TOPOLOGY II

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SYLLABUS

Topology II

Objectives: For some time now, topology has been firmly established as one of basic disciplines of pure mathematics. It's ideas and methods have transformed large parts of geometry and analysis almost beyond recognition. In this course we will study not only introduce to new concept and the theorem but also put into old ones like continuous functions. Its influence is evident in almost every other branch of mathematics.In this course we study an axiomatic development of point set topology, connectivity, compactness, separability, metrizability and function spaces.

Sr. No.	Content
1	The Urysohn Lemma, The Urysohn Metrization Theorem, The Tietze Extension Theorem, The Tychonoff Theorem
2	The Stone-Cech Compactification, Local Finiteness, Paracompactness
3	The Nagata-Smirnov Metrization Theorem, The Smirnov Metrization Theorem
4	Complete Metric Spaces, Compactness in Metric Spaces, Pointwise and Compact Convergence, Ascoli's Theorem
5	Baire Spaces, Introduction to Dimension Theory

CONTENT

Unit 1: The Urysohn Lemma

CONTENTS

Objectives

After studying this unit, you will be able to:

- State Urysohn's lemma;
- Understand the proof of Urysohn's lemma;
- Solve the problems on Urysohn's lemma.

Introduction

Saying that a space X is normal turns out to be a very strong assumption. In particular, normal spaces admit a lot of continuous functions. Urysohn's lemma is sometimes called "the first non-trivial fact of point set topology" and is commonly used to construct continuous functions with various properties on normal spaces. It is widely applicable since all metric spaces and all compact Hausdorff spaces are normal. The lemma is generalized by (and usually used in the proof of) the Tietze Extension Theorem.

1.1 Urysohn's Lemma

In topology, Urysohn's lemma is a lemma that states that a topological space is normal iff any two disjoint closed subsets can be separated by a function.

This lemma is named after the mathematician Pavel Samuilovich Urysohn.

1..1.1 Proof of Urysohn's Lemma

Urysohn's Lemma: Consider the set R with usual topology where $R = \{x \in R : 0 \leq x \leq 1\}$

A topological space (X, T) is normal iff given a pair of disjoint closed sets $A, B \subset X$, there is a continuous functions.

$$f : X \to R \text{ s.t. } f(A) = \{0\} \text{ and } f(B) = \{1\}.$$

Notes

Proof:

(1) Let R denote the set of all real numbers lying in the closed interval [0, 1] with usual topology. Let (X, T) be a topological space and let given a pair of disjoint closed sets A, B $\subset$ X ; $\exists$ a continuous map $f : X \to R$ s.t.

$f(A) = \{0\}, f(B) = \{1\}$.

To prove that (X, T) is a normal space.

Let a, b $\in$ R be arbitrary s.t. $a \le b$

write G = [0, a), H = (b, 1].

Then G and H are disjoint open sets in R.

Continuity of f implies that $f^{-1}(G)$ and $f^{-1}(H)$ are open in X.

Then our assumption says that

$f(A) = \{0\}, f(B) = \{1\}$

$f(A) = \{0\} \Rightarrow f^{-1}(\{0\} = f^{-1}(f(A)) \supset A$

$\Rightarrow f^{-1}(\{0\}) \supset A \Rightarrow A \subset f^{-1}(\{0\})$

Similarly $B \subset f^{-1}\{1\}$.

Evidently

$$\{0\} \subset [0, a) \Rightarrow f^{-1}(\{0\}) \subset f^{-1}([0, a))$$

$$\Rightarrow A \subset f^{-1}(\{0\}) \subset f^{-1}([0, a)]$$

$$\Rightarrow A \subset f^{-1}([0, a)) \Rightarrow A \subset f^{-1}(G)$$

$$\{1\} \subset (b, 1] \Rightarrow B \subset f^{-1}(\{1\}) \subset f^{-1}((b, 1])$$

$$\Rightarrow B \subset f^{-1}((b, 1]) \Rightarrow B \subset f^{-1}(H)$$

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$$

Given a pair of disjoint closed sets A, B $\subset$ X, we are able to discover a pair of disjoint open sets,

$$f^{-1}(G), f^{-1}(H) \subset X \text{ s.t. } A \subset f^{-1}(G), B \subset f^{-1}(H).$$

This proves that (X, T) is a normal space.

(2) Conversely, suppose that R is a set of real numbers lying in the interval [0, 1] with usual topology. Also suppose that A, B are disjoint closed subsets of a normal space (X, T).

To prove that $\exists$ a continuous map.

$$f : (X, T) \to R \text{ s.t. } f(A) = \{0\}, f(B) = \{1\}.$$

Step (i): Firstly, we shall prove that $\exists$ a map

$$f : (X, T) \to R \text{ s.t. } f(A) = \{0\}, f(B) = \{1\}.$$

Write $T = \left\{t : t = \dfrac{m}{2^n}, \text{where } m, n \in N \text{ s.t. } m \le 2^n\right\}$

Throughout the discussion we treat $t \in T$.

Making use of the fact that m takes 2^n values for a given value of n, we have

$$\sup(T) = \sup(t) = \sup\left(\frac{m}{2^n}\right) = \frac{1}{2^n}\sup(m) = \frac{2^n}{2^n} = 1 \ \sup(T) = 1$$

Notes

$t = \frac{1}{2}, 1,$ for n = 1

$t = \frac{1}{2^2}, \frac{2}{2^2}, \frac{3}{2^2}, \frac{4}{2^2},$ for n = 2

$A \cap B = \phi \Rightarrow A \subset X - B$

$\because$ X – B is an open set containing a closed set A. Using the normality, we can find an open set $G \subset X$ s.t.

$$A \subset G \subset \overline{G} \subset X - B \quad ...(1)$$

Writing $G = H_{1/2}$, $X - B = H_1$, we get

$$A \subset H_{1/2} \subset \overline{H}_{1/2} \subset H_1$$

This is the first stage of our construction

Consider the pairs of sets $(A, H_{1/2})$, $(\overline{H}_{1/2}, H_1)$

Using normality, we obtain open sets, $H_{1/4}$, $H_{3/4} \subset X$ s.t.

$$A \subset H_{1/4} \subset \overline{H}_{1/4} \subset H_{1/2}$$

$$\overline{H}_{1/2} \subset H_{3/4} \subset \overline{H}_{3/4} \subset H_1$$

Combining the last two relations, we have

$$A \subset H_{1/4} \subset \overline{H}_{1/4} \subset H_{1/2} \subset \overline{H}_{1/2} \subset H_{3/4} \subset \overline{H}_{3/4} \subset H_1$$

This is the second stage of our construction.

If we continue this process of each dyadic rational m of the function $t = m/2^n$, where

$$n = 1, 2, ... \text{ and } m = 1, 2 ... 2^n - 1,$$

Then open sets H_t will have the following properties:

(i) $A \subset H_t \subset \overline{H}_t \subset H_1 \ \forall \ t \in T$

(ii) Given $t_1, t_2 \in T$ s.t.

$t_1 < t_2 \Rightarrow A \subset H_{t_1} \subset \overline{H}_{t_1} \subset H_{t_2} \subset \overline{H}_{t_2} \subset H_2$

Construct a function $f : X \to R$ s.t. $f(x) = 0 \ \forall \ x \in H_t$

and $f(x) = \{t : x \notin H_t\}$ otherwise

In both cases $x \in X$.

$f(x) = \sup\{t : x \in H_t\} = \sup\{t\} = \sup(T) = 1$

Thus $f(x) = 0 \ \forall \ x \in H_t$

and $f(x) = 1 \ \forall \ x \notin H_t$ otherwise

$f(x) = 0 \ \forall \ x \in H_t$, $A \subset H_t \ \forall \ t \in T \Rightarrow f(x) = 0 \ \forall x \in A \Rightarrow f(A) = \{0\}$

$f(x) = 1 \ \forall \ x \notin H_t$, $H_t \subset H_1 \ \forall \ t \in T \Rightarrow f(x) = 1 \ \forall \ x \notin H_1$

$\Rightarrow f(x) = 1 \ \forall \ x \in X - B \ (\because X - B = H_1)$

$\Rightarrow f(x) = 1 \ \forall \ x \in B$

$\Rightarrow f(B) = \{1\}$

Thus we have shown that f(A) = {0}, f(B) = {1}.

Step (ii): Secondly, we shall prove that f is continuous. Let $a \in R$ be arbitrary then [0, a) and (a, 1] are open sets in R with usual topology.

Write $G_1 = f^{-1}([0, a))$, $G_2 = f^{-1}((a, 1])$.

Then G_1, G_2 can also be expressed as

$$G_1 = \{x \in X : f(x) \in [0, a)\}$$
$$= \{x \in X : 0 \le f(x) < a\}$$
$$= \{x \in X : f(x) < a\}$$

$\because$ According to the construction of f

$0 \le f(x) \le 1 \ \forall \ x \in X$

$$G_2 = \{x \in X : f(x) \in (a, 1)\}$$
$$= \{x \in X : a < f(x) \le 1\}$$
$$= \{x \in X : a < f(x)\}$$
$$= \{x \in X : f(x) > a\}$$

Finally, $G_1 = \{x \in X : f(x) < a\}$, $G_2 = \{x \in X : f(x) > a\}$

We claim $G_1 = \bigcup_{t<a} H_1, \quad G_2 = \bigcup_{t>a} \left(\overline{H}_t\right)$

Any $x \in G_1 \Rightarrow f(x) < a \Leftrightarrow x \in H_t$ for some $t < a$

This proves that $G_1 = \bigcup_{t<a} H_t$

$x \in G_2 \Rightarrow f(x) > a \Leftrightarrow x$ is out side of $\overline{H}_t$ for $t > a$

$\Leftrightarrow x \in \bigcup_{t>a} \left(\overline{H}_t\right)'$

Hence we get $G_2 = \bigcup_{t>a} \left(\overline{H}_t\right)'$

Since an arbitrary union of open sets is an open set and hence

$$\bigcup_{t<a} H_t, \quad \bigcup_{t>a} \left(\overline{H}_t\right)' \subset X$$

are open i.e., $G_1, G_2 \subset X$ are open, i.e.,

$f^{-1}([0, a))$, $f^{-1}((a, 1])$ are open in X.

$\therefore$ f is continuous

1 .1.2 Solved Examples

Example 1: If F_1 and F_2 are T-closed disjoint subsets of a normal space (x, T), then there exist a continuous map g of X into [0, 1] such that

$$g(x) = \begin{cases} 0 & \text{if } x \in F_1 \\ 1 & \text{if } x \in F_2 \end{cases}$$

$f(F_1) = \{0\}$ and $g(F_2) = \{1\}$.

Solution: Here write the proof of step II of the Urysohn's Theorem.

Example 2: If F_1 and F_2 are T-closed disjoint subsets of a normal space (X, T) and [a, b] is any closed interval on the real line, then there exists a continuous map f of X into [a, b] such that

$$f(x) = \begin{cases} a & \text{if } x \in F_1 \\ b & \text{if } x \in F_2 \end{cases}$$

i.e., $f(F_1) = \{a\}$, $f(F_2) = \{b\}$

This problem in known as **general form of Urysohn's lemma.**

Solution: Let F_1 and F_2 be disjoint closed subset of (X, T).

To prove that ∃ a continuous map

$$f : X \rightarrow [a, b] \text{ s.t. } f(F_1) = \{a\}, f(F_2) = \{b\}$$

By Urysohn's lemma, ∃ a continuous map

$$g : X \rightarrow [0, 1] \text{ s.t. } g(F_1) = \{0\}, g(F_2) = \{1\}.$$

Define a map $h : [0, 1] \rightarrow [a, b]$ s.t.

$$h(x) = \frac{(b-a)x}{1-0} + a$$

i.e., $h(x) = x(b - a) + a$

[This is obtained by writing the equation of the straight line joining (0, a) and (1, b) and then putting y = h(x)].

Evidently $h(0) = a$, $h(1) = b - a + a = b$

Also h is continuous

Write f = hg

$g : X \rightarrow [0, 1]$, $h : [0, 1] \rightarrow [a, b]$

$\Rightarrow hg : X \rightarrow [a, b] \Rightarrow f : X \rightarrow [a, b]$

Product of continuous functions is continuous

Therefore $f(F_1) = (hg)(F_1) = h[g(F_1)] = h(\{0\}) = \{a\}$

$f(F_2) = (hg)(F_2) = h[g(F_2)] = h(\{1\}) = \{b\}$

Thus ∃ a continuous map.

$f : X \to [a, b]$ s.t. $f(F_1) = \{a\}$, $f(F_2) = \{b\}$

1.2 Summary

- Urysohn's lemma is a lemma that states that a topological space is normal iff any two disjoint closed subsets can be separated by a function.
- Urysohn's lemma is sometimes called "the first non-trivial fact of point set topology."
- Urysohn's lemma: If A and B are disjoint closed sets in a normal space X, then there exists a continuous function $f : X \to [0, 1]$ such that $\forall\ a \in A$, $f(a) = 0$ and $\forall\ b \in B$, $f(b) = 1$.

1.3 Keywords

Continuous map: A continuous map is a continuous function between two topological spaces.

Disjoint: A and B are disjoint if their intersection is the empty set.

Normal: A topological space X is a normal space if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint.

Separated sets: A and B are separated in X if each is disjoint from the other's closure. The closures themselves do not have to be disjoint from each other.

1.4 Review Questions

1. Prove that every continuous image of a separable space is separable.
2. (a) Prove that the set of all isolated points of a second countable space is countable.

   (b) Show that any uncountable subset A of a second countable space contains at least one point which is a limit point of A.
3. (a) Let f be a continuous mapping of a Hausdorff non-separable space (X, T) onto itself.

   Prove that there exists a proper non-empty closed subset A of X such that f(A) = A.

   (b) Is the above result true if (X, T) is separable?
4. Examine the proof of the Urysohn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q,$$

   p, q rational.
5. Give a direct proof of the Urysohn lemma for a metric space (X, d) by setting

$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$$

6. Show that every locally compact Hausdorff space is completely regular.
7. Let X be completely regular, let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

1.5 Further Readings

Notes

Books

G. F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw Hill.

J. L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

Online links

www.planetmath.org.

www.amazon.ca/lemmas-pumping...urysohns

Unit 2: The Urysohn Metrization Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the Metrization;
- Explain the Urysohn Metrization Theorem;
- Solve the problems on Metrizability;
- Solve the problems on Urysohn Metrization Theorem.

Introduction

With Urysohn's lemma, we now want to prove a theorem regarding the metrizability of topological space. The idea of this proof is to construct a sequence of functions using Urysohn's lemma, then use these functions as component functions to embed our topological space in the metrizable space.

2.1 Metrization

Given any topological space (X, T), if it is possible to find a metric ρ on X which induces the topology T i.e. the open sets determined by the metric ρ are precisely the members of T, then X is said to the metrizable.

Example 1: The set $\mathbb{R}$ with usual topology is metrizable. For the usual metric on $\mathbb{R}$ induces the usual topology on $\mathbb{R}$. Similarly $\mathbb{R}^2$ with usual topology is metrizable.

Example 2: A discrete space (X, T) is metrizable. For the trivial metric induces the discrete topology T on X.

Example 3: Prove that if a set is metrizable, then it is metrizable in an infinite number of different ways.

Solution: Let X be a metrizable space with metric d.

Then ∃ a metric d on X which defines a topology T on X.

Notes

write $d_1(x, y) = \frac{d(x,y)}{1+d(x,y)} \quad \forall\, x, y \in X$

Then d_1 is a metric on X.

Again $d_2(x, y) = \frac{d_1(x,y)}{1+d_1(x,y)} \quad \forall\, x, y \in X$

Then d_2 is also a metric on X.

Continuing like this, we can define an infinite number of metrics on X.

Urysohn Metrization Theorem

Statement: Every regular second countable T_1-space is metrizable.

or

Every second countable normal space is metrizable.

Proof: Let (X, T) be regular second countable T_1-space.

To prove: (X, T) is metrizable.

X is regular and second countable.

$\Rightarrow$ X is normal.

Since (X, T) is second countable and hence there exists countable base $\mathcal{B}$ for the topology T on X. The elements of $\mathcal{B}$ can be enumerated as $B_1, B_2, B_3, \ldots$, where $\phi \neq B_n \in T$. Let $x \in X$ be arbitrary and $x \in U \in B$.

By normality of X,

$\exists\, V \in \mathcal{B}$ s.t. $x \in \overline{U} \subset V$

Write $C = \{(U, V) : U \times V \in \mathcal{B} \times \mathcal{B} \text{ s.t. } \overline{V} \subset V\}$

$\mathcal{B}$ is countable. $\Rightarrow \mathcal{B} \times \mathcal{B}$ is countable.

$\Rightarrow$ every subset of $\mathcal{B} \times \mathcal{B}$ is countable.

$\Rightarrow \mathcal{C}$ is countable.

For $\mathcal{C} \subset \mathcal{B} \times \mathcal{B}$

$\overline{U} \subset V \Rightarrow \overline{U} \cap (X - V) = \phi$

Also $\overline{U}$ and $X - V$ are closed in the normal space (X, T).

Hence, by Urysohn's lemma,

$\exists$ continuous map $f : X \to [0, 1] = I$, s.t.

$f(\overline{U}) = \{0\}$, $f(X - V) = \{1\}$

This implies $\quad f(x) = 0 \quad$ iff $\quad x \in \overline{U}$

and $\quad f(x) = 1 \quad$ iff $\quad x \in X - V$

Notes

Since continuous map f can be determined corresponding to every element (U, V) of $\mathcal{C}$. Take $\mathcal{F}$ as the collection of all such continuous maps.

$\mathcal{C}$ is countable $\Rightarrow \mathcal{F}$ is countable.

To prove that $\mathcal{F}$ distinguishes points and closed sets. For this, let H is closed subset of X and $x \in X - H$.

Now X – H is a nhd of x so that

$$\exists B_j \in \mathcal{B} \text{ s.t. } x \in B_j \subset X - H$$

Regularity of X $\Rightarrow \exists G \in B$ s.t. $x \in G \subset \overline{G} \subset B_j$.

By definition of base, we can choose $B_j \in \mathcal{B}$ s.t. $x \in B_j \subset G$

Thus, $x \in B_i \subset \overline{B_i} \subset B_j \subset X - H$

or $\quad x \in \overline{B_i} \subset B_j \subset X - H$

This implies $(B_i, B_j) \subset \mathcal{C}$

If f be corresponding member of $\mathcal{F}$, then

$f(\overline{B_i}) = \{0\}$, $f(X - B_j) = \{1\}$

$B_j \subset X - H$

$\Rightarrow H \subset X - B_j$

$\Rightarrow f(H) \subset f(X - B_j) = \{1\}$

$\Rightarrow f(H) \subset \{1\}$

$\Rightarrow \overline{f(H)} \subset \overline{\{1\}} = \{1\}$.

{For {1} is closed in I = [0, 1] for the usual topology on I and so $\overline{\{1\}} = \{1\}$].

This implies $\overline{f(H)} \subset \{1\} \Rightarrow \overline{f(H)} = \{1\}$

Also $f(X - B_j) = \{1\}$.

Hence, $f(X - B_j) = \{1\} = \overline{f(H)}$

Also $f(\overline{B_i}) = \{0\}$

$$f(x) = 0 \notin \{1\} = f(X - B_j) = \overline{f(H)}$$

$\Rightarrow \quad f(x) \notin \overline{f(H)}$ ...(1)

$\overline{f(H)}$ is closed subset of X.

Equation (1) shows that $\mathcal{F}$ distinguishes points and closed sets. Also, we have seen that $\mathcal{F}$ is countable family of continuous maps $f : X \to [0, 1]$.

If follows that X can be embedded as a subspace of the Hilbert Cube I^N which is metrizable.

Also, every subspace of metrizable space is metrizable.

This proves that (X, T) is metrizable.

Notes

Example 4: A compact Hausdorff space is separable and metrizable if it is second countable.

Solution: Let (X, T) be a compact Hausdorff space which is second countable.

To prove that X is separable and metrizable.

Firstly, we shall show that X is regular.

X is a Hausdorff space. ⇒ X is a T_2-space.

⇒ X is also a T_1-space.

⇒ $\{x\}$ is closed $\forall\ x \in X$.

Let $F \subset X$ be closed and $x \in X$ s.t. $x \notin F$.

Then F and $\{x\}$ are disjoint closed subsets of X.

X is a compact Hausdorff space.

⇒ X is a normal space.

As we know that "A compact Hausdorff space is normal".

By definition of normality,

We can find a pair of open set $G_1, G_2 \subset X$

s.t. $\{x\} \subset G_1$, $F \subset G_2$, $G_1 \cap G_2 = \phi$

i.e. $x \in G_1$, $F \subset G_2$, $G_1 \cap G_2 = \phi$

∴ Given a closed set f and a point $x \in X$ s.t. $x \notin F$ implies that ∃ disjoint open sets $G_1, G_2 \subset X$ s.t. $x \in G_1$, $F \subset G_2$.

This implies X is a regular space. ...(2)

X is a second countable. [A second countable space is always separable] ...(3)

⇒ X is separable.

From (1), (2) and (3), it follows that (X, T) is a regular second countable T_1-space.

And so by Urysohn's theorem, it will follow that X is metrizable. ...(4)

From (3) and (4), it follows that X is separable and metrizable.

Hence the result.

Theorem 1: Every metrizable space is a normal Frechet space.

Proof: Let X be a metrizable space so that ∃ a metric d on X which defines a topology T on X.

Step (i): To prove that (X, T) is a Frechet space i.e. T_1 space.

Let (X, d) be a metric space. Let $x, y \in X$ be arbitrary s.t. $d(x, y) = 2r$. Let T be a metric topology.

We know that every open sphere is T open. Then $S_{r(x)}$, $S_{r(y)}$ are open sets s.t.

$x \in S_{r(x)}$, $y \notin S_{r(x)}$

$x \in S_{r(y)}$, $x \notin S_{r(y)}$

Hence (x, d) is a T_1-space.

Step (ii): To prove (X, T) is a normal space.

It follows by the theorem.

"Every metric space is normal space" proved in Unit -17.

Example 5: Every subspace of a metrizable is metrizable.

Solution: Let (Y, ρ) be a subspace of a metric space (X, d) which is metrizable so that

(i) $\exists$ a topology T on X defined by the metric d on X.

(ii) $Y \subset X$ and $\rho(x, y) = d(x, y) \quad \forall\ x, y \in X$

Then the map ρ is a restriction of the map 'd' of Y. Consequently ρ defines the relative topology $\mathcal{U}$ on Y, showing thereby Y is metrizable.

2.2 Summary

- Given any topological space (X, T), if it is possible to find a metric ρ on X which induces the topology T then X is said to be the metrizable.
- The set R with usual topology is metrizable.
- Urysohn metrization theorem: Every second countable normal space is metrizable.
- Every metrizable space is a normal Frechet space.

2.3 Keywords

Compact: X is compact iff every open cover of X has a finite subcover.

Hausdorff : A topological space (X, T) is a Hausdorff space if given any two points $x, y \in X$, $\exists$ G, $H \in T$ s.t. $x \in G$, $y \in H$, $G \cap H = \phi$.

Normal: Let X be a topological space where one-point sets are closed. Then X is normal if two disjoint sets can be separated by open sets.

Regular: Let X be a topological space where one-point sets are closed. Then X is regular if a point and a disjoint closed set can be separated by open sets.

T_1 space: A topological space X is a T_1 if given any two points $x, y \in X$, $x \neq y$, there exists neighbourhoods U_x of x such that $y \notin Ux$.

2.4 Review Questions

1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.

2. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

3. Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Is it true that if Y is metrizable, then X has a countable basis?

4. Let X be a compact Hausdorff space that is the union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable.

5. A space X is locally metrizable if each point x of X has a neighbourhood that is metrizable in the subspace topology. Show that a compact Hausdorff space X is metrizable if it is locally metrizable.
6. Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable, then X has a countable basis?
7. Prove that the topological product of a finite family of metrizable spaces is metrizable.
8. Prove that every metrizable space is first countable.

2.5 Further Readings

Books

Robert Canover, *A first Course in Topology*, The Williams and Wilkins Company 1975.

Michael Gemignani, *Elementary Topology*, Dover Publications 1990.

Unit 3: The Tietze Extension Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- State the Tietze Extension Theorem;
- Understand the proof of Tietze Extension Theorem.

Introduction

One immediate consequence of the Urysohn lemma is the useful theorem called the Tietze extension theorem. It deals with the problem of extending a continuous real-valued function that is defined on a subspace of a space X to a continuous function defined on all of X. This theorem is important in many of the applications of topology.

3.1 Tietze Extension Theorem

Suppose (X, T) is a topological space. The space X is normal iff every continuous real function of defined point a closed subspace F of X into a closed interval [a, b] has a continuous extension.

$$f^* : X \to [a, b]$$

Proof:

(i) Suppose (X, T) is a topological space s.t. Every continuous real valued function $f : F \to [a, b]$ has a continuous extended function $g : X \to [a, b]$ where F is a closed subset of X, [a, b] being closed interval.

To prove X is a normal space.

Let F_1 and F_2 be two closed disjoint subsets of X.

Define a map $f : F_1 \cup F_2 \to [a, b]$

s.t. $f(x) = a$ if $x \in F_1$ and $f(x) = b$ is $x \in F_2$.

This map f is certainly continuous over the subspace $F_1 \cup F_2$. By assumption, f can be extended to a continuous map

$$g : X \to [a, b] \text{ s.t.}$$

$$g(x) = \begin{cases} a & \text{if } x \in F_1 \\ b & \text{if } x \in F_2 \end{cases}$$

The map g satisfies Urysohn's lemma and hence (X, T) is normal.

Notes

(ii) Conversely, suppose that (X, T) is a normal space.

Let f : F → [a, b] be a continuous map. F being a closed subset of X.

To prove that ∃ a continuous extension of f over X. For convenience, we take a = – 1, b = 1

Now we define a map f_0 = F → [–1, 1] s.t.

$$f_0(x) = f(x) \ \forall \ x \in F.$$

Suppose A_0 and B_0 are two subsets of F. s.t.

$$A_0 = \left\{x : f_0(x) \le -\frac{1}{3}\right\}, B_0 = \left\{x : f_0(x) \ge \frac{1}{3}\right\}$$

Then A_0 and B_0 are closed in X.

For F is closed in X. Applying general from of Urysohn's lemma, ∃ a continuous function

$$g_0 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right] \text{s.t. } g_0(A_0) = -\frac{1}{3}, g_0(B_0) = \frac{1}{3}$$

Write $$f_1 = f_0 - g_0$$

Then $$|f_1(x)| = |(f_0 - g_0)(x)| = |f_0(x) - g_0(x)| \le \frac{2}{3}$$

Let $$A_1 = \left\{x : f_1(x) \le \left(-\frac{1}{3}\right), \left(\frac{2}{3}\right)\right\},$$

$$B_1 = \left\{x : f_1(x) \ge \frac{1}{3}, \frac{2}{3}\right\},$$

Then A_1, B_1 are non-empty disjoint closed sets in X and hence ∃ a continuous function s.t.

$$g_1 : X \to \left[-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]$$

$$g_1(A_1) = -\frac{1}{3}, \frac{2}{3}, g_1(B_1) = \frac{1}{3}, \frac{2}{3}$$

Again we define a function f_2 and F s.t.

$$f_2 = f_1 - g_1 = f_0 - g_0 - g_1 = f_0 - (g_0 + g_1)$$

Then $$|f_2(x)| = |f_0(x) - (g_0 + g_1)(x)| \le \left(\frac{2}{3}\right)^2$$

Continuing this process, we get a sequence of function.

$$\langle f_0, f_1, f_2, \ldots, f_n, \ldots \rangle$$

defined on F s.t. $$|f_n(x)| \le \left(\frac{2}{3}\right)^n$$

and a sequence $\langle g_0, g_1, g_2, \ldots \rangle$

defined on X s.t. $$|g_n(x)| \le \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n$$

$$f_n = f_0 - (g_0 + g_1 + ... + g_{(n-1)})$$

Write $$S_n = \sum_{r=0}^{n-1} g_r$$

Now S_n can be regarded as partial sums bounded continuous function defined on X. Since the space of bounded real valued function is complete and

$$|g_n(x)| \leq \frac{1}{3}.\left(\frac{2}{3}\right)^n \text{ and } \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1,$$

the sequence $\langle S_n \rangle$ converges confirmly on X to g (say) when $|g(x)| \leq 1$.

$|f_n(x)| \leq \left(\frac{2}{3}\right)^n \Rightarrow \langle S_n \rangle$ converges uniformly on F to f_0 say

Hence g = f on F.

Thus g is a continuous extension of f to X which satisfies the given conditions.

3.2 Summary

- Tietze extension theorem:

  Suppose (X, ρ) is a topological space. The space X is normal iff every continuous real function f defined on a closed subspace F of X into a closed interval [a, b] has a continuous extension f^* $X \to [a, b]$

3.3 Keywords

Closed Set: A subset A of a topological space X is said to be closed if the set X – A is open.

Continuous Map: A function f : R → R is said to be continuous if for each a ∈ R and each positive real number ∈, there exists a positive real number δ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \in$.

Normal Space: A topological space (X, T) is said to be a normal space iff it satisfies the following axioms of Urysohn: If F_1 ad F_2 are disjoint closed subsets of X then there exists a two disjoint subsets one containing F_1 and the other containing F_2.

3.4 Review Questions

1. Show that the Tietze extension theorem implies the Urysohn lemma.
2. Let X be metrizable. Show that the following are equivalent:
   (a) X is bounded under every metric that gives the topology of X.
   (b) Every continuous function $\phi : X \to R$ is bounded.
   (c) X is limit point compact.

3.5 Further Readings

Notes

Books

J.F. Simmons, *Introduction to Topology and Modern Analysis*. McGraw Hill International Book Company, New York 1963.

A.V. Arkhangel'skii, V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, Reidel (1984).

Online links

www.mathword. wolfram.com

http://www.answers.com/topic/planetmath

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Unit 4: The Tychonoff Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define finite intersection property;
- Solve the problems on finite intersection;
- Understand the proof of Tychonoff's theorem.

Introduction

Like the Urysohn Lemma, the Tychonoff theorem is what we call a "deep" theorem. Its proof involves not one but several original ideas; it is anything but straightforward. We shall prove the Tychonoff theorem, to the effect that arbitrary products of compact spaces are compact. The proof makes use of Zorn's lemma. The Tychonoff theorem is of great usefulness to analysts we apply it to construct the Stone-Cech compactification of a completely regular space and in proving the general version of Ascoli's theorem.

4.1 Finite Intersection Property

Let X be a set and f a family of subsets of X. Then f is said to have the finite intersection property if for any finite number F_1, F_2, ..., F_n of members of f.

$$F_1 \cap F_2 \cap .... F_n \neq \phi$$

Proposition: Let (X, T) be a topological space. Then (X, T) is compact if any only if every family f of closed subsets of X with the finite intersection property satisfies $\cap_{F \in f} F \neq \phi$.

Proof: Assume that every family f of closed subsets of X with the finite intersection property satisfies $\cap_{F \in f} F \neq \phi$. Let $\mathcal{U}$ be any open covering of X. Put f equal to the family of complements of members of $\mathcal{U}$. So each $F \in f$ is closed in (X, f). As $\mathcal{U}$ is an open covering of X, $\cap_{F \in f} F = \phi$. By our assumption, then, f does not have the finite intersection property. So for some F_1, F_2,...F_n in f, $F_1 \cap F_2 \cap ...., \cap F_n = \phi$.

Thus $\bigcup_1 \cup \bigcup_2 \cup ... \cup \bigcup_n = X$, where

$\cup_i = X \setminus F_i$, i = 1,...., n.

So $\mathcal{U}$ has a finite subcovering. Hence, (X, T) is compact.

The converse statement is proved similarly.

Example 1: Let X be a topological space and let $\mathcal{B}$ be a closed sub-base for X and let $\{B_i\}$ be its generated closed base i.e. the class of all finite union of members of $\mathcal{B}$ if every class of B_i's with the finite intersection property (FIP) has a non-empty intersection then X is compact.

Solution: Under the given hypothesis, we shall prove that X is compact. In order to prove the required result it is sufficient to show that every basic cover of X has a finite sub-cover.

Let $\{O_j\}$ be any basic open cover of X. Then $X = \bigcup_j O_j$.

Now, $\{B_i^C\}$ being an open base for X implies that each O_j is a union of certain B_i^C's and the totality of all such B_i^C's that arise in this way is a basic open cover of X. By De-Morgan's law, the totality of corresponding B_i's has empty intersection and therefore by the given hupothesis this totality does not have FIP. This implies that there exist finitely many B_i's, say,

$$B_{i_1}, B_{i_2} \ldots B_{i_n} \text{ such that } \bigcap_{K=1}^{n} B_{i_K} = \phi .$$

Taking complements on both sides, we set

$$\bigcup_{K=1}^{n} B_{i_K}^C = X. \qquad \text{(By De-Morgan's Law)}$$

For each $B_{i_K}^C$ (K = 1, 2, …, n) we can find a O_{j_K} such that $B_{i_K}^C \subseteq O_{j_K}$.

Thus $X = \bigcup_{K=1}^{n} O_{j_K}$.

Thus, we have shown that every basic open cover of X has a finite sub-cover.

Example 2: Let X be a non-empty set. Then every class $\{B_j\}$ of subsets of X with the FIP is contained in some maximal class with the FIP.

Solution: Let $\{B_j\}$ be a class of subsets of X with the FIP and let P be the family of all classes of subsets of X that contains $\{B_j\}$ and have the FIP.

For any F_λ, $F_\mu \in P$, define $F_\lambda \leq F_\mu$ so that $F_\lambda \subseteq F_\mu$.

Then (P, ≤) is a partially ordered set. Let $\top$ be any totally ordered subset of (P, ≤). Then, the union of all classes in $\top$ has an upper bound for $\top$ in P.

Thus (P, ≤) is a partially ordered set in which every totally ordered subset has an upper bound.

Hence by Zern's lemma, P possesses a maximal element i.e., there exist a class $\{B_K\}$ of subsets of X such that $\{B_j\} \subseteq \{B_K\}$, $\{B_K\}$ has the FIP and any class of subsets of X which properly contains $\{B_K\}$ does not have the FIP.

Tychonoff's Theorem

Before proving Tychonoff's theorem, we shall prove two important lemmas.

Lemma 1: Let X be a set; Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. Then there is a collection D of subsets of X such that D contains $\mathcal{A}$ and D has the finite intersection property, and no collection of subsets of X that properly contains D has this property.

We often say that a collection D satisfying the conclusion of this theorem is maximal with respect to the finite intersection property.

Notes

Proof: As you might expect, we construct D by using Zorn's lemma. It states that, given a set A that is strictly partially ordered, in which every simply ordered subset has an upper bound, A itself has a maximal element.

The set A to which we shall apply Zorn's lemma is not a subset of X, nor even a collection of subsets of X, but a set whose elements are collections of subsets of X. For purpose of this proof, we shall call a set whose elements are collections of subsets of X a "superset" and shall denote it by an outline letter. To summarize the notation:

c is an element of X.

C is a subset of X.

$\mathcal{C}$ is collection of subset of X.

$\mathbb{C}$ is a superset whose elements are collections of subsets of X.

Now by hypothesis, we have a collection $\mathcal{A}$ of subsets of X that has the finite intersection property. Let $\mathbb{A}$ denote the superset consisting of all collections $\mathcal{B}$ of subsets of X such that $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{B}$ has the finite intersection property. We use proper inclusion $\not\subseteq$ as our strict partial order of $\mathbb{A}$. To prove our lemma, we need to show that $\mathbb{A}$ has a maximal element D.

In order to apply Zorn's lemma, we must show that if $\mathbb{B}$ is a "sub-superset" of $\mathbb{A}$ that is simply ordered by proper inclusion, then $\mathbb{B}$ has a upper bound in $\mathbb{A}$. We shall show in fact that the collection

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathrm{B}} \mathcal{B},$$

which is the union of the collections belonging to $\mathbb{B}$, is an element of $\mathbb{A}$; the it is the required upper bound on $\mathbb{B}$.

To show that $\mathcal{C}$ is an element of $\mathbb{A}$, we must show that $\mathcal{C} \in \mathcal{A}$ and the $\mathcal{C}$ has the finite intersection property. Certainly $\mathcal{C}$ contains $\mathcal{A}$, since each element of B contains $\mathcal{A}$. To show that $\mathcal{C}$ has the finite intersection property, let $C_1, ..., C_n$ be elements of $\mathcal{C}$. Because $\mathcal{C}$ is the union of the elements of $\mathbb{B}$, there is, for each i, an element $\mathcal{B}_i$ of $\mathbb{B}$ such that $C_i \in \mathcal{B}_i$. The superset $\{\mathcal{B}_i, ..., \mathcal{B}_n\}$ is contained in $\mathbb{B}$. So it has a largest element; that is, there is an index K such that $B_i \subset B_K$ for i = 1, ..., n. then all the sets $C_1...., C_n$ are elements of $\mathcal{B}_k$. Since $\mathcal{B}_k$ has the finite intersection property, the intersection of the sets $C_1, ..., C_n$ is non-empty, as desired.

Lemma 2: Let X be a set; Let D be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) Any finite intersection of elements of D is a element of D.

(b) If A is a subset of X that intersects every element of D, then A is an element of D.

Proof:

(a) Let B equal the intersection of finitely many elements of D. Define a collection of E by adjoining B to D, so that $E = D \cup \{B\}$. We show that E has the finite intersection property; then maximality of D implied that E = D, so that $B \in D$ as desired.

Take finitely many elements of E. If none of them is the set B, then their intersection is non-empty because D has the finite intersection property. If one of them is the set B, then their intersection is of the form

$$D_1 \cap ,... D_m \cap B.$$

Since B equals a finite intersection of elements of D, this set is non-empty.

(b) Given A, define $E = D \cap \{A\}$. We show that E has the finite intersection property from which we conclude that A belongs to D. Take finitely many elements of E. If none of them is the set A, their intersection is automatically non-empty. Otherwise, it is of the form

$$D_1 \cap ... \cap D_n \cap A.$$

Now $D_1 \cap ... \cap D_n$ belongs to D, by (a); therefore this intersection is non-empty, by hypothesis.

***Theorem 1:* (*Tychonoff theorem*):** An arbitrary product of compact spaces is compact is the product topology:

Proof: Let

$$X = \prod_{\alpha \in J} X_\alpha,$$

where each space X_α is compact. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection

$$\bigcap_{A \in \mathcal{A}} \bar{A}$$

is non-empty. Compactness of X follows:

Applying *Lemma 1*, choose a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and $\mathcal{D}$ is maximal with respect to the finite intersection property. It will suffice to show that the intersection $\cap_{D \in \mathcal{D}} \bar{D}$ is non-empty.

Given $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map, as usual. Consider the collection

$$\{\pi_\alpha (D) \mid D \in \mathcal{D}\}$$

of subset of X_α. This collection has the finite intersection property because $\mathcal{D}$ does. By compactness of X_α, we can for each α choose a point x_α of X_α such that

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}.$$

Let x be the point $(x_\alpha)_\alpha \in J$ of X. We shall show that for $x \in \bar{D}$ for every $D \in \mathcal{D}$; then our proof will be finished.

First we show that if $\pi_\beta^{-1} (\cup_\beta)$ is any sub-basis element (for the product topology on X) containing x, then $\pi_\beta^{-1} (\cup_\beta)$ intersects every element of $\mathcal{D}$. The set $\cup_\beta$ is a neighbourhood of x_β in X_β. Since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, $\cup_\beta$ intersects π_β (D) in some point π_β (y), where $y \in D$. Then it follows that $y \in \pi_\beta^{-1} (\cup_\beta) \cap D$.

It follows from (b) of Lemma 2, that every sub-basis element containing x belongs to D. And then it follows (a) of the same lemma that every basis element containing x belongs to $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, this means that every basis element containing x intersects every element of $\mathcal{D}$; hence $x \in \bar{D}$ for every $D \in \mathcal{D}$ as desired.

4.2 Summary

- Let X be a set and f a family of subsets of X. Then f is said to have the finite intersection property if for any finite number $F_1, F_2, ... F_n$ of members of f. $F_1 \cap F_2 \cap ... \cap F_n \neq \phi$.
- Let (X, T) be a topology space. Then (X, T) is compact iff every family f of closed subsets of X with the finite intersection property satisfies $\cap_{F \in f} F \neq \phi$.
- An arbitrary product of compact spaces is compact in the product topology.

4.3 Keywords

Compact Set: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to finite sub-covering.

Maximal: Let (A, ≤) be a partially ordered set. An element $a \in A$ is called a maximal element of A if ∃ no element in A which strictly dominates a, i.e.

$$x \leq a \text{ for every comparable element } x \in A.$$

Projection Mappings: The mappings

$$\pi_x ; X \times Y \to X \text{ s.t. } \pi_x(x, y) = x \ \forall (x, y) \in X \times Y$$

$$\pi_y ; X \times Y \to Y \text{ s.t. } \pi_y(x, y) = y \ \forall (x, y) \in X \times Y$$

are called projection maps of X × Y onto X and Y space respectively.

Tychonoff Space: It is a completely regular space which is also a T_1-space i.e. $T_{3\frac{1}{2}} = [CR] + T_1$.

Upper bound: Let $A \subset R$ be any given set. A real number b is called an upper bound for the set A if.

$$x \leq b \ \forall \ x \in A.$$

4.4 Review Questions

1. Let X be a space. Let D be a collection of subsets of X that is maximal with respect to the finite intersection property.

   (a) Show that $x \in \bar{D}$ for every $D \in \mathcal{D}$ if any only if every neighbourhood of x belongs to $\mathcal{D}$. Which implication uses maximality of $\mathcal{D}$?

   (b) Let $D \in \mathcal{D}$. Show that if $A \supset D$, then $A \in \mathcal{D}$.

   (c) Show that if X satisfies the T_1 axion, there is at most one point belonging to $\cap_{D \in \mathcal{D}} \bar{D}$.

2. A collection $\mathcal{A}$ of subsets of X has the countable intersection property if every countable intersection of elements of $\mathcal{A}$ is non-empty. Show that X is a Lindelöf space if any only if for every collection $\mathcal{A}$ of subsets of X having the countable intersection property,

   $$\bigcap_{A \in \mathcal{A}} \bar{A}$$

   is non-empty.

4.5 Further Readings

Books

Bimmons, *Introduction to Topology and Modern Analysis*.

Nicolas Bourbaki, *Elements of Mathematics*.

Online links

www.planetmath.org

www.jstor.org

Unit 5: The Stone-Cech Compactification

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the compactification;
- Define the Stone-Cech compactification;
- Explain the related theorems.

Introduction

We have already studied one way of compactifying a topological space X, the one-point compactification; it is in some sense the minimal compactification of X. The Stone-Cech compactification of X, which we study now, is in some sense the maximal compactification of X. It was constructed by M. Stone and E. Cech, independently, in 1937. It has a number of applications in modern analysis. The Stone-Cech compactification is defined for all Tychonoff Spaces and has an important extension property.

5.1 Compactification

A compactification of a space X is a compact Hausdorff space Y containing X as a subspace such that $\bar{X} = Y$. Two compactifications Y, and Y_2 of X are said to be equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that h(x) = x for every $x \in X$.

Remark: If X has a compactification Y, then X must be completely regular, being a subspace of completely regular space Y. Conversely, if X is completely regular, then X has a compactification.

Lemma 1: Let X be a space; suppose that $h : X \to X$ is an imbedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X; it has the property that there is an imbedding $H : Y \to Z$ that equals h on X. The compactification Y is uniquely determined up to equivalence.

We call Y the compactification induced by the imbedding h.

Proof: Given h, let X_0 denote the subspace h(X) of Z, and let Y_0 denote its closure of Z.

Notes

Then Y_0 is a compact Hausdorff space and $\bar{X}_0 = Y_0$; therefore, Y_0 is a compactification of X_0.

We now construct a space Y containing X such that the pair (X, Y) is homeomorphic to the pair (X_0, Y_0). Let us choose a set A disjoint from X that is in bijective correspondence with set $Y_0 - X_0$ under map $K : A \rightarrow Y_0 - X_0$.

Define $Y = X \cup A$, and define a bijective correspondence $H : Y \rightarrow Y_0$ by the rule

$$H(x) = h(x) \quad \text{for } x \in X,$$

$$H(a) = k(a) \quad \text{for } a \in A.$$

Then topologize Y by declaring $\cup$ to be open in Y if and only if $H(\cup)$ is open in Y_0. The map H is automatically a homeomorphism; and the space X is a subspace of Y because H equals the homeomorphism 'h' when restricted to the subspace X of Y. By expanding the range of H, we obtain the required imbedding of Y into Z.

Now suppose Y_i is a compactification of X and that $H_i : Y_i \rightarrow Z$ is an imbedding that is an extension of h, for i = 1, 2. Now H_i maps X onto $h(X) = X_0$. Because H_i is continuous, it must map Y_i into $\bar{X}_0$; because $H_i(Y_i)$ contains X_0 and is closed (being compact), it contains $\bar{X}_0$. Hence, $H_i(Y_i) = \bar{X}_0$ and H_2^{-1} o H_1 defines a homeomorphism of Y_1 with Y_2 that equals the identity on X.

Theorem 1: The collection of all compactifications of a topological space is partially ordered by $\geqq$. If (f, Y) and (g, Z) are Hausdorff compactifications of a space and (f, Y) $\geqq$ (g, Z) $\geqq$ (f, Y), then (f, Y) and (g, Z) are topologically equivalent.

Proof: If (f, Y) $\geqq$ (g, Z) $\geqq$ (h, U), where these are compactification of a space X, then there are continuous functions j on Y to Z and K on Z to U such that g = j o f and h = k o g and hence h = k o j o f and (f, Y) $\geqq$ (h, U). Consequently $\geqq$ partially orders the collection of all compactifications of X. If (f, Y) and (g, Z) are Hausdorff compactifications each of which follows the other relative to the ordering $\geqq$, then both f o g^{-1} and g o f^{-1} have continuous extensions j and k to all of Z and Y respectively.

Since k o j is the identity map on the dense subset g [X] of Z and Z is Hausdorff k o j is the identity map of Z onto itself and similarly j o k is the identity map of Y onto Y. Consequently (f, Y) and (g, Z) are topologically equivalent.

5.1.1 One Point Compactification

Definition: Let X be a locally compact Hausdorff space.

Take some objects outside X, denoted by the symbol ∞ for convenience and adjoin it to X, forming the set

$$Y = X \cup \{\infty\}.$$

Define topology $\cup$ on Y as follows:

(i) $G \in \cup$ if T

(ii) $Y - C \in \cup$ if C is a compact subset of X.

The space Y is called one point compactification of X.

Theorem 2: Let X be a locally compact Hausdorff space which is not compact. Let Y be one point compactification of X. Then Y is compact Hausdorff space : X is a subspace of Y : the set Y – X consists of a single point and $\bar{X} = Y$.

Notes

Proof:

1. To show that X is a subspace of Y and $\bar{X} = Y$.

   Let ∪ be a topology on Y. Let H ∈ ∪, then

   H ∩ X = H

   and so H ∈ T. Also (Y – C) ∩ X = X – C

   and so X – C ∈ T. Conversely any open set in X is of the type (1) and therefore open in Y. Since X is not compact, each open set Y – C containing ∞ intersects X, meaning thereby ∞ is a limit point of X, so that $\bar{X} = Y$.

2. To show that Y is compact.

   Let G be an ∪-open covering of Y. The collection G must contain an open set of the type Y – C. Also G contains set of the type G, where G ∈ T, each of these sets does not contain the point ∞. Take all such sets of G different from Y – C, intersect them with X, they form a collection of open sets in X covering C.

   As C is compact, hence a finite number of these members will cover C; the corresponding finite collection of elements of G along with the elements of Y – C cover all of Y.

   Hence Y is compact.

3. To show that y is Hausdorff.

   Let x, y ∈ Y.

   If both of them lie in X and X is known to be compact so that ∃ disjoint open sets ∪, V in X

   s.t. x ∈ ∪, y ∈ V.

   On the other hand if

   x ∈ X

   and y = ∞.

We can chose compact set C and X containing a nbd ∪ of x.

The ∪ and Y – C are disjoint nbds of x and ∞ respectively in Y.

Theorem 3: If (X^*, T^*) be a one point compactification of a non-compact topological space (X, T), then (X^*, T^*) is a Hausdorff space iff (X, T) is locally compact.

Proof: Assuming that X is a Hausdorff space, each pair of distinct points in X^*, all of which belong to X can be separated by open subsets of X. Thus it is sufficient to show that any pair (x, ∞) ∈ X, can be separated by open subsets of X^*. Now X is locally compact

⇒ any x ∈ X, has a nbd N whose closure N in X is compact

⇒ N and $\bar{N}'$ are disjoint open subsets of X^* s.t. x ∈ N and ∞ ∈ N′

⇒ distinct points x, ∞ of X^* have disjoint nbds

⇒ (X^*, T^*) is Hausdorff.

Conversely if (X^*, T^*) is Hausdorff, then

X is a subspace of X^* ⇒ X is Hausdorff, since Hausdorffness is hereditary.

Notes

Now we claim that X is locally compact. It will be so if every point of it has a nbd whose closure is compact.

$x \in X$ is fixed and distinct $x, \infty \in X^*$ (Hausdorff) $\Rightarrow \exists$ disjoint open sets A_1^*, A_2^* in X^* s.t. $x \in A_1^*$ and $\infty \in A_2^*$.

But an open set containing ∞ must be of the form

$$A_2^* = \{\infty\} \cup A$$

where A is an open set in X containing x s.t. its complement is compact.

Also $\infty \in A_1^* = A_1^*$ is an open set in X containing x, whose closure is contained in A

$\Rightarrow A_1^*$ is compact

$\Rightarrow$ every point of X has a nbd whose closure is compact

$\Rightarrow$ X is locally compact.

5.1.2 Stone-Cech Compactification

The pair (e, $\beta(X)$), where X is a Tychonoff space and $\beta(X)\left(=\overline{e(x)}\right)$ is called Stone-Cech compactification of X. e is a map from X into $\beta(X)$.

For each completely regular space X, let us choose, once and for all, a compactification of X satisfying the extension condition i.e. For a completely regular space X, $\exists$ a compactification Y of X having the property that every bounded continuous map $f : X \to \mathcal{R}$ extends uniquely to a continuous map of Y into $\mathcal{R}$.

We will denote this compactification of X by $\beta(X)$ and call it the Stone-Cech compactification of X. It is characterized by the fact that any continuous map $f : X \to C$ of X into a compact Hausdorff space C extends uniquely to a continuous map $g : \beta(X) \to C$.

Theorem 4: Let X be a Tychonoff space, (e, $\beta(X)$) its stone-cech compactification and suppose $f : X \to [0, 1]$ is continuous. Then there exists a map $g : \beta(X) \to [0, 1]$ such that g o e = f, i.e. g is an extension of f to $\beta(X)$, if we identify X with e(X).

Proof: Let $\exists$ be the family of all continuous functions from X into [0, 1]. Then $\beta(X) \subset [0, 1]^{\exists}$ we define g on the entire cube $[0, 1]^{\exists}$ by $g(\lambda) = \lambda(f)$ for $\lambda \in [0, 1]^{\exists}$.

This is well defined because an element of $[0, 1]^{\exists}$ is a function from $\exists$ into [0, 1] and can be evaluated at f since $f \in \exists$. Equivalently, g is nothing but the projection f from $[0, 1]^{\exists}$ onto [0, 1], and hence is continuous. Now if $x \in X$ then, by definition of the evaluation map, $e(x) \in [0, 1]^{\exists}$ is the function $e(x) : \exists \to [0, 1]$ such that

$$g \text{ o } e(x)\,(h) = h(x) \quad \text{for } h \in \exists.$$

Now $\quad g \text{ o } e(x) = g(e(x)) = e(x)\,(f) = f(x)\ \forall\ x \in X$

So $\quad g \text{ o } e = f.$

Thus, we extended f not only to $\beta(x)$ but to the entire cube $[0, 1]^{\exists}$. Its restriction to $\beta(X)$ proves the theorem.

Theorem 5: A continuous function from a Tychonoff space into a compact Hausdorff space can be extended continuously over the stone-cech compactification of the domain. Moreover such an extension is unique.

Proof: Let X be a Tychonoff space, $\beta(X)$ its stone-cech compactification and $f : X \to Y$ a map where Y is a compact Hausdorff space.

Notes

Let $\exists_1$, $\exists_2$ be respectively the families of all continuous functions from X, Y respectively to the unit interval [0, 1] and let e, e′ be the embedding of X, Y into $[0,1]^{\exists_1}$ and $[0,1]^{\exists_2}$ respectively. For any $g \in \exists_2$ let $\pi g : [0,1]^{\exists_2} \to [0, 1]$ be the corresponding projection.

Then π_g o e′ of is a map from X into [0, 1] and so it has an extension say θ_g to β(X). Then θ_g o e π_g o e′ o f.

Now consider the family $\{\theta_g = g \in \exists_2\}$ of maps from β(X) into [0, 1]. Let $\theta : \beta(X) \to [0, 1]^{\exists}$ be the evaluation map determined by this family. We claim that θ o e = e′ o f. Let x ∈ X. Them θ(e(x)) is an element of $[0,1]^{\exists_2}$ given by

$$\theta(e(x))(g) = \theta g(e(x)) \qquad \text{[by the definition of the evaluation functions]}$$

$$\text{But } \theta_g(e(x)) = \pi_g(e'f(x)) = e'(f(x))(g)$$

Thus for all $g \in \exists_2$

$$[\theta_o e(x)](g) = [e'_o f(x)](g) \quad \text{and} \quad \text{so}$$

$$\theta \text{ o } g = e'_o f \text{ as claimed.}$$

Now $\theta(e(x)) = e'(f(X)) \subset e'(Y)$.

Since Y is compact, e′(Y) compact and hence a closed subset of $[0,1]^{\exists_2}$.

$$\text{So} \quad \overline{\theta(e(X))} \subset e'(Y).$$

But since θ is continuous,

$$\theta(B(X)) = \overline{\theta(e(X))} \subset \overline{\theta(e(X))}$$

Thus we see that θ maps β(X) into e′(Y). Since e′ is an embedding, there exists a map $e_1 : e'(Y) \to Y$ which is an inverse to e′ regarded as a map from Y onto e′(Y). Then e_1 o e′ o f = f.

Uniqueness of the extension is immediate in view of the fact that Y is a Hausdorff space and e(X) is dense in β(X).

5.2 Summary

- A compactification of a space X is a compact Hausdorff space Y containing X as a subspace such that $\bar{X} = Y$.
- Two compactifications Y_1 and Y_2 of X are said to be equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that h(x) = x for every x ∈ X.
- If X has a compactification Y, then X must be completely regular, being a subspace of completely regular space Y.
- If X is completely regular, then X has a compactification.
- The pair (e, β(X)), where X is a Tychonoff space and β(X) $(= \overline{e(x)})$ is called Stone-Cech compactification of X, e is a map from X into β(x).
- The Stone-Cech compactification is defined for all Tychonoff spaces.

5.3 Keywords

Connected Spaces: A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Hausdorff Space: It is a topological space in which each pair of distinct points can be separated by disjoint neighbourhoods.

Homeomorphism: A map f : (X, T) $\rightarrow$ (Y, $\cup$) is said to be homeomorphism if:

(i) f is one-one onto.

(ii) f and f^{-1} are continuous.

5.4 Review Questions

1. Let (X, T) be a Tychonoff space and (βX, T′) its stone-cech compactification. Prove that (X, T) is connected if and only if (βX, T′) is connected.

   [***Hint:*** Firstly verify that providing (X, T) has at least 2 points it is connected if and only if there does not exist a continuous map of (X, T) onto the discrete space {0, 1}.]

2. Let (X, T) be a Tychonoff space and (βX, T′) its stone-cech compactification. If (A, T_1) is a subspace of (βX, T′) and A $\supseteq$ X, prove that (βX, T′) is also the stone-cech compactification of (A, T_1).

3. Let (X, T) be a dense subspace of a compact Hausdorff space (Z, T_1). If every continuous mapping of (X, T) into [0, 1] can be extended to a continuous mapping of (Z, T_1) into [0, 1], prove that (Z, T_1) is the Stone-Cech compactification of (X, T).

4. Let Y be an arbitrary compactification of X; let β(X) be the Stone-Cech compactification. Show that there is a continuous surjective closed map g : β(X) $\rightarrow$ Y that equals the identity on X.

5. Under what conditions does a metrizable space have a metrizable compactification?

5.5 Further Readings

Books

S. Lang, *Algebra* (Second Edition), Addison-Wesley, Menlo Park, California 1984.

S. Willard, *General Topology*, MA : Addison-Wesley.

Online links

www.planetmath.org

www.jstor.org

Unit 6: Local Finiteness and Paracompactness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define local finiteness and solve problems on it;
- Define countably locally finite, open refinement and closed refinement;
- Understand the paracompactness and theorems on it.

Introduction

In this unit we prove some elementary properties of locally finite collections and a crucial lemma about metrizable spaces.

The concept of paracompactness is one of the most useful generalization of compactness that has been discovered in recent years. It is particularly useful for applications in topology and differential geometry. Many of the spaces that are familiar to us already are paracompact. For instance, every compact space is paracompact; this will be an immediate consequence of the definition. It is also true that every metrizable space is paracompact; this is a theorem due to A.H. Stone, which we shall prove. Thus the class of paracompact space includes the two most important classes of spaces we have studied. It includes many other spaces as well.

6.1 Local Finiteness

Definition: Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be a locally finite in X if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

Example 1: The collection of intervals

$$\mathcal{A} = \{(n, n + 2) \mid n \in \mathbb{Z})\}$$

is locally finite in the topological space $\mathbb{R}$, on the other hand, the collection

$$\mathcal{B} = \{0, 1/n\} \mid n \in \mathbb{Z}_+\}$$

is locally finite in (0, 1) but not in $\mathbb{R}$, as in the collection

$$\mathcal{C} = \{(1/(n+1), 1/n) \mid n \in \mathbb{Z}_+\}.$$

Lemma 1: Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any sub collection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\bar{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\cup_{A \in \mathcal{A}} A = \cup_{A \in \mathcal{A}} \bar{A}$.

Proof: Statement (a) is trivial. To prove (b), note that any open set U that intersects the set $\bar{A}$ necessarily intersects A. Therefore, if U is a neighbourhood of x that intersects only finitely many elements A of $\mathcal{A}$, then U can intersect at most the same number of sets of the collection $\mathcal{B}$. (It might intersect fewer sets of $\mathcal{B}$, $\bar{A}_1$ and $\bar{A}_2$ can be equal even though A_1 and A_2 are not).

To prove (c), let Y denote the union of the elements of $\mathcal{A}$:

$$\bigcup_{A \in \mathcal{A}} A = Y.$$

In general, $\cup \bar{A} \subset \bar{Y}$; we prove the reverse inclusion, under the assumption of local finiteness. Let $x \in \bar{Y}$; let U be a neighbourhood of x that intersects only finitely many elements of $\mathcal{A}$, say $A_1, \ldots, A_k$. We assert that x belongs to one of the sets $\bar{A}_1, \ldots, \bar{A}_k$ and hence belongs to $\cup \bar{A}$. For otherwise, the set $U - \bar{A}_1 - \ldots - \bar{A}_k$ would be a neighbourhood of x that intersect no element of $\mathcal{A}$ and hence does not intersect Y, contrary to the assumption that $x \in \bar{Y}$.

6.1.1 Countably Locally Finite

Definition: A collection $\mathcal{B}$ of subsets of X is said to be countably locally finite of $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

6.1.2 Open Refinement and Closed Refinement

Definition: Let $\mathcal{A}$ be a collection of subsets of the space X. A collection $\mathcal{B}$ of subsets of X is said to be a refinemet of $\mathcal{A}$ (or is said to refine $\mathcal{A}$) if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an open refinement of $\mathcal{A}$; if they are closed sets, we call $\mathcal{B}$ a closed refinement.

Lemma 2: Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering E of X refining $\mathcal{A}$ that is countably locally finite.

Proof: We shall use the well-ordering theorem in proving this theorem. Choose a well-ordering, < for collection $\mathcal{A}$. Let us denote the elements of $\mathcal{A}$ generically by the letters U, V, W,.... .

Choose a metric for X. Let n be a positive integer, fixed for the moment. Given an element U of $\mathcal{A}$, let us define S_n (U) to be the subset of U obtained by "shrinking" U a distance of 1/n. More precisely, let

$$S_n(U) = \{x \mid B(x, 1/n) \subset U\}.$$

(It happens that $S_n(U)$ is a closed set, but that is not important for our purposes.) Now we use the well-ordering < of $\mathcal{A}$ to pass to a still smaller set. For each U in $\mathcal{A}$, define

$$T_n(U) = S_n(U) - \bigcup_{V<U} V.$$

Notes

The situation where $\mathcal{A}$ consists of the three sets $U < V < W$. The sets we have formed are disjoint. In fact, they are separated by a distance of at least $1/n$. This means that if V and W are distinct elements of $\mathcal{A}$, then $d(x, y) \geq 1/n$ whenever $x \in T_n(V)$ and $y \in T_n(W)$.

To prove this fact, assume the notation has been so chosen that $V < W$. Since x is in $T_n(V)$, then x is in $S_n(V)$, so the $1/n$-neighbourhood of x lies in V. On the other hand since $V < W$ and y is in $T_n(W)$, the definition of the latter set tells us that y is not in V. If follows that y is not in the $1/n$-neighbourhood of x.

The sets $T_n(U)$ are not yet the ones we want, for we do not know that they are open sets. (In fact, they are closed.) So let us expand each of them slightly to obtain an open set $E_n(\cup)$. Specifically, let $E_n(U)$ be the $1/3$ n-neighbourhood of $T_n(U)$; that is, let $E_n(U)$ be the union of the open balls $B(x, 1/3n)$, for $x \in T_n(U)$.

In case $U < V < W$, we have the situation. The sets we have formed are disjoint. Indeed, if V and W are distinct elements of $\mathcal{A}$, we assert that $d(x, y) \geq 1/3n$ whenever $x \in E_n(V)$ and $y \in E_n(W)$; this fact follows at once from the triangle inequality. Note that for each $V \in \mathcal{A}$, the set $E_n(V)$ is contained in V.

Now let us define

$$\varepsilon_n = \{E_n(\cup) \mid \cup \in \mathcal{A}\}.$$

We claim that E_n is a locally finite collection of open sets that refines $\mathcal{A}$. The fact that E_n refines $\mathcal{A}$ comes from the fact that $E_n(V) \subset V$ for each $V \in \mathcal{A}$. The fact E_n is locally finite comes from the fact that for any x in X, the $1/6n$ - neighbourhood of x can intersect at most one element of E_n.

Of course, the collection ε_n, will not cover X. But we assert that the collection

$$E = \bigcup_{n \in \mathbb{Z}_+} \varepsilon_n$$

does cover X.

Let x be a point of X. The collection $\mathcal{A}$ with which we began covers X; let us choose U to be the first element of $\mathcal{A}$ (in the well-ordering <) that contains x. Since U is open, we can choose n so that $B(x, 1/n) \subset U$. The, by definition, $x \in S_n(U)$. Now because U is the first element of $\mathcal{A}$ that contains x, the point x belongs to $T_n(U)$. Then x also belongs to the element $E_n(U)$ of E_n, as desired.

Self Assessment

1. Many spaces have countable bases; but no T_1 space has a locally finite basis unless it is discrete. Prove this fact.

2. Find a non-discrete space that has a countably locally finite basis but does not have a countable basis.

6.2 Paracompactness

Definition: A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement B that covers X.

Example 2: The Space $\mathbb{R}^n$ is paracompact. Let $X = \mathbb{R}^n$. Let $\mathcal{A}$ be an open covering of X. Let $B_0 = \phi$, and for each positive integer m, let B_m denote the open ball of radius m centered at the origin. Given m, choose finitely many elements of $\mathcal{A}$ that cover $\overline{B}_m$ and intersect each one with the open set $X - \overline{B}_{m-1}$; let this finite collection of open sets be denoted $\mathcal{C}_m$. Then the collection $\mathcal{C} = \cup\mathcal{C}_m$ is a refinement of $\mathcal{A}$. It is clearly locally finite, for the open set B_m intersects only finitely many elements of $\mathcal{C}$, namely those elements belonging to the collection $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$. Finally,

$\mathcal{C}$ covers X. For, given x let m be the smallest integer such that $x \in \overline{B}_m$. Then x belongs to a element of $\mathcal{C}_m$, by definition.

Note Some of the properties of a paracompact space are similar to those of a compact space. For instance, a subspace of a paracompact space is not necessarily paracompact; but a closed subspace is paracompact. Also, a paracompact Hausdorff space is normal. In other ways, a paracompact space is not similar to a compact space; in particular, the product of two paracompact spaces need not be paracompact.

Theorem 1: Every paracompact Hausdorff space X is normal.

Proof: The proof is somewhat similar to the proof that a compact Hausdorff space is normal. First one proves regularity. Let a be a point of X and let B be a closed set of X disjoint from a. The Hausdorff condition enables is to choose for each b in B, an open set $\cup_b$ about b whose closure is disjoint from a. Cover X by the open sets $\cup_b$, along with the open set X – B; take a locally finite open refinement $\mathcal{C}$ that covers X. Form the subcollection D of $\mathcal{C}$ consisting of every element of $\mathcal{C}$ that intersects B. The $\mathcal{D}$ covers B. Furthermore, if $D \in \mathcal{D}$, then $\overline{D}$ is disjoint from a. For D intersect B, so it lies in some set $\cup_b$, whose closure is disjoint from a. Let

$$V = \bigcup_{D \in \mathcal{D}} D;$$

then V is an open set in X containing B. Because $\mathcal{D}$ is locally finite,

$$\overline{V} = \bigcup_{D \in \mathcal{D}} \overline{D},$$

so that $\overline{V}$ is disjoint from a. Thus regularity is proved.

To prove normality, one merely repeats the same argument, replacing a by the closed set A throughout and replacing the Hausdorff condition by regularity.

Theorem 2: Every closed subspace of a paracompact space is paracompact.

Proof: Let Y be a closed subspace of the paracompact space X; let $\mathcal{A}$ be a covering of Y by sets open in Y.

For each $A \in \mathcal{A}$, choose an open set A′ of X such that $A' \cap Y = A$. Cover X by the open sets A′, along with the open set X – Y.

Let $\mathcal{B}$ be a locally finite open refinement of this covering that covers X.

The collection $\mathcal{C} = \{B \cap Y : B \in \mathcal{B}\}$

is the required locally finite open refinement of $\mathcal{A}$.

Example 3: A paracompact subspace of a Hausdorff space X need not be closed in X.

Solution: Indeed, the open interval (0, 1) is pracompact, being homeomorphic to $\mathbb{R}$, but it is not closed in $\mathbb{R}$.

Lemma 3: Let X be regular. Then the following conditions on X are equivalent:

Every open covering of X has a refinement that is:

1. An open covering of X and countably locally finite.
2. A covering of X and locally finite.

Notes

3. A closed covering of X and locally finite.

4. An open covering of X and locally finite.

Proof: It is trivial that (4) ⇒ (1).

What we need to prove our theorem is the converse. In order to prove the converse, we must go through the steps (1) ⇒ (2) ⇒ (3) ⇒ (4)

anyway, so we have for convenience listed there conditions in the statement of the lemma.

(1) ⇒ (2).

Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B}$ be an open refinement of $\mathcal{A}$ that covers X and is countably locally finite; let

$$\mathcal{B} = \cup \mathcal{B}_n$$

where each $\mathcal{B}_n$ is locally finite.

Now we apply essentially the same sort of shrinking trick, we have used before to make sets from different $\mathcal{B}_n'$ disjoint. Given i, let

$$V_i = \bigcup_{U \in \mathcal{B}_i} U$$

Then for each $n \in Z_+$ and each element U of $\mathcal{B}_n$, define

$$S_n(U) = U - \bigcup_{i<n} V_i$$

[Note that $S_n(U)$ is not necessarily open, nor closed.]

Let $\mathcal{C}_n = \{S_n(U) : U \in \mathcal{B}_n\}$

Then $\mathcal{C}_n = \cup \mathcal{C}_n$. We assert that $\mathcal{C}$ is the required locally finite refinement of $\mathcal{A}$, covering X.

Let x be a point of X. We wish to prove that x lies in an element of $\mathcal{C}$, and that x has a neighbourhood intersecting only finitely many elements of $\mathcal{C}$. Consider the covering $\mathcal{B} = \cup \mathcal{B}_n$; let N be the smallest integer such that x lies in an element of $\mathcal{B}_N$. Let $\cup$ be an element of $\mathcal{B}_N$ containing x. First, note that since x lies in no element of $\mathcal{B}_i$ for i < N, the point x lies in the element $S_N(\cup)$ of $\mathcal{C}$. Second, note that since each collection $\mathcal{B}_n$ is locally finite, we can choose for each n = 1, ..., N a neighbourhood W_n of x that intersects only finitely many elements of $\mathcal{B}_n$. Now if W_n intersects the element $S_n(V)$ of $\mathcal{C}_n$, it must intersect the element V of $\mathcal{B}_n$, since $S_n(V) \subset V$. Therefore, W_n intersects only finitely many elements of $\mathcal{C}_n$. Furthermore, because U is in $\mathcal{B}_N$, U intersects no element of $\mathcal{C}_n$ for n > N. As a result, the neighbourhood

$$W_1 \cap W_2 \cap ... \cap W_n \cap \cup$$

of x intersects only finitely many elements of $\mathcal{C}$.

(2) ⇒ (3). Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B}$ be the collection of all open sets U of X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. By regularity, $\mathcal{B}$ covers X. Using (2), we can find a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X and is locally finite. Let

$$\mathcal{D} = \{\overline{C} : C \in C\}$$

Then $\mathcal{D}$ also covers X; it is locally finite by lemma (1) and it refines $\mathcal{A}$.

(3) ⇒ (4): Let $\mathcal{A}$ be an open covering of X. Using (3), choose $\mathcal{B}$ to be a refinement of $\mathcal{A}$ that covers X and is locally finite. (We can take $\mathcal{B}$ to be closed refinement if we like, but that is irrelevant.) We seek to expand each element B of $\mathcal{B}$ slightly to an open set, making the expansion slight enough that the resulting collection of open sets will still be locally finite and will still refine $\mathcal{A}$.

This step involve a new trick. The previous trick, used several times, consisted of ordering the sets in some way and forming a new set by subtracting off all the previous ones. That trick shrinks the sets; to expand them we need something different. We shall introduce an auxiliary locally finite closed covering $\mathcal{C}$ of X and use it to expand the element of $\mathcal{B}$.

For each point x of X, there is a neighbourhood of x that intersects only finitely many elements of $\mathcal{B}$. The collection of all open sets that intersect only finitely many element of $\mathcal{B}$ is thus an open covering of X. Using (3) again, let $\mathcal{C}$ be a closed refinement of this covering that covers X and is locally finite. Each element of $\mathcal{C}$ intersect only finitely many elements of $\mathcal{B}$.

For each element B of $\mathcal{B}$, let

$$\mathcal{C}(B) = \{C : C \in \mathcal{C} \text{ ad } C \subset X - B\}$$

Then define

$$E(B)X = X - \bigcup_{C \in \mathcal{C}(B)} C$$

Because $\mathcal{C}$ is locally finite collection of closed sets, the union of the elements of any subcollection of $\mathcal{C}$ is closed by lemma, therefore the set E(B) is an open set. Furthermore, E(B) $\supset$ B by definition.

Now we may have expanded each B too much; the collection {E(B)} may not be a refinemet of $\mathcal{A}$. This is easily remedied. For each B $\in$ $\mathcal{B}$, choose an element F(B) of $\mathcal{A}$ containing B. Then define

$$\mathcal{D} = \{E(B) \cap F(B) \mid B \in \mathcal{B}\}.$$

The collection $\mathcal{D}$ is a refinement of A. Because B $\subset$ (E(B) $\cap$ F(B)) and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ also covers X.

We have finally to prove that $\mathcal{D}$ is locally finite. Given a point x of X, choose a neighbourhood W of x that intersects only finitely may elements of $\mathcal{C}$, say C_1, ..., C_k. We show that W intersects only finitely many elements of $\mathcal{D}$. Because $\mathcal{C}$ covers X, the set W is covered by C_1,...C_K. thus, it suffices to show that each element C of $\mathcal{C}$. Now if C intersects the set E (B) $\cap$ F(B), then it intersects E(B), so by definition of E(B) it is not contained in X–B; hence C must intersect B. Since C intersects, only finitely many elements of $\mathcal{B}$, it can intersect at most the same number of elements of the collection $\mathcal{D}$.

Theorem 3: Every metrizable space is paracompact.

Proof: Let X be a metrizable space. We already know from Lemma 2 that, given an open covering $\mathcal{A}$ of X, it has an open refinement that covers X and is countably locally finite. The preceding lemma then implies that $\mathcal{A}$ has an open refinement that covers X and is locally finite.

Example 4: The product of two paracompact spaces need not be paracompact. The space $\mathbb{R}_\ell$ is paracompact, for it is regular and Lindelöf. However, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not paracompact, for it is Hausdorff but not normal.

Self Assessment

3. Show that Paracompactness is a topological property.
4. If every open subset of a paracompact space is paracompact, then every subset is paracompact. Prove it.

6.3 Summary

- Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be locally finite in X if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

- A collection $\mathcal{B}$ of subsets of X is said to be countably locally finite is $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.
- Let $\mathcal{A}$ be a collection of subsets of space X. A collection $\mathcal{B}$ of subsets of X is said to be a refinement of $\mathcal{A}$ if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an open refinement of $\mathcal{A}$; if they are closed sets, we call $\mathcal{B}$ a closed refinement.
- A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

6.4 Keywords

Metrizable: Any topological space (X, T) if it a possible to find a metric ρ on X which induces the topology T i.e. the open sets determined by the metric ρ are precisely the members of T, then X is said to the metrizable.

Open Cover: Let (X, T) be a topological space and $A \subset X$ let G denote a family of subsets of X. G is called a cover of A if $A \subset \bigcup \{G : G \in G\}$.

6.5 Review Questions

1. Give an example to show that if X is paracompact, it does not follow that for every open covering $\mathcal{A}$ of X, there is a locally finite subcollection of $\mathcal{A}$ that covers X.
2. (a) Show that the product of a paracompact space and a compact space is paracompact. [***Hint:*** Use the tube lemma.]

   (b) Conclude that S_Ω is not paracompact.
3. Is every locally compact Hausdorff space paracompact?
4. (a) Show that if X has the discrete topology, then X is paracompact.

   (b) Show that if $f : X \to Y$ is continuous and X is paracompact, the subspace f(X) of Y need not be paracompact.
5. (a) Let X be a regular space. If X is a countable union of compact subspace of X, then X is paracompact.

   (b) Show $\mathbb{R}^\infty$ is paracompact as a subspace of $\mathbb{R}^w$ in the box topology.
6. Let X be a regular space.

   (a) Show that if X is a finite union of closed paracompact subspaces of X, then X is paracompact.

   (b) If X is a countable union of closed paracompact subspaces of X whose interiors cover X, show X is paracompact.
7. Find a point-finite open covering $\mathcal{A}$ of $\mathbb{R}$ that is not locally finite (The collection $\mathcal{A}$ is point finite if each point of $\mathbb{R}$ lies in only finitely many elements of $\mathcal{A}$).
8. Give an example of a collection of sets $\mathcal{A}$ that is not locally finite, such that the collection $\mathcal{B} = \{\bar{A} / A \in \mathcal{A}\}$ is locally finite.
9. Show that if X has a countable basis, a collection $\mathcal{A}$ of subsets of X is countably locally finite if and only if it is countable.
10. Consider $\mathbb{R}^w$ in the uniform topology. Given n, let $\mathcal{B}_n$ be the collection of all subsets of $\mathbb{R}^w$ of the form ΠA_i; where $A_i = \mathbb{R}$ for $i \leq n$ and A_i equals either {0} or {1} otherwise. Show that collection $\mathcal{B} = \bigcup \mathcal{B}_n$ is countably locally finite, but neither countable nor locally finite.

6.6 Further Readings

Books

J.L.. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Unit 7: The Nagata-Smirnov Metrization Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define G_δ set;
- State "The Nagata-Smirnov Metrization Theorem";
- Understand the proof of "The Nagata Smirnov Metrization Theorem".

Introduction

Although Urysohn solved the metrization problem for separable metric spaces in 1924, the general metrization problem was not solved until 1950. Three mathematicians, J. Nagata, Yu. M. Smirnov, and R.H. Bing, gave independent solutions to this problem. The characterizations of Nagata and Smirnov are based on the existence of locally finite base, while that of Bing requires a discrete base for the topology.

We will prove the regularity of X and the existence of a countably locally finite basis for X are equivalent to metrizability of X.

7.1 The Nagata Smirnov Metrization Theorem

7.1.1 G_δ Set

A subset A of a space X is called a G_δ set in X if it equals the intersection of a countable collection of open subsets of X.

Example 1: In a metric space X, each closed set is a G_δ set- Given $A \subset X$, let $U(A, \in)$ denote the $\in$ – neighbourhood of A. If A is closed, you can check that

$$A = \bigcap_{n \in Z_+} U(A, 1/n)$$

Notes

Lemma 1: Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.

Proof: *Step I:* Let W be open in X. We show there is a countable collection $\{U_n\}$ of open sets of X such that

$$W = \bigcup U_n = \bigcup \overline{U}_n$$

since the basis $\mathcal{B}$ for X is countable locally finite, we can write $\mathcal{B} = \bigcup \mathcal{B}_n$, where each collection $\mathcal{B}_n$ is locally finite. Let $\mathcal{C}_n$ be the collection of those basis elements $\mathcal{B}$ such that $\mathcal{B} \in \mathcal{B}_n$ and $\overline{B} \subset W$. Then $\mathcal{C}_n$ is locally finite, being a subcollection of $\mathcal{B}_n$.

Define $U_n = \bigcup\limits_{B \in \mathcal{C}_n} B$

Then U_n is an open set, and by Lemma "Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\overline{U_{A \in \mathcal{A}} A} = U_{A \in \mathcal{A}} \overline{A}$.

$$\overline{U}_n = \bigcup_{B \in C_n} \overline{B}$$

Therefore, $\overline{U}_n \subset W$, so that

$$\bigcup U_n \subset \bigcup \overline{U}_n \subset W.$$

We assert that equality holds. Given $x \in W$, there is by regularity a basis element $B \in \mathcal{B}$ such that $x \in B$ and $\overline{B} \subset W$. Now $B \in \mathcal{B}_n$ for some n. Then $B \in \mathcal{C}_n$ by definition, so that $x \in U_n$. Thus $W \subset \bigcup U_n$, as desired.

Step II: We show that every closed set C in X is a G_δ set in X. Given C, let W = X – C, by Step I, there are sets U_n in X such that $W = \bigcup \overline{U}_n$. Then

$$C = \bigcap (X - \overline{U}_n),$$

so that C equals a countable intersection of open sets of X.

Step III: We show X is normal. Let C and D be disjoint closed sets in X. Applying step I to the open set X – D, we construct a countable collection $\{U_n\}$ of open sets such that $\bigcup U_n = \bigcup \overline{U}_n = X - D$.

Then $\{U_n\}$ covers C and each set $\overline{U}_n$ is disjoint from D. Similarly there is a countable covering $\{V_n\}$ of D by open sets whose closures are disjoint from C.

Now we are back in the situation that arose in the proof that a regular space with a countable basis is normal. We can repeat that proof. Define

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V'_n = V_n - \bigcup_{i=1}^{n} \overline{U}_i$$

Then the sets

$$U' = \bigcup_{n \in Z_+} U'_n \text{ and } V' = \bigcup_{n \in Z_+} V'_n$$

are disjoint open sets about C and D, respectively.

Lemma 2: Let X be normal, let A be a closed G_δ set in X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.

Proof: Write A as the intersection of the open sets U_n, for $n \in Z_+$. For each n, choose a continuous function $f_n : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in X - U_n$. Define $f(x) = \Sigma f_n(x)/2^n$. The series converges uniformly, by comparison with $\Sigma 1/2^n$, so that f is continuous. Also, f vanishes on A and is positive on X – A.

7.1.2 Nagata–Smirnov Metrization Theorem

Statement: A space X is metrizable if and only if X is regular and has a basis that is countably locally finite.

Proof: *Step I:* Assume X is regular with a countably locally finite basis $\mathcal{B}$. Then X is normal, and every closed set in X is a G_δ set in X. We shall show that X is metrizable by imbedding X in the metric space $(\mathcal{R}^J, \bar{\rho})$ for some J.

Let $\mathcal{B} = \cup\mathcal{B}_n$, where each collection $\mathcal{B}_n$ is locally finite. For each positive integer n, and each basis element $\mathcal{B} \in \mathcal{B}_n$, choose a continuous function

$$f_{n,B} : X \to \left[0, \frac{1}{n}\right]$$

such that $f_{n,B}(x) > 0$ for $x \in B$ and $f_{n,B}(x) = 0$ for $x \notin B$. The collection $[f_{n,B}]$ separates points from closed sets in X: Given a point x_0 and a neighbourhood U of x_0, there is basis element B such that $x_0 \in B \subset U$. Then $B \in \mathcal{B}_n$ for some n, so that $f_{n,B}(x_0) > 0$ and $f_{n,B}$ vanishes outside U.

Let J be the subset of $Z_+ \times \mathcal{B}$ consisting of all pairs (n, B) such that B is an element of $\mathcal{B}_n$.

Define $F : X \to [0, 1]^J$

by the equation $F(x) = (f_{n,B}(x))_{(n,b) \in J}$.

Relative to the product topology on $[0, 1]^J$, the map F is an imbedding.

Now we give $[0, 1]^J$ the topology induced by the uniform metric and show that F is an imbedding relative to this topology as well. Here is where the condition $f_{n,B(x)} < \frac{1}{n}$ comes in. The uniform topology is finer (larger) than the product topology. Therefore, relative to the uniform metric, the map $\mathcal{F}$ is injective and carries open sets of X onto open sets of the image space $\mathcal{Z} = F(x)$. We must give a separate proof that F is continuous.

Note that on the subspace $[0, 1]^J$ of $\mathcal{R}^J$, the uniform metric equals the metric

$$\rho((x_\alpha), (y_\alpha)) = \sup\{|x_\alpha - y_\alpha|\}$$

To prove continuity, we take a point x_0 of X and a number $\in > 0$, and find a neighbourhood W of x_0 such that

$$x \in W \Rightarrow \rho(F(x), F(x_0)) < \in$$

Let n be fixed for the moment. Choose a neighbourhood U_n of x_0 that intersects only finitely many elements of the collection $\mathcal{B}_n$. This means that as B ranges over $\mathcal{B}_n$, all but finitely many of the functions $f_{n,B}$ are identically equal to zero on U_n. Because each function $f_{n,B}$ is continuous, we can now choose a neighbourhood V_n of x_0 contained in U_n on which each of the remaining functions $f_{n,B}$ for $B \in \mathcal{B}_n$, varies by at most $\in/2$.

Choose such a neighbourhood V_n of x_0 for each $n \in Z_+$. Then choose N so that $\frac{1}{N} \leq \frac{\in}{2}$, and define $W = V_1 \cap ... \cap V_n$. We assert that W is the desired neighbourhood of x_0. Let $x \in W$. If $n \leq \mathcal{N}$, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \in/2$$

because the function $f_{n,B}$ either vanishes identically or varies by at most $\in/2$ on W. If n > N, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq Y_n < \in/2$$

because $f_{n,B}$ maps X into $\left[0, \frac{1}{n}\right]$. Therefore,

$$\rho(F(x), F(x_0)) \leq \in/2 < \in,$$

as desired.

Step II: Now we prove the converse.

Assume X is metrizable. We know X is regular; let us show that X has a basis that is countably locally finite.

Choose a metric for X. Given m, let $\mathcal{A}_m$ be the covering of X by all open balls of radius $\frac{1}{m}$. There is an open covering $\mathcal{B}_m$ of X refining $\mathcal{A}_m$ such that $\mathcal{B}_m$ is countably locally finite. Note that each element of $\mathcal{B}_m$ has diameter at most $\frac{2}{m}$. Let $\mathcal{B}$ be the union of the collections $\mathcal{B}_m$, for $m \in \mathcal{Z}_+$. Because each collection $\mathcal{B}_m$ is countably locally finite, so is $\mathcal{B}$. We show that $\mathcal{B}$ is a basis for X.

Given $x \in X$ and given $\in > 0$, we show that there is an element B of $\mathcal{B}$ containing x that is contained in $B(x, \in)$. First choose m so that $\frac{1}{m} < \frac{\in}{2}$. Then, because $\mathcal{B}_m$ covers X, we can choose an element B of $\mathcal{B}_m$ that contains x. Since B contains x and has diameter at most $\frac{2}{m} < \in$, it is contained in $B(x, \in)$, as desired.

7.2 Summary

- A subset A of a space X is called a G_δ set in X if it equals the intersection of a countable collection of open subsets of X.
- Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.
- A space X is metrizable iff X is regular and has a basis that is countably locally finite.

7.3 Keywords

Basis: A collection of subsets B of X is called a basis for a topology if:

(1) The union of the elements of B is X.

(2) If $x \in B_1 \cap B_2$, $B_1, B_2 \in B$, then there exists a B_3 of B such that $x \in B_3 \subset B_1 \cap B_2$.

Notes

Metrizable: A topological X is metrizable if there exists a metric d on set X that induces the topology of X.

Neighbourhood: An open set containing x is called a neighbourhood of x.

Product topology: Let X, Y be sets with topologies T_x and T_y. We define a topology $T_{X\times Y}$ on X × Y called the product topology by taking as basis all sets of the form U × W where $U \in T_X$ and $W \in T_Y$.

7.4 Review Questions

1. Many spaces have countable bases; but no T_1 space has a locally finite basis unless it is discrete. Prove this fact.
2. Find a non-discrete space that has a countably locally finite basis does not have a countable basis.
3. A collection $\mathcal{A}$ of subsets of X is said to be locally discrete if each point of X has a neighbourhood that intersects at most one elements of $\mathcal{A}$. A collection $\mathcal{B}$ is countably locally discrete if it equals a countable union of locally discrete collections. Prove the following:

   Theorem (Being Metrization Theorem):

   A space X is metrizable if and only if it is regular and has a basis that is countably locally discrete.
4. A topological space is called locally metrizable iff every point is contained in an open set which is metrizable. Prove that if a normal space has a locally finite covering by metrizable subsets, then the entire space is metrizable.

7.5 Further Readings

Books

Lawson, Terry, *Topology: A Geometric Approach*, New York, NY: Oxford University Press, 2003.

Patty. C. Wayne (2009), *Foundations of Topology* (2nd Edition) Jones and Barlett.

Robert Canover, *A First Course in Topology*, The Willams and Wilkins Company 1975.

Unit 8: The Smirnov Metrization Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the locally metrizable space;
- Explain the Smirnov Metrization theorem.

Introduction

The Nagata-Smirnov metrization theorem gives one set of necessary and sufficient conditions for metrizability of a space. In this, unit we prove a theorem that gives another such set of conditions. It is a corollary of the Nagata-Smirnov theorem and was first proved by Smirnov. This unit starts with the definitions of paracompact and locally metrizable space. After explaining these terms, proof of "The Smirnov Metrization Theorem" is given.

8.1 Locally Metrizable Space

A space X is locally metrizable if every point x of X has a neighborhood U that is metrizable in the subspace topology.

The Smirnov Metrization Theorem

Statement: A space X is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

Proof: Suppose that X is metrizable.

Then X is locally metrizable; it is also paracompact. [Every metrizable space is paracompact].

Conversely, suppose that X is a paracompact Hausdorff space that is locally metrizable.

We shall show that X has a basis that is countably locally finite. Since X is regular, it will then follow from the Nagata – Smirnov theorem that X is metrizable.

Cover X by open sets that are metrizable; then choose a locally finite open refinement $\mathcal{C}$ of this covering that covers X. Each element C of $\mathcal{C}$ is metrizable, let the function $d_C : C \times C \to R$ be a metric that gives the topology of C. Given $x \in C$, let $B_C(x, \in)$ denote the set of all points y of C such that $d_C(x, y) < \in$. Being open in C, the set $B_C(x, \in)$ is also open in X.

Notes

Given $m \in Z_+$, let $\mathcal{A}_m$ be the covering of X by all these open balls of radius $\frac{1}{m}$; that is, let

$$\mathcal{A}_m = \left\{B_C\left(x, \frac{1}{m}\right) : x \in C \quad \text{and} \quad C \in \mathcal{C}\right\}$$

Let $\mathcal{D}_m$ be a locally finite open refinement of $\mathcal{A}_m$ that covers X. (Here we use paracompactness).

Let $\mathcal{D}$ be the union of the collections $\mathcal{D}_m$.

Then $\mathcal{D}$ is countably locally finite.

We assert that $\mathcal{D}$ is a basis for X; our theorem follows.

Let x be a point of X and let $\cup$ be a neighbourhood of x. We seek to find an element D of $\mathcal{D}$ such that $x \in D \subset \cup$.

Now x belongs to only finitely many elements of $\mathcal{C}$ say to $C_1, \ldots, C_K$. Then $\cup \cap C_i$ is a neighbourhood of x in the set C_i, so there is an $\in_i > 0$ such that

$$B_{C_i}(x, \in) \subset (\cup \cap C_i).$$

Choose m so that $\frac{2}{m} < \min.\{\in_1, \ldots, \in_K\}$.

Because the collection $\mathcal{D}_m$ covers X, there must be an element D of $\mathcal{D}_m$ containing x.

Because $\mathcal{D}_m$ refines $\mathcal{A}_m$, there must be an element $B_C\left(y, \frac{1}{m}\right)$ of $\mathcal{A}_m$, for some $C \in \mathcal{C}$ and some $y \in C$ that contains D. Because $x \in D \subset B_C\left(y, \frac{1}{m}\right)$, the point $x \in C$, so that C must be one of the sets $C_1, \ldots, C_K$. Say $C = C_i$. Since $B_C\left(y, \frac{1}{m}\right)$ has diameter at most $\frac{2}{m} < \in_i$, it follows that

$$x \in D \in B_{C_i}\left(y, \frac{1}{m}\right) \subset B_{C_i}(x, \in_i) \subset \cup, \text{ as desired.}$$

8.2 Summary

- A space X is locally metrizable if every point x of X has a neighbourhood $\cup$ that is metrizable in the subspace topology.
- A space X is metrizable iff it is a paracompact Hausdorff space that is locally metrizable.

8.3 Keywords

Hausdorff Space: A topological space X is a Hausdorff space if given any two points $x, y \in X$, $x \neq y$, there exists neighbourhoods $\cup_x$ of x, $\cup_y$ of y such that $\cup_x \cap \cup_y \neq \phi$.

Metrizable: A topological X is metrizable if there exists a metric d on set X that induces the topology of X.

Paracompact: A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

Regular: Let X be a topological space where one-point sets are closed. Then X is regular if a point and a disjoint closed set can be separated by open sets.

8.4 Review Questions

1. If a separable space is also metrizable, then prove that the space has a countable base.
2. Show that any finite subset of metrizable space is always discrete.
3. Show that a topological space X is metrizable ⇔ there exists a homeomorphism of X onto a subspace of some metric space Y.
4. A compact Hausdorff space is separable and metrizable if it is:

   (a) second countable (b) not second countable

   (c) first countable (d) none

8.5 Further Readings

Books

Lawson, Terry, *Topology : A Geometric Approach*. New York, NY: Oxford University Press, 2003.

Robert Canover, *A first course in topology*. The Williams and Wilkins Company, 1975.

Unit 9: Complete Metric Spaces

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Cauchy's sequence;
- Solve the problems on Cauchy's sequence;
- Define complete metric space;
- Solve the problems on complete metric spaces.

Introduction

The concept of completeness for a metric space is basic for all aspects of analysis. Although completeness is a metric property rather than a topological one, there are a number of theorems involving complete metric spaces that are topological in character. In this unit, we shall study the most important examples of complete metric spaces and shall prove some of these problems.

9.1 Cauchy's Sequence

A sequence $<x_n>$ in a metric space X is said to be a Cauchy sequence in X if given $\in > 0$ there exists a positive integer n_o such that

$$d(x_m, x_n) < \in \quad \text{where } m, n \geq n_o.$$

Alternative definition: A sequence $<x_n>$ is Cauchy if given $\in > 0$, there exists a positive integer n_o such that

$$d(x_{n+p}, x_n) < \in \quad \text{for all } n \geq n_o \text{ and for all } p \geq 1.$$

Theorem 1: Every convergent sequence in a metric space is a Cauchy sequence.

Proof: Let (X, d) be a metric space.

Let $<x_n>$ be a convergent sequence in X.

Suppose $\underset{n\to\infty}{\text{lt}} x_n = x$.

Notes

Then given $\in > 0$, there exist a positive integer n_o such that $m, n \geq n_o \Rightarrow d(x_m, x) < \frac{\in}{2}$ and $d(x_n, x) < \frac{\in}{2}$. Therefore, $m, n \geq n_o \Rightarrow d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\in}{2} + < \frac{\in}{2} = \in$.

Hence, $<x_n>$ is a Cauchy sequence.

Note The converse of this theorem is not true i.e., Cauchy sequence need not be convergent.

To prove this, consider the following example.

Let $$X = \mathcal{R} - \{0\}.$$

Let $$d(x, y) = |x - y|$$

Consider the sequence $x_n = \frac{1}{n}, n \in \mathcal{N}$

We shall show that

$<x_n>$ is a Cauchy sequence but it does not converge in X. Let $\in > 0$ be given and n_o be a positive integer such that $n_o > \frac{2}{\in}$.

Now
$$\begin{aligned} d(x_m, x_n) &= |x_m - x_n| \\ &= |x_m + (-x_n)| \\ &= |x_m| + |x_n| \\ &= \frac{1}{m} + \frac{1}{n} \end{aligned}$$

If $m \geq n_o \Rightarrow m > \frac{2}{\in}$ and so $\frac{1}{m} < \frac{\in}{2}$

Similarly, $\frac{1}{n} < \frac{\in}{2}$

$$\therefore \quad d(x_m, x_n) \leq \frac{1}{m} + \frac{1}{n} < \frac{\in}{2} + \frac{\in}{2} = \in$$

Thus $d(x_m, x_n) < \in$.

Hence $<x_n>$ is a Cauchy sequence.

Clearly, the limit of this sequence is 0 (zero) which does not belong to X.

Thus x_n does not converge in X.

Example 1: Let $<a_n>$ be a Cauchy sequence in a metric space (X, ρ) and let $<b_n>$ be any sequence in X s.t. $\rho(a_n, b_n) < \frac{1}{n} \; \forall \; n \in N$.

Show that

(i) $<b_n>$ is a Cauchy sequence.

(ii) $<a_n>$ converges to a point $p \in X$ iff $<b_n>$ converges in p.

Solution: Let $<a_n>$ be a Cauchy sequence in a metric space (X, ρ) so that

given $\in, K > 0 \quad \exists\, n_o \in N$ s.t.

$$n, m \geq n_o \Rightarrow \rho(a_n, a_m) < \in K \qquad \ldots(1)$$

Also let $<b_n>$ be a sequence in X s.t.

$$\rho(a_n, b_n) < \frac{1}{n} \;\; \forall\, n \in N \qquad \ldots(2)$$

Step (i): To prove that $<b_n>$ is a Cauchy sequence.

Let ε, K > 0 any given real numbers.

Then $\exists\, m_o \in N$ s.t. $\frac{1}{m_o} < \varepsilon k$. …(3)

Set $K_o = \max. (n_o, m_o)$.

Then $K_o \geq n_o, m_o$, so that

$$\frac{1}{K_o} \leq \frac{1}{n_o}, \frac{1}{m_o} \qquad \ldots(4)$$

$$\frac{1}{m_o} < \varepsilon K, \; \frac{1}{K_o} \leq \frac{1}{m_o} \Rightarrow \frac{1}{K_o} \leq \frac{1}{m_o} < \varepsilon K \Rightarrow \frac{1}{K_o} < \varepsilon K. \qquad \ldots(5)$$

If $n \geq K_o$, then $\rho(a_n, b_n) < \frac{1}{n} \leq \frac{1}{K_o} < \varepsilon K$,

i.e., $\rho(a_n, b_n) < \varepsilon K \;\forall\, n, m \geq K_o$ …(6)

For $n, m \geq K_o$, we have

$$\rho(b_n, b_m) \leq \rho(b_n, a_n) + p(a_n, a_m) + p(a_m, b_m)$$

$$< \varepsilon K + \varepsilon K + \varepsilon K = 3\, \varepsilon K.$$

Choosing initially $K = \frac{1}{3}$, we get

$$\rho(b_n, b_m) < \varepsilon \;\forall n \geq K_o.$$

This proves that $<b_n>$ is a Cauchy sequence.

Step (ii): Let $a_n \to p \in X$.

To prove that $b_n \to p$.

$$a_n \to p \Rightarrow \text{given } \varepsilon, K > 0, \exists\, m_o \in N \text{ s.t.}$$

$$n \geq m_o \Rightarrow \rho(a_n, p) < \varepsilon K.$$

We have seen that $<a_n>$ and $<b_n>$ are Cauchy Sequences and therefore given $\varepsilon, K > 0, \exists\, n_o \in N$ s.t.

$$\forall\, m, n \geq n_o \Rightarrow \rho(a_n, a_m) < \varepsilon K, \rho(b_n, b_m) < \varepsilon K.$$

Choose $K_o = \max.(n_o, m_o)$

$$\rho(b_n, p) \le \rho(b_n, b_m) + \rho(b_m, a_m) + \rho(a_m, p)$$

$$< \varepsilon K + \varepsilon K + \varepsilon K = 3\,\varepsilon K \quad \forall m, n \ge K_o.$$

Choosing initially $K = \frac{1}{3}$, we get

$$\rho(b_n, p) < \varepsilon \;\forall n \ge K_o$$

This $b_n \to p$.

Conversely if $b_n \to p$, then by making parallel arguments, we can show that $a_n \to p$. Hence the result.

Self Assessment

1. In any metric space, prove that every Cauchy sequence is totally bounded.
2. Let a subsequence of a sequence $\langle a_n \rangle$ converge to a point p. Prove that $\langle a_n \rangle$ also converges to p.

9.2 Complete Metric Space

A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

Example 2: The complex plane C is complete.

Solution: Let $\langle z_n \rangle$ be a Cauchy sequence of complex numbers, where $Z_n = x_n + i\,y_n$.

Here $\langle x_n \rangle$ and $\langle y_n \rangle$ are themselves Cauchy sequences of real numbers,

$$|x_m - x_n| \le |z_m - z_n|$$

and

$$|y_m - y_n| \le |z_m - z_n|$$

But the real line being a complete metric space, there exists real numbers x and y such that $x_n \to x$ and $y_n \to y$.

Thus, taking z = x + iy, we find $z_n \to z$ as

$$|z_n - z| = |(x_n + i\,y_n) - (x + i\,y)|$$

$$= |(x_n - x) + i(y_n - y)|$$

$$\le |x_n - x| + |y_n - y|$$

$$\to 0 \text{ as } n \to \infty$$

$\therefore$ $|z_n - z| = 0 \Rightarrow z_n \to z$.

Hence if the real line is a complete metric space, then the complex plane is also a complete metric space.

9.3 Theorems and Solved Examples

Notes

Theorem 2: Let X be a complete metric space and Y be a subspace of X. Show that Y is closed iff it is complete.

Proof: Let Y be closed.

Let $<x_n>$ be a Cauchy sequence in Y. This implies that it is a Cauchy sequence in X.

Since X is complete, $<x_n>$ converges to some point $x \in X$.

Let A be the range of $<x_n>$.

If A is finite, then x is that term of $<x_n>$ which is infinitely repeated and therefore $x \in X$. If A is infinite, then x, being limit of $<x_n>$, is a limit point of its range A. Since $A \subset Y$, so, x is a limit point of Y. But Y is closed, therefore, $x \in Y$.

This implies that $<x_n>$ is convergent in Y. Hence Y is complete.

Conversely, let Y be complete.

Here we are to prove that Y is closed.

Let x be a limit point of Y.

Then, for each positive integer n, $\exists$ an open sphere $S\left(x, \frac{1}{n}\right)$ containing at least one point x_n of Y, other than x.

Let $\in > 0$ be given.

$\exists$ a positive integer n_o such that $\frac{1}{n_o} < \in$. We have $\frac{1}{n} < \in$ for all $n \geq n_o$.

Since $x_n \in S\left(x, \frac{1}{n}\right)$,

$$d(x_n, x) < \frac{1}{n}.$$

Therefore $d(x_n, x) < \in \ \forall\ n \geq n_o$.

This implies that $<x_n>$ converges to x in X. Therefore $<x_n>$ is a Cauchy sequence in X, So it is a Cauchy sequence in Y.

But Y is complete.

Therefore $<x_n>$ is convergent in Y.

This implies that $x \in Y$, because limit of convergent sequence is unique. Hence, Y is closed.

Theorem 3: Cantor's Intersection Theorem.

Let X be a complete metric space. Let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof: Let $F = \bigcap_{n=1}^{\infty} F_n$.

For $n \in N$, let $x_n \in F_n$, we prove that $<x_n>$ is a Cauchy sequence.

Let $\in > 0$ be given.

$d(F_n) \to 0$, therefore there exists a positive integer n_o of such that $d(F_{n_o}) < \in$.

Since $<F_n>$ is a decreasing sequence,

$\therefore \quad m, n \geq n_o \Rightarrow F_m, F_n \subseteq F_{n_o}$

$\Rightarrow x_m, x_n \in F_{n_o}$

$\Rightarrow d(x_m, x_n) < d(F_{n_o})$

$\Rightarrow d(x_m, x_n) < \in \qquad [\because \; d(F_{n_o}) < \in]$

$\Rightarrow <x_n>$ is a Cauchy sequence.

Since the space X is complete, $<x_n>$ must converge to some point, say x in X i.e. $x_n \to x \in X$.

We shall prove that

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

If possible, let $x \notin \bigcap_{n=1}^{\infty} F_n$.

$\Rightarrow \quad x \notin F_k$ for some $k \in N$.

Since each F_n is a closed set, F_k is also a closed set, therefore x cannot be a cluster point of F_k, and so $d(x, F_k) \neq 0$.

Let $\quad d(x, F_k) = r > o$ so that

$$d(x, y) \geq r \; \forall \; y \in F_k.$$

This shows that $\quad F_k \cap S\left(x, \frac{1}{2}r\right) = \phi$.

Now, $\quad n > k \Rightarrow F_n \subset F_k$

$\Rightarrow x_n \in F_k \qquad (\because x_n \in F_n \subset F_k)$

$\Rightarrow x_n \notin S\left(x, \frac{1}{2}r\right) \qquad [\because F_k \cap S\left(x, \frac{1}{2}r\right) = \phi]$

This contradicts the fact that $x_n \to x$.

Therefore $x \in \bigcap_{n=1}^{\infty} F_n$ and hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Example 3: Show that every compact metric is complete.

Solution: Let (X, d) be a compact metric space.

To prove : X is complete.

Let $<a_n>$ be an arbitrary Cauchy sequence in X. If we show that $<a_n>$ converges to a point in X, the result will follow.

X is compact $\Rightarrow$ X is sequentially compact.

$\Rightarrow$ Every sequence in X has a convergent subsequence.

$\Rightarrow$ In particular, every Cauchy sequence in X has a convergent subsequence.

$\Rightarrow <a_n>$ has a subsequence $<a_{in} : n \in N>$ which converges to a point $a_{io} \in X$

$\Rightarrow <a_n>$ also converges to the point $a_{io} \in X$.

Theorem 4: A metric space is compact iff it is totally bounded and complete.

Proof: Let (X, d) be a compact metric space.

To prove that X is complete and totally bounded.

X is compact. $\Rightarrow$ X is sequentially compact. …(1)

$\Rightarrow$ X is totally bounded. …(2)

X is sequentially compact. $\Rightarrow$ every sequence in X has convergent subsequence.

$\Rightarrow$ In particular, every Cauchy sequence in X has a convergent subsequence

$\Rightarrow$ Every Cauchy sequence in X converges to some point in X.

$\Rightarrow$ X is complete. …(3)

From (2) & (3) the required result follows.

Conversely, suppose that a metric space (X, d) is complete and totally bounded.

To prove that X is compact.

Consider an arbitrary sequence

$$S_1 = \langle x_{11}, x_{12}, x_{13}, \ldots \rangle$$

X is totally bounded $\Rightarrow \exists$ finite class of open spheres, each of radius 1, whose union is X.

From this we can deduce that S_1 has a subsequence

$$S_2 = \langle x_{21}, x_{22}, x_{23}, \ldots \rangle$$

all of whose points be in some open sphere of radius $\frac{1}{2}$.

Similarly we can construct a subsequence S_3 of S_2 s.t.

$$S_3 = \langle x_{31}, x_{32}, x_{33}, \ldots \rangle$$

all of whose points be in some open sphere of radius $\frac{1}{3}$.

We continue this process to from successive subsequences. Now we suppose that

$$S = \langle x_{11}, x_{22}, x_{33}, \ldots \rangle.$$

Then S is a diagonal subsequence to form successive subsequence. Now we suppose that $S = \langle x_{11}, x_{22}, x_{33} \ldots \rangle$. Then S is a diagonal subsequence of S_1. By nature of this construction, S is clearly Cauchy subsequence of S_1.

X is complete $\Rightarrow$ every Cauchy sequence in X is convergent.

$\Rightarrow$ in particular, the Cauchy sequence S is convergent.

Finally, the sequence, S_1 has a convergent subsequence S. Since the sequence S_1 in X is arbitrary and hence every sequence in X has a convergent subsequence, meaning thereby X is sequentially compact and hence X is compact.

Theorem 5: Let A be a subset of a complete metric space (X, d). Prove that A is compact $\Leftrightarrow$ A is closed and totally bounded.

Proof: Let A be a compact subset of complete metric space (X, d).

To prove that A is closed and totally bounded.

X is a metric space ⇒ X is a Hausdroff space w.r.t. the metric topology.

Being a compact subset of a Hausdroff space, A is closed.

A is compact. ⇒ A is sequentially compact.

⇒ A is totally bounded.

Finally, we have shown that A is closed and totally bounded.

Conversely, suppose that A is closed and totally bounded subset of complete metric space (X, d).

To prove that A is compact.

A is complete, being a closed subset of a complete metric space (X, d). Thus A is complete and totally bounded.

Self Assessment

3. Let X be a metric space and Y is a complete metric space, and let A be dense subspace of X. If f is a uniformly continuous mapping of A into Y, then f can be extended uniquely to a uniformly continuous map of X into Y.

4. Let A be subspace of a complete metric space and show that $\bar{A}$ is compact ⇔ A is totally founded.

5. If $<A_n>$ is a sequence of nowhere dense sets in a complete metric space X, then there exist a point in X which is not in any of the A_n's.

9.4 Summary

- A sequence $<x_n>$ is Cauchy if given $\in > 0$, ∃ a positive integer n_o such that

  $d(x_{n+p}, x_n) < \in$ for all $n \geq n_o$ and for all $p \geq 1$.

- A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

- A metric space is compact iff it is totally bounded and complete.

9.5 Keywords

Closed Set: A set A is said to be closed if every limiting point of A belongs to the set A itself.

Cluster Point: Let (X, T) be a topological space and $A \subset X$. A point $x \in X$ is said to be the cluster point if each open set containing x contains at least one point of A different from x.

Convergent Sequence: A sequence $<a_n>$ is said to converge to a, if $\varepsilon > 0$, $\exists\, n_o \in N$, s.t. $n \geq n_o \Rightarrow |a - a_n| < \varepsilon$.

Sequentially Compact: A metric space (X, d) is said to be sequentially compact if every sequence in X has a convergent subsequence.

9.6 Review Questions

1. If a Cauchy sequence has a convergent subsequence, then prove that it is itself convergent.

2. Show that every compact metric space is complete.

3. Show that the metric space ($\mathcal{R}$, d) is complete, where d is usual metric on $\mathcal{R}$.
4. Show that the set $\mathcal{C}$ of complex numbers with usual metric is complete metric space.
5. Prove that every closed subset of a complete metric space is complete.
6. Prove that Frechet space is complete.
7. Show that a metric space is complete iff every infinite totally bounded subset has a limit point.

9.7 Further Readings

Books

Dmitre Burago, Yu D Burago, Sergei Ivanov, *A Course in Metric Geometry*, American Mathematical Society, 2004.

Victor Bryant, *Metric Spaces; Iteration and Application*, Cambridge University Press, 1985.

Unit 10: Compactness in Metric Spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Know the Bolzano Weierstrass theorem and BWP;
- Define sequentially compact and lebesgul measure;
- Define totally bounded set;
- Describe the compactness in metric spaces;
- Solve the related problems.

Introduction

We have already shown that compactness, limit point compactness and sequentially compact are equivalent for metric spaces. There is still another formulation of compactness for metric spaces, one that involves the notion of completeness. We study it in this unit. As an application, we shall prove a theorem characterizing those subspaces of $\mathcal{C}(X, R^n)$, that are compact in the uniform topology.

10.1 Bolzano Weierstrass Theorem

A closed and bounded infinite subset of R contains a limit point.

Bolzano Weierstrass Property: A metric space (X, d) is said to have the Bolzano weierstrass property if every infinite subset of X has a limit point.

In brief, 'Bolzano Weierstrass Property' is written as B.W.P. A space with B.W.P. is also called Frechet compact space.

10.1.1 Sequentially Compact

A metric space (X, d) is said to be sequentially compact if every sequence in X has a convergent sub-sequence.

Example 1: The set of all real numbers in (0, 1) is not sequentially compact.

For the sequence $\left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$ in (0, 1) converges to $0 \notin (0, 1)$, on the other hand [0, 1] is sequentially compact.

10.1.2 Lebesgue Number

Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d). A real number $\delta > 0$ is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < \delta \Rightarrow A \subset G_{i_0}$ for at least one index $i_0 \in \Delta$.

Lebesgue Covering Lemma

Every open covering of a sequentially compact space has a lebesgue number.

10.1.3 Totally Bounded Set

Let (X, d) be a metric space. Let $\in > 0$ be any given real number. A set $A \subset X$ is called an $\in$ - net if

(i) A is finite set

(ii) $X = U\{S_{\varepsilon(a)} : a \in A\}$

The metric space (X, d) is said to be topology bounded if it contains an $\in$ – net for every $\in > 0$. Here (ii) $\Rightarrow$ given any point $p \in X$, $\exists$ at least one point $a \in A$ s.t. $d(p, a) < \in$.

10.1.4 Compactness in Metric Spaces

If (X, d) be a metric space and $A \subset X$, then the statement that A is compact, A is countably compact and A is sequentially compact are equivalent.

10.2 Theorems and Solved Examples

Theorem 1: A metric space is sequentially compact iff it has the Bolzano Weierstrass Property.

Proof: Let X be a metric space.

Let us suppose that it is sequentially compact.

Let A be an infinite subset of X.

Since A is infinite so let $\langle x_n \rangle$ be any sequence of distinct points of A. Since X is sequentially compact, so there exists a convergent subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$. Let x be its limit and B be its range.

Since $\langle x_n \rangle$ is a sequence of distinct points, B is infinite.

We know that if the range of a convergent sequence is infinite then its limit point is the limit point of the range.

Notes

Thus, x is the limit point of B.

$\Rightarrow$ x is a limit point of A, as $B \subset A$.

Hence X has the Bolzano Weierstrass Property.

Conversely, let X has the Bolzano Weierstrass Property. Let $\langle x_n \rangle$ be a sequence in X. Let A be the range of $\langle x_n \rangle$. If A is infinite, then there is some term of $\langle x_n \rangle$ which is infinitely repeated and that gives us a convergent subsequence of $\langle x_n \rangle$. If A is infinite then by our assumption the set A has a limit point, say x.

Since A is infinite and x is a limit point of A, therefore there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \to x$.

Thus proves that X is sequentially compact.

Theorem 2: Every compact metric space has the Bolzano Weierstrass Property.

Proof: Let X be a compact metric space.

To prove: X has Bolzano Weierstrass Property.

Let A be an infinite subset of X. Suppose that A has no limit point. Then to each $x \in X$, there exists an open sphere S_x which contains no other point of A other than its centre x.

Thus, the class $\{S_x\}$ of all such open spheres is an open cover of X.

But X is compact, therefore its open cover is reducible to a finite subcover say

$\{S_{x_i} : i = 1, 2, ..., n\}$, so that

$$A \subset \bigcup_{i=1}^{n} S_{x_i}.$$

Each S_{x_i} contains no point of A other than its centre x_i, i = 1,2,...,n

$\therefore$ $A = \{x_1, x_2, ..., x_n\}$

$\Rightarrow$ A is finite.

This contradicts the fact that A is infinite.

Hence A must have a limit point.

Thus, the compact metric space X has BWP.

Theorem 3: A compact metric space is separable.

Proof: Let (X, d) be a compact metric space.

To prove that (X, d) is separable.

Fix a positive integer n.

Each open sphere forms an open set.

Consider the family $\{S_{(x, 1/n)} : x \in X\}$

Clearly it is an open cover of X which is known to be compact.

Hence this cover must be reducible to a finite sub cover, say

$\{(S_{x_{nr}}, 1/n) : r = 1, 2, ..., K_n\}$

Notes

Write $A_n = \{(x_{nr} : r = 1, 2, ..., K_n\}$.

The set A_n can be constructed for each $n \in N$.

A_n has the following properties:

(i) A_n is a finite set,

(ii) given $x \in X$; $\exists\, x_{nr} \in A_n$ s.t. $d(x, x_{nr}) < \frac{1}{n}$.

Write $A = \bigcup_{n \in N} A_n$

Being a countable union of countable sets, A is enumerable

Clearly $A \subset X$

Taking closure of both sides, $\overline{A} \subset \overline{X} = X$ i.e.

$\overline{A} \subset X$ [$\because$ X is closed in X]

We claim $\overline{A} = X$

For this it is enough to show that $X \subset \overline{A}$.

Let $x \in X$ be arbitrary and let $G \subset X$ be an open set s.t. $x \in G$.

By the property (ii) of A_n,

Given, $x \in X$, $\exists\, x_{nr} \in A_n \subset A$ s.t. $d(x_{nr}, x) < \varepsilon$ on taking $\frac{1}{n} < \varepsilon$. By the definition of open set in a metric space.

$X \in G$, G is open $\Rightarrow \exists$ positive real number r, $S_{(x,r)} \subset G$

$\Rightarrow$ in particular $S_{(x, \in)} \subset G$

$d(x, x_{nr}) < \varepsilon \Rightarrow x_{nr} \in S_{(x, \in)} \subset G$

$\Rightarrow x_{nr} \in G$

$\Rightarrow$ G contains some points of A other than x.

$\Rightarrow (G - \{x\}) \cap A \neq \phi$

$\Rightarrow x \in D(A) \subset \overline{A}$

$\Rightarrow x \in \overline{A}$

Thus we have shown that

any $x \in X \Rightarrow x \in \overline{A}$

This proves that $X \subset \overline{A}$

Finally we have shown that

$\exists\, A \subset X$ s.t. A is enumerable and $\overline{A} = X$.

This proves that X is separable.

Notes

Example 2: If a metric space (X, d) is totally bounded, then X is bounded.

Solution: Let (X, d) be a totally bounded metric space so that it contains an ε - net for every ε > 0. Let $A \subset X$ be an ε - net then:

(i) A is finite

(ii) $X = \bigcup \{S_\varepsilon(a) : a \in A\}$

(i) ⇒ A is bounded ⇒ d(A) is finite.

(ii) ⇒ $d(X) \leq d(A) + 2\varepsilon$ = a finite quantity.

⇒ $d(X) \leq$ a finite quantity

⇒ X is a bounded set. Hence proved.

Example 3: Every totally bounded metric space is separable.

Solution: Let (X, d) be totally bounded metric space so that X contains an ε - net A_n $\forall$ $\varepsilon_n > 0$.

To prove that X is separable.

A_n is ε - net ⇒ A_n is finite and $X = \bigcup\{S(a, \varepsilon_n) : a \in A_n\}$.

Write $A = \bigcup\{A_n : n \in N\}$

Being an enumerable union of finite sets A is enumerable.

$A \subset X \Rightarrow \overline{A} \subset \overline{X} = X \Rightarrow \overline{A} \subset X.$ ...(1)

Let $x \in X$ be arbitrary and let G be an open set s.t. $x \in G$.

By definition of open set

$G \subset S_{(x, \in_n)}$ ...(2)

Also $A_n \cap S_{(x, e_n)} \neq \phi$. For A_n is ε - net.

This $S_{(x, e_n)} \cap A \neq \phi$

⇒ $G \cap A \neq \phi$ [by (2)]

⇒ $x \in \overline{A}$

∴ Any $x \in X \Rightarrow x \in \overline{A}$

Consequently $X \subset \overline{A}$

In view of (1), this $X = \overline{A}$

This leads to the conclusion that X is separable.

Theorem 4: Lebesgue covering lemma: Every open cover of sequentially compact metric space has a Lebesgue number.

Proof: Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d). A real number δ > 0 is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < \delta \Rightarrow A \subset G_{i_0}$ for at least one index $i_0 \in \Delta$.

Let $\{G_i : i \in \Delta\}$ be an open cover of a sequentially compact metric space (X, d).

To prove that the cover $\{G_i\}_{i \in \Delta}$ has a Lebesgue number.

Notes

Suppose the contrary.

Then ∃ no Lebesgue number for the cover $\{G_i\}_{i \in D}$. Then for each $n \in N$, ∃ a set $B_n \subset X$ with the property that $0 < d(B_n) < \frac{1}{n}$

and $B_n \not\subseteq G_i \ \forall \ i \in \Delta$ ...(1)

Choose a point $b_n \in B_n \ \forall \ n \in N$ and consider the sequence $<b_n>$. By the assumption of sequential compactness, the sequence $<b_n : n \in N>$ contains a subsequence $<b_{in} : n \in N>$ which converges to $b \in X$.

But $\{G_i\}$ is an open cover of X so that

∃ open set G_{i_0} s.t. $b \in G_{i_0}$. By definition of open set

$$S_{\varepsilon(b)} \subset G_{i_0} \quad ...(2)$$

$\because \quad b_{i_n} \to b$

$\therefore$ Given any $\varepsilon > 0$, $\exists \ n_0 \in N$ s.t. $\forall \ i_n \geq n_0 \Rightarrow b_{i_n} \in S\frac{\varepsilon}{2}(b)$. ...(3)

Choosing a positive integer K_0 ($\geq n_0$) such that

$$\frac{1}{K_0} < \frac{\varepsilon}{2} \quad ...(4)$$

From (3), $i_n \geq K_0 \Rightarrow b_{i_n} \in S_{\varepsilon/2}(b)$

$\Rightarrow$ In particular $bK_0 \in S_{\varepsilon/2}(b)$ ...(5)

In accordance with (1)

$$b_{K_0} \in B_{K_0}, 0 < d(B_{K_0}) < \frac{1}{K_0} \quad ...(6)$$

On using (4)

$0 < d(B_{K_0}) < \varepsilon/2$ ...(7)

From (5) and (6), if follows that

$B_{K_0} \cap S_{\varepsilon/2}(b) \neq \phi$ ...(8)

From (7) and (8), if follows that B_{K_0} is a set of diameter $< \frac{\varepsilon}{2}$ and it intersects $S\frac{\varepsilon}{2}.2.(b)$, Showing thereby

$$B_{K0} \subset S\frac{\varepsilon}{2}.2.(b)$$

i.e., $B_{K0} \subset S_\varepsilon(b)$.

In view of (2), this gives $B_{K_0} \subset G_{i_0}$ ...(9)

In accordance with (1), $B_{K_0} \not\subset G_{i_0}$, $i_0 \in \Delta$

In particular, $B_{K_0} \subseteq G_{i_0}$, $i_0 \in \Delta$

Contrary to (9).

Hence the required results follows.

Notes

Theorem 5: Every compact subset of a metric space is closed and bounded.

Proof: Let Y be a compact subset of a metric space (X, d). If f is finite, then it is certainly bounded and closed.

Consider the case in which Y is not finite.

Y is compact $\Rightarrow$ Y is sequentially compact.

To prove that Y is bounded. Suppose not. Then Y is not bounded. Then it is possible to find a pair of points of Y at large distance apart. Let $y_1 \in Y$ be arbitrary.

Then we take $y_2 \in Y$

s.t. $d(y_1, y_2) > 1$

Now we can select a point y_3 s.t.

$d(y_1, y_3) > 1 + d(y_1, y_2)$

Continuing this process, we get a sequence

$\langle y_n \rangle \in Y$

with the property that $d(y_1, y_m) > 1 + d(y_1, y_{m-1}) \; \forall \; n \in N$

$\therefore \quad d(y_1, y_m) > 1 + d(y_1, y_n)$ for $m > n$ (1)

This $d(y_m, y_n) \geq |d(y_1, y_m) - d(y_1, y_n)| > 1$

Above relation shows that $\langle y_n \rangle$ has no convergent subsequence contrary to the fact that Y is sequentially compact. Hence Y is bounded.

Aim: Y is closed.

Let y be a limit point of Y, $\exists$ sequence

$\langle y_n \rangle \in Y$ s.t. $\lim y_n = y$

Every sequence of $\langle y_n \rangle$ converges to y. For Y is sequentially compact and so every sequence in Y must converge in Y.

Hence $y \in Y$

Thus $y \in D(Y) \Rightarrow y \in Y$

or $D(Y) \subset Y$ or Y is closed.

Theorem 6: Every sequentially compact metric space is compact.

Proof: Let (X, d) be a sequentially compact metric space. To prove that X is compact.

Since X is sequentially compact metric space.

X is totally bounded. Let $\varepsilon > 0$ be an arbitrary real number fixed.

X is totally bounded $\Rightarrow$ X has ε - net.

Let us denote the set $\in$ net by A.

Then A is finite subset of X with the property

$$X = \cup\{S_{\varepsilon(a)} : a \in A\} \quad ...(1)$$

Since A is finite and hence we can write

$$A = \{x_1, x_2, x_3, \ldots, x_n\}$$

Notes

In this event (1) takes the form

$$X = \bigcup_{i=1}^{n} S_{\varepsilon(x_i)} \qquad ...(2)$$

Let $\{G_i : i \in \Delta\}$ be an open cover of X which is known to be sequentially compact so that, by theorem (lebesgue covering lemma), ∃ a Lebesgue number, say, δ for the cover $\{G_i\}_{i \in \Delta}$. Set δ = 3ε.

The diameter of an open sphere of radius r is less than 2r.

i.e., $d(S_\varepsilon(x_i)) < 2\varepsilon = 2.\frac{\delta}{3} < \delta$

∴ $d(S_\varepsilon(x_i) < \delta$

By definition of Lebesgue number, ∃ an open set

$G_{ik} \in \{G_i : i \in \Delta\}$ s.t. $S_\varepsilon(x_k) \subset G_{ik}$ for $1 \le k \le n$.

From which we get $\bigcup_{k=1}^{n} S_{\varepsilon(x_k)} \subset \bigcup_{k=1}^{n} G_{ik}$

On using (2), $X \subset \bigcup_{k=1}^{n} G_{ik}$ ...(3)

But X is a universal set,

$$\bigcup_{k=1}^{n} G_{ik} \subset X \qquad ...(4)$$

Combining (3) and (4), we get $X = \bigcup_{k=1}^{n} G_{ik}$.

This implies that the family $\{G_{ik} : 1 \le K \le n\}$ is an open cover of X.

Thus the open cover $\{G_i : i \in \Delta\}$ of X is reducible to a finite the subcover $\{G_{ik} : 1 \le K \le n\}$ showing thereby X is compact.

Sequentially compact ⇒ compact ⇒ Countably compact

Theorem 7: A metric space (X, d) is compact iff it is complete and totally bounded.

Proof: If X is a compact metric space then X is complete. The fact that X is totally bounded is a consequence of the fact that the covering of X by all open ∈ - balls must contain a finite subcovering.

Conversely, Let X be complete and totally bounded.

To prove: X is sequentially compact.

Let $<x_n>$ be sequence of points of X. We shall construct a subsequence of $<x_n>$ i.e. a Cauchy sequence, so that it necessarily converges.

First cover X by finitely many balls of radius 1. At least one of these balls, say B_1, contains x_n for infinitely many values of n. Let J_1 be the subset of Z_+ consisting of those indices n for which $x_n \in B_1$.

Next, cover X by finitely many balls of radius $\frac{1}{2}$. Because J_1 is infinite, at least one of these balls, say B_2, must contain x_n for infinitely many values of n in J_1. Choose J_2 to be the set of those indices n for which $n \in J_1$ and $x_n \in B_2$. In general, given on infinite set J_k of positive integers, choose J_{k+1} to be an infinite subset of J_k such that there is a ball B_{k+1} of radius $\frac{1}{k+1}$ that contains x_n for all $n \in J_{k+1}$.

Choose $n_1 \in J_1$. Given n_k, choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$; this we can do because J_{k+1} is an infinite set. Now for i, j ≥ k, the indices n_i and n_j both belong to J_k (because $J_1 \supset J_2 \supset ...$ is nested sequence of sets). Therefore, for all i, j ≥ k, the points x_{n_i} and x_{n_j} are contained in a ball B_k of radius $\frac{1}{k}$. It follows that the sequence $\langle x_{n_i} \rangle$ is a Cauchy sequence, as desired.

Theorem 8: Let X be a space; let (Y, d) be a metric space. If the subset $\mathcal{F}$ of $\mathcal{C}$(X, Y) is totally bounded under the uniform metric corresponding to d, then $\mathcal{F}$ is equicontinuous under d.

Proof: Assume $\mathcal{F}$ is totally bounded. Give $0 < \in < 1$, and given x_0, we find a nhd U of x_0 such that $d(f(x), f(x_0)) < \in$ for $x \in U$ and $f \in F$.

Set δ = ε/3 ; Cover $\mathcal{F}$ by finitely many open δ - balls.

$B(f_1, \delta)$, ..., $B(f_n, \delta)$ in $\mathcal{C}$(X, Y). Each function f_i is continuous; therefore, we can choose a nhd of x_0 such that for i = 1 ,.., n.

$$d(f_i(x), f_i(x_0)) < \delta$$

whenever x ∈ U.

Let f be an arbitrary element of F. Then f belongs to at least one of the above δ balls say to $B(f_i, \delta)$. Then for x ∈ U, we have

$\bar{d}(f(x), f_i(x)) < \delta,$

$d(f_i(x), f_i(x_0)) < \delta$

$\bar{d}(f_i(x_0), f(x_0)) < \delta.$

The first and third inequalities hold because $\bar{p}(f, f_i) < \delta$, and the second holds because x ∈ U.

Since δ > 1, the first and third also hold if $\bar{d}$ is replaced by d; then the triangle inequality implies that for all x ∈ U, we have d(f(x)), f(x0) < ε, as desired.

Example 4: Let E be a subspace of a metric space X. Show that E is totally bounded ⇔ $\bar{E}$ is totally bounded.

Solution: Let E be totally bounded and ε > 0 be given.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be an $\frac{\varepsilon}{2}$ net for E so that

$$E \subseteq \bigcup_{i=1}^{n} S\left(a_i, \frac{\varepsilon}{2}\right) \qquad \ldots(1)$$

Let y be any element of $\bar{E}$.

Then there exists x ∈ E such that

$$d(x, y) < \frac{\varepsilon}{2} \qquad \ldots(2)$$

Notes

$$x \in E \Rightarrow x \in S\left(a_i, \frac{\varepsilon}{2}\right) \text{ for some i, } 1 \le i \le n \text{ by (1)}$$

$$\Rightarrow d(x, a_i) < \frac{\varepsilon}{2} \ (1 \le i \le n) \qquad \ldots(3)$$

Hence $d(y, a_i) \le d(y, x) + d(x, a_i)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ by (2) and (3).}$$

$\Rightarrow y \in S(a_i, \varepsilon)$ $(1 \le i \le n)$

Thus $y \in \bar{E} \Rightarrow y \in S(a_i, \varepsilon)$ for some i, $1 \le i \le n$.

$\Rightarrow \bar{E} \subseteq \bigcup_{i=1}^{n} S(a_i, \varepsilon)$

$\Rightarrow A = \{a_1, a_2, \ldots, a_n\}$ is an ε-net for $\bar{E}$

$\Rightarrow \bar{E}$ is totally bounded.

Conversely, let $\bar{E}$ be totally bounded. Then since $E \subseteq \bar{E}$, E is totally bounded since every subspace of a totally bounded metric space is totally bounded.

Example 5: Let A be a compact subset of a metric space (X, d). Show that for any $B \subset X$ there is a point $p \in A$ such that

$$d(p, B) = d(A, B).$$

Solution: By the definition, we have

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}.$$

Let $d(A, B) = \varepsilon$.

$\therefore \varepsilon = \inf \{d(a, b) : a \in A, b \in B\} \le d(a, b)$,

$a \in A$, $b \in B$ being arbitrary which follows that

$\forall n \in N$, $a_n \in A$ and $b_n \in B$ such that

$\varepsilon \in d(a_n, b_n) < \varepsilon + \frac{1}{n}$.

Since A is compact, it is also sequentially compact and so the sequence $\langle a_n \rangle$ has a subsequence $\langle a_{n_i} \rangle$ which converges to a point $p \in A$.

We claim that $d(p, B) = \varepsilon$

Let, if possible, $d(p, B) > \varepsilon$

Let $d(p, B) = \varepsilon + \varepsilon'$ where $\varepsilon' > 0$

Since $\langle a_{n_i} \rangle$ converges to p there must exist a natural number n_0 such that

$$d(p, a_{n_0}) < \frac{\varepsilon'}{2}$$

and $$d(a_{n_0}, b_{n_0}) < \varepsilon + \frac{1}{n_0}$$

$$< \varepsilon + \frac{\varepsilon'}{2}$$

$$\therefore \quad d\left(p, a_{n_0}\right) + d\left(a_{n_0}, b_{n_0}\right) < \frac{1}{2}\varepsilon' + \varepsilon + \frac{1}{2}\varepsilon'$$

$$< \varepsilon + \varepsilon' = d\,(p, B)$$

$$\leq d\left(p, b_{n_0}\right) \quad \text{since } b_{n_0} \in B.$$

or $\quad d\left(p, a_{n_0}\right) + d\left(a_{n_0}, b_{n_0}\right) < d\left(p, b_{n_0}\right)$

This contradicts the triangle inequality.

Thus d(p, B) = d(A, B).

10.3 Summary

- A closed and bounded infinite subset of R contains a limit point.
- A metric space (X, d) is said to have the BWP if every infinite subset of X has a limit point.
- A metric space (X, d) is said to sequentially compact if every sequence in X has a convergent subsequence.
- Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d). A real number $\delta > 0$ is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < d \Rightarrow A \subset G_{i_0}$ for at least one index $i_0 \in \Delta$.
- Every open covering of a sequentially compact space has a lebesgue number.
- If (X, d) be a metric space and $A \subset X$, then the statement that A is compact, A is countably compact and A is sequentially compact are equivalent.

10.4 Keywords

Cauchy sequence: Let $<x_n>$ be a sequence in a metric space (X, d). Then $<x_n>$ is called a cauchy sequence if given $\varepsilon > 0$, $\exists\, n_0 \in N$ s.t. $m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$.

Compact: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to finite sub covering.

Complete metric space: Let (X, d) a metric space then (X, d) is complete if cauchy sequence of elements of X converges to some elements (belonging to X).

Equicontinuous: A collection of real valued functions.

$A = \{f_n : f_n : X \to R\}$ defined on a metric space (X, d) is said to be equicontinuous if

given $\varepsilon > 0$, $\exists\, \delta = \delta(\in) > 0$ s.t.

$d(x_0, x_1) < \delta \Rightarrow |\, f(x_0) - f(x_1) \,| < \varepsilon \;\; \forall \; f \in A.$

Finite subcover: If $\exists\, G_1 \subset G$ s.t. G_1 is a finite set and that $\{G : G \in G_1\}$ is a cover of A, then G_1 is called a finite subcover of the original cover.

Open cover: If every member of G is an open set, then the cover G is called an open cover.

Notes

10.5 Review Questions

1. A finite subset of a topological space is necessarily sequentially compact. Prove it.
2. Prove that if X is sequentially compact, then it is countably compact.
3. Let A be a compact subset of a metric space (X, d). Show that for every $B \subset X$, $\exists\ p \in A$ s.t. $d(p, B) = d(A, B)$.
4. Let A be a compact subset of a metric space (X, d) and let $B \subset X$, be closed. Show that $d(A, B) > 0$ if $A \cap B = \phi$.

10.6 Further Readings

Books

John Kelley (1955), *General Topology*, *Graduate Texts in Mathematics*, Springer-Verlag.

Dmitre Burago, Yu D Burgao, Sergei Ivanov, *A course in Metric Geometry*, American Mathematical Society, 2004.

Unit 11: Pointwise and Compact Convergence

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define pointwise convergence and solve related problems;
- Understand the concept of compact convergence and solve problems on it;
- Discuss the compact open topology.

Introduction

There are other useful topologies on the spaces Y^X and $\mathcal{C}(X, Y)$, in addition to the uniform topology. We shall consider three of them here: they are called the topology of pointwise convergence, the topology of compact convergence, and the compact-open topology.

11.1 Pointwise and Compact Convergence

11.1.1 Pointwise Convergence

Definition: Given a point x of the set X and an open set U of the space Y, let

$S(x, U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}$

The sets S(x, U) are a sub-basis for topology on Y^X, which is called the topology of pointwise convergence (or the point open topology).

Example 1: Consider the space $\mathbb{R}^I$, where I = [0, 1]. The sequence (f_n) of continuous functions given by $f_n(x) = x^n$ converges in the topology of pointwise convergence to the function f defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

This example shows that the subspace $\mathcal{C}(I, \mathbb{R})$ of continuous functions is not closed in $\mathbb{R}^I$ in the topology of pointwise convergence.

11.1.2 Compact Convergence

Definition: Let (Y, d) be a metric space; let X be a topological space. Given an element f of Y^X, a compact subspace C of X, and a number $\in > 0$, let $B_C(f, \in)$ denote the set of all those elements g of Y^X for which

$$\sup \{d(f(x), g(x)) \mid x \in C\} < \in$$

The sets $B_C(f, \in)$ form a basis for a topology on Y^X. It is called the topology of compact convergence (or sometimes the "topology of uniform convergence on compact sets").

It is easy to show that the sets $B_C(f, \in)$ satisfy the conditions for a basis. The crucial step is to note that if $g \in B_C(f, \in)$, then for

$$\delta = \in - \sup\{d(f(x), g(x)) \mid x \in C\},$$

we have $B_C(g, \delta) \subset B_C(f, \in)$

Note The topology of compact convergence differs from the topology of pointwise convergence in that the general basis element containing f consists of functions that are "close" to f not just at finitely many points, but at all points of some compact set.

11.1.3 Compactly Generated

Definition: A space X is said to be compactly generated if it satisfies the following condition. A set A is open in X if $A \cap C$ is open in C for each compact subspace C of X.

This condition is equivalent to requiring that a set B be closed in X if $B \cap C$ is closed in C for each compact C. It is a fairly mild restriction on the space; many familiar spaces are compactly generated.

Lemma 1: If X is locally compact, or if X satisfies the first countability axiom, then X is compactly generated.

Proof: Suppose that X is locally compact. Let $A \cap C$ be open in C for every compact subspace C of X. We show A is open in X. Given $x \in A$, choose a neighbourhood U of x that lies in a compact subspace C of X. Since $A \cap C$ is open in C by hypothesis, $A \cap U$ is open in U, and hence open in X. Then $A \cap U$ is a neighbourhood of x contained in A, so that A is open in X.

Suppose that X satisfies the first countability axiom. If $B \cap C$ is closed in C for each compact subspace C of X, we show that B is closed in X. Let x be a point of $\overline{B}$; we show that $x \in B$. Since X has a countable basis at x, there is a sequence (x_n) of points of B converging to x. The subspace

$$C = \{x\} \cup \{x_n \mid n \in \mathbb{Z}_+]$$

is compact, so that $B \cap C$ is by assumption closed in C. Since $B \cap C$ contains x_n for every n, it contains x as well. Therefore, $x \in B$, as desired.

Lemma 2: If X is compactly generated, then a function $f : X \rightarrow Y$ is continuous if for each compact subspace C of X, the restricted function $f|C$ is continuous.

Proof: Let V be an open subset of Y; we show that $f^{-1}(V)$ is open in X. Given any subspace C of X.

$$f^{-1}(V) \cap C = (f \mid C)^{-1}(V)$$

If C is compact, this set is open in C because f|C is continuous. Since X is compactly generated, it follows that $f^{-1}(V)$ is open in X.

Theorem 1: Let X be a compactly generated space. Let (Y, d) be a metric space. Then $\mathcal{C}(X, Y)$ is closed in Y^X in the topology of compact convergence.

Proof: Let $f \in Y^X$ be a limit point of $\mathcal{C}(X, Y)$; we wish to show f is continuous. It suffices to show that f |C is continuous for each compact subspace C of X. For each n_1 consider the neighbourhood $B_c(f, 1/n)$ of f; it intersects $\mathcal{C}(X, Y)$, so we can choose a function $f_n \in C(X, Y)$ lying in this neighbourhood. The sequence of functions $f_n \mid C : C \to Y$ converges uniformly to the function f|C, so that by the uniform limit theorem, f | C is continuous.

11.1.4 Compact-open Topology

Definition: Let X and Y be topological spaces. If C is a compact subspace of X and U is an open subset of Y, define

$$S(C, U) = \{f \mid f \in \mathcal{C}(X, Y) \text{ and } f(C) \subset U\}$$

The sets S(C, U) form a sub-basis for a topology on $\mathcal{C}(X, Y)$ that is called the compact-open topology.

Theorem 2: Let X be a space and let (Y, d) be a metric space. On the set $\mathcal{C}(X, Y)$, the compact-open topology and the topology of compact convergence coincide.

Proof: If A is a subset of Y and $\in > 0$, let $U(A, \in)$ be the $\in$ - neighbourhood of A. If A is compact and V is an open set containing A, then there is an $\in > 0$ such that $U(A, \in)$. Indeed, the minimum value of the function d(a, X – V) is the required $\in$.

We first prove that the topology of compact convergence is finer than the compact-open topology. Let S(C, U) be a sub-basis element for the compact-open topology, and let f be an element of S(C, U). Because f is continuous, f(C) is a compact subset of the open set U. Therefore, we can choose $\in$ so that $\in$ - neighbourhood of f(C) lies in U. Then, as desired.

$$B_C(f, \in) \subset S(C, U)$$

Now we prove that the compact-open topology is finer than the topology of compact convergence. Let $f \in \mathcal{C}(X, Y)$. Given an open set about f in the topology of compact convergence, it contains a basis element of the form $B_C(f, \in)$. We shall find a basis element for the compact-open topology that contains f and lies in $B_C(f, \in)$.

Each point x of X has a neighbourhood V_x such that $F(V_x)$ lies in an open set U_x of Y having diameter less than $\in$. [For example, choose V_x so that $f(V_x)$ lies in the $\in/4$-neighbourhood of f(x). Then $f(V_x)$ lies in the $\in/3$-neighbourhood of f(x), which has diameter at most $2\in/3$]. Cover C by finitely many such sets V_x, say for $x = x_1, ..., x_n$. Let $C_x = V_x \cap C$. Then C_x is compact, and the basis element.

$$S(C_{x_1}, U_{x_1}) \cap ... \cap S(C_{x_n}, U_{x_n})$$

Theorem 3: Let X be locally compact Hausdorff; let e (X, Y) have the compact-open topology. Then the map

$$e : X \times e(X, Y) \to Y$$

defined by the equation

$$e(x, f) = f(x)$$

is continuous.

The map e is called the evaluation map.

Proof: Given a point (X, f) of X × *e* (X, Y) and an open set V in Y about the image point e (x, f) = f (x), we wish to find an open set about (x, f) that e maps into V. First, using the continuity of f and the fact that X is locally compact Hausdorff, we can choose an open set ⋃ about x having compact closure $\bar{\cup}$, such that f carries $\bar{\cup}$ into V. Then consider the open set ⋃ × S ($\bar{\cup}$, V) in X × *e* (X, Y). It is an open set containing (x, f). And if (x′, f′) belongs to this set, then e (x′, f′) = f′ (x′) belongs to V, as defined.

Theorem 4: Let X and Y be spaces, give *e* (X, Y) the compact-open topology. If f : X × Z → Y is continuous, then so is the induced function F : Z → *e* (X, Y). The coverse holds if X is locally compact Hausdorff.

Proof: Suppose first that F is continuous and that X is locally compact Hausdorff. It follows that f is continuous, since f equals the composite.

$$X \times Y \xrightarrow{i_x \times F} X \times e\,(X \times Y) \xrightarrow{e} Y,$$

where i_x is the identity map of X.

Now suppose that f is continuous. To prove continuity of F, we take a point Z_0 of Z and a sub-basic element S (*e*, ⋃) for C (X, Y) containing F (Z_0) and find a neighborhood W of Z_0 that is mapped by F into S (C, ⋃). This will suffice.

The stalement that F (Z_0) lies in S (C, ⋃) means simply that (F (Z_0)) (x) = f (x, Z_0) is in ⋃ for all x ∈ C. That is, f (C × Z_0) ⊂ ⋃. Continuity of f implies that f^{-1} (⋃) is an open set in X × Z containing C × Z_0. Then

$$f^{-1}\,(\cup) \cap (C \times Z)$$

is an open set in the subspace C × Z containing the slice C × Z_0.

The tube lemma implies that there is a neighborhood W of Z_0 in Z such that the entire tube C × W lies in f^{-1} (⋃). Then for Z ∈ W and x ∈ C, we have f (x, z) ∈ ⋃. Hence F (W) ⊂ S (C, ⋃), as desired.

11.2 Summary

- Give a point x of the set X and an open set U of the space Y, let

  $$S(x, U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}$$

  The sets S(x, U) are a sub-basis for topology on Y^X, which is called the topology of pointwise convergence.

- Let (Y, d) be a metric space; let X be a topological space. Given an element f of Y^X, a compact subspace C of X, and a number ∈ > 0, let $B_C(f, \in)$ denote the set of all those elements g of Y^X for which

  $$\sup\{d(f(x), g(x)) \mid x \in C\} < \in$$

  The sets $B_C(f, \in)$ form a basis for a topology of Y^X. It is called the topology of compact convergence.

- A space X is said to be compactly generated if it satisfies the following condition. A set A is open in X if A ∩ C is open in C for each compact subspace C of X. This condition is equivalent to requiring that a set B be closed in X if B ∩ C is closed in C for each compact C. It is a fairly mild restriction on the space; many familiar spaces are compactly generated.

- Let X and Y be topological spaces if C is a compact subspace of X and U is an open subset of Y, define S(C, U) = {f | f ∈ C(x, y) and f(C) ⊂ U}.

11.3 Keywords

Compact set: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to fine sub-covering.

Locally compact: Let (X, T) be a topological space and let $x \in X$ be arbitrary. Then X is said to be locally compact at x if the closure of any neighbourhood of x is compact.

Subbase: Let (X, T) be a topological space. Let $S \subset T$ s.t. $S \neq \phi$

S is said to be a sub-base or open sub-base for the topology T on X if finite intersections of the members of S form a base for the topology T on X i.e. the unions of the members of S give all the members of T. The elements of S are referred to as sub-basic open sets.

11.4 Review Questions

1. Show that the set $\mathcal{B}(\mathbb{R}, \mathbb{R})$ of founded functions $f : \mathbb{R} \to \mathbb{R}$ is closed in $\mathbb{R}^{\mathbb{R}}$ in the uniform topology, but not in the topology of compact convergence.

2. Consider the sequence of functions

   $f_n : (-1, 1) \to \mathbb{R}$, defined by

   $$f_n(x) = \sum_{k=1}^{n} Kx^k$$

   (a) Show that (f_n) converges in the topology of compact convergence, conclude that the limit function is continuous.

   (b) Show that (f_n) does not converge in the uniform topology.

3. Show that in the compact-open topology, $\mathcal{C}(X, Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular.

   [***Hint:*** If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(U, V)$]

4. Show that if Y is locally compact Hausdorff then composition of maps

   $$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$

   is continuous, provided the compact open topology is used throughout.

11.5 Further Readings

Books

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

J. Dugundji, *Topology*, Prentice Hall of India, New Delhi, 1975.

Unit 12: Ascoli's Theorem

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define equicontinuous and uniformly equicontinuous;
- Understand the proof of Ascoli's theorem;
- Solve the problems on Ascoli's theorem.

Introduction

Ascoli's theorem deals with continuous functions and states that the space of bounded, equicontinuous functions is compact. The space of bounded "equimeasurable functions," is compact and it contains the bounded equicontinuous functions as a subset. Giulio Ascoli is an Italian Jewish mathematician. He introduced the notion of equicontinuity in 1884 to add to closedness and boundedness for the equivalence of compactness of a function space. This is what is called Ascoli's theorem.

12.1 Ascoli's Theorem

12.1.1 Equicontinuous

A family F of functions on a metric space (X, d) is called equicontinuous if

$\forall\ x \in X, \forall \in > 0, \exists\ \ \delta > 0$ s.t. $\forall\ y \in X$ with $d(x, y) < \delta$ we have $|f(x) - f(y)| < \in$ for all $f \in F$.

12.1.2 Uniformly Equicontinuous

A family F of functions on a metric space (X, d) is called uniformly equicontinuous if $\forall \in > 0$, $\exists\ \delta > 0$ s.t. $\forall\ x, y \in X$ with $d(x, y) < \delta$. We have $|f(x) - f(y)| < \in$ for all $f \in F$.

Theorem 1: Let f_n be an equicontinuous sequence of functions on (X, d). Suppose that $f_n(x) \to f(x)$ pointwise. Then f(x) is continuous.

Proof: Let $x \in X$ and $\in > 0$, choose $\delta > 0$ so that $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \in/2$ for any n.

Then $$|f(x) - f(y)| = \underset{n\to\infty}{\text{lt}} |f_n(x) - f_n(y)|$$

$$\leq \sup_n |f_n(x) - f_n(y)|$$

$$\leq \frac{\in}{2} < \in.$$

12.1.3 Statement and Proof of Ascoli's Theorem

Statement: Let $\mathcal{A}$ be a closed subset of the function space C [0, 1]. Then $\mathcal{A}$ is compact iff $\mathcal{A}$ is uniformly bounded and equicontinuous.

Proof: Let $\mathcal{A}$ be closed subset of the function space C [0, 1].

Step I: Let $\mathcal{A}$ be compact.

To prove : $\mathcal{A}$ is uniformly bounded and equicontinuous.

$\mathcal{A}$ is compact $\Rightarrow$ $\mathcal{A}$ is totally bounded

$\Rightarrow$ $\mathcal{A}$ is bounded.

Now $\mathcal{A}$ is a bounded subset of C [0, 1] and each member of C [0, 1] is uniformly continuous. It means that $\mathcal{A}$ is uniformly bounded as a set of functions. Remains to show that $\mathcal{A}$ is equicontinuous.

By definition of totally bounded, $\mathcal{A}$ has an $\in$-net Denote this $\in$-net by $\mathcal{B}$. We can take

$B = \{f_1, f_2, \ldots, f_m\}$ s.t. for any

$f \in \mathcal{A}$, $\exists f_{i_o} \in \mathcal{B}$ s.t. $\|f - f_{i_o}\| < \in k$, where $k > 0$

where $$\|f - f_{i_o}\| = \sup\{|f(x) - f_{i_o}(x)| : x \in [0,1]\}$$

$$\Rightarrow \quad |f(x) - f_{i_o}(x)| < \in k \ \forall \ x \in [0, 1]. \qquad \ldots(1)$$

Let $x, y \in [0, 1]$ and $f \in \mathcal{A}$ be arbitrary.

$$|f(x) - f(y)| = |f(x) - f_{i_o}(x) + f_{i_o}(x) - f_{i_o}(y) + f_{i_o}(y) - f(y)|$$

$$< |f(x) - f_{i_o}(x)| + |f_{i_o}(x) - f_{i_o}(y)| + |f_{i_o}(y) - f(y)|$$

Using (1), $$|f(x) - f(y)| < \in k + |f_{i_o}(x) - f_{i_o}(y)| + \in k \qquad \ldots(2)$$

$f_{i_o} \in \mathcal{B} \Rightarrow f_{i_o} \in \mathcal{A} \Rightarrow f_{i_o}$ is uniformly continuous on [0, 1].

$$\therefore \quad \exists\, \delta_i > 0 \text{ s.t. } |x - y| < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \in k \qquad \ldots(3)$$

Take $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$. Then, by (3), we get

$|x - y| < \delta \Rightarrow |f_{i_o}(x) - f_{i_o}(y)| < \in k$, Using this in (2), or $|f(x) - f(y)| < \in' k$, for $|x - y| < \delta$, $f \in \mathcal{A}$ where $k = \frac{1}{3}$.

This proves that $\mathcal{A}$ is equicontinuous.

Step II: Suppose $\mathcal{A}$ is uniformly bounded and equicontinuous.

To prove: $\mathcal{A}$ is compact.

Since C [0, 1] is complete and $\mathcal{A}$ is a closed subset of it and so $\mathcal{A}$ is complete. Hence we need only to show that $\mathcal{A}$ is totally bounded.

[As we know that "A metric space is compact iff it is totally bounded and complete."]

Given $\in > 0$, $\exists$ positive integer n_o s.t.

$$|x - y| < \frac{1}{n_o} \Rightarrow |f(x) - f(y)| < \frac{\in}{5} \ \forall f \in \mathcal{A}$$

for each $f \in \mathcal{A}$, we can construct a polygon arc p_f s.t. $\|f - p_f\| < \in$ and p_f connects points belonging to

$$P = \left\{(x, y) : x = 0, \frac{1}{n_o}, \frac{2}{n_o}, \ldots, 1, y = \frac{n \in}{5}, n \text{ is an integer}\right\}.$$

Write $\mathcal{B} = \{p_f : f \in \mathcal{A}\}$

We want to show that $\mathcal{B}$ is finite and hence an $\in$-net for $\mathcal{A}$.

A is uniformly bounded.

$\Rightarrow$ B is uniformly bounded.

Hence a finite number of points in $\mathcal{A}$ will appear in the polygonal arcs in $\mathcal{B}$. It means that there can only be a finite number of arcs in $\mathcal{B}$, showing thereby $\mathcal{B}$ is an $\in$-net for $\mathcal{A}$ and so $\mathcal{A}$ is totally bounded. Also $\mathcal{A}$ is complete. Consequently $\mathcal{A}$ is compact.

Remark: Ascoli's theorem is also sometimes called Arzela-Ascoli's theorem.

Theorem 2: Every compact metric space is separable.

Proof: Let (X, d) be a compact metric space.

Let m be a fixed positive number.

Let $\mathcal{C} = \left\{S\left(x, \frac{1}{m}\right) : x \in X\right\}$ be a collection of open spheres.

($\because$ each open sphere forms an open set.)

Then $\mathcal{C}$ is clearly an open cover of X. Since X is compact and hence its open cover is reducible to a finite sub cover say

$$\mathcal{C}' = \left\{S\left(x_{m_i}, \frac{1}{m}\right) : i = 1, 2, \ldots, k\right\}$$

Let $A_m = \{x_{m_i} : i = 1, 2, \ldots, k\}$.

Thus for each $m \in N$, we can construct A_m in above defined manner.

Also, each such set is finite and for each x ∈ X, there is an element $x_{m_i} \in A_m$ such that $d(x, x_{m_i}) < \frac{1}{m}$.

Then $A = \bigcup_{m \in N} A_m \subset X$ is countable as it is the union of countable sets.

Now $\quad A \subset X \Rightarrow \bar{A} \subset \bar{X}$

$\Rightarrow \quad \bar{A} \subset X$ since X is closed $\Rightarrow \bar{X} = X$.

In order to show that (X, d) is separable, it is sufficient to show that $\bar{A} = X$, for which it is sufficient to show that each point of X is an adherent point of A.

So, let x be an arbitrary point of X and G be any open nhd. of x, ∃ an open sphere $S\left(x, \frac{1}{m}\right)$ for some positive integer m such that,

$$x \in S\left(x, \frac{1}{m}\right) \subset G \quad \ldots(1)$$

But for each x ∈ X, ∃ $x_{m_i} \in A_m \subset A$ such that $d(x, x_{m_i}) < \frac{1}{m}$

or $$x_{m_i} \in S\left(x, \frac{1}{m}\right) \quad \ldots(2)$$

Then from (1) and (2), we get

$$x_{m_i} \in S\left(x, \frac{1}{m}\right) \subset G.$$

Thus, every open nhd. of x contains at least one point of A and therefore, x is an adherent point of A.

This shows that every point of X is an adherent point of A.

$\therefore \quad X \subset \bar{A}$ and therefore

$\bar{A} = X$

which follows that A is countable dense subset of X and hence X is separable.

12.2 Summary

- A family $\mathcal{F}$ of functions on a metric space (X, d) is called equicontinuous if ∀ x ∈ X, ∀ ∈ > 0, ∃ δ > 0 s.t. ∀ y ∈ X with d(x, y) < δ , we have

  $|f(x) - f(y)| < \in$ for all f ∈ $\mathcal{F}$.

- A family $\mathcal{F}$ of functions on a metric space (X, d) is called uniformly equicontinuous if ∀ ∈ > 0, ∃ δ > 0, s.t. x, y ∈ X with d(x, y) < δ, we have

  $|f(x) - f(y)| < \in$ for all f ∈ $\mathcal{F}$.

- ***Ascoli's Theorem:*** Let $\mathcal{A}$ be a closed subset of the function space C [0, 1]. Then $\mathcal{A}$ is compact iff $\mathcal{A}$ is uniformly bounded and continuous.

12.3 Keywords

Adherent Point: A point $x \in X$ is called an adherent point of A iff every nhd of x contains at least one point of A.

Compact Metric Space: If (X, d) be a metric space and $A \subset X$, then the statement that A is compact, A is countably compact and A is sequentially compact are equivalent.

Complete Metric Space: A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

Open Sphere: Let (X, ρ) be a metric space. Let $x_o \in X$ and $r \in R^+$. Then set $\{x \in X : \rho(x_o, x) < r\}$ is defined a open sphere with centre x_o and radius r.

Separable Space: Let X be a topological space and $A \subset X$, then X is said to be separable if

(i) $\bar{A} = X$ (ii) A is countable

Totally Bounded: A metric space (X, d) is said to be totally bounded if for every $\in > 0$, there is a finite covering of X by $\in$-balls.

12.4 Review Questions

1. Prove that A subset T of $\mathcal{C}$ (X) is compact if and only if it is closed, bounded and equicontinuous.

2. Prove the following:

   Theorem: If X is locally compact Hausdorff space, then a subspace T of $\mathcal{C}$ (X, R^n) in the topology of compact convergence has compact closure if and only if T is pointwise bounded and equicontinuous under either of the standard metric on R^n.

3. Let (Y, d) be a metric space; let $f_n : X \to Y$ be a sequence of continuous functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose f_n converges to f in the topology of pointwise convergence. Show that if $\{f_n\}$ is equicontinuous, then f is continuous and f_n converges to f in the topology of compact convergence.

4. Prove the following:

   Theorem (Arzela's theorem, general version). Let X be a Hausdorff space that is σ-compact; let f_n be a sequence of functions $f_n : X \to R^k$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence f_n has a subsequence that converges, in the topology of compact convergence, to a continuous function.

12.5 Further Readings

Books

H.F. Cullen, *Introduction to General Topology*, Boston, M.A.

Stephen Willard, *General Topology*, (1970).

Richa Nandra, Lovely Professional University

Unit 13: Baire Spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Know about the Baire spaces;
- Understand the Baire's category theory;
- Understand the Baire's category theorem.

Introduction

In this unit, we introduce a class of topological spaces called the Baire spaces. The defining condition for a Baire space is a bit complicated to state, but it is often useful in the applications, in both analysis and topology. Most of the spaces we have been studying are Baire spaces. For instance, a Hausdorff space is a Baire space if it is compact, or ever locally compact. And a metrizable space X is a Baire space if it is topologically complete, that is, if there is a metric for X relative to which X is complete.

Then we shall give some applications, which ever if they do not make the Baire condition seem any more natural, will at least show what a useful tool it can be in feet, it turns out to be a very useful and fairly sophisticated tool in both analysis and topology.

13.1 Baire Spaces

13.1.1 Definition – Baire Space

A space X is said to be a Baire space of the following condition holds. Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.

Example 1: The space Q of rationals is not a Baire space. For each one-point set in Q is closed and has empty interior in Q; and Q is the countable union of its one-point subsets. The space Z_+, on the other hand, does form a Baire space. Every subset of Z_+ is open, so that there exist no subsets of Z_+ having empty interior, except for the empty set. Therefore Z_+ satisfies the Baire condition vacuously.

Notes

Lemma 1: X is a Baire space iff gives any countable collection $\{ \bigcup_n \}$ of open sets in X, each of which is dense in X their intersection $\cap \bigcup_n$ is also dense in X.

Proof: Recall that a set $\mathcal{C}$ is dense in X if $\bar{C} = X$. The theorem now follows at once from the two remarks.

1. A is closed in X iff X-A is open in X.
2. B has empty interior in X if and only if X-B is dense in X.

Lemma 2: Any open subspace Y of a Baire space X is itself a Baire space.

Proof: Let A_n be a countable collection of closed set of Y that have empty interiors in Y. We show that $\cup A_n$ has empty interior in Y.

Let $\bar{A}_n$ be the closure of A_n in X; then $\bar{A}_n \cap Y = A_n$. The set $\bar{A}_n$ has empty interior in X. For it $\bigcup$ is a non empty open set of X contained in $\bar{A}_n$, then $\bigcup$ must intersect A_n. Then $\bigcup \cap Y$ is a non-empty open set of Y contained in A_n, contrary to hypothesis.

If the union of the sets A_n contains the non empty open set W of Y, then the union of the sets $\bar{A}_n$ also contains the set W, which is open in X because Y is open in X. But each set $\bar{A}_n$ has empty interior in X, contradicting the fact that X is a Baire space.

13.1.2 Baire's Category Theory

Let (X, d) be a metric space and $A \subset X$. The set A is called of the **first category** if it can be expressed as a countable union of non dense sets. The set A is called of the second category if it is not of the first category.

Definition: A metric space is said to be totally of **second category** if every non empty closed subset of X is of the second category.

Example 2: Let $q \in \mathcal{Q}$ be arbitrary.

$$\{\bar{q}\} = \{q\} \cup D(\{q\}), \qquad [\because \bar{A} = A \cup D(A)]$$

$$= \{q\} \cup \phi = \{q\}$$

$$\therefore \quad \text{int}\{\bar{q}\} = \text{int}\{q\}$$

$$= \cup \{G \subset \mathcal{R} : G \text{ is open}, G \subset \{q\}\} = \phi.$$

For every subset of $\mathcal{R}$ contains rational as well irrational numbers.

Thus, $\quad \text{int}\{\bar{q}\} = \phi$.

This proves that {q} is a non-dense subset of $\mathcal{Q}$.

$$\mathcal{Q} = \cup \{\{q\} : a \in \mathcal{Q}\}.$$

Furthermore $\mathcal{Q}$ is enumerable.

$\therefore$ $\mathcal{Q}$ is an enumerable union of non-dense sets.

From what has been done it follows that $\mathcal{Q}$ is of the first category.

Example 3: Consider a sequence $\langle f_n(x) \rangle$ of continuous functions defined from I = [0, 1] into $\mathcal{R}$ s.t. $f_n(x) = x_n \ \forall \ x \in N$.

Then $<f_n>$ converges pointwise to $g : \mathcal{I} \to \mathcal{R}$ s.t.

$$g(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

Evidently g is not continuous.

13.1.3 Baire Category Theorem

Theorem 1: Every complete metric space is of second category.

Proof: Let (X, d) be a complete metric space.

To prove that X is of second category.

Suppose not. Then X is not of second category so that X is of first category. By def., X is expressible as a countable union of nowhere dense sets arranged in a sequence $<A_n>$. Since A_1 is non-dense and so $\exists$ a closed sphere K_1 of radius $r_1 < \frac{1}{2}$ s.t. $K_1 \cap A_1 = \phi$.

Let the open sphere with same centre and radius as r_1 be denoted by S_1. In S_1, we can find a closed sphere K_2 of radius $r_2 < \left(\frac{1}{2}\right)^2$ s.t.

$$K_1 \cap A_2 = \phi \quad \text{and so } K_2 \cap A_1 = \phi$$

Continuing like this we construct a nested sequence $<K_n>$ of closed spheres having the following properties:

(i) For each positive integer n, K_n does not intersect
$A_1, A_2, \ldots, A_n$.

(ii) The radius of K_n tends to zero as $n \to \infty$. For $\frac{1}{2^n} \to 0$ as $n \to \infty$.

Since (X, d) is complete and so by Cantor's intersection theorem, $\bigcap_n K_n$ contains a single point x_o.

$$\therefore \quad x_o \in \bigcap_{n=1}^{\infty} K_n \Rightarrow x_o \in K_n \ \forall \ n$$

$$\Rightarrow x_o \notin A_n \ \forall \ n \text{ (according to (i))}$$

$$\Rightarrow x_o \notin \bigcup_{n=1}^{\infty} A_n = X$$

$$\Rightarrow x_o \notin X. \text{ A contradiction}$$

For X is universal set.

Hence X is not of first category. A contradiction. Hence the required result follows.

Remarks: The theorem 1 can also be expressed in the following ways:

1. If $<A_n>$ is a sequence of nowhere dense sets in a complete metric space (X, d), then $\exists$ a point in X, which is not in A_n's.
2. If a complete metric space is the union of a sequence of its subsets, then the closure of at least one set in the sequence must have non-empty interior.

Theorem 2: Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.

Proof: Given a positive integer N and given ε > 0, define

$$A_N(\varepsilon) = \{x | d(f_n(x), f_m(x)) \leq \varepsilon \text{ for all } n, m \geq N\}.$$

Note that $A_N(\varepsilon)$ is closed in X. For the set of those x for which $d(f_n(x), f_m(x)) \leq \varepsilon$ is closed in X, by continuity of f_n and f_m and $A_N(\in)$ is the intersection of these sets for all n, m ≥ N.

For fixed ∈, consider the sets $A_1(\in) \subset A_2(\in) \subset \ldots$. The union of these sets is all of X. For, given $x_0 \in X$, the fact that $f_n(x_0) \to f(x_0)$ implies that the sequence $f_n(x_0)$ is a Cauchy sequence; hence $x_0 \in A_N(\in)$ for some N.

Now let

$$\cup(\in) = \bigcup_{N \in \mathbb{Z}_+} \text{Int} A_N(\in).$$

We shall prove two things:

(1) $\cup(\in)$ is open and dense in X.

(2) The function f is continuous at each point of the set

$$\mathcal{C} = \cup(1) \cap \cup(1/2) \cap \cup(1/3) \cap \ldots$$

Our theorem then follows from the fact that X is a Baire space. To show that $\cup(\in)$ is dense in X, it suffices to show that for any non-empty open set V of X, there is an N such that the set $V \cap \text{Int}\, A_N(\in)$ is non-empty. For this purpose, we note first that for each N, the set $V \cap A_N(\in)$ is closed in V. Because V is a Baire space by the preceding lemma, at least one of these sets, say $V \cap A_M(\in)$, must contain a non-empty open set W of V. Because V is open in X, the set W is open in X; therefore, it is contained in Int $A_M(\varepsilon)$.

Now we show that if $x_0 \in \mathcal{C}$, then f is continuous at x_0. Given ε > 0, we shall find a neighborhood W of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for $x \in W$.

First, choose K so that $1/K < \varepsilon/3$. Since $x_0 \in \mathcal{C}$, we have $x_0 \in \cup(1/K)$ therefore, there is an N such that $x_0 \in \text{Int}\, A_N(1/K)$. Finally, continuity of the function f_N enables us to choose a neighborhood W of x_0, contained in $A_N(1/K)$, such that

(*) $d(f_N(x), f_N(x_0)) \leq \varepsilon/3$ for $x \in W$.

The fact that $W \subset A_N(1/K)$ implies that

(**) $d(f_n(x), f_N(x)) \leq 1/K$ for $n \geq N$ and $x \in W$.

Letting $n \to \infty$, we obtain the inequality

(***) $d(f(x), f_N(x)) \leq 1/K < \varepsilon/3$ for $x \in W$.

In particular, since $x_0 \in W$, we have

$$d(f(x_0), f_N(x_0)) < \varepsilon/3$$

Applying the triangle inequality (*), (**) and (***) gives us our desired result.

Theorem 3: If Y is a first category subset of a Baire space (X, T) then the interior of Y is empty.

Proof: As Y is first category, $Y = \bigcup_{n=1}^{\infty} Y_n$, where each Y_n, $n \in N$ is nowhere dense.

Let $\cup \in T$ be such that $\cup \subseteq Y$. Then $\cup \subseteq \bigcup_{n=1}^{\infty} Y_n \subseteq \bigcup_{n=1}^{\infty} \overline{Y}_n$. So $X \setminus \cup \geq \bigcap_{n=1}^{\infty} (X \setminus \overline{Y}_n)$, and each of the sets $X \setminus \overline{Y}_n$ is open and dense in (X, T). As (X, T) is Baire. $\bigcap_{n=1}^{\infty} (X \setminus \overline{Y}_n)$ is dense in (X, T). So the closed set $X \setminus \cup$ is dense in (X, T). This implies $X \setminus \cup = \overset{n=1}{X}$. Hence $\cup = \phi$. This completes the proof.

13.2 Summary

- A space X is said to be a Baire space if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.
- Let (X, d) be a metric space and $A \subset X$. The set A is called of the first category if it can be expressed as a countable union of non dense sets. The set A is called of the second category if it is not of the first category.

13.3 Keywords

Complete Metric Space: A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

Dense: A said to be dense in X if $\bar{A} = X$.

Nowhere Dense: A is said to be nowhere dense if $(\bar{A})^\circ = \phi$.

13.4 Review Questions

1. Show that if every point x of X has a neighborhood that is a Baire space, then X is a Baire space.

   [***Hint:*** Use the open set formulation of the Baire Condition].
2. Show that every locally compact Hausdorff space is a Baire space.
3. Show that the irrationals are a Baire space.
4. A point x in a topological space (X, T) is said to be an isolated point if $\{x\} \in T$. Prove if (X, T) is a countable T_1-space with no isolated points. Then it is not a Baire space.
5. Let (X, T) be any topological space and Y and S dense subsets of X. If S is also open in (X, T), prove that $S \cap Y$ is dense in both X and Y.
6. Let (X, T) and (Y, T_1) be topological space and $f : (X, T) \to (Y, T_1)$ be a continuous open mapping. If (X, T) is a Baire space. Show that an open continuous image of a Baire space is a Baire space.
7. Let (Y, T_1) be an open subspace of the Baire space (X, T). Prove that (Y, T) is a Baire space. So an open subspace of a Baire space is a Baire space.
8. Let B be a Banach space where the dimension of the underlying vector space is countable. Using the Baire Category Theorem, prove that the dimension of the underlying vector space is, in fact, finite.

13.5 Further Readings

Books

A.V. Arkhangal'skii, V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, Reidel (1984).

J. Dugundji, *Topology*, Prentice Hall of India, New Delhi.

Online link

www.springer.com/978-3642-00233-5

Unit 14: Introduction to Dimension Theory

CONTENTS

Objectives

After studying this unit, you will be able to:

- Know about the dimensional theory;
- Define Hausdorff dimension of measures;
- Define pointwise dimension;
- Solve the problems on the dimensional theory.

Introduction

For many familiar objects there is a perfectly reasonable intuitive definition of dimension: A space is d-dimensional if locally it looks like a patch R^d. This immediately allows us to say: The dimension of a point is zero; the dimension of a line is 1; the dimension of a plane is 2; the dimension of R^d is d.

There are several different notions of dimension for more general sets, some more easy to compute and others more convenient in applications. We shall concentrate on Hausdorff dimension. Hausdorff introduced his definition of dimension in 1919. Further contributions and applications, particularly to number theory, were made by Besicovitch.

Hausdorff's idea was to find the value at which the measurement changes from infinite to zero. Dimension is at the heart of all fractal geometry, and provides a reasonable basis for an invariant between different fractal objects.

14.1 Introduction to Dimension Theory

Before we begin defining Hausdorff and other dimensions, it is a good idea to clearly state our objectives. What should be the features of a good definition of dimension? Based on intuition, we would expect that the dimension of an object would be related to its measurement at a certain scale. For example, when an object is scaled by a factor of 2.

- for a line segment, its measure will increase by $2^1 = 2$
- for a rectangle, its measures will increase by $2^2 = 4$
- for a parallelepiped, its measures will increase by $2^3 = 8$

In each case, we extract the exponent and consider this to be the dimension. More precisely, dim F = log Δμ(F)/log 1/p where p is the precision (1/p is the scaling factor) and Δμ(F) is the change in the 'measure' of F when scaled by 1/p. Falconer suggests that most of following criteria also be net [Falc[2]], by any thing called a dimension:

1. ***Smooth manifolds:*** If F is any smooth, n-dimensional manifold, dim F = n.
2. ***Open Sets:*** For an open subset $F \subset \mathcal{R}^n$, dim F = n.
3. ***Countable Sets:*** dim F = 0 if F is finite or countable.
4. ***Monotonicity:*** $E \subset F \Rightarrow \dim E \le \dim F$.
5. ***Stability:*** $\dim (E \cup F) = \max (\dim E, \dim F)$.
6. ***Countable Stability:*** $\dim\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_i\{\dim F_i\}$.
7. ***Lipschitz Mapping:*** If $f : E \to \mathcal{R}^m$ is lipschitz, then $\dim f(E) \le \dim (E)$.
8. ***Bi-lipschitz Mapping:*** If $f : E \to \mathcal{R}^m$ is Bi-lipschitz, then dim f(E) = dim (E).
9. ***Geometric Invariance:*** dim f(F) = dim F, if f is a similarity or affine transformation.

Recall that $f : E \to \mathcal{R}^m$ is ***Lipschitz*** iff ∃ c such that

$$|f(x) - f(y)| \le c|x - y| \quad \forall\, x, y \in E;$$

and that f is ***Bi-lipschitz*** iff ∃ c_1, c_2 such that

$$c_1|x - y| \le |f(x) - f(y)| \le c_2|x - y| \quad \forall\, x, y \in E;$$

and f is a ***Similarity*** iff ∃ c such that

$$|f(x) - f(y)| = c|x - y| \quad \forall\, x, y \in E;$$

14.1.1 Hausdorff Dimension of Measures

Let μ denote a probability measure on a set of X. We can define the Hausdorff dimension μ in terms of the Hausdorff dimension of subsets of A.

Definition: For a given probability measure μ we define the Hausdorff dimension of the measure by

$$\dim_H(\mu) = \inf\{\dim_H(X) : \mu(X) = 1\}.$$

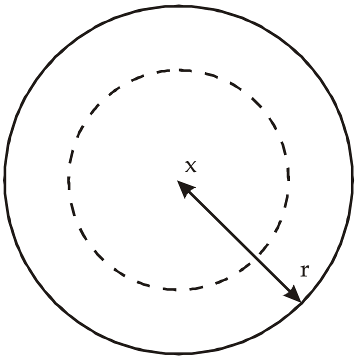
We next want to define a local notion of dimension for a measure μ at a typical point $x \in X$.

14.1.2 Pointwise Dimension

Definition: The upper and lower pointwise dimensions of a measure μ are measurable functions, $\overline{d}_\mu, \underline{d}_\mu : X \to R \cup \{\infty\}$ defined by $\overline{d}_\mu(x) = \lim_{r\to 0}\sup \frac{\log \mu(B(x,r))}{\log r}$ and $\underline{d}_\mu(x) = \lim_{r\to 0}\inf \frac{\log \mu(B(x,r))}{\log r}$

where B(x, r) is a ball of radius r > 0 about x.

Notes



The pointwise dimensions describe how the measure μ is distributed. We compare the measure of a ball about x to its radius r, as r tends to zero.

These are interesting connections between these different notions of dimension for measure.

Theorem 1: If $\underline{d}_\mu(x) \geq d$ for a.e.(μ) $x \in X$ then $\dim_H(\mu) \geq d$.

Proof: We can choose a set of full μ measure $X_o \subset X$ (i.e. $\mu(X_o) = 1$). Such that $\underline{d}_\mu(x) \geq d$ for all $x \in X_o$. In particular for any $\in > 0$ and $x \in X$ we have $\lim\sup_{r\to 0} \mu(B(x, r))/r^{d-\in}$ Fix C > 0 and δ > 0, and let us denote

$$X_\delta = \{x \in X_i : \mu(B(x,r)) \leq C_r^{d-\in}, \forall 0 < r < \delta\}.$$

Let $\{U_i\}$ be any δ-cover for X. Then if $x \in U_i$, $\mu(U_i) \leq C \text{ diam } (U_i)^{d-\in}$. In particular

$$\mu(X_\delta) \leq \sum_{\cup_i \cap X_\delta} \mu(U_i) \leq C\sum_i \text{diam}(U_i)^{d-\in}.$$

Thus, taking the infimum over all such cover we have $\mu X_\delta \leq CH^{d-\in}(X_\delta) \leq CH^{d-\in}(X)$. Now letting $\delta \to 0$ we have that $1 = \mu(X_o) \leq CH^{d-\in}(X)$. Since C > 0 can be chosen arbitrarily large we deduce that $H^{d-\in}(X) = +\infty$. In particular $\dim_H(X) \geq d - \in$ for all $\in > 0$. Since $\in > 0$ is arbitrary, we conclude that $\dim_H(X) \geq d$.

We have the following simple corollary, which is immediate from the definition of $\dim_H(\mu)$.

Corollary: Given a set $X \in \mathcal{R}^d$, assume that there is a probability measure μ with $\mu(X) = 1$ and $\underline{d}_\mu(x) \geq d$ for a. ∈ (μ) $x \in X$. Then $\dim_H(X) \geq d$.

In the opposite direction we have that a uniform bound on pointwise dimensions leads to an upper on the Hausdorff Dimension.

Theorem 2: If $\overline{d}_\mu(x) \leq d$ for a. ∈ (μ) $x \in X$ then $\dim_H(\mu) \leq d$. Moreover, if there is a probability measure μ with $\mu(X) = 1$ and $\overline{d}_\mu(x) \leq d$ for every $x \in X$ then $\dim_H(X) \leq d$.

Proof: We begin with the second statement. For any $\in > 0$ and $x \in X$ we have $\lim\sup_{r\to 0} \mu(B(x, r))/r^{d+\in} = \infty$. Fix C > 0. Given δ > 0, consider the cover μ for X by the balls

$$\{B(x, r) : 0 < r \leq \delta \quad \text{and} \quad \mu(B(x, r)) > C_r^{d+\in}\}.$$

We recall the following classical result.

14.1.3 Besicovitch Covering Lemma

There exists $N = N(d) \geq 1$ such that for any cover by balls we can choose a sub-cover $\{U_i\}$, such that any point x lies in at most N balls.

Thus we can bound

$$H_\delta^{d+\epsilon}(X) \leq \sum_i \text{diam}(U_i)^{d+\epsilon} \leq \frac{1}{C}\sum_i \mu(B_i) \leq \frac{N}{C}.$$

Letting $\delta \to 0$ we have that $H^{d+\epsilon}(X) \leq \frac{N}{C}$. Since $C > 0$ can be chosen arbitrarily large we deduce that $H^{d+\epsilon}(X) = 0$. In particular, $\dim_H(X) \leq d + \epsilon$ for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we deduce that $\dim_H(X) \leq d$.

The proof of the first statement is similar, except that a replace X by a set of full measure for which $\overline{d}_\mu(x) \leq d$.

Example 1: If $L : X_1 \to X_2$ is a surjective Lipschitz map i.e. $C > 0$ such that

$$|L(x) - L(y)| \leq C|x - y|,$$

then $\dim_H(X_1) \leq \dim_H(X_2)$.

Example 2: If $L : X_1 \to X_2$ is a bijective bi-Lipschitz map i.e. $\exists$ $C > 0$ such that

$$\left(\tfrac{1}{C}\right)|x - y| \leq |L(x) - L(y)| \leq C|x - y|,$$

then $\dim_H(X_1) = \dim_H(X_2)$.

Solution: For part 1, consider an open cover $\mathcal{U}$ for X_1 with $\dim(U_i) \leq \epsilon$ for all $U_i \in \mathcal{U}$. Then the images $\mathcal{U}' = \{L(U) : U \in \mathcal{U}\}$ are a cover for X_2 with $\dim(L(U_i)) \leq L_\epsilon$ for all $U \in \mathcal{U}'$. Thus, from the definitions, $H^\delta_{L_\epsilon}(X_2) \geq H^\delta_\epsilon(X_1)$. In particular, letting $\epsilon \to 0$ we see that $H^\delta(X_1) \geq H^\delta(X_2)$. Finally, from the definitions $\dim_H(X_1) \leq \dim_H(X_2)$.

For part 2, we can apply the first part a second time with $\mathcal{L}$ replaced by L^{-1}.

14.1.4 Bernoulli's Measures

Example 3: For an iterated function scheme $T_1, \ldots\ldots, T_k : U \to U$ we can denote as before

$$\Sigma = \left\{\underline{x} = (x_m)_{m=0}^\infty : x_m \in \{1,\ldots,k\}\right\}$$

with the Tychonoff product topology. The shift map $\sigma : \Sigma \to \Sigma$ is a local homeomorphism defined by $(\sigma x)_m = x_{m+1}$. The kth level cylinder is defined by,

$$[x_0,\ldots,x_{k-1}] = \left\{(i_m)_{m=0}^\infty \in \Sigma : i_m = x_m \text{ for } 0 \leq m \leq k-1\right\}$$

(i.e., all sequence which begin with $x_0, \ldots, x_{k-1}$). We denote by $W_k = \{(x_0, \ldots, x_{k-1}]\}$ the set of all kth level cylinders (of which there are precisely k^n).

Notation: For a sequence $\underline{i} \in \Sigma$ and a symbol $r \in \{1, \ldots, k\}$ we denote by $k_r(\underline{i}) = \text{card}\{0 \leq m \leq k-1 : i_m = r\}$ the number of occurrences of r in the first k terms of $\underline{i}$.

Consider a probability vector $\underline{p} = (p_0, \ldots, p_{n-1})$ and define the Bernoulli measure of any level cylinder to be,

$$\mu([i_0,\ldots,i_{k-1}]) = p_0^{k_0(\underline{i})} p_1^{k_1(\underline{i})} \ldots p_{n-1}^{k_{n-1}(\underline{i})}.$$

A ***probability measure*** μ on σ is said to be ***invariant*** under the shift map if for any Borel set $B \subset X$, $\mu(B) = \mu(\sigma^{-1}(B))$. We say that μ is ergodic if any Borel set $B \subseteq \Sigma$ such that $\sigma^{-1}(X) = X$ satisfies $\mu(X) = 0$ or $\mu(X) = 1$. A Bernoulli measure is both invariant and ergodic.

14.2 Summary

- Criteria for defining a dimension
  - (i) When X is a manifold then the value of the dimension is an integer which coincides with the usual notion of dimension;
  - (ii) For more general sets X we can have "fractional" dimensional; and
  - (iii) Points and countable unions of points, have zero dimension.
- For a given probability measure μ, we define the Hausdorff dimension of the measure by

  $$\dim_H(\mu) = \inf \{\dim_H(X) : \mu(X) = 1\}.$$
- The upper and lower pointwise dimensions of a measure μ are measurable functions, $\overline{d}_\mu$, $\underline{d}_\mu : X \to \mathcal{R} \cup \{\infty\}$ defined by

$$\overline{d}_\mu(x) = \lim_{r\to 0} \sup \frac{\log \mu(B(x,r))}{\log r} \text{ and}$$

$$\underline{d}_\mu(x) = \lim_{r\to 0} \inf \frac{\log \mu(B(x,r))}{\log r}$$

14.3 Keywords

Countable Set: A set is countable if it is non-empty and finite or if it is countably infinite.

Hausdorff Space: A topological space (X, T) is called Hausdorff space if given a pair of distinct points $x, y \in X$,

$$\exists\, G, H \in T \quad \text{s.t.} \quad x \in G, y \in H, G \cap H = \phi.$$

Iterated Function Scheme: An iterated function scheme on an open set $\cup \subset R^d$ consists of a family of contractions $T_1, \ldots, T_k : \cup \to \cup$.

Open Set: Any set $A \in T$ is called an open set.

Subcover: Let (X, T) be a topological space and $A \subset X$. Let G denote a family of subsets of X. If $\exists$ $G_1 \subset G$ s.t. G_1 is a finite set and that $\{G : G \in G_1\}$ is a cover of A then G_1 is called a finite subcover of the original cover.

14.4 Review Questions

1. Write a short note on Dimension Theory.
2. State Besicovitch covering lemma.

3. If $\dim_H(X) < d$ then show that the (d-dimensional) Lebesgue measure of X is zero.

4. Let $\Lambda_1, \Lambda_2 \subset \mathcal{R}$ and let

$$\Lambda_1 + \Lambda_2 = \{\lambda_1 + \lambda_2 : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2\}$$

then prove that $\dim_H(\Lambda_1 + \Lambda_2) \leq \dim_H(\Lambda_1) + \dim_H(\Lambda_2)$.

5. If we can find a probability measure μ satisfying the above hypothesis then prove that $\dim_H(X) \geq d$.

14.5 Further Readings

Books

Rogers, M. (1998), *Hausdorff Measures*, Cambridge University Press.

Lapidus, M. (1999), *Math 209A – Real Analysis Mid-term*, UCR Reprographics.

Online links

en.wikipedia.org/wiki/E8-mathematics

en.wikipedia.org/wiki/M-theory