi}\left[1 - hx \pm h\sqrt{(x^2 - 1)}\cos\phi\right]^{-1} d\phi$$

$$= \int_0^{\pi}\left[1 - h\left(x \pm \sqrt{x^2 - 1}\right)\cos\phi\right]^{-1} d\phi$$

$$= \int_0^{\pi}(1 - ht)^{-1} d\phi \qquad \text{where } t = x \pm \sqrt{x^2 - 1}\cos\phi$$

or $$\pi\sum h^n P_n(x) = \int_0^{\pi}(1 - ht + h^2t^2 + ... + h^n t^n + ...)\, d\phi$$

Equating coefficient of h^n we get

$$\pi p_n(x) = \int_0^{\pi} t^n d\phi = \int_0^{\pi}\left[x \pm \sqrt{x^2 - 1}\cos\phi\right]^n d\phi$$

$\therefore$ $$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2 - 1)}\cos\phi\right]^n d\phi$$

Deductions

(i) Putting $x = \cos\theta$ in above relation, we get

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n d\phi.$$

(ii) If we take n = 1 and +ve sign, then we get

$$P_1(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + \sqrt{[(x^2 - 1)]}\cos\phi\right] d\phi.$$

Laplace's Second Integral for $P_n(x)$: When n is a Positive Integer. Show that

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left[x \pm \sqrt{(x^2 - 1)}\cos\phi\right]^{n+1}}$$

Proof: From integral calculus, we have

$$\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{(a^2 - b^2)}}, \text{ where } a^2 > b^2. \qquad ...(i)$$

Let $a = xh - 1$ and $b = h\sqrt{(x^2 - 1)}$

so that $$a^2 - b^2 = 1 - 2xh + h^2$$

By putting these values in (i) we have

$$\pi(1-2xh+h^2)^{-1/2} = \int_0^{\pi}\left[-1+xh \pm h\sqrt{(x^2-1)}\cos\phi - 1\right]^{-1} d\phi$$

or $$\frac{\pi}{h}\left[1-2x.\frac{1}{h}+\frac{i}{h^2}\right]^{-1/2} = \int_0^{\pi}\left[h\{x \pm \sqrt{(x^2-1)}\cos\phi - 1\right]^{-1} d\phi$$

or $$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^{\pi}(t-1)^{-1}d\phi \qquad \text{where } t = h\left\{x \pm \sqrt{(x^2-1)}\cos\phi\right\}$$

$$= \int_0^{\pi}\frac{1}{t}\left[1-\frac{1}{t}\right]^{-1} d\phi$$

$$= \int_0^{\pi}\frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\ldots+\frac{1}{t^n}+\ldots\right]d\phi$$

$$= \int_0^{\pi}\left[\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots+\frac{1}{t^{n+1}}\right]d\phi$$

$$= \int_0^{\pi}\sum_{n=0}^{\infty}\frac{1}{t^{n+1}}d\phi$$

$$= \sum_{n=0}^{\infty}\int_0^{\pi}\frac{d\phi}{h^{n+1}\left[x \pm \sqrt{(x^2-1)}\cos\theta\right]^{n+1}}$$

Equating the coefficient of $\frac{1}{h^{n+1}}$, we get

$$\pi P_n(x) = \int_0^{\pi}\frac{d\phi}{\left\{x \pm \sqrt{(x^2-1)}\cos\phi\right\}^{n+1}}\ ]$$

$$P_n(x) = \frac{1}{x}\int_0^{\pi}\frac{d\phi}{\left\{x \pm \sqrt{(x^2-1)}\cos\phi\right\}^{n-1}}$$

Deductions: Replacing n by $-(n+1)$ in above relation, we get

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left\{x \pm \sqrt{(x^2-1)}\cos\phi\right\}^{-n}}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left\{x \pm \sqrt{(x^2-1)}\cos\phi\right\}^{x} d\phi$$

$$= P_n(x)$$

$\therefore$ $$P_n(x) = P_{-n-1}(x)$$

2.4 Recurrence Relations for Legendre Polynomials

I. ***Prove that***

$$\mathbf{(2n+1)xP_n(x) = (n+1)P_{n+1}(x)+nP_{n-1}(x)}$$

We now have from generating function

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Differentiating both sides w.r.t. h we have

$$-\frac{1}{2}(-2x+2h)(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

Multiplying both sides by $(1-2hx+h^2)$; we get

$$(x-h)(1-2hx+h^2)^{-1/2} = (1-2hx+h^2)\sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

or

$$(x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)\sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

Expanding

$$(x-h)\left[P_0(x)+hP_1(x)+h^2P_2(x)+...\right] \equiv (1-2hx+h^2)\left[P_1(x)+2hP_2(x)+3h^2P_3(x)+...\right]$$

Comparing the coefficients of h^n on both sides, we have

$$xP_n(x)-P_{n-1}(x) = (n+1)P_{n+1}(x)-2xnP_n(x)+(n-1)P_{n-1}(x)$$

Rearranging terms, we have

$$xP_n(x)+2xnP_n(x) = (n+1)P_{n+1}(x)+(n-1+1)P_{n-1}(x)$$

or

$$\mathbf{(2n+1)\times P_n(x) = (n+1)P_{n+1}(x)+nP_{n-1}(x)}$$

II. ***Prove that***

$$\mathbf{nP_n(x) = xP_n'(x)-P_{n-1}'(x)}$$

Proof:

Consider the relation

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^x P_n(x) \qquad ...(A)$$

Differentiating w.r.t. h, we have

$$\left(-\frac{1}{2}\right)(-2x+2h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

or

$$(x-h)(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \qquad \text{...(B)}$$

Differentiating (A) again by x, we have

$$(-2h)(-1/2)(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

or

$$h(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x) \qquad \text{...(C)}$$

Multiplying (B) by h and (C) by $(x-h)$, and subtracting we get

$$h\sum_{n=0}^{\infty} n\, h^{n-1}P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \qquad \text{...(D)}$$

Now comparing the coefficients of h^n on both sides we have

$$\boldsymbol{(n)P_n(x) = xP_n'(x) - P_{n-1}'(x)}$$

which is the recurrence relation II

III. ***Prove that***

$$\boldsymbol{(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)}$$

Proof:

From recurrence relation I

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating w.r.t. x we have

$$(2n+1)xP_n'(x) + (2n+1)P_n(x) = (n+1)P_{n+1}'(x) + nP_n'(x) \qquad \text{...(A)}$$

From recurrence formula II

$$xP_n'(x) = nP_n(x) + P_{n-1}'(x) \qquad \text{...(B)}$$

Substituting in (A) we have

$$(2n+1)\left[nP_n(x) + P_{n-1}'(x)\right] + (2n+1)P_n(x) = (n+1)P_{n+1}'(x) + nP_n'(x)$$

$$(2n+1)\left[(n+1)P_n(x) + P_{n-1}'(x)\right] = (n+1)P_{n+1}'(x) + np_{n-1}'(x)$$

or rearranging

$$(2n+1)(n+1)P_n(x) = (n+1)P_{n+1}'(x) - (n+1)P_{n-1}'(x)$$

Removing common factor we have

$$\boldsymbol{(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)}$$

Self Assessment

Notes

3. ***Prove***

$$(n+1)P_n(x) = P'_{n+1}(x) - xP_n(x)$$

4. ***Prove that***

$$(1-x)^2 P'_n(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$$

2.5 Orthogonal Properties of Legendre Polynomials

Prove that

(i) $\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0$ if $m \neq n$ and

(ii) $\int_{-1}^{+1} [P_n(x)]^2 dx = \dfrac{2}{2n+1}$

Proof:

From Legendre equation $P_n(x)$ being solution of it so we have

$$(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dt} + n(n+1)P_n(x) = 0$$

or
$$\frac{d}{dx}\left[(1-x)^2 \frac{dP_n(x)}{dx}\right] + n(n+1)\,P_n(x) = 0 \quad \text{...(A)}$$

In the same way, we have

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right] + m(m+1)\,P_m(x) = 0 \quad \text{...(B)}$$

Multiplying equation (A) by $P_m(x)$ and (B) by P_n and subtracting

$$P_m \frac{d}{dx}\left[(1-x^2)\frac{d\,P_n(x)}{dx}\right] - P_n\left\{\frac{d}{dx}\left[(1-x^2)\frac{d\,P_m}{dx}(x)\right]\right\} + [n(n+1) - m(m+1)]\,P_m(x)\,P_n(x) = 0 \quad \text{...(C)}$$

Integrating equation (C) between the limits –1 to 1, we have

$$\int_{-1}^{+1} P_m(x)\,\frac{d}{dx}\left[(1-x^2)\frac{d\,P_n(x)}{dx}\right]dx - \int_{-1}^{+1} P_n(x)\frac{d}{dx}\left[(1-x^2)\frac{d\,P_m}{dx}\right]dx + (n-m)(n+m+1)$$

$$\times\int_{-1}^{+1} P_m(x)P_n(x)dx = 0$$

Integrating by parts we have

$$\left[P_m(x)(1-x^2)\frac{d\,P_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{dP_m(x)}{dx}(1-x^2)\frac{d\,P_n(x)}{dx}dx - \left[P_n(x)(1-x^2)\frac{dP_m(x)}{dx}\right]_{-1}^{+1} +$$

$$+\int_{-1}^{+1}\frac{d}{dx}P_n(x)(1-x^2)\frac{d\,P_m(x)}{dx}dx + (n-m)(n+m+1)\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0$$

Notes

or

$$0-\int_{-1}^{+1}\frac{dP_m(x)}{dx}\frac{dP_n(x)}{dx}(1-x^2)\,dx-0+\int_{-1}^{+1}\frac{d\,P_n(x)}{dx}\frac{dP_m(x)}{dx}(1-x^2)dx+$$

$$+(n-m)(n+m+1)\int_{-1}^{+1}P_m(x)\,P_n(x)dx \;=\; 0$$

or

$$(n-m)(n+m+1)\int_{-1}^{+1}P_m(x)\;P_n(x)dx \;=\; 0$$

Thus

$$\int_{-1}^{+1}P_m(x)P_n(x)dx \;=\; 0 \qquad \text{for } m\neq n \qquad \text{...(D)}$$

This proves the first part.

To prove (ii)

We have

$$(1-2xh+h^2)^{-1} \;=\; (1-2xh+h^2)^{-1/2}.(1-2x+h^2)^{-1}$$

$$=\left[\sum_{n=0}^{\infty}h^nP_n(x)\right]\left[\sum_{m=0}^{\infty}h^m\,P_m(x)\right]$$

$$=\sum_{n=0}^{\infty}h^{2n}\,P_n^2(x)+2\sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty}h^{m+n}\,P_n(x)\,P_m(x) \qquad \text{...(E)}$$

Integrating between the limits −1 to +1, we have

$$\int_{-1}^{+1}\sum_{n=0}^{\infty}h^{2n}P_n^2(x)\,dx+2\int_{-1}^{+1}\sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty}h^{m+n}P_n(x)P_m(x)dx \;=\; \int_{-1}^{+1}\frac{dx}{(1-2hx+h^2)}$$

Thus

$$\sum_{n=0}^{\infty}h^{2n}\int_{-1}^{+1}P_n^2(x)\,dx \;=\; \int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)^{1/2}} \qquad \text{...(F)}$$

$$=-\frac{1}{2h}\log\left(\frac{1-h}{1+h}\right)^2=\frac{1}{h}\log\left(\left|\frac{1+h}{1-h}\right|\right)$$

Expanding the R.H.S. in powers of h, we have

$$\sum_{n=0}^{\infty}h^{2n}\int_{-1}^{+1}P_n^2(x)\,dx \;=\; \frac{1}{h}\left\{h+\frac{h^2}{2}+\frac{h^3}{3}+\frac{h^4}{4}+\ldots-\left(-h+\frac{h^2}{2}-\frac{h^3}{3}+\frac{h^4}{4}\ldots\right)\right\}$$

$$=\frac{2}{h}\left\{h+\frac{h^3}{3}+\frac{h^5}{5}+\frac{h^7}{7}+\ldots\right\}$$

$$=2\sum_{n=0}^{\infty}h^{2n}\left(\frac{1}{2n+1}\right)$$

So comparing the coefficients of h^{2n} on both sides we have

$$\int_{-1}^{+1} P_n^2(x)\,dx = \frac{2}{2n+1} \qquad \text{...(G)}$$

Thus
$$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{2}{2n+1} & \text{for } m = n \end{cases}$$

From the properties of Legendre's polynomials we can prove certain results.

2.6 Expansion of a *f*(*x*) in terms of Legendre's Polynomials

Since $P_0(x), P_1(x), P_2(x)...$ a set Legendre polynomials are orthogonal in the range of $x, (-1,1)$, any function $f(x)$ can be expressed in terms an expansion series involving $P_n(x)$ i.e.

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x) \quad \text{for } x \text{ in the range } -1 \leq x \leq 1 \qquad \text{...(i)}$$

Multiplying equation (i) by $P_m(x)$ and integrating over the limit −1 to 1, we have

$$\int_{-1}^{+1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} C_n \int_{-1}^{+1} P_m(x)P_n(x)\,dx \qquad \text{...(ii)}$$

Now

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases} \qquad \text{...(iii)}$$

Substituting in (i) we have

$$\int_{-1}^{+1} f(x)P_m(x)\,dx = C_m\left(\frac{2}{2m+1}\right) \qquad \text{...(iv)}$$

Example: Expand $f(x)$ in the form

$$\sum_{r=0}^{\infty} C_r P_r(x),$$

Where

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases} \qquad \text{...(i)}$$

We know

$$f(x) = \sum_{r=0}^{\infty} C_r P_r(x) \qquad \text{...(ii)}$$

Notes

where

$$C_r = \left(\frac{2r+1}{2}\right)\int_{-1}^{+1} f(x)\,P_r(x)\,dx$$

$$\therefore \qquad C_r = \frac{(2r+1)}{2}\int_0^1 1.\,P_r(x)\,dx \qquad \text{for } r = 1, 2, ... \qquad ...\text{(iii)}$$

Putting $r = 0, 1, 2, 3, ...$

$$C_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 1.\,dx = \frac{1}{2}x\int_0^1 = \frac{1}{2}$$

$$C_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{2}.\frac{x^2}{2}\int_0^1 = \frac{3}{4}$$

$$C_2 = \frac{5}{2}\int_0^1 P_2(x)\ dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2 - 1)dx$$

$$= \frac{5}{4}\left[\frac{\cancel{3}x^3}{\cancel{3}} - x\right]\int_0^1 = 0$$

$$C_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx$$

$$= \frac{7}{2}\int_0^1 \frac{1}{2}(5x^3 - 3x)dx$$

$$= \frac{7}{4}\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]\int_0^1$$

$$= \frac{7}{4}\left[\frac{5}{4} - \frac{3}{2}\right]$$

$$= \frac{7}{4}\left[\frac{5-6}{4}\right] = -\frac{7}{6}$$

So

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + ...$$

Self Assessment

Notes

5. Obtain the first three terms in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

in terms of Legendre's Polynomials and show that

$$f(x) = \frac{1}{4}P_0(x) + \frac{3}{4}P_1(x) + \frac{25}{48}P_2(x) + \ldots$$

Prove that all the roots of $P_n(x) = 0$ are distinct

Solution: If the roots of $P_n(x) = 0$ are not all different, then at least two of them must be equal.

Let α be their common value. Then

$$P_n(\alpha) = 0 \quad \text{(i)}$$

and $$P_n'(\alpha) = 0 \quad \left[\text{Here} \frac{dp}{dx} = P'\right]$$

Since $P_n(x)$ is the solution of Legendre's equation

$$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d\,P_n(x)}{dx} + n(n+1)P_n(x) = 0 \quad \text{...(ii)}$$

Differentiating (ii) r times by Leibnitz's theorem, we get

$$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x^r c_1 \frac{d^{n+1}}{dx^{n+1}}P_n(x) - 2^r c_2 \frac{dr}{dx^r}P_n(x)$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + 1.c_1^r \frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{dr}{dx^r}P_n(x) = 0$$

or $$(1-x)^2 \frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x\left(r_{C_1} + 1\right)\frac{d^{r+1}}{dx^{r+1}}P_n(x) - \left[2r_{C_2} + 2r_{C_1} - n(n+1)\right]\frac{d^r P_n(x)}{dr} = 0 \quad \text{...(iii)}$$

Putting $r = 0, x = \alpha$

$$(1-\alpha^2)\left[\frac{d^2}{dx^2}P_n(x)\right]_{x=\alpha} - 2\alpha\left[\frac{d}{dx}P_n(x)\right]_{x=\alpha} + n(n+1)P_n(\alpha) = 0 \quad \text{...(iv)}$$

Since $\frac{d}{dx}P_n(x)\Big|_{x=\alpha} = 0$ and $P_n(\alpha) = 0, so$

$$\left[\frac{d^2 P_n(x)}{dx^2}\right]_{x=\alpha} = 0 \quad \text{...(v)}$$

Notes

Similarly putting r = 1, 2, ... in (iii) and simplifying stepwise, we have

$$P_n'''(\alpha) = 0 = P_n^{iv}(\alpha) = 0 = 0... = P_n^n(\alpha) = 0 \quad ...(vi)$$

But since

$$P_n^n(x)\Big|_{x=\alpha} = \frac{1.3...(2n-1)}{\lfloor n}.n! \neq 0 \quad ...(vii)$$

Therefore our assumption that $P_n(\alpha) = 0$ has a repeated root is not correct.

Hence all the roots of $P_n(x) = 0$ are distinct.

Example: Find the roots of $P_2(x) = 0$

As

$$P_2(x) = 0 = \frac{1}{2}(3x^2 - 1)$$

$$P_2(\alpha) = 0 = \frac{1}{2}(3x^2 - 1)$$

$$\therefore \quad 3\alpha^2 = 1$$

$$\alpha = \pm 1/\sqrt{3}$$

So the roots are

$$\alpha_1 = -1/\sqrt{3}, d_2 = \frac{1}{\sqrt{3}}$$

Self Assessment

6. Show that the roots of $P_3(x) = 0$ are

$$-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}$$

2.7 Summary

- Legendre's Differential equation is obtained from Laplace equation in spherical polar co-ordinates.
- Legendre's Differential equation has $x = \pm 1$, as well as $x = \infty$ as regular singular points.
- So Legendre's Differential equation is solved as a power series.
- It is found that Legendre polynomial $P_n(x)$ is a finite power series having x^n as the highest power of x.
- The generating function for $P_n(x)$ is found to be $(1 - 2h + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Notes

- Rodrigue's formula for Legendre polynomials help us to find a few $P_n(x)$ i.e. $P_0(x), P_1(x),$ $P_2(x),......$
- Orthogonal properties of $P_n(x)$ are obtained. It is seen that $\{P_n(x)\} n = 0, 1, ...$ form a complete set in the range $-1 \le x \le 1$.
- Just as Fourier series we show that a function in the range $-1 \le x \le 1$ is expanded in terms of $P_n(x)$'s.

2.8 Keywords

Regular singular points of Legendre equations are $x = \pm 1$ and $x = \infty$.

Legendre polynomial $P_n(x)$ is a terminating series with highest power of x as x^n.

Generating function of the Legendre polynomial is $(1 - 2hx + h^2)^{-1} = \sum_{n=0}^{\infty} h^n P_n(x)$

Rodrigue's formula has been obtained and certain properties of $P_n(x)$ are obtained in a straight forward manner.

Recurrence relations between various Legendre's polynomials obtained are useful in expressing higher polynomials in terms of $P_0(x)$ and $P_1(x)$.

Orthogonality properties of the Legendre Polynomials obtained, help us in evaluating certain integrals easily.

2.9 Review Questions

Show that

1. $P_n'(x) - P_{n-2}(x) = (2n-1)P_{n-1}(x)$

2. $\int_{-1}^{+1} x\, P_n(x)\, P_{n-1}(x)\, dx = \frac{2n}{4n^2 - 1}$

3. $x\, P_9'(x) = P_8'(x) + 9\, P_9(x)$

4. Show that all the roots of $P_n(x) = 0$ are real and lie between −1 and +1.

5. Prove that

$$x^4 - 3x^2 + x \equiv \frac{8}{35} P_4(x) + \frac{6}{35} P_2(x) + P_1(x)$$

2.10 Further Readings

Books

Piaggio H.T.H., Differential Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 3: Hermite Polynomials

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Solve second order differential equation like Hermite equation.
- Familiarize yourself with the properties of Hermite Polynomials through generating function.
- Obtain certain relations involving Hermite polynomials with the help of Rodrigue formula.
- Solve certain integrals. You can express any function $f(x)$ in terms of Hermite polynomials $H_n(x)$.
- Relate some Hermite polynomials in terms of others with the help of recurrence relations.

Introduction

In the previous two units you have learnt the method of Frobenius in solving second order differential equations in power series. This method will help us to solve Hermite differential equation. In this unit we will be able to solve the equation for $-\infty < x < \infty$ range.

Just as the generating functions were introduced in the previous chapter, here in this chapter also it will be introduced for Hermite polynomials. Also orthogonal properties and recurrence relations are very important in understanding the properties of Hermite polynomials.

3.1 Power Series Solution of Hermite Polynomials

Consider the following equation, containing a parameter λ,

$$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right)+2\lambda\, e^{-x^2}y = 0 \qquad ...(A)$$

Notes

On the infinite open interval $(-\infty,\infty)$. Here we take as boundary conditions the following: as $x \to -\infty$, and as $x \to +\infty$, $y(x)$ tends to infinity of an order not greater than a certain finite power of x, i.e.

$$y(x) = 0\left(x^k\right) \text{ as } x \to \pm\infty \quad \text{...(B)}$$

The equation (i) is written as

$$e^{-x^2}\frac{d^2y}{dx^2} - 2x\,e^{-x^2}\frac{dy}{dx} + 2\lambda\,e^{-x^2}y = 0$$

or

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0 \quad \text{...(i)}$$

From the coefficients of $\frac{dy}{dx}$ and y, it is clear that there are no singular points except $x = \pm\infty$.

Hence its solution can be given by a power series by Frobenius method

$$y(x) = \sum_{r=0}^{\infty} a_r x^{r+k} \quad \text{...(ii)}$$

Which converges for $|x| < \infty$.

$$\frac{dy}{dx} = \sum_{r=0}^{\infty}(r+k)a_r x^{k+r-1}$$

and

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting in (i), we have

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r)x^{k+r} + 2\lambda x^{k+r}\right] = 0,$$

or

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r-\lambda)x^{k+r}\right] = 0 \quad \text{...(iii)}$$

Now (iii) being an identity, we can equate to zero the coefficients of various powers of x.

Equating to zero the coefficient of lowest power of x, i.e., of x^{k-2}, we get

$$a_0\,k(k-1) = 0.$$

Now $a_0 \neq 0$, as it is the coefficient of the first term with which the series is started.

$\therefore$ either $k = 0$

or $k = 1$ ...(iv)

Equating the coefficient of x^{k-1} in (iii) to zero, we get

$$a_1(k+1)k = 0 \quad \text{...(v)}$$

Notes

which implies that $a_1 = 0$ or $k = 0$ or both are zero, since $k+1 \neq 0$ for any value of k given by (iv).

Now equating to zero the coefficient of general term, i.e., x^{k+r} in (iii), we get

$$a_{r+2}(k+r+2)(k+r+1) = 2a_r(k+r-\lambda) = 0$$

or
$$a_{r+2} = \frac{2(k+r-\lambda)}{(k+r+2)(k+r+1)}.a_r$$

or
$$a_{r+2} = \frac{2(k+r)-2\lambda}{(k+r+2)(k+r+1)}.a_r \quad \text{...(vi)}$$

Now two cases arise—

Case I: when $k = 0$, then from (vi), we have

$$a_{r+2} = \frac{2r-2\lambda}{(r+2)(r+1)}a_r \quad \text{...(vii)}$$

Putting $r = 0, 2, 4$, etc. in (vii), we have

$$a_2 = \frac{-2\lambda}{2.1}a_0 = -\frac{2\lambda}{2!}a_0$$

$$a_4 = \frac{4-2\lambda}{4.3}a_2 = -\frac{(4-2\lambda).2\lambda}{4.3.2!}a_0$$

$$= \frac{2^2(-2+\lambda)\lambda}{4!}a_0 = \frac{2^2\lambda(\lambda-2)}{4!}a_0$$

and so on.

$$\therefore \quad a_{2m} = \frac{(-2)^m\lambda(\lambda-2)\ldots(\lambda-2m+2)}{(2m)!}a_0.$$

Again putting $r = 1, 3, 5$, etc.

$$a_3 = \frac{2-2\lambda}{3.2}a_1 = -\frac{2(\lambda-1)}{3!}a_1$$

$$a_5 = \frac{6-2\lambda}{5.4}a_3$$

$$= \frac{-2(6-2\lambda)(\lambda-1)}{5.4.3.2}a_1$$

$$= (-2)^2\frac{(\lambda-1)(\lambda-3)}{5!}a_1$$

and so on.

$$\therefore \quad a_{2m+1} = \frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)!}a_1$$

Notes

Now if $a_1 \neq 0$, then we have

$$y = \sum_{r=0}^{\infty} a_r x^r$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0\left[1 - \frac{2\lambda}{2!}x^2 + \frac{2^2\lambda(\lambda-2)}{4!}x^4 + \ldots + \frac{(-2)^m\lambda(\lambda-2)\ldots(\lambda-2m+2)}{(2m)!}x^{2m} + \ldots\right]$$

$$+a_1\left[x - \frac{2(\lambda-1)}{3!}x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}x^5 + \ldots + \right.$$

$$\left.\frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)!}x^{2m+1} + \ldots\right] \qquad \ldots(viii)$$

and if $a_1 = 0$, then we have

$$y = a_0\left[1 - \frac{2\lambda}{2!}x^2 + \frac{2^2\lambda(\lambda-2)}{4!}x^4 + \ldots + \frac{(-2)^m\lambda(\lambda-2)\ldots(\lambda-2m+2)}{(2m)!}x^{2m} + \ldots\right] \qquad \ldots(ix)$$

$$= y_1 \text{ (say).}$$

Case II: When $k = 1$, from (vi), we have

$$a_{r+2} = \frac{2(r+1)-2\lambda}{(r+3)(r+2)}a_r.$$

Putting $r = 1, 3, \ldots$ etc.

$$a_3 = a_5 = \ldots = 0 \text{ (each).}$$

Since in this case from (iv), $a_1 = 0$

Putting $r = 0, 2, 4, \ldots$ etc.

$$a_2 = \frac{2-2\lambda}{3.2}a_0 = -\frac{2(\lambda-1)}{3!}a_0$$

$$a_4 = \frac{6-2\lambda}{5.4}a_2 = \frac{2(\lambda-1)(\lambda-3)}{3!}a_0$$

and so on.

$\therefore$
$$a_{2m} = \frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)!}a_0$$

$\therefore$ we have

$$y = \sum_{r=0}^{\infty} a_r x^{r+1}$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \ldots + a_{2m}x^{2m+1} + \ldots$$

Notes

$$= a_0\left[x - \frac{2(\lambda-1)}{3!}x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}x^5 + \ldots + \right.$$

$$\left.\frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)!}x^{2m+1} + \ldots\right]$$

$$= y_2 \text{ (say)} \qquad \ldots\text{(x)}$$

From (viii) and (x) it is obvious that (x) is the part of solution, given by (viii). But as the two are the solutions of the same equations so (x) must not be the part of solution (viii).

$\therefore$ $a_1 = 0$ and the solution in the case $k = 0$ must be given by (ix).

Hence the general solution of Hermite's equation is

$$y = Ay_1 + By_2,$$

where A and B are arbitrary constants and y_1, y_2 are given by (ix) and (x).

Hermite's Polynomials

When λ is an even integer, equation (ix) gives an even polynomial of degree n.

Let $\lambda = n$, n being an even integer and let

$$a_0 = (-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}$$

$\therefore$ Coefficient of x^n in (ix) is

$$(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}\cdot\frac{(-2)^{n/2}n(n-2)\ldots(n-n+2)}{n!} = \frac{2^n.\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots1}{(n/2)!} = 2^n.$$

Similarly coefficient of x^{n-2}

$$= (-1)^{n/2}\frac{n!}{(n/2)!}\frac{(-2)^{(n-2)/2}n(n-2)\ldots(n-n+2+2)}{(n-2)!}$$

$$= -\frac{2^{n-2}n(n-1)n/2(n/2-1)\ldots2}{(n/2)!}$$

$$= -\frac{n(n-1)}{1!}2^{n-2}$$

and so on.

So value of y is given by

$$y_n = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \ldots + (-1)^{n/2}\frac{n!}{(n/2)!}$$

This value of y_n is known as the Hermite's polynomial of degree n and is written as

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \ldots + (-1)^{n/2}\frac{n!}{(n/2)!}$$

or $$H_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}$$

where $$\left(\frac{n}{2}\right) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

A first few $H_n(x)$ are given as follows

$$H_0(x) = 1, H_1(x) = 2x$$

$$H_2(x) = (2x)^2 - 2 = 4x^2 - 2$$

$$H_3(x) = (2x)^3 - \frac{3.2}{1}(2x) = 4x\left(2x^2 - 3\right)$$

$$H_4(x) = (2x)^4 - \frac{12}{1}(2x)^2 + \frac{4.3.2.1}{2}(1)$$

$$= 16x^4 - 48x^2 + 12$$

Self Assessment

Fill in the blanks:

1. Hermite polynomial $H_n(x)$ is a .... series.
2. As $x \to \infty, H_4(x)$ tends to infinity of an order not greater than .... power of x.
3. $H_3(x)$ satisfies equation (i) for λ = ...
4. The value of $H_4(o)$ is ....

We now give some of the properties of Hermite polynomials like generating functions, Rodrigue formula, orthogonality relations and the recurrence formulae.

3.2 Generating Functions of Hermite Polynomials $H_n(x)$

To prove that

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

or

show that $\frac{H_n(x)}{\lfloor n}$ are the coefficients of t^n in the expansion of the function e^{2xt-t^2} (known as generating function for $H_n(x)$), We have

$$e^{2tx-t^2} = e^{2tx}.\, e^{-t^2}$$

$$= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \sum_{s=0}^{\infty} \frac{\left(-t^2\right)^s}{s!}$$

$$= \sum_{r=0,s=0}^{\infty} (-1)^s \frac{(2x)^r}{r!s!}.t^{r+2s}$$

Coefficient of t^n (for fixed value of s)

[obtained by putting $r + 2s = n$, i.e., $r = n - 2s$]

$$= (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!}$$

The total value of t^n is obtained by summing over all allowed values of s, and since $r = n - 2s$

$\therefore$ $$n - 2s \geq 0 \text{ or } s \leq n/2$$

Thus if n is even s goes from 0 to $n/2$ and if n is odd, s goes from 0 to $(n - 1)/2$.

$\therefore$ $$\text{Coefficient of } t^n = \sum_{s=0}^{(n/2)} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!}$$

$$= \frac{H_n(x)}{n!}$$

$$\text{Hence } e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

Other form for the Hermite Polynomials

Prove

$$H_n(x) = 2^n \left\{ \exp\left(-\frac{1}{4} \frac{d^2}{dx^2} \right) x^n \right\} \qquad \text{...(i)}$$

We have

$$\frac{1}{2} \frac{d}{dx} e^{2tx} = te^{2tx}$$

$$\frac{d}{dx}\left(\frac{1}{2} \frac{d}{dx} e^{2tx} \right) = 2t^2 e^{2tx}$$

$$\therefore \qquad \frac{1}{2}\frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right) = t^2e^{2tx}$$

or $$\left(\frac{1}{2}\frac{d}{dx}\right)^2 e^{2tx} = t^2e^{2tx}$$

$$\therefore \qquad \left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx} = t^ne^{2tx}$$

Hence

$$\left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}e^{2tx} = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n\right]e^{2tx}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n}e^{2tx}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}e^{2tx} \text{ [from (ii)]}$$

$$= e^{2tx}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}$$

$$= e^{2tx}.e^{-t^2} = e^{(2tx-t^2)}$$

or $$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\sum_{n=0}^{\infty}\frac{1}{n!}(2tx)^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)$$

Equating the coefficient of t^n from the two sides, we have

$$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\frac{1}{n!}2^nx^n = \frac{1}{n!}H_n(x)$$

or $$H_n(x) = 2^n\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}x^n.$$

Self Assessment

5. Obtain the expression for $H_2(x)$ from generating function e^{2xt-t^2}.

6. Obtain the expression for $\frac{d}{dx}H_n(x)$ from the generating function e^{2xt-t^2}.

7. Show that for odd n.

$$H_n(0) = 0$$

8. $H_1(x) - 2xH_0(x)$ is
   (a) positive
   (b) zero
   (c) negative
   (d) none of the above

3.3 The Rodrigue's Formula for $H_n(x)$

To Prove

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad ...(i)$$

Proof:

We have $$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

or $$e^{x^2-(t-x)^2} = \frac{H_0(x)}{0!} t^0 + \frac{H_1(x)}{1!} t + \frac{H_2(x)}{2!} t^2 + \ldots + \frac{H_n(x)}{n!} t^n + \frac{H_{n+1}(x)}{(n+1)!} t^{n+1} + \ldots$$

Differentiating both sides, partially with respect to t, n times and then putting $t = 0$, we have

$$\frac{H_n(x)}{n!} n! = \left[\frac{\partial^n}{\partial t^n} e^{-(t-x)^2}\right]_{t=0} e^{x^2} \quad ...(ii)$$

Now let $t - x = \mu$, i.e., at $t = 0, x = -\mu$

$$\therefore \quad \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial \mu}$$

or $$\left[\frac{\partial^n}{\partial t^n} e^{-(t-x)^2}\right]_{t=0} = \frac{\partial^n}{\partial \mu^n}\left(e^{-\mu^2}\right)$$

$$= (-1)^n \frac{\partial^n}{\partial x^n}\left(e^{-x^2}\right)$$

$$= (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\therefore \quad H_n(x) = (-1)^n . e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad ...(A)$$

First few Hermite Polynomials from Rodrigue's Formula

From Rodrigue's Formula for $H_n(x)$

$$H_n(x) = (-1)^n e^{x^2} . \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Putting n = 0, 1, 2, 3, ... we get

$$H_0(x) = e^{x^2} . e^{-x^2} = 1$$

$$H_1(x) = (-1)e^{x^2} \frac{d}{dx}\left(e^{-x^2}\right) = 2x$$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2} \frac{d}{dx}\left(-2xe^{-x^2}\right)$$

$$= e^{x^2}\left(4x^2 e^{-x^2} - 2e^{-x^2}\right)$$

$$= \left(4x^2 - 2\right).$$

$$H_3(x) = (-1)^3 e^{x^2} \frac{d^3}{dx^3}\left(e^{-x^2}\right)$$

$$= -e^{x^2} \frac{d}{dx}\left\{\left(4x^2 - 2\right)e^{-x^2}\right\}$$

$$= -e^{x^2}\left\{-2x\left(4x^2 - 2\right)e^{-x^2} + 8xe^{-x^2}\right\}$$

$$= -e^{x^2}\left[\left(-8x^3 + 12x\right)e^{-x^2}\right] = 8x^3 - 12x.$$

Similarly, $H_4(x) = 16x^4 - 48x^2 + 12$ etc.

3.4 Orthogonal Properties of Hermite Polynomials

Prove

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ \sqrt{\pi}\, 2^2 (n)! & \text{if } m = n \end{cases}$$

We have $$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

and $$e^{-s^2+2sx} = \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$

$\therefore$ $$e^{-t^2+2tx} e^{-s^2+2sx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$

Notes

$\therefore \qquad \frac{1}{n!m!} H_n(x) H_m(x) = \text{Coeff. of } t^n s^m \text{ in the expansion of } e^{-t^2+2tx} e^{-s^2+2sx}$

$\therefore$ $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx$ is equal to $n!\ m!$ times the coefficient of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2} . e^{-t^2+2tx} . e^{-s^2+2sx} dx$$

Now, $\int_{-\infty}^{\infty} e^{-x^2} . e^{-t^2+2tx} . e^{-s^2+2sx} dx$

$$= e^{-t^2-s^2} \int_{-\infty}^{\infty} e^{-x^2+2tx+2sx} dx$$

$$= e^{-t^2-s^2} \int_{-\infty}^{\infty} e^{-\left[x^2-(t+s)^2+(t+s)^2\right] dx}$$

$$= e^{2ts} \int_{-\infty}^{\infty} e^{-[x-(t+x)]2 dx}$$

$$= e^{2ts} \int_{-\infty}^{\infty} e^{-u^2} du, \text{ putting } x-(t+s) = u$$

$$= e^{2ts} \sqrt{\pi}, \qquad \left(\text{since } \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}\right)$$

$$= \sqrt{\pi}\left[1 + 2ts + \frac{(2ts)^2}{2!} + \ldots + \frac{(2ts)^n}{n!} + \ldots\right]$$

Coefficient of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-t^2+2tx} e^{-s^2+2sx} dx$$

is 0 if $m \neq n$

and $\frac{2^n \sqrt{\pi}}{n!}$. if $m = n$.

We can also write it as follows

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi}\, 2^n n! \delta_{mn},$$

where δ_{mn} is Kronecker delta defined as

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

Self Assessment

9. Using Rodrigue's Formula derive the Hermite's polynomials $H_2(x)$ and $H_3(x)$

10. Evaluate

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_2(x)\, H_1(x) dx$$

11. Evaluate

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_2(x) dx$$

3.5 Recurrence Formula for Hermite Polynomials

(I) Prove

$$\boldsymbol{\frac{d}{dx} H_n(x) = 2n\, H_{n-1}(x) \qquad \text{for } n \geq 1}$$

We have from generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} = e^{2xt-t^2} \qquad \text{...(i)}$$

Differentiating both sides with respect to x, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \frac{dH_n(x)}{dx} = 2t\, e^{2xt-t^2}$$

$$= 2t \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!}$$

$$= 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} \qquad \text{Let } n+1 = n'$$

$$= 2 \sum_{n'=1}^{\infty} \frac{H_{n'-1}(x) t^{n'}}{(n'-1)!}$$

or

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \frac{dH_n(x)}{dx} = 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x) t^n}{(n-1)!} \qquad \text{...(ii)}$$

Comparing t^n on both sides we have

$$\frac{H_n'(x)}{n!} = 2 \frac{H_{n-1}(x)}{(n-1)!} \qquad \left[\text{Here } \frac{dH_n(x)}{dx} = H_n'(x)\right]$$

or

Notes

or $$\boldsymbol{H_n'(x) = 2n\, H_{n-1}(x)}$$

(II) $$\boldsymbol{2x\, H_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)}$$

we have $$\sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = e^{-t^2+2^t x}$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n\, t^{n-1} = e^{-t^2+2tx}(-2t+2x)$$

or $$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = 2t\, e^{-t^2+2tx} + 2x\, e^{-t^2+2tx}$$

or $$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = -2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

(Since term of L.H.S. Corresponding to $n = 0$ is zero)

or $$2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} + \sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

or $$2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n + \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} t^n$$

Equating the coefficient of t^n, on both sides, we have

$$2x \frac{H_n(x)}{n!} = 2\frac{H_{n-1}(x)}{(n-1)!} + \frac{H_{n+1}(x)}{n!}$$

or $$\boldsymbol{2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x)}$$

(III) $$\boldsymbol{H_n'(x) = 2x\, H_n(x) - H_{n+1}(x)}$$

Writing recurrence formulae I and II, we have

$$\boldsymbol{H_n'(x) = 2n\, H_{n-1}(x)} \qquad \text{...(i)}$$

and $$2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x) \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we have

$$\boldsymbol{H_n'(x) = 2x\, H_n(x) - H_{n+1}(x)}$$

(IV) $$\boldsymbol{H_n''(x) - 2x\, H_n'(x) + 2nH_n(x) = 0}$$

Hermite's differential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

Since $H_n(x)$ is the solution of (i), hence, we have

$$\boldsymbol{H_n''(x) - 2x\, H_n'(x) + 2nH_n(x) = 0}$$

Illustrative Examples

Example 1: Evaluate

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x) dx$$

Solution: From recurrence formula II, we have

$$xH_n(x) = nH_{n-1}(x) + \frac{1}{2} H_{n+1}(x)$$

$$\therefore \quad \int_{-\infty}^{\infty} xe^{-x^2} H_n H_m(x) dx = \int_{-\infty}^{\infty} e^{-x^2} \left\{ nH_{n-1}(x) + \frac{1}{2} H_{n+1}(x) \right\} H_m(x) dx$$

$$= n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x) H_m(x) dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x) H_m(x) dx$$

$$= n\sqrt{\pi}2^{n-1}(n-1)!\delta_{n-1,m} + \frac{1}{2}\sqrt{\pi}2^{n+1}(n+1)!\delta_{n+1,m}$$

$$= \sqrt{\pi}2^{n-1}n!\delta_{n-1,m} + \sqrt{\pi}\left(2^n\right)(n+1)!\delta_{n+1,m}$$

where δ is Kronecker delta.

Example 2: Prove that $H_n'' = 4n(n-1)H_{n-1}$

Solution: From recurrence formula I, we have

$$H_n' = 2nH_{n-1} \quad \text{...(i)}$$

Differentiating with respect to x, we have

$$H_n'' = 2nH_{n-1}' \quad \text{...(ii)}$$

Replacing n by $(n - 1)$ in (i), we have

$$H_{n-1}' = 2(n-1)H_{n-2} \quad \text{...(iii)}$$

∴ From (ii) and (iii), we have

$$\boldsymbol{H_n'' = 4n(n-1)H_{n-1}}$$

Example 3: Prove that, if $m < n$

$$\frac{d^n}{dx^m}\{H_n(x)\} = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

Notes

Solution: We have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x) = e^{-t^2+2tx} \qquad ...(i)$$

$$\therefore \qquad \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d^m}{dx^m}\{H_n(x)\} = \frac{d^m}{dx^m}e^{-t^2+2tx}$$

$$= (2t)^m e^{-t^2+2tx}$$

$$= (2t)^m \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x) \qquad \text{(from (i))}$$

$$= 2^m \sum_{n=0}^{\infty}\frac{1}{n!}t^{n+m}H_n(x)$$

Putting $n+m=r, n=r-m,$ for $n=0;$

$r=m,$ for $n=\infty, r=\infty,$

$$\sum_{n=0}\frac{t^n}{n!}\frac{d^m}{dx^m}H_n(x) = 2^m\sum_{r=m}^{\infty}\frac{1}{(r-m)!}t^r\, H_{r-m}(x) \qquad ...(ii)$$

Equating the coefficient of t^n from the two sides, we have

$$\frac{1}{n!}\frac{d^n}{dx^m}\{H_n(x)\} = 2^m\frac{1}{(n-m)!}H_{n-m}(x)$$

$$\therefore \qquad \frac{d^n}{dx^m}\{H_n(x)\} = \frac{2^m n!}{(n-m)!}H_{n-m}(x) \text{ Q.E.D.}$$

Example 4: Prove that $H_{2n}(0)=(-1)^n.\frac{(2n)!}{n!}$ and (ii) $H_{2n+1}(0)=0$

Solution: We have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x) = e^{-t^2+2tx}$$

Putting x = 0

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(0) = e^{-t^2}$$

$$= \left\{1-t^2+\frac{(t^2)^2}{2!}+...+(-1)^n\frac{(t^2)^n}{n!}+...\right\} \qquad ...(1)$$

(i) Equating the coefficients of t^{2n}, on both sides, we have

$$\frac{1}{(2n)!}H_{2n}(0) = (-1)^n \frac{1}{n!}$$

or

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$$

(ii) Again equating the coefficients of t^{2n+1}, on both sides of (i), we have

$$\frac{1}{(2n+1)!}H_{2n+1}(0) = 0$$

[Since R.H.S. of (i) does not involve odd powers of t]

Hence

$$H_{2n+1}(0) = 0.$$

Example 5: Prove that

$$P_n(x) = \frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2} H_n(xt)\,dt.$$

Solution: We have

$$H_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)}(2x)^{n-2r}$$

$$\therefore \quad H_n(xt) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)}(2xt)^{n-2r}$$

$$\therefore \quad \frac{2}{\sqrt{\pi}n!}\int_0^\infty t^n e^{-t^2} H_n(xt)dt = \frac{2}{\sqrt{\pi}.n!}\int_0^\infty t^n e^{-t^2} \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)}(2xt)^{n-2r}.dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}(-1)^r x^{n-2r}}{\sqrt{\pi}\, r!(n-2r)!}\int_0^\infty e^{-t^2} t^{2\left(n-r+\frac{1}{2}\right)-1} dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}(-1)^r x^{n-2r}}{\sqrt{\pi}\, r!(n-2r)!}\frac{1}{2}\Gamma\left(n-r+\frac{1}{2}\right)$$

$$\left[\text{Since } 2\int_0^\infty e^{-t^2} t^{(2n-1)}dt = \Gamma(n)\right]$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r}(-1)^r x^{n-2r}\left[2(n-r)\right]!}{\sqrt{\pi}\, r!(n-2r)!2^{2(n-r)}(n-r)!}\sqrt{\pi}$$

$$\left[\text{Since } \Gamma\left(x+\frac{1}{2}\right) = \frac{(2x)!}{2^{2x}}\frac{\sqrt{\pi}}{x!}\right]$$

Notes

$$= \sum_{n=0}^{(n/2)} (-1)^r \frac{(2n-2r)!x^{n-2\pi}}{2^n (r)!(n-2r)!(n-r)!} = P_n(x)$$

Hence,

$$P_n(x) \frac{2}{\sqrt{\pi}n!} \int_0^{\infty} t^n e^{-t^2} H_n(xt)dt.$$

Self Assessment

12. From recurrence relation II Obtain the value of $H_3(x)$. Given that

$$H_2(x) = 4x^2 - 2; H_1(x) = 2x$$

13. Prove that

$$H_n''(x) - 4nx\, H_{n-1}(x) + 2n\, H_n(x) = 0$$

14. Prove that

$$\frac{dH_3(x)}{dx} = 6H_2(x)$$

3.6 Summary

- Hermite differential equation has no finite singular points except $x = \pm\infty$. Therefore Frobenius method involving a power series solution is obtained.
- There are two independent solutions corresponding to two different values of indicial power.
- For $\lambda = n$ a polynomial solution called Hermite polynomial is obtained.
- Hermite polynomials are seen to be generated by a generating function.
- Orthogonal properties of Hermite polynomials are obtained. It helps in expressing any polynomial in terms of $H_n(x)$.
- Recurrence relations established help in expressing every polynomial as well as its derivatives in terms of two or three Hermite polynomials.

3.7 Keywords

Boundary Conditions are the behaviour of the solution of the differential equations in the initial value of the independent variable as well as at the final value of independent variable.

Frobenius Method: At an ordinary point as well as at regular singular point, helps in evaluating the solution as a power series.

Orthogonality relations of Hermite polynomials are relations involving integrals of two Hermite polynomials. These relations help us to see that $H_n(x)$ form a complete set.

Recurrence Relations are relations between two or three polynomials for all values of n and x.

Rodrigue Formula Expresses $H_n(x)$ in an alternative way than that of finding a solution of differential equations.

3.8 Review Questions

1. Use the Rodrigue's formula to drive the Hermite polynomials $H_2(x)$ and $H_3(x)$
2. Evaluate

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_2(x) H_3(x) dx$$

3. Show that

$$H_1(x) = 2xH_0(x)$$

4. For what value of $n, H_n(0) = 0$?
5. From generating function show that

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

Answers: Self Assessment

1. Terminating
2. Finite
3. n
4. 12
5. $H_2(x) = (4x^2 - 2)$
6. $\dfrac{dH_n(x)}{dx} = 2nH_{n-1}(x)$
8. b
9. $H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x$
10. $4\sqrt{\pi}$
11. Zero

3.9 Further Readings

Books K. Yosida, Lectures on Differential and Integral Equations

L.D. Landau and E.M. Lifshitz, Quantum Mechanics

Unit 4: Laguerre Polynomials

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Use generating function which helps you to familiarise with more properties of Laguerre polynomials.
- Use Rodrigue formula which is quite helpful in making you more familiar with properties of Laguerre polynomials.
- Employ of orthogonal properties to evaluate certain integrals.
- Use recurrence relations to correct one set of polynomials into another.

Introduction

Laguerre polynomials are shown to satisfy Laguerre differential equation. This equation has $x = 0$ as **regular singular** point whereas $x = \infty$ is an **irregular singular point.** A power series solution is obtained by Frobenius method.

Generating function is obtained wherein it will be seen that most properties of Laguerre polynomials are obtained orthogonal properties, recurrence relations Rodrigue's formula for Laguerre polynomials are very important and almost all properties of $L_n(x)$ are obtained from the above relations.

4.1 Solution of Laguerre's Differential Equation

Consider the following differential equation containing a parameter λ.

$$\left(x\, e^{-x} y'\right)' + \lambda\, e^{-x} y = 0$$

On the infinite interval $(0,\infty)$, we take as boundary conditions the following:

$y(x)$ remains finite as $x \to 0$,

$y(x)$ tends to infinity as $0(x^{\alpha})$ as $x \to \infty$.

The above equation when expanded is equal to

$$x\, e^{-x} y'' - e^{-x}(x-1)y' + \lambda\, e^{-x} y = 0$$

or

$$x\, y'' + (1-x)y' + \lambda\, y = 0 \qquad \text{...(i)}$$

Here $$y' = \frac{dy}{dx}.$$

Equation (i) has only one finite regular singular point $x = 0$ whereas $x \to \infty$ is irregular singular point. So we can apply Frobenius method to express the solution of (i) as a power series:

$$y = \sum_{n=0}^{\infty} a_r\, x^{k+r} \qquad \text{...(ii)}$$

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k=r-2}$$

Substituting in (i), we get

$$\sum_{r=0}^{\infty} a_r \left[(k+r)(k+r-1)x^{k+r-1} + (1-x)(k+r)x^{k+r-1} + \lambda x^{k+r} \right] = 0$$

or

$$\sum_{r=0}^{\infty} a_r \left[(k+r)^2 x^{k+r-1} - (k+r-\lambda)x^{k+r} \right] = 0 \qquad \text{...(iii)}$$

Now (iii) being an identity, we can equate the coefficients of various powers of x to zero.

Equating to zero the coefficient of lowest power of x, i.e., of x^{k-1}, we have

$$a_0\, k^2 = 0$$

Now, $a_0 \neq 0$, as it is coefficient of the first term with which the series is started.

$$\therefore \quad k = 0.$$

Equating to zero the coefficient of general term, i.e., of x^{k+r}, we have

$$a_{r+1}(k+r+1)^2 - a_r(k+r-\lambda) = 0$$

Notes

$$\therefore \qquad a_{r+1} = \frac{(k+r-\lambda)}{(k+r+1)^2} a_r$$

for $\qquad k = 0$

$$a_{r+1} = \frac{r-\lambda}{(r+1)^2} a_r \qquad \text{...(iv)}$$

Putting r = 0, 1, 2, ..., in (iv), we have

$$a_1 = -\frac{\lambda}{1} a_0 = (-1)\lambda a_0$$

$$a_2 = \frac{1-\lambda}{2^2} a_1 = (-1)^2 \frac{\lambda(\lambda-1)}{(2!)^2} a_0$$

$$a_3 = \frac{2-\lambda}{3^2} \cdot a_2 = (-1)^3 \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} a_0 \text{ etc.}$$

Hence $\qquad a_r = (1)^r \frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-r+1)}{(r!)^2} a_0$

$\therefore \quad$ From (ii), we have

$$y = \sum_{r=0}^{\infty} a^r x^r = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots$$

$$= a_0 \left[1 - \lambda x + \frac{\lambda(\lambda-1)}{(2!)^2} x^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} x^3 \right.$$

$$\left. + \ldots + (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda-r+1)}{(r!)^2} x^r + \ldots \right] \qquad \text{...(v)}$$

If $\lambda = n$

$$y = a_0 \left[1 - \frac{n}{1^2}.x + \frac{n(n-1)}{(2!)^2} x^2 + \ldots + (-1)^2 \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} \right]$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} x^r$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r$$

Laguerre Polynomials

The standard solution of Laguerre equation for which $a_0 = 1$ is called the Laguerre polynomial of order n and is denoted by $L_n(x)$.

Notes

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r \quad \text{...(vi)}$$

The first few Laguerre polynomials are:

$$L_0(x) = 1, L_1(x) = 1 - x$$

$$L_2(x) = \frac{1}{2}\left(2 - 4x + x^2\right)$$

$$L_3(x) = \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)$$

Self Assessment

1. The value of $L_n(0)$ is

   (a) 0 (b) 1

   (c) −1 (d) None of these

2. $L_2(x)$ satisfies Laguerre's differential equation for λ equal to

   (a) −1 (b) 3

   (c) 2 (d) 1

3. Fill in the blanks:

   The Laguerre polynomial tends to infinity as a .... power of x as $x \to \infty$.

4. Laguerre polynomial $L_n(x)$ is a polynomial having a leading power of x equal to

   (a) n (b) Zero

   (c) One (d) None of the above

4.2 Generating Function for Laguerre Polynomials $L_n(x)$

To prove $\frac{1}{1-t} e^{-tx/(1-t)} = \sum_{r=0}^{\infty} t^n L_n(x)$.

We have

$$\frac{1}{1-t} e^{-tx/(1-t)} = \frac{1}{1-t} \sum_{r=0}^{\infty} \frac{1}{r!}\left(-\frac{xt}{1-t}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{x^r t^r}{(1-t)^{r+1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r (1-t)^{-(r+1)}$$

Notes

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \left[1 + (r+1)t + \frac{(r+1)(r+2)}{2!} t^2 + \ldots\right]$$

$$= \sum_{r=0}^{\infty} \left[\frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!s!}.t^s\right]$$

$$= \sum_{r,\, s=0}^{\infty} (-1)^r \frac{(r+s)!}{(r!)^2 s!} x_t^r r + s$$

Putting $s + r = n$, or $s = n - r$, we get the coefficient of t^n, for a fixed value of r as

$$(-1)^r \frac{n!}{(r!)^2 (n-r)!}.x^r$$

Therefore the total coefficient of t^n is obtained by summing over all allowed values of r, since $s = n - r$ and $s \geq 0$

$\therefore \quad n - r \geq 0$ or $r \leq n$.

Hence the coefficient of t^n is

$$\sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r = L_n(x)$$

Hence $$\frac{1}{(1-t)} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x).$$

Self Assessment

5. Obtain the expression for $L_1(x)$ and $L_2(x)$ from the generating function

$$\frac{1}{(1-t)} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x)$$

6. Show that from the generating function

$$L_n(0) = 1 \text{ for } n = 0, 1, 2, \ldots$$

7. Obtain the expression for $L_3(x)$ from the generating function

$$\frac{1}{(1-t)} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x)$$

8. Whether $2L_2(x) - x^2 + 4x$ is equal to

   (a) 0 (b) 1

   (c) 2 (d) −2

4.3 Rodrigue's Formula for Laguerre Polynomials $L_n(x)$

To prove

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) \text{ for } n = 0, 1, 2, \ldots \quad \text{...(i)}$$

Proof: Using Leibnitz's theorem we have

$$\frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) = \frac{e^x}{n!}\Big[x^n(-1)^n e^{-x} + n\cdot n\cdot x^{n-1}(-1)^{n-1}e^{-x} +$$

$$+\frac{n(n-1)}{2}\cdot n(n-1)x^{n-2}(-1)^{n-2}e^{-x} + \ldots + n!e^{-x}\Big]$$

$$= \frac{e^x e^{-x}}{n!}\left[(-1)^n x^n + (-1)^{n-1}\frac{n\cdot n!}{(n-1)!}x^{n-1} + \ldots + n!\right] \quad \text{...(ii)}$$

$$= (-1)^n\frac{n!}{(n!)^2}x^n + (-1)^{n-1}\frac{n!}{\{(n-1)!\}^2 .i!}x^{n-1} + \ldots + \frac{n!}{n!}$$

$$= \sum_{r=0}^{n}(-1)^r\frac{n!}{(r!)^2(n-r)!}x^r = L_n(x) \quad \text{...(iii)}$$

Hence $$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

First few Laguerre Polynomials from Rodrigue's Formula

We have from Rodrigue's formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

Putting $n = 0$

$$L_0(x) = \frac{e^x}{0!}\frac{d^0}{dx^0}(x^0 e^{-x}) = 1$$

Putting $n = 1$

$$L_1(x) = \frac{e^x}{1!}\frac{d}{dx}(xe^{-x}) = e^x\left(e^{-x} - xe^{-x}\right) = 1 - x$$

Putting $n = 2$

$$L_2(x) = \frac{e^x}{2!}\frac{d^2}{dx^2}\left(x^2e^{-x}\right) = \frac{e^x}{2!}\frac{d}{dx}\left(2xe^{-x} - x^2e^{-x}\right)$$

$$= \frac{e^x}{2!}\left(2e^{-x} - 4x\,e^{-x} + x^2e^{-x}\right)$$

Notes

$$= \frac{1}{2!}\left(2-4x+x^2\right)$$

Similarly,

$$L_3(x) = \frac{1}{3!}\left(6-18x+9x^2-x^3\right)$$

$$L_4(x) = \frac{1}{4!}\left(24-96x+72x^2-16x^3+x^4\right), \ldots etc.$$

Self Assessment

9. Show that

$$L_2(x) = \frac{e^x}{2}\frac{d^2}{dx^2}\left(x^2 e^{-x}\right)$$

10. Show that x^3 is given by

$$x^3 = 6\left[L_0(x) - 3L_1(x) + 3L_2(x) - L_3(x)\right]$$

11. From Rodrigue's formula show that

$$\frac{dL_2(x)}{dx} = -L_1(x) - L_0(x)$$

4.4 Orthogonality Property of Laguerre Polynomials $L_n(x)$

To prove

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\,dx = \delta_{mn} = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n \end{cases} \qquad \ldots\text{(i)}$$

We have from the generating function of Laguerre polynomial, that

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{\left(-tx/(1-t)\right)}$$

and

$$\sum_{m=0}^{\infty} s^m L_m(x) = \frac{1}{1-s} e^{-xs/(1-s)}$$

$$\therefore \quad \sum_{m,n=0}^{\infty} e^{-x} t^n s^m L_n(x) L_m(x) = e^{-x} \frac{1}{(1-t)(1-s)} e^{-tx/(1-t) - \frac{sx}{(1-s)}} \qquad \ldots\text{(ii)}$$

Thus

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\,dx = \text{Coeff. of } s^m t^n \text{ in the expansion of } \int_0^\infty e^{-x} \frac{1}{(1-t)(1-s)} e^{-tx/(1-t) e^{-sx/(1-s)}}\,dx$$

$$= \frac{1}{(1-t)(1-s)}\int_0^{\infty}\left[e^{\left[-x\left(1+\frac{t}{1-t}+\frac{s}{1-s}\right)\right]}\right]dx$$

$$= \frac{1}{(1-t)(1-s)}\frac{1}{\left[1+\frac{t}{(1-t)}+\frac{s}{(1-s)}\right]}\times\left[e^{\left[-x\left(1+\frac{t}{1-t}+\frac{s}{1-s}\right)\right]}\right]_0^{\infty}$$

$$= -\frac{1}{(1-t)(1-s)}\frac{(1-t)(1-s)}{\{(1-t)(1-s)+t(1-s)+s(1-t)\}}[-1]$$

$$= \frac{1}{1-st} = (1-st)^{-1} = \left[1+st+(st)^2+(st)^3+\ldots(st)^n+\ldots\right] \qquad \text{...(iii)}$$

In which coefficient of $s^m\ t^n$

is 0 if $m \neq n$ ...(iv)

and is 1 if $m = n$

Hence

$$\int_0^{\infty} e^{-x}L_m(x)L_n(x)dx = \begin{cases}0 & \text{if } m \neq n\\ 1 & \text{if } m = n\end{cases}$$

or

$$\int_0^{\infty} e^{-x}L_m(x)L_n(x)dx = \delta_{mn} \text{ (where } m, n, = 1, 2, 3, \ldots) \qquad \text{...(v)}$$

Self Assessment

12. Whether $\int_0^{\infty} e^{-x}L_2(x)L_3(x)dx$ is equal to

(a) 1 (b) 5

(c) –1 (d) 0

13. Find out

$$\sum_{m=0}^{\infty}\delta_{mn}L_m(x)$$

14. Prove that

$$\int_0^{\infty} e^{-x}L_1(x)L_2(x)dx = 0$$

Notes

4.5 Recurrence Formulae for Laguerre Polynomials $L_n(x)$

I. $$(n+1)L_{n+1}(x) = (2n+1-x)\,L_n(x) - n\,L_{n-1}(x)$$

We have $$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{(1-t)} e^{\left[-\frac{-tx}{(1-t)}\right]}$$

Differentiating both sides with respect to t, we have

$$\sum_{n=0}^{\infty} n\, t^{n-1} L_n(x) = \frac{1}{(1-t)^2}\left(1-\frac{x}{1-t}\right) e^{\left[-\frac{-tx}{(1-t)}\right]}$$

or $$(1-t)^2 \sum_{n=0}^{\infty} n\, t^{n-1} L_n(x) = (1-t)\frac{e^{-tx/(1-t)}}{(1-t)} - x.\frac{1}{1-t} e^{-tx/(1-t)}$$

or $$(1-t)^2 \sum_{n=0}^{\infty} n\, t^{n-1} L_n(x) = (1-t)\sum_{n=0}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\left(1-2t+t^2\right)\sum_{n=1}^{\infty} n\, t^{n-1} L_n(x) = (1-t)\sum_{n=1}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\sum_{n=1}^{\infty} n\, t^{n-1} L_n(x) - 2\sum_{n=1}^{\infty} n\, t^n L_n(x) + \sum_{n=1}^{\infty} n\, t^{n+1} L_n(x)$$

$$= \sum_{n=0}^{\infty} t^n L_n(x) - \sum_{n=0}^{\infty} t^{n+1} L_n(x) - x\sum_{n=0}^{\infty} t^n L_n(x)$$

Equating the coefficient of t^n on both sides, we have

$$(n+1)L_{n+1}(x) - 2n - L_n(x) + (n-1)L_{n-1}(x)$$

$$= L_n(x) - L_{n-1}(x) - xL_n(x)$$

or $$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - n\,L_{n-1}(x)$$

II. $$xL_n'(x) = nL_n(x) - nL_{n-1}(x)$$

We have

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{(1-t)} e^{\left[\frac{-tx}{(1-t)}\right]}$$

Differentiating with respect to x, we have

$$\sum_{n=0}^{\infty} t^n L_n'(x) = \frac{1}{(1-t)} e^{-tx/(1-t)}\left(-\frac{t}{1-t}\right)$$

or $$(1-t)\sum_{n=0}^{\infty} t^n L_n'(x) = -t.\frac{1}{1-t} e^{-tx/(1-t)}$$

or $$(1-t)\sum_{n=0}^{\infty} t^n L_n'(x) = -t\sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\sum_{n=0}^{\infty} t^n L_n'(x) - \sum_{n=0}^{\infty} t^{n+1} L_n'(x) = -\sum_{n=0}^{\infty} t^{n+1} L_n(x)$$

Equating the coefficients of t^n, on both sides, we get

$$L_n'(x) - L_{n-1}'(x) = -L_{n-1}(x)$$

or $$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x) \quad \text{...(i)}$$

Differentiating recurrence formula I with respect to x, we get

$$(n+1)L_{n+1}'(x) = (2n+1-x)\, nL_n'(x) - L_n(x) - nL_{n-1}'(x) \quad \text{...(ii)}$$

Replacing n by $(n + 1)$ in (i), we get

$$L_{n+1}'(x) = L_n'(x) - L_n(x)$$

Also from (i) $$L_{n-1}'(x) = L_n'(x) + L_{n-1}(x)$$

Substituting these values in (ii), we have

$$(n+1)\{L_n'(x) - L_n(x)\} = (2n+1-x)L_n'(x) - L_n(x) - n\{L_n(x) + L_{n-1}(x)\}$$

or $$\boldsymbol{xL_n'(x) = nL_n(x) - n\,L_{n-1}(x)}$$

III $$\boldsymbol{L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)}$$

We have $$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$$

Differentiating with respect to x, we have

$$\sum_{n=0}^{\infty} t^n L_n'(x) = \frac{-t}{1-t}\sum_{r=0}^{\infty} t^r L_r(x) \quad \text{(as in II)}$$

$$= -t(1-t)^{-1}\sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -t\left(1+t+t^2+\ldots\right)\sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -t\sum_{s=0}^{\infty} t^s \sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -\sum_{s=0,\, r=0}^{\infty} t^{r+s+1} L_r(x) \quad \text{...(i)}$$

For fixed values of r, the coefficient of t^n on the R.H.S. is $-L_r(x)$, obtained by putting $r+s+1=n$ or $s=n-r-1$.

Total Coefficient of t^n is obtained by summing over all allowed values of r.

Since $s=n-r-1$ and $r \geq 0$

Therefore $n-r-1 \geq 0$ or $r \leq (n-1)$.

$\therefore$ Coefficient of t^n on the R.H.S. $= -\sum_{r=0}^{n-1} L_r(x)$

Therefore equating coefficient of t^n, on both sides of (i), we have

$$\mathbf{L_n'(x) = -\sum_{r=0}^{n-1} L_r(x).}$$

Illustrative Examples

Example 1: Prove that $L_n(0)=1$.

Solution: We have

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{\left[\frac{-tx}{(1-t)}\right]}$$

Putting $x=0$, we have

$$\sum_{n=0}^{\infty} t_n L_n(0) = \frac{1}{(1-t)} = (1-t)^{-1}$$

$$= 1+t+t^2+\ldots+t^n+\ldots$$

$$= \sum_{n=0}^{\infty} t^n$$

$$L_n(0) = 1$$

Example 2: Expand x^3+x^2-3x+2 in a series of Laguerre polynomials.

Solution: We know that $L_n(x)$ is a polynomial of degree n. Since x^3+x^2-3x+2 is a polynomial of degree 3, we may write

$$x^3+x^2-3x+2 = \sum_{r=0}^{3} C_r L_r(x) \qquad \text{...(i)}$$

Putting values of $L_0(x), L_1(x), L_2(x)$ and $L_3(x)$ from section 4.3, we have

$$x^3+x^2-3x+2 = c_0 + c_1(1-x) + c_2 . \frac{1}{2!}\left(2-4+x^2\right) + \frac{c_3}{3!}\left(6-18x+9x^2-x^3\right)$$

or $$x^3+x^2-3x+2 = (c_0+c_1+c_2+c_3)-(c_1+2c_2+3c)x + \left(\frac{c_2}{2}+\frac{3}{2}c_3\right)x^2-\frac{c_3}{6}x^3 \quad \text{...(ii)}$$

Equating coefficients of like powers of x on both sides of (ii), we get

$$c_0+c_1+c_2+c_3 = 2$$

$$c_1+2c_2+3c_3 = 3$$

$$\frac{1}{2}c_2+\frac{3}{2}c_3 = 1 \text{ and } -\frac{c_3}{6}=1$$

Solving these, we get,

$$c_3 = -6, C_2=20, c_1=-19, c_0=7 \quad \text{...(iii)}$$

Putting these values in (i) we get

$$x^3+x^2-3x+2 = 7L_0(x)-19L_1(x)+20L_2(x)-6L_3(x).$$

Example 3: Prove that

$$xL_n''(x)+(1-x)L_n'(x)+nL_n(x) = 0$$

and hence deduce that

$$L_n'(0) = -n$$

Solution: Since $L_n(x)$ satisfies the Laguerre's equation

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0, \text{ for } \lambda=n$$

$$\therefore \quad xL_n''(x)+(1-x)L_n'(x)+nL_n(x)=0.$$

Putting $x=0$, we have

$$L_n'(0)=-nL_n(0)$$

or $$L_n'(0)=-n \qquad (\text{since } L_n'(0)=-1)$$

Self Assessment

15. Express $L_4(x)$ in terms of $L_3(x)$ and $L_2(x)$

16. Show that

$$L_n'(x)-L_{n+1}'(x)=L_n(x)$$

17. Show that

$$L_n''(1)+nL_n(1)=0$$

4.6 Summary

- Laguerre differential equation has $x = 0$ as a regular singular point. Thus Frobenius method is applied to get a power series.
- For $\lambda = n, n$ being a positive integer we obtain a finite power series solution known as Laguerre polynomials $L_n(x)$. The highest power of $L_n(x)$ is x^n.
- Like in the previous units here we show a generating function, Rodrigue formula for $L_n(x)$.
- $L_n(x)$ for $n = 0,1,2\ldots$ form an orthogonal set of functions and satisfy orthogonality property.
- Various recurrence relations are obtained that help in understanding Laguerre polynomials.

4.7 Keywords

Laguerre Polynomials are a finite power series in x.

Frobenius Method: Laguerre differential equation has $x = 0$ as regular singular point. So Frobenius method on application gives a power series solution.

Orthogonal Relations of Laguerre polynomials are relations involving integrals of two Hermite polynomials. Due to these relations $L_n(x)$ for $n = 0,1,2,\ldots$ form an orthogonal set of functions.

4.8 Review Questions

1. Discuss the nature of singularities of the differential equation

$$xy'' + y' - xy = 0$$

2. Find all the singular points of the differential equation

$$\left(1 - x^2\right)y'' - xy' + x^2 y = 0$$

3. Show from recurrence relation III

$$L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$$

   Prove that

$$\int_0^\infty e^{-x} \frac{dL_n(x)}{dx} \cdot L_n(x) dx = 0, \text{ for } n = 1,2,\ldots$$

4. Show that $L_3(x), L_2(x)$ and $L_1(x)$ are related as

$$3L_3(x) = (5 - x)L_2(x) - L_1(x)$$

Answers: Self Assessment

1. (b)
2. (c)
3. finite
4. n
5. $L_1(x) = 1 - x, L_2(x) = \frac{1}{2}(2 - 4x + x^2)$
7. $L_3(x) = \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)$
8. (c)
12. (d)
13. $L_n(x)$

4.9 Further Readings

Books K. Yosida, Lectures on differential and Integral Equations

L.D. Landau and E.M. Lifshitz, Quantum Mechanics

Unit 5: Integral Equations and Algebraic System of Linear Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Remind ourselves that in unit six we studied Picard's method of showing the existence of the solution of first order differential equations which let us to integral equations.
- Study how to express a differential equation with boundary conditions or initial conditions into an integral equation.
- See the connection between an integral equation and an algebraic system of linear equations.

Introduction

In the next few units we are interested in studying various types of integral equations and see how to solve them.

You will learn how to express a differential equation with initial conditions into an integral equation.

In the case of boundary value problem of a differential equation we are let to Fredholm type of integral equations.

By dividing the interval into segments we will see how the solution of an integral equation reduced to an algebraic system to equations.

5.1 Connection between a First Order Differential Equation and Integral Equation

In unit 6 we studied the existence and uniqueness of the solution of the first order differential equation of the type

$$\frac{dy}{dx} = f(x, y) \quad ...(1)$$

with the initial conditions that at $x = x_0$, $y = y_0$. We assume that

1. The function $f(x, y)$ is real valued and continuous on a domain D of the xy plane given by

   $$x_0 - a \leqq x \leqq x_0 + a,\ y_0 - b \leqq y \leqq y_0 + b \qquad ...(2)$$

   where a and b are positive numbers

2. $f(x, y)$ satisfies the Lipschitz condition with respect to y in D, that is, there exists a positive constant k such that

   $$|f(x, y_1) - f(x, y_2)| \leqq K|y_1 - y_2| \qquad ...(3)$$

   for every pair of points $(x, y_2), (x, y_2)$ of D,

   with the help of Picard's method of successive approximation, it is then seen that $y(x)$ satisfies the integral equation

   $$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \qquad ...(4)$$

The integrand $f(t, y(t))$ on the right hand side of (4) is a continuous function, hence $y(x)$ is differentiable with respect to x, and its derivatives is equal to $f(x, y(x))$. Here the integral equation (4) can be solved by the method of successive approximation.

Uniqueness of the Solution: We have obtained the integral equations (4) for the solution $y(x)$ of (1) satisfying the initial conditions $x = x_0$, $y = y_0$. There remains an other important problem, the problem of uniqueness. Is there any other solution satisfying the same initial condition. Fortunately under our two assumptions, we can prove the uniqueness of the solution. To see this let $z(x)$ be another solution of (1) such that $x = x_0$, $z(x_0) = y_0$. Then

$$z(x) = y_0 + \int_{x_0}^{x} f(t, z(t))dt.$$

By the assumption 2, we obtain for $|x - x_0| \leqq b$

$$|y(x) - z(x)| \leqq K\left|\int_{x_0}^{x} |y(t) - z(t)|dt\right| \qquad ...(5)$$

Therefore, we also obtain $|x - x_0| \leqq h$

$$|y(x) - z(x)| \leqq KN|x - x_0|$$

where

$$N = Sup_{|x - x_0| \leqq h} |y(x) - z(x)|$$

Substituting the above estimate for $|y(t) - z(t)|$ on the right side of (5), we obtain further

$$|y(x) - z(x)| \leqq NK|x - x_0|^2/\underline{|2},$$

for $(x - x_0) \leqq h$. Substituting this estimate for $|y(t) - z(t)|$ once more on the right side of (5) we have

$$|y(x) - z(x)| \leqq KN|x - x_0|^3 / \lfloor 3$$

for $|x - x_0| \leqq h$. Repeating this substitution we obtain

$$|y(x) - z(x)| \leqq NK|x - x_0|^m / m!, \quad m = 1, 2, ...$$

for $|x - x_0| \leqq h$. The right side of the above inequality tends to zero as $m \to \infty$. This means that

$$N = \sup_{|x - x_0| \leqq h} |y(x) - z(x)|$$

is equal to zero. So the solution of $y(x)$ by the integral equal is unique also.

5.2 Conversion of a Differential Equation of Second Order to an Integral Equation

Example: Convert the differential equation

$$\frac{d^2y}{dx} + 2\frac{dy}{dx} - 8y = 5x^2 - 3x \qquad ...(1)$$

with the initial conditions

$$x = 0, \; y(x_0) = -2, \; \left.\frac{dy}{dx}\right|_{x=0} = 3. \qquad ...(2)$$

Solution 1: Let

$$y'' = \frac{d^2y}{dx^2} = G(x) \qquad ...(3)$$

Integrating (3) once yields the result

$$y'(x) = \int_0^x G(t)\, dt + C_1$$

For $x = 0$, this gives

$$y'(0) = 0 + C_1 = 3$$

therefore

$$y'(x) = \int_0^x G(t)\, dt + 3 \qquad ...(4)$$

Again integrating (4),

$$y(x) = \int_0^x \int_0^t G(t')\, dt'\, dt + 3\int_0^x dx + C_2$$

Notes

Integrating the first term on the right by parts we have

$$y(x) = \left[t\int_0^t G(t')dt'\right]\Bigg|_0^x - \int_0^x tG(t)dt + 3x + C_2$$

$$= x\int_0^x G(t')\,dt' - \int_0^x t'G(t')dt' + 3x + c_2$$

or

$$y(x) = \int_0^x (x - t')G(t')\,dt' + 3x + c_2$$

Subjecting this to the condition

$$y(x) = -2 \text{ at } x = 0$$

we get

$$-2 = 0 + 0 + c_2 \text{ or } c_2 = -2$$

so

$$y(x) = \int_0^x (x - t)G(t)\,dt + 3t - 2 \quad ...(5)$$

Writing (1) with the help of (3), (4) and (5), we have

$$G(x) + 2\int_0^x G(t)\,dt + 6 - 8\int_0^x (x - t)G(t)\,dt - 24t + 16 = 5x^2 - 3x$$

or

$$G(x) + \int_0^x (2 - 8x + 8t)G(t)\,dt - 5x^2 - 21t + 22 = 0 \quad ...(6)$$

Where

$$G(x) = \frac{d^2y}{dx^2} \quad ...(7)$$

Solution 2: We follow an other method. In this method we integrate equation (1) from 0 to x,

$$[y'(t)]\big|_0^x + 2[y(t)]\big|_0^x - 8\int_a^x y(t)\,dt = \frac{5}{3}x^3 - \frac{3}{2}x^2$$

or

$$y'(x) - y'(0) + 2y(x) - 2y(0) - 8\int_0^x y(t)\,dt = \frac{5}{3}x^3 - \frac{3}{2}x^2$$

but

$$y'(0) = 3,\ y(0) = -2$$

$$-y'(0) - 2y(0) = 1$$

∴

$$y'(x) + 2y(x) - 8\int_0^x y(t)\,dt = \frac{5}{3}x^3 - \frac{3}{2}x^2 - 1$$

Again integrating, we get

$$[y(t)]\big|_0^x + 2\int_0^x y(t)\,dt - 8\int_0^x (x - t)\,y(t)\,dt = \frac{5}{12}x^4 - \frac{x^3}{2} - x$$

or $$y(x) - y(0) + \int_0^x (-8x + 8t + 2)\, y(t)\, dt = \frac{5}{12}x^4 - \frac{x^3}{2} - x$$

or $$y(x) + \int_0^x (-8x + 8t + 2)\, y(t)\, dt = \frac{5}{12}x^4 - \frac{x^3}{2} - x - 2 \quad \text{...(8)}$$

Note: In this problem we have two answers, i.e. one for $\frac{d^2y}{dx^2}$ another for y for the same problem, but they lead to the same conclusion.

Self Assessment

1. Express the differential equation

   $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x^2 y(x) = 1 + x$ with the condition at $x = 0$, $y(0) = 4$, $\left.\frac{dy}{dx}\right|_{x=0} = 2$, into integral equation.

5.3 Fredholm Integral Equations and Boundary Value Problem

Let us consider the following example of a second order differential equation with the given boundary conditions and establish the integral equation

Example 1: Express the differential equation

$$\frac{d^2y}{dx^2} + a\,y(x) = 0,$$

with the boundary conditions

$$x = 0,\ y(0) = 0,\ x = 1,\ y(1) = 0,$$

as an integral equation

Solution: We have

$$\frac{d^2y}{dx^2} + a\,y(x) = 0 \quad \text{...(i)}$$

with $$y(0) = 0 = y(1) \quad \text{...(ii)}$$

Method 1: Let

$$\frac{d^2y}{dx^2} = G(x)$$

Integrating, we get

$$\frac{dy}{dx} = \int_0^x G(t)\, dt + c_1 \quad \text{...(iii)}$$

Notes

Again Integrating

$$y(x) = \int_0^x dt' \int_0^{t'} G(t)\,dt + c_1 + c_2$$

$$= \left[t'\int_0^{t'} G(t)\,dt\right]\Bigg|_0^x - \int_0^x t'G(t')\,dt' + c_1 x + c_2$$

$$= x\int_0^x G(t)\,dt - \int_0^x tG(t)\,dt + c_1 x + c_2$$

or $$y(x) = \int_0^x (x-t)G(t)\,dt + c_1 x + c_2 \qquad \text{...(iv)}$$

For x = 0, equation (iv) gives

$$0 = y(0) = 0 + c_1 0 + c_2 \text{ or } c_2 = 0$$

Now (iv) becomes

$$y(x) = \int_0^x (x-t)G(t)\,dt + c_1 x \qquad \text{...(v)}$$

For $t = 1$

$$y(1) = 0 = \int_0^1 (1-t)G(t)\,dt + c_1.1$$

or $$c_1 = -\int_0^1 (1-t)G(t)\,dt$$

Now equation (v) becomes

$$y(x) = \int_0^x (x-t)G(t)\,dt - \int_0^1 x(1-t)G(t)\,dt$$

$$= \int_0^x (x-t)G(t)\,dt + \int_0^1 (xt-x)G(t)\,dt$$

$$= \int_0^x (x-t)G(t)\,dt + \int_0^x (xt-x)G(t)\,dt + \int_x^1 (xt-x)G(t)\,dt$$

$$= \int_0^x (x-1) + G(t)\,dt + \int_x^1 x(t-1)G(t)\,dt$$

or $$y(x) = \int_0^1 K(x,t)G(t)\,dt \qquad \text{...(vi)}$$

Notes

with $$K(x, t) = \begin{cases} (x-1)t & \text{if} \quad t < x \\ x(t-1) & \text{if} \quad t > x \end{cases} \qquad \text{...(vii)}$$

Using this in (i),

$$G(x) + a\int_0^1 K(x,t)G(t)\,dt = 0$$

where $$G(x) = \frac{d^2y}{dx^2},\; K(x,t) = \begin{cases} (x-1)t, & t < x \\ (t-1)x, & t > x \end{cases}$$

Method 2:

Integrating (i) from 0 to x

$$\int_0^x y''(t)\,dt + a\int_0^x y(t)\,dt = 0$$

$$\left[y'(t)\right]\Big|_0^x + a\int_0^x y(t)\,dt = 0$$

or $$y'(x) - y'(0) + a\int_0^t y(t)\,dt = 0$$

Again integrating,

$$\left[y(t)\right]\Big|_0^x - y'(0)[t]\Big|_0^x + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) - y(0) - y'(0)x + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) - y'(0)x + a\int_0^x (x-t)\,y(t)\,dt = 0 \qquad \text{...(viii)}$$

Putting $x = 1$, this gives

$$y(1) - y'(0) + a\int_0^1 (1-t)\,y(t)\,dt = 0$$

or as $y(1) = 0$, we have

$$y'(0) = a\int_0^1 (1-t)\,y(t)\,dt$$

Notes

Substituting this in (viii) we have

$$y(x) - x\,a\int_0^1 (1-t)\,y(t)\,dt + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) + a\int_0^1 x(t-1)\,y(t)\,dt + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) + a\int_0^x x(t-1)\,y(t)\,dt + a\int_x^1 x(t-1)\,y(t)\,dt + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) + a\int_0^x t(x-1)\,y(t)dt + a\int_x^1 x(t-1)\,y(t)\,dt = 0$$

Taking

$$K(t, x) = \begin{cases} t(x-1) & , \quad t < x \\ x(t-1) & , \quad t > x \end{cases}$$

So we get

$$y(x) + a\int_0^1 K(t,x)\,y(t)\,dt = 0 \qquad \text{...(ix)}$$

Example 2: Express the differential equation

$$\frac{d^2y(x)}{dx^2} + \lambda\,y(x) = f(x) \qquad \text{...(1)}$$

into an integral equation. Here y, y' and f are continuous differentiable on the interval $0 < x < 1$ with the boundary conditions.

$$y(0) = 0 = y(1)$$

Following the method 2, let us integrate (1) from 0 to x, we have

$$\int_0^x y''(u)\,du + \lambda\int_0^x y(u)\,du - \int_0^x f(u)\,du = 0$$

or $$y'(x) - y'(0) + \lambda\int_0^x y(u)\,du - \int_0^x f(u)\,du = 0 \qquad \text{...(2)}$$

Integrating once again, we have

$$\int_0^x y'(x)\,dx - y'(0)\,x + \lambda\int_0^x (x-u)\,y(u)\,du - \int_0^x (x-u)\,f(u)\,du = 0$$

or $$y(x) - y(0) - y'(0)x + \lambda\int_0^x (x-u)y(u)\,du - \int_0^x (x-u)f(u)\,du = 0$$

or $$y(x) - y'(0)x + \lambda\int_0^x (x-u)y(u)\,du - \int_0^x (x-u)f(u)\,du = 0 \quad ...(3)$$

To find the value of $y'(0)$, put $x = 1$ in equation (3), we get

$$0 - y'(0).1 + \lambda\int_0^1 (1-u)\,y(u)\,du - \int_0^1 (1-u)f(u)\,du = 0$$

so $y'(0)$ is given by

$$y'(0) = \lambda\int_0^1 (1-u)y(u)\,du - \int_0^1 (1-u)f(u)\,du \quad ...(4)$$

Substituting this value of $y'(0)$ in equation (3) and rearranging terms we get

$$y(x) = \lambda\int_0^1 x(1-u)y(u)\,du - \lambda\int_0^x (x-u)y(u)\,du - \int_0^1 (1-u)f(u)\,du + \int_0^x (x-u)f(u)\,du$$

or $$y(x) = \lambda\int_0^x x(1-u)y(u)\,du - \lambda\int_0^x (x-u)y(u)\,du + \lambda\int_x^1 x(1-u)y(u)\,du + \int_0^x (x-u)f(u)\,du -$$

$$\int_0^x x(1-u)f(u)\,du - \int_x^1 (1-u)f(u)x\,du = 0$$

Simplifying the above equation we have

$$y(x) = \lambda\int_0^x u(1-x)y(u)\,du + \lambda\int_x^1 x(1-u)y(u)du + \int_0^x u(x-1)f(u)\,du - \int_x^1 x(1-u)f(u)\,du = 0 \quad ...(5)$$

Defining

$$K(u, x) = \begin{cases} u(x-1) & u < x \\ x(u-1) & u > x \end{cases}$$

We write equation (4) as

$$y(x) = -\lambda\int_0^1 K(u,x)y(u)\,du + \int_0^1 K(u,x)f(u)\,dx$$

Knowing $K(u, x)$ and $f(u)$, we know the second integral on the right hand side. Let us put

$$\int_0^1 K(u,x)f(u)\,du = \phi(x) \quad ...(6)$$

Notes

Thus $y(x)$ is given by the integral equation

$$y(x) = -\lambda\int_0^1 K(u,x)\,y(u)\,du + \phi(x) \qquad ...(7)$$

Self Assessment

2. Express

   $2y''(x) - 3y'(x) - 2y(x) = 4e^{-t} + 2\cos t$ with initial conditions $y(0) = 4, y'(0) = -1,$ into integral equation.

   (**Hint:** Integrate the differential equation twice and use initial conditions.)

5.4 Relation between Integral Equations and Algebraic System of Linear Equations

Consider the general linear Fredholm integral of the second kind for a function $\phi(x)$ of the type

$$\phi(x) - \lambda\int_0^1 K(x,y)\,dy = f(x) \qquad (0 \le x \le 1) \qquad ...(1)$$

and the linear Fredholm equation of the first kind is given by

$$\int_0^1 K(x,y)\,\phi(y)\,dy = f(x) \qquad (0 \le x \le 1) \qquad ...(2)$$

The problem of solving (1) and (2) can be considered as a generalization of the problem of solving a set of n linear algebraic equations in n unknown:

$$\sum_{s=1}^{n} a_{rs}x_s = b_r. \qquad (r = 1, 2, ...n) \qquad ...(3)$$

For this purpose we divide the interval $(0 \le x \le 1)$ into n segments and define

$$K(x, y) = K_{rs} \qquad (r, s = 1, 2, ...\, n)$$

and

$$f(x) = f_r$$

Here, x, y are divided into strips as ...(4)

$$\frac{r-1}{n} < x \le \frac{r}{n} \qquad (r = 1, 2, ...\, n)$$

$$\frac{s-1}{n} < y \le \frac{s}{n} \qquad (s = 1, 2, 3, ...\, n)$$

Then equation (1) becomes

$$\phi(x) = f_r + \lambda\sum_{s=1}^{n}\int_{(s-1)/n}^{s/n} K_{rs}\,\phi(y)\,dy \qquad \left(\frac{r-1}{n} < x \le \frac{r}{n}\right) \qquad ...(5)$$

Equation (5) shows that if a function $\phi(x)$ exists it must be a step function, i.e.

$$\phi(x) = \phi_r \qquad \left(\frac{r-1}{n} < x \le \frac{r}{n}\right), \quad (r = 1, 2, ... n)$$

then equation (1) can be written in the form

$$\phi_r - \lambda\sum_{s=1}^{n} K_{rs}\phi_s = f_r \qquad (r = 1, 2, 3, ... n) \qquad ...(6)$$

Define the determinant A with elements

$$\delta_{rs} - \frac{\lambda}{n}K_{rs} \qquad \text{for } r, s = 1, 2, ... \text{ n}$$

and
$$\delta_{rs} = \begin{Bmatrix} 0 & , & r \ne s \\ 1 & , & r = s \end{Bmatrix}$$

If the determinant A does not vanish, then (6), and therefore (5) has a unique solution for any given step function $f(x)$.

In the same way if we take up equation (2) and use equations (3) and (4) then equation (2) takes up the form

$$f_r = \sum_{s=1}^{n} K_{rs} \int_{(s-1)/n}^{s/m} \phi(y)\, dy \qquad \left(\frac{r-1}{n} < x \le \frac{r}{n}\right), \qquad ...(7)$$

This case of (7) is different than that of (5) as here one cannot conclude that $\phi(x)$ is necessarily a step function. All that can be said is that if we set

$$n \int_{(s-1)/n}^{s/n} \phi(y)\, dy = x_s,$$

then (7) becomes

$$F_r = \frac{\lambda}{n}\sum_{s=1}^{n} K_{rs}\, x_s \qquad (r = 1, 2, ... n) \qquad ...(8)$$

Here $x_1, x_2, ... x_n$ give the mean values of $\phi(x)$ is the successive intervals $\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right) ... \left(\frac{n-1}{n}, \frac{n}{n}\right),$

So there are infinitely many solutions of $\phi(x)$.

5.5 Summary

- In this unit we have seen how to convert a differential equation with conditions into an integral equation.
- The existence and uniqueness of the solution of the integral equation is based on Picard's method which puts some conditions on the Kernel as well on the function.
- It is seen that the integral equation is reduced to a algebraic system of equations if we divide the interval into segment.

5.6 Keywords

Integral equation is an equation in which the unknown variable appears under the integral sign.

The conversion of a ***differential equation*** into an integral equation is possible if the function and its first derivatives are continuous in the interval.

5.7 Review Questions

1. Express the differential equation

   $$y'' + 2y' - 8y = 0$$

   with boundary conditions $y(0) = 0 = y(1)$ as in integral equation.

2. Convert the differential equation

   $$y'' + 2y' - 8y = 5x^2 - 3x,$$

   with $y(0) = -2,\ y'(0) = 3$ into integral equation.

Answers: Self Assessment

1. $$G(x) + \int_0^x (x^3 - x - 4x^2)\, G(t)dt = 1 + 3x - 4x^2 - 2, \text{ with } G(x) = \frac{d^2y}{dx^2}$$

2. $$2y(x) + \int_0^t (-2t + 2u - 3)\, y(u)\, du = 4\, e^{-x} - 2\cos x - 10t + 6$$

5.8 Further Readings

Books

Louis A. Pipes and Lawrence R. Harnvill, Applied Mathematics for Engineers and Physicists

Tricomi, P.G., Integral Equation

Yosida, K., Lectures in Differential and Integral Equations

Unit 6: Volterra Equations and L_2 Kernels and Functions

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that integral equations can be of Volterra type equations of first or second kind or they can be Fredholm type of first or second kind.
- See that in the case of Volterra integral equations the upper limit depends upon the independent variable while in the case of Fredholm integral equations the limits are fixed.
- Understand that there are certain conditions on the Kernels as well on the functions for the existence of the solution. Here it is seen that the Kernels as well as the functions are L_2 class and so the solution does exist.

Introduction

L_2 class Kernels as well as functions are square integrable. So if the iteration procedure is applied one can see that product of two L_2 class Kernels is also L_2-class.

This method enables us to find the resolvent Kernels by L_2-class method and the solution of the integral equation is obtainable.

6.1 Classification of Integral Equations

In the last unit we studied the integral equations by converting a differential equation with boundary conditions or initial conditions. We see that the boundary conditions lead us to integral equations of the type

$$y(x) = f(x) + \int_a^b K(x, u)\, y(u) du \qquad ...(1)$$

or $$\int_a^b K(x,u)\, y(u) = f(x) \qquad ...(2)$$

In these cases the limits of integrations are fixed by some constants and the unknown variable appears inside the integral sign. These equations are known as Fredholm integral equations of the second kind (1) and the first kind (2) respectively.

We can also have integral equations of the following type.

$$y(x) = f(x) + \int_a^x K(x,u)\, y(u)du \qquad ...(3)$$

or $$\int_a^x K(x,u)\, y(u)du = f(x) \qquad ...(4)$$

In the equations (3) and (4) the limits of integration depends on the independent variable. Equations (3) and (4) are known as Volterra integral equations of the second kind and the first kind respectively.

We can take up the various types of integral equations and study them and devise methods of solving them. The solution of the integral equation is based on the properties of the Kernels $K(u, x)$ as well as the function $f(x)$.

In this unit we concentrate on the Volterra integral equations and in particular see how the solution of the Volterra integral equations are carried out along with the discussion of the L_2-Kernel.

6.2 Volterra Integral Equations

In the previous unit we had seen some difficulties in the solutions of the integral equation by converting them into an algebraic system of equations. It is seem there that when dealing with integral equation of the first kind we find the mean values of the function in the successive intervals $\left(0, \frac{1}{n}\right)\left(\frac{1}{n}, \frac{2}{n}\right)$, ... and so therefore the equation (2) of that section will possess infinite many solutions.

To avoid these difficulties, Vito Volterra investigated the solution of the integral equations in which the Kernel satisfies the conditions

$$K(x, y) = 0 \qquad \text{if } u > x \qquad ...(1)$$

This corresponds (in the sense of the previous unit) to the simple case of a system of algebraic linear equations where the elements of the determinant above the main diagonal are all zero.

We rewrite the integral equations of Volterra type of the second kind and first kind as follows:

$$y(x) - \lambda \int_0^t K(x,u)\, y(u)du = f(x) \qquad ...(2)$$

and $$\int_0^t K(x,u)\, y(u)du = f(x) \qquad ...(3)$$

In this section we shall study the Volterra integral equation of the second kind (2) that we can readily solve by Picard's process of successive approximation as discussed in unit 6. We state by setting $y_0(x) = f(x)$ and then determine $y_1(x)$:

$$y_1(x) = f(x) + \lambda \int_0^k K(x, u)\, f(u) du$$

Continuing in this manner we obtain an infinite sequence of functions

$$y_0(x), y_1(x), y_2(x)......y_n(x)...... \qquad ...(4)$$

satisfying the recurrence relations

$$y_n(x) = f(x) + \lambda \int_0^k K(x, u)\, y_{n-1}(u) du, \qquad (n = 1, 2, 3....) \quad ...(5)$$

Setting

$$y_n(x) - y_{n-1}(x) = \lambda^n\, \Psi_n(x) \qquad (n = 1, 2, 3....) \quad ...(6)$$

and putting

$\Psi_0(x) = f(x)$, we get

$$y_n(x) = \sum_{v=0}^{n} \lambda^v \psi_2\,(x) \qquad ...(7)$$

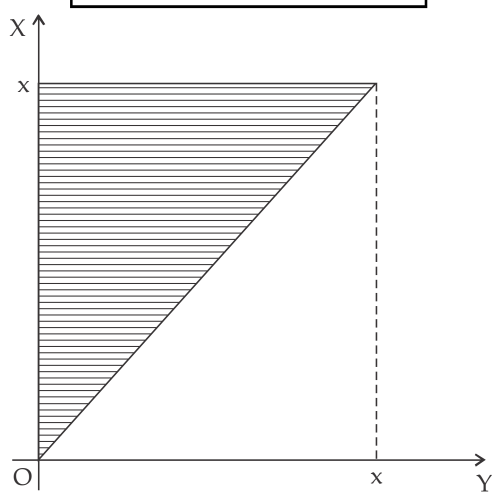
Also $\quad \psi_n(x) = \int_0^x K(x, u)\, \psi_{n-1}(u) du \qquad (n = 1, 2, 3, ...)$

Hence $\quad \psi_1(x) = \int_0^t K(x, u)\, f(u) du$

and $\quad \psi_2(x) = \int_0^t K(x, u_1)\, du_1 \int_0^{u_1} K(u_1, u) f(u) du$

This repeated integral be considered as a double integral over the triangular region indicated in the figure 25.1 thus interchanging the order of integration, we obtain

Figure 6.1



$$\psi_2(x) = \int_0^x f(u)\, du \int_u^x K(x, u_1)\, K(u_1, u) du_1$$

or $\quad \psi_2(x) = \int_0^x K_2(x, u) f(u) du$

Notes

where $K_2(x,u) = \int_u^x K(x,u_1)\, K(u_1,u) du_1$

Similarly, we find in general

$$\psi_n(x) = \int_0^x K_n(x,u)\, f(u) du \qquad \text{(n = 1, 2, 3, ...)} \qquad ...(8)$$

Where the integrated Kernels are defined as

$K_1(x, u) \equiv K(x, u)$, $K_2(x, u)$, $K_3(x, u)$......

are defined by the recurrence formula

$$K_{n+1}(x,u) = \int_0^x K(x,u_1)\, K_n(u_1,u) du_1 \qquad (n = 1, 2, 3, ...) \quad ...(9)$$

Moreover, it is easily seen that we also have

$$\begin{aligned} K_{n+1}(x,u) &= \int_0^x K_1(x,u_1)\, K_n(u_1,u) du_1 \\ &= \int_0^x K_{r_0}(x,u_1)\, K_{s_0}(u_1,u) du_1 \qquad r_0 + s_0 = n+1 \end{aligned} \qquad ...(9)$$

where $r_0 = 1$, $s_0 = n$.

Now $K_{n+1}(x,u) = \int_0^x K_1(x,u_1) \int_{u_1}^x K_1(u_1,u_2)\, K_{n-1}(u_2,u) du_2 du_1$

Interchanging the integrals we have

$$\begin{aligned} K_{n+1}(x,u) &= \int_0^x K_{n-1}(u_2,u) du_2 \int_{u_2}^x K_1(x,u_1)\, K_1(u_1,u_2) du_1 \\ &= \int_0^x K_{n-1}(u_2,u)\, K_2(x,u_2) du_2 \\ &= \int_0^x K_2(x,u_2)\, K_{n-1}(u_2,u) du_2 \end{aligned}$$

In the same way we get

$$K_{n+1}(x,u) = \int_0^x K_3(x,u_2)\, K_{n-2}(u_2,u) du_2$$

and so on. So we may write

$$K_{n+1}(x,u) = \int_0^x K_r(x,u_2)\, K_s(u_2,u) du_2 \qquad \text{where } (r = 1, 2, ...n,\ s = n - r + 1) \quad ...(10)$$

Now from equation (7)

$$\begin{aligned} y_n(x) &= \sum_{v=0}^{n} \lambda^v \psi_v(x) \\ &= \sum_{v=0}^{n} \lambda^v \int_0^x K_v(x,u) f(u) du \\ &= f(x) + \sum_{v=1}^{n} \lambda^v \int_0^x K_v(x,u) f(u) du \end{aligned}$$

or $y_n(x) = f(x) + \int_0^x \left[\sum_{v=1}^{n} K_v(x,u) \right] f(u) du$

Hence if the solution exists, it should be given by letting $n \to \infty$ and given by

$$f(x) = f(x) - \lambda \int_0^x H(x, u, \lambda)\, f(u) du \qquad ...(11)$$

where $H(x, u, \lambda)$ is the resolvent Kernel given by the series

$$H(x, u, \lambda) = -\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, u) \qquad ...(12)$$

This method of successive approximation cannot only be applied to those of Volterra type integral equations but a whole lot of other equations including the Fredholm integral equations.

Example: Let the Volterra integral equation be given by

$$y(x) = \overset{x}{X} - \int_0^x (x-t)\, y(t) dt \qquad \text{for } 0 \le x \le 1$$

The interacted Kernels are

$$K_1(x, t) = x - t$$

$$K_2(x, t) = \int_t^x (x-r)(r-t) dr = \frac{(x-t)^3}{1.2.3.}$$

$$K_3(x, t) = \int \frac{(x-r)^3 (r-t) dr}{1.2.3} = \frac{(s-t)^5}{\lfloor 5}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$K_n(x, t) = \frac{(x-t)^{2n-1}}{(2n-1)!}$$

Hence

$$y(x) = x - \int_0^s \left[(x-t) + \frac{(x-t)^3}{3!} + \frac{(x-t)^5}{5!} + ... \right] t.dt$$

$$= x + \left[\frac{(x-t)^3}{3!} - \frac{(x-t)^5}{5!} + \frac{(x-t)^7}{7!} .... \right]_0^x = \sin x$$

So the answer is

$y(x) = \sin x$

6.3 L_2-Kernels and Functions

In the case of Volterra integral equation

$$y(x) = f(x) + \lambda \int_0^x K(x, u)\, f(u) du \qquad ...(1)$$

The Kernel $K(x, u)$ and the $f(x)$ are supposed to be continuous and differentiable in the double interval $0 \le x \le h$ and $0 \le u \le h$. They are consequently bounded in the L_2-space. Namely the Kernel and the function $f(x)$ are quadratically integrable in the L_2-space i.e. $0 \le x \le h$ and $0 \le u \le h$ where h is constant i.e. the integrals

$$\|K\| = \int_0^h \int_0^h K^2(x, u) dx du \le N^2 \qquad ...(2)$$

$$\|t\| = \int_0^h f^2(x) dx \qquad ...(3)$$

Notes

exist and are finite in the Lebesgue sense while N is finite. Such a Kernel as well as the function will be called L_2 Kernel and *L2-function,* respectively.

The consequences of the Kernel being L_2-Kernel are many. One of them is as follows: The functions

$$A(x) = \left[\int_0^h K^2(x, u)du\right]^{1/2}, B(u) = \left[\int_0^h K^2(x, u)dx\right]^{1/2} \quad ...(4)$$

exist almost everywhere for $0 \le x \le h$ an $0 \le u \le h$ respectively. Also $A(x)$, $B(u)$ belong to L_2 class.

and finally that

$$\|K\|^2 = \int_0^h A^2(x)dx = \int_0^h B^2(u)du \quad ...(5)$$

Secondly, if $\phi(x)$ is any L_2-function in (0, h) then the two functions

$$\psi(x) = \int_0^h K(x, u)\phi(u)du, \psi(u) = \int_0^h K(x, u)\phi(x)dx \quad ...(6)$$

are also L_2-functions. This is an immediate consequence of the Schwarz inequality

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

From (6) it follows that

$$\|\psi\| \le \|K\|\,\|\phi\|, \|\chi\| \le \|K\|\,\|\phi\| \quad ...(7)$$

In the same way, it is easy to show that the composition of two L_2 Kernels $K(x, u)$ and $H(u, t)$ i.e. the formation of two new Kernels

$$G_1(x, u) = \int_0^h K(x, u_1)\, H(u_1\ u)du_1$$
$$G_2(x, u) = \int_0^h H(x, u_1)\, K(u_1\ u)du_1 \quad ...(8)$$

yields two new L_2-Kernels, such that

$$\|G_1\| \leqq \|K\|\,\|H\|, \ \|G_2\| \le \|H\|\,\|K\| \quad ...(9)$$

and so on. In fact this last formula give us useful bounds for the norms of the iterated Kernels

$$\|K_n\| \leqq \|K\|^n \quad ...(10)$$

Self Assessment

1. Show that the nth iterated Kernel $K_n(x, u)$ satisfies the bound

$$\|K_n\| \leqq \|K\|^n$$

6.4 Solution of Volterra Integral Equation of Second Kind

In the section we want to prove the existence and uniqueness of the solution of the Volterra integral equation of the second kind

$$y(x) - \lambda\int_0^x K(x, u)\, y(u)du = f(x) \quad (0 \le x \le h) \quad ...(1)$$

where the Kernel $K(x, u)$ and the function $f(x)$ belong to the class L_2. In the last sections we had seen that the solution is given by the formula

$$y(x) = f(x) - \lambda\int_0^x H(x, u, \lambda)\, f(u)du \qquad ...(2)$$

where the resolvent Kernel $H(x, u, \lambda)$ is given by the series of iterated Kernels

$$-H(x, y, \lambda) = \sum_{v=0}^{\infty} \lambda^v K_{v+1}(x, u) \qquad ...(3)$$

The series (3) converges almost everywhere. $H(x, u, \lambda)$ satisfies the integral equation

$$\begin{aligned} K(x, u) + H(x, u, \lambda) &= \lambda\int_y^x K(x, z)\, H(z, u, \lambda)dz \\ &= \lambda\int_y^x H(x, z, \lambda)\, K(z, u)dz \end{aligned} \qquad ..(4)$$

Proof: with the help of the Schwarz inequality, we first find

$$\begin{aligned} K_2^2(x, u) &= \left[\int_y^x K(x, u_1)\, K(u_1, u)du_1\right]^2 \\ &\le \int_y^x K^2(x, u_1)\, du_1 \int_y^x k^2(u_1, y)du_1 \\ &\le \int_0^h K^2(x, u_1)du_1 \int_0^h K^2(u_1, u)\, du_1 = A^2(x)\, B^2(y), \end{aligned}$$

and successively

$$\begin{aligned} K_3^2(x, u) &\le \int_y^x K^2(x, u_1)du_1 \int_y^x K_2^2(u_1, u)du_1 \\ &\le \int_0^h K^2(x, u_1)du_1 \int_y^x A^2(u_1)\, B^2(u)du_1 \\ &= A^2(x)B^2(u)\int_y^x A^2(u_1)du_1 \end{aligned}$$

$$\begin{aligned} K_4^2(x, u) &\le \int_y^x K^2(x, u_1)du_1 \int_y^x K_3^2(u_1, u)du_1 \\ &\le \int_0^h K^2(x, u_1)du_1 \int_y^x A^2(u_1)\, B^2(u)du_1 \int_u^x A^2(u_2)du_2 \\ &= A^2(x)B^2(u)\int_y^x A^2(u_1)du_1 \int_u^x A^2(u_2)du_2 \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned}$$

In general, we can write

$$K_{n+2}^2(x, u) \le A^2(x)B^2(u)\, F_n(x, u) \qquad (n = 1, 2, 3, ...) \quad ...(5)$$

where

$$F_1(x, u) = \int_y^x A^2(u_1)du_1,\ F_2(x, u) = \int_u^x A^2(u_1)F_1(u_1, u)dz, ...$$

or generally

$$F_n(x, u) = \int_y^x A^2(z)\, F_{n-1}(z, u)dz, \qquad (n = 2, 3, ...) \quad ...(6)$$

Notes

Now we state that

$$F_n(x,u) = \frac{1}{n!}F_1^n(x,u) \qquad (n = 1, 2, 3, ...) \quad ...(7)$$

This formula is obviously valid for n = 1. If it is assumed true for n = 1, it also remains valid for n, since it follows from (6) that

$$F_n(x,u) = \frac{1}{(n-1)!}\int_y^x A^2(z)\, F_1^{n-1}(z,u)dz$$
$$= \frac{1}{(n-1)!}\int_y^x F_1^{n-1}(z,u)\frac{\partial F_1(z,u)}{\partial z}dz$$
$$= \frac{1}{(n-1)!}\left[\frac{1}{n}F_1^n(z,u)\right]_{z=u}^{z=x} = \frac{1}{n!}F_1^n(x,u)$$

On the other hand from equation (2) of the section it follows that

$$0 \le F_1(x,u) \le \int_0^h A^2(z)dz \le N^2$$

hence

$$0 \le F_n(x,u) \le \frac{1}{n!}N^{2n}$$

and by substituting into (5) we obtain

$$\left|K_{n+2}(x,u)\right| \le A(x)\, B(u)\frac{N^n}{\sqrt{n!}}, (n = 0, 1, 2, ...)$$

Neglecting the first term, this shows that the infinite series (3) or that of equation (12) of section (25.2) which gives the resolvent Kernel H, has the majorant

$$M(x,u) = |\lambda|\, A(x)\, B(u)\sum_{n=0}^{\infty}\frac{(N|\lambda|)^n}{\sqrt{n!}},$$

where the last series always converges because the power series

$$\sum_{n=0}^{\infty}\frac{Z^n}{\sqrt{n!}}$$

has an infinite radius of convergence. This is not sufficient to insure that the series (3) be uniformly and absolutely convergent everywhere, but it is sufficient to ensure its uniform convergence almost every where, because the functions $A(x)$ and $B(u)$ may become infinite in a subset of $(0, h)$ of measure zero. However a fundamental theorem of Lebesgue allows the integration of the series term-by-term, because $M(x, u)$ is a L_2 function. In such a case, we will say that the series is almost uniformly convergent.

If follows that term-by-term integration can be used to evaluate

$$\int_u^x K(x,t)\, H(t,u,\lambda)dt, \int_u^x H(x,u_1,\lambda)\, K(u_1,u)du_1$$

Remembering that

$$K_n(x,u) = \int_y^x K_v(x,z)K_{n-v}(z,u)dz, \qquad (h = 1, 2, ...n - 1) \quad ...(8)$$

we obtain the basic equation (4). Here the interchange of order improving (8) is allowed under our hypothesis that K and hence K_n and H belong to L_2-class.

Notes

With the help of (4), it is easy to prove that the function $y(x)$ given by (2) satisfies (1). Also

$$y_0(x) = f(x) - \lambda\int_0^x H(x, u, \lambda) f(u) du \qquad ...(9)$$

certainly belongs to L_2 proved that $f(x)$ belongs to the same class. But then we have

$$y_0(x) - \lambda\int_0^x K(x, u) y_0(u) du$$

$$= f(x) - \lambda\int_0^x H(x, u, \lambda) f(u) du - \lambda\int_0^x K(x, u) f(u) du + \lambda^r \int_0^x K(x, z) dz \int_0^z H(z, u, \lambda) f(u) du$$

$$= f(x) - \lambda\int_0^x \left[K(x, u) + H(x, u, \lambda) - \lambda\int_0^x K(x, z) H(z, u, \lambda) dz\right] f(u) du$$

$$= f(x) + 0 = f(x)$$

So the function $y_0(x)$ from (9) is the only function of class L_2 of the given equation, neglecting the function $y(x)$ given by

$$y(x) - \lambda \int_0^x K(x, u) \phi(u) du \qquad ...(10)$$

known as a zero function in L_2-space. For this we observe that let v be the norm of $y(x)$ in the basic interval $(0, h)$

$$v^2 = \int_0^h y^2(x) dx$$

then from (10) using Schwarz inequality, it follows that

$$y^2(x) \le |\lambda|^2 \int_0^x K^2(x, u) du \int_0^x y^2(z) dz \le |\lambda|^2 A^2(x) v^2$$

and successively

$$y^2(x) \le |\lambda|^4 v^2 \int_0^x K^2(x, u) du \int_0^x A^2(z) dz = |\lambda|^4 v^2 A^2(x) \int_0^x A^2(u) du$$

$$y^2(x) \le |\lambda|^6 v^2 \int_0^x K^2(x, u) du \int_0^x A^2(u) du \int_0^u A^2(z) dz$$

$$= |\lambda|^6 v^2 A^2(x) \int_0^x A^2(u) du \int_0^y A^2(z) dz$$

By analogy to (7) we have

$$\int_0^x A^2(u_1) du_1 \int_0^{u_1} ...... \int_0^{u_{n-1}} A^2(u_n) du_n = \frac{1}{n!}\left[\int_0^x A^2(u) du\right]^n \le \frac{N^{2n}}{n!} \qquad ...(11)$$

hence we can write

$$\phi^2(x) \le |\lambda|^2 v^2 A^2(x) \frac{(|\lambda|^2 N^2)^n}{n!}, (n = 0, 1, 2, ...)$$

and this shows that $y(x) = 0$ at any point where $A(x)$ is finite. So we have shown that $y_0(x)$ is a unique solution of (1).

An alternative approx of proving the existence and uniqueness of the solution of the Volterra integral equation is by Picard's process of successive approximation method. It is advisable to try it as an alternative as given in Yosida book.

Self Assessment

2. Show that for L_2 Kernel $K(x, t)$ the nth iterated Kernel of Volterra integral equation $K_n(x, t)$ is also L_2 class.

6.5 Summary

- Volterra integral equations are obtained by converting a differential equation with initial conditions.
- For L_2-Kernels the resolvent Kernel can be found by iterated Kernel in the limit of $n \to \infty$.
- For degenerate type of Kernels the resolvent Kernel can be obtained in a simpler way.

6.6 Keywords

Kernel that is L_2 class has the same properties as a square integrable integral.

The ***L_2 class*** nature of the Kernel as well as the function of L_2 class helps finding the solution by iteration.

6.7 Review Questions

1. What ae integral equation. Give examples.
2. How will you classify integral equations?
3. Account for volterra integral equations.
4. What are L_2 Kernel and functions? Explain with suitable examples.
5. Consider the volterra equation with Kernel function

   $$\hat{K}(t) = K_k(t) + \theta$$

   where $k = 2$, $\theta = 10^{-3}$ and k_k indefined by

   $$k_k(t) = \frac{1}{2t^{3/2}\sqrt{k\pi}}\exp\left[\frac{-1}{4kt}\right]$$

   construct a solution function.

6.8 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 7: Volterra Integral Equation of the First Kind

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know various types of Kernels and how they help in solving the integral equations.
- Understand that it is difficult to solve Volterra integral equation of the first kind. It can first be converted to Volterra integral equation of the second kind and the methods discussed earlier in units can be employed to solve it.

Introduction

Volterra integral equations of the first kind is by suitable method converted into Volterra integral equations to solve it by suitable method.

The resolvent Kernel can be found easily in the case of Volterra integral equation of the second kind.

7.1 Volterra Equations of First Kind, function and Kernel Classes

In this unit we present methods for solving Volterra linear equations of the first kind which have the form

$$\int_0^x K(x,u)\,y(u)\,du = f(x) \qquad \text{...(1)}$$

Here $y(x)$ is unknown function on the interval $a \le x \le b$ $K(x, u)$ is the Kernel of the equation and $f(x)$ is a given known function. The functions $y(x)$, $f(x)$ are usually assumed to be continuous or square integrable on (a, b). The Kernel $K(x, u)$ is assumed to either continuous or the square $a \le x \le b$, $a \le u \le b$ or it satisfies the condition

$$\int_a^b\int_a^b K^2(x,u)\,dx\,du = N^2 < \infty \qquad \text{...(2)}$$

i.e. $K(x, u)$ is of class L_2. Also $K(x, u) = 0$ for $u > x$.

Notes

We now classify some of the Kernels as follows:

1. ***Degenerate Kernels or Poincere Goursat Kernels***

   The Kernel $K(x, u)$ of the integral equation is said to be degenerate if it can be represented in the form

$$K(x, u) = g_1(x)h_1(u) + g_2(x)h_2(u) + \ldots \qquad \ldots(3)$$

2. ***Difference Kernel***

   The Kernel of the integral equation is said to be difference Kernel if it depends upon the difference of the arguments,

$$K(x, u) = K(x - u).$$

3. ***Polar Kernels***

   They are of the form

$$K(x, u) = \frac{L(x,u)}{(x-u)^{\beta}} + M(x,u) \qquad 0 < \beta < 1 \qquad \ldots(4)$$

   where $L(x, u)$ and $M(x, u)$ are continuous on the square

   $a \le x \le b$, $a \le u \le b$ and $L(x, x) \ne 0$

4. ***Logarithmic Kernels***

   They are of the form

$$K(x, u) = L(x, u) \log (x - u) + \text{M} (x - u) \qquad \ldots(5)$$

   The following generalized Abel equation is a special case of equation (1) with the Kernel of the form (4)

$$\int_0^x \frac{y(u)\,du}{(x-u)^{\beta}} = f(x) \qquad 0 \le \beta \le 1 \qquad \ldots(6)$$

Example: In case the Kernel $K(x, u)$ and $f(x)$ are continuous then $f(x)$ must satisfy the following conditions:

(i) If $K(a,a) \ne 0$, then $f(\alpha) = 0$

(ii) If $K(a,a) = K_x^1(a,a) = K_x^2(a,a) \ldots = K_x^{n-1}(a,a) = 0$, and

$$0 \le K_x^n(a,a) \le \infty,$$

then $$f(a) = f^1(a) = \ldots = f^n(a) = 0.$$

7.2 Reduction of Volterra Equations of the First Kind to Volterra Equations of the Second Kind

Consider Volterra integral equation of the first kind

$$\int_0^x K(x,u)\, y(u)\,du = f(x) \qquad \ldots(1)$$

Notes

Also suppose that the Kernel $K(x, u)$ and the function $f(x)$ have continuous derivatives on the interval $a \le x \le b$ and $a \le u \le b$ i.e.

$$\frac{d}{dx} f(x), \frac{\partial K(x,u)}{\partial x}, \frac{\partial K(x,u)}{\partial u}$$

exist and continuous, the equation (1) can be reduced to that of second kind provided $k(x, x) \neq 0$.

To see that differentiate (1) with respect to x,

$$K(x,x)y(x) + \int_0^x \frac{\partial K(x,u)}{\partial x}.y(u)\,du = \frac{df}{dx}$$

or
$$y(x) + \int_0^a \frac{\frac{\partial}{\partial x}K(x,u)}{K(x,x)} y(u)\,du = \frac{\frac{dt}{dx}}{K(x,x)} \quad ...(2)$$

which is the Volterra equation of the second kind with Kernel

$$\frac{\partial}{\partial x}[K(x,u)] \Big/ K(x,x)$$

and the function

$$\frac{df}{dx} \Big/ K(x,x).$$

If $K(x, x) = 0$ then we have to differentiate twice to reduce the equation to that of second kind.

There is a second method of reducing the Volterra equation of the first kind to Volterra equation of the second kind. For this consider the equation (1)

$$\int_0^x K(x,u)\, y(u)\,du = f(x) \quad ...(1)$$

If we set
$$\int_0^x y(u)\,du = Z(x) \quad ...(2)$$

Clearly
$$Z(0) = 0$$

Now integrate by parts of L.H.S. of the integral i.e.

$$\int_0^x K(x,u)\frac{dz}{du}(u)\,du = f(x)$$

or
$$K(x,u)Z(u)\Big|_{u=0}^{u=x} - \int_0^x \frac{\partial K(r,u)}{\partial u} Z(u)\,du = f(x)$$

or
$$K(x,x)Z(x) - \int_0^x \frac{\partial K(x,u)}{\partial u} Z(u)\,du = f(x)$$

or $$Z(x) - \int_0^x \frac{\frac{\partial K(x,u)}{\partial u}}{K(x,x)} Z(u)\,du = \frac{f(x)}{K(x,x)} \quad ...(3)$$

which is Volterra equation of second kind with Kernel

$$\frac{\frac{\partial K(x,u)}{\partial u}}{K(x,x)}$$

and the function $f(x)/K(x,x)$. Here it is assumed that $K(x, x) \neq 0$. Applying the techniques of last unit we can write the solution of $Z(x)$ as

$$Z(x) = \frac{f(x)}{K(x,x)} - \int_0^x H^{\alpha}(x,u,1)\frac{f(u)}{K(u,u)}du \quad ...(4)$$

where H^{α} $(x, u, 1)$ is the resolvent Kernel corresponding to the Kernel $\frac{d}{du}K(x,u)/K(x,x)$.

Example 1: Consider the Volterra integral equation of the first kind

$$\int_0^x K(x,u)y(u)\,du = f(x) \quad ...(1)$$

with the Kernel $K(x, u)$ given by

$$K(x, u) = e^{x-u}$$

So the equation (1) becomes

$$\int_0^x e^{x-u}y(u)\,du = f(x) \quad ...(2)$$

Let us put

$$\int_0^x y(u)\,du = Z(x) \quad ...(3)$$

So that $Z(0) = 0$ and $y(x) = \frac{dz(x)}{dx}$.

Substituting this value of y in (2) we have

$$\int_0^x e^{x-u}\frac{dz}{du}(u)\,du = f(x)$$

Integrating L.H.S. by parts once we have

$$e^{x-u}Z(u)\Big|_0^x + \int_0^x e^{x-u}Z(u)\,du = f(x)$$

or $$z(x)+\int_0^x e^{x-u}Z(u)\,du = f(x) \qquad ...(4)$$

Which is the Volterra integral equation of the second kind. The equation (4) can be solved by the method developed in the last unit. Here

$$K(x,u) = e^x.e^{-u} \qquad ...(5)$$

$$= K_1(x,u)$$

Example 2: Consider the Volterra equation of the first kind

$$\int_0^x K(x,u)y(u) = f(x) \qquad ...(1)$$

Where $K(x, u)$ is a degenerate Kernel of the form

$$K(x,u) = g_1(x)g_2(u)+h_1(x)h_2(u) \qquad ...(2)$$

Substituting in (1) we get

$$\int_0^x g_1(x)g_2(u)y(u)\,du+\int_0^x h_1(x)h_2(x)y(u)\,du = f(x)$$

or

$$g_1(x)\int_0^x g_2(u)y(u)du+h_1(x)\int_0^x h_2(u)y(u)\,du = f(x) \qquad ...(3)$$

Let us introduce an other variable $Z(x)$ by the relation

$$Z(x) = \int_0^x g_2(u)y(u)\,du \qquad ...(4)$$

where $$Z(0) = 0$$

and $$\frac{dz(x)}{dx} = g_2(x)y(x) \qquad ...(5)$$

So equation (3) becomes

$$g_1(x)Z(x)+h_1(x)\int_0^x \frac{h_2\dfrac{dz}{dx}}{g_z(u)}du = f(x)$$

Now integrating by parts the integral on L.H.S. we have

$$g_1(x)Z(x)+h_1(x)\frac{h_2(u)}{g_2(u)}Z(u)\Bigg|_0^x - h_1(x)\int_0^x \frac{d}{du}\left[\frac{h_2(u)}{g_2(u)}\right]Z(u)\,du = f(x)$$

or

$$\left[g_1(x)g_2(x)+h_1(x)h_2(x)\right]Z(x)-h_1(x)g_2(x)\int_0^x \frac{d}{dx}\left[\frac{h_2(u)}{g_2(u)}\right]Z(u)\,du \quad = \quad f(x)g_2(x).$$

Simplifying equation (6) we have

$$Z(x)-\int_0^x \frac{h_1(x)g_2(x)}{\left[g_1(x)g_2(x)+h_1(x)h_2(x)\right]}\frac{d}{du}\left[\frac{h_2(u)}{g_2(u)}\right]Z(u)\,du \quad = \quad \frac{f(x)g_2(x)}{\left[g_1(x)g_2(x)+h_1(x)h_2(x)\right]} \quad ...(7)$$

So equation (7) is Volterra equation of the second kind. Putting

$$K^N(x,u) \quad = \quad \frac{h_1(x)g_2(x)}{[g_1(x)g_2(x)+h_1(x)h_2(a)]}\frac{d}{du}\left[\frac{h_2(u)}{g_2(u)}\right]$$

and

$$f^N(x) \quad = \quad \frac{f(x)g_2(x)}{\left[g_1(x)g_2(x)+h_1(x)h_2(x)\right]}$$

equation (7) can be put into the form

$$Z(x)-\int_0^x K^N(x,u)\,Z(u)\,du \quad = \quad f^N(x) \quad ...(8)$$

Knowing $g_1(x), g_2(x), h_1(x), h_2(x)$ and $f(x)$ we can then solve equation (8) by the methods of the last unit.

Self Assessment

1. Solve the integral equation

$$y(x)=f(x)+\lambda\int_0^x e^{2(x-u)}\,y(u)\,du$$

7.3 Summary

- Volterra integral equation of the first kind may have a number of different kinds of Kernels.
- It is sometimes useful to convert Volterra integral equation into Volterra integral equation of the second kind.
- By converting Volterra integral equation into that of second order the method of solving the Volterra integral equation of second kind may be employed.

7.4 Keyword

Volterra integral equation of the first kind is related to Volterra integral equation of the second kind and the solution of Volterra integral equation of the first kind can be found by the methods already used.

7.5 Review Question

Convert the Volterra integral equation of the first kind

$$\int_0^x K(x,t)\, y(t)\, dt = x^2$$

where $K(x, t)$ is a degenerate Kernel of the form

$$K(x,t) = xt + (x+1)(t+1).$$

into integral equation of the second kind.

Answer: Self Assessment

1. $y(x) = f(x) + \lambda \int_0^x e^{(2+\lambda)(x-u)} f(u)\, du$

7.6 Further Readings

Books

Tricomi, F.G., Integral Equation

Yosida, K., Lectures in Differential and Integral Equation

Notes

Unit 8: Volterra Integral Equations and Linear Differential Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that the existence and uniqueness of the solution of differential equations leads us to the integral equations
- See the relation between the integral equations and the linear differential equations with initial conditions.
- Understand that the solution of the integral equation also satisfies a certain differential equation with boundary conditions.

Introduction

The connection between a differential equation and integral equation should be seen clearly.

This connection helps us to solve certain differential equations by converting it into an integral equation and vice versa.

8.1 Relation between Linear Differential Equations and Volterra Integral Equations

In the unit 24 we had seen that a differential equation of first order or second order under certain conditions is converted into an integral equation. This idea can be further explained in details in this unit. Let us consider an nth order linear differential equation as follows:

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + ... + a_n y = f(x) \quad ...(1)$$

It is assumed that the unknown functions $y(x)$, $f(x)$, $a_1(x)$ $a_2(x)$,...$a_n(x)$ are continuous and differentiable on the interval (a, b). The function $y(x)$ satisfies the following initial conditions:

$$y(0) = y_0, y'(0) = y_0', y''(0) = y_0''......\overset{(n-1)}{y}(0) = \overset{(n-1)}{y_0} \; ... \quad ...(2)$$

Notes

To convert the linear differential equation (1) into an integral equation we introduce a function $\phi(x)$ by the relation

$$\frac{d^n y}{dx^n} = \phi(x) \qquad ...(3)$$

Integrating once we have by taking into account (2),

$$\left.\frac{d^{n-1}y}{dx^{n-1}}\right]_0^x = \int_0^x \phi(u)du$$

$$\frac{d^{n-1}y(x)}{dx^{n-1}} = y_0^{n-1} + \int_0^x \phi(u)du$$

Integrating once more we have

$$\frac{d^{n-2}}{dx^{n-2}}y(x) = y_0^{n-2} + y_0^{n-1}x + \int_0^x du \int_0^u \phi(u_1)du_1$$

$$= y_0^{n-2} + y_0^{n-1}x + \int_0^x (x-u)\phi(u)du \qquad ...(4)$$

In general integrating up to n times we have

$$y(x) = y_0 + y_0'x + y_0''\frac{x^2}{\underline{|2}} + y_0'''\frac{x^3}{\underline{|3}} + ... + y_0^{n-2}\frac{x^{n-2}}{\underline{|n-2}} + y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1}\phi(u)du... \qquad ...(5)$$

Writing (1) with the help of (3), (4) and (5) we have

$$\phi(x) + a_1(x)\left[y_0^{n-1} + \int_0^x \phi(u)du\right] + a_2(x)\left[y_0^{n-2} + xy_0^{n-1} + \int_0^x (x-u)\phi(u)du\right]$$

$$+ a_3(x)\left[y_0^{n-3} + xy_0^{n-2} + \frac{x^2}{\underline{|2}}y_0^{n-1} + \frac{1}{\underline{|2}}\int_0^x (x-u)^2\,\phi(u)du\right] + ... + ...$$

$$+ a_n\left[y_0 + y_0'\,x + y_0''\frac{x^2}{\underline{|2}} + ... + y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-n)^{n-1}\phi(u)du\right] = f(x) \qquad ...(6)$$

Defining

$$F(x) = f(x) - a_1(x)y_0^{n-1} - a_2(x)\left[y_0^{n-2} + xy_0^{n-1}\right] - a_3\left[y_0^{n-3} + xy_0^{n-2} + \frac{x^2}{2}y_0^{n-2}\right]$$

$$+ ...... - a_n\left[y_0 + xy_0' + \frac{x^2}{\underline{|2}}y_0'' + ... + y_0^{n-1}\frac{x^{n-1}}{(n-1)!}\right] \qquad ...(7)$$

and

$$K(x,u) = a_1(x) + a_2(x)\,(x-u) + \frac{a_3(x)}{2!}(x-u)^2 + ... + a_n\frac{(x-u)^{n-1}}{(x-1)!}$$

or $$K(x,u) = \sum_{k=1}^{n} a_k\frac{(x-u)^{k-1}}{(k-1)!} \qquad ...(8)$$

Substituting (7) and (8) into (6) we get

$$\phi(x) + \int_0^x K(x,u)\,\phi(u)du = F(x) \qquad ...(9)$$

So we have converted a differential equation (1) into the Volterra integral equation of the second kind with Kernel given by (8) and the function given by (7). The unknown function being given by (3).

Example: Convert

$$2y''(x) - 3y'(x) - 2y(x) = 4e^{-x} + 2\cos x \qquad ...(1)$$

with $y(0) = 4$, $y'(0) = -1$, into integral equation

Let us put

$$y''(x) = G(x) \qquad ...(2)$$

Integrating (1) with respect to x, we have

$$y'(x)\Big|_0^x = \int_0^x G(u)du$$

or $$\frac{dy}{dx} - y'(0) = \int_0^x G(u)du$$

$$\frac{dy}{dx} = -1 + \int_0^x G(u)du \qquad ...(3)$$

Integrating with respect to x again we have

$$y(x)\Big|_0^x = -x + \int_0^x du_1 \int_0^{u1} G(u)du$$

or $$y(x) = y(0) - x + \int_0^x (x-u)G(u)du$$

$$y(x) = 4 - x + \int_0^x (x-u)\,G(u)du \qquad ...(4)$$

Substituting from equations (2), (3) and (4) into (1) we have

$$2G(x) - 3\left[-1 + \int_0^x G(u)du\right] - 2\left[4 - x + \int_0^x (x-u)\,G(u)du\right] = 4e^{-x} + 2\cos x$$

Rearranging we have

$$2G(x) + \int_0^x G(u)du\left[-3 - 2(x-u)\right] = 4e^{-x} + 2\cos x - 3 + 2\,(4-x) \qquad ...(5)$$

$$2G(x) + \int_0^x K(x,u)\,G(u)du = F(x) \qquad ...(6)$$

where

$$\begin{aligned} K(x,u) &= -3 - 2(x-u) \\ F(x) &= 4e^{-x} + 2\cos x + 5 - 2x \end{aligned} \qquad ...(7)$$

So we get Volterra integral equation of the second kind.

Self Assessment

1. Convert the linear differential equation

$$\frac{d^3y}{dx^3} + 6y(x) = 0 \;\text{ with }\; y(0) = 4,\, y'(0) = -3,\, y''(0) = 2$$

8.2 Conversion of Volterra Integral Equation of Second Kind into a Differential Equation

We have seen that a linear differential equation with initial conditions can be expressed into a Volterra integral equation. In this section we can show that an integral equation can also be converted into a linear differential equation. To see that we take up the following example.

Example: Convert the integral equation

$$y(x) = 3x - 4 - 2\sin x + \int_0^x \left[(x-u)^2 - 3(x-u) + 2\right] y(u)du \qquad ...(1)$$

into the linear differential equation.

Before attempting the problem we know that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} K(t,u)y(u)du = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}K(t,u)y(u)du + K[t,b(t)]\frac{db}{dt} - K[t,a(t)]\frac{da}{dt} \qquad ...(2)$$

using equation (2), differentiate (1) with respect to x, we have

$$y'(x) = 3 - 2\cos x + \left[(x-x)^2 - 3(x-x) + 2\right]y(x) + \int_0^x \left[2(x-u) - 3\right]y(u)du$$

or $$y'(x) = 3 - 2\cos x + 2y(x) + \int_0^x \left[2(x-u) - 3\right]y(u)du \qquad ...(3)$$

Differentiating (3) again, we have

$$y''(x) = 2\sin x + 2y'(x) + \left[2(x-x) - 3\right]y(x) + \int_0^x (2)y(u)du$$

or $$y''(x) = 2\sin x + 2y'(x) - 3y(x) + 2\int_0^x y(u)du \qquad ...(4)$$

Differentiating equation (4) again, we have

$$y'''(x) = 2\cos x + 2y''(x) - 3y'(x) + 2y(x)$$

or $$y'''(x) - 2y''(x) + 3y'(x) - 2y(x) = 2\cos x \qquad ...(5)$$

Self Assessment

2. Convert the integral equation

$$y(x) = 2x^2 - 3x + 3\cos x + \int_0^x \left[2(x-u)^3 + 3(x-u)^2 + 6\right]y(u)du$$

8.3 Summary

- We have taken up the case of nth order differential equation and have seen how an integral equation can be established.
- There is a strong connection between the initial value differential equation and the Volterra integral equation of the second type or of first type.

8.4 Keywords

The relation between the ***Volterra integral equation*** and ***linear differential equation*** with initial condition has to be understood.

Method of conversion of differential equation to integral equation shows that the solution is unique as we show that the new integral equation satisfies the original differential equation.

8.5 Review Questions

1. Convert the differential equation

   $$y''(x) - xy'(x) + x^2 y(x) = 1 + x$$

   with $y(0) = 4$, $y'(0) = 2$, into integral equation.

2. Convert the differential equation

   $$y''(x) - 3y'(x) + 2y(x) = 4\left(x - \frac{x^3}{6}\right)$$

   with $y(0) = 1$, $y'(0) = -2$

Answers: Self Assessment

1. $G(x) + 3\int_0^x (x-u)^2 G(u)du = 18x - 24 - 3x^2$ with $G(x) = \dfrac{d^3y}{dx^3}$

2. $y^{iv}(x) - 6y'''(x) - 6y'(x) - 12y(x) = 3\cos x$

8.6 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 9: Integral Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Classify the type of Fredholm integral equations.
- Classify the Kernel of any integral equation i.e. is it symmetric or Poincere Goursat type or of different type?
- Choose the right method of solving the integral equation.

Introduction

You have learnt in the previous few units the Volterra integral equation of the second and first kind.

You will find similarities and differences in approach between the two types of integral equations.

9.1 Fredholm Equations

In the last three units we studied one type of integral equation known as Volterra integral equation. In the next few units we are interested in studying an other integral equation known as Fredholm integral equation.

In the case of Volterra integral equation we saw that linear differential equations with initial condition lead us to Volterra integral equation. In the case of boundary value problem, the differential equations can be converted into Fredholm integral equation.

Now the Fredholm equations can be of the form

$$Q(x) = f(x) + \lambda \int_a^b K(x,t)\, Q(t)\, dt \quad ...(1)$$

or

$$f(x) = \lambda \int_a^b K(x,t) Q(t)\, dt \quad ...(2)$$

Here $K(x, t)$ the Kernel and $f(x)$ the function are known and $Q(x)$ is an unknown function on the interval $a \le x \le b$.

Let $\Psi(x)$ be a function which satisfies the Fredholm integral equation

$$\Psi(t) = g(t) + \lambda \int_a^b \overline{K}(t,x) Q(x)\, dx \qquad ...(3)$$

Here $\overline{K}(t,x) = K(x, t)$

9.2 Types of Kernels

Just like in Volterra integral equation in the case of Fredholm integral equations are a variety of Kernels as follows:

1. ***Symmetric Kernels:*** Kernels having properties

   as $K(x, t) = K(t, x)$

   are called symmetric Kernels.

2. ***Degenerate Kernels or Poincere Goursat type of Kernels.*** The Kernels of the type

   $$K(x, t) = \sum_{i=1}^{n} g_i(x)\, h_i(t)$$

   These Kernels play an important part in the development of Fredholm theory of integral equation like the eigenvalue and eigenfunction problems.

3. ***Difference Kernels:*** The Kernels of the type

   $$K(x, t) = K(x - t)$$

   are known as difference Kernels. These types of Kernels do arise while converting a differential equation with boundary conditions.

The conditions on Kernels are that they should be continuous and its partial derivatives should be continuous. Also they should be square integrable.

9.3 Methods of Solving Fredholm Integral Equations

There are various methods of solving integral equations which can briefly summarized as follows:

(a) We can reduce integral equation to a differential equation which can be solved easily.

(b) The Fredholm integral equations can be solved by transform method. In this method the Laplace transformation helps in writing an integral equation into an algebraic equation and then by inverse Laplace transformation get the final solution.

(c) ***The Iteration Method:*** The most important method of solving the Fredholm integral equation is the iterative method. In this method the unknown function is expanded in powers of the iterated parameter. This series is known as Neumann series. There is an other alternate approach in which the Kernels are iterated up to *n*th times and then solved the integral equations. The famous iterative method are that of Picard's methods or by using the idea of L_2 class Kernels in the iterative approaches.

(d) ***Numerical Methods:*** Sometimes the Kernel of the Fredholm equations is approximated by a suitable Poincere Goursat Kernel on step functions, then the integral equations can be reduced to an algebraic system of linear equations. If the integral of the given integral equation is replaced by a suitable sum then instead of dividing the basic integral into sub-intervals of the same size, it may be useful to divide it according to the zeros of a certain polynomial of Legendre. This method is developed in most books on numerical methods.

9.4 Description of Some Methods used in the solution of Fredholm Integral Equation

In the next few units we are interested in studying the Fredholm integral equations. In the unit 29, we study the Fredholm equations by the method of *successive approximation*. In this iteration method either the unknown function or the Kernel is iterated into a series known as Neumann's series. The convergence of the series depends upon the iterative parameter and the nature of the Kernel as well as the function in the domain $a \le x \le b$, $a \le t \le b$ when the Kernel $K(x, t)$ and the function $f(x)$ are square integrable. For this purpose the function as well as Kernel has continuous derivations. Then in the unit 31 we will study the solution of Fredholm equations with a special type of Kernels known as Poincere Goursat Kernels (P.G.). In the light of P.G. Kernels the existence and uniqueness of the solution of Fredholm equations of both kinds. In the unit 32 the final unit the famous Fredholm theorem on the existence and uniqueness of the solutions is described along with the conditions put on the functions.

Example: Express the differential equation

$$\frac{d^2y}{dx^2} + 9y = 18x \qquad y(0) = y\left(\frac{\pi}{2}\right) = 0 \qquad \text{...(1)}$$

as an integral equation.

Solution: Integrating (1) from 0 to x,

$$\int_0^x \frac{d^2y}{dx^2}dx + 9\int_0^x y(u)\,du = \int_0^x 18x\,dx$$

or

$$\frac{dy}{dx} - \left.\frac{dy}{dx}\right|_{x=0} + 9\int_0^x y(u)\,du = 9x^2$$

or

$$y'(x) - y'(0) + 9\int_0^x y(u)\,du = 9x^2$$

Again integrating

$$y(x)\Big|_0^x - y'(0)x + 9\int_0^x dt\int_0^t y(u)\,du = \frac{9x^3}{3}$$

or

$$y(x) - y(0) - y'(0)x + 9\int_0^x y(u)\,du\int_0^x dt = 3x^3$$

or $$y(x) - y'(0)x + 9\int_0^x y(u)\,du\,(x-u) = 3x^3 \qquad ...(2)$$

Putting $x = \pi/2$ in (2) and using boundary condition we have

$$-y'(0)\frac{\pi}{2} + 9\int_0^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u)\,du = 3\pi^3$$

Solving for $y'(0)$, we have

$$y'(0) = \frac{18}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u)\,du - 6\pi^2 \qquad ...(3)$$

Substituting in equation (3) we have

$$y(x) - \frac{18x}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u)\,du - 6\pi^2 x + 9\int_0^x (x-u)\,y(u) = 3x^3$$

or $$y(x) - \frac{18x}{\pi}\int_0^x\left(\frac{\pi}{2} - u\right)y(u)\,du - \frac{18x}{\pi}\int_x^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u) + 9\int_0^x (x-u)\,y(u) = 3x^3 + 6\pi^2 x \qquad ...(4)$$

or letting

$$F(x) = 3x\,(x^2 + 2\pi^2) \qquad ...(5)$$

and $$K(x, u) = \begin{cases} \dfrac{18x}{\pi}\left(\dfrac{\pi}{2} - u\right) - 9(x-u) & \text{for } u < x \\ \dfrac{18x}{\pi}\left(\dfrac{\pi}{2} - u\right) & \text{for } u \geq x \end{cases} \qquad ...(6)$$

Equation (4) then becomes

$$y(x) - \int_0^{\pi/2} K(x,u)\,y(u) = F(x) \qquad ...(7)$$

which is the required integral equation of the second kind.

Self Assessment

1. Convert the differential equation

   $y'' + 4y = \sin 3x$ with $y(0) = y(1) = 0$

   into integral equation.

9.5 Summary

- In Fredholm integral equations of first kind and second kind the upper limit of integration is fixed.
- Fredholm Integral equation can be obtained from linear differential equations by applying certain boundary conditions.
- Types of Kernels appearing in Fredholm equations are of the type; symmetric Kernels, difference Kernels, Poincere Goursat Kernels.

9.6 Keywords

Degenerate Kernels or Poincere Goursat Kernels are of the type $K(x,t) = \sum_{i=1}^{n} g_i(x) h_i(t)$ where $g_i(x)$, $h_i(t)$ are known functions.

Symmetric Kernels: The Kernels $K(x, t)$ having the property $K(x, t) = K(t, x)$ are known as symmetric Kernels.

9.7 Review Question

1. Express the differential equation

   $$y''(x) - y'(x) - 6y = x^2 + 1$$

   with $y(0) = y(1) = 0$ into Fredholm integral equation.

Answer: Self Assessment

1. $$G(x) - 4\int_0^1 K(x,t) G(t)\, dt = \sin 3x$$

   where $G(x) = 4''(x)$,

   $$K(x,t) = \begin{cases} t(1-x) & t < x \\ x(1-t) & t > x \end{cases}$$

9.8 Further Readings

Books

Erwin Kreyzig, Introductory Functional Analysis with Application

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equation

Notes

Unit 10: Fredholm Equations Solution by the Method of Successive Approximation

CONTENTS

Objectives

After studying this unit, you should be able to:

- Realize that when the expansion parameter is small the unknown function is iterated in powers of this parameter.
- Describe the Kernel iteratively in powers of the expansion parameter.
- Explain and calculate the iterated function $\psi_n(x)$ or iterated Kernel $K_n(x, t)$.
- Estimate the lower bound for the radius of convergence of Neumann series.

Introduction

You have learnt the method of successive approximation in the case of Volterra integral equations.

The method of successive approximation becomes all the more easy as upper limit of integration is fixed.

10.1 The Method of Successive Approximation

The method of successive approximation in the earlier unit has been applied to the solution of Volterra integral equation. This method can be applied even more easily to the basic Fredholm equation of the second kind. Let us consider the Fredholm integral equation of the second kind.

$$y(x) - \lambda \int_0^1 K(x, u)\, y(u) du = f(x) \qquad ...(1)$$

However, the solution obtained in this way has some difficulty in case $|\lambda|$ is not small and hence may no longer converge. The method of successive approximation can be used more easily because now all integrations are to be performed between the limits 0 and 1.

Now let

$$y(x) = f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + ... \qquad ...(2)$$

Notes

This is called Neumann Series. Substituting (2) into (1) we obtain

$$f(x)+\lambda\psi_1(x)+\lambda^2\psi_2(x)+....-\lambda\int_0^1 K(x,u)\left[f(u)+\lambda\psi_1(u)+\lambda^2\psi_2(x)+....\right]du \equiv f(x) \quad ...(3)$$

Comparing the powers of λ on both sides we have

$$\psi_1(x)-\int_0^1 K(x,u)f(u)du=0$$

$$\psi_1(x) = \int K(x,u)f(u)du$$

$$\psi_2(x) = \int_0^1 K(x,u)\psi_1(u)du=\int_0^1 K(x,u)\int_0^1 K(u,u_1)f(u_1)du_1$$

$$= \int_0^1 K_2(x,y_1)\, f(u_1)\, du$$

$$\psi_3 = \int_0^1 K(x,u)\psi_2(u)du=\int_0^1 K_3(x,u)f(u)du$$

………………………………………………………………

$$\psi_n = \int_0^1 K(x,u)\psi_{n-1}(u)du=\int_0^1 K_n(x,u)du \qquad (\text{for } n=1,2,...)$$

In the above we have

$$\left.\begin{aligned} K_2(x,u)&=\int K(x,u_1)\,K(u_1,u)du_1\\ K_3(x,u)&=\int K(x,u_1)\,K_2(u_1,u)du_1\\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}. \quad ...(4)$$

and so on.

More generally

$$K_n(x,u)=\int K_r(x,u_1)K_{n-r}(u_1,u)du_1 \qquad [n=2,3,4,...;\ r=1,2,...,n-1;\ K_1=K \quad ...(5)$$

Thus the series for the resolvent Kernel $H(x, u, \lambda)$ is given by

$$-H(x,u,\lambda)=K(x,u)+\lambda K_2(x,u)+\lambda^2K_3(x,u)+...+\lambda^nK_n(x,u) \quad ...(6)$$

The solution then is given by

$$y(x)=f(x)=-\lambda\int H(x,u,\lambda)f(u)du \quad ...(7)$$

The main difference from the Volterra case is that the series for the resolvent Kernel (6) now converges only for sufficiently small values of $|\lambda|$. In other words, although $H(x, u, \lambda)$ is still analytic function of λ it is no longer an entire function of λ.

10.2 Lower Bound for the Radius of Convergence

We shall now determine a lower bound for the radius of convergence of the power series (6). We observe that if we preserve the basic hypothesis i.e. that the Kernel $K(x, y)$ is an L_2 Kernel,

i.e. $$\|K\|^2=\iint K^2(x,u)dxdu=\int A^2(x)dx=\int B^2(u)du\le N^2 \quad ...(8)$$

where

$$A(x)=\left[\int_0^1 K^2(x,u)du\right]^{1/2},\ B(u)=\left[\int_0^1 K^2(x,u)dx\right]^{1/2} \quad ...(9)$$

Notes

we have successively

$$K_2^2(x,u) = \left[\int K(x,u_1)\,K(u_1,u)du_1\right]^2 \le A^2(x)\,B^2(u)$$

$$K_3^2(x,u) = \int K^2(x,u_1)du_1 \int K_2^2(u_1,u)du_1 \le A^2(x)\,B^2(u)\int A^2(u_1)du_1 \le A^2(x)\,B^2(u)N^2$$

$$K_4^2(x,u) \le \int K^2(x,z)dz \int K_3^2(z,u)dz \le A^2(x)\,B^2(u)\,N^2 \int A^2(z)dz \le A^2(x)\,B^2(u)N^4$$

and hence in general

$$\left|K_{n+2}(x,u)\right| \le A(x)\,B(x)N^n \qquad (n = 0, 1, 2, ...) \quad ...(10)$$

If we neglect the first term of (6), this process that (6) has the majorant

$$A(x)\,B(x)\,|\lambda|\sum_{n=0}^{\infty}(|\lambda|N)^n;$$

This is a geometric series with the common ratio $|\lambda|N$, hence it converges for $|\lambda|N < 1$, i.e. for

$$|\lambda| < \|K\|^{-1} \qquad ...(11)$$

We thus see that under the condition (11), the partial sums of (6) have a majorant of the type

$$C\,A(x)\,B(x)$$

where C is a constant i.e., a majorant which is L_2 function of both x and u. In other words (6) is an almost uniformly convergent series, hence a series which can be integrated term-by-term in either x or u (by Lebesgue fundamental theorem). Now the resolvent Kernel is an analytic function whose singular points are outside or on the boundary of the circle (11).

Since term-by-term integration is permitted, we see that by using (5) under condition (11) we have

$$\begin{aligned}-\int_0^1 K(x,u_1)H(u_1,u,\lambda)du_1 &= -\int H(x,u_1,\lambda)\,K(u_1,u)du_1 \\ &= K_2(x,u) + \lambda K_3(x,u) + ... \\ &= \lambda^{-1}[H(x,u,\lambda) + K(x,u)]\end{aligned}$$

that is,

$$\left.\begin{aligned}K(x,u) + H(x,u,\lambda) &= \lambda\int K(x,u_1)\,H(u_1,u,\lambda)du_1 \\ &= \lambda\int H(x,u_1,\lambda)\,K(u_1,u)du_1\end{aligned}\right\} \qquad ...(12)$$

Now considering that all the terms of this double equality are analytic functions of λ, we can thus assert that the basic equation (12) for the resolvent Kernel are valid not only in the circle (11), but in the whole domain of existence of the resolvent Kernel H in the λ plane. If now $f(x)$ belongs to the class L_2, then the given equation (1) has at least one solution of the same class L_2, this solution is

$$y(x) = f(x) - \lambda\int H(x,u,\lambda)f(u)du \qquad ...(13)$$

in the domain of existence H, of H.

Moreover, it is easy to see that the solution (13) is the unique L_2-solution of our equation, not only inside the circle $|\lambda| < \|K\|^{-1}$ but also in the whole domain of existence H.

Example: Let us consider the following integral equation

$$y(x) - \lambda\int_0^1 e^{x-u} y(u)du = f(x) \qquad ...(1)$$

we now have

$$K_2(x, u) = \int_0^1 e^{x - u_1 + u_1 - u} du_1 = e^{x-u}\int_0^1 du_1 = e^{x-u} = K(x, u)$$

with this consequence that all the iterated Kernels K_n coincide with the given Kernel $K(x, u)$ and the series (6) becomes

$$-H(x, u, \lambda) = K(x, u)\,(1 + \lambda + \lambda^2 + ...) \qquad ...(2)$$

Hence we have

$$H(x, u, \lambda) = \frac{K(x, u)}{(\lambda - 1)} \qquad ...(3)$$

and we see that the resolvent Kernel is analytic function of λ. So we have one and only one solution for $\lambda \neq 1$.

$$y(x) = f(x) - \frac{\lambda e^x}{(\lambda - 1)}\int_0^1 e^{-u} f(u)du \qquad ...(4)$$

We started with the integral equation (1)

$$y(x) - \lambda\int_0^1 K(x, u)\, y(u) = f(x) \qquad ...(1)$$

and arrived at the equation (13)

$$y(x) = f(x) - \lambda\int_0^1 H(x, u, \lambda)\, f(u)du \qquad ...(13)$$

where the resolvent Kernel satisfies the equation (12)

$$K(x, u) + H(x, u, \lambda) = \lambda\int H(x, u_1, \lambda)\,(u_1, u)du \qquad ...(11)$$

Substituting (13) into L.H.S. of (1)

$$f(x) - \lambda\int_0^1 H(x, u, \lambda)\, f(u)du - \lambda\int_0^1 K(x, u)\left[f(u) - \lambda\int_0^1 H(u, u_1, \lambda)\, f(u_1)du_1\right]du$$

$$= f(x) - \lambda\int_0^1 du\ f(u)\left[K(x, u) + H(x, u, \lambda)\right] - \lambda\int_0^1\int_0^1 H(u, u_1, \lambda) f(u_1) du_1 K(x,u)du$$

$$= f(x) - \lambda\int_0^1 du\left[K(x, u) + H(x, u, \lambda) - \lambda\int_0^1 H(x, u_1, \lambda)K(u_1, u)\right]f(u)$$

$= f(x) + 0 =$ R.H.S.

Self Assessment

1. Solve the following integral equation

   $$y(x) = m\int_0^1 y(t)dt = 1.$$

   Also find the Neumann series for $y(x)$

   $$\left[\textbf{Hint: } \int_0^1 y(t)dt = \text{constant}.\right]$$

10.3 Summary

- The iterative method gives the solution of the function in terms of the powers of the parameter of the equation.
- We can either get an iterative power series in the wave function or the iterated Kernel.
- After iterating it nth times we get the solution as limiting as n tends to ∞.
- In this way we get the Resolvent Kernel in the nth iteration when n is very large.

10.4 Keyword

The successive method helps in getting the solution of the problem as a power series in terms of powers of the parameter known as ***Neumann series***. The estimate of the radius of convergence of the Neumann series gives an estimate of the accuracy of the solution.

10.5 Review Question

The Fredholm integral equation is

$$y(x) - \lambda\int_0^{2\pi} K(x, t)\, y(t)\, dt = f(x)$$

where $K(x, t) = \sum_{v=1}^{\infty} v^{-2} \sin(vs) \sin[(v+1)t]$

find $K_3(x, t)$, the third iterative Kernel.

Answer: Self Assessment

1. $y(x) = \dfrac{1}{1-u}$, the Neumann series is $y(x) = 1 + \mu^2 + \mu^3 + \ldots$.

10.6 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 11: Neumann's Series

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Find lots of similarities of the description of the successive approximation approach in regard to getting Neumann Series.
- Observe that the unknown function can either be expanded in power series of λ or the resolvent Kernel is expanded in power series in λ.
- Understand the convergence of the Neumann Series as given in unit 29.

Introduction

For small values of λ the solution of the Fredholm equation can be determined as power series known as Neumann's Series.

The resolvent kernel is an analytic function of the parameter λ but it is not an entire function of the whole complex plane.

11.1 Fredholm Integral Equations, Successive Approximation Neumann's Series

Consider the Fredholm integral equations of the first kind and second kind:

$$f(x) = \lambda \int_a^b K(x, t)\, y(t) dt \qquad ...(1)$$

and

$$y(x) = f(x) + \lambda \int_a^b K(x, t)\, y(t) dt \qquad ...(2)$$

In these equations $y(x)$ is an unknown function that has to be found and $f(x)$ and $K(x, t)$ are given as function and the Kernel of the integral equations. Unless in the case of Volterra integral equation, here the limits of the integral are fixed as constants a and b. The range of x and t are given as $a \le x \le b$ and $a \le t \le b$. Depending upon the nature of Kernel $K(x, t)$ a suitable method of

Notes

solving the integral equation is to be chosen. Here the parameter λ also plays an important part. So if λ is small as well as the Kernel $K(x, t)$ is continuous along with its partial derivatives, we can use the method of successive approximation.

Let us consider first the equation (2) of Fredholm integral equation of the second kind. To a zero approximation

$$y(x) \simeq f(x).$$

If we substitute this value of $y(x)$ in the integral (2) we get

$$f(x) \simeq f(x) + \lambda \int_a^b K(x, t) f(t) dt \quad ...(3)$$

or $\quad y(x) \cong f(x) + \lambda \psi_1(x)$

where $\quad \psi_1(x) = \int_a^b K(x, t) f(t) dt \quad ...(4)$

So to a first approximation $y(x)$ is given by (2). To get an improvement over the above approximation we put this new value of $y(x)$ given by (3) into (2) to improve the solution as follows:

$$y(x) \cong f(x) + \lambda \int_a^b K(x, t) \left[f(t) + \lambda \int_a^b K(t, u) f(u) du \right] dt$$

$$= f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) dt \int_a^b K(t, u) f(u) du$$

or $\quad y(x) \cong f(x) + \lambda \psi_1(x) + \lambda^2 \psi_2(x) \quad ...(5)$

where

$$\psi_2(x) = \int_a^b K(x, t) dt \int_a^b K(t, u)\, f(u) du$$

$$= \int_a^b du\, f(u) \int_a^b K(x, t)\, K(t, u) dt$$

or $\quad \psi_2(x) = \int_a^b du\, K_2(x, u) f(u) \quad ...(6a)$

where $\quad K_2(x, u) = \int_a^b K(x, t)\, K(t, u) dt \quad ...(6b)$

We can improve the accuracy by taking more powers of λ in $y(x)$ i.e. we may write

$$y(x) \cong f(x) + \lambda \psi_1 + \lambda^2 \psi_2 + \lambda^3 \psi_3 + ... + \lambda^n \psi_n + ... \quad ...(7)$$

where ψ_1, ψ_2 are given by (4) and (6a) and other ψ's are given by

$$\psi_n(x) = \int_a^b du\, K_n(x, u)\, f(u) \qquad \text{for } n = 1, 2, ... \quad ...(8)$$

and the n^{th} Kernel $K_n(x, u)$ given by

$$K_n(x, u) = \int_a^b K_r(x, u_1) K_{n-r}(u_1, u) du_1 \qquad [n = 2, 3, 4,...; r = 1, 2, ...n - 1 \quad ...(9)$$

while $\quad K_1(x, u) = K(x, u)$

Thus $\quad y(x) \cong f(x) + \sum_{i=1}^{n} \lambda^i \psi_i(x)$ .... for any $n \quad ...(10)$

Notes

This series for the solution of the Fredholm integral equation of the second kind is known as Neumann Series.

11.2 Successive Approximation for the Resolvent Kernel

Writing in full the expression for the function $y(x)$, we have

$$y(x) \cong f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \lambda^3\ \Psi_3(x) + ... \qquad ...(10)$$

Making use of (4) (6a) and (8) for ψ_1, ψ_2, ψ_3,...... into (10) we get

$$y(x) = f(x) + \lambda\int_a^b K(x, t)\ f(t)dt + \lambda^2\int_a^b K_2(x, t)f(t)dt + \lambda^3\int_a^b K_3(x, t)\ f(t)dt + ...$$

$$= f(x) + \int\left[\lambda K_1(x, t) + \lambda^2 K_2(x, t) + \lambda^3 K_3(x, t) + ...\right] f(t)dt$$

$$y(x) = f(x) + \lambda\int_a^b H(x, t, \lambda)f(t)\ dt \qquad ..(11)$$

where the resolvent Kernel $H(x, t, \lambda)$ is given by the series

$$-H(x, t, \lambda) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + ... \qquad ...(12)$$

Equation (12) is now the power series known again as Neumann Series.

As discussed in unit 29, we see that the resolvent Kernel is still analytic function of λ but is no longer an entire function of λ. Also the resolvent Kernel satisfies the integral equation

$$-H(x, u, \lambda) = K(x, u) - \lambda\int H(x, u_1, \lambda)\ K(u_1, u)du_1 \qquad ...(13)$$

Now the solution (11) is the unique L_2-solution of the equation (2), as $f(x)$ and $K(x, t)$ are L_2-class and it exists in the whole domain of $C(a, b)$. We now show that if the homogeneous equation (1) for $\lambda = \lambda_0$ has a certain non-trivial solution then with the help of equation (13) we obtain

$$\phi_0(x) = \lambda_0\int K(x, t)\phi_0(t)dt$$

$$= -\lambda_0\int H(x, t, \lambda_0)\phi_0(t)dt + \lambda_0^2\int\phi_0(t)dt\int H(x, z, \lambda_0)\ K(z, t)dz$$

$$= -\lambda_0\int H(x, t, \lambda_0)\phi_0(t)dt + \lambda_0^2\int H(x, z, \lambda_0)dz\int K(z, t)\phi_0(t)dt$$

$$= -\lambda_0\int H(x, t, \lambda_0)\phi_0(t)dt + \lambda_0\int H(x, z, \lambda_0)dz\ \phi_0(z)$$

$$\equiv 0$$

This shows that if equation (2) has a unique non-trivial solution of the form (12) then the non-trivial solution of the homogeneous equation (1) is $\phi_0(x)$, vanishes almost everywhere.

The above analysis process the following theorem to each quadratically integrable Kernel $K(x, t)$ there corresponds a resolvent Kernel $H(x, t, \lambda)$ which is analytic function of λ, regular at least inside the circle $|\lambda| < \|K\|^{-1}$ and represented these by the power series (12). Let the domain of existence of the resolvent Kernel in the complex plane λ be H. Then if $f(x)$ also belongs to the class L_2, the unique quadratically integrable solution of Fredholm's equation (2) valid in H is given by (11).

For the proof of this theorem please refer to the treatment in the unit 29.

Example: Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x, t)\, y(t) dt \qquad ...(1)$$

Find the solution when $f(x) = e^x$, $K(x, t) = 2e^{x+t}$, $a = 0$, $b = 1$.

Substitute the value of $f(x)$ and $K(x, t)$ in (1) we have

$$y(x) = e^x + 2\lambda e^x \int_0^1 e^t y(t) dt$$

$$= e^x \left[1 + 2\lambda \int_0^1 e^t y(t) dt\right]$$

Let $\quad C = \int_0^1 e^t y(t) dt = \text{constant} \qquad ...(2)$

then $\quad y(x) = e^x (1 + 2\lambda C) \qquad ...(3)$

Substituting this value of y in (2) we have

$$C = (1 + 2\lambda C)\int_0^1 e^t . e^t dt = (1 + 2\lambda C)\frac{(e^2 - 1)}{2}$$

Solving for C i.e.

$$2C - 2\lambda C(e^2 - 1) = (e^2 - 1)$$

$$C = \frac{(e^2 - 1)}{2\left[1 - \lambda(e^2 - 1)\right]} \qquad ....(4)$$

Substituting in (3) we have

$$y(x) = e^x / [1 - \lambda(e^2 - 1)] \qquad ...(5)$$

The denominator is non-zero.

Self Assessment

1. Solve the Fredholm integral equation

   $$y(x) = f(x) + \lambda \int_a^b K_0 y(t) dt$$

   where K_0 is a constant and show that for $|\lambda| < 1/K_0(b - a)$ the corresponding Neumann Series is convergent.

11.3 Summary

- In case the parameter λ is small one gets the solution of Fredholm equation of the second kind as a power series in λ called Neumann series.
- The Resolvent Kernel can also be expanded in powers of λ provided the Kernel $K(x, t)$ is of L_2-class. The resolvent Kernel is though an analytic function of λ but is not an entire function in whole of complex λ-plane.

Notes

11.4 Keywords

The $C(a, b)$ is a space of all ***continuous functions*** defined on the interval (a, b).

The ***unknown functions*** $y(x)$ and $f(x)$ are of $C(a, b)$ type while $K(x, t)$ is of $C^2(a, b) \to C(a, b)$ type on the square $a \le x \le b$ and $a \le t \le b$.

11.5 Review Question

Solve the Fredholm integral equation of the second kind

$$Y(x) = f(x) + \lambda \int_0^1 x(1+t)y(t)dt$$

when λ is not an eigenvalue.

Answer: Self Assessment

1. $$y(x) = f(x) + \frac{\lambda K_o C_o}{1 - \lambda K_0(b-a)]}, C_0 = \int_a^b f(x)dx$$

   Expand $\frac{1}{1 - \lambda K_0(b-a)}$ in powers of λ to get Neumann Series.

11.6 Further Readings

Books

Erwin Kreyzig, Introductory Functional Analysis with Applications

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 12: Fredholm Equations with Poincere Goursat Kernels

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that Fredholm equations may have varieties of Kernels. Among them the Poincere-Goursat Kernel also plays an important part.
- Observe that in this type of Fredholm equation the resolvent Kernel is a quotient of two polynomials of the nth degree in λ and the denominator is independent of the variables of the Kernel.
- Understand the nature of singular points of resolvent Kernel in terms of zeros of the denominator polynomial $D(\lambda)$.

Introduction

In this unit we saw that resolvent Kernel has a structure that helps in understanding the nature of the solution of non-homogeneous as well as homogeneous equations.

Fredholm integral equation as well as its conjugate equation can be studied together to understand the structure of the solutions.

12.1 The Poincere Goursat Kernels

In the unit we consider again the Fredholm integral equation of the second kind i.e.

$$y(x) - \lambda \int_0^1 K(x, u) y(u) du = f(x) \qquad ...(1)$$

Here we take the structure of the Kernel to be of the form

$$K(x, u) = \sum_{i=i}^{n} g_i(x) h_i(u) \qquad ...(2)$$

Notes

Such a Kernel has been mentioned in the case of Volterra integral equations and are called degenerate Kernels or Poincere-Goursat Kernels i.e. P-G Kernels. In equation (2) the functions $g_i(x)$ and $h_i(u)$, for $i = 1, 2, ...n$, are L_2-class. We will see that in this case the Fredholm integral equation can be reduced to an algebraic system of a linear equation in n unknown. Here $g_i(x)$ and $h_i(x)$ $i = 1, 2, ...n$ are independent in the basic interval (0, 1).

Substituting (2) in one we have

$$y(x) - \lambda \int_0^1 \sum_{i=1}^{n} g_i(x)\, h_i(u) y(u) du = f(x)$$

or
$$y(x) - \sum_{i=1}^{n} (\lambda)\, g_i(x) \int_0^1 h_i(u) y(u) du = f(x)$$

If we put

$$\varepsilon_k = \int_0^1 h_k(u) g(u) du \qquad (x = 1, 2, ...n) \qquad ...(3)$$

then equation (1) becomes

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} \varepsilon_i g_i(x) \qquad ...(4)$$

From equation (4) it is already seen that the difference $y(x)$ -$f(x)$ must necessarily coincide with a suitable linear combination of the functions $g_i(x)$. Now multiply equation (4) by $h_k(x)$ $i = 1, 2,...n$ and integrate between 0 and 1 we have

$$\int_0^1 h_k(x) y(x) dx = \int_0^1 h_k(x) f(x) dx + \lambda \sum_{i=1}^{n} \varepsilon_i \int_0^1 g_i(x) h_k(x) dx$$

Defining

$$\left.\begin{aligned} \int g_i(x) h_k(x) dx &= a_{ik} \\ \int h_i(x) f(x) dx &= b_i \end{aligned}\right\} \qquad ...(5)$$

We have

$$\varepsilon_k - \lambda \sum_{i=1}^{n} a_{ik} \varepsilon_i = b_k \qquad ...(6)$$

We thus see that the unknowns ε_1, ...ε_n must satisfy the following system of linear equations

$$\left.\begin{aligned} &(1 - a_{11})\varepsilon_1 - \lambda a_{12}\varepsilon_2 - \lambda a_{13}\varepsilon_3 ...... - \lambda a_{1n}\varepsilon_n = b_1 \\ &-\lambda a_{21}\varepsilon_1 + (1 - a_{22})\varepsilon_2 - \lambda a_{23}\varepsilon_3 .... - \lambda a_{2n}\varepsilon_n = b_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &-\lambda a_{n1}\varepsilon_1 - \lambda a_{n2}\varepsilon_2 ...... + (1 - \lambda a_{nn})\varepsilon_n = b_n \end{aligned}\right\} \qquad ...(7)$$

To each set of solution $\varepsilon^\circ_1, \varepsilon^\circ_2 ... \varepsilon^\circ_n$ of this system there corresponds a solution of equation (1) given by (4). Now the solution of equation exist if the determinant formed by the coefficients ε_i in equation (7) defined by

$$D(\lambda) = \begin{vmatrix} 1 - \lambda a_{11} - \lambda a_{12} - \lambda a_{13} ...... - \lambda a_{1n} \\ -\lambda a_{21}(1 - \lambda a_{22}) .... - \lambda a_{2n} \\ -\lambda a_{31} .... \\ .... \\ -\lambda a_{n1} - \lambda a_{n2} ....(1 - \lambda a_{nn}) \end{vmatrix} \qquad ...(8)$$

is non-zero. So there is one and only one solution of the system of n simultaneous equations ε_1, ε_2, ...ε_n. Thus if $D(\lambda) \neq 0$, then the system (7) has one and only one solution given by Gamer's rule i.e.

$$\varepsilon_k = \frac{1}{D(\lambda)}\left(D_{1k}b_1 + D_{2k}b_2 + ... + D_{nk}b_n\right) \ (k = 1, 2, 3, ...n)$$

where D_{hk} denotes co-factor of (h, k) the elements of the determinant (8), correspondingly, the solution (1) has the unique solution

$$y(x) = f(x) + \frac{\lambda\sum_{k=1}^{n}}{D(\lambda)}\left[D_{1k}b_1 + D_{2k}b_2 + ... + D_{nk}b_n\right]g_k(x) \qquad ...(9)$$

As $D(\lambda) \neq 0$, the corresponding Fredholm equation of the first kind

$$y(x) - \lambda\int_0^1 K(x, u)y(u)du \qquad ...(10)$$

has only the trivial solution $y(x) \equiv 0$ as $D(\lambda) \neq 0$.

12.2 Resolvent Kernel $H(x, u, \lambda)$

If we now substitute the expression of b_i in (5), the solution (9) can also be written as

$$y(x) = f(x) + \frac{\lambda}{D(\lambda)}\int_0^1 \left[D_{1k}h_1(u) + D_{2k}h_2(u) + D_{3k}h_3(u) + ... + D_{nk}h_n(u)\right]f(u)g_k(x)du$$

but the sum under the integral sign can be considered as the expansion of the negative of a determinant of the $(n + 1)$ order i.e.

$$-\left[(D_{ik}h_1(u) + D_{2k}h_2(u) + D_{3k}h_3(4) + ... + D_{nk}b_n(u)g_k(x)\right]$$

$$= D(x,u,\lambda) = \begin{vmatrix} 0 \ g_1(x)g_2(x) \ ....g_n(x) \\ h_1(u) \ 1 - \lambda a_{11} - \lambda a_{12} .... \lambda a_{1n} \\ \vdots .... - \lambda a_{2n} \\ .... \\ h_n(u) \qquad\qquad (1 - \lambda a_{nn}) \end{vmatrix} \qquad ...(11)$$

Hence we can write equation (9) as

$$y(x) = f(x) - \frac{\lambda}{D(\lambda)}\int_0^1 D(x, u, \lambda)f(u)du \qquad ...(12)$$

Defining the resolvent Kernel $H(x, u, \lambda)$ by

$$H(x, u, \lambda) = \frac{D(x, u, \lambda)}{D(\lambda)} \qquad ...(13)$$

so equation (12) becomes

$$y(x) = f(x) - \lambda\int_0^1 H(x, u, \lambda)f(u)du \qquad ...(14)$$

In the equation (13) the resolvent Kernel $H(x, u, \lambda)$ is the quotient of two polynomials of the nth degree in λ and the denominator is independent of x and u and this has important consequences.

Notes

At this point it is to be noticed that the only singular points of $H(x, u, \lambda)$ in the λ-plane are the roots of the equation

$$D(\lambda) = 0 \qquad ...(15)$$

which will be called the eigenvalues of our Kernel $K(x, u)$

12.3 Eigenvalues and Eigenvectors

If $D(\lambda) = 0$ the non-homogeneous equation (1) has no solution in general, because an algebraic linear system with vanishing determinant can only be solved for certain values of the quantities on the right hand side of equation (7).

Furthermore, from each non-trivial solution $\varepsilon_1^0, \varepsilon_2^0, ..., \varepsilon_n^0$ of the homogeneous algebraic system we obtain a non-trivial solution of the homogeneous equation (10), which we call an eigenfunction and vice versa.

To be precise, from the theory of algebraic systems of linear equations. We infer that, if λ coincides with a certain eigenvalue λ_0 for which the determinant $D(\lambda_0)$ has the characteristic $P(1 \le p \le n - 1)$, and we put $n - p = r$, then there are ∞^r solutions of the homogeneous system (7). Furthermore, these solutions can be represented by formulae of the type

$$\xi_k = B_{1k}C_1 + B_{2k}C_2 + ... + B_{rk}C_r \quad (k = 1, 2, ..., n) \qquad ...(16)$$

where C_1, C_2,..., C_r denote r arbitrary constants and

$$\left.\begin{array}{l} B_{11}, B_{12}, ... B_{1n} \\ .... \\ B_{r1}, B_{r2} . ... B_{rn} \end{array}\right\} \qquad ...(17)$$

are r arbitrarily fixed but linearly independent solutions of the system in question.

This shows that to each eigenvalue λ_0 of index $r = n - p$ there corresponds a solution of the homogeneous equation (10) of the form

$$\phi_0(x) = C_1\phi_{01}(x)\, C_2\phi_{02}(x) + ... + C_r\phi_{0r}(x) \qquad ...(18)$$

where C_1, C_2,...C_r are r arbitrary constants and

$$\phi_{01}(x), \phi_{02}(x), ..., \phi_{0r}(x)$$

are r linearly independent functions, which can be expressed in terms of the B_{hk} as follows:

$$\phi_{0h}(x) = \sum_{k=1}^{n} B_{hk}\, g_k(x) \qquad (h = 1, 2, ..., r) \qquad ...(19)$$

Moreover, we can assume that these functions are normalized, i.e., that their norms are all equal to unity,

$$\int \phi_{0h}^2(x)dx = 1 \quad (h = 1, 2, ..., r) \qquad ...(20)$$

Al these eigenfunctions are annihilated by the Fredholm operator

$$F_s[\phi_{oh}(y)] \equiv 0. \qquad ...(21)$$

Using elementary transformations on the determinant (7), we can see that the index $r = n - p$ of an eigenvalue is never larger than its multiplicity m as a root of the equation $D(\lambda) = 0$. Moreover, in the important case $a_{hk} = a_{kh}$ we have

$$r = m$$

Another important fact is that to the given Kernel (1) and to the associated one

$$K(y, x) = \sum_{k=1}^{n} g_k(y)\, h_k(x) \qquad \text{...(21)}$$

there corresponds the same function $D(\lambda)$ and consequently the same eigenvalues. This is because the interchange of g_k and h_k carries a_{hk} into a_{kh} and hence only interchanges the rows and columns of determinant (8).

However, the eigenfunctions of the associated Kernel, i.e. the non-trivial solutions of the associated homogeneous equation

$$\psi(x) - \lambda \int K(y, x)\psi(y)dy = 0 \qquad \text{...(22)}$$

for $\lambda = \lambda_0$ are not the previous function (16) but other ones,

$$\psi_{0h}(x) - \sum_{k=1}^{n} B^*_{hk} h_k(x) \;\; (h = 1, 2, ..., r), \qquad \text{...(23)}$$

where

$$\left.\begin{matrix} B^*_{11}, B^*_{12}, ... B^*_{1n} \\ \text{.......................} \\ B^*_{r1}, B^*_{r2} . ... B^*_{rn} \end{matrix}\right\} \qquad \text{...(24)}$$

are any r linearly independent solutions of the associated homogeneous system

$$\left.\begin{matrix} (1 - \lambda a_{11})\xi_1 - \lambda a_{21}\xi_2 - ... - \lambda a_{n1}\xi_n = 0 \\ -\lambda a_{21}\xi_1 + (1 - \lambda a_{22})\xi_2 - ... - \lambda a_{n2}\xi_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\lambda a_{1n}\xi_1 - \lambda a_{2n}\xi_2 - ... + (1 - \lambda a_{nn})\xi_n = 0 \end{matrix}\right\} \qquad \text{...(25)}$$

Any eigenfunction $\phi_{0h}(x)$ corresponding to the eigenvalue λ_0 and any associated eigenfunction $\psi_{1k}(x)$ corresponding to a different eigenvalue λ_1 are always orthogonal in the basic interval (0, 1).

In fact we have

$$I = \int \phi_{oh}(x)\psi_{1k}(x)dx = \lambda_0 \int \psi_{1k}(x)\,dx \int K(x, y)\phi_{oh}(y)dy$$
$$= \lambda_0 \int \phi_{0h}(y)dy \int K(x, y)\psi_{1k}(x)dx = \frac{\lambda_0}{\lambda_1}\int \phi_{oh}(y)\psi_{1k}(y)dy = \frac{\lambda_0}{\lambda_1} I,$$

and this equality can be true only if $\lambda_0 = \lambda_1$ or if $I = 0$.

We now return to the non-homogeneous equation (1) for the case $D(\lambda) = 0$. We prove that for $\lambda = \lambda_0$ the non-homogeneous equation can be solved if and only if the r orthogonality conditions

$$(f, \psi_{oh}) \equiv \int f(x)\psi_{oh}(x)dx = 0 \qquad (h = 1, 2, ..., r) \qquad \text{...(26)}$$

are satisfied. In this case the non-homogeneous equation has ∞^r solutions of the form

$$\phi(x) = \Phi(x) + C_1\phi_{01}(x) + C_2\phi_{02}(x) + ... + C_r\phi_{or}(x), \qquad \text{...(27)}$$

where $\Phi(x)$ is a suitable linear combination of $g_1(x)$, $g_2(x)$,...$g_n(x)$.

Notes

In fact, conditions (26) are necessary because if equation (1) for $\lambda = \lambda_0$ admits a certain solution $\Phi(x)$, then from the equation itself, it follows that

$$\int f(x)\psi_{0h}(x)dx = \int \Phi(x)\psi_{0h}(x)dx - \lambda_0 \int \psi_{0h}(x)dx \int K(x, y)\Phi(y)dy$$
$$= \int \Phi(x)\psi_{0h}(x)dx - \lambda_0 \int \Phi(y)dy \int K(x, y)\psi_{0h}(x)dx.$$

But, since λ_0 and $\psi_{0h}(x)$ are eigenvalue and corresponding eigenfunction of the associated Kernel, we have

$$\lambda_0 \int K(x, y)\psi_{0h}(x)dx = \psi_{0h}(y);$$

hence

$$\int f(x)\psi_{0h}(x)dx = 0$$

Furthermore, conditions (26) are also sufficient, since from them it can be easily deduced that the non-homogeneous system (7), which we shall write briefly as

$$\Xi_1 = b_1\ \Xi_2 = b_2\ ..., \Xi_n = b_n,$$

reduces to only $n - r$ independent equations. Consequently we can now solve it readily (carrying r unknowns on the right hand side), since the characteristic of matrix of the coefficients is exactly $p = n - r$.

We can reduce the system for the following reason: Let us multiply the previous equations by B^*_{h1}, B^*_{h2}, ...B^*_{hr}, respectively and add. Bearing in mind equations (25), we have

$$\begin{aligned}\sum_{k=1}^{n} B^*_{hk}\Xi_k &= [(1 - \lambda a_{11})B^*_{h1} - \lambda a_{21}B^*_{h2} - ... - \lambda a_{n1}B^*_{hn}]\ \xi_1 \\ &+ [-\lambda a_{12}B^*_{h1} + (1 - \lambda a_{22})B^*_{h2} - ... - \lambda a_{n2}B^*_{hn}]\xi_2 \\ &+ .... \\ &+ [-\lambda a_{1n}B^*_{h1} - \lambda a_{2n}B^*_{h2} - ... + (1 - \lambda a_{nn})B^*_{hn}]\xi_n \equiv 0,\end{aligned}$$

while on the other side, by virtue of (26), we also have

$$\sum_{k=1}^{n} B^*_{hk}b_k = \int \left[\sum_{k=1}^{n} B^*_{hk}Y_k(x)\right] f(x)dx = \int \psi_{oh}(x)f(x)dx = 0.$$

Among other things, form (27) of the solution demonstrates the following obvious fact: the general solution of equation (1) when $D(\lambda) = 0$ can be considered as the sum of any particular solution $\Phi(x)$ and of the general solution (18) of the homogeneous equation.

Thus we have proved for PG Kernels the following basic Fredholm theorem, which will be extended to general Kernels in the next section:

Fredholm's integral equation of the second kind

$$\phi(x) - \lambda \int K(x, y)\phi(y)dy = f(x)$$

has, in general, one and only one solution of the class L_2 given by the formula

$$\phi(x) = f(x) - \lambda \int H(x, y; \lambda)f(y)dy,$$

where $H(x, y; \lambda)$ is the resolvent Kernel. $H(x, y; \lambda)$ is an analytic function λ, and if $|\lambda| < ||K||^{-1}$ it is given by the Neumann series

$$-H(x, y; \lambda) = K(x, y) + \lambda K_2(x, y) + \lambda^2 K_3(x, y) + ...,$$

where K_2, K_3,... are the iterated Kernels. The only exceptions are the singular points of $H(x, y; \lambda)$ which coincide with the zeros (called eigenvalues) of an analytic function $D(\lambda)$ of λ. In the case of a PG Kernel, $D(\lambda)$ is a polynomial.

If $\lambda = \lambda_0$ is a root of multiplicity $m \geq 1$ of the equation $D(\lambda) = 0$, then the homogeneous equation

$$\phi(x) - \lambda \int K(x, y)\phi(y)dy = 0$$

has r linearly independent non-trivial solutions, called eigenfunctions, where r, the index of the eigenvalue, satisfies the condition $1 \leq r \leq m$. The same is true of the associated homogeneous equation.

$$\psi(x) - \lambda \int K(x, y)\psi(y)dy = 0.$$

However, if $\lambda = \lambda_0$ the non-homogeneous equation has solutions (exactly ∞^r solutions) if and only if the given function $f(x)$ is orthogonal to all the eigenfunctions of the associated homogeneous equation.

A very important alternative theorem can immediately be deduced as a corollary:

Alternative Theorem: If the homogeneous Fredholm integral equation has only the trivial solution, then the corresponding non-homogeneous equation always has one and only one solution. On the contrary, if the homogeneous equation has some non-trivial solutions, then the non-homogeneous integral equation has either no solution or an infinity of solutions, depending on the given function $f(x)$.

But even this corollary has been proved only for PG Kernels.

Self Assessment

1. The Kernel of Fredholm integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} K(x, t)y(t)dt$$

is given by

$$k(x, t) = \sum_{v=1}^{\infty} \frac{1}{V^2} \sin(vx) \sin[(v+1)t]$$

Find the iterated Kernel.

$$K_2(x, t)$$

$$\left[\textbf{Hint :} \text{Use the relation } \lim_{\alpha \to 0} \frac{\sin \alpha\, u}{\alpha} = u\right].$$

12.4 Summary

- Fredholm integral equation of the second kind is studied with the help of Poincere Goursat Kernels.

- It is seen that the resolvent Kernel can be expressed in terms of quotient of two polynomials of the *n*th degree in λ and denominator is independent of the independent variables.
- Also conditions are discussed when λ is an eigenvalue and the corresponding eigenfunctions are discussed with respect to P.G. Kernel only.

12.5 Keywords

In this unit the resolvent Kernel of the ***Fredholm integral equation*** of the second kind as well as corresponding conjugate equation is discussed.

In the next unit we shall be studying ***Fredholm theorem*** for the existence and uniqueness of the eigenvalue solution of the problem with only general Kernel.

12.6 Review Question

The Kernel of Fredholm integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} K(x, t)\, y(t) dt$$

is given by

$$K(x, t) = \sum_{v=1}^{\infty} \frac{1}{v^2} \sin(vx) \sin[(v+1)t]$$

Find the iterated Kernel

$K_3(x, t)$

$$\left[\textbf{Hint: } \text{Use the relation } \lim_{\alpha \to 0} \frac{\sin \alpha u}{\alpha} = u\right].$$

Answer: Self Assessment

1. $$K_2(x, t) = \sum_{v=1}^{\infty} \pi \frac{\sin(vx) \sin[(v+2)t]}{v^2 (v+1)^2}$$

12.7 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 13: The Fredholm Theorem

CONTENTS

Objectives

After studying this unit, you should be able to:

- Learn that Fredholm integral equations are of two types – of first kind and of second kind
- Prove that if λ is not an eigenvalue then the Fredholm Integral equation has a solution for the second kind and the solution for the homogeneous equation is zero.
- Show that for an eigenvalue problem the Fredholm integral equation of second kind has a solution which also contains a set of *r*-constants in addition to one of its solution.

Introduction

The proof of the Fredholm theorem consists of two parts. In the first part the solution is unique and λ is not an eigenvalue.

The second part explains the eigenvalue problem of the homogeneous Fredholm integral equation and explains the structure of the main integral equation and the conjugate one.

13.1 Fredholm Alternate Theorem

The theorem states that:

Either the integral equation of the second kind

$$f(s) = Q(s) - \lambda \int_a^b K(s, t) Q(t) dt \quad ...(1)$$

with fixed λ, admits a unique continuous solutions $Q(s)$ for any continuous function $f(s)$, in particular $Q(s) = 0$ for $f(s) \equiv 0$, or the associated homogeneous equation

$$\bar{Q}(s) - \lambda \int_a^b K(s, t)\, \bar{Q}(t) dt = 0 \quad ...(2)$$

admits a number $r(r \geqq 1)$ of linearly independent continuous solutions $\bar{Q}_1(s), \bar{Q}_2(s)....\bar{Q}_n(s)$. In the first case, the conjugate equation

Notes

$$g(s) = \psi(s) - \lambda\int_a^b K(t, s)\psi(t)dt \qquad ...(3)$$

also admits a unique continuous solution $\psi(s)$ for any continuous function $g(s)$. In the second case the associated homogeneous equation

$$\bar{\psi}(s) = \lambda\int_a^b K(t, s)\bar{\psi}(t)dt \qquad ...(4)$$

admits a number r of linearly independent continuous solutions $\bar{\psi}_1(s), \bar{\psi}_2(s), \bar{\psi}_3(s)...\bar{\psi}_r(s)$. In the second case, the equation (1) admits a solution if and only if

$$\int_a^b t(s)\bar{\psi}_i(s)ds = 0 \qquad (i = 1, 2, ...r) \quad ...(5)$$

If condition (5) is satisfied, the general solution of (1) is written as

$$Q(s) = Q^{(1)}(s) + \sum_{j=1}^{r} C_j\bar{Q}_j(s) \qquad ...(6)$$

by means of a particular solution $Q^{(1)}(s)$ of (1) and r arbitrary constants $C_1, C_2, ..., C_r$. Similarly, the conjugate equation (3) admits a solution if and only if

$$\int_a^b g(s)\bar{Q}_j(s) = 0 \qquad (j = 1, 2, 3....r) \qquad ...(7)$$

If condition (7) is satisfied, the general solution of (3) is written as

$$\psi(s) = \psi^{(1)}(s) + \sum_{j=1}^{r} C_j\bar{\psi}_j(s)$$

by means of a particular solution $\psi^{(1)}(s)$ of (3) and r arbitrary constants $C_1, C_2, ...C_r$.

The theorem also shows that the unique solution of (1) exists for any continuous function $f(x)$ if and only if λ is not an eigenvalue.

The proof of the Fredholm's alternative theorem is given in two parts for continuous Kernel $K(s, t)$ on the domain $a \le s \le b,\ a \le t \le b$. We shall start proving the theorem by Schmidt's method instead of L_2-class method. Of course both the methods had to the same conclusion.

13.2 Proof of Fredholm Theorem

The case when $\int_a^b\int_a^b |K(s, t)|^2\, ds\, dt < 1$

For the sake of simplicity, we take $\lambda = 1$ and consider the equation

$$\varphi(s) - \int_a^b K(s, t)\varphi(t)dt = f(s) \qquad ...(1)$$

An equation in the unknown $\phi(t)$, of the form

$$\phi(t) - \int_a^b K(s, t)\phi(s)ds = g(t) \qquad ...(2)$$

$g(t)$ being a given continuous function on the interval $a \leq t \leq b$, is called the conjugate equation of (1).

Notes

Theorem 1: Under the assumption

$$\int_a^b\int_a^b |K(s,t)|^2\, ds\, dt < 1 \qquad ...(3)$$

the equation (1) [(2)] admits one and only one solution $\varphi(s)[\phi(t)]$ for any $f(s)[g(t)]$; in particular $\varphi(s) \equiv 0[\varphi(t) \equiv 0]$ for the homogeneous equation

$$\varphi(s) - \int_a^b K(s,t)\varphi(t)dt = 0 \qquad ...(4)$$

$$\phi(t) - \int_a^b K(s,t)\phi(s)ds = 0 \qquad ...(5)$$

Proof: Starting with the Kernel $K(s, t)$, we define the iterated Kernels $K^{(1)}(s, t)$, $K^{(2)}(s, t)$, ..., $K^{(n)}(s, t)$, ... as follows:

$$\begin{aligned} K^{(1)}(s,t) &= K(s,t) \\ K^{(2)}(s,t) &= \int_a^b K(s,r)K(r,t)dr \\ &\dots\ \dots\ \dots \\ K^{(n)}(s,t) &= \int_a^b K(s,r)K^{(n-1)}(r,t)dr \end{aligned} \qquad ...(6)$$

The following relation clearly holds for the iterated Kernels

$$K^{(n+m)}(s,t) = \int_a^b K^{(n)}(s,r)K^{(m)}(r,t)dr \qquad ...(7)$$

By (6) and the Schwarz inequality, we have

$$|K^{(n)}(s,t)|^2 \leqq \int_a^b |K(s,r)|^2\, dr \int_a^b |K^{(n-1)}(r,t)|^2\, dr$$

hence

$$\int_a^b\int_a^b |K^{(n)}(s,t)|^2\, ds\, dt$$

$$\leqq \int_a^b\int_a^b |K(s,r)|^2\, ds\, dt \int_a^b\int_a^b |K^{(n-1)}(r,t)|^2\, dr\, dt$$

Repeating this procedure, we finally obtain

$$\int_a^b\int_a^b |K^{(n)}(s,t)|^2\, ds\, dt \leqq \left[\int_a^b\int_a^b |K(s,t)|^2\, ds\, dt\right]^n \qquad ...(8)$$

On the other hand, according to (6) and (7), we see that for $n \geqq 3$.

$$K^{(n)}(s,t) = \int_a^b\int_a^b K(s,r)K^{(n-2)}(r,r_1)K(r_1,t)dr\, dr_1$$

Hence by the Schwarz inequality we have

$$|K^{(n)}(s,t)|^2 \leqq \int_a^b\int_a^b |K^{(n-2)}(r,r_1)|^2\, dr\, dr_1 \int_a^b\int_a^b |K(s,r)K(r_1,t)|^2\, dr\, dr_1$$

Accordingly, by making use of (8), we obtain

$$|K^{(n)}(s,t)|^2 \leqq \left[\int_a^b\int_a^b |K(s,t)|^2\, ds\, dt\right]^{n-2} \left\{\int_a^b |K(s,r)|^2 dr \int_a^b |K(r_1,t)|^2\, dr_1\right\}$$

The term in braces on the right side is continuous on the domain $a \leqq s \leqq b, a \leqq t \leqq b$; hence bounded. Therefore, according to the assumption (3), the series.

$$\Gamma'(s, t) = \sum_{n-1}^{\infty} K^{(n)}(s, t) \qquad ...(9)$$

converge uniformly on the domain $a \leqq s \leqq b, a \leqq t \leqq b$. Hence by term-by-term integration and by using (7) we obtain

$$\Gamma(s, t) = K(s, t) + \int_a^b K(s, r)\Gamma(r, t)dr \qquad ...(10)$$

$$\Gamma(s, t) = K(s, t) + \int_a^b \Gamma(s, r)K(r, t)dr \qquad ...(11)$$

The series (9) is known as the Neumann series for the Kernel $K(s, t)$.

Now, by making use of (10), we can prove that

$$\varphi(s) = f(s) + \int_a^b \Gamma(s, t)f(t)dt \qquad ...(12)$$

satisfies the equation (1). In fact, substituting (12) in (1) and using (10), we have

$$\begin{aligned} &\varphi(s) - \int_a^b K(s, t)\varphi(t)dt \\ &= f(s) + \int_a^b \Gamma(s, t)f(t)dt - \int_a^b K(s, t)\left\{f(t) + \int_a^b \Gamma(t, r)f(r)dr\right\}dt \\ &= f(s) + \int_a^b \left\{\Gamma(s, t) - K(s, t) - \int_a^b K(s, r)\Gamma(r, t)dr\right\}f(t)dt \\ &= f(s) \end{aligned}$$

Conversely, we can prove that if $\varphi(s)$ satisfies the equation (1), then $\varphi(s)$ satisfies (12). In fact, substituting $f(s) = \varphi(s) - \int_a^b K(s, t)\varphi(t)dt$ in (12) and using (11), we see that

$$\begin{aligned} &\varphi(s) - \int_a^b K(s, t)\varphi(t)dt \\ &+ \int_a^b \Gamma(s, t)\left\{\varphi(t) - \int_a^b K(t, r)\varphi(r)dr\right\}dt \\ &= \varphi(s) + \int_a^b \left\{\Gamma(s, t) - K(s, t) - \int_a^b \Gamma(s, r)K(r, t)dr\right\}\varphi(t)dt \\ &= \varphi(s) \end{aligned}$$

Accordingly, we see that the equation (1) is equivalent to the equation (12). Similarly, we can prove that conjugate equation (2) is equivalent to the equation

$$\phi(t) = g(t) + \int_a^b \Gamma(s, t)g(s)ds \qquad ...(13)$$

Example: Under the assumption (3), every solution $\varphi(s)$ of the equation (1) is given by (12) by means of the Kernel $\Gamma(s, t)$ and every solution $\phi(t)$ of the conjugate equation (2) is given by (13) by means of the conjugate Kernel $\Gamma(s, t)$ of $\Gamma(s, t)$, defined by

$$\Gamma(s, t) = \Gamma(t, s) \qquad ...(14)$$

Notes

For this reason, the Kernels $\Gamma(s, t)$ and $\Gamma'(s, t)$ are called the resolvent Kernels of the equation (1) and (2) respectively.

The foregoing theorem shows that λ is not an eigenvalue of either the Kernel $K(s, t)$ or its conjugate Kernel $K'(s, t)$,

$$K'(s, t) = K(t, s) \qquad ...(15)$$

The General Case

We shall prove that there exist two sets of linearly independent continuous functions

$$\begin{matrix} \alpha_1(s), \alpha_2(s), ..., \alpha_m(s) \\ B_1(t), \beta_2(t), ..., \beta_m(t) \end{matrix} \qquad ...(15)$$

defined on the interval $[a, b]$, such that

$$\int_a^b \int_a^b \left| K(s, t) - \sum_{v=1}^{m} \alpha_v(s)\beta_v(t) \right|^2 ds\, dt < 1 \qquad ...(16)$$

To prove this, let ε be an arbitrary positive number. Then we divide the interval (a, b) into a finite number of sub-intervals $I_1, I_2, ..., I_n$, such that

$$\sup_{a \leqq s \leqq v} |K(s, t') - K(s, t'')| \leqq \varepsilon$$

for any pair of points t', t'' in each I_v. This is possible, because of the uniform continuity of $K(s, t)$ on the domain $a \leqq s \leqq b, a \leqq t \leqq b$. Let t_v be an inner point of I_v. Let I'_v be an interval contained in the interior of I_v and containing the point t_v. Then we define $\beta_v(t)$ as follows:

$$\beta_v(t) \equiv \begin{cases} 0 & \text{outside of } I_v \\ 1 & \text{on } I'_v \end{cases}$$

such that the function $\beta_v(t)$ is continuous and $0 \leqq \beta_v(t) \leqq 1$ on the interval $[a, b]$. We now set $\alpha_r(s) \equiv K(s,t)$ and

$$N(s, t) = \left| K(s, t) - \sum_{v=1}^{n} \alpha_v(s)\beta_v(t) \right|$$

Then we see that

$$|N(s, t)| = |K(s, t) - K(s, t_v)| \leqq \varepsilon$$

for t in I'_v, and

$$|N(s, t)| = |K(s, t) - K(s, t_v)\beta_v(t)| \leqq 2M$$

for t in $I_v - I'_v$ where

$$M = \sup_{a \leqq s \leqq b \leqq a \leqq t \leqq b \leqq} |K(s, t)|$$

Since ε and the sum of lengths of $I_v - I'_v$ are both arbitrary, we can choose the values of them so small that

$$\int_a^b \int_a^b \left| K(s, t) - \sum_{v=1}^{n} \alpha_v(s)\beta_v(t) \right|^2 ds\, dt < 1$$

Clearly, the function $\beta_1(t), \beta_2(t), ..., \beta_n(t)$ are linearly independent. Hence, if $\alpha_1(s), \beta_2(s), ..., \alpha_n(s)$ are linearly independent, then our proof is completed. If otherwise, say, $\alpha_n(s)$ is written as a linear combination of $\alpha_1(s), \alpha_2(s), ..., \alpha_{n-1}(s)$, then

Notes

$$R(s,t) \equiv \sum_{v=1}^{n} \alpha_v(s)\beta_v(t)$$

is also written in the form

$$R(s,t) \equiv \sum_{v=1}^{n-1} \alpha_v(s)\beta_v^{(1)}(t)$$

If $\beta_1^{(1)}(t), \beta_2^{(1)}(t), ..., \beta_{n-1}^{(1)}(t)$ are linearly independent, then, by setting $\beta_v^{(1)}(t) = \beta_v(t)$, the number n is diminished. If otherwise, say, $\beta_{n-1}^{(1)}(t)$ is written as a linear combination of $\beta_1^{(1)}(t), \beta_2^{(2)}(t), ..., \beta_{n-2}^{(1)}(t)$ then $R(s, t)$ is also written as

$$R(s,t) \equiv \sum_{v=1}^{n-2} \alpha_v^{(1)}(s)\beta_v^{(1)}(t)$$

Repeating this argument alternatively for α and β, we finally obtain two sets of linearly independent functions

$$\gamma_1(s), \gamma_2(s), ..., \gamma_m(s) \text{ and } \delta_1(t), \delta_2(t), ..., \delta_m(t)$$

in terms of which $R(s, t)$ is written as

$$R(s,t) \equiv \sum_{v=1}^{m} \gamma_v(s)\delta_v(t)$$

provided that $K(s,t) \not\equiv 0$ and $R(s,t) \not\equiv 0$. Then by setting $\gamma_v(s) = \alpha_v(s)$, and $\delta_v(t) = \beta_v(t)$, the proof is completed

we now set

$$K_1(s,t) = K(s,t) - \sum_{v=1}^{m} \alpha_v(s)\beta_v(t)$$

and denote the resolvent Kernel of $K_1(s, t)$ by

$$\Gamma_1(s,t) = \sum_{n=1}^{\infty} K_1^{(n)}(s,t)$$

Then, the equation (1) is written as

$$\varphi(s) - \int_a^b K_1(s,t)\varphi(t)dt$$

$$= f(s) + \int_a^b \left(\sum_{v=1}^{m} \alpha_v(s)\beta_v(t) \right) \varphi(t)dt \qquad ...(17)$$

and we can prove in the same way as in last that φ(s) is determined by

$$\varphi(s) - \int_a^b \left[\sum_{v=1}^{m} \left(\alpha_v(s) + \int_a^b \Gamma_1(s,r)\alpha_v(r)dr \right) \beta_v(t) \right] \varphi(t)dt$$

$$= f(s) + \int_a^b \Gamma_1(s,r) f(r)\, dr \qquad ...(18)$$

Notes

From this follows the fact that to solve the equation (1) is equivalent to finding a solution $\varphi(s)$ of the equation (18) with the term in brackets as the Kernel and for the right side the given function

$$f(s) + \int_a^b \Gamma_1(s, r) f(r)\, dr$$

We shall prove incidentally that

$$\alpha_v(s) + \int_a^b \Gamma_1(s, r)\alpha_v(r) dr \qquad (v = 1, 2, ..., m) \quad ...(19)$$

are linearly independent. To prove this, suppose

$$0 \equiv \sum_{v=1}^{m} c_v \left(\alpha_v(s) + \int_a^b \Gamma_1(s, r)\, \alpha_v(r) dr \right)$$

$$= \sum_{v=1}^{m} c_v \alpha_v(s) + \int_a^b \Gamma_1(s, r) \left(\sum_{v=1}^{m} c_v \alpha_v(r) \right) dr$$

and $\sum_{v=1}^{m} | c_v | \neq 0.$ Then, by the properties of the resolvent Kernel $\Gamma_1(s, t)$, we have

$$\sum_{v=1}^{m} c_v \alpha_v(s) \equiv 0 - \int_a^b K_1(s, r) 0. dr \equiv 0$$

This contradicts the linear independence of $\alpha_v(s)$.

The equation (18) is reduced to the system of equations

$$\varphi(s) = f(s) + \int_a^b \Gamma_1(s, r) f(r) + \sum_{v=1}^{m} \rho_v \left(\alpha_v(s) + \int_a^b \Gamma_1(s, r)\alpha_v(r) dr \right) \quad ...(20)$$

$$\rho_\mu = \int_a^b \beta_\mu(t)\varphi(t) dt \qquad (\mu = 1, 2, ..., m) \quad ...(21)$$

Hence, substituting (20) in (21), we have a system of linear equations in unknowns, $\rho_1, \rho_2, ... \rho_m,$

$$\rho_\mu - \sum_{v=1}^{m} \rho_v \left[\int_a^b \alpha_v(s)\beta_\mu(s) ds + \int_a^b \Gamma_1(s, r)\, \alpha_v(r)\beta_\mu(s) dr\, ds \right]$$

$$= \int_a^b \left(\beta_\mu(t) + \int_a^b \Gamma_1(r, t)\, \beta_\mu(r) dr \right) f(t) dt \qquad (\mu = 1, 2, ..., m) \quad ...(22)$$

Accordingly to solve the equation (1) is equivalent to finding the solutions ρ_μ of (22); indeed, substituting the solution ρ_μ in (20), we obtain the solution of (1).

Similarly, we see that to solve the equation (2) is equivalent to solving the following system of linear equations in the unknowns

$\rho'_1, \rho'_2, ..., \rho'_m,$

$$\rho'_\mu - \sum_{v=1}^{m} \rho'_v \left[\int_a^b \alpha_\mu(t)\beta_v(t) dt + \int_a^b \Gamma_1(r, t)\alpha_\mu(t)\beta_v(r) dr\, dt \right] \quad ...(23)$$

$$= \int_a^b \left(a_\mu(s) + \int_a^b \Gamma_1(s, r)\, \alpha_\mu(r) dr \right) g(s) ds \qquad (\mu = 1, 2, ..., m)$$

and the solution $\phi(t)$ of (2) is given by

$$\phi(t) = g(t) + \int_a^b \Gamma_1(r, t) g(r) dr + \sum_{v=1}^{m} \rho'_v \left(\beta_v(t) + \int_a^b \Gamma_1(r, t)\beta_v(r) dr \right) \quad ...(24)$$

Notes

where the ρ'_v are the solution of (23).

Let Δ be the matrix of the equations (22), in the unknowns ρ, and Δ' that of the equations (23), in the unknowns ρ'. Then

Δ' is the transposed matrix of Δ ...(25)

Hence $\det \Delta' \neq 0$ if and only if $\det \Delta \neq 0$.

We first consider the case when $\det \Delta \neq 0$, and hence, $\det \Delta' \neq 0$. In this case, the equation (22)[(23)] for any function $f(s)[g(t)]$, admits a unique solution

$$\rho = (\rho_1, \rho_2, ..., \rho_m)\ [\rho' = (\rho'_1, \rho'_2, ..., \rho'_m)]$$

Therefore, for the given function $f(s)[g(t)]$, the equation (1) [2] admits a unique solution $\varphi(s)[\phi(t)]$. In particular, if $f(s) \equiv 0$ $[g(t) \equiv 0]$, then

$$\rho = (\rho_1, \rho_2, ..., \rho_m) = (0, 0, ..., 0)$$
$$[\rho' = (\rho'_1, \rho'_2, ..., \rho'_m) = (0, 0, ..., 0)]$$

hence, $\varphi(s) \equiv 0[\phi(t) \equiv 0]$

We next consider the case when $\det \Delta = 0$, and hence $\det \Delta' = 0$. For the sake of simplicity we write (22), (23) as

$$\rho_\mu - \sum_{v=1}^{m} c_{\mu v}\rho_v = f_\mu \qquad (\mu = 1, 2, ..., m) \quad ...(23')$$

$$\rho'_\mu - \sum_{v=1}^{m} c_{v\mu}\rho'_v = g_\mu \qquad (\mu = 1, 2, ..., m) \quad ...(24')$$

respectively. The matrices Δ, Δ' are of course written as

$$\Delta = (\delta_{\mu\nu} - c_{\mu\nu}),\ \Delta' = (\delta_{\nu\mu} - c_{\nu\mu})$$

where $\delta_{\mu\nu} = 0$ for $\mu \neq \nu$, and $\delta_{\mu\nu} = 1$ for $\mu = \nu$. For the case when $\det \Delta = \det \Delta' = 0$, the following facts are known:

The associated systems of linear homogeneous equations

$$\rho_\mu - \sum_{v=1}^{m} c_{\mu v}\rho_v = 0 \qquad (\mu = 1, 2, ..., m) \quad ...(22'')$$

and

$$\rho'_\mu - \sum_{v=1}^{m} c_{v\mu}\rho'_v = 0 \qquad (\mu = 1, 2, ..., m) \quad ...(23'')$$

admit a number $r(r \geqq 1)$ of linearly independent solutions

$$\rho(1) = (\rho_{11}, \rho_{12}, ..., \rho_{1m}), ...,$$
$$\rho(r) = (\rho_{r1}, \rho_{r2}, ..., \rho_{rm})$$

and

$$\rho'(1) = (\rho'_{11}, \rho'_{12}, ..., \rho'_{1m}), ...,$$
$$\rho'(r) = (\rho'_{r1}, \rho'_{r2}, ..., \rho'_{rm})$$

respectively. The inhomogeneous system (22′) admits a solution for given $f_1, f_2, ..., f_m$ if and only if

$$\sum_{\mu=1}^{m} f_\mu \rho'_{j\mu} = 0 \qquad (j = 1, 2, ..., m)$$

Notes

in other words, for the general solution of (23′′).

$$\sum_{j=1}^{r} c_j \rho'(j) = \left(\sum_{j=1}^{r} c_j \rho'_{j1}, \sum_{j=1}^{r} c_j \rho'_{j2}, ..., \sum_{j=1}^{r} c_j \rho'_{jm} \right)$$

which contains a number r of arbitrary constants c_1, c_2, ..., c_r, there hold the following relations

$$\sum_{\mu=1}^{m} f_\mu = \left(\sum_{j=1}^{r} c_j \rho'_{j\mu} \right) = 0 \qquad ...(26')$$

If the condition (26′) is satisfied, then the general solution of (22′) is given by the sum of a particular solution $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, ..., \bar{\rho}_m)$ of (22′) and the general solution $\sum_{j=1}^{r} c_j \rho(j)$ of (22′), that is, by the following expression containing r arbitrary constants c_1, c_2, ..., c_r.

$$\rho = \bar{\rho} + \sum_{j=1}^{r} c_j \rho(j) \qquad ...(27)$$

$$= \left(\bar{\rho}_1 + \sum_{j=1}^{r} c_j \rho_{j1}, \bar{\rho}_2 + \sum_{j=1}^{r} c_j \rho_{j2}, ..., \bar{\rho}_m + \sum_{j=1}^{r} c_j \rho_{jm} \right)$$

Similarly, the equations (23′) admit a solution for given g_1, g_2, ...g_m if and only if the following relations

$$\sum_{\mu=1}^{m} g_\mu \left(\sum_{j=1}^{r} c_j \rho_{j\mu} \right) = 0 \qquad ...(28)$$

hold; and under the condition (28), the general solution of (23′) is given by the sum of a particular solution $\bar{\rho}' = (\bar{\rho}'_1, \bar{\rho}'_2, ..., \bar{\rho}'_m)$ of (23′) and the general solution $\sum_{j=1}^{r} c_j p'(j)$ of (23′), that is, by the following expression containing r arbitrary constants c_1, c_2,.... c_r.

$$\rho' = \bar{\rho}' + \sum_{j=1}^{r} c_j \rho'(j)$$

$$= \left(\bar{\rho}'_1 + \sum_{j=1}^{r} c_j \rho'_{j1}, \bar{\rho}'_2 + \sum_{j=1}^{r} c_j \rho'_{j2}, ..., \bar{\rho}'_m + \sum_{j=1}^{r} c_j \rho'_{jm} \right) \qquad ...(29)$$

Accordingly, substituting the solution-ρ given by (27), if any, in (20), we obtain the general solution φ(s) of the equation (1). The function φ(s) contains r arbitrary constants. In fact, if

$$0 = \sum_{v=1}^{m} \left(\alpha_v(s) + \int_a^b \Gamma_1(s, r) \alpha_v(r) dr \right) \left(\sum_{j=1}^{r} c_j \rho_{jv} \right)$$

then, by the linear independence of (19)

$$0 = \sum_{j=1}^{r} c_j \rho_{jv} \qquad (v = 1, 2, ..., m,)$$

This contradicts the fact that ρ(1), ρ(2), ... ρ(r) are linearly independent solution of (22′′). We can also obtain, substituting (29) in (24), the general solution ϕ(t) of (2) which contains a number r of arbitrary constants.

Finally, we shall reduce the solvability condition (26′) to a more readable and usual form as follows:

The term on the left side of (26) is, by (22) and (22′)

$$\sum_{\mu=1}^{m} f_\mu \rho'_{j\mu} = \int_a^b \left[\sum_{\mu=1}^{m} \rho'_{j\mu} \left(\beta_m(t) + \int_a^b \Gamma_1(r, t)\beta_\mu(r)dr \right) \right] f(t)dt$$

From (24), it is easily seen that the function in brackets on the right side is a solution of (2) with $g(t) \equiv 0$, that is, of

$$\phi(t) - \int_a^b K(s, t)\phi(s)ds = 0 \quad \text{...(30)}$$

On the other hand, the general solution of (30) is given by linear combinations of the functions in brackets. Therefore (26) is equivalent to the following for every solution $\phi(t)$ of (30).

$$\int_a^b f(t)\phi(t)dt = 0 \quad \text{...(31)}$$

Similarly, we see that the condition (28) is equivalent to the following: for every solution $\varphi(s)$ of the equation

$$\varphi(s) - \int_a^b K(s, t)\varphi(t)dt = 0$$
$$\int_a^b g(s)\varphi(s)ds = 0 \quad \text{...(32)}$$

Self Assessment

1. The Fredholm equation is given by

$$y(x) = f(x) + \lambda \int_0^1 (xt - x^2t^2)y(t)dt$$

solve for $y(x)$ when $f(x) = x^3$.

13.3 Summary

- We have seen that Fredholm integral equation has solutions that depend on the nature of the resolvent Kernel as well on the function $f(s)$.
- If the parameter λ is not an eigenvalue then the non-homogeneous equation has one and only one solution and the homogeneous equation has a solution $Q(x) = 0$.
- For λ to be one of the eigenvalues, the homogeneous equation admits a number of independent solutions.

13.4 Keywords

The nature of the solution of the ***Fredholm integral equation*** of the second kind as well as on the first kind depends upon the constant parameter λ as well as on the function $f(x)$.

The ***eigenvalue problem*** puts certain conditions on the function $f(s)$ for the solutions to exist. Fredholm theorem elaborates on these points.

13.5 Review Question

Show that for the unsymmetric Kernel

$$K(x, t) = \sum_{v=1}^{\infty} v^{-2} \sin(vx) \sin[(v+1)t]$$

defined on the domain $0 \le x \le 2\pi, 0 \le t \le 2\pi$ has the iterated Kernel given by

$$K_n(x, t) = \sum_{v=1}^{\infty} \pi^{n-1}[v^2(v+1)^2(v+2)^2 \ldots (v+n-1)^2]^{-1} \sin xv \sin[(n+v)t]$$

[**Hint:** Integrate term-by-term and use the relation $\lim_{\alpha \to 0} \frac{\sin \alpha\mu}{\alpha} = u.$]

Answer: Self Assessment

1. $y(x) = x^3 + \lambda x c_1 - \lambda x^2 c_2$

   where $C_1 = \dfrac{(120 - \lambda)}{(600 - 80\lambda - \frac{5}{2}\lambda^2)}$,

   $C_2 = \dfrac{4 - 5C_1(1 - \lambda/3)}{5\lambda}$.

13.6 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations