LOVELY PROFESSIONAL UNIVERSITY

MEASURE THEORY AND FUNCTIONAL ANALYSIS

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SYLLABUS

Measure Theory and Functional Analysis

Objectives: This course is designed for the of analysis of various types of spaces like Banach Spaces, Hilbert Space, etc. and also

Sr. No	Description
1	Differentiation and Integration: Differentiation of monotone functions, Functions of bounded variation,
2	Differentiation of an integral, Absolute continuity
3	Spaces, Holder, Minkowski inequalities, Convergence and Completeness
4	Bounded linear functional on the *Lp* spaces, Measure spaces, Measurable Functions, Integration
5	General Convergence Theorems, Signed Measures, Radon-Nikodym theorem.

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Unit 1: Differentiation and Integration: Differentiation of Monotone Functions

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Objectives

After studying this unit, you will be able to:

- Understand differentiation and integration
- Describe Lipschitz condition and Lebesgue point of a function
- State Vitali's Lemma and understand its proof.
- Explain four Dini's derivatives and its properties
- Describe Lebesgue differentiation theorem.

Introduction

Differentiation and integration are closely connected. The fundamental theorem of the integral calculus is that differentiation and integration are inverse processes. The general principle may be interpreted in two different ways:

1. If f is a Riemann integrable function over [a, b], then its indefinite integral i.e.

   F : [a, b] → R defined by F (x) = $\int_a^x f(t)\,dt$ is continuous on [a, b]. Furthermore if f is continuous at a point $x_o \in$ [a, b], then F is differentiable thereat and F′(xo) = f (x_o).

2. If f is Riemann integrable over [a, b] and if there is a differentiable function F on [a, b] such that $F' = f(x)$ for x ∈ [a, b], then

$$\int_a^x f(t)\,dt = F(x) - F(a) \quad [a \le x \le b].$$

Notes

1.1 Differentiation and Integration

1.1.1 Lipschitz Condition

Definition: A function f defined on [a, b] is said to satisfy Lipschitz condition (or Lipschitzian function), if ∃ a constant M > 0 s.t.

$$|f(x) - f(y)| \le M\ |x - y|,\ \forall x, y \in [a, b].$$

1.1.2 Lebesgue Point of a Function

Definition: A point x is said to be a Lebesgue point of the function f (t), if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+n} |f(t) - f(x)|\, dt = 0.$$

1.1.3 Covering in the Sense of Vitali

Definition: A set E is said to be covered in the sense of Vitali by a family of intervals (may be open, closed or half open), M in which none is a singleton set, if every point of the set E is contained in some small interval of M i.e., for each x ∈ E, ∃ and ε > 0, an interval I ∈ M s.t. x ∈ I and $\ell(I) < \varepsilon$.

The family M is called the *Vitali Cover* of set E.

Example: If E = {q : q is a rational number in the interval [a, b]}, then the family $[I_{qi}]$ where $[I_{qi}] = \left[q - \frac{1}{i}, q + \frac{1}{i}\right]$, i ∈ N is a vitali cover of [a, b].

Vitali's Lemma

Let E be a set of finite outer measure and M be a family of intervals which cover E in the sense of Vitali; then for a given ε > 0, it is possible to find a finite family of disjoint intervals $\{I_k, k = 1, 2, \ldots n\}$ of M, such that

$$m^*\left[E - \bigcup_{k=1}^{n} I_k\right] < \varepsilon.$$

Proof: Without any loss of generality, we assume that every interval of family M is a closed interval, because if not we replace each interval by its closure and observe that the set of end points of $I_1, I_2, \ldots\ldots I_n$ has measure zero.

[Due to this property some authors take family M of closed intervals in the definition of Vitali's covering].

Suppose 0 is an open set containing E s.t. m* (0) < m* (E) + 1 < ∞ we assume that each interval in M is contained in 0, if this can be achieved by discarding the intervals of M extending beyond 0 and still the family M will cover the set E in the sense of Vitali.

Now we shall use the induction method to determine the sequence $< I_k : k = 1, 2, \ldots n >$ of disjoint intervals of M as follows:

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Let I_1 be any interval in M and let ℓ_1 be the supremum (least upper bound of the lengths of the intervals in M disjoint from I_1 (i.e., which do not have any point common with I_1).

Obviously $\ell_1 < \infty$ as $\ell_1 \leq m\,(0) < \infty$.

Now we choose an interval I_2 from M, disjoint from I_1, such that $\ell(I_2) > \frac{1}{2}\ell_1$. Let ℓ_2 be the supremums of lengths of all those intervals of M which do not have any point common with I_2 or I_2 obviously $\ell_2 < \infty$.

In general, suppose we have already chosen r intervals $I_1, I_2, \ldots I_r$ (mutually disjoint). Let ℓ_r be the supremums of the length of those intervals of M which do not have any point in common with $\bigcup_{i=1} I_i$ (i.e., which do not meet any of the intervals $I_1, I_2, \ldots I_r$. Then $\ell_r \leq m\,(0) < \infty$.

Now if E is contained in $\bigcup_{i=1}^{r} I_i$, then Lemma established. Suppose $\bigcup_{i=1}^{r} I_i \subset E$. Then we can find interval I_{r+1} s.t. $\ell(I_{r+1}) > \frac{1}{2}\ell_r$ which is disjoint from $I_1, I_2, \ldots I_r$.

Thus at some finite iteration either the Lemma will be established or we shall get an infinite sequence $<I_r>$ of disjoint intervals of M s.t. $\ell(I_{r+1}) > \frac{1}{2}\ell_r$ and $\ell_r < \infty$, n = 1, 2, 3 ….

Note that $< \ell_r >$ is a monotonically decreasing sequence of non-negative real numbers.

Obviously, we have that $\bigcup_{i=1}^{\infty} I_r \subset 0 \Rightarrow \sum_{r=1}^{\infty} \ell(\ell_r) \leq m(0) < \infty$ hence for any arbitrary $\varepsilon > 0$, we can find an integer N s.t.

$$\sum_{r=N+1}^{\infty} \ell(I_r) < \frac{1}{5}\varepsilon .$$

Let a set $F = \bigcup_{r=1}^{N} I_r$.

The lemma will be established if we show that $m^*(F) < \varepsilon$. For, let $x \in F$, then $x \notin \bigcup_{r=1}^{N} I_r \Rightarrow x$ is an element of E not belonging to the closed set $\bigcup_{r=1}^{N} I_r \Rightarrow \exists$ an interval I in M s.t. $x \in I$ and $\ell(I)$ is so small that I does not meet the $\bigcup_{r=1}^{N} I_r$, i.e.

$$I \cap I_r = \phi, \ \forall\, r = 1, 2, \ldots N.$$

Therefore we shall have $\ell(I) \leq \ell_N < 2\ell(I_{n+1})$ as by the method of construction we take $\ell(I_{n+1}) \leq \frac{1}{2}\ell_N$.

It also I $\cap$ I_{N+1} = ϕ, we should have $\ell(I) \le \ell_{N+1}$. Further if the interval I does not meet any of the intervals in the sequence $<I_r>$, we must have

$$\ell(I) \le \ell_r, \forall r$$

which is not true as $\ell_r < 2\ell(I_{r+1}) \to 0$ as $r \to \infty$.

$\Rightarrow$ I must meet at least one of the intervals of the sequence $<I_r>$. Let p be the least integer s.t. I meets I_p. Then p > N and $\ell(I) \le \ell_{p-1} < 2\ell(I_p)$. Further let $x \in$ I as well $x \in I_p$, then the distance of x from the mid point of I_p is at most

$$\ell(I) + \frac{1}{2}\ell(\ell_p) < 2\ell(I_p) + \frac{1}{2}\ell(I_p) = \frac{5}{2}\ell(I_p)$$

Thus if I_p is an interval having the same mid point as I_p but length 5 times the length of I_p, i.e. $\ell(J_p) = 5\ell(I_p)$. Then $x \in J_p$ also.

Thus for every $x \in$ F, $\exists$ an integer p > N s.t. $x \in J_p$

and $\ell(J_p) = 5\ell(I_p)$. Also

$$F \subset \bigcup_{p=N+1}^{\infty} J_p$$

$$\Rightarrow \qquad m^*(F) \le \sum_{p=N+1}^{\infty} \ell(J_p) = 5 \sum_{p=N+1}^{\infty} \ell(I_p) < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

and hence the Lemma holds good.

1.1.4 Four Dini's Derivatives

The usual condition of differentiability of a function f (x) is too strong. Here we are studying the functions under slightly weaker condition (measurability). So why we define four quantities, called as Dini's Derivatives, which may be defined even at the points where the function is not differentiable.

1. $D^+ f(x) = \overline{\lim_{n \to 0_+}} \frac{f(x+h) - f(x)}{h}$, called *upper right derivative*

2. $D_+ f(x) = \underline{\lim_{h \to 0_+}} \frac{f(x+h) - f(x)}{h}$, called *lower right derivative*

3. $D^- f(x) = \overline{\lim_{h \to 0_-}} \frac{f(x+h) - f(x)}{h}$,

   or $\overline{\lim_{h \to 0_+}} \frac{f(x-h) - f(x)}{-h}$, called *upper left derivative*

4. $D_- f(x) = \underline{\lim_{h \to 0_-}} \frac{f(x+h) - f(x)}{h}$

   or $\underline{\lim_{h \to 0_+}} \frac{f(x-h) - f(x)}{-h}$, called *lower left derivative*

Notes

Notes

1. $D^+f(x) \geq D_+ f(x)$ and $\overline{D}f(x) \geq \underline{D}f(x)$

   If $D^+ f(x) = D_+ f(x)$, then we conclude that right hand derivative of f (x) exists at the point x and denoted by $f'(x^+)$. Similarly if $D^- f(x) = D_- f(x)$, we say that f (x) is left differentiable at x and denote this common value by $f'(x^-)$.

2. The function is said to be differentiable at x if all the four Dini's derivatives are equal but different than $\pm \infty$, i.e. if

   $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \pm \infty$

   and their common value is denoted by $f'(x)$.

Properties of Dini's Derivatives

1. Dini's derivatives always exist, may be finite or infinite for every function f.
2. $D^+(f + g) \leq D^+ f + D^+ g$ with similar properties for the other derivatives.
3. If f and g are continuous at a point 'x', then

   $D^+(f \cdot g)(x) \leq f(x) D^+ g(x) + g(x) D^+ f(x)$.
4. $D_+ f(x) = -D^+(-f(x))$

   and $D_- f(x) = -D^-(-f(x))$.
5. If f is a continuous function on [a, b] and one of its derivatives (say D^+) is non-negative on (a, b). Then f is non-decreasing on [a, b] i.e.

   $f(x) \leq f(y)$ whenever $x \leq y$, $y \in [a, b]$.
6. If f is any function on an interval [a, b], then the four derivatives if exist are measurable.

1.1.5 Lebesgue Differentiation Theorem

Statement: Let $f : [a, b] \to R$ be a finite valued monotonically increasing function, then f is differentiable. Also $f : [a, b] \to R$ is L-integrable and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

Proof: Define a sequence $<f_n>$ of non-negative functions, where $f_n : [a, b] \to R$ such that,

$$f_n(x) = n\left[f\left(x + \frac{1}{n}\right) - f(x)\right], \forall x \in [a, b] \quad \ldots (1)$$

and set $f(x) = f(b)$, for $x \geq b$.

By hypothesis, $f : [a, b] \to R$ is an increasing function, therefore $f_n : [a, b] \to R$ is also an increasing function and hence integrable in the Lebesgue sense.

Again from (i) we have

$$\lim_{n\to\infty} f_n(x) = \lim_{1/n \to 0} \frac{f\{x + (1/n)\} - f(x)}{(1/n)}, \forall x \in [a, b],$$

$$= f'(x), \text{ a.e.}$$

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Thus, the sequence <f_n> converges to f′ (x), a.e.

Using Fatou's Lemma, we have

$$\int_a^b f'(x)\,dx \le \lim_{n\to\infty}\inf\left\{\int_a^b f_n(x)\,dx\right\} \quad \dots \text{(ii)}$$

Again

$$\lim_{n\to\infty}\inf\int_a^b f_n(x)\,dx = \lim_{n\to\infty}\inf n\int_a^b\left[f\left(x-\frac{1}{n}\right)-f(x)\right]dx$$

$$= \lim_{n\to\infty}\inf n\left[\int_a^b f\left(x+\frac{1}{n}\right)dx-\int_a^b f(x)\,dx\right]$$

Putting t = x + (1/n), we get

$$\int_a^b f\left(x+\frac{1}{n}\right)dx = \int_{a+(1/n)}^{b+(1/n)} f(t)\,dt = \int_{a+(1/n)}^{b+(1/n)} f(x)\,dx$$

[By the first property of definite integrals]

$$\therefore \quad \lim_{n\to\infty}\inf\int_a^b f_n(x)\,dx = \lim_{n\to\infty}\inf n\left[\int_{a+(1/n)}^{b+(1/n)} f(x)\,dx-\int_a^b f(x)\,dx\right]$$

$$= \lim_{n\to\infty}\inf n\left[\int_b^{b+(1/n)} f(x)\,dx-\int_a^{a+(1/n)} f(x)\,dx\right] \quad \dots \text{(iii)}$$

Now extend the definition of f by assuming

$$f(x) = f(b),\ \forall\, x \in [b, b+1/n].$$

$$\Rightarrow \quad \int_b^{b+(1/n)} f(x)\,dx = \int_b^{b+(1/n)} f(b)\,dx = \frac{1}{n}f(b)$$

Also f (a) ≤ f (x), for $x \in \left(a, a+\frac{1}{n}\right)$, therefore

$$\int_a^{a+(1/n)} f(x)\,dx \ge \int_a^{a+(1/n)} f(a)\,dx = \frac{1}{n}f(a)$$

$$\Rightarrow \quad -\int_a^{a+(1/n)} f(x)\,dx \le -\frac{1}{n}f(a)$$

$$\text{(iii)} \Rightarrow \quad \lim_{n\to\infty}\inf\int_a^b f_n(x)\,dx = \lim_{n\to\infty}\inf n\left[\int_b^{b+(1/n)} f(b)\,dx+\left(-\int_a^{a+(1/n)} f(x)\,dx\right)\right]$$

$$\le \lim_{n\to\infty}\inf n\left[f(b)\cdot\frac{1}{n}+\left(-\frac{1}{n}\right)f(a)\right] \le f(b)-f(a)$$

Thus from (ii), we get

$$\int_a^b f'(x)\,dx \le f(b) - f(a)$$

⇒ f′(x) is integrable and hence finite a.e. thus f is differentiable a.e.

Example: Let f be a function defined by f (0) = 0 and f (x) = x sin (1/x) for x ≠ 0. Find $D^+ f(0)$, $D_+ f(0)$, $D^- f(0)$, $D_- f(0)$.

$$D^+ f(0) = \overline{\lim_{h\to o^+}} \frac{f(0+h)-f(o)}{h} = \overline{\lim_{h\to o}} \frac{h\sin\frac{1}{h}-0}{h}$$

$$= \overline{\lim_{h\to o}} \sin\frac{1}{h} = 1, \text{ as } -1 \le \sin\frac{1}{h} \le 1$$

Also

$$D_+ f(0) = \underline{\lim_{h\to o^+}} \frac{f(0+h)-f(0)}{h} = \underline{\lim_{h\to o^+}} \sin\frac{1}{h} = -1$$

$$D^- f(0) = \overline{\lim_{h\to o^-}} \frac{f(0-h)-f(0)}{0-h} = \overline{\lim_{h\to o}} \frac{(-h)\sin\left(\frac{-1}{h}\right)-0}{-h}$$

$$= \overline{\lim_{h\to o}} -\sin\frac{1}{h} = 1$$

and

$$D_- f(0) = \underline{\lim_{h\to o}} \frac{f(0-h)-f(0)}{-h} = \underline{\lim_{h\to o}} \left(-\sin\frac{1}{h}\right) = -1$$

Theorem: Let x be a Lebesgue point of a function f (t); then the indefinite integral

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

is differentiable at each point x and F′ (x) = f (x).

Proof: Given that x is a Lebesgue point of f (t), so that

$$\lim_{h\leftarrow o} \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt = 0 \qquad \dots \text{(i)}$$

Now

$$\frac{1}{h}\int_x^{x+h} f(x)\,dt = \frac{1}{h} f(x)\int_x^{x+h} 1\,dt = \frac{1}{h} f(x)\,[t]_x^{x+h}$$

$$= \frac{1}{h} f(x).h = f(x)$$

Thus

$$f(x) = \frac{1}{h}\int_x^{x+h} f(x)\,dt \qquad \dots \text{(ii)}$$

Also

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

⇒

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \qquad \dots \text{(iii)}$$

Notes

From (ii) and (iii) we have

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_x^{x+h} f(x)\,dt\right|$$

$$= \left|\frac{1}{h}\int_x^{x+h}\left[f(t)-f(x)\right]dt\right| \le \frac{1}{h}\int_x^{x+h}\left|f(t)-f(x)\right|dt$$

$$\therefore \quad \operatorname*{Lim}_{h\leftarrow o}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| \le \operatorname*{Lim}_{h\to o}\frac{1}{h}\int_x^{x+h}\left|f(t)-f(x)\right|dt \le 0 \qquad \text{[Using (i)]}$$

$$\text{or} \quad \operatorname*{Lim}_{h\to o}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| \le 0 \qquad \ldots \text{(iv)}$$

Since modulus of any quantity is always positive, therefore

$$\operatorname*{Lim}_{h\to o}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| \ge 0 \qquad \ldots \text{(v)}$$

Combining (iv) and (v), we obtain

$$\operatorname*{Lim}_{h\to o}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| = 0$$

$$\Rightarrow \quad \operatorname*{Lim}_{h\to o}\frac{F(x+h)-F(x)}{h} = f(x)$$

$$\Rightarrow \quad F'(x) = f(x).$$

Theorem: Every point of continuity of an integrable function f (t) is a Lebesgue point of f (t).

Proof: Let f (t) be integrable over the closed interval [a, b] and let f (t) be continuous at the point x_o.

f (t) is continuous at t = x_o implies that $\forall\ \varepsilon > 0, \exists$ a $\delta > 0$ such that,

$|f(t) - f(x_o)| < \varepsilon$, whenever $|t - x_o| < \delta$.

$$\Rightarrow \quad \int_{x_o}^{x_o+h}|f(t)-f(x_o)|\,dt < \varepsilon\int_{x_o}^{x_o+h}dt + \varepsilon h \text{ whenever } |h| < \delta.$$

$$\therefore \quad \frac{1}{h}\int_{x_o}^{x_o+h}|f(t)-f(x_o)|\,dt < \varepsilon \qquad \ldots \text{(i)}$$

Now $h \to 0 \Rightarrow \varepsilon \to 0$. So from (i), we have

$$\operatorname*{Lim}_{h\to o}\frac{1}{h}\int_{x_o}^{x_o+h}|f(t)-f(x_o)|\,dt < 0 \qquad \ldots \text{(ii)}$$

$$\text{Now} \quad \operatorname*{Lim}_{h\to o}\left|\frac{1}{h}\int_{x_o}^{x_o+h}|f(t)-f(x_o)|\,dt\right|$$

$$\le \operatorname*{Lim}_{h\to o}\frac{1}{h}\int_{x_o}^{x_o+h}|f(t)-f(x_o)|\,dt \le 0 \qquad \text{[Using (ii)]}$$

Notes

or $\lim_{h \to o} \left| \frac{1}{h} \int_{x_o}^{x_o+h} | f(t) - f(x_o) | dt \right| = 0$ [∵ Modulus of any quantity is always non-negative]

i.e. $\lim_{h \to o} \frac{1}{h} \left| \int_{x_o}^{x_o+h} | f(t) - f(x_o) | dt \right| = 0$.

This shows that x_o is a Lebesgue point of f (t).

1.2 Summary

- A function f defined on [a, b] is said to satisfy Lipschitz condition if ∃ a constant M > 0 such that

$$| f(x) - f(y) \leq M |x - y|, \ \forall \ x, y \in [a, b].$$

- A point x is said to be a Lebesgue point of the function f (t), if

$$\lim_{h \to o} \frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)| dt \leq 0 \quad = 0$$

- Let E be a set of finite outer measure and M be a family of intervals which cover E in the sense of Vitali; then for a given ∈ > 0, it is possible to find a finite family of disjoint intervals

  $\{I_k, k = 1, 2, \ldots, n\}$ of M, such that

$$m^* \left[E - \bigcup_{k=1}^{n} I_k \right] < \in .$$

- Lebesgue differentiation theorem: Let f : [a, b] → R be a finite valued monotonically increasing function, then f is differentiable. Also

  f : [a, b] → R is L-integrable and

$$\int_a^b f'(x)\, dx \leq f(b) - f(a) .$$

1.3 Keywords

Dinni's Derivatives: These are the ways to define the quantities to judge the measurability of the functions even at the points where it is not differentiable.

Fundamental Theorem of the Integral: The fundamental theorem of the integral calculus is that differentiation and integration are inverse processes.

Measurable functions: An extended real valued function f defined over a measurable set E is said to be measurable in the sense of Lebesgue if set

E (f > a) = {x ∈ E : f (x) > a} is measurable for all extended real numbers a.

Vitali's Lemma: Let E be a set of finite outer measure and M be a family of intervals which cover E in the sense of Vitali; then for a given ε > 0, it is possible to find a finite family of disjoint intervals $\{I_k, k = 1, 2, \ldots n\}$ of M, such that

$$m^* \left[E - \bigcup_{k=1}^{n} I_k \right] < \varepsilon.$$

Notes

1.4 Review Questions

1. If the function f assumes its maximum at c, show that $D+ f(c) \leq 0$ and $D_- f(c) \geq 0$.
2. Give an example of functions such that $D^+ (f + g) \neq D^+ f + D^+ g$.
3. Find the four Dini's derivatives of function $f : [0, 1] \to R$

   such that $f(x) = 0$, if $x \in 0$, if $x \in Q$ and $f(x) = 1$, if $x \neq Q$.
4. Evaluate the four Dini's derivative at $x = 0$ of the function f (x) given below:

$$f(x) = \begin{cases} ax \sin^2 \frac{1}{x} + bx \cos^2 \frac{1}{x}, x > 0 \\ px \sin^2 \frac{1}{x} + qx \cos^2 \frac{1}{x}, x < 0 \end{cases}$$

   and $f(0) = 0$, given that $a < b$, $p < q$.
5. Every point of continuity of an integrable function f (t) is a Lebesgue point of f (t). Elucidate.

1.5 Further Readings

Books

J. Yeh, *Real Analysis: Theory of Measure and Integration*

Bartle, Robert G. (1976). *The Elements of Real Analysis* (second edition ed.)

Online links

www.solitaryroad.com/c756.html

www.public.iastate.edu/.../Royden_Real_Analysis_Solutions.pdf

Unit 2: Functions of Bounded Variation

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Objectives

After studying this unit, you will be able to:

- Define absolute continuous function.
- Define monotonic function.
- Understand functions of bounded variation.
- Solve problems on functions of bounded variation.

Introduction

Functions of bounded variation is a special class of functions with finite variation over an interval. In Mathematical analysis, a function of bounded variation, also known as a BV function, is a real-valued function whose total variation is bounded: the graph of a function having this property is well behaved in a precise sense. Functions of bounded variation are precisely those with respect to which one may find Riemann – Stieltjes integrals of all continuous functions.

In this unit, we will study about absolute continuous function, Monotonic function and functions of bounded variation.

2.1 Functions of Bounded Variation

2.1.1 Absolute Continuous Function

A real-valued function f defined on [a,b] is said to be absolutely continuous on [a,b], if for an arbitrary $\in > 0$, however small, $\exists$ a,$\delta > 0$, such that

$$\sum_{r=1}^{n} \left| f(b_r) - f(a_r) \right| < \in, \text{wherever} \sum_{r=1}^{n} (b_r - a_r) < \delta,$$

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where $a_1 < b_1 \le a_2 < b_2 \le ... \le a_n < b_n$ i.e., a_i's and b_i's are forming finite collection $\{(a_i, b_i): i = 1,2,...,n\}$ of pair-wise disjoint (non-overlapping) intervals (or of disjoint closed intervals).

Obviously, every absolutely continuous function is continuous.

Notes

- If a function satisfies $\left|\sum \{f(b_r) - f(a_r)\}\right| < \varepsilon$, even then it is absolutely continuous.
- The condition $\sum_{r=1}^{n} (b_r - a_r) < \delta$, means that total length of all the intervals must be less than δ.

2.1.2 Monotonic Function

Recall that a function f defined on an interval I is said to be monotonically non-increasing, iff

$$x > y \Rightarrow f(x) \le f(y), \forall x, y \in I$$

and monotonically non-decreasing, iff

$$x > y \Rightarrow f(x) \ge f(y), \forall x, y \in I$$

Also f is said to be strictly decreasing, iff

$$x > y \Rightarrow f(x) < f(y)$$

and strictly increasing, iff

$$x > y \Rightarrow f(x) > f(y)$$

2.1.3 Functions of Bounded Variation – Definition

Let f be a real-valued function defined on [a,b] which is divided by means of points

$$a = x_0 < x_1 < x_2 < ... < x_n = b.$$

Then the set $P = \{x_0, x_1, x_2, ..., x_n\}$ is termed as subdivision or partition of [a,b].

Let us take $\overset{b}{\underset{a}{V}}(f,P) = \sum_{r=0}^{n-1} \left|f(x_{r+1}) - f(x_r)\right|$, and $\overset{b}{\underset{a}{V}}(f,P) = \sup \overset{b}{\underset{a}{V}}(f,P)$ for all possible subdivisions P of [a,b]. (This is called total variation of f over [a,b] and also denoted by $\overset{b}{\underset{a}{T}}(f)$).

Notes

If $\overset{b}{\underset{a}{V}}(f)$ is finite, then f is called a function of bounded variation or function of finite variation over [a,b].

Set of all the functions of bounded variation on [a,b] is denoted by BV [a,b].

> *Notes*
>
> If f is defined on R, then we define
>
> $$\overset{\infty}{\underset{-\infty}{V}}(f) = \underset{a\to\infty}{\text{lin}}\overset{a}{\underset{-a}{V}}(f).$$

Some important observations about the functions of bounded variations.

Let f: $[a,b] \to R$ and P be any subdivision of [a,b]. Then:

(i) $|f(x) - f(a)| \leq \overset{x}{\underset{a}{V}}(f), x \in [a,b]$

(ii) $\overset{a}{\underset{a}{V}}(f) = 0$

(iii) $P_1 \subset P_2 \Rightarrow \overset{b}{\underset{a}{V}}(f,P_1) \leq \overset{b}{\underset{a}{V}}(f,P_2)$, where P_1 and P_2 are any two subdivisions of [a,b].

(iv) $\overset{b}{\underset{a}{V}}(f,P) \leq \overset{b}{\underset{a}{V}}(f)$, for all subdivisions P of [a,b].

(v) For each $\varepsilon > 0$, however small, $\exists$ at least one subdivision P' of [a,b] such that

$$\overset{b}{\underset{a}{V}}(f,P') + \varepsilon > \overset{b}{\underset{a}{V}}(f).$$

(vi) $\overset{b}{\underset{a}{V}}(f) \geq 0.$

(vii) $a < b < c \Rightarrow \overset{b}{\underset{a}{V}}(f) \leq \overset{c}{\underset{a}{V}}(f).$

2.1.4 Theorems and Solved Examples

Theorem 1: A monotonic function on [a,b] is of bounded variation.

Proof: Divide the interval [a,b] by means of points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

without any loss of generality, we can take f(x) as increasing function on [a,b]. Since if f is a decreasing function, –f is an increasing function and so by taking –f = g, we see that g is an increasing function and so we are allowed to consider only increasing functions. Thus

$$x_r < x_{r+1} \Rightarrow f(x_r) \leq f(x_{r+1})$$

$$\Rightarrow f(x_{r+1}) - f(x_r) \geq 0$$

$$\Rightarrow |f(x_{r+1}) - f(x_r)| = f(x_{r+1}) - f(x_r) \quad \text{...(i)}$$

Now $V = \sum_{r=0}^{n-1} \left| f(x_{r+1}) - f(x_r) \right| = \sum_{r=0}^{n-1} \left\{ f(x_{r+1}) - f(x_r) \right\}$[using (i)]

$\therefore \quad V = f(x_n) - f(x_0) = f(b) - f(a).$

Now f is monotonic $\Rightarrow f(b)$ and $f(a)$ are finite quantities.

$\Rightarrow V = a$ finite quantity independent of the mode subdivision. Hence f is of bounded variation.

Notes

If f is a monotonic function on [a,b], then

$$\mathop{T}_{a}^{b}(f) = \left| f(b) - f(a) \right|$$

Theorem 2: Let V, P, N denote total, positive and negative variations of a bounded function f on [a,b]; then prove that

V = P+N, and P–N= f(b)–f(a).

Proof: Let the interval [a,b] be divided by means of points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

$$v = \sum_{r=0}^{n-1} \left| f(x_{r+1}) - f(x_r) \right|$$

If P denotes the sum of those differences $f(x_{r+1}) - f(x_r)$ which are +n for positive and –n for negative, then obviously,

$$v = p + n,\ f(b) - f(a) = p - n \qquad \text{...(i)}$$

Let $\quad P = \sup p, V = \sup v, N = \sup n, \qquad$ ...(ii)

where suprema are taken over all subdivisions of [a,b]. From (i), we have

$$v = 2p + f(a) - f(b), \qquad \text{...(iii)}$$

$$v = 2n + f(b) - f(a). \qquad \text{...(iv)}$$

Taking supremum in (iii) and (iv) and using (ii), we get

$$V = 2P + f(a) - f(b), \qquad \text{...(v)}$$

$$V = 2N + f(b) - f(a). \qquad \text{...(vi)}$$

By adding and subtracting, (v) and (vi) give

V = P+N and f(b) – f(a) = P–N.

Theorem 3: If f_1 and f_2 are non-decreasing functions on [a,b], then $f_1 - f_2$ is of bounded variation on [a,b].

Proof: Let $f = f_1 - f_2$ defined on [a,b].

Then for any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$, we have

Notes

$$\sum\left|f(x_i)-f(x_{i-1})\right| \le \sum\left|f_1(x_i)-f_1(x_{i-1})\right| + \sum\left|f_2(x_i)-f_2(x_{i-1})\right|$$

$$\le \left[f_1(b)-f_1(a)\right]+\left[f_2(b)-f_2(a)\right]$$

as f_1 and f_2 are monotonically increasing.

$\Rightarrow \quad \overset{b}{\underset{a}{V}}(f) \le f_1(b)+f_2(b)-f_2(a)-f_1(a)$, which is a finite quantity.

$\Rightarrow \quad \overset{b}{\underset{a}{V}}(f) < \infty$ and hence f is of bounded variation.

Theorem 4: If $f \in BV\ [a,b]$ and $c \in (a,b)$, then $f \in BV\ [a,c]$ and $f \in BV\ [c,b]$. Also

$$\overset{b}{\underset{a}{V}}(f) = \overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f)$$

Proof: Since $f \in BV[a,b]$ and $[a,c] \subset [a,b]$ we get

$$\overset{c}{\underset{a}{V}}(f) < \overset{b}{\underset{a}{V}}(f) < \infty$$

$\Rightarrow f \in BV\ [a,c]$ and similarly $f \in BV\ [c,b]$.

Now if P_1 and P_2 are any subdivisions of [a,c] and [c,b] respectively, then $P = P_1 \cup P_2$ is a subdivision of [a,b].

$$\Rightarrow \overset{c}{\underset{a}{V}}(f,P_1) + \overset{b}{\underset{c}{V}}(f,P_2) = \overset{b}{\underset{a}{V}}(f,P) \le \overset{b}{\underset{a}{V}}(f).$$

But P_1 and P_2 are any subdivisions. So taking supremums on P_1 and P_2, we get

$$\overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f) \le \overset{b}{\underset{a}{V}}(f). \qquad ...(1)$$

Now let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a subdivision of [a,b] and $c \in [x_{r-1}, x_r]$

$\Rightarrow P_1 = \{x_0, x_1, x_2, ..., x_{r-1}, c\}$ and

$P_2 = \{c, x_r, x_{r+1}, x_{r+2}, ..., x_n\}$ are the subdivisions of [a,c] and [c,b] respectively.

$$\text{Now } \overset{b}{\underset{a}{V}}(f,P) = \sum_{i=1}^{r-1}\left|f(x_i)-f(x_{i-1})\right| + \left|f(x_r)-f(x_{r-1})\right| + \sum_{i=r+1}^{n}\left|f(x_i)-f(x_{i-1})\right|$$

$$= \sum_{i=1}^{r-1}\left|f(x_i)-f(x_{i-1})\right| + \left|f(x_r)-f(c)+f(c)-f(x_{r-1})\right| + \sum_{i=r+1}^{n}\left|f(x_i)-f(x_{i-1})\right|$$

$$\le \left[\sum_{i=1}^{r-1}\left|f(x_i)-f(x_{i-1})\right| + \left|f(c)-f(x_{r-1})\right|\right] + \left[\left|f(x_r)-f(c)\right| + \sum_{i=r+1}^{n}\left|f(x_i)-f(x_{i-1})\right|\right]$$

$$\le \overset{c}{\underset{a}{V}}(f,P_1) + \overset{b}{\underset{c}{V}}(f,P_2) \le \overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f)$$

(i) and (ii) $\Rightarrow \overset{b}{\underset{a}{V}}(f) = \overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f).$ ...(ii)

Notes

Notes

- This theorem enables us to define a new function (called variation function) say

  $V(x) = \overset{x}{\underset{a}{V}}(f), \forall\ x \in [a,b]$.

- If x > y in [a,b], then $\overset{y}{\underset{a}{V}}(f) = \overset{x}{\underset{a}{V}}(f) + \overset{y}{\underset{x}{V}}(f)$.

  i.e. v(y) = v(x) + $\overset{y}{\underset{x}{V}}(f)$.

  $\Rightarrow v(x)$ is an increasing function.

- If $a < c_1 < c_2 < ... < c_n < b$, then

  $\overset{b}{\underset{a}{V}}(f) = \overset{c_1}{\underset{a}{V}}(f) + \overset{c_2}{\underset{c_1}{V}}(f) + ... + \overset{b}{\underset{c_n}{V}}(f)$

Corollary:

$f \in BV[a,b] \Leftrightarrow f \in BV[a,c]$,

$f \in BV[c,b]$ for each $c \in [a,b]$.

Theorem 5: If a function f of bounded variation in [a,b] is continuous at $c \in [a,b]$, then the function defined by v(x) = $\overset{x}{\underset{a}{V}}(f)$, is also continuous at x = c and vice versa.

Proof: Suppose f is continuous at x = c. Hence for arbitrary $\in / 2 > 0$, we can find a δ_1 such that

$$a \leq c - \delta_1 < x < c \text{ or } |x - c| < \delta_1 \Rightarrow |f(x) - f(c)| < \in / 2 \qquad ...(i)$$

Also we know by remark (v) after the definition (2.1.3), for above $\in$, we can get a subdivision

P = $\{a = x_0, x_1, x_2, ..., x_n = c\}$ of [a,c]

$$\text{s.t. } \overset{c}{\underset{a}{V}}(f) < \overset{c}{\underset{a}{V}}(f,P) + \frac{\in}{2} \qquad ...(ii)$$

Now choosing positive $\delta > \min[\delta_1, c - x_{n-1}]$, we get that for any x such that $c - \delta < x < c$, we also have $x_{n-1} < x < x_n$.

$$\text{(ii)} \Rightarrow \overset{c}{\underset{a}{V}}(f) < \sum_{r=1}^{n-1} |f(x_r) - f(x_{r-1})| + |f(x_n) - f(x_{n-1})| + \frac{\in}{2}$$

$$< \sum_{r=1}^{n-1} |f(x_r) - f(x_{r-1})| + |f(x_n) - f(x) + f(x) - f(x_{n-1})| + \frac{\in}{2}$$

Notes

$$< \sum_{r=1}^{n-1}\left|f(x_r)-f(x_{r-1})\right|+\left|f(x)-f(x_{n-1})\right|+\left|f(x_n)-f(x)\right|+\frac{\in}{2}$$

$$< \overset{x}{\underset{a}{V}}(f)+\left|f(c)-f(x)\right|+\frac{\in}{2}$$

$\Rightarrow \overset{c}{\underset{a}{V}}(f) < \overset{x}{\underset{a}{V}}(f)+\in,$ [by (i)]

$\Rightarrow 0 = \overset{c}{\underset{a}{V}}(f)-\overset{x}{\underset{a}{V}}(f) <\in,$

$\Rightarrow \forall x$ s.t. $c-\delta < x < c,$ we have $v(c)-v(x) <\in$

$\Rightarrow \lim\limits_{x\to c-0} v(x) = v(c).$

$\Rightarrow v(x)$ is continuous on the left at x = c.

Similarly considering the partition of [c,b], one can show that v(x) is right continuous also at x = c and hence v(x) is also continuous at x = c.

Converse of the above Theorem

If v(x) is continuous at $x = c =\in [a,b]$ so is f also at x – c.

Proof: Since v(x) is continuous at x = c, for arbitrary small $\in > 0, \exists$ a $\delta > 0$ such that

$$\left|v(x)-v(c)\right| <\in, x \in (c-\delta, c+\delta) \qquad ...(i)$$

Now let $c < x < c+\delta.$ Then by Note (ii) of Theorem 4, we get

$$\overset{x}{\underset{a}{V}}(f) = \overset{c}{\underset{a}{V}}(f)+\overset{x}{\underset{c}{V}}(f)$$

$\Rightarrow \qquad v(x) = v(c)+\overset{x}{\underset{c}{V}}(f)$

$\Rightarrow \qquad v(x)-v(c) = \overset{x}{\underset{c}{V}}(f) \geq \left|f(x)-f(c)\right|$

$\Rightarrow \qquad \left|f(x)-f(c)\right| \leq \left|v(x)-v(c)\right| <\in$ [by (i)] ...(ii)

Similarly, we can show that $\left|f(c)-f(x)\right| <\in,$ if $c-\delta < x < c.$ ...(iii)

(ii) and (iii) show that f(x) is also continuous at x = c.

Theorem 6: Let f and g be functions of bounded variation on [a,b] ; then prove that f+g, f-g, fg and $f/g \ \left(\left|g(x)\right| \geq \sigma > 0, \forall x\right)$ and cf are functions of bounded variation, c being constant.

Proof:

(i) Set f + g = h, then

$$\left|h(x_{r+1})-h(x_r)\right| = \left|\left[f(x_{r+1})+g(x_{r+1})\right]-\left[f(x_r)+g(x_r)\right]\right|$$

where $a = x_0 < x_1 < x_2 < ... < x_n = b$

Notes

$= \left|\left[f(x_{r+1}) - f(x_r)\right] + \left[g(x_{r+1}) - g(x_r)\right]\right|$

$\le \left|\left[f(x_{r+1}) - f(x_r)\right] + \left[g(x_{r+1}) - g(x_r)\right]\right|.$

$\therefore \sum_{r=0}^{n-1} \left|h(x_{r+1}) - h(x_r)\right| \le \sum_{r=0}^{n-1} \left|h(x_{r+1}) - h(x_r)\right| + \sum_{r=0}^{n-1} \left|g(x_{r+1}) - g(x_r)\right|.$

or $\overset{b}{\underset{a}{V}}(h) \le \overset{b}{\underset{a}{V}}(f) + \overset{b}{\underset{a}{V}}(g).$

Now by hypothesis, f, g are functions of bounded variations.

$\Rightarrow \quad \overset{b}{\underset{a}{V}}(f) \text{ and } \overset{b}{\underset{a}{V}}(g) \text{ are finite.}$

$\Rightarrow \quad \overset{b}{\underset{a}{V}}(h) = \text{a finite quantity.}$

Hence h = f + g is of bounded variation in [a,b].

(ii) Let h = f – g. Then as above,

$\left|h(x_{r+1}) - h(x_r)\right| \le \left|\left[f(x_{r+1}) - f(x_r)\right] + \left[g(x_{r+1}) - g(x_r)\right]\right|.$

$\Rightarrow \quad \overset{b}{\underset{a}{V}}(h) \le \overset{b}{\underset{a}{V}}(f) + \overset{b}{\underset{a}{V}}(g)$

$\Rightarrow \quad \overset{b}{\underset{a}{V}}(h) = \text{a finite quantity.}$

Hence h = f – g is of bounded variation in [a,b].

(iii) Let h(x) = f(x).g(x). Then

$\left|h(x_{r+1}) - h(x_r)\right| = \left|f(x_{r+1}).g(x_{r+1}) - f(x_r).g(x_r)\right|$

$= \left|f(x_{r+1}).g(x_{r+1}) - f(x_r)g(x_{r+1}) + f(x_r)g(x_{r+1}) - f(x_r)g(x_r)\right|$

$\le \left|g(x_{r+1})\left[f(x_{r-1}) - f(x_r)\right]\right| + \left|f(x_r)\left[g(x_{r+1}) - g(x_r)\right]\right|.$

$\le \left|g(x_{r+1})\right|\left|f(x_{r-1}) - f(x_r)\right| + \left|f(x_r)\right|.\left|g(x_{r+1}) - g(x_r)\right|.$

Let A = $\sup\left\{\left|f(x)\right| : x \in [a,b]\right\},$

B = $\sup\left\{\left|g(x)\right| : x \in [a,b]\right\},$

$\therefore \left|h(x_{r+1}) - h(x_r)\right| \le B.\left|f(x_{r+1}) - f(x_r)\right| + A.\left|g(x_{r+1}) - g(x_r)\right|.$

$\therefore \sum_{r=0}^{n-1} \left|h(x_{r+1}) - h(x_r)\right| \le B.\sum_{r=0}^{n-1} \left|h(x_{r+1}) - h(x_r)\right| + A \cdot \sum_{r=0}^{n-1} \left|g(x_{r+1}) - g(x_r)\right|.$

Notes

i.e. $\overset{b}{\underset{a}{V}}(h) \leq B.\overset{b}{\underset{a}{V}}(f) + A.\overset{b}{\underset{a}{V}}(g).$

= a finite quantity.

Hence h(x) = f(x).g(x) is of bounded variation in [a,b].

(iv) First, we shall show that 1/g is of bounded variation, where $g(x) \geq \sigma > 0, \forall x \in [a,b]$.

Now, $g(x) \geq \sigma > 0, \forall x \in [a,b]$

$$\Rightarrow \frac{1}{g(x)} \leq \frac{1}{\sigma} > 0, \forall x \in [a,b].$$

Again, we observe that

$$\left|\frac{1}{g(x_{r+1})} - \frac{1}{g(x_r)}\right| = \left|\frac{g(x_r) - g(x_{r+1})}{g(x_r).g(x_{r+1})}\right| \leq \frac{1}{\sigma^2}\left|g(x_r) - g(x_{r+1})\right|$$

$$\therefore \sum_{r=0}^{n-1}\left|\frac{1}{g(x_{r+1})} - \frac{1}{g(x_r)}\right| \leq \frac{1}{\sigma^2}\sum_{r=0}^{n-1}\left|g(x_r) - g(x_{r+1})\right|$$

$$\Rightarrow \overset{b}{\underset{a}{V}}\left(\frac{1}{g}\right) \leq \frac{1}{\sigma^2}\overset{b}{\underset{a}{V}}(g) = \text{a finite quantity.}$$

Hence $\frac{1}{g}$ is of bounded variation in [a,b].

Now f and $\frac{1}{g}$ are of bounded variation in [a,b].

$\Rightarrow f.\frac{1}{g}$ is of bounded variation in [a,b] [by case (iii)]

$\Rightarrow \frac{f}{g}$ is of bounded variation in [a,b].

(v) Do yourself. Note that $\overset{b}{\underset{a}{V}}(cf) = |c|\overset{b}{\underset{a}{V}}(f).$

> *Notes*
>
> Since BV [a,b] is closed for all four algebraic operations, it is a linear space.

Theorem 7: Every absolutely continuous function f defined on [a,b] is of bounded variation.

Proof: Since f is absolutely continuous on [a,b]; for $\in = 1, \exists$ a $\delta > 0$

s.t. $\sum_{i=1}^{n} \left| f(b_i) - f(a_i) \right| < 1,$

whenever $\sum_{i=1}^{n} (b_i - a_i) < \delta,$

and $a = a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n = b.$

Now consider another subdivision of [a,b] or say refinement of P by adjoining some additional points to P in such a way that all the intervals can be divided into r parts each of total length less than δ.

Let the r sub-intervals be $[c_0,c_1]$, $[c_1,c_2]$,...,$[c_{r-1},c_r]$ such that

$$a = c_0,\ c_r = b \text{ and } (c_{k+1} - c_k) < \delta, \forall K = 0,1,2,\ldots,(r-1)$$

Obviously, $\sum_{i} \left| f(x_{i+1}) - f(x_{i+1}) \right| < 1,$ where $x_{i+1}, x_i. \in [c_k, c_{k+1}]$

or $\quad \overset{c_{k+1}}{\underset{c_k}{V}} (f) < 1,$ $\qquad$ [Using (i)]

Hence $\overset{b}{\underset{a}{V}}(f) = \overset{c_1}{\underset{c_0}{V}}(f) + \overset{c_2}{\underset{c_1}{V}}(f) + \ldots + \overset{c_r}{\underset{c_{r-1}}{V}}(f) < 1 + 1 + 1 + \ldots + 1 = r =$ finite quantity.

Hence, f is of bounded variation.

> *Notes*
>
> Converse of above theorem is not necessarily true. These exists functions of bounded variation but not absolutely continuous.

Theorem 8: Jordan Decomposition Theorem

A function f is of bounded variation, if and only if it can be expressed as a difference of two monotonic functions both non-decreasing.

Proof: Let f be the function of $f : [a,b] \to R$.

Case I. $f \in BV[a,b]$. Then we can write

$$f = v - (v - f), \qquad \ldots(i)$$

so that $f(x) = v(x) - (v(x) - f(x)), x \in [a,b].$

Now if x, y ∈ [a, b] such that x < y, then by the remark (ii) of theorem 4, we get

$$\overset{y}{\underset{a}{V}}(f) = \overset{x}{\underset{a}{V}}(f) + \overset{y}{\underset{x}{V}}(f).$$

$$\Rightarrow v(y) - v(x) = \overset{y}{\underset{x}{V}}(f) \ge 0$$

$\Rightarrow v(x) \le v(y)$ and hence v is a non-decreasing function on [a,b].

Notes

Again, if x<y in [a,b], then as above

$$v(y)-v(x)=\overset{y}{\underset{x}{V}}(f)\geq|f(y)-f(x)|\geq f(y)-f(x)$$

$$\Rightarrow v(y)-f(y)\geq v(x)-f(x)\Rightarrow(v-f)y\geq(v-f)x$$

$\Rightarrow v-f$ is also a non-decreasing function on [a,b].

Thus (i) shows that f is expressible as a difference of two monotonically non-decreasing functions.

Case II. Set g (x) and h (x) be increasing functions such that f(x) = g(x) – h(x).

Divide the closed interval [a,b] by means of points

$a = x_0 < x_1 < x_2 < ... < x_n = b$.

Let $V=\sum_{r=0}^{n-1}|f(x_{r+1})-f(x_r)|$

Now, we have that

$$\begin{aligned}|f(x_{r+1})-f(x_r)| &= |g(x_{r+1})-h(x_{r+1})-\{g(x_r)-h(x_r)\}| \\ &= |[g(x_{r+1})-g(x_r)]+[h(x_r)-h(x_{r+1})]| \\ &\leq |g(x_{r+1})-g(x_r)|+|h(x_r)-h(x_{r+1})| \\ &\leq |g(x_{r+1})-g(x_r)|+|h(x_{r+1})-h(x_r)|\end{aligned}$$

Now, g(x) and h(x) are monotonically increasing functions, so that $g(x_{r+1})-g(x_r)\geq 0$ and $h(x_{r+1})-h(x_r)\geq 0$

$$\Rightarrow |g(x_{r+1})-g(x_r)|=g(x_{r+1})-g(x_r)$$

and $|h(x_{r+1})-h(x_r)|=h(x_{r+1})-h(x_r)$.

Hence $|f(x_{r+1})-f(x_r)|\leq[g(x_{r+1})-g(x_r)]+[h(x_{r+1})-h(x_r)]$

$$\therefore \sum_{r=0}^{n-1}|f(x_{r+1})-f(x_r)|\leq\sum_{r=0}^{n-1}[g(x_{r+1})-g(x_r)]+\sum_{r=0}^{n-1}[h(x_{r+1})-h(x_r)]$$

Now $\sum_{r=0}^{n-1}[g(x_{r+1})-g(x_r)]=[g(x_1)-g(x_0)]+[g(x_2)-g(x_1)]+.....+....+[g(x_n)-g(x_{n-1})]$

$$\begin{aligned}&= g(x_n)-g(x_0) \\ &= g(b)-g(a) \qquad (\because\ x_n=b, x_0=a)\end{aligned}$$

Similarly, $\sum_{r=0}^{n-1}[h(x_{r+1})-h(x_r)]=h(b)-h(a)$.

Hence $\sum_{r=0}^{n-1}[f(x_{r+1})-f(x_r)]\leq g(b)-g(a)+h(b)-h(a)$.

Notes

since f is finite in [a,b] Now $\Rightarrow g(b), g(a) h(b), h(a)$ are finite numbers.

$$\therefore \sum_{r=0}^{n-1}\left[f(x_{r+1}) - h(x_r)\right] < \infty$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(f) < \infty.$$

$\Rightarrow$ f is a function of bounded variation. Alternatively, since g (x) and h(x) are both non-decreasing, so by theorem 3, g(x) – h(x) and hence f(x) is of bounded variation.

Corollary: A continuous function is of bounded variation iff it can be expressed are as a difference of two continuous monotonically increasing functions. It follows from the results of Theorems 5 and 8.

Theorem 9: An indefinite integral is a function of bounded variation, i.e. if $f \in L[a,b]$ and $F(x)$ is indefinite integral of $f(x)$ i.e. F (x) = $\int_a^x f(t)dt,$ then $F \in BV[a,b].$ Also show that

$$\overset{b}{\underset{a}{V}}(f) \le \int_a^x |f|.$$

Proof: Since $f \in L[a,b],$ also $|f| \in L[a,b].$

Let P = $\{x_i : i = 0,1,2,...,n\}$ be a subdivision of the interval [a,b]. Then

$$\sum_{r=0}^{n}\left|F(x_i) - F(x_{i-1})\right| = \sum_{i=1}^{n}\left|\int_a^{x_i} f - \int_a^{x_{i-1}} f\right|$$

$$= \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} f\right| \le \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |f|$$

$$= \int_a^b |f| < \infty.$$

$$\Rightarrow f \in BV[a,b] \text{ and } \overset{b}{\underset{a}{V}}(f,p) \le \int^b |f|.$$

Further above result is true for any subdivision of P of [a,b]. Therefore taking supremum, we get

$$\int_a^b (f) \le \int_a^b |f|.$$

Example: A function f of bounded variation on [a,b] is necessarily bounded on [a,b] but not conversely.

Solution: If $x \in [a,b]$, then $|f(x)-f(a)| \geq \overset{x}{\underset{a}{V}}(f) \leq \overset{b}{\underset{a}{V}}(f) < \infty$

$$\Rightarrow -\overset{b}{\underset{a}{V}}(f) \leq f(x)-f(a) \leq \overset{b}{\underset{a}{V}}(f)$$

$$\Rightarrow f(a)-\overset{b}{\underset{a}{V}}(f) \leq f(x) \leq \overset{b}{\underset{a}{V}}(f)+f(a)$$

$\Rightarrow f(x)$ is bounded on [a,b]

For the converse, define the function f on [0,1] by

$$f(x)=\begin{cases}0, \text{if } x=0\\ x.\sin\left(\dfrac{\pi}{x}\right), \text{if } 0<x\leq 1\end{cases}$$

since $0 \leq x \leq 1$ and $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$, the function f is obviously bounded. Now consider the partition

$$P=\left\{0, \frac{2}{2n+1}, \frac{2}{2n-1}, \ldots, \frac{2}{7}, \frac{2}{5}, \frac{2}{3}, 1\right\} \text{of } [0,1]$$

Where $n \in N.$ Then we get

$$\overset{1}{\underset{0}{V}}(f,P)=\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|+\ldots+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+\left|f(1)-f\left(\frac{2}{3}\right)\right|$$

$$=\left|\frac{2}{2n+1}(-1)^n-0\right|+\ldots+\left|\frac{2}{3}(-1)-\frac{2}{5}.1\right|+\left|0-\frac{2}{3}(-1)\right|$$

$$=\frac{2}{2n+1}+\ldots+\left(\frac{2}{3}+\frac{2}{5}\right)+\frac{2}{3}$$

$$=4.\left(\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n+1}\right).$$

But we know that series $\sum \frac{1}{2n+1}$ is divergent. Therefore letting $n \to \infty$ we get that

$$\overset{1}{\underset{0}{V}}(f)=\underset{n\to\infty}{\text{lt}}\overset{1}{\underset{0}{V}}(f,P)=\infty$$

$\Rightarrow$ f is not of bounded variation.

Example: Show that the function

$$f(x)=\begin{cases} x\sin\frac{\pi}{x}\text{ if }(0<x\le 2) \\ 0,\text{ if } x=0\end{cases}\text{ is continuous}$$

without being of bounded variation.

or

show that there exists a continuous function without being of bounded variation.

Solution: We know that $\underset{x\to 0}{\text{lt}}\, f(x)=0=f(0)$

$\Rightarrow f(x)$ is continuous but not of bounded variation (see converse of above example.)

Hence the result.

Problem: Show that if f′ exists and is bounded on [a, b], then f ∈ BV [a, b].

Solution: According to given, let $|f'| \le M$ on [a, b].

Then for any X_{i-1}, $x_i \in$ [a, b], we get

$$\left|\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right|\le M \Rightarrow |f(x_i)-f(x_{i-1})|\le M(x_i-x_{i-1})$$

⇒ for any partition P of [a, b],

$$\overset{b}{\underset{a}{V}}(f)\le M\sum(x_i-x_{i-1})=M(b-a)$$

⇒ f ∈ B V [a, b].

Problem: Show that the function f defined as

$$f(x)=x^p\sin\frac{1}{x}\text{ for }0<x\le 1,\ f(o)=0,\ p\ge 2.$$

is of bounded variation [0, 1].

Solution: Note that RF′(0) $= \underset{h\to o}{\text{Lim}}\dfrac{(0+h)^p\sin\frac{1}{h}-0}{h}$

$$= \underset{h\to o}{\text{Lim}}\, h^{(p-1)}\sin\frac{1}{h}=0$$

and $Lf'(0)=\lim\limits_{h\to o}\dfrac{(-h)^p\sin\left(-\frac{1}{h}\right)-0}{-h}=0$

Notes

$\Rightarrow$ $f'(0) = 0$ and $f'(x) = x^p \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + px^{p-1}\sin\frac{1}{x}$

$\Rightarrow$ $f'(x) = x^{p-2}\left[px\sin\frac{1}{x} - \cos\frac{1}{x}\right]$, for $0 < x \le 1$

$\Rightarrow$ $f'(x)$ is bounded for $0 \le x \le 1$.

According to above problem, $f \in BV\,[0, 1]$.

2.2 Summary

- A real-valued function f defined on [a,b] is said to be absolutely continuous on [a,b], if for an arbitrary $\in > 0$, however small, $\exists\ a, \delta > 0$, s.t.

  $$\sum_{r=1}^{n} \left|f(b_r) - f(a_r)\right| < \in \text{ whenever} \sum_{r=1}^{n} (b_r - a_r) < \delta,$$

  where $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$

- A function f defined on an interval I is said to be monotonically non-increasing, iff

  $x > y \Rightarrow f(x) \le f(y), \forall x, y \in I.$

  and monotonically non-decreasing, iff $x > y \Rightarrow f(x), \ge f(y)\ \forall\ x, g \in I$.

- Let $\overset{b}{\underset{a}{V}}(f,P) = \sum_{r=0}^{n-1} \left|f(x_{r+1}) - f(x_r)\right|$, and $\overset{b}{\underset{a}{V}}(f) = \text{Sup}\overset{b}{\underset{a}{V}}(f,P)$ for all possible subdivisions P of [a,b]. If $\overset{b}{\underset{a}{V}}(f)$ is finite, then f is called a function of bounded variation over [a,b].

2.3 Keywords

Absolute Continuous Function: A real valued function f defined on [a, b] is said to be absolutely continuous on [a, b], if for an arbitrary $\varepsilon > 0$, however small, $\exists$ a, $\delta > 0$, such that

$$\sum_{r=1}^{n} |f(b_r) - f(a_r)| < \varepsilon, \text{ wherever} \sum_{r=1}^{n} (b_r - a_r) < \delta$$

where $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ i.e. a_1's and b_1's are forming finite collection $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$ of pair-wise disjoint intervals.

Continuous: A continuous function is a function $f : X \to Y$ where the pre-image of every open set in Y is open in X.

Disjoint: Two sets A and B are said to be disjoint if they have no common element, i.e. $A \cap B = \phi$.

Monotonic Decreasing Function: A monotonic decreasing function is a function that either decreases or remains the same, never increases i.e. a function f(x) such that $f(x_2) \leq f(x_1)$ for $x_2 > x_1$.

Monotonic Function: A monotonic function is a function that is either a monotonic increasing or monotonic decreasing.

Monotonic Increasing Function: A monotonic increasing function is a function that either increases or remains the same, never decreases i.e. a function f(x) such that $f(x_2) \geq f(x_1)$ for $x_2 > x_1$.

2.4 Review Questions

1. Show that sum and product of two functions of bounded variation is again a function of bounded variation.
2. Show that the function f defined on [0,1] by

   $$f(x) = \begin{cases} x\cos\left(\frac{\pi x}{2}\right) \text{for } 0 < x \leq 1 \\ 0 \text{ for } x = 0 \end{cases}$$

   is continuous but not of bounded variation on [0,1].
3. Show that the function f defined on [0,1] as f(x) = $x\sin\left(\frac{\pi}{x}\right)$ for x > 0, f(0)=0 is continuous but is not of bounded variation on [0,1].
4. Define a function of bounded variation on [a,b]. Show that every increasing function on [a,b] is of bounded variation and every function of bounded variation on [a,b] is differentiable on [a,b].
5. Show that a continuous function may not be of bounded variation.
6. Show that a function of bounded variation may not be continuous.
7. If f is a function such that its derivative f′ exists and is bounded. Then prove that the function f is of bounded variation.

2.5 Further Readings

Books

Halmos, Paul (1950), *Measure Theory*, Van Nostrand and Co.

Kolmogorov, Andrej N.; Fomin, Sergej V. (1969). *Introductory Real Analysis*, New York: Dovers Publications.

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Unit 3: Differentiation of an Integral

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define differentiation of an integral.
- Solve problems related to it.

Introduction

If f is an integrable function on [a, b], we define its indefinite integral to be the function F defined on [a, b] by

$$F(x) = \int_a^x f(t)\,dt$$

Here, it is shown that the derivative of the indefinite integral of an integrable function is equal to the integrand almost everywhere. We begin by establishing some lemmas.

3.1 Differentiation of an Integral

If f is an integrable function on [a, b] then f is integrable on any interval $[a, x] \subset [a, b]$. The function F given by

$$F(x) = \int_a^x f(t)\,dt + c,$$

where c is a constant, called the indefinite integral of f.

Lemma 1: If f is integrable on [a, b] then the indefinite integral of f namely the function F on [a, b] given by $F(x) = \int_a^x f(t)$ is a continuous function of bounded variation on [a, b].

Proof: Let x_o be any point of [a, b].

Notes

Then $$|F(x)-F(x_o)| = \left|\int_a^x f(t)\,dt - \int_a^{x_o} f(t)\,dt\right|$$

$$= \left|\int_a^x f(t)\,dt + \int_{x_o}^a f(t)\,dt\right|$$

$$= \left|\int_{x_o}^a f(t)\,dt + \int_a^x f(t)\,dt\right|$$

$$= \left|\int_{x_o}^x f(t)\,dt\right|$$

$$\leq \left|\int_{x_o}^x |f(t)|\,dt\right|$$

But f is integrable on [a, b]

$\Rightarrow$ $|f|$ is integrable on [a, b]

[Since we know that measurable function f is integrable over E iff $|f|$ is integrable over E]

$\therefore$ Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that for every measurable set $A \subset [a, b]$ with m (A) < δ, we have $\int_A |f| < \varepsilon$ by theorem, "if f is a non-negative function which is integrable over a set E, then given $\varepsilon > 0$, there is a $\delta > 0$ such that for every set $A \subset E$ with m (A) < δ, $\int_A f < \varepsilon$."

$\Rightarrow$ $\left|\int_{x_o}^x |f(t)|\,dt\right| < \varepsilon$, for $|x - x_o| < \delta$.

$\Rightarrow$ $|F(x) - F(x_o)| = \left|\int_{x_o}^x f(t)\,dt\right| \leq \left|\int_{x_o}^x |f(t)|\,dt\right| < \in$

whenever $|x - x_o| < \delta$.

$\Rightarrow$ $|F(x) - F(x_o)| < \varepsilon$ wherever $|x - x_o| < \delta$

$\Rightarrow$ F is continuous at x_o and hence in [a, b].

Now we shall show that F is a function of bounded variation.

Let P = $\{a = x_o < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of [a, b].

Then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} f(t)\,dt\right|$$

Notes

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,dt$$

$$= \int_a^b |f(t)|\,dt$$

$$\Rightarrow \qquad T_a^b(F) \leq \int_a^b |f(t)|\,dt$$

But $|f|$ is integrable therefore.

$$\int_a^b |f|\,dt < \infty$$

$$\Rightarrow \qquad T_a^b(F) < \infty$$

$$\Rightarrow \qquad F \in BV[a, b]$$

Hence the Proof.

Theorem 1: Let f be an integrable on [a, b].

If $\int_a^x f(t)\,dt = 0 \;\forall\; x \in [a, b]$ then f = 0 a.e. in [a, b].

Proof: Let if possible, $f \neq 0$ a.e. in [a, b].

Let f (t) > 0 on a set E of positive measure, then there exists a closed set $F \subset E$ with m (F) > 0.

Let A = (a, b) – F.

Then A is an open set.

Now $\int_a^b f(t)\,dt = \int_{A \cup F} f(t)\,dt$

But $\int_a^b f(t)\,dt = 0$

$$\Rightarrow \qquad \int_{A \cup F}^{b} f(t)\,dt = 0$$

$$\Rightarrow \qquad \int_A f(t)\,dt + \int_F f(t)\,dt = 0$$

$$\Rightarrow \qquad \int_A f(t)\,dt + \int_F f(t)\,dt = 0 \Rightarrow \int_A f(t)\,dt = -\int_F f(t)\,dt$$

Notes

But f (t) > 0 on F with m (F) > 0 implies

$$\int_F f(t)\,dt \neq 0$$

Therefore $\int_A f(t)\,dt \neq 0$

Now, A being as open set, it can be expressed as a union of countable collection $\{(a_n, b_n)\}$ of disjoint open intervals as we know that an open set can be expressed as a union of countable collection of disjoint open intervals.

Thus $\int_A f(t)\,dt = \sum_n \int_{a_n}^{b_n} f(t)\,dt$

But $\int_A f(t)\,dt \neq 0$

$\Rightarrow$ $\sum_n \int_{a_n}^{b_n} f(t)\,dt \neq 0$

$\Rightarrow$ $\int_{a_n}^{b_n} f(t)\,dt \neq 0$ for some n

$\Rightarrow$ either $\int_a^{a_n} f(t)\,dt \neq 0$

Or $\int_a^{b_n} f(t)\,dt \neq 0$

In either case, we see that if f is positive on a set of positive measure, then for some x ∈ [a, b] we have

$$\int_a^x f(t)\,dt \neq 0\,.$$

Similarly if f is negative on a set of positive measure we have

$$\int_a^x f(t)\,dt \neq 0\,.$$

But it leads to the contradiction of the given hypothesis. Hence our supposition is wrong.

f = 0 a.e. in [a, b].

Hence the proof.

Theorem 2: First fundamental theorem of calculus statement: If f is bounded and measurable on [a, b] and F (x) = $\int_a^x f(t)\,dt + F(a)$, then F′(x) = f (x) a.e. in [a, b].

Proof: Since every indefinite integral is a function of bounded variation, therefore F (x) is a function of bounded variation over [a, b]. Thus F (x) can be expressed as a difference of two monotonic functions and since every monotonic function has a finite differential coefficient at every point of a set of non-zero measure, therefore F (x) has a finite differential coefficient a.e. in [a, b]. Now F is given to be bounded;

∴ $$|f| \leq M \text{ (say)} \quad \ldots (1)$$

Let $$f_n(x) = \frac{F(x+h) - F(x)}{h}.$$

with h = $\frac{1}{x}$.

Then $$|f_n(x)| = \left|\frac{1}{h}(F(x+h) - F(x)\right|$$

$$= \left|\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)\right|$$

$$= \left|\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right)\right|$$

$$= \left|\frac{1}{h}\left(\int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt\right)\right|$$

$$= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt\right|$$

But $$|f| \leq M$$

∴ $$|f_n(x)| \leq \frac{M}{h}\int_x^{x+h} dt = \frac{M}{h}(x + h - x)$$

⇒ $$|f_n(x)| \leq \frac{M}{h}(h)$$

⇒ $$|f_n(x)| \leq M$$

Since f_n (x) → F′ (x) a.e.,

Notes

then the bounded convergence theorem implies that

$$\int_a^x F'(x)\,dx = \lim_{h\to\infty}\int_a^x F_x(x)\,dx$$

$$= \lim_{h\to 0}\frac{1}{h}\int_a^x [F(x+h)-F(x)]\,dx$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\int_x^{x+h}[F(x)dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx\right]$$

$$= F(x) - F(a)$$

$$= \int_a^x f(t)\,dt, \text{ by hypothesis}$$

$$\left[\because F(x) = \int_a^x f(t)\,dt + F(a) \Rightarrow F(x) - F(a) = \int_a^x f(t)\,dt\right]$$

or $\int_a^x [F'(t) - f(t)]\,dt = 0, \forall x$

$\Rightarrow$ $F'(x) - f(x) = 0$ a.e. in [a, b]

Hence $F'(x) = f(x)$ a.e. in [a, b] by the theorem, "If f is integrable on [a, b] and $\int_a^x f(t)\,dt = 0, \forall x \in [a, b]$ then f = 0 a.e. in [a, b]".

Hence $F'(x) = f(x)$ a.e. in [a, b].

Hence the proof.

Theorem 3: If f is an integrable function on [a, b] and if $F(x) = \int_a^x f(t)\,dt + F(a)$ then $F'(x) = f(x)$ a.e. in [a, b].

Proof: Without loss of generality, we may assume that $f(x) \geq 0\ \forall x$

Let us define a sequence $\{f_n\}$ of functions

$f_n : [a, b] \to R$, where

$$f_n(x) = \begin{cases} f(x) \text{ if } f(x) \leq n, \\ n \text{ if } f(x) > n \end{cases}$$

Clearly, each f_n is bounded and measurable function and so, by the theorem,

Let f be a bounded and measurable function defined on [a, b]. If $F(x) = \int_a^x f(t)\,dt + F(a)$, then $F'(x) = f(x)$ a.e. in [a, b]", we have

$$\frac{d}{dx}\int_a^x f_n = f_n(x) \text{ a.e.}$$

Notes

Also, $f - f_n > 0$ $\forall$ n, and therefore, the function G_n defined by

$$G_n(x) = \int_a^x (f - f_n)$$

is an increasing function of x, which must have a derivative almost everywhere by Lebesgue theorem and clearly, this derivative must be non-negative.

Since
$$G_n(x) = \int_a^x (f - f_n)$$
$$= \int_a^x f(t)\,dt - \int_a^x f_n(t)\,dt$$

$$\Rightarrow \quad \int_a^x f(t)\,dt = G_n(x) + \int_a^x f_n(t)\,dt$$

Now the relation

$$F(x) = \int_a^x f(t)\,dt + F(a) \text{ becomes}$$

$$F(x) = G_n(x) = \int_a^x f_n(t)\,dt + F(a),$$

$$\Rightarrow \quad F'(x) = G_n'(x) + f_n(x) \text{ a.e.}$$
$$\geq f_n(x) \text{ a.e. } \forall n.$$

since n is arbitrary, we have

$$F'(x) \geq f(x) \text{ a.e.}$$

$$\Rightarrow \quad \int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx \qquad \ldots (1)$$

Also by the Lebesgue's theorem, i.e. "Let f be an increasing real-valued function defined on [a, b].

Then f is differentiable a.e. and the derivative f′ is measurable.

and $\int_a^b f'(x)\,dx \leq f(b) - f(a)$", we have

$$\int_a^b F'(x)\,dx \leq F(b) - F(a) \qquad \ldots (2)$$

But $$F(x) = \int_a^b f(t)\,dt + F(a)$$

$$\Rightarrow \quad F(b) - F(a) = \int_a^b f(x)\,dx$$

Therefore (2) becomes

$$\int_a^b F'(x)\,dx \le \int_a^b f(x)\,dx \qquad \ldots (3)$$

From (1) and (3), we get

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad \int_a^b F'(x)\,dx - \int_a^b f(x)\,dx = 0$$

$$\Rightarrow \quad \int_a^b \left[F'(x) - f(x)\right] dx = 0$$

since $F'(x) - f(x) \ge 0$ a.e., which gives that

$F'(x) - f(x) = 0$ a.e. and

so $F'(x) = f(x)$ a.e.

3.2 Summary

- If f is an integrable function on [a, b] then f is integrable on any interval $[a, x] \subset [a, b]$. The function F given by

  $$F(x) = \int_a^x f(t)\,dt + c,$$

  where c is a constant, called the indefinite integral of F.

- Let f be an integrable on [a, b]. If $\int_a^x f(t)\,dt = 0 \;\forall\, x \in [a, b]$ then f = 0 a.e. in [a, b].

3.3 Keyword

Differentiation of an Integral: If f is an integrable function on [a, b] then f is integrable on any interval $[a, x] \subset [a, b]$. The function F given by

$$F(x) = \int_a^x f(t)\,dt + c,$$

where c is a constant, called the indefinite integral of f.

Notes

3.4 Review Questions

1. If f is an integrable function on [a, b] and if $F(x) = \int_a^x f(t)\,dt + F(a)$ then check whether $F'(x) = f(x)$ is absolute continuous function in [a, b] or not.

2. If F is an absolutely continuous function on [a, b], then prove that $F(x) = \int_a^x f(t)\,dt + C$ where $f = F'$ a.e. on [a, b] and C is constant.

3.5 Further Readings

Books

Flanders, Harley. *Differentiation under the Integral Sign*

Frederick S. Woods, *Advanced Calculus*, Ginn and Company

David V. Widder, *Advanced Calculus*, Dover Publications Inc., New Edition (Jul 1990).

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Unit 4: Absolute Continuity

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Absolute Continuous function.
- Solve problems on absolute continuity
- Understand the proofs of related theorems.

Introduction

It may happen that a continuous function f is differentiable almost everywhere on [0,1], its derivative f′ is Lebesgue integrable, and nevertheless the integral of f′ differs from the increment of f. For example, this happens for the Cantor function, which means that this function is not absolutely continuous. Absolute continuity of functions is a smoothness property which is stricter than continuity and uniform continuity.

4.1 Absolute Continuity

4.1.1 Absolute Continuous Function

A real-valued function f defined on [a,b] is said to be absolutely continuous on [a,b], if for an arbitrary $\in > 0$, however small, $\exists\ a, \delta > 0$, such that

$$\sum_{r=1}^{n} \left| f(b_r) - f(a_r) \right| < \in \text{ whenever } \sum_{r=1}^{n} (b_r - a_r) < \delta,$$

where $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ i.e. a_i's and b_i's are forming finite collection $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$ of pair-wise disjoint intervals.

Obviously, every absolutely continuous function is continuous.

Notes

Notes

1. If a function satisfied $\left|\sum f(b_r) - f(a_r)\right| < \in$, even then it is absolutely continuous.

2. The condition $\sum_{r=1}^{n}(b_r - a_r) < \delta$ means that total length of all the intervals must be less than δ.

4.1.2 Theorems and Solved Examples

Theorem 1: Every absolutely continuous function f defined on [a,b] is of bounded variation.

Proof: Since f is absolutely continuous on [a,b]; for $\in = 1$, $\exists$ a $\delta > 0$ such that

$$\sum_{r=1}^{n}\left|f(b_i) - f(a_i)\right| < 1,$$

whenever $\sum_{r=1}^{n}(b_i - a_i) < \delta,$

and $a = a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n = b.$

Now consider another subdivision of [a,b] or say refinement of P by adjoining some additional points to P in such a way that all the intervals can be divided into r parts each of total length less than δ.

Let the r-sub-intervals be $[c_0, c_1], [c_1, c_2], \ldots, [c_{r-1}, c_r]$ such that

$a = c_0, c_r = b$ and $(c_{k+1} - c_k) < \delta, \forall k = 0, 1, 2, \ldots, (r-1)$

Obviously, $\sum_i \left|f(x_{i+1}) - f(x_i)\right| < 1,$

where $x_{i+1}, x_i \in [c_k, c_{k+1}]$

or $\overset{c_{k+1}}{\underset{c_k}{V}}(f) < 1,$

Hence $\overset{b}{\underset{a}{V}}(f) = \overset{c_1}{\underset{c_0}{V}}(f) + \overset{c_2}{\underset{c_1}{V}}(f) + \ldots + \overset{c_r}{\underset{c_{r-1}}{V}}(f) < 1 + 1 + \ldots + 1 = r =$ finite quantity.

Hence f is of bounded variation.

Notes

Converse of above theorem is not necessarily true. There exists functions of bounded variation but not absolutely continuous.

Theorem 2: Let f(x) and g(x) be absolutely continuous functions, then prove that $f(x) \pm g(x)$ and $f(x).g(x)$ are also absolutely continuous functions. Hence show that $\frac{f(x)}{g(x)}\left(\text{if}\left|g(x)>0, \forall x\right|\right)$ is also absolutely continuous function.

Proof: Given $f(x)$ and $g(x)$ are absolutely continuous functions on the closed interval [a,b], therefore for each $\in > 0$, there exists $\delta > 0$ such that

$$\sum_{r=1}^{n}\left|f(b_r) - f(a_r)\right| < \in \text{ and}$$

$$\sum_{r=1}^{n}\left|g(b_r) - g(a_r)\right| < \in,$$

whenever $\sum_{r=1}^{n}(b_r - a_r) < \delta$, for all the points $a_1, b_1, a_2, b_2, ..., a_n, b_n$ such that

$a_1 < b_1 \leq a_2 < b_2 \leq ... \leq a_n < b_n$.

(i) We have, $\sum_{r=1}^{n}\left|\left[f(b_r) \pm g(b_r)\right] - \left[f(a_r) \pm g(a_r)\right]\right| \leq \sum_{r=1}^{n}\left|f(b_r) - f(a_r)\right| + \sum_{r=1}^{n}\left|g(b_r) - g(a_r).\right|$

Now if $\sum_{r=1}^{n}(b_r - a_r) < \delta$, then

$$\sum_{r=1}^{n}\left|f(b_r) - f(a_r)\right| < \frac{\in}{2} \text{ and} \sum_{r=1}^{n}\left|g(b_r) - g(a_r)\right| < \frac{\in}{2}.$$

$$\therefore \sum_{r=1}^{n}\left|\left[f(b_r) \pm g(b_r)\right] - \left[f(a_r) \pm g(a_r)\right]\right| < \frac{\in}{2} + \frac{\in}{2} = \in,$$

whenever $\sum_{r=1}^{n}(b_r - a_r) < \delta$.

This show that $\left[f(x) \pm g(x)\right]$ are also absolutely continuous functions over [a,b].

Notes

(ii) We have $\sum_{r=1}^{n}\left|f(b_r)g(b_r)-f(a_r)g(a_r)\right|$

$$=\sum_{r=1}^{n}\left|f(b_r)g(b_r)-f(b_r)g(a_r)+f(b_r)g(a_r)-f(a_r)g(a_r)\right|$$

$$=\sum_{r=1}^{n}\left|f(b_r)\left[g(b_r)-g(a_r)\right]+g(a_r)\left[f(b_r)-f(a_r)\right]\right|$$

$$\leq\sum_{r=1}^{n}\left|f(b_r)\left[g(b_r)-g(a_r)\right]\right|+\sum_{r=1}^{n}\left|g(a_r)\left[f(b_r)-f(a_r)\right]\right|$$

$$\leq\sum_{r=1}^{n}\left|f(b_r)\right|\left|g(b_r)-g(a_r)\right|+\sum_{r=1}^{n}\left|g(a_r)\right|\left|f(b_r)-f(a_r)\right|.$$

Now every absolutely continuous function is bounded therefore f(x) and g(x) are bounded in the closed interval [a,b].

Let $\left|f(x)\right|\leq K_1, \left|g(x)\right|\leq K_2, \forall x\in[a,b]$.

Then we have

$$\sum_{r=1}^{n}\left|f(b_r)g(b_r)-f(a_r)g(a_r)\right|\leq\left|K_1\right|\in+\left|K_2\right|\in=\in\left(\left|K_1\right|+\left|K_2\right|\right),$$

Whenever $\sum_{r=1}^{n}\left|(b_r-a_r)\right|<\delta.$

Setting $\in\left(\left|K_1\right|+\left|K_2\right|\right)= \ \in^*,$

We have $\sum_{r=1}^{n}\left|f(b_r)g(b_r)-f(a_r)g(a_r)\right|< \ =\in^*,$

Whenever $\sum_{r=1}^{n}(b_r-a_r)<\delta,$

where $a_1<b_1\leq a_2<b_2\leq\ldots\leq a_n<b_n;$

$\Rightarrow$ Product of two absolutely continuous functions is also absolutely continuous.

(iii) We have $\left|g(x)>0\forall x\in[a,b]\right|$; therefore

$$\left|g(x)\right|\geq\rho, \text{where } \rho>0, \forall x\in[a,b].$$

Now, $\sum_{r=1}^{n}\left|\frac{1}{g(b_r)}-\frac{1}{g(a_r)}\right| = \sum_{r=1}^{n}\left|\frac{g(a_r)-g(b_r)}{g(b_r)g(a_r)}\right| < \frac{\in}{\rho^2},$

Whenever $\sum_{r=1}^{n}(b_r - a_r) < \delta$. Setting $\frac{\in}{\rho^2} = \in *$, we get

$$\sum_{r=1}^{n}\left|\frac{1}{g(b_r)}-\frac{1}{g(a_r)}\right| < \in *.$$

This show that $\frac{1}{g(x)}$ is absolutely continuous function over [a,b].

Now f(x), $\frac{1}{g(x)}$ are absolutely continuous.

$\Rightarrow f(x).\frac{1}{g(x)}$ is absolutely continuous.

$\Rightarrow \frac{f(x)}{g(x)}$ is also absolutely continuous over [a,b].

Hence the theorem is true.

Note

By Theorem 1, its remark and above theorem it follows that set of all absolutely continuous functions on [a,b] is a proper subspace of the space BV [a,b] of all functions of bounded variation on [a,b].

Theorem 3: If $\in BV[a,b]$, then f is absolutely continuous on [a,b], iff the variation function $v(x) = \overset{x}{\underset{a}{V}}(f)$ is absolutely continuous on [a,b].

Proof: Case I: Given v(x) is absolutely continuous.

$\Rightarrow$ For arbitrary $\in >0, \exists\ \delta > 0$ s.t.

$$\sum_{r=1}^{n}\left|v(b_r)-v(a_r)\right| < \in, \text{whenever } \sum_{r=1}^{n}(b_r - a_r) < \delta.$$

Also, we know that $\left|f(x)-f(a)\right| \le \overset{x}{\underset{a}{V}}(f) = v(x)$

$$\Rightarrow \sum_{r=1}^{n} \left|f(b_r) - f(a_r)\right| = \sum_{r=1}^{n} \left|f(b_r) - f(a) + f(a) - f(a_r)\right|$$

$$= \sum_{r=1}^{n} \left[\left|f(b_r) - f(a)\right| + \left|f(a) - f(a_r)\right|\right]$$

Now taking supremum over all collections of P_i of $[a_i, b_i]$ for i = 2,...,n, we get

$$\sum_{r=1}^{n} \overset{b_i}{\underset{a_i}{V}}(f) < \in.$$

But $\overset{b_i}{\underset{a}{V}}(f) = \overset{a_i}{\underset{a}{V}}(f) + \overset{b_i}{\underset{a_i}{V}}(f)$

$$\Rightarrow \overset{b_i}{\underset{a_i}{V}}(f) = \overset{b_i}{\underset{a}{V}}(f) + \overset{a_i}{\underset{a}{V}}(f)$$

$$\Rightarrow \overset{b_i}{\underset{a_i}{V}}(f) = v(b_i) - v(a_i)$$

$$\Rightarrow \sum_{i=1}^{n} \left|v(b_i) - v(a_i)\right| < \in$$

$\Rightarrow v(x)$ is absolutely continuous.

Theorem 4: A necessary and sufficient condition that a function should be an indefinite integral is that it should be absolutely continuous.

Proof: Condition is sufficient.

Let f(x) be an absolutely continuous function over the closed interval [a,b].

Therefore f is of bounded variation and hence we can express f(x) as

$$f(x) = f_1(x) - f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are monotonically increasing functions and hence both are differentiable.

$$\le \sum_{r=1}^{n} \left|v(b_r) - v(a_r)\right| < \in \text{ whenever} \sum_{r=1}^{n} (b_r - a_r) < \delta$$

$\Rightarrow$ f is also absolutely continuous on [a,b].

Case II: Given f is absolutely continuous on [a,b].

$\Rightarrow$ for a given $\in > 0, \exists$ a $\delta > 0$ s.t.

$$\sum_{i=1}^{n} \left|f(b_i) - f(a_i)\right| < \in, \qquad \text{...(i)}$$

Notes

for every finite collection $P\{]a_i, b_i[, i = 1,2,...,n\}$ of pairwise disjoint sub-intervals of [a,b] such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Now, let $P_i\{]x_{k-1}^i, bx_{ki}^i[, k = 1,...,m_i\}$ be a finite collection of non-overlapping intervals of the interval $[a_i, b_i]$.

Then the collection $\{]x_{k-1}^i, bx_{ki}^i[: i = 1,2,...,n, k = 1,...,m_i\}$ is a finite collection of non-overlapping sub-intervals of [a,b] such that

$$\sum_{i=1}^{n}\left[\sum_{k=1}^{m_i}\left(x_k^i - x_{k-1}^i\right)\right] = \sum_{i=1}^{n}(b_i - a_i) < \delta.$$

and hence by (i), $\sum_{i=1}^{n}\sum_{k=1}^{m_i}\left|f\left(x_k^i\right) - f\left(x_{k-1}^i\right)\right| = \in.$

Hence f′(x) exists and $|f'(x)| \le f_1'(x) + f_2'(x)$

$\Rightarrow \int_a^b |f'(x)| \le f_1(b) + f_2(b) - f_1(a) - f_2(a) < \infty,$

$\Rightarrow f'(x)$ is integrable also.

Now let F(x) be an definite integral of f′(x) i.e.

$$F(x) = F(a) + \int_a^x f'(t)dt,\ x \in [a,b] \qquad ...(ii)$$

Using fundamental theorem of integral calculus,

We get

F′(x) = f′(x)

or F(x) = f(x) + constant (say c) ...(iii)

From (ii), we have F(a) = f(a),

Using this in (iii), we get c = 0 and hence F (x) = f(x).

Thus every absolutely continuous function f(x) is an indefinite integral of its own derivative.

Condition is necessary: Let f(x) be an indefinite integral of f(x) defined on the closed interval [a,b], so that

$F(x) = \int_a^x f(t)dt + f(a), \forall x \in [a,b]$ and f(x) is integrable over [a,b].

Corresponding to arbitrary small $\in > 0$, let δ >0 be such that if $m(A) < \delta$, then $\int_A |f| < \in$,

Notes

Now select 2n real numbers such that

$a_i < b_i \le a_2 < b_2 \le a_3 < \ldots \le a_n < b_n$

such that A = $\bigcup_{i=1}^{n} [a_i, b_i]$ and $\sum_{i=1}^{n} (b_i - a_i) < \delta$.

Then $\sum_{i=1}^{n} |F(b_i) - F(a_i)| = \sum_{i=1}^{n} \left| \int_a^{b_i} f - \int_a^{a_i} f \right|$

$$= \sum_{i=1}^{n} \left| \int_{a_i}^{b_i} F \right| \le \sum_{i=1}^{n} \left| \int_{a_i}^{b_i} f \right| = \int_A |f| < \in.$$

Thus, we have shown that for arbitrary small $\in > 0, \exists$ a $\delta > 0$ s.t. $\sum_{i=1}^{n} (b_i - a_i) < \delta$.

$$\Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \in.$$

$\Rightarrow$ F is absolutely continuous.

Thus every indefinite integral is absolutely continuous.

Theorem 5: If a function f is absolutely continuous in an interval [a,b] and if f′(x) = 0. a.e. in [a,b], then f is constant.

Proof: Let $c \in [a, b]$ be arbitrary. If we show that f(c) = f(a), then the theorem will be proved.

Let E = $\{x \in]a, c[: f'(x) = 0\}$.

since c is arbitrary, therefore set $E \subset]a, c[$. This implies any $x \in E \Rightarrow f'(x) = 0$.

Let $\in$, η > 0 arbitrary. Now $f'(x) = 0, \forall x \in E \Rightarrow \exists$ an arbitrary small interval $[x, x+h] \subset [a, c]$

such that $\frac{|f(x+h) - f(x)|}{h} < \eta \Rightarrow |f(x+h) - f(x)| < \eta h.$

This implies that corresponding to every $x \in E, \exists$ an arbitrary small closed interval $[x, x+h]$ contained in [a,c] s.t.

$$|f(x+h) - f(x)| < \eta h.$$

Thus the interval $[x, x+h], \forall x \in E,$ over E in Vitali's sense. Thus by Vitali's Lemma, we can determine a finite number of non-overlapping intervals I_k, where

$$I_k = [x_k, y_k] \forall\ k = 1, 2, 3, \ldots, n$$

such that this collection covers all of E except for a set of measure less than $\delta > 0$ where δ is pre-assigned number which corresponds to $\in$ occurring in the definition of absolute continuity of f.

Suppose $x_k < x_{k+1}$; then adjoining the points y_0, x_{n+1}.

We have $a = y_0 \le x_1 < y_1 \le x_2 < y_2 \le ... \le x_n < y_n \le x_{n+1} = c$.

Now since f is absolutely continuous, therefore for above subdivision of [a,c], we have

$$\sum_{k=0}^{n} \left|f(x_{k+1}) - f(y_k)\right| < \in, \text{ whenever} \sum_{k=0}^{n} (x_{k+1} - y_k) < \delta.$$

(i) $$\Rightarrow \sum_{k=1}^{n} \left|f(y_k) - f(x_k)\right| \le n \sum_{k=1}^{n} (y_k - x_k) < n(c-a).$$

$$\text{Now } |f(c) - f(a)| = \left|\sum_{k=0}^{n}\left[f(x_{k+1}) - f(y_k)\right] + \sum_{k=1}^{n}\left[f(y_k) - f(x_k)\right]\right|$$

$$\le \sum_{k=0}^{n}\left|f(x_{k+1}) - f(y_k)\right| + \le \sum_{k=1}^{n}\left|f(y_k) - f(x_k)\right|$$

$$< \in + n(c-a)$$

But $\in, n$ and hence $\in + n(c-a)$ are arbitrary small positive numbers. So letting $\in \to 0, n \to 0$

We get f(c) = f(a)

$\Rightarrow f(x)$ is a constant function.

Corollary: If the derivatives of two absolutely continuous functions are equivalent, then the functions differ by a constant.

Proof: Let f and g be two absolutely continuous functions and f′ = g′ $\Rightarrow (f-g)' = 0 \Rightarrow$ by above theorem f – g = constant and hence the result.

Example: If f is an absolutely continuous monotone function on [a,b] and E a set of measure zero, then show that f (E) has measure zero.

Proof: Let the function f be monotonically increasing. By the definition of absolute continuity of f, for $\in > 0, \exists\ \delta > 0$ and non-overlapping intervals $\{I_n = [a_n, b_n]\}$ such that

$$\sum (b_n - a_n) < \delta \Rightarrow \sum \left|f(b_n) - f(a_n)\right| < \in$$

or $$\sum \left[f(b_n) - f(a_n)\right] < \in$$

Now, $E \subseteq [a,b] \Rightarrow E \subseteq \cup I_n$

$\Rightarrow$ $f(E) \subseteq f(\cup I_n) = \cup f(I_n)$

$\Rightarrow$ $m^*(f(E)) \le \sum m^*(f(I_n)) \le \sum \left[\bar{f}(x_n) - \underline{f}(x_n)\right] < \in,$

Notes

where $\bar{f}(x_n)$ and $\underline{f}(x_n)$ are the maximum and maximum values of f(x) in the interval $[a_n, b_n]$.

Also note that $\sum \left|\bar{x}_n - \underline{x}_n\right| \le \sum (b_n - a_n) < \delta$

$\Rightarrow m^*(f(E)) \le \in, \in$ being arbitrary.

$\Rightarrow m^*(f(E)) = 0 \Rightarrow m(f(E)) = 0.$

Example: Give an example which is continuous but not absolutely continuous.

Solution: Consider the function $f: F \to R$, where F is the Cantor's ternary set.

Let $x \in F \Rightarrow x = x_1 x_2 x_3 \ldots = \sum_{K=1}^{\infty} \frac{x_k}{3^k}, x_k = 0$ or 2

Define $f(x) = \sum_{K=1}^{\infty} \frac{r_k}{2^k}$, where $r_k = \frac{1}{2} x_k$.

$= 0.\ r_1, r_2, r_3 \ldots..$

This function is continuous but not absolutely continuous.

(i) Note that this function is constant on each interval contained in the complement of the Cantor's ternary set.

For, let (a,b) be one of the countable open intervals contained in F^c. Then in ternary notation,

$a = 0.a_1 a_2 \ldots a_{n-1}\ 0\ 2\ 2\ 2$

and $b = 0.a_1 a_2 \ldots a_{n-1}\ 2\ 0\ 0\ 0$,

where $a_i = 0$ or 2, for $i \le n-1$.

$\Rightarrow f(a) = 0.r_1, r_2, \ldots, r_{n-1}\ 0\ 1\ 1\ 1\ 1 \ldots$, where $r_i = \left(\frac{a_i}{2}\right)$,

$f(b) = 0.r_1, r_2, \ldots, r_{n-1}\ 1\ 0\ 0\ 0\ 0 \ldots$

But in binary notation

$0.r_1, r_2, \ldots, r_{n-1}\ 0\ 1\ 1\ 1\ 1 \ldots = 0.r_1, r_2, \ldots, r_{n-1}\ 1\ 0\ 0\ 0\ 0 \ldots$

$\Rightarrow f(a) = f(b).$

Thus, we extend the function f overall of the set [0,1] instead of F by defining $f(x) = f(b), \forall x \in (a,b) \subset F^c$. Thus, the Cantor's function is defined over [0,1] and maps it onto [0,1].

It is clearly a non-decreasing function.

Notes

(ii) To show that f(x) is a continuous function. Note that if $c', c'' \in F$, then we have

$$\left.\begin{aligned} c' &= 0.(2p_1)(2p_2)(2p_3)\ldots \\ c'' &= 0.(2q_1)(2q_2)(2q_3)\ldots \end{aligned}\right\} \text{each } p_i, q_i = 0 \text{ or } =1$$

If $|c' - c''| < \left(\frac{1}{3^n}\right)$, then $p_i = q_i$, for $1 \le i \le n+1$ and hence

$$|f(c') - f(c'')| < \left(\frac{1}{2^n}\right) \qquad \ldots(i)$$

$\Rightarrow$ as $n \to \infty, c' \to c'', f(c') \to f(c'')$,

Hence if $c_0 \in F$ and $\langle c_n \rangle$ is a sequence in F such that $c_n \to c_0$, when $n \to \infty$, then $f(c_n) \to f(c_0)$, when $n \to \infty$.

Now let $x_0 \in [0,1]$ and let $\langle x_n \rangle$ be a sequence in [0,1] such that $x_n \to x_0$ as $n \to \infty$.

Case I: Let $x_0 \notin F \Rightarrow x_0 \in I, \text{say} (a,b) \subset F^c$

$$\Rightarrow x_n \in I \text{ and hence } f(x_n) = f(x) = f(a)$$

and hence $f(x_n) \to f(x_0)$ as $n \to \infty$.

Case II: Let $x_0 \in F$. Now for each n such that $x_n \in F$, set $x_n = c_n$ and hence $f(x_n) \to f(x_0)$.

If $x_n \notin F$, then $\exists$ an open interval $I \supseteq F^c$.

(i) if $x_n < x_0$, then set c_n as the upper end point of I.

(ii) If $x_0 < x_n$, then set c_n as the lower end point of I.

$\Rightarrow$ in any case $f(x_n) \to f(x_0)$ as $n \to \infty$.

But the sequence $\langle x_n \rangle$ was any sequence satisfying the stated conditions.

$\Rightarrow$ f is a continuous function.

(iii) To show f(x) is not absolutely continuous. Note that f′(x) = 0 at each $x \in F^c$.

$\Rightarrow f'(x)$ exists and is zero on [0,1] and is summable on [0,1].

We know that for f(x) to be absolutely continuous, we must have

$$f(x) = \int_0^x f'(x)dx + f(0).$$

Particularly, we must have

$$f(1) - f(0) = \int_0^1 f'(x)dx.$$

But f(1) – f(0) = 1 and $\int_0^x f'(1)dx = 0$ as $f'(x) = 0$

$\Rightarrow f(1) = f(0) \neq \int_0^x f'(1)dx = 0$ as $f'(x) = 0$

$\Rightarrow f(x)$ is not absolutely continuous.

Theorem 6: Prove that an absolutely continuous function on [a, b] is an indefinite integral.

Proof: Let f(x) be an absolutely continuous function in a closed interval [a, b] so that f′(x) & $\int_a^x f'(t)\,dt$ exists finitely $\forall x \in [a, b]$.

Let F(x) be an indefinite integral of f′(x), so that

$$F(x) = f(a) + \int_a^x f'(t)\,dt,\ x \in [a,b] \qquad \ldots (1)$$

We shall prove that F(x) = f(x).

Since an indefinite integral is an absolutely continuous function.

Therefore F(x) is absolutely continuous in [a, b].

Then from (1),

$$F'(x) = f'(x) \text{ a.e.}$$

$$\Rightarrow \quad \frac{d}{dx}[F(x) - f(x)] = 0.$$

Integrating, we get

$$F(x) - f(x) = c \text{ (constant)} \qquad \ldots (2)$$

Taking x = a in (1), we get

$$F(a) = f(a) + \int_a^a f'(t)\,dt$$

$$\Rightarrow \quad F(a) - f(a) = 0$$

or $\quad F(x) - f(x) = 0$ for x = a

Then from (2), we get c = 0.

Thus (2) reduces to

$$F(x) - f(x) = 0 \text{ a.e.}$$

$$\Rightarrow \quad F(x) = f(x) \text{ a.e.}$$

which shows that f(x) is indefinite integral of its own derivative.

4.2 Summary

- A real-valued function f defined on [a,b] is said to be absolutely continuous on [a,b], if for an arbitrary $\in > 0$, however small, $\exists\ a, \delta > 0$, such that

$$\sum_{r=1}^{n} \left| f(b_r) - f(a_r) \right| < \in, \text{ whenever } \sum_{r=1}^{n} (b_r - a_r) < \delta.$$

- Every absolutely continuous function is continuous.
- Every absolutely continuous function f defined on [a,b] is of bounded variation.

4.3 Keywords

Absolute Continuity of Functions: Absolute continuity of functions is a smoothness property which is stricter than continuity and uniform continuity.

Absolute Continuous Function: A real-valued function f defined on [a,b] is said to be absolutely continuous on [a,b], if for an arbitrary $\in > 0$, however small, $\exists\ a, \delta > 0$, such that

$$\sum_{r=1}^{n} \left| f(b_r) - f(a_r) \right| < \in \text{ whenever } \sum_{r=1}^{n} (b_r - a_r) < \delta,$$

where $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ i.e. a_i's and b_i's are forming finite collection $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$ of pair-wise disjoint intervals.

4.4 Review Questions

1. Define absolute continuity for a real variable. Show that f(x) is an indefinite integral, if F is absolutely continuous.
2. If f,g: [0,1] $\to R$ are absolutely continuous, prove that f + g and fg are also absolutely continuous.
3. Show that the set of all absolutely continuous functions on an interval I is a linear space.
4. If g is a non-decreasing absolutely continuous function on [a,b] and f is absolutely continuous on [g(a), g(b)], show that fog is also absolutely continuous on [a,b].
5. If f is absolutely continuous on [a,b] and $f'(x) \ge 0$ for almost all $x \in [a, b]$, show that f is non-decreasing on [a,b].

4.5 Further Readings

Books

Krishna B Athreya, N Soumendra Lahiri, *Measure Theory and Probability Theory*, Springer (2006).

Aole Nielsen, *An Introduction to Integration and Measure Theory*, Wiley-Interscience, (1997).

N.L. Royden, *Real Analysis* (third ed.), Collier MacMillan, (1988).

Online links

dl.acm.org

mrich.maths.org

Unit 5: Spaces, Hölder

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand L^p-spaces, conjugate numbers and norm of an element of L^p-space
- Understand the proof of Hölder's inequality.

Introduction

In this unit, we discuss an important construction, which is extremely useful in virtually all branches of analysis. We shall study about L^p-spaces and Hölder's inequality.

5.1 Spaces, Hölder

5.1.1 L^p-Spaces

The class of all measurable functions f (x) is known as L^p-spaces over [a, b], if Lebesgue – integrable over [a, b] for each p exists, $0 < p < \infty$, i.e.

$$\int_a^b |f|^p \, dx < \infty, (p > 0)$$

and is denoted by L^p [a, b].

Notes

Note The symbol L^p is used for such classes when limits of integration are known and mentioning of interval is not necessary.

5.1.2 Conjugate Numbers

Let p, q be any two n on-negative extended real numbers s.t. $\frac{1}{p}+\frac{1}{q}=1$, then p, q are called (mutually) conjugate numbers.

Obviously, 2 is self-conjugate number.

Also if $p \neq 2$, then $q \neq 2$. Further, if $p = \infty$, then $q = 1 \Rightarrow 1, \infty$ are conjugate numbers.

Note Non-negativity $\Rightarrow p \geq 1, q \geq 1$.

5.1.3 Norm of an Element of L^p-space

The p-norm of any $f \in L^p [a, b]$, denoted by $\| f \|_p$, is defined as

$$\| f \|_p = \left[\int_a^b | f |^p\right]^{\frac{1}{p}}, 0 < p < \infty.$$

Theorem 1: If $f \in L^p [a, b]$ and $g \leq f$, then $g \in L^p [a, b]$.

Proof: Let α be any positive real number.

$$\therefore \quad \{x \in [a, b] : g(x) > \alpha\} = \{x \in [a, b] : \alpha < g(x) \leq f(x)\} \quad (\because g \leq f)$$

$$= \{x \in [a, b] : f(x) > \alpha\}$$

Again $f \in L^p [a, b]$

$\Rightarrow$ f is measurable over [a, b].

$\Rightarrow$ $\{x \in [a, b] : f(x) > \alpha\}$ is a measurable set.

$\Rightarrow$ $\{x \in [a, b] : g(x) > \alpha\}$ is a measurable set.

$\Rightarrow$ g is a measurable function over [a, b]

Again since $g(x) \leq f(x), \ \forall \ x \in [a, b]$

$$\Rightarrow \int_a^b | g |^p \, dx \leq \int_a^b | f |^p \, dx < \infty \quad (\because | f |^p \in L [a, b])$$

or $$\int_a^b | g |^p \, dx < \infty$$

Thus $| g |^p \in L [a, b]$.

Thus we have proved that g is a measurable function over [a, b] such that

$$|g|^p \in L[a, b]$$

Hence $\quad g \in L^p[a, b]$

Theorem 2: If $f \in L^p[a, b]$, $p > 1$, then $f \in L[a, b]$

Proof: $f \in L^p[a, b] \Rightarrow f$ is measurable over [a, b]

Let $\quad A_1 = \{x \in [a, b] : |f(x) \geq 1\}$

and $\quad A_2 = \{x \in [a, b] : |f(x) < 1\}$

Then $\quad [a, b] = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$

Using countable additive property of the integrals, we have

$$\int_a^b |f|\,dx = \int_{A_1} |f|\,dx + \int_{A_2} |f|\,dx \qquad \ldots \text{(i)}$$

Now $\because \quad |f(x)| \geq 1, x \in A_1$

$\therefore \quad |f| \leq |f|^p$ on A_1 as $p > 1$

$$\therefore \quad \int_{A_1} |f|\,dx \leq \int_{A_1} |f|^p\,dx < \infty \text{ as } f \in L^p[a, b] \qquad \ldots \text{(ii)}$$

Now $|f(x)| <|, \forall\ x \in A_2$

Using first mean value theorem, we get

$$\int_{A_2} |f|\,dx < m(A_2) = \text{A finite quantity} \qquad \ldots \text{(iii)}$$

Combining (ii) and (iii) and making use of (i), we get

$$\int_a^b |f|\,dx < \infty$$

Thus f is a measurable function over [a, b], such that

$$\int_a^b |f|\,dx < \infty$$

$\Rightarrow \quad |f| \subset L[a, b]$ and hence $f \in L[a, b]$.

Theorem 3: If $f \in L^p[a, b]$, $g \in L^p[a, b]$; then $f + g \in L^p[a, b]$

Proof: Since $f, g \in L^p[a, b] \Rightarrow f, g$ are measurable over [a, b]

$\Rightarrow f + g$ is measurable over [a, b]

Let $\quad A_1 = \{x \in [a, b] : |f(x)| \geq |g(x)|\}$

and $\quad A_2 = \{x \in [a, b] : |f(x)| < |g(x)|\}$

Then $[a, b] = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$

Notes

Therefore $\int |f+g|^p\, dx = \int_{A_1} |f+g|^p + \int_{A_2} |f+g|^p\, dx$.

Again, $|f + g|^p \le (|f| + |g|)^p \le (|g| + |g|)^p$ on A_2 and $\le (|f| + |f|)^p$ on A_1

$\le 2P\ |g|^p$ on A_2 and $\le 2^p\ |f|^p$ on A_1

Integrating, we have

$$\int_{A_1} |f+g|^p \le 2^p \int_{A_1} |f|^p$$

and

$$\int_{A_2} |f+g|^p \le 2^p \int_{A_2} |g|^p$$

Since f, g $\in L^p$ [a, b] $\Rightarrow \int_{A_1} |f|^p < \infty$ and $\int_{A_2} |g|^p < \infty$

$\because$ $\int_{A_1} |f+g|^p < \infty$ and $\int_{A_1} |f+g|^p < \infty$ [by (i) and (ii)]

$\Rightarrow$ $\int_a^b |f+g|^p\, dx < \infty \Rightarrow f + g \in L^p$ [a, b]

5.1.4 Simple Version of Hölder's Inequality

Lemma 1: Let p, q > 1 be such that $\frac{1}{p}+\frac{1}{q}=1$, and let u and v be two non-negative numbers, at least one being non-zero. Then the function f : [0, 1] $\to$ R defined by

$$f(t) = ut + v(1-t^q)^{\frac{1}{q}},\ t \in [0, 1],$$

has a unique maximum point at

$$s = \left[\frac{u^p}{u^p + v^p}\right]^{\frac{1}{q}} \qquad \dots (1)$$

The maximum value of f is

$$\max_{t\in[0,1]} f(t) = (u^p + v^p)^{\frac{1}{p}} \qquad \dots (2)$$

Proof: If v = 0, then f (t) = tu, $\forall\ t \in [0, 1]$ (with u > 0), and in this case, the Lemma is trivial.

Likewise, if u = 0, then

$$f(t) = v(1-t^q)^{\frac{1}{q}},\ \forall\ t \in [0, 1] \text{ (with } v > 0), \qquad \dots (3)$$

Notes

and using the inequality

$$(1-t^q)^{\frac{1}{q}} < 1, \forall\, t \in (0,1],$$

We immediately get f (t) < f (0), $\forall$ t $\in$ (0, 1],

and the Lemma again follows.

For the remainder of the proof we are going to assume that u, v > 0.

Obviously f is differentiable on (0, 1) and the solutions of the equation (3)

$$f'(t) = 0$$

Let s be defined as in (1), so under the assumption that u, v > 0, we clearly have 0 < s < 1.

We are going to prove first that s is the unique solution in (0, 1) of the equation (3).

We have

$$f'(t) = u + v \cdot \frac{1}{q}(1-t^q)^{\frac{1}{q}-1} \cdot q \cdot t^{q-1} \qquad \dots (4)$$

$$= u - v\left(\frac{t^q}{1-t^q}\right), \; t \in (0,1)$$

so the equation (3) reads

$$u - v\left(\frac{t^q}{1-t^q}\right) = 0.$$

Equivalently, we have

$$\left(\frac{t^q}{1-t^q}\right)^{\frac{1}{p}} = u/v,$$

$$\Rightarrow \qquad \frac{t^q}{1-t^q} = (u/v)^p,$$

$$t^2 = \frac{(u/v)^p}{1-(u/v)^p} = \frac{u^p}{u^p+v^p},$$

Having shown that the "candidates" for the maximum point are 0, 1 and s let us show that s is the only maximum point.

For this purpose, we go back to (4) and we observe that f′ is also continuous on (0, 1).

Since $\lim_{t \to 0^+} f'(t) = u > 0$ and

$$\lim_{t \to 1^-} f'(t) = -\infty$$

and the equation (3) has exactly one solution in (0, 1), namely s, this forces

$$f'(t) > 0 \;\; \forall\; t \in (0, s)$$

Notes

and $$f'(t) < 0 \ \forall \ t \in (s, 1).$$

This means that, f is increasing on [0, s] and decreasing on [s, 1], and we are done.

The maximum value of f is then given by

$$\max_{t\in[0,1]} f(t) = f(s),$$

and the fact that f (s) equals the value in (2) follows from an easy computation.

5.1.5 Hölder's Inequality

Statement: Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be non-negative numbers. Let $p, q > 1$ be real number with the property $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\sum_{j=1}^{n} a_j b_j \le \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{n} b_j^q\right)^{\frac{1}{q}} \qquad \ldots (5)$$

Moreover, one has equality only when the sequences $\left(a_1^p, \ldots, a_n^p\right)$ and $\left(b^q, \ldots, b_n^q\right)$ are proportional.

Proof: The proof will be carried on by induction on n. The case n = 1 is trivial.

Case n = 2.

Assume $(b_1, b_2) \neq (0, 0)$. (otherwise everything is trivial).

Define the number

$$r = \frac{b_1}{\left(b_1^q + b_2^q\right)^{\frac{1}{q}}}.$$

Notice that $r \in [0, 1]$ and we have

$$\frac{b_2}{\left(b_1^q + b_2^q\right)^{\frac{1}{q}}} = \left(1 - r^q\right)^{\frac{1}{q}}$$

Notice also that, upon dividing by $\left(b_1^q + b_2^q\right)^{\frac{1}{q}}$, the desired inequality

$$a_1 b_1 + a_2 b_2 \le \left(a_1^q + a_2^q\right)^{\frac{1}{p}} \left(b_1^q + b_2^q\right)^{\frac{1}{q}} \qquad \ldots (6)$$

reads

$$a_1 r + a_2 \left(1 - r^q\right)^{\frac{1}{q}} \le \left(a_1^q + a_2^q\right)^{\frac{1}{p}} \qquad \ldots (7)$$

It is obvious that this is an equality when $a_1 = a_2 = 0$. Assume $(a_1, a_2) \neq (0, 0)$, and set up the function.

$$f(t) = a_1 t + a_2 \left(1 - t^q\right)^{\frac{1}{q}}, \ t \in [0, 1].$$

Notes

We now apply Lemma (1) stated above, which immediately gives us (7).

Let us examine when equality holds.

If $a_1 = a_2 = 0$, the equality obviously holds, and in this case (a_1, a_2) is clearly proportional to (b_1, b_2). Assume $(a_1, a_2) \neq (0, 0)$.

Again by Lemma (1), we know that equality holds in (7), exactly when

$$r = \left[\frac{a_1^p}{a_1^p + a_2^p}\right]^{\frac{1}{q}}$$

that is
$$\frac{b_1}{\left(b_1^q + b_2^q\right)^{\frac{1}{q}}} = \left[\frac{a_1^p}{a_1^p + a_2^p}\right]^{\frac{1}{q}},$$

or equivalently

$$\frac{b_1^q}{b_1^q + b_2^q} = \frac{a_1^p}{a_1^p + a_2^p}.$$

Obviously this forces

$$\frac{b_1^q}{b_1^q + b_2^q} = \frac{a_1^q}{a_1^p + a_2^p},$$

so indeed $\left(a_1^p, a_2^p\right)$ and $\left(b_1^q, b_2^q\right)$ are proportional.

Having proven the case n = 2, we now proceed with the proof of:

The implication: Case n = k $\Rightarrow$ case n = k + 1, start with two sequences $(a_1, a_2, \ldots, a_k, a_{k+1})$ and $(b_1, b_2, \ldots, a_k, b_{k+1})$.

Define the numbers

$$a = \left(\sum_{j=1}^{k} a_j^p\right)^{\frac{1}{p}} \text{ and } b = \left(\sum_{j=1}^{k} b_j^q\right)^{\frac{1}{q}}.$$

Using the assumption that the case n = k holds, we have

$$\sum_{j=1}^{k+1} a_j b_j \leq \left(\sum_{j=1}^{k} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{k} b_j^q\right)^{\frac{1}{q}} + a_{k+1}\, b_{k+1}$$

$$= ab + a_{k+1}\, b_{k+1} \qquad \ldots (8)$$

Using the case n = 2, we also have

$$ab + a_{k+1}\, b_{k+1} \leq \left(a^p + a_{k+1}^p\right)^{\frac{1}{p}} \cdot \left(b^q + b_{k+1}^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{j=1}^{k+1} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{k+1} b_j^q\right)^{\frac{1}{q}}, \qquad \ldots (9)$$

so combining with (8) we see that the desired inequality (5) holds for n = k + 1.

Assume now we have equality. Then we must have equality in both (8) and in (9).

On one hand, the equality in (8) forces $\left(a_1^p, a_2^p, \ldots, a_k^p\right)$ and $\left(b_1^q, b_2^q, \ldots, b_k^q\right)$ to be proportional (since we assume the case n = k). On the other hand, the equality in (9) forces $\left(a^p, a_{k+1}^p\right)$ and $\left(b^q, b_{k+1}^q\right)$ to be proportional (by the case m = 2). Since

$$a^p = \sum_{j=1}^{k} a_j^p \text{ and } b^q = \sum_{j=1}^{k} b_j^q,$$

it is clear that $\left(a_1^p, a_2^p, \ldots, a_k^p, a_{k+1}^p\right)$ and $\left(b_1^q, b_2^q, \ldots, b_k^q, b_{k+1}^q\right)$ are proportional.

5.1.6 Riesz-Hölder's Inequality

Statement: Let p and q be conjugate indices or exponents (numbers) and $f \in L^p [a, b]$, $g \in L^q [a, b]$; then show that

(i) $f \cdot g \in L[a, b]$

(ii) $\| fg \| \leq \| f \|_p \| g \|_q$ i.e.

$$\int | fg | \leq \left(\int | f |^p\right)^{\frac{1}{p}} \left(\int | g |^q\right)^{\frac{1}{q}}$$

with equality only when $\alpha \ |f|^p = \beta \ |g|^q$ a.e. for some non-zero constants α and β.

Lemma: If A and B are any two non-negative real numbers and $0 < \lambda < 1$, then

$A^\lambda B1^{-\lambda} \leq \lambda A + (1 - \lambda) B$, with equality when A = B.

Proof: If either A = 0 or B = 0, then the result is trivial.

Let A > 0, B > 0

Consider the function

$\phi (x) = x^\lambda - \lambda x$, where $0 \leq x < \infty$ and $0 < \lambda < 1$

$$\Rightarrow \quad \frac{d\phi}{dx} = \lambda x^{\lambda-1} - \lambda \text{ and } \frac{d^2\phi}{dx^2} = \lambda(\lambda - 1)x^{\lambda-1}.$$

Now solving $\frac{d\phi}{dx} = 0$, we get x = 1.

Also at x = 1, $\frac{d^2\phi}{dx^2} < 0$ as $0 < \lambda < 1$.

By calculus, ϕ (x) is maximum at x = 1, so

$$\phi (x) \leq \phi (1) \text{ i.e. } x^\lambda - \lambda x \leq l^\lambda - \lambda. \qquad \ldots (1)$$

Now, putting $x = \frac{A}{B}$, we get

$$\left(\frac{A}{B}\right)^\lambda - \lambda\left(\frac{A}{B}\right) \leq 1 - \lambda \text{ or } A^\lambda B^{-\lambda} - \lambda\frac{A}{B} \leq 1 - \lambda$$

Notes

Or $$A^{\lambda} B^{1-\lambda} - \lambda A \le (1-\lambda) B \text{ or}$$

$$A^{\lambda} B^{1-\lambda} \le \lambda A + B(1-\lambda) \quad \ldots (2)$$

Obviously equality holds good only for x = 1, i.e. only when A = B.

Proof of Theorem

Note that when p = 1, q = ∞, the proof of theorem is obvious. Let us assume that $1 < p < \infty$ and $1 < q < \infty$.

Now set $$\lambda = \frac{1}{p};\ p > 1 \quad \Rightarrow \quad \lambda < 1$$

Therefore $$\frac{1}{q} = 1 - \lambda \qquad \left(\because \frac{1}{p} + \frac{1}{q} = 1\right)$$

Putting these values of λ and 1 – λ in (2), we get

$$A^{\frac{1}{p}} B^{\frac{1}{q}} \le \frac{A}{p} + \frac{B}{q} \quad \ldots (3)$$

If one of the functions f (x) and g (x) is zero a.e. then the theorem is trivial. Thus, we assume that $f \neq 0$, $g \neq 0$ a.e. and hence the integrals

$$\int_a^b |f|^p \, dx \quad \text{and} \quad \int_a^b |g|^q \, dx$$

are strictly positive and hence $\| f \|_p > 0$, $\| g \|_q > 0$.

Set $$f(x) = \frac{f(x)}{\|f\|_p}, \quad g(x) = \frac{g(x)}{\|g\|_q}$$

and $$A^{\frac{1}{p}} = |f(x)|, \; B^{\frac{1}{q}} = |g(x)|.$$

Then (3) gives

$$|f(x)\, g(x)| \le \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$$

Integrating, we get

$$\int_a^b |f(x)\, g(x)| \, dx \le \frac{1}{p} \int_a^b |f(x)|^p \, dx + \frac{1}{q} \int_a^b |g(x)|^p \, dx$$

$$= \frac{1}{p} \int_a^b \frac{|f(x)|^p}{\int_a^b |f|^p \, dx} \, dx + \frac{1}{q} \int_a^b \frac{|g(x)|^q}{\int_a^b |g|^q \, dx} \, dx$$

$$= \frac{1}{p}\frac{\int_a^b |f|^p\,dx}{\int_a^b |f|^p\,dx} + \frac{1}{q}\frac{\int_a^b |g|^q\,dx}{\int_a^b |g|^q\,dx}$$

$$= \frac{1}{p} + \frac{1}{q} = 1\,.$$

Hence $\int_a^b |f(x)\,g(x)|\,dx \le 1.$

Putting the values of f (x) and g (x), we get

$$\int_a^b \frac{|f(x)\,g(x)|\,dx}{\|f\|_p\ \|g\|_q} \le 1 \text{ or } \|fg\| \le \|f\|_p \|g\|_q \qquad \ldots (4)$$

Now $f \in L^p$ [a, b], $g \in L^q$ [a, b]

$\Rightarrow \quad \int_a^b |f|^p\,dx < \infty$ and $\int_a^b |g|^q\,dx < \infty$

$\Rightarrow \quad \|f\|_p < \infty$ and $\|g\|_q < \infty$

Therefore, from (4), we have

$$\|fg\|_1 < \infty \Rightarrow f_g \in L' \text{ [a, b]}$$

Also the equality will hold when A = B

i.e. $\quad |f(x)|^p = |g(x)|^q$, a.e.

i.e. $\quad$ if $\dfrac{|f|^p}{\|f\|_p^p} = \dfrac{|g|^q}{\|g\|_q^q}$, a.e.

or $\quad$ if $\|g\|_q^q\ |f|^p = \|f\|_p^p\ |g|^q$, a.e.

or if we have got some non-zero constants α, β

$$\alpha\ |f|^p = \beta\,|g|^q, \text{ a.e.}$$

Hence the theorem.

5.1.7 Riesz-Hölder's Inequality for 0 < p < 1

If $0 < p < 1$ and p and q are conjugate exponents, and $f \in L^p$ and $g \in L^q$, then

$$\int |fg| \ge \|f\|_p \|g\|_q, \text{ provided } \int |g|^q \ne 0\,.$$

(In this case, the inequality is reversed than that of the case for $1 \le p < \infty$.)

Proof: Conjugacy of p, q $\Rightarrow \dfrac{1}{p} + \dfrac{1}{q} = 1$

$$\Rightarrow \frac{1}{q} = 1 - \frac{1}{p}$$

$$\Rightarrow \frac{-p}{q} = 1 - p$$

$$\Rightarrow p + \left(-\frac{p}{q}\right) = 1$$

If we take $p = \frac{1}{P}$ and $\frac{-p}{P} = \frac{1}{Q}$, then $\frac{1}{p} + \frac{1}{Q} = 1$ and since $0 < p < 1 \Rightarrow \frac{1}{0} < \frac{1}{P} < 1 \Rightarrow P > 1$,

i.e. 1< P < ∞ < ∞ and also $\frac{1}{Q} = \frac{-p}{q} = 1 - p \Rightarrow 0 < \frac{1}{Q} < 1$ as 0 < p < 1 ⇒ Q > 1.

P, Q are conjugate numbers with 1 < P < ∞.

If we take $|fg| = F^p$ and $|g|^q = G^Q$.

Then $$fg = |f_g|^{\frac{1}{p}} \cdot |g|^{\frac{q}{Q}} = |f|^p\, |g|^{\left(\frac{1}{p}\right)+\left(\frac{q}{Q}\right)\left(\frac{-p}{q}\right)}$$

$$= |f|^p.$$

f, g are non-negative measurable functions s.t.

Also $f \in L^p$ and $g \in L^q$.

Applying the Hölder's inequality for P, Q to the functions f and g, we get

$$\int |FG| \le \|F\|_P \, \|G\|_Q$$

$$\Rightarrow \int |f|^p \le \left(\int |F|^P\right)^{\frac{1}{P}} \left(\int |G|^Q\right)^{\frac{1}{Q}} \text{ as } |fg| = fg = |f|^p$$

$$\Rightarrow \int |f|^p \le \left(\int |fg|\right)^p \left(\int |g|^q\right)^{\frac{-p}{q}}$$

$$\Rightarrow \left(\int |f|^p\right)^{\frac{1}{p}} \le \left(\int |fg|\right)\left(\int |g|^q\right)^{\frac{-1}{q}}$$

$$\Rightarrow \left(\int |f|^p\right)^{\frac{1}{p}} \le \frac{\int |fg|}{\left(\int |g|^q\right)^{\frac{1}{q}}}, \text{ provided } \int |g|^q \ne 0$$

$$\Rightarrow \int |fg| \ge \left(\int |f|^p\right)^{\frac{1}{p}} \left(\int |q|^q\right)^{\frac{1}{q}} \ge \|f\|_p \, \|g\|_q$$

Theorem 4: SCHWARZ or CAUCHY-SCHWARZ INEQUALITY statement: Let f and g be square integrable, i.e.

$f, g \in L^2[a, b]$; then $fg \leq L\,[a, b]$ and $\| fg \| \leq \| f \|_2 \, \| g \|_2$.

Proof: Let $x \in [a, b]$ be arbitrary, then

$$[\,|f(x)| - |g(x)|\,]^2 \geq 0$$

or $\quad 2|f(x)|.|g(x)| \leq |f(x)^2 + |g(x)|^2$.

On integrating, we get

$$2\int_a^b |f(x)g(x)|\,dx \leq \int_a^b |f(x)^2 dx + \int_a^b |g(x)|^2\,dx \quad \ldots \text{(i)}$$

Now $f, g \in L^2[a, b] \Rightarrow$ f and g are measurable over [a, b] and $\int_a^b |f(x)|^2\,dx < \infty, \int_a^b |g(x)|^2\,dx < \infty$.

Using in (i), we get $\int_a^b |f(x)\,g(x)|\,dx < \infty$

Thus $fg \in L\,[a, b]$.

Let $a \in R$ be arbitrary. Then

$$(\alpha|f| + |g|)^2 \geq 0$$

$$\therefore \quad \int_a^b (\alpha|f| + |g|)^2 \geq 0$$

or $\alpha^2 \int_a^b |f|^2\,dx + 2\alpha\int_a^b |fg|\,dx + \int_a^b |g|^2\,dx \geq 0$

Write $A = \int_a^b |f|^2\,dx, B = 2\int_a^b |fg|\,dx, C = \int_a^b |g|^2\,dx$

Then we have $\alpha^2 A + \alpha B + C \geq 0 \quad \ldots \text{(ii)}$

Now, if A = 0, then f(x) = 0 a.e. in [a, b] and hence B = 0 and both sides of the inequality to be proved are zero. Thus when A = 0, the inequality is trivial.

Again, let $A \neq 0$. Writing $\alpha = -\dfrac{B}{2A}$ in (ii), we get

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C \geq 0.$$

which gives $B^2 \leq 4AC$.

Now putting the values of A, B, C in last inequality, we have

$$4\left[\int_a^b |fg|\,dx\right]^2 \le 4\left[\int_a^b |f|^2\right]\left[\int_a^b |g|^2\right]$$

or
$$\int_a^b |f(x)\,(g(x)|\,dx \le \left(\int_a^b |f(x)|^2\right)^{1/2}\left(\int_a^b |g(x)|^2\right)^{1/2}$$

or
$$\| fg \| \le \| f \|_2 \; \| g \|_2.$$

Note: The above theorem is a particular case of Hölder's inequality.

Example: Let f, g be square integrable in the Lebesgue sense then prove f + g is also square integrable in the Lebesgue sense, and $\| f + g \|_2 \le \| f \|_2 + \| g \|_2$.

Solution: By hypothesis $f^2 \in L\,[a, b]$, $g^2 \in L\,[a, b]$.

$$f^2, g^2 \in L\,[a, b] \Rightarrow fg \in L\,[a, b]. \qquad \text{[by Schwarz inequality]}$$

Again
$$(f + g)^2 = f^2 + g^2 + 2fg \in L\,[a, b].$$

Hence (f + g) is square integrable, again, we have

$$\int_a^b (f+g)^2 = \int_a^b f^2 + \int_a^b g^2 + 2\int_a^b fg$$

$$\le \int_a^b f^2 + \int_a^b g^2 + 2\left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2} \qquad \text{(by Schwarz inequality)}$$

$$= \left[\left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}\right]^2$$

∴
$$\left(\int_a^b (f+g)^2\right)^{1/2} \le \left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}$$

or
$$\| f + g \|_2 \le \| f \|_2 + \| g \|_2.$$

Example: Prove that $\| f + g \|_1 \le \| f \|_1 + \| g \|_1$.

Solution: We know that $|f + g| \le |f| + |g|$.

Integrating both the sides.

$$\int |f+g| \le \int |f| + \int |g|$$

⇒
$$\| f + g \|_1 \le \| f \|_1 + \| g \|_1.$$

5.2 Summary

- The class of all measurable functions f (x) is known as L^p – space over [a, b], if Lebesgue-integrable over [a, b] for each p exists, $0 < p < \infty$, i.e.

$$\int_a^b |f|^p \, dx < \infty, \ (p > 0)$$

- The p-norm of any $f \in L^p$ [a, b], denoted by $\| f \|_p$, is defined as

$$\| f \|_p = \left[\int_a^b |f|^p \right]^{\frac{1}{p}}, \ 0 < p < \infty$$

- Let p, q > 1 be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let u and v be two non-negative numbers, at least one being non-zero. Then the function f : [0, 1] → R defined by

$$f(t) = ut + v\left(1 - t^q\right)^{\frac{1}{q}}, \ t \in [0, 1],$$

has a unique maximum point at

$$s = \left[\frac{u^p}{u^p + v^p} \right]^{\frac{1}{q}}$$

- Let p and q be conjugate indices or exponents and $f \in L^p$ [a, b], $g \in L^q$ [a, b], then it is evident that

  (i) f, g ∈ L [a, b]

  (ii) $\| fg \| \le \| f \|_p \ \| g \|_q$ i.e.

$$\int |fg| \le \left(\int |f|^p \right)^{\frac{1}{p}} \left(\int |g|^q \right)^{\frac{1}{q}}$$

5.3 Keywords

Conjugate Numbers: Let p, q be any two n on-negative extended real numbers s.t. $\frac{1}{p} + \frac{1}{q} = 1$, then p, q are called (mutually) conjugate numbers.

Hölder's Inequality: Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be non-negative numbers. Let p, q > 1 be real number with the property $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{j=1}^{n} a_j b_j \le \left(\sum_{j=1}^{n} a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{n} b_j^q \right)^{\frac{1}{q}}$$

Moreover, one has equality only when the sequences $\left(a_1^p, \ldots, a_n^p\right)$ and $\left(b^q, \ldots, b_n^q\right)$ are proportional.

Notes

L^p-Spaces: The class of all measurable functions f (x) is known as L^p-spaces over [a, b], if Lebesgue – integrable over [a, b] for each p exists, $0 < p < \infty$, i.e.

$$\int_a^b |f|^p \, dx < \infty, (p > 0)$$

and is denoted by L^p [a, b].

p-norm: The p-norm of any $f \in L^p$ [a, b], denoted by $\|f\|_p$, is defined as

$$\|f\|_p = \left[\int_a^b |f|^p\right]^{\frac{1}{p}}, 0 < p < \infty.$$

5.4 Review Questions

1. If f and g are non-negative measurable functions, then show that in Hölder's inequality, equality occurs iff ∃ some constants s and t (not both zero) such that $sf^p + tg^q = 0$.
2. State and prove Hölder's Inequality.

5.5 Further Readings

Books

G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, (1934)

L.P. Kuptsov, *Hölder inequality*, Springer (2001)

Kenneth Kuttler, *An Introduction of Linear Algebra*, BRIGHAM Young University, 2007

Online links

www.m–hiKari.com

www.math.Ksu.edu

www.tandfonline.com

Unit 6: Minkowski Inequalities

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define L^p-space, conjugate numbers and norm of an element of L^p-space.
- Understand Minkowski inequality.
- Solve problems on Minkowski inequality.

Introduction

In mathematical analysis, the Minkowski inequality establishes that the L^p spaces are normed vector spaces. Let S be a measure space, let $1 \le p \le \infty$ and let f and g be elements of L^p (s). Then f + g is in L^p (s), we have the triangle inequality

$$\| f + g \|_p \le \| f \|_p + \| g \|_p$$

with equality for $1 < p < \infty$ if and only if f and g are positively linearly dependent, i.e. $f = \lambda_g$ for some $\lambda \ge 0$. In this unit, we shall study Minkowski's inequality for $1 \le p < \infty$ and for $0 < p < 1$. We shall also study almost Minkowski's inequality in integral form.

6.1 Minkowski Inequalities

Here, the norm is given by:

$$\| f \|_p = \left(\int | f |^p \, d\mu \right)^{1/p}$$

if $p < \infty$, or in the case $p = \infty$ by the essential supremum

$$\| f \|_\infty = \text{ess sup}_{x \in S} \; | f (x) | .$$

The Minkowski inequality is the triangle inequality in $L^p(S)$. In fact, it is a special case of the more general fact

$$\| f \|_p = \sup_{\|g\|_q = 1} \int |fg| \, d\mu, \quad 1/p + 1/q = 1$$

where it is easy to see that the right-hand side satisfies the triangular inequality.

Like Holder's inequality, the Minkowski inequality can be specialized to sequences and vectors by using the counting measure:

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p \right)^{1/p} \le \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p}$$

for all real (or complex) numbers $x_1, ..., x_n, y_1, ..., y_n$ and where n is the cardinality of S (the number of elements in S).

Thus, we may conclude that

If p > 1, then Minkowski's integral inequality states that

$$\left| \int_a^b |f(x) + g(x)|^p \, dx \right|^{1/p} \le \left| \int_a^b |f(x)|^p \, dx \right|^{1/p} + \left| \int_a^b |g(x)|^p \, dx \right|^{1/p}$$

Similarly, if p >1 and $a_k, b_k > 0$, then Minkowski's sum inequality states that

$$\left| \sum_{k=1}^{n} |a_k + b_k|^p \right|^{1/p} \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p}$$

Equality holds iff the sequences $a_1, a_2, ...$ and $b_1, b_2, ...$ are proportional.

6.1.1 Proof of Minkowski Inequality Theorems

Theorem 1: State and prove Minkowski inequality. If f and g ∈ L^p ($1 \le p < \infty$), then f + g ∈ L^p and

$\| f + g \|_p \le \| f \|_p + \| g \|_p$.

or

Let $1 \le p \le \infty$. Prove that for every pair f, g ∈ L^p {0, 1}, the function f + g ∈ L^p {0, 1} and that $\| f + g \|_p \le \| f \|_p + \| g \|_p$. When does equality occur?

Or

Suppose $1 \le p < \infty$. Prove that for any two functions f and g in L^p [a, b]

$$\left(\int_a^b |f + g|^p \right)^{\frac{1}{p}} \le \left(\int_a^b |f|^p \, dx \right)^{\frac{1}{p}} + \left(\int_a^b |g|^p \, dx \right)^{\frac{1}{p}}$$

Proof: When p = 1, the desired result is obvious.

If p = ∞, then

$$|f| \le \| f \|_\infty \text{ a.e.}$$

$$|g| \le \| g \|_\infty \text{ a.e.}$$

Notes

$\Rightarrow \qquad |f+g| \le |f| + |g|$

$\qquad\qquad \le \|f\|_\infty + \|g\|_\infty$ a.e.

$\Rightarrow \qquad \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

Hence the result follows in this case also. Thus, we now assume that $1 < p < \infty$.

Since L^p is a linear space, $f + g \in L^p$.

Let q be conjugate to p, then $\frac{1}{p}+\frac{1}{q}=1$.

Now $\quad (f+g) \in L^p$

$\Rightarrow \quad (f+g)^{p/q} \in L^q$

Since $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \frac{1}{q}=1-\frac{1}{p} \Rightarrow \frac{1}{q}=\frac{p-1}{p}$

$$\Rightarrow (p-1)^q = p, \int \left(|f+g|^{p-1}\right)^q = \int |f+g|^p$$

and therefore $|f+g|^{p-1} \in L^p \Rightarrow (f+g)^{p/q} \in L^p$ because $p-1 = \frac{p}{q}$.

On applying Hölder's inequality for f and $(f+g)^{p/q}$, we get

$$\int |f|\cdot|f+g|^{p/q}\,dx \le \left(\int |f|^p)\,dx\right)^{1/p}\left(\int |f+g|^{p/q\,q}\,dx\right)^{1/p}$$

or

$$\int |f|\cdot|f+g|^{p/q}\,dx \le \left(\int |f|^p)\,dx\right)^{1/p}\left(\int |f+g|^p\,dx\right)^{1/q} \qquad \ldots (1)$$

Since $g \in L^p$, therefore interchanging f and g in (1), we get

$$\int |g|\cdot|f+g|^{p/q}\,dx \le \left(\int |g|^p)\,dx\right)^{1/p}\left(\int |f+g|^p\,dx\right)^{1/q} \qquad \ldots (2)$$

Adding, we get

$$\int |f|\cdot|f+g|^{p/q}\,dx + \int |g|\cdot|f+g|^{p/q}\,dx \le \left[\left(\int |f|^p\,dx\right)^{1/p} + \left(\int |g|^p\,dx\right)^{1/p}\right]\cdot\left(\int |f+g|^p\,dx\right)^{1/q} \qquad \ldots (3)$$

Now $\qquad |f+g|^p = |f+g|\cdot|f+g|^{p-1}$

But $\qquad \frac{1}{p}+\frac{1}{q}=1 \Rightarrow 1+\frac{p}{q}=p \Rightarrow p-1=\frac{p}{q}$

$\therefore \qquad |f+g|^p = |f+g|\cdot|f+g|^{p/q}$

$\qquad\qquad \le (|f|+|g|)\cdot|f+g|^{p/q}$

Notes

or $$|f+g|^p \leq |f|\cdot|f+g|^{p/q} + |g|\cdot|f+g|^{p/q}$$

Integrating, we get

$$\int |f+g|^p\,dx \leq \int |f|\cdot|f+g|^{p/q} + |g|\cdot|f+g|^{p/q}\,dx \qquad \dots (4)$$

Using (3), relation (4) becomes

$$\int |f+g|^p\,dx \leq \left[\left(\int |f|^p\,dx\right)^{1/p} + \left(\int |g|^p\,dx\right)^{1/p}\right]\left(|f+g|^p\,dx\right)^{1/q}$$

Dividing each term by $\int\left(|f+g|^p\,dx\right)^{1/q}$, we get

$$\left(\int |f+g|^p\,dx\right)^{1-1/q} \leq \left(\int |f|^p\,dx\right)^{1/p} + \left(\int |g|^p\,dx\right)^{1/p}$$

But $$\frac{1}{p}+\frac{1}{q}=1 \Rightarrow 1-\frac{1}{q}=\frac{1}{p}$$

So $$\left(\int |f+g|^p\,dx\right)^{1/p} \leq \left(\int |f|^p\,dx\right)^{1/p} + \left(\int |g|^p\,dx\right)^{1/p}$$

or $$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Hence the proof.

Note Equality hold in Minkowski's inequality if and only if one of the functions f and g is a multiple of the other.

Theorem 2: Minkowski's inequality for $0 < p < 1$. If $0 < p < 1$ and f, g are non-negative functions in L^p, then

$$\|f+g\|_p \geq \|f\|_p + \|g\|_p.$$

Proof: For this proceed as in theorem Minkowski's inequality and applying the Hölder's inequality for $0 < p < 1$ for the functions $f \in L^p$ and $(f+g)^{p/q} \in L^q$, we get

$$\int |f||f+g|^{p/q} \geq \left(\int |f|^p\right)^{1/p}\left(\int (|f+g|^{p/q})^q\right)^{1/q}$$

$$\Rightarrow \quad \int |f||f+g|^{p/q} \geq \left(\int |f|^p\right)^{1/p}\left(\int |f+g|^p\right)^{1/q} \qquad \dots (i)$$

Also $g \in L^p$, proceeding as above, we get

$$\int |g||f+g|^{p/q} \geq \left(\int |g|^p\,dx\right)^{1/p}\left(\int |f+g|^p\right)^{1/q} \qquad \dots (ii)$$

Adding these two,

Notes

$$\int |f+g|^{p/q}(|f|+|g|) \geq \left[\{|f|^p\}^{\frac{1}{p}}+\{|g|^p\}^{\frac{1}{p}}\right]\times\{|f+g|^p\}^{\frac{1}{q}} \quad \ldots \text{(iii)}$$

Also $\quad \dfrac{1}{p}+\dfrac{1}{q}=1 \Rightarrow 1+\dfrac{p}{q}=p$

$\Rightarrow \quad |f+g|^p = |f+g|\cdot|f+g|^{p/q} = (|f|+|g|)\cdot|f+g|^{p/q}$, as $f \geq 0, g \geq 0$

$\Rightarrow \quad \displaystyle\int |f+g|^p = \left[\{|f|^p\}^{1/p}+\{|g|^p\}^{1/p}\right]\left(\int |f+g|^p\right)^{1/q}$

Dividing by $\left(\int |f+g|^p\right)^{1/q}$, we get

$$\left(\int |f+g|^p\right)^{1-(1/q)} \geq \left[\|f\|_p+\|g\|_p\right]$$

$\Rightarrow \quad \left(\int |f+g|^p\right)^{1/q} \geq \|f\|_p+\|g\|_p$

$\Rightarrow \quad \|f+g\|_p \geq \|f\|_p+\|g\|_p$

6.1.2 Minkowski Inequality in Integral Form

Statement: Suppose $f : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is Lebesgue measurable and $1 \leq p < \infty$. Then

$$\left(\int\left|\int h(x,y)\,dy\right|dx\right)^{1/p} \leq \int\left(\left|\int h(x,y)^p\,dx\right|\right)^{1/p} dy$$

Proof: By an approximation argument we need only consider h of the form

$$h(x,y)=\sum_{j=1}^{N} f_j(x)1\,f_j(y), (x,y)\in\mathcal{R}\times\mathcal{R},$$

where N is a positive integer, f_j is Lebesgue measurable, and $F_j \in L_n$, $j = 1, \ldots N_i$ and $F_i \cap F_j = \phi$ if $1 \leq i < j \leq N$. We use Minkowski's inequality to estimate

$$\left(\int\left|\int h(x,y)\,dy\right|dx\right)^{1/p} = \left(\left|\sum_{j=1}^{N}\|F_j\|\,f_j(x)\right|_{dx}^{p}\right)^{1/p} \leq \sum_{j=1}^{N}\|F_j\|\left(\int |f_i(x)^p dx\right)^{1/p}$$

But

$$\int\left(\left|\int h(x,y)\right|^p dx\right)^{1/p} dy = \sum_{j=1}^{N}\int_{F_j}\left(\int |h(x,y)|^p\,dx\right)^{1/p} = \sum_{j=1}^{N}\int_{F_j}\left(\int |f_j(x)|^p\,dx\right)^{1/p}$$

Example: If $<f_n>$ is a sequence of functions belonging to $L^2(a, b)$ and also $f \in L^2(a, b)$ and $\text{Lim} \| f_n - f \|_2 = 0$, then prove that

$$\int_a^b f^2 dx = \text{Lim} \int_a^b f_n^2 dx$$

Solution: By Minkowski's inequality, we get

$$\left| \| f_n \|_2 - \| f_2 \| \right| \le \| f_n - f \|_2$$

$$\Rightarrow \quad \text{Lim} \left| \| f_n \|_2 - \| f \|_2 \right| \le \text{Lim} \| f_n - f \|_2 = 0$$

$$\Rightarrow \quad \text{Lim} \left| \| f_n \|_2 - \| f \|_2 \right| = 0 \Rightarrow \text{Lim} \| f_n \|_2 = \| f \|_2$$

$$\Rightarrow \quad \text{Lim} \left(\int_a^b (f_n)^2 dx \right)^{1/2} = \left(\int_a^b f^2 dx \right)^{1/2} \Rightarrow \text{Lim} \int_a^b f_n^2 \, dx = \int_a^b f^2 dx \, .$$

6.2 Summary

- The class of all measurable function f (x) is known as L^p space over [a, b], if Lebesgue integrable over [a, b] for each p exists, $0 < p < \infty$.
- If f and $g \in L^p$ $(1 \le p \le \infty)$, then $f + g \in L^p$ and $\| f + g \|_p \le \| f \|_p + \| g \|_p$.

6.3 Keywords

L^p-space: The class of all measurable functions f (x) is known as L^p-space over [a, b], if Lebesgue-integrable over [a, b] for each exists, $0 < p < \infty$, i.e.,

$$\int_a^b | f |^p \, dx < \infty, (p > 0)$$

and is denoted by L^p [a, b].

Minkowski Inequality in Integral Form: Suppose $f : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is Lebesgue measurable and $1 \le p < \infty$. Then

$$\left(\int \left| \int h(x, y) \, dy \right| dx \right)^{1/p} \le \int \left(\left| \int h(x, y)^p \, dx \right| \right)^{1/p} dy$$

Minkowski Inequality: Minkowski inequality establishes that the L^p spaces are normed vector spaces. Let S be a measure space, let $1 \le p \le \infty$ and let f and g be elements of L^p (s). Then f + g is in L^p (s), we have the triangle inequality

$$\| f + g \|_p \le \| f \|_p + \| g \|_p$$

with equality for $1 < p < \infty$ if and only if f and g are positively linearly dependent.

Notes

6.4 Review Questions

1. If f, g are square integrable in the Lebesgue sense, prove that f + g is also square integrable and

$$\| f + g \|_2 \leq \| f \|_2 + \| g \|_2.$$

2. If $| < p < \infty$, then show that equality can be true, iff there are non-negative constants α and β, such that $\beta f = \alpha g$.

6.5 Further Readings

Books

Books: Stein, Elias (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press.

Hardy, G.H.; Littlewood, J.E.; Polya, G. (1952). *Inequalities,* Cambridge Mathematical Library (second ed.). Cambridge: Cambridge University Press.

Online links

Mathworld.wolfram.com>Calculus and Analysis>Inequalities

Planet math.org/Minkowski In-equality.html

Unit 7: Convergence and Completeness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand convergence and completeness.
- Understand Riesz-Fischer theorem.
- Solve problems on convergence and completeness.

Introduction

Convergence of a sequence of functions can be defined in various ways, and there are situations in which each of these definitions is natural and useful. In this unit, we shall start with the definition of convergence and Cauchy sequence and proceed with the topic completeness of L^p.

7.1 Convergence and Completeness

7.1.1 Convergent Sequence

Definition: A sequence $<x_n>$ in a normal linear space X with norm $\|\cdot\|$ is said to converge to an element $x \in X$ if for arbitrary $\in > 0$, however small, $\exists\, n_0 \in N$ such that $\| x_n - x \| < \in, \forall\ n > n_0$.

Then we write $\lim\limits_{n\to\infty} x_n = x$.

7.1.2 Cauchy Sequence

Notes

Definition: A sequence $<x_n>$ in a normal linear space $(X, \|\cdot\|)$ is said to be a Cauchy sequence if for arbitrary $\in > 0$, $\exists\, n_0 \in N$ s.t.

$$\| x_n - x_m \| < \in, \ \forall \ n, m \geq n_0.$$

7.1.3 Complete Normed Linear Space

Definition: A normed linear space $(X, \|\cdot\|)$ is said to be complete if every Cauchy sequence $<x_n>$ in it converges to an element $x \in X$.

7.1.4 Banach Space

Definition: A complete normed linear space is also called Banach space.

7.1.5 Summable Series

Definition: A series $\sum_{n=1}^{\infty} u_n$ in N_1 is said to be summable to a sum u if $u \in N_1$ and $\lim_{n\to\infty} S_n = u$, where

$$S_n = u_1 + u_2 + \ldots + u_n$$

In this case, we write

$$u = \sum_{n=1}^{\infty} u_n .$$

Further, the series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely summable if $\sum_{n=1}^{\infty} \|u_n\| < \infty$.

7.1.6 Riesz-Fischer Theorem

Theorem: The normed L^p-spaces are complete for $(p \geq 1)$.

Proof: In order to prove the theorem, we shall show that every Cauchy sequence in L^p [a, b] space converges to some element f in L^p-space. Let $<f_n>$ be one of such sequences in L^p-space. Then for given $\in > 0$, $\exists$ a natural number n_0, such that

$$m, n \geq n_0 \Rightarrow \| f_m - f_n \|_p < \in,$$

since $\in$ is arbitrary therefore taking $\in = \frac{1}{2}$, we can find a natural number n_1 such that

$$\text{for all } m, n \geq n_1 \Rightarrow \| f_m - f_n \|_p < \frac{1}{2}$$

Similarly, taking $\in = \frac{1}{2^k}, \forall\, k \in N$, we can find a natural number n_k, such that

$$\text{for all } m, n \geq n_k \Rightarrow \| f_m - f_n \|_p < \frac{1}{2^k}$$

In particular, $m > n_k \Rightarrow \| f_m - f_{n_k} \| < \frac{1}{2^k}$.

Obviously $n_1 < n_2 < n_3 \dots < n_k < \dots$

i.e. $\langle n_k \rangle$ is a monotonic increasing sequence of natural numbers.

Set $g_k = f_{n_k}$, then from above, we have

$$\| g_2 - g_1 \|_p = \left\| f_{n_2} - f_{n_2} \right\|_p < \frac{1}{2},$$

$$\| g_3 - g_2 \|_p = \left\| f_{n_3} - f_{n_2} \right\|_p < \frac{1}{2^2},$$

… … … … …

… … … … …

$$\| g_{k+1} - g_k \|_p = \left\| f_{n_{k+1}} - f_{n_k} \right\|_p < \frac{1}{2^k}.$$

… … … … …

… … … … …

Adding these inequalities, we get

$$\sum_{k=1}^{\infty} \left\| g_{k+1} - g_k \right\|_p < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \qquad \dots \text{(i)}$$

Thus $\sum_{k=1}^{\infty} \left\| g_{k+1} - g_k \right\|_p$ is convergent. Define g such that

$$g(x) = \left| g_1(x) \right| + \sum_{k=1}^{\infty} \left| g_{k+1} - g_k \right|_p \text{ if R.H.S. is convergent} \qquad \dots \text{(ii)}$$

and g (x) = ∞, if right hand side is divergent.

Now,
$$\left(\int_a^b |g(x)|^p \, dx \right) = \lim_{n \to \infty} \left\{ \int_a^b | g_1(x) + \sum_{k=1}^{n} \left| g_{k+1} - g_k \right|^p_{dx} \right\}^{\frac{1}{p}}$$

or
$$\| g \|_p = \lim_{n \to \infty} \left(\left\| g_1 \right\|_p + \sum_{k=1}^{n} \left\| g_{k+1} - g_k \right\|_p \right) \text{ (By Minkowski's inequality)}$$

$$= \| g_1 \|_p + \sum_{k=1}^{\infty} \left\| g_{k+1} - g_k \right\|_p < \| g_1 \|_p + 1, \qquad \text{[by (i)]}$$

⇒ $\| g \|_p < \infty \Rightarrow g \in L^p [a, b]$.

Let $E = \{x \in [a, b] : g(x) = \infty\}$.

Now we define a function f such that **Notes**

$$f(x) = 0, \ \forall \ x \in E$$

and $$f(x) = g_1(x) + \sum_{k=1}^{\infty}\left(g_{k+1} - g_k\right), \text{ for } x \in [a, b] \text{ but } x \notin E,$$

or $$f(x) = \lim_{m\to\infty}\left(g_1 + \sum_{k=1}^{m-1}\left(g_{k+1} - g_k\right)\right), \text{ for } x \notin E$$

$$= \lim_{m\to\infty} g_m(x)$$

Thus $$f(x) = 0, \text{ for } x \in E \text{ and}$$

$$f(x) = \lim_{m\to\infty} g_m(x) \ \text{ for } x \notin E.$$

$\therefore$ $$f(x) = \lim_{m\to\infty} g_m(x) \ \text{ a.e. in } [a, b]$$

or $$\lim_{m\to\infty}\left|g_m - f\right| = 0 \ \text{ a.e. in } [a, b] \qquad \dots \text{(iii)}$$

Also, $$g_m(x) = g_1 + \sum_{k=1}^{m-1}\left(g_{k+1} - g_k\right)$$

$\Rightarrow$ $$|g_m| \le |g_1| + \sum_{k=1}^{m-1}\left(g_{k+1} - g_k\right)$$

$$\le |g_1| + \sum_{k=1}^{\infty}\left(g_{k+1} - g_k\right) = g,$$

$\Rightarrow$ $$|g_m| \le g, \ \forall \ m \in N$$

$\Rightarrow$ $$\lim_{m\to\infty}\left|g_m(x)\right| \le g$$

$\therefore$ $$\text{(iii)} \Rightarrow |f| \le g.$$

Again, $$|g_m - f| \le |g_m| + |f| \le g + g = 2g.$$

$\therefore$ $$|g_m - f| \le 2\,g.$$

Thus there exists a function $g \in L^p[a, b]$ s.t.

$$|g_m - f| \le 2g, \ \forall \ m$$

and $$\lim_{m\to\infty}\left|g_m - f\right| = 0 \text{ a.e. in } [a, b] \qquad \dots \text{(iv)}$$

Applying Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty}\int_a^b |g_m - f|^p\,dx = \int_a^b \lim_{m\to\infty}|g_m - f|^p\,dx = \int_a^b 0 \cdot dx = 0 \qquad \text{[Using (iv)]}$$

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$$\Rightarrow \quad \left(\lim_{m\to\infty}\int_a^b |g_m - f|^p\,dx\right)^{\frac{1}{p}} = 0$$

$$\Rightarrow \quad \lim_{m\to\infty}\|g_m - f\|_p = 0$$

$$\Rightarrow \quad \lim_{m\to\infty}\|f_{n_m} - f\|_p = 0 \text{ as } g_m = f_{n_m}$$

$$\Rightarrow \quad \|f_{n_m} - f\|_p < \in'.$$

Also $\|f_m - f_{n_m}\|_p < \in.$

$$\therefore \quad \| f_m - f \|_p = \|f_m - f_{n_m} + f_{n_m} - f\|_p$$

$$\leq \|f_m - f_{n_m}\|_p + \|f_{n_m} - f\|_p$$

$$< (\in + \in') = \in''.$$

Hence $\lim_{m\to\infty}\|f_m - f\|_p = 0$

or $\lim_{m\to\infty} f_m = f \in L^p [a, b].$

This proves the theorem.

Alternative Statement of this Theorem

A convergent sequence $<f_n>$ in L^p-spaces has a limit in L^p-space.

Or

Every Cauchy sequence $<f_n>$ in the L^p-space converges to a function in L^p-space.

Theorem: Prove that a normed linear space is complete iff every absolutely summable sequence is summable.

Proof: *Necessary part*

Let X be a complete normed linear space with norm $\|\cdot\|$ and $<f_n>$ be an absolutely summable sequence of elements of X

$$\Rightarrow \quad \sum_{n=1}^{\infty}\|f_n\| = M < \infty,$$

$\Rightarrow$ For arbitrary $\in > 0$, however small, $\exists n_o \in N$

$$\text{s.t.} \quad \sum_{n=n_o}^{\infty}\|f_n\| < \in, \qquad \ldots \text{(i)}$$

Now, if $S_n = \sum_{i=1}^{n} f_i$, then $\forall\ n \geq m \geq n_o$, we get

Notes

$$\| S_n - S_m \|_p = \left\| \sum_{i=m+1}^{n} f_i \right\| \leq \sum_{i=m+1}^{n} \| f_i \|$$

$$\leq \sum_{i=n_0}^{\infty} \| f_i \| < \in .$$

⇒ Sequence $<S_n>$ of partial sums is a Cauchy sequence

⇒ $<S_n>$ converges.

⇒ Sequence $<f_n>$ is summable to some element $S \in X$.

But X is a complete space. Therefore $<S_n>$ will converge to some element $S \in X$.

Sufficient part: Given that every absolutely summable sequence in the space X is summable.

To show that X is complete.

Let $<f_n>$ be a Cauchy sequence in X.

For each positive integer k, we can choose a number $n_k \in N$ such that

$$\| f_n - f_m \| < \frac{1}{2^k}, \ \forall \ n, m \geq n_k \qquad \text{… (ii)}$$

We can choose these n_k's such that $n_{k+1} > n_k$.

Then $\left\langle f_{n_{k=1}} \right\rangle^{\infty}$ is a subsequence of $<f_n>$.

Setting $g_1 = f_{n_1}$ and $g_k = f_{n_k} - f_{n_{k-1}}$, (k > 1), we get a sequence $<g_k>$ s.t. its k^{th} partial.

Sum = $S_k = g_1 + g_2 + … + g_k = f_{n_1} + (f_{n_2} - f_{n_1}) + … + (f_{n_k} - f_{n_{k-1}}) = f_{n_k}$.

Now, $\| g_k \| = \left\| f_{n_k} - f_{n_{k-1}} \right\| < \frac{1}{2^{k-1}}$, [by (ii)], $\forall$ k > 1

⇒ $$\sum_{k=1}^{\infty} \| g_k \| \leq \| g_1 \| + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \| g_1 \| + 1 \quad \text{(a finite quantity)}$$

⇒ The sequence $<g_k>$ is absolutely summable and hence by the hypothesis, it is a summable sequence.

⇒ The sequence of partial sums of this sequence converges to some $S \in X$.

⇒ The sequence $<S_k>$ converges and hence $\left\langle f_{n_k} \right\rangle$ converges to some $f \in X$.

Now, we shall show that the limit $f_n = f$.

Again, since $<f_n>$ is a Cauchy sequence, we get that for each $\in > 0$, however small, $\exists n' \in N$ s.t. $\forall$ n, m > n′.

$$\| f_n - f_m \| < \frac{\varepsilon}{2}.$$

Also since $f_{n_k} \to f$, $\exists n'' \in N$ such that $\forall k > n''$,

$$\| f_{n_k} - f \| < \frac{\varepsilon}{2}.$$

Choosing a number k, as large that k > n″ and n_k > n′, we get

$$\| f_n - f \| \le \| f_n - f_{n_k} \| + \| f_{n_k} - f \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \in,$$

⇒ $\forall$ n > n′, we obtain $\| f_n - f \| < \varepsilon$, where ε is an arbitrary quantity.

⇒ $f_n \to f \in X$ and hence X is a complete space.

Theorem: Let $\{f_n\}$ be a sequence in L^p, $1 \le p < \infty$, such that $f_n \to f$ a.e. and that $f \in L^p$.

If $\lim_{n\to\infty} \| f_n \|_p = \| f \|_p$, then

$$\lim_{n\to\infty} \| f_n - f \|_p = 0.$$

Proof: Without any loss of generality, we may assume that each $f_n \ge 0$ a.e. so that f is also ≥ 0 a.e. since the result in general case follows by considering $f = f^+ - f^-$.

Now, let a and b be any pair of non-negative real numbers, we have

$$| a - b |^p \le 2^p (| a |^p + | b |^q),$$

$$1 \le p < \infty$$

So, we get

$$2^p (| f_n |^p + (| f |^p) - | f_n - f |^p \ge 0 \text{ a.e.}$$

Thus, by Fatou's Lemma and by the given hypothesis,

We get

$$\begin{aligned} 2^{p+1} \int | f |^p &= \int \lim_{n\to\infty} \left[2^p \left(|f_n|^p + |f|^p \right) - |f_n - f|^p \right] \\ &\le \liminf_{n\to\infty} \int \left[2^p \left(|f_n|^p + |f|^p \right) - |f_n - f|^p \right] \\ &= 2^{p-1} \lim_{n\to\infty} \int |f_n|^p + 2^p \int |f|^p + \liminf_{n\to\infty} \left(-\int |f_n - f|^p \right) \\ &= 2^{p+1} \int |f|^p - \limsup_{n\to\infty} \int |f_n - f|^p . \end{aligned}$$

Since $\int | f |^p < \infty$ it follows that

$$\limsup_{n\to\infty} \int | f_n - f |^p \le 0.$$

Therefore $\limsup_{n\to\infty} \int | f_n - f |^p = \liminf_{n\to\infty} \int | f_n - f |^p = 0$,

So that $\lim_{n\to\infty} \int | f_n - f |^p = 0$

Notes

$$\Rightarrow \qquad \lim_{n\to\infty}\left(\int |f_n - f|^p \, dt\right)^{\frac{1}{p}} = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty}\|f_n - f\|_p = 0.$$

Theorem: In a normed linear space, every convergent sequence is a Cauchy sequence.

Proof: Let the sequence $<x_n>$ in a normed linear space N, converges to a point $x_o \in N_1$. We shall show that it is a Cauchy sequence.

Let $\in > 0$ be given. Since the sequence converges to x_o $\exists$ a positive integer m_o s.t.

$$n \geq m_o \Rightarrow \| x_n - x_o \| < \in/2 \qquad \dots (1)$$

Hence for all m, $n \geq m_o$, we have

$$\| x_m - x_n \| = \| x_m - x_o + x_o - x_n \|$$

$$\leq \| x_m - x_o \| + \| x_o - x_n \|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in \text{ by (1)}$$

It follows that the convergent sequence $<x_n>$ is a Cauchy sequence.

Theorem: Prove that L^∞ [0, 1] is complete.

Proof: Let (f_n) be any Cauchy sequence in L^∞, and let

$$A_k = \{x : |f_k(x)| > \| f_k \|_\infty\},$$

$$B_{m,n} = \{x : |f_k(x) - f_m(x)| > \| f_k - f_m \|_\infty\}.$$

Then $m(A_k) = 0 = m(B_{m,n})$ (k, m, n = 1, 2, 3, …),

So that if E is the union of these sets, we have m (E) = 0.

Now, if $x \in F = [0, 1] - E$, then

$$|f_k(x) \leq \| f_k \|_\infty$$

$$|f_n(x) - f_m(x) \leq \| f_n - f_m \|_\infty \to 0 \text{ as } n, m \neq \infty.$$

Hence the sequence (f_n) converges uniformly to a bounded function on F.

Define $f : [0, 1] \to R$ by

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if } x \in F \\ 0, & \text{if } x \in E \end{cases}$$

Then $f \in L^\infty$ and $\| f_n - f \|_\infty \to 0$ as $n \to \infty$.

Thus L^∞ is

Hence proved.

7.2 Summary

- A sequence $<x_n>$ in a normal linear space X with norm $\| \cdot \|$ is said to converge to an element $x \in X$ if for arbitrary $\in > 0$, however small, $\exists$ n_o N s.t. $\| x_n - x \| < \in$, $\forall$ $n > n_o$. Then we write $\lim\limits_{n\to\infty} x_n = x$.

- A complete normed linear space is also called Banach space.
- The normed L^p-spaces are complete for ($p \geq 1$).
- A convergent sequence $<f_n>$ in L^p-spaces has a limit in L^p-space.
- A normed linear space is complete iff every absolutely summable sequence is summable.
- In a normed linear space, every convergent sequence is a Cauchy sequence.

7.3 Keywords

Banach Space: A complete normed linear space is also called Banach space.

Cauchy Sequence: A sequence $<x_n>$ in a normal linear space $(X, \|\cdot\|)$ is said to be a Cauchy sequence if for arbitrary $\in\; > 0$, $\exists\; n_0 \in N$ s.t.

$$\| x_n - x_m \| < \in, \;\forall\; n, m \geq n_0.$$

Complete Normed Linear Space: A normed linear space $(X, \|\cdot\|)$ is said to be complete if every Cauchy sequence $<x_n>$ in it converges to an element $x \in X$.

Convergence almost Everywhere: Let $<f_n>$ be a sequence of measurable functions defined over a measurable set E. Then $<f_n>$ is said to converge almost everywhere in E if there exists a subset E_o of E s.t.

(i) $f_n(x) \to f(x), \;\forall\; x \in E - E_o$.

and (ii) $m(E_o) = 0$.

Convergent Sequence: A sequence $<x_n>$ in a normal linear space X with norm $\|\cdot\|$ is said to converge to an element $x \in X$ if for arbitrary $\in\; > 0$, however small, $\exists\; n_0 \in N$ such that $\| x_n - x \| < \in, \;\forall\; n > n_0$.

Then we write $\lim\limits_{n\to\infty} x_n = x$.

Normed Linear Space: A linear space N together with a norm defined on it, i.e., the pair $(N, \|\;\|)$ is called a normed linear space.

Summable Series: A series $\sum\limits_{n=1}^{\infty} u_n$ in N_1 is said to be summable to a sum u if $u \in N_1$ and $\lim\limits_{n\to\infty} S_n = u$, where

$$S_n = u_1 + u_2 + \ldots + u_n$$

7.4 Review Questions

1. Prove that ℓ_p^n is complete.
2. Prove that the vector space L^∞ equipped with $\|\cdot\|_{L^\infty}$ is a complete vector space.
3. Suppose $f \in L^\infty$ is supported on a set of finite measure.

   Then $f \in L^p$ for all $p < \infty$, and

   $$\| f \|_{L^\infty} \to \| f \|_{L^\infty} \text{ as } p \to \infty.$$

Notes

If $f \in L^p$ $(p > 0)$, $f \geq 0$ and $f_n = \min(f, n)$ $(n \in N)$, show that $f_n \in L^p$ and $\lim_{n\to\infty} \| f_n - f \|_p = 0$.

7.5 Further Readings

Books

H.L. Royden, *Real analysis*.

Walter Rudin, *Real and Complex Analysis*, Third, McGraw-Hill Book Co., New York, 1987.

Online links

www.public.iastate.edu

www.scribd.com/doc/49732162/103.Convergence-and-Completeness.

Unit 8: Bounded Linear Functional on the L^p-spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand bounded linear functional on L^p-spaces
- Understand related theorems.
- Solve problems on bounded linear functionals.

Introduction

In this unit, we obtain the representation of bounded linear functionals on L^p-space. We shall also study about linear functional, continuous linear functionals and norm of $f \in \ell_p^*$. Further, we shall prove important theorems on bounded linear functionals.

8.1 Bounded Linear Functionals on L^p-spaces

8.1.1 Linear Functional

Definition: Let N_1 be a normed space over a field R (or C). A mapping $f : N_1 \to R$ (or C) is called a linear functional on N_1 if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\forall\ x, y \in N_1$ and $\alpha, \beta \in R$ (or C).

8.1.2 Bounded Linear Functional

Definition: A linear functional f on a normed space N_1 is said to be bounded if there is a constant $k > 0$ such that

$$|f(x)| \le k \|x\|,\ \forall\ x \in N_1 \qquad \ldots (1)$$

The smallest constant k for which (1) holds is called the norm of f, written $\| f \|$.

Thus $\| f \| = \sup\left\{\dfrac{|f(x)|}{\|x\|} : x \neq 0 \text{ and } x \in N_1\right\}$ or equivalently

$$\| f \| = \sup \{ | f (x) | : x \in X \text{ and } \| x \| = 1\}.$$

Also $\quad | f (x) | \leq \| f \| \; \| x \| \;\; \forall \; x \in N_1.$

Definition: Let $p \in R$, $p > 0$. We define $L^p = L^p [0, 1]$ to be the set of all real-valued functions on [0, 1] such that

(i) f is measurable and (ii) $\| f \|_p = \left(\int_0^1 | f |^p\right)^{\frac{1}{p}} < \infty.$

> *Note* L^1 or simply L denotes the class of measurable function f (x) which are also L-integrable.

8.1.3 Bounded Linear Functional on L^p-spaces

If $x \in \ell_p$ and f is bounded linear functional on ℓ_p, then f has the unique representation of the form as an infinite series

$$f (x) = \sum_{k=1}^{\infty} x_k f(e_k)$$

8.1.4 Norm

The norm of $f \in \ell_p^*$ is given by

$$\| f \| = \left\{\sum_{k=1}^{\infty} | f(e_k) |^q\right\}^{\frac{1}{q}}$$

Likewise in finite dimensional case, the bounded linear functionals are characterised by the values they assume on the set e_k, k = 1, 2, 3, … .

8.1.5 Continuous Linear Functional

A linear functional f is continuous if given $\in > 0$ there exists $\delta > 0$ so that

$$| f (x) - f (y) | \leq \in \text{ whenever } \| x - y \| \leq \delta.$$

8.1.6 Theorems

Theorem 1: Suppose $1 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then, with $\mathcal{B} = L^p$ we have

$$\mathcal{B}^* = L^q ,$$

Notes

in the following sense: For every bounded linear functional ℓ on L^p there is a unique $g \in L^q$ so that

$$\ell(f) = \int_x f(x)g(x)\,d\mu(x), \text{ for all } f \in L^p$$

Moreover, $\|\ell\|_{B^*} = \|g\|_{L^q}$.

This theorem justifies the terminology where by q is usually called the dual exponent of p.

The proof of the theorem is based on two ideas. The first, as already seen, is Hölder's inequality; to which a converse is also needed. The second is the fact that a linear functional ℓ on L^p, $1 \le p < \infty$, leads naturally to a (signed) measure υ. Because of the continuity of ℓ the measure υ is absolutely continuous with respect to the underlying measure μ, and our desired function g is then the density function of υ in terms of μ.

We begin with:

Lemma: Suppose $1 \le p, q \le \infty$, are conjugate exponents.

(i) If $g \in L^q$, then $\|g\|_{L^q} = \sup_{\|f\|_{L^p} \le 1} \left|\int fg\right|$.

(ii) Suppose g is integrable on all sets of finite measure and

$$\sup_{\substack{\|f\|_{L^p} \le 1 \\ f \text{ simple}}} |fg| = M < \infty$$

Then $g \in L^q$, and $\|g\|_{L^q} = M$.

For the proof of the lemma, we recall the signum of a real number defined by

$$\text{sign}(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Proof: We start with (i). If g = 0, there is nothing to prove, so we may assume that g is not 0 a.e., and hence $\|g\|_{L^q} \neq 0$. By Hölder's inequality, we have that

$$\|g\|_{L^q} \ge \sup_{\|f\|_{L^p} \le 1} \left|\int fg\right|.$$

To prove the reverse inequality we consider several cases.

- First, if q = 1 and p = ∞, we may take f (x) = sign g (x). Then, we have $\|f\|_{L^\infty} = 1$ and clearly $\int fg = \|g\|_{L^1}$.

- If $1 < p, q < \infty$, then we set $f(x) = |g(x)|^{q-1} \text{ sign } g(x) / \|g\|_{L^q}^{q-1}$. We observe that $\|f\|_{L^p}^p = \int |g(x)|^{p(q-1)} d\mu / \|g\|_{L^q}^{p(q-1)} = 1$ since p (q – 1) = q, and that $\int fg = \|g\|_{L^q}$.

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- Finally, if $q = \infty$ and $p = 1$, let $\in > 0$, and E a set of finite positive measure, where $|g(x)| \geq \|g\|_{L^\infty} - \in$. (Such a set exists by the definition of $\|g\|_{L^\infty}$ and the fact that the measure μ is σ-finite). Then, if we take $f(x) = \mathcal{X}E(x)\ \text{sign}\ g(x)/\mu(E)$, where $\mathcal{X}E$ denotes the characteristic function of the set E, we see that $\|f\|_{L^1} = 1$, and also

$$\left|\int fg\right| = \frac{1}{\mu(E)}\int_E |g| \geq \|g\|_\infty - \in .$$

This completes the proof of part (i).

To prove (ii) we recall that we can find a sequence $\{g_n\}$ of simple functions so that $|g_n(x) \leq |g(x)|$ while $g_n(x) \to g(x)$ for each x. When $p > 1$ (so $q < \infty$), we take $f_n(x) = |g_n(x)|^{q-1}\ \text{sign}\ g(x)/\|g_n\|_{L^q}^{q-1}$. As before, $\|f_n\|_{L^p} = 1$. However

$$\int f_n g = \frac{\int |g_n(x)|^q}{\|g_n\|_{L^q}^{q-1}} = \|g_n\|_{L^q},$$

and this does not exceed M. By Fatou's Lemmas if follows that $\int |g|^q \leq M^q$, so $g \in L^q$ with $\|g\|_{L^q} \leq M$. The direction $\|g\|_{L^q} \geq M$ is of course implied by Hölder's inequality. When p = 1 the argument is parallel with the above but simpler. Here we take $f_n(x) = (\text{sign}\ g(x))\ \mathcal{X}E_n(x)$, where E_n is an increasing sequence of sets of finite measure whose union is X. The details may be left to the reader.

With the lemma established we turn to the proof of the theorem. It is simpler to consider first the case when the underlying space has finite measure. In this case, with ℓ the given functional on L^p, we can then define a set function υ by

$$\upsilon(E) = \ell(\mathcal{X}E),$$

where E is any measurable set. This definition make sense because $\mathcal{X}_E$ is now automatically in L^p since the space has finite measure. We observe that

$$|\upsilon(E)| \leq C(\mu(E))^{1/p} \qquad \dots (1)$$

where C is the norm of the linear functional, taking into account the fact that $\|\mathcal{X}_E\|_{L^p} = (\mu(E))^{1/p}$.

Now the linearity of ℓ clearly implies that υ is finitely-additive. Moreover, if $\{E_n\}$ is a countable collection of disjoint measurable sets, and we put $E = \bigcup_{n=1}^{\infty} E_n$, $E_N^* = \bigcup_{n=N+1}^{\infty} E_n$, then obviously

$$\mathcal{X}_E = \mathcal{X}_{E_N^*} + \bigcup_{n=1}^{N} \mathcal{X}E_n .$$

Then $\upsilon(E) = \upsilon(E_N^*) + \sum_{n=1}^{N} \upsilon(E_n)$. However $\upsilon(E_N^*) \to 0$, as $N \to \infty$ because of (1) and the assumption $p < \infty$. This shows that υ is countably additive and moreover (1) also shows us that υ is absolutely continuous with respect to μ.

We can now invoke the key result about absolutely continuous measures, the Lebesgue-Radon – Nykodin theorem. It guarantees the existence of an integrable function g so that $\upsilon(E) = \int_E g\ du$

Notes

for every measurable set E. Thus we have $\ell(\mathcal{X}E) = \int \mathcal{X}Eg\, d\mu$. The representation $\ell(f) = \int fg\, d\mu$ then extends immediately to simple function f, and by a passage to the limit, to all $f \in L^p$ since the simple functions are dense in L^p, $1 \le p < \infty$. Also by lemma, we see that $\|g\|_{L^q} = \|\ell\|$.

To pass from the situation where the measure of X is finite to the general case, we use an increasing sequence $\{E_n\}$ of sets of finite measure that exhaust X, that is, $X = \bigcup_{n=1}^{\infty} E_n$. According to what we have just proved, for each n there is an integrable function g_n on E_n (which we can set to be zero in E_n^c) so that

$$\ell(f) = \int fg_n\, d\mu \qquad \dots (2)$$

whenever f is supported in E_n and $f \in L^p$. Moreover by conclusion (ii) of the lemma $\|g_n\|_{L^p} \le \|\ell\|$.

Now it is easy to see because of (2) that $g_n = g_m$ a.e. on E_m, whenever $n \ge m$. Thus $\lim_{n \to \infty} g_n(x) = g(x)$ exists for almost every x, and by Fatou's lemma, $\|g\|_{L^q} \le \|\ell\|$. As a result we have that $\ell(f) = \int f g\, du$ for each $f \in L^p$ supported in E_n, and then by a simple limiting argument, for all $f \in L^p$ supported in E_n. The fact that $\|\ell\| \le \|g\|_{L^q}$ is already contained in Hölder's inequality and therefore the proof of the theorem is complete.

Theorem 2: Let f be a linear functional defined on a normed linear space N, then f is bounded $\Leftrightarrow$ f is continuous.

Proof: Let us first show that continuity of f $\Rightarrow$ boundedness of f.

If possible let f is continuous but not bounded. Therefore, for any natural number n, however large, there is some point x_n such that

$$|f(x_n)| \ge n \|x_n\| \qquad \dots (1)$$

Consider the vector $y_n = \frac{x_n}{n\|x_n\|}$ so that

$$\|y_n\| = \frac{1}{n}.$$

$\Rightarrow$ $\|y_n\| \to 0$ as $n \to \infty$.

$\Rightarrow$ $y_n \to \infty$ 0 in the norm.

Since any continuous functional maps zero vector into zero, and f is continuous $f(y_n) \to f(0) = 0$.

But $$|f(y_n)| = \frac{1}{n\|x_n\|} f(x_n) \qquad \dots (2)$$

It now follows from (1) and (2) that

$|f(y_n)| > 1$, a contradiction to the fact that

$f(y_n) \to 0$ as $n \to \infty$.

Thus if f is bounded then f is continuous.

Conversely, let f is bounded. Then for any sequence (x_n), we have

$|f(x_n)| \leq k \| x_n \| \; \forall \, n = 1, 2, \ldots$ and $k \geq 0$.

Let $x_n \to 0$ as $n \to \infty$ then

$f(x_n) \to 0$

⇒ f is continuous at the origin and consequently it is continuous everywhere.

This completes the proof of the theorem.

Theorem 3: If L is a linear space of all n-tuples, then

(i) $\left(\ell_p^n\right)^* = \ell_q^n$

(ii) $\left(\ell_1^n\right)^* = \ell_\infty^n$

(iii) $\left(\ell_\infty^n\right)^* = \ell_1^n$

Proof: Let $(e_1, e_2, \ldots, e_n)$ be a standard basis for L so that any $x = (x_1, x_2, \ldots, x_n) \in L$ can be written as $x = x_1e_1 + x_2e_2 + \ldots + x_ne_n$.

If f is a scalar valued linear function defined on L, then we get

$$f(x) = x_1 f(e_1) + x_2 f(e_2) + \ldots + x_n f(e_n) \qquad \ldots (1)$$

⇒ f determines and is determined by n scalars $y_i = f(e_i)$.

Then the mapping

$y = (y_1, y_2, \ldots y_n) \to f$ where $f(x) = \sum_{i=1}^{n} x_i y_i$ is an isomorphism of L onto the linear space L′ of all function f. We shall establish (i) – (iii) by using above given facts.

(i) If we consider the space $L = \ell_p^n$ $(1 \leq p < \infty)$ with the p^{th} norm, then f is continuous and L′ represents the set of all continuous linear functionals on ℓ_p^n so that $L' = \left(\ell_p^n\right)^*$.

Now for $y \to f$ as an isometric isomorphism we try to find the norm of y's.

For $1 < p < \infty$, we show that

$$\left(\ell_p^n\right)^* = \ell_p^n .$$

For $x \in \ell_p^n$, we have defined,

$$\|x\| = \left\{\sum_{i=1}^{n} |x_i|^p\right\}^{\frac{1}{p}}$$

Now $|f(x)| = \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n} |x_i||y_i|$

By using Holder's inequality, we get

$$\sum_{i=1}^{n} |x_i y_i| \leq \left\{\sum_{i=1}^{n} |x_i|^p\right\}^{\frac{1}{p}} \left\{\sum_{i=1}^{n} |y_i|^q\right\}^{\frac{1}{q}} \text{ so that}$$

Notes

$$|f(x)| \le \left\{\sum_{i=1}^{n}|y_i|^q\right\}^{\frac{1}{q}}\left\{\sum_{i=1}^{n}|x_i|^p\right\}^{\frac{1}{p}}$$

Using the definition of norm for f, we get

$$\|f\| \le \left\{\sum_{i=1}^{n}|y_i|^q\right\}^{\frac{1}{q}} \quad \ldots (2)$$

Consider the vector, defined by

$$x_i = \frac{|y_i|^q}{y_i}, y_i \ne 0 \text{ and } x_i = 0 \text{ if } y_i = 0 \quad \ldots (3)$$

Then, $$\|x\| = \left\{\sum_{i=1}^{n}|x_i|^p\right\}^{\frac{1}{p}} = \left[\sum_{i=1}^{n}\left\{\frac{|y_i|^q}{|y_i|}\right\}^p\right]^{\frac{1}{p}} \quad \ldots (4)$$

Since q = p (q – 1) we have from (4),

$$\|x\| = \left\{\sum_{i=1}^{n}|y_i|^q\right\}^{\frac{1}{q}} \quad \ldots (5)$$

Now $$|f(x)| = \left|\sum_{i=1}^{n}x_i y_i\right| = \left|\sum_{i=1}^{n}\frac{|y_i|^q}{y_i}y_i\right| \quad \ldots (4)$$

$$= \sum_{i=1}^{n}|y_i|^q \quad \text{(By (3))}$$

So that

$$\sum_{i=1}^{n}|y_i|^q = |f(x) \le \|f\|\ \|x\| \quad \ldots (6)$$

From (5) and (6) we get,

$$\left\{\sum_{i=1}^{n}|y_i|^q\right\}^{1-\frac{1}{p}} \le \|f\|$$

$$\Rightarrow \quad \left\{\sum_{i=1}^{n}|y_i|^q\right\}^{\frac{1}{q}} \le \|f\| \quad \ldots (7)$$

Also from (2) and (7) we have

$$\|f\| = \left\{\sum_{i=1}^{n}|y_i|^q\right\}^{\frac{1}{q}}, \text{ so that}$$

$y \to f$ is an isometric isomorphism.

Hence $\left(\ell_p^n\right)^* = \ell_q^n$.

(ii) Let $L = \ell_1^n$ with the norm defined by

$$\|x\| = \sum_{i=1}^{n} |x_i|$$

Now f defined in (1), above is continuous as in (i) and L′ here represents the set of continuous linear functional on ℓ_1^n so that

$$L' = \left(\ell_1^n\right)^*.$$

We now determine the norm of y's which makes $y \to f$ an isometric isomorphism.

Now, $$|f(x)| = \left|\sum_{i=1}^{n} x_i y_i\right|$$

$$\leq \sum_{i=1}^{n} |x_i|\,|y_i|$$

But $$\sum_{i=1}^{n} |x_i|\,|y_i| \leq \max.\{|y_i|\} \sum_{i=1}^{n} |x_i| \text{ so that}$$

$$|f(x)| \leq \max.\{|y_i|\} \sum_{i=1}^{n} |x_i|.$$

From the definition of the norm for f, we have

$$\|f\| = \max.\{|y_i| : i = 1, 2, \ldots, n\} \quad \ldots (8)$$

Now consider the vector defined as follows:

If $|y_i| = \max\limits_{1\leq i\leq n}\{|y_i|\}$, let us consider the vector x as

$$x_i = \frac{|y_i|}{y_i} \text{ when } |y_i| = \max_{1\leq i\leq n}\{|y_i|\} \text{ and } x_i = 0 \quad \ldots (9)$$

otherwise

From the definition, $x_k = 0 \ \forall k \neq i$, so that we have

$$\|(x)\| = \left|\frac{y_i}{y}\right| = 1$$

Further $|f(x)| = \left|\sum_{i=1}^{n} (x_i y_i)\right| = |y_i|$.

Hence $|y_i| = |f(x)| \leq \|f\|\ \|x\|$

$$\Rightarrow \quad |y_i| \leq \|f\| \text{ or } \max\{|y_i|\} \leq \|f\| \qquad [\because \|x\| = 1] \quad \ldots (10)$$

From (8) and (10), we obtain

$$\|f\| = \max.\{|y_i|\} \text{ so that}$$

$y \to f$ is an isometric isomorphism of L′ to $\left(\ell_1^n\right)^*$.

Hence $\left(\ell_1^n\right)^* = \ell_\infty^n$.

(iii) Let L = ℓ_∞^n with the norm

$$\|x\| = \max\{|x_i| : 1, 2, 3, \ldots, n\}.$$

Now f defined in (1) above is continuous as in (1). Let L′ represents the set of all continuous linear functionals on ℓ_∞^n so that

$$L' = \left(\ell_\infty^n\right)^*.$$

Now we determine the norm of y's which makes $y \to f$ as isometric isomorphism.

$$|f(x)| = \left|\sum_{i=1}^n x_i y_i\right| \leq \sum_{i=1}^n |x_i|\,|y_i|.$$

$$\text{But } \sum_{i=1}^n |x_i|\,|y_i| \leq \max(|x_i|) \sum_{i=1}^n |y_i|$$

Hence we have

$$|f(x)| \leq \left\{\sum_{i=1}^n |y_i|\right\}(\|x\|) \text{ so that } \|f\| \leq \sum_{i=1}^n |y_i| \quad \ldots (11)$$

Consider the vector x defined by

$$x_i = \frac{|y_i|}{y_i} \text{ when } y_i \neq 0 \text{ and } x_i = 0 \text{ otherwise.} \quad \ldots (12)$$

Hence $\|x\| = \max\left\{\dfrac{|y_i|}{|y_i|}\right\} = 1$.

and $|f(x)| = \left|\sum_{i=1}^n |x_i y_i|\right| = \sum_{i=1}^n |y_i|$

Therefore

$$\sum_{i=1}^n |y_i| = |f(x)| \leq \|f\|\,\|x\| = \|f\|.$$

$$\Rightarrow \quad \sum_{i=1}^n |y_i| \leq \|f\| \quad \ldots (13)$$

Notes

It follows now from (11) and (13) that

$$\|f\| = \sum_{i=1}^{n} |y_i| \text{ so that } y \to f \text{ is an isometric isomorphism.}$$

Hence, $\left(\ell_\infty^n\right)^* = \ell_1^n$.

This completes the proof of the theorem.

Note We need the signum function for finding the conjugate spaces of some infinite dimensional space which we define as follows:

If γ is a complex number, then

$$\left.\begin{aligned} \operatorname{sgn} \gamma &= \frac{\gamma}{|\gamma|} \text{ if } \gamma \neq 0 \\ &= 0 \text{ if } \gamma = 0 \end{aligned}\right\}$$

$\therefore$ (i) $|\operatorname{sgn} \gamma| = 0$ if $\gamma = 0$ and $|\operatorname{sgn} \gamma| = 1$ if $\gamma \neq 0$

(ii) $\gamma \operatorname{sgn} \bar{\gamma} = 0$ if $\gamma = 0$ and $\gamma \operatorname{sgn} \bar{\gamma} = \dfrac{\gamma\bar{\gamma}}{|\gamma|} = |\gamma|$, if $\gamma \neq 0$.

Theorem 4: The conjugate space of ℓ_p is ℓ_q, where

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 < p < \infty.$$

or $\ell_p^* = \ell_q$.

Proof: Let $x = (x_n) \in \ell_p$ so that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ … (1)

Let $e_n = (0, 0, 0, \ldots, 1, 0, 0, \ldots)$ where 1 is in the n^{th} place.

$$e_n \in \ell_p \text{ for } n = 1, 2, 3, \ldots .$$

We shall first determine the form of f and then establish the isometric isomorphism of ℓ_p^* onto ℓ_q.

By using (e_n), we can write any sequence

$(x_1, x_2, \ldots x_n, 0, 0, 0, \ldots)$ in the form $\sum_{k=1}^{n} x_k e_k$ and

$$x - \sum_{k=1}^{n} x_k e_k = (0, 0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots).$$

Now $\left\| x - \sum_{k=1}^{n} x_k \, e_k \right\| = \left\{ \sum_{k=n+1}^{\infty} | x_k |^p \right\}^{\frac{1}{p}}$ … (2)

The R.H.S. of (2) gives the remainder after n terms of a convergent series (1).

Hence $\left\{ \sum_{k=n+1}^{\infty} | x_k |^p \right\}^{\frac{1}{p}} \to 0$ as $n \to \infty$. … (3)

From (2) and (3) if follows that

$$x = \sum_{k=1}^{\infty} x_k \, e_k \quad \text{… (4)}$$

Let $f \in \ell^*_p$ and $S_n = \sum_{k=1}^{n} x_k \, e_k$ then

$$S_n \to x \text{ as } n \to \infty \quad \text{(Using (4))}$$

Since f is linear, we have

$$f(S_n) = \sum_{k=1}^{n} x_k f(e_k) .$$

Also f is continuous and $S_n \to x$, we have

$$f(S_n) \to f(x) \text{ as } n \to \infty.$$

$$\Rightarrow \quad f(x) = \sum_{k=1}^{n} x_k f(e_k) \quad \text{… (5)}$$

which gives the form of the functional on ℓ_p .

Now we establish the isomeric isomorphism of ℓ^*_p onto ℓ_q , for which proceed as follows:

Let $f(e_k) = \alpha_k$ and show that the mapping

$T : \ell^*_p \to \ell_q$ given by

$T(f) = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots)$ is an isomeric isomorphism of ℓ^*_p onto ℓ_q .

First, we show that T is well defined.

For let $x \in \ell_p$, where $x = (\beta_1, \beta_2, \ldots, \beta_n, 0, 0, \ldots)$ where

$$\beta_k = \begin{cases} | \alpha_k |^{g-1} \ \text{sgn}\ \bar{\alpha}_k , & 1 \le k \le n \\ 0 & \forall n > k \end{cases}$$

$$\Rightarrow \quad | \beta_k | = | \alpha_k |^{q-1} \text{ for } 1 \le k \le n.$$

$$\Rightarrow \quad | \beta_k |^p = | \alpha_k |^{(q-1)p} = | \alpha_k |^q. \quad \left(\because \frac{1}{p} + \frac{1}{q} = q \Rightarrow p(q-1) = q \right)$$

Notes

Now $\alpha_k\beta_k = \alpha_k\,|\,\alpha_k\,|^{q-1}\ \text{sgn}\ \overline{\alpha}_k = |\,\alpha_k\,|^{q-1}\ \alpha_k\ \text{sgn}\ \overline{\alpha}_k$

$\Rightarrow$ $$\alpha_k\ \beta_k = |\,\alpha_k\,|^q = |\,\beta_k\,|^p \qquad \ldots (7)$$

(Using property of sgn function)

$\Rightarrow$ $$\|\,x\,\| = \left\{\sum_{k=1}^{n}|\,\beta_k\,|^p\right\}^{\frac{1}{p}}$$

$\Rightarrow$ $$\|\,x\,\| = \left\{\sum_{k=1}^{n}|\,\beta_k\,|^p\right\}^{\frac{1}{p}}$$

$$= \left\{\sum_{k=1}^{n}|\,\alpha_k\,|^q\right\}^{\frac{1}{q}} \qquad \ldots (8)$$

Since we can write

$$x = \sum_{k=1}^{n}\beta_k e_k \text{ , we get}$$

$$f\,(x) = \sum_{k=1}^{n}\beta_k f\,(e_k) = \sum_{k=1}^{n}\alpha_k\beta_k$$

$\Rightarrow$ $$f\,(x) = \sum_{k=1}^{n}|\,\alpha_k\,|^q \qquad \text{(Using (7))} \quad \ldots (9)$$

We know that for every $x \in \ell_p$

$$|\ f\,(x)\ |\ \leq \|\,f\,\|\ \|\,x\,\|,$$

which upon using (8) and (9), gives

$$|\,f\,(x)\,|\ \leq \sum_{k=1}^{n}|\,\alpha_k\,|^q \leq \|\,f\,\|\left\{\sum_{k=1}^{n}|\,\alpha_k\,|^q\right\}^{\frac{1}{p}}$$

which yields after simplification.

$$\left\{\sum_{k=1}^{n}|\,\alpha_k\,|^q\right\}^{\frac{1}{p}} \leq \|\,f\,\| \qquad \ldots (10)$$

Since the sequence of partial sum on the L.H.S. of (10) is bounded; monotonic increasing, it converges. Hence

$$\left\{\sum_{k=1}^{n}|\,\alpha_k\,|^q\right\}^{\frac{1}{p}} \leq \|\,f\,\| \qquad \ldots (11)$$

So the sequence (α_k) which is the image of f under T belongs to ℓ_q and hence T is well defined.

Notes

We next show that T is onto ℓ_q.

Let $(\beta_k) \in \ell_q$, we shall show that is a $g \in \ell_p^*$ such that T maps g into (β_k).

Let $x \in \ell_p$ so that

$$x = \sum_{k=1}^{n} x_k e_k$$

We shall show that

$$g(x) = \sum_{k=1}^{n} x_k \beta_k \text{ is the required } g.$$

Since the representation for x is unique, g is well defined and moreover it is linear on ℓ_p. To prove it is bounded, consider

$$|g(x)| = \left|\sum_{k=1}^{n} \beta_k x_k\right| \le \sum_{k=1}^{n} |\beta_k x_k|$$

$$\le \left\{\sum_{k=1}^{n} |x_k|^p\right\}^{\frac{1}{p}} \left\{\sum_{k=1}^{n} |\beta_k|^q\right\}^{\frac{1}{q}} \qquad \text{(Using Hölder's inequality)}$$

$\Rightarrow$ $$|g(x)| \le \|x\| \left\{\sum_{k=1}^{n} |\beta_k|^q\right\}^{\frac{1}{q}}$$

$\Rightarrow$ g is bounded linear functional on ℓ_p.

Since $e_k \in \ell_p$ for k = 1, 2, …, we get

$g(e_k) = \beta_k$ for any k so that

$T_g = (\beta_k)$ and T is on ℓ^*_p onto ℓ_q.

We next show that

$\| Tf \| = \| f \|$ so that T is an isometry.

Since $Tf \in \ell_q$, we have from (6) and (10) that

$$\left\{\sum_{k=1}^{\infty} |\alpha_k|^q\right\}^{\frac{1}{q}} = \| Tf \| \le \| f \|$$

Also, $x \in \ell_p \Rightarrow x = \sum_{k=1}^{\infty} x_k e_k$. Hence

$$f(x) = \sum_{k=1}^{\infty} x_k (e_k) = \sum_{k=1}^{\infty} x_k \alpha_k$$

Notes

$\Rightarrow$ $$|f(x)| \le \sum_{k=1}^{\infty} |x_k|\,|\alpha_k|$$

$$\le \left\{\sum_{k=1}^{n} |\alpha_k|^q\right\}^{\frac{1}{q}} \left\{\sum_{k=1}^{\infty} |x_k|^p\right\}^{\frac{1}{p}} \quad \text{(Using Hölder's inequality)}$$

or $$|f(x)| \le \left\{\sum_{k=1}^{n} |\alpha_k|^q\right\}^{\frac{1}{q}} \|x\| \ \forall\, x \in \ell_p .$$

Hence we have

$$\sup_{x\neq 0}\left\{\frac{|f(x)|}{\|x\|}\right\} \le \left\{\sum_{k=1}^{\infty} |\alpha_k|^q\right\}^{\frac{1}{q}} = \|Tf\| \quad \text{(Using (6))}$$

which upon using definition of norm yields

$$\|f\| \le \|Tf\| \quad \dots (13)$$

Thus $$\|f\| = \|Tf\| \quad \text{(Using (12) and (13))}$$

From the definition of T, it is linear. Also since it is an isometry, it is one-to-one and onto (already shown). Hence T is an isometric isomorphism of ℓ^*_p onto ℓ_q, i.e.,

$$\ell^*_p = \ell_q .$$

This completes the proof of the theorem.

Theorem 5: Let p > 1 with $\frac{1}{p}+\frac{1}{q}=1$ and let $g \in L_p$ (X). Then the function defined by

$$F(f) = \int_X fg\, d\mu \text{ for } f \in L_p(X)$$

is a bounded linear functional on L_p (X) and

$$\|F\| = \|g\|_q \quad \dots (1)$$

Proof: We first note that

F is linear on L_p (X). For if $f_1, f_2 \in L_p$ (X), then we get

$$F(f_1 + f_2) = \int_X (f_1+f_2)g\, d\mu = \int_X f_1 g\, d\mu + \int_X f_2 g\, d\mu$$

$$= F(f_1) + F(f_2)$$

So that

$$F(f_1 + f_2) = F(f_1) + F(f_2)$$

and $$F(\alpha F) = \alpha \int_X fg\, d\mu = \alpha F(f) .$$

Now $$|F(f)| = \left|\int_X fg\, d\mu\right| \le \int_X |fg|\, d\mu \quad \dots (2)$$

Notes

Making use of Hölder's inequality, we get

$$\int_X |fg|\, d\mu \le \left\{\int_X |f|^p\, d\mu\right\}^{\frac{1}{p}} \left\{\int_X |g|^q\, d\mu\right\}^{\frac{1}{q}}$$

$$= \|f\|_p \; \|g\|_q \qquad \dots (3)$$

From (2) and (3) it follows that

$$|F(f)| \le \|f\|_p \; \|g\|_q.$$

$$\text{Hence } \sup\left\{\frac{|F|f||}{\|f\|_p} : f \in Lp(X) \text{ and } f \neq 0\right\} \le \|g\|_q$$

$$\Rightarrow \qquad \|F\| \le \|g\|_q \qquad \text{(Using definition of the norm)} \dots (4)$$

Further, $\quad$ let $f = |g|^{q-1} \operatorname{sgn} \bar{g}$ $\qquad \dots (5)$

Since $\quad \left|\operatorname{sgn} \bar{g}\right| = 1$, we get

$$|f|^p = |g|^{p(q-1)} = |g|^q \qquad (\because p(q-1) = q)$$

Thus, $\quad f \in L_p(X)$ and $\left\{\int_X |f|^p\, d\mu\right\}^{\frac{1}{p}} = \left\{\int_X |g|^q\, d\mu\right\}^{\frac{1}{q}}$ $\qquad \dots (6)$

But $\quad \left\{\int_X |g|^q\, d\mu\right\}^{\frac{1}{p}} \left\{\int_X |g|^p\, d\mu\right\} = \|g\|_q^{q/p}$

which implies on using (6) that

$$\|f\|_p = \|g\|_q^{q/p} \qquad \dots (7)$$

Now $\quad F(f) = \int_X fg\, d\mu = \int_X |g|^{q-1}\, g \operatorname{sgn} \bar{g}\, d\mu$

$$= \int_X |g|^q\, d\mu = \|g\|_q^q$$

Hence $\quad \|g\|_q^q \, \|g\| = F(f) \le \|F\| \; \|f\|_p.$

and this on using (7) yields that

$$\|g\|_q^q = F(f) \le \|F\| \; \|g\|_q^{q/p}$$

$$\Rightarrow \qquad \|g\|_q^{q-q/p} = \|g\|_q \le \|F\| \qquad \dots (8)$$

$$(\because g \neq 0)$$

Notes

From (4) and (8) it finally follows that

$$\| F \| = \| g \|_q.$$

This completes the proof of the theorem.

Approximation by Continuous Function

Theorem 6: If f is a bounded measurable function defined on [a, b], then for given ε > 0, ∃ a continuous function g on [a, b], such that

$$\| f - g \|_2 < \varepsilon$$

Proof: Let $F(x) = \int_a^x f(t)\,dt$ where $x \in [a, b]$.

Then
$$|F(x+h) - F(x)| = \left| \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_x^{x+h} f(t)\,dt \right| \le \int_x^{x+h} f(t)\,dt$$

$$\le Mh, \text{ where } |f(x)| \le M, \ \forall x \in [a, b].$$

Taking $h < \delta$, and $Mh < \varepsilon_1$, we get

$$|x + h - x| < \delta \Rightarrow |F(x+h) - F(x)| < \varepsilon_1$$

$$\Rightarrow F(x) \text{ is continuous on } [a, b].$$

Let
$$G_n(x) = n \int_x^{x+h} f(t)\,dt : x \in [a, b] \text{ and } n \in N;$$

then
$$G_n(x) = n\left[F\left(x + \frac{1}{n}\right) - F(x)\right]$$
($\because$ F (x) is continuous on [a, b] ⇒ $G_n(x)$ is continuous on [a, b] ∀ n)

Again, since
$$F(x) = \int_a^x f(t)\,dt,\ x \in [a, b].$$

$$\therefore \quad F'(x) = f(x) \text{ a.e. in } [a, b].$$

Now,
$$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} \frac{F(x + (1/n) - F(x)}{1/n}$$

$$= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h},\ h = \frac{1}{n}$$

$$= F'(x) = f(x) \text{ a.e. in } [a, b]$$

and hence
$$\lim_{n\to\infty} \left[G_n(x) - f(x)\right]^2 = 0.$$

Notes

Also $$|G_n(x)| = \left| \int_x^{x+(1/n)} f(t)\,dt \right| \le n \int_x^{x+(1/n)} |f(t)|\,dt = M.$$

Hence $|G_n(x)| \le M,\ \forall\ n \in N$ and $\forall\ x \in [a, b]$.

$\therefore$ $$[G_n(x) - f(x)]^2 \le (M + M)^2 = 4M^2, x \in [a, b].$$

On applying Lebesgue bounded convergence theorem, we get

$$\lim_{n\to\infty} \int_a^b (G_n - f)^2 = \int_a^b \lim (G_n - f)^2 = 0$$

$\Rightarrow$ $$\lim_{n\to\infty} [G_n - f]_2^2 = 0$$

$\Rightarrow$ $$\lim_{n\to\infty} \|G_n - f\|_2 = 0$$

or $$\lim_{n\to\infty} \|f - G_n\| = 0$$

$\Rightarrow$ for given $\varepsilon > 0$, $\exists\ n_o \in N$, such that $n \ge n_o$

$\Rightarrow$ $$\|f - G_n\|_2 < \varepsilon$$

Particularly for $n = n_o$.

$\Rightarrow$ $$\|f - G_{n_o}\|_2 < \varepsilon$$

$\Rightarrow$ $$\|f - g\|_2 < \varepsilon \qquad \text{(Taking } G_{n_o} = g\text{)}$$

Thus there exists a continuous function $G_{n_o}(x) = g(x)$

$$= n_o \int_x^{x+(1/n_o)} f(t)\,dt,\ x \in [a, b],$$

which satisfies the given condition.

8.2 Summary

- A linear functional f on a normed space N_1 is said to be bounded if there is a constant $k > 0$ such that

  $$|f(x)| \le k\,\|x\|,\ \forall\ x \in N_1$$

- If $x \in \ell_p$ and f is bounded linear functional on ℓ_p, then f has the unique representation of the form as an infinite series.

  $$f(x) = \sum_{k=1}^{\infty} x_k\, f(e_k)$$

- The norm of $f \in \ell_p^*$ is given by

  $$\|f\| = \left\{ \sum_{k=1}^{\infty} |f(e_k)|^q \right\}^{\frac{1}{q}}$$

8.3 Keywords

Bounded Linear Functional on L^p-spaces: If $x \in \ell_p$ and f is bounded linear functional on ℓ_p, then f has the unique representation of the form as an infinite series

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k)$$

Bounded Linear Functional: A linear functional f on a normed space N_1 is said to be bounded if there is a constant k > 0 such that

$$|f(x)| \leq k \| x \|, \ \forall \ x \in N_1$$

Continuous Linear Functional: A linear functional f is continuous if given $\in > 0$ there exists $\delta > 0$ so that

$$|f(x) - f(y)| \leq \in \text{ whenever } \| x - y \| \leq \delta.$$

Linear Functional: Let N_1 be a normed space over a field R (or C). A mapping $f : N_1 \rightarrow R$ (or C) is called a linear functional on N_1 if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \ \forall \ x, y \in N_1$ and $\alpha, \beta \in R$ (or C).

Norm: The norm of $f \in \ell_p^*$ is given by

$$\| f \| = \left\{ \sum_{k=1}^{\infty} | f(e_k) |^q \right\}^{\frac{1}{q}}$$

8.4 Review Questions

1. Account for bounded linear functionals on L^p-space.
2. State and prove different continuous linear functional theorems.
3. Describe approximation by continuous function.
4. How will you explain norms of bounded linear functional on L^p-space?
5. What is Isometric Isomorphism?

8.5 Further Readings

Books

Rudin, Walter (1991), *Functional Analysis*, Mc-Graw-Hill Science/Engineering/Math

Kreyszig, Erwin, *Introductory Functional Analysis with Applications*, WILEY 1989.

T.H. Hilderbrandt, *Transactions of the American Mathematical Society*. Vol. 36, No. = 4, 1934.

Online links

www.math.psu.edu/yzheng/m597k/m597kLIII4.pdf

www.public.iastate.edu/…/Royden_Real_Analysis_Solutions.pdf

Richa Nandra, Lovely Professional University

Unit 9: Measure Spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define measure space.
- Define null set in a measure space.
- Understand theorems based on measure spaces.
- Solve problems on measure spaces.

Introduction

A measurable space is a set S, together with a non-empty collection, S, of subsets of S, satisfying the following two conditions:

1. For any A, B in the collection S, the set[1] A – B is also in S.
2. For any $A_1, A_2, \ldots \in S$, $\cup A_i \in S$.

The elements of S are called measurable sets. These two conditions are summarised by saying that the measurable sets are closed under taking finite differences and countable unions.

9.1 Measure Space

Measurable Space: Let $\mathcal{U}$ be a σ-algebra of subsets of set X. The pair $(X, \mathcal{U})$ is called a measurable space. A subset E of X is said to be $\mathcal{U}$-measurable if $E \in \mathcal{U}$.

(a) If μ is a measure on a σ-algebra $\mathcal{U}$ of subsets of a set X, we call the triple $(X, \mathcal{U}, u)$ a measure space.

(b) A measure μ on a σ-algebra $\mathcal{U}$ of subsets of a set X is called a finite measure if $m(X) < \infty$. In this case $(X, \mathcal{U}, \mu)$ is called a finite measure space.

(c) A measure μ on a σ-algebra $\mathcal{U}$ of subsets of a set X is called a *σ-finite measure* if there exists a sequence $(E_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that $\mathcal{U}_{n \in \mathcal{N}} E_n = X$ and $\mu(E_n) < \infty$ for every $n \in \mathcal{N}$. In this case $(X, \mathcal{U}, \mu)$ is called a σ-finite measure space.

(d) A set $D \in \mathcal{U}$ in an arbitrary measure space $(X, \mathcal{U}, \mu)$ is called a *σ-finite set* if there exists a sequence $(D_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that $\cup_{n \in \mathcal{N}} D_n = D$ and $\mu(D_n) < \infty$ for every $n \in \mathcal{N}$.

Lemma 1: (a) Let $(X, \mathcal{U}, \mu)$ be a measure space. If $D \in \mathcal{U}$ is a σ-finite set, then there exists an increasing sequence $(F_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that $\lim_{n\to\infty} F_n = D$ and $\mu(F_n) < \infty$ for every $n \in \mathcal{N}$ and there exists a disjoint sequence $(G_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that $\mathcal{U}_{n\in\mathcal{N}} G_n = D$ and $\mu(G_n) < \infty$ for every $n \in \mathcal{N}$.

(b) If $(X, \mathcal{U}, \mu)$ is a σ-finite measure space then every $D \in \mathcal{U}$ is a σ-finite set.

Proof 1: Let $(X, \mathcal{U}, \mu)$ be a measure space. Suppose $D \in \mathcal{U}$ is a σ-finite set. Then there exists a sequence $(D_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that $\cup_{n \in \mathcal{N}} D_n = D$ and $\mu(D_n) < \infty$ for every $n \in \mathcal{N}$. For each $n \in \mathcal{N}$, let $F_n = \cup_{k=1}^{n} D_k$. Then $(F_n : n \in \mathcal{N})$ is an increasing sequence in $\mathcal{U}$ such that $\lim_{n\to\infty} F_n = \cup_{n\in\mathcal{N}} F_n$

$$= \cup_{n\in\mathcal{N}} D_n = D \text{ and } \mu(F_n) = \mu\left(\bigcup_{k=1}^{n} D_k\right) \le \sum_{k=1}^{n} \mu(D_k) < \infty \text{ for every } n \in \mathcal{N}.$$

Let $G_1 = F_1$ and $G_n = F_n \setminus \cup_{k=1}^{n-1} F_k$ for $n \ge 2$. Then $(G_n : n \in \mathcal{N})$ is a disjoint sequence in $\mathcal{U}$ such that $\cup_{n\in\mathcal{N}} G_n = \cup_{n\in\mathcal{N}} F_n = D$ as in the proof of Lemma "let $(E_n : n \in \mathcal{N})$ be an arbitrary sequence in an algebra $\mathcal{U}$ of subsets of a set X. Then there exists a disjoint sequence $(F_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that

(1) $\bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} F_n$ for every $N \in \mathcal{N}$,

and

(2) $\bigcup_{n\in\mathcal{N}} E_n = \bigcup_{n\in\mathcal{N}} F_n$ ".

$\mu(G_1) = \mu(F_1) < \infty$ and $\mu(G_n) = \mu\left(F_n \middle| \bigcup_{k=1}^{n-1} F_k\right) \le \mu(F_n) < \infty$ for $n \ge 2$. This proves (a).

2. Let $(X, \mathcal{U}, \mu)$ be a σ-finite measure space. Then there exists a sequence $(E_n : n \in \mathcal{N})$ in $\mathcal{U}$ such that $\cup_{n \in \mathcal{N}} E_n = X$ and $\mu(E_n) < \infty$ for every $n \in \mathcal{N}$. Let $D \in \mathcal{U}$. For each $n \in \mathcal{N}$, let $D_n = D \cap E_n$. Then $(D_n : n \in \mathcal{N})$ is a sequence in $\mathcal{U}$ such that $\cup_{n\in\mathcal{N}} D_n = D$ and $m(D_n) \le \mu(E_n) < \infty$ for every $n \in \mathcal{N}$. Thus D is a σ-finite set. This proves (b).

9.1.1 Null Set in a Measure Space

Definition: Given a measure μ on a σ-algebra $\mathcal{U}$ of subsets of a set X · A subset E of X is called a null set with respect to the measure μ if $E \in \mathcal{U}$ and $\mu(E) = 0$. In this case we say also that E is a null set in the measure space $(X, \mathcal{U}, \mu)$. (Note that ϕ is a null set in any measure space but a null set in a measure space need not be ϕ.)

Theorem 1: A countable union of null sets in a measure space is a null set of the measure space.

Proof: Let $(E_n : n \in \mathcal{N})$ be a sequence of null sets in a measure space $(X, \mathcal{U}, \mu)$. Let $E = \cup_{n\in\mathcal{N}} E_n$. Since $\mathcal{U}$ is closed under countable unions,

we have $E \in \mathcal{U}$.

By the countable subadditivity of μ on $\mathcal{U}$,

we have $\mu(E) \leq \Sigma_{n\in \mathcal{N}} \mu(E_n) = 0$.

Thus $\mu(E) = 0$.

This shows that E is a null set in $(X, \mathcal{U}, \mu)$.

9.1.2 Complete Measure Space

Definition: Given a measure μ on a σ-algebra $\mathcal{U}$ of subsets of a set X. We say that the σ -algebra $\mathcal{U}$ is complete with respect to the measure μ if an arbitrary subset E_0 of a null set E with respect to μ is a member of $\mathcal{U}$ (and consequently has $\mu(E_0) = 0$ by the Monotonicity of μ). When $\mathcal{U}$ is complete with respect to μ, we say that $(X, \mathcal{U}, \mu)$ is a complete measure space.

Example: Let X = {a, b, c}. Then $\mathcal{U} = \{\phi, \{a\}, \{b, c\}, X\}$ is a σ-algebra of subsets of X. If we define a set function μ on $\mathcal{U}$ by setting $\mu(\phi) = 0$, $\mu(\{a\}) = 1$, $\mu(\{b, c\}) = 0$, and $\mu(X) = 1$, then μ is a measure on $\mathcal{U}$. The set {b, c} is a null set in the measure space $(X, \mathcal{U}, \mu)$, but its subset {b} is not a member of $\mathcal{U}$. Therefore, $(X, \mathcal{U}, \mu)$ is not a complete measure space.

9.1.3 Measurable Mapping

Let f be a mapping of a subset D of a set X into a set Y. We write D (f) and R (f) for the domain of definition and the range of f respectively. Thus

$$D(f) = D \subset X,$$

$$R(f) = \{y \in Y : y = f(x) \text{ for some } x \in D(f)\} \subset Y.$$

For the image of D (f) by f, we have f (D (f)) = R (f). For an arbitrary subset E of y we define the preimage of E under the mapping f by

$$F^{-1}(E) := \{x \in X : f(x) \in E\} = \{x \in D(f) : f(x) \in E\}.$$

Notes

1. E is an arbitrary subset of Y and need not be a subset of R (f). Indeed E may be disjoint from $\mathcal{R}(f)$, in which case $f^{-1}(E) = \phi$. In general, we have $f(f^{-1}(E)) \subset E$.
2. For an arbitrary collection C of subsets of Y, we let $f^{-1}(C) := \{f^{-1}(E) : E \in C\}$.

Theorem 2: Given sets X and Y. Let f be a mapping with $D(f) \subset X$ and $\mathcal{R}(f) \subset Y$. Let E and E_α be arbitrary subsets of Y. Then

1. $f^{-1}(Y) = D(f)$,
2. $f^{-1}(E^C) = f^{-1}(Y\backslash E) = f^{-1}(Y)\backslash f^{-1}(E) = D(f) \backslash f^{-1}(E)$,
3. $f^{-1}(\cup_{\alpha\in \mathcal{A}} E_\alpha) = \cup_{\alpha\in \mathcal{A}} f^{-1}(E_\alpha)$,
4. $f^{-1}(\cap_{\alpha\in \mathcal{A}} E_\alpha) = \cap_{\alpha\in \mathcal{A}} f^{-1}(E_\alpha)$.

Theorem 3: Given sets X and Y. Let f be a mapping with $D(f) \subset X$ and $R(f) \subset Y$. If $\mathcal{B}$ is a σ-algebra of subsets of Y then $f^{-1}(\mathcal{B})$ is a σ-algebra of subsets of the set D (f). In particular, if D (f) = X then $f^{-1}(\mathcal{B})$ is a σ-algebra of subsets of the set X.

Proof: Let $\mathcal{B}$ be a σ-algebra of subsets of the set Y. To show that $f^{-1}(\mathcal{B})$ is a σ-algebra of subsets of the set D (f) we show that $D(f) \in f^{-1}(\mathcal{B})$; if $A \in f^{-1}(\mathcal{B})$ then $D(f)\backslash A \in f^{-1}(\mathcal{B})$; and for any sequence $(A_n : n \in \mathcal{N})$ in $f^{-1}(\mathcal{B})$ we have $U_{n \in \mathcal{N}} A_n \in f^{-1}(B)$.

1. By (1) of above theorem, we have $D(f) = f^{-1}(Y) \in f^{-1}(B)$ since $Y \in \mathcal{B}$.

2. Let $A \in f^{-1}(\mathcal{B})$. Then $A = f^{-1}(\mathcal{B})$ for some $B \in \mathcal{B}$. Since $B^C \in \mathcal{B}$ we have $f^{-1}(B^C) \in f^{-1}(\mathcal{B})$. On the other hand by (2) of above theorem, we have $f^{-1}(B^C) = D(f)\backslash f^{-1}(B) = D(f)\backslash A$. Thus $D(f)\backslash A \in f^{-1}(\mathcal{B})$.

3. Let $(A_n : n \in \mathcal{N})$ be a sequence in $f^{-1}(\mathcal{B})$. Then $A_n = f^{-1}(B_n)$ for some $B_n \in \mathcal{B}$ for each $n \in \mathcal{N}$. Then by (3) of above theorem, we have

$$\bigcup_{n\in\mathcal{N}} A_n = \bigcup_{n\in\mathcal{N}} f^{-1}(B_n) = f^{-1}\left(\bigcup_{n\in\mathcal{N}} B_n\right) \in f^{-1}(\mathcal{B}),$$

since $\bigcup_{n\in\mathcal{N}} B_n \in (\mathcal{B})$.

Measurable Mapping

Definition: Given two measurable spaces $(X, \mathcal{U})$ and $(Y, \mathcal{B})$. Let f be a mapping with $D(f) \subset X$ and $\mathcal{R}(f) \subset Y$. We say that f is a $\mathcal{U}/\mathcal{B}$ measurable mapping if $f^{-1}(B) \in \mathcal{U}$ for every $B \in \mathcal{B}$, that is, $f^{-1}(\mathcal{B}) \subset \mathcal{U}$.

Theorem 4: Given two measurable spaces $(X, \mathcal{U})$ and $(Y, \mathcal{B})$. Let f be a $\mathcal{U}/\mathcal{B}$-measurable mapping.

(a) If $\mathcal{U}$, is a σ-algebra of subsets of X such that $\mathcal{U}, \supset \mathcal{U}$, then f is $\mathcal{U}_1/\mathcal{B}$-measurable.

(b) If $\mathcal{B}_0$ is a σ-algebra of subsets of Y such that $\mathcal{B}_0 \subset \mathcal{B}$, then f is $\mathcal{U}/\mathcal{B}_0$-measurable.

Proof: (a) Follows from $f^{-1}(\mathcal{B}) \subset \mathcal{U} \subset \mathcal{U}_1$ and (b) from $f^{-1}(B_0) \subset f^{-1}(\mathcal{B}) \subset \mathcal{U}$.

Composition of two measurable mappings is a measurable mapping provided that the two measurable mappings from a chain.

Theorem 5: Given two measurable spaces $(X, \mathcal{U})$ and $(Y, \mathcal{B})$, where $\mathcal{B} = \sigma(\mathcal{C})$ and $\mathcal{C}$ is arbitrary collection of subsets of Y. Let f be a mapping with $D(f) \in \mathcal{U}$ and $\mathcal{R}(f) \subset Y$. Then f is a $\mathcal{U}/\mathcal{B}$-measurable mapping of D (f) into Y if and only if $f^{-1}(\mathcal{C}) \subset \mathcal{U}$.

Proof: If f is a $\mathcal{U}/\mathcal{B}$-measurable mapping of D (f) into Y, then $f^{-1}(\mathcal{B}) \subset \mathcal{U}$ so that $f^{-1}(\mathcal{C}) \subset \mathcal{U}$. Conversely if $f^{-1}(\mathcal{C}) \subset \mathcal{U}$, then $\sigma(f^{-1}(\mathcal{C}) \subset \sigma(\mathcal{U}) = \mathcal{U}$. Now by theorem,

"Let f be a mapping of a set X into a set Y. Then for an arbitrary collection C of subsets of Y, we have $\sigma(f^{-1}(\mathcal{C})) = f^{-1}(\sigma(\mathcal{C})$."

$\sigma(f^{-1}(\mathcal{C}) = f^{-1}(\sigma(\mathcal{C})) = f^{-1}(\mathcal{B})$. Thus $f^{-1}(\mathcal{B}) \subset \mathcal{U}$ and f is a $\mathcal{U}/\mathcal{B}$- measurable mapping of D (f).

Theorem 6: If X_0 is a thick subset of a measure space (X, S, μ), if $S_0 = S \cap X_0$, and if, for E in S, $\mu_0(E \cap X_0) = \mu(E)$, then (X_0, S_0, μ_0) is a measure space.

Proof: If two sets, E_1 and E_2, in S are such that $E_1 \cap X_0 = E_2 \cap X_0$, then $(E_1 \Delta E_2) \cap X_o = 0$, so that $\mu(E_1 \cap E_2) = 0$ and therefore $\mu(E_1) = \mu(E_2)$. In other words μ_0 is indeed unambiguously defined on S_0.

Suppose next that $\{F_n\}$ is a disjoint sequence of sets in S_0, and let E_n be a set in S such that

$$F_n = E_n \cap X_0, \; n = 1, 2, \ldots .$$

If $\tilde{E}_n = E_n - \cup\{E_i : 1 \leqq i < n\}$, n = 1, 2, ... , then

$$\left(\tilde{E}_n \,\Delta\, E_n\right) \cap X_0 = \left(F_n - \cup\left\{F_i : 1 \leqq i < n\right\}\right) \Delta F_n$$

$$= F_n \,\Delta\, F_n = 0,$$

So that $\mu\left(\tilde{E}_n \Delta E_n\right) = 0$, and therefore

$$\Sigma_{n=1}^{\infty} \mu_0(F_n) = \Sigma_{n=1}^{\infty} \mu_0(E_n) = \Sigma_{n=1}^{\infty} \mu_0(\tilde{E}_n) = \mu\left(\Sigma_{n=1}^{\infty} \tilde{E}_n\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu_0\left(\bigcup_{n=1}^{\infty} F_n\right)$$

In other word μ_0 is indeed a measure, and the proof of the theorem is complete.

9.2 Summary

- Let $\mathcal{U}$ be a σ-algebra of subsets of a set X. The pair (X, $\mathcal{U}$) is called a measurable space. A subset E of X is said to be $\mathcal{U}$-measurable if $E \in \mathcal{U}$.
- If μ is a measure on a σ-algebra $\mathcal{U}$ of subsets of a set X, we call the triple (X, $\mathcal{U}$, μ) a measure space.
- A subset E of X is called a null set with respect to the measure μ if $E \in \mathcal{U}$ and μ (E) = 0.
- Two measurable spaces (X, $\mathcal{U}$) and (Y, $\mathcal{B}$). Let f be a mapping with D (f) ⊂ X and $\mathcal{R}$ (f) ⊂ Y. We say that f is a $\mathcal{U}/\mathcal{B}$-measurable mapping if f^{-1} (B) ∈ $\mathcal{U}$ for every B ∈ $\mathcal{B}$, that is f^{-1} ($\mathcal{B}$) ⊂ $\mathcal{U}$.

9.3 Keywords

Complete Measure Space: Given a measure μ on a σ-algebra $\mathcal{U}$ of subsets of a set X. We say that the σ -algebra $\mathcal{U}$ is complete with respect to the measure μ if an arbitrary subset E_0 of a null set E with respect to μ is a member of $\mathcal{U}$ (and consequently has μ (E_0) = 0 by the Monotonicity of μ). When $\mathcal{U}$ is complete with respect to μ, we say that (X, $\mathcal{U}$, μ) is a complete measure space.

Measurable Mapping: Given two measurable spaces (X, $\mathcal{U}$) and (Y, $\mathcal{B}$). Let f be a mapping with D (f) ⊂ X and $\mathcal{R}$ (f) ⊂ Y. We say that f is a $\mathcal{U}/\mathcal{B}$ measurable mapping if f^{-1} (B) ∈ $\mathcal{U}$ for every B ∈ $\mathcal{B}$, that is, f^{-1} ($\mathcal{B}$) ⊂ $\mathcal{U}$.

Measurable Space: A measurable space is a set S, together with a non-empty collection, S, of subsets of S.

Null Set in a Measure Space: A subset E of X is called a null set with respect to the measure μ if E ∈ $\mathcal{U}$ and μ (E) = 0. In this case we say also that E is a null set in the measure space (X, $\mathcal{U}$, μ).

Sigma Algebra: $\mathcal{F}$ is sigma algebra which establishes following relations:

(i) $A_k \in \mathcal{F}$ for all k implies $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

(ii) $A \in \mathcal{F}$ implies $A^C \in \mathcal{F}$

(iii) $\phi \in \mathcal{F}$

Notes

9.4 Review Questions

1. Let $\mathcal{U}$ be a σ-algebra of subsets of a set X and let Y be an arbitrary subset of X. Let $\mathcal{B} = \{A \cap Y : A \in \mathcal{U}\}$. Show that $\mathcal{B}$ is a σ-algebra of subsets of Y.

2. Let $(X, \mathcal{U}, \mu)$ be a measure space. Show that for any $E_1, E_2 \in \mathcal{U}$ we have the equality: $\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2)$.

9.5 Further Readings

Books

Paul Halmos, (1950). *Measure Theory*. Van Nostrand and Co.

Bogachev, V.I. (2007), *Measure Theory*, Berlin : Springer

Online links

planetmath.org/measurable space.html

mathworld.wolfram.com > Calculus and Analysis > Measure Theory

Unit 10: Measurable Functions

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand measurable functions.
- Define equivalent functions and characteristic function.
- Describe Egoroff's theorem and Riesz theorem.
- Define simple function and step function.

Introduction

In this unit, we shall see that a real valued function may be Lebesgue integrable even if the function is not continuous. In fact, for the existence of a Lebesgue integral, a much less restrictive condition than continuity is needed to ensure integrability of f on [a, b]. This requirement gives rise to a new class of functions, known as measurable functions. The class of measurable functions plays an important role in Lebesgue theory of integration.

10.1 Measurable Functions

10.1.1 Lebesgue Measurable Function/Measurable Function

Definition: Let E be a measurable set and R* be a set of extended real numbers. A function $f : E \to R^*$ is said to be a Lebesgue measurable function on E or a measurable function on E iff the set

$E(f > \alpha) = \{x \in E : f(x) > \alpha\} = f^{-1}\{\alpha, \infty)\}$ is a measurable subset of E $\forall\ \alpha \in R$.

Notes

1. The definition states that f is a measurable function if for every real number α, the inverse image of (α, ∞) under f is a measurable set.
2. The measure of the set $E(f > \alpha)$ may be finite or infinite.
3. A function whose values are in the set of extended real numbers is called an extended real valued function.
4. If E = R, then the set $E(f > \alpha)$ becomes an open set.

Example: A constant function with measurable domain is measurable.

Sol: Let f be a constant function defined over a measurable set E so that $f(x) = \in\ \forall x \in E$.

Then for any real number α,

$$E(f > \alpha) = \begin{cases} E, \text{ if } & c > \alpha \\ \phi, \text{ if } & c \leq \alpha \end{cases}$$

The sets E and ϕ are measurable and hence $E(f > \alpha)$ is measurable i.e. the function f is measurable.

Theorem 1: Let f and g be measurable real valued functions on E, and c is a constant. Then each of the following functions is measurable on E.

(a) $f \pm c$ (b) $c f$

(c) $f + g$ (d) $f - g$

(e) $|f|$ (f) f^2

(g) fg (h) f/g (g vanishes no where on E)

Proof: Let α be an arbitrary real number.

(a) Since f is measurable and

$E(f \pm c > \alpha) = E(f > \alpha \mp c)$,

the function $f \pm c$ is measurable.

(b) To prove c f is measurable over E.

If c = 0, then cf is constant and hence measurable because a constant function is measurable.

Notes

Consider the case in which $c \neq 0$, then

$$E(cf > \alpha) = \begin{cases} E\left(f > \frac{\alpha}{c}\right) \text{if } c > 0 \\ E\left(f < \frac{\alpha}{c}\right) \text{if } c < 0 \end{cases}$$

Both the sets on R.H.S. are measurable.

Hence $E(cf > \alpha)$ is measurable and so cf is measurable $\forall c \in R$.

(c) Before proving f + g is measurable, we first prove that if f and g are measurable over E then the set E (f > g) is also measurable.

Now $f > g \Rightarrow \exists$ a rational number r such that

$$f(x) > r > g(x)$$

Thus $$E(f > g) = \bigcup_{r \in Q} [(E(f > r) \cap (E(g < r))]$$

= an enumerable union of measurable sets.

= measurable set, since Q is an enumerable set.

Now, we shall prove that f + g is measurable over E. Let a be any real number.

Now $$E(f + g > a) = E(f > a - g) \quad \ldots (1)$$

Again, g is measurable

$\Rightarrow$cg is measurable, c is constant.

($\because$ We know that if f is a measurable function and c is constant then cf is measurable)

$\Rightarrow$ a + cg is measurable $\forall$ a, c $\in$ R

$\Rightarrow$ a – g is measurable by taking c′ = –1,

since f and a – g are measurable

$\Rightarrow$ E (f > a – g) is measurable.

$\Rightarrow$ E (f + g > a) is a measurable set.

$\Rightarrow$ f + g is a measurable function.

(d) To prove that f – g is measurable. Before proving f – g is measurable, we first prove that if f and g are measurable over E then the set E (f > g) is also measurable.

Now $f > g \Rightarrow \exists$ a rational number r, such that f (x) > r > g (x).

Thus $$E(f > g) = \bigcup_{r \in Q} [(E(f > r) \cap (E(g < r))]$$

= an enumerable union of measurable sets.

= measurable sets, since Q is an enumerable set.

Now we shall prove that f – g is measurable over E.

Let a be any real number.

Now $E(f - g > a) = E(f > a + g)$

since g is measurable.

⇒ cg is measurable, c is constant.

⇒ a + cg is measurable $\forall a, c \in R$

⇒ a + g is measurable by taking c = 1,

since f and a + g are measurable

⇒ $E(f > a + g)$ is measurable.

⇒ $E(f - g > a)$ is a measurable set.

⇒ f – g is a measurable function.

(e) To prove $|f|$ is measurable.

We have

$$E(|f| > \alpha) = \begin{cases} E \text{ if } \alpha < 0 \\ [E(f > \alpha)] \cup [E(f < -\alpha)] \text{ if } \alpha \geq 0 \end{cases}$$

[because we know that $|x| > a \Rightarrow x > a$ or $x < -a$]

since f is measurable therefore $E(f > \alpha)$ and $E(f < -\alpha)$ are measurable by definition.

Also we know that finite union of two measurable sets is measurable.

⇒ $E(f > \alpha) \cup E(f < -\alpha)$ is measurable.

⇒ $E(|f| > \alpha)$ is measurable.

⇒ $|f|$ is measurable.

(f) To prove f^2 is measurable.

We have $E(f^2 > \alpha) = \begin{cases} E \text{ if } \alpha < 0 \\ E(|f| > \sqrt{\alpha})] \text{ if } \alpha \geq 0 \end{cases}$

But $E(|f| > \sqrt{\alpha}) = [E(f > \sqrt{\alpha})] \cup [E(f < -\sqrt{\alpha})]$, if $\alpha \geq 0$ $(\because |x| > a \Rightarrow x > a \text{ or } x < -a)$

∴ $E(f^2 > \alpha) = \begin{cases} E \text{ if } \alpha < 0 \\ [E(f > \sqrt{\alpha})] \cup [E(f < -\sqrt{\alpha})] \text{ if } \alpha \geq 0 \end{cases}$

But f is measurable over E.

⇒ $E(f > \sqrt{\alpha})$ and $E(f < -\sqrt{\alpha})$ are measurable sets.

⇒ $[E(f > \sqrt{\alpha})] \cup [E(f < -\sqrt{\alpha})]$ is measurable.

($\because$ union of two measurable sets is measurable)

⇒ $E(f^2 > \alpha)$ is measurable because both the sets on RHS are measurable.

⇒ f^2 is measurable over E.

(g) To prove fg is measurable.

Clearly, f + g and f – g are measurable functions over E.

Notes

$\Rightarrow$ $(f + g)^2$, $(f - g)^2$ are measurable functions over E.

$\Rightarrow$ $(f + g)^2 - (f - g)^2$ is a measurable function over E.

$\Rightarrow$ $\frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ is a measurable function over E.

$\Rightarrow$ fg is a measurable function over E.

(h) To prove f/g is measurable.

Let g vanish nowhere on E, so that

$g(x) \neq 0 \quad \forall\ x \in E.$

$\Rightarrow$ $\frac{1}{g}$ exists.

Now we shall show that $\frac{1}{g}$ is measurable.

We have

$$E\left(\frac{1}{g} > \alpha\right) = \begin{cases} E(g > 0) \text{ if } \alpha = 0 \\ [E(g < 0)] \cap \left[E\left(g < \frac{1}{\alpha}\right)\right] \text{ if } \alpha > 0 \\ [E(g < 0)] \cup [E(g < 0)] \cap \left[E\left(g < \frac{1}{\alpha}\right)\right] \end{cases}$$

Also finite union and intersection of measurable sets are measurable. Hence $E\left(\frac{1}{g} > \alpha\right)$ is measurable in every case.

Since f and $\frac{1}{g}$ are measurable.

$\Rightarrow$ $(f)\Big/\left(\frac{1}{g}\right)$ is measurable over E.

$\Rightarrow$ $\frac{f}{g}$ is measurable over E.

10.1.2 Almost Everywhere (a.e.)

Definition: A property is said to hold almost everywhere (a.e.) if the set of points where it fails to hold is a set of measure zero.

Example: Let f be a function defined on R by

$$f(x) = \begin{cases} 0, & \text{if x is irrational} \\ 1, & \text{if x is rational} \end{cases}$$

Then f (x) = 0 a.e.

10.1.3 Equivalent Functions

Definition: Two functions f and g defined on the same domain E are said to be equivalent on E, written as $f \sim g$ on E, if f = g a.e. on E, i.e. f (x) = g (x) for all $x \in E - E_1$, where $E_1 \subset E$ with $m(E_1) = 0$.

Theorem 2: If f, g : $E \to R$ ($E \in M$) such that $g \in \Omega(E)$.

Proof: Let α be any real number and let $E_1 = E(f > \alpha)$ and $E_2 = E(g > \alpha)$

Then
$$E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1)$$
$$= \{x \in E : f(x) \neq g(x)\}$$

so that by given hypothesis we have

$$m(E_1 \Delta E_2) = 0.$$

This together with the fact that E_1 is measurable

$\Rightarrow$ E_2 is measurable.

Hence $g \in \Omega(E)$.

10.1.4 Non-negative Functions

Definition: Let f be a function, then its positive part, written f^+ and its negative part, written f^{-1}, are defined to be the non-negative functions given by

$$f^+ = \max(f, 0) \text{ and } f^{-1} = \max(-f, 0) \text{ respectively.}$$

Note

$$f = f^+ - f^{-1}$$
$$\text{and } |f| = f^+ + f^{-1}$$

Theorem 3: A function is measurable iff its positive and negative parts are measurable.

Proof: For every extended real valued function f, we may write

$$f^+ = \frac{1}{2}[f + |f|]$$

and
$$f^{-1} = \frac{1}{2}[|f| - f]$$

But f is measurable then |f| is measurable and hence positive and negative parts of f i.e. f^+ and f^- are measurable.

Conversely, let f^+ and f^{-1} be measurable.

Since $f = f^+ - f^{-1}$

Since we know that if f and g are measurable functions defined on a measurable set E then f – g is measurable on E.

Here $f^+ - f^{-1}$ is measurable.

and hence f is measurable.

Notes

Theorem 4: If f is a measurable function and f = g a.e. then g is measurable.

Proof: Let E = $\{x : f(x) \neq g(x)\}$.

Then m (E) = 0

Let α be a real number.

$$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} - \{x \in E : g(x) \leq \alpha\}$$

since f is measurable, the first set on the right is measurable i.e. $\{x : f(x) > \alpha\}$ is measurable.

The last two sets on the right are measurable since they are subsets of E and m (E) = 0.

Thus, $\{x : g(x) > \alpha\}$ is measurable.

So, g is measurable.

Example: Give an example of function for which f is not measurable but |f| is measurable.

Sol: Let k be a non-measurable subset of E = [0, 1).

Define a function f : E → R by

$$f(x) = \begin{cases} 1 \text{ if } x \in k \\ -1 \text{ if } x \notin k \end{cases}$$

The function f is not measurable, since E (f > 0) (=k) is a non-measurable set. But |f| is measurable as the set

$$E(|f| > \alpha) = \begin{cases} E \text{ if } \alpha < 1 \\ \phi \text{ if } \alpha \geq 1 \end{cases} \text{ is measurable}$$

10.1.5 Characteristic Function

Definition: Let A be subset of real numbers. We define the characteristic function χ_A of the set A as follows:

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Note The characteristic function χ_A of the set A is also called the indicator function of A.

Theorem 5: Show that the characteristic function χ_A is measurable iff A is measurable.

Proof: Let χ_A be measurable.

Since A = $\{x : \chi_A(x) > 0\}$ is measurable.

But χ_A is measurable, therefore the set $\{x : \chi_A(x) > 0\}$ is measurable.

⇒ A is measurable.

Conversely, let A be measurable and α be any real number.

$$\text{then } E(\chi_A > \alpha) = \begin{cases} \phi \text{ if } \alpha \geq 1 \\ A \text{ if } 0 \leq \alpha < 1 \\ E \text{ if } \alpha < 0 \end{cases}$$

Notes

Every set on R.H.S. is measurable.

Therefore E ($\chi_A > \alpha$) is measurable.

Hence χ_A is measurable.

> *Note* The above theorem asserts that the characteristic function of non-measurable sets are non-measurable even though the domain set is measurable.

10.1.6 Simple Function

A real valued function ϕ is called simple if it is measurable and assumes only a finite number of values.

If ϕ is simple and has the values $\alpha_1, \alpha_2, \ldots \alpha_n$, then

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $A_i = \{x : \phi(x) = \alpha_i\}$

and $A_i \cap A_j$ is a null set.

Thus we can always express a simple function as a linear combination of characteristic function.

> *Notes*
>
> (i) ϕ is simple $\Leftrightarrow$ A'_is are measurable.
>
> (ii) sum, product and difference of simple functions are simple.
>
> (iii) the representation of ϕ as given above is not unique.
>
> But if ϕ is simple and $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ is the set of non-zero values of f, then
>
> $$\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$
>
> where $A_i = \{x : \phi(x) = \alpha_i\}$
>
> This representation of ϕ is called the *Canonical representation*. Here A'_is are disjoint and α'_is are distinct and non-zero.
>
> (iv) Simple function is always measurable.

10.1.7 Step Function

A real valued function S defined on an interval [a, b] is said to be a step function if these is a partition $a = x_o < x_1 \ldots < x_n = b$ such that the function assumes one and only one value in each interval.

Notes

(i) Step function also assumes finite number of values like simple functions but the sets $\{x : S(x) = C_i\}$ are intervals for each i.

(ii) Every step function is also a simple function but the converse is not true.

e.g. $f : R \to R$ such that $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is a simple function but not step as the sets of rational and irrational are not intervals.

Theorem 6: If f and g are two simple functions then $\alpha f + \beta g$ is also a simple function.

Proof: Since f and g are simple functions and we know that every simple function can be expressed as the linear combination of characteristic function.

$\therefore$ f and g can be expressed as the linear combination of characteristic function.

$$\therefore \quad f = \sum_{i=1}^{m} \alpha_i \chi_{A_i}$$

$$\text{and} \quad g = \sum_{j=1}^{m'} \beta_j \chi_{B_j}$$

where A'_is and B'_js are disjoint.

$$A_i = \{x : f(x) = \alpha_i\}$$

$$B_j = \{x : g(x) = \beta_j\}$$

The set E_k obtained by taking all intersections $A_i \cap B_j$ from a finite disjoint collection of measurable sets and we may write

$$f = \sum_{k=1}^{n} a_k \chi_{E_k}$$

$$\text{and} \quad g = \sum_{k=1}^{n} b_k \chi_{E_k}$$

where $n = mm'$.

$$\alpha f + \beta g = \alpha \sum_{k=1}^{n} a_k \chi_{E_k} + \beta \sum_{k=1}^{n} b_k \chi_{E_k}$$

$$= \sum_{k=1}^{n} (\alpha a_k + \beta_{b_k}) \chi_{E_k}$$

which is a *linear combination* of characteristic functions, therefore it is simple.

Notes

Similarly $fg = \sum_{k=1}^{n} a_k + b_k \ \chi_{E_k}$

which is again a linear combination of characteristic function, therefore fg is simple.

Theorem 7: Let E be a measurable set with m (E) < ∞ and $\{f_n\}$ a sequence of measurable functions converging a.e. to a real valued function defined on E. Then, given ε > 0 and δ > 0, there is a set $A \subset E$ with m (A) < δ and an integer N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in E - A$ and all $n \geq N$.

Proof: Let F be the set of points of E for which $f_n \rightarrow f$. Then m (F) = 0 and $f_n(x) \rightarrow f(x)$ for all $x \in E - F = E_1$ (say). Then by the previous theorem for the set E_1, we get $A_1 \subset E_1$ with $m(A_1) < \delta$ and an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N \text{ and } x \in E_1 - A_1.$$

We get the required result by taking

$$A = A_1 \cup F \text{ since } m(F) = 0 \text{ and } E - A = E_1 - A_1$$

Note Before proving this theorem first prove the previous theorem.

10.1.8 Convergent Sequence of Measurable Function

Definition: A sequence $\{f_n\}$ of measurable functions is said to converge almost uniformly to a measurable function f defined on a measurable set E if for each ∈ > 0 there exists a measurable set $A \subset E$ with m (A) < ∈ such that $\{f_n\}$ converges to f uniformly an E – A.

10.1.9 Egoroff's Theorem

Statement: Let E be a measurable set with m (E) < ∞ and $\{f_n\}$ a sequence of measurable functions which converge to f a.e. on E. Then, given η > 0 there is a set $A \subset E$ with m (A) < η with that the sequence $\{f_n\}$ converges to f uniformly on E – A.

Proof: Applying the theorem, "Let E be a measurable set with m (E) < ∞ and $\{f_n\}$ a sequence of measurable function converging a.e. to real valued function f defined on E. Then given ε > 0 and δ > 0 there is a set $A \subset E$ with m (A) < δ and an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in E - A \text{ and all } n \geq N\text{"}$$

with ε = 1, δ = η/2, we get a measurable set

$A_1 \subset E$ with $m(A_1) < \eta/2$ and a positive integer N_1, such that

$$|f_n(x) - f(x)| < 1 \text{ for all } x \geq N_1$$

and $x \in E_1$, where $E_1 = E - A_1$.

Again, taking ε = 1/2 and $\delta = n/2^2$,

we get a measurable set $A_2 \subset E_1$ with $m(A_2) < \eta/2^2$, and a positive integer N_2 such that

$$|f_n(x) - f(x)| < \frac{1}{2} \ \forall \ n \geq N_2 \text{ and } x \in E_2 \text{ where } E_2 = E_1 - A_2, \text{ and so on.}$$

Notes

At the p^{th} stage, we get a measurable set

$A_p \subset E_{p-1}$ with $m(A_p) < \frac{n}{2^p}$ and a positive integer N_p such that

$|f_n(x) - f(x)| < \frac{1}{p}$ $\forall n \geq N_p$ and $x \in E_p$ where

$$E_p = E_p - 1 - A_p.$$

Let $$A = \sum_{p=1}^{\infty} A_p,$$

then $$m(A) \leq \sum_{p=1}^{\infty} m(A_p)$$

But $$m(A_p) < \frac{n}{2p}$$

$\therefore$ $$m(A) < \sum_{p=1}^{\infty} \frac{n}{2^p}$$

But $\sum_{p=1}^{\infty} \frac{1}{2^p}$ is a G.P. series so $$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$\therefore$ $$m(A) < \eta$$

Also, $$E - A = E - \bigcup_p A_p$$

$$= \bigcap_p (E - A_p)$$

$$= \bigcap_p (E_{p-1} - A_p)$$

$$= \bigcap_p E_p$$

Let $x \in E - A$. Then $x \in E_p$ $\forall p$ and so

$$|f_n(x) - f(x)| < \frac{1}{p}, \ \forall n \geq N_p.$$

Let us choose p such that $\frac{1}{p} < \varepsilon$, we get

$$|f_n(x) - f(x)| < \varepsilon \ \forall x \in E - A \text{ and } n \geq N_p = N$$

> *Note* Egoroff's theorem can be stated as: almost every where convergence implies almost uniform convergence.

10.1.10 Riesz Theorem

Let $\{f_n\}$ be a sequence of measurable functions which converges in measure to f. Then there is a subsequence $\{f_{n_k}\}$ which converges in measure to f a.e.

Proof: Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be two sequences of positive real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$.

Let us now choose an increasing sequence $\{n_k\}$ of positive integers as follows.

Let n be a positive integer such that

$$m\left(\left\{x \in E \middle| f_n(x) - f(x)\right| \geq \varepsilon\}\right) < \delta$$

Since $f_n \to f$ in measure for a given $\varepsilon_1 > 0$ and $\delta_1 > 0$, $\exists$ a positive integer n_1 such that

$$m\left(\left\{x \in E, \left|f_{n_1}(x) - f(x)\right| \geq \varepsilon_1\right\}\right) < \delta_1, \forall\, n_1 \geq n$$

Similarly, let n_2 be a positive number such that $m\left(\left\{x \in E, \left|f_{n_2}(x) - f(x)\right| \geq \varepsilon_2\right\}\right) < \delta_2, \forall\, n_2 \geq n_1$ and so on.

In general let n_k be a positive number such that

$$m\left(\left\{x : x \in E, \left|f_{n_k}(x) - f(x)\right| \geq \varepsilon_k\right\}\right) < \delta_k \text{ and that } n_k \geq n_{k-1}.$$

We shall now prove that the subsequence $\{f_{n_k}\}$ converges to f a.e.

Let $A_k = \bigcup_{i=k}^{\infty} \left\{x : x \in E, \left|f_{n_i}(x) - f(x)\right| \geq \varepsilon_i\right\}, k \in N$ and $A = \bigcap_{k=1}^{\infty} A_k$.

Clearly, $\{A_k\}$ is a decreasing sequence of measurable sets and $m(A_1) < \infty$.

Therefore, we have

$m(A) = \lim_{k\to\infty} m(A_k)$

But $m(A_k) \leq \sum_{i=k}^{\infty} \delta_i \to 0$ as $k \to \infty$.

Hence $m(A) = 0$.

Now it remains to show that $\{f_n\}$ converges to f on $E - A$. Let $x_o \in E - A$.

Then $x_o \notin A_{k_o}$ for some positive integer k_o.

or $x_o \notin \{x \in E : |f_n(x) - f(x)| \geq \varepsilon_k\}, k \geq k_o$

which gives $|f_n(x_o) - f(x_o)| < \varepsilon_k, k \geq k_o$

But $\varepsilon_k \to 0$ as $k \to \infty$.

Hence $\lim_{k\to\infty} f_{n_k}(x_o) = f(x_o)$.

10.2 Summary

- Let E be a measurable set and R* be a set of extended real numbers. A function $f : E \to R^*$ is said to be a Lebesgue measurable function on E or a measurable function on E iff the set E $(f > \alpha) = \{x \in E : f(x) > \alpha\} = f^{-1}\{(\alpha, \infty)\}$ is a measurable subset of E $\forall\, \alpha \in R$.
- A property is said to hold almost everywhere (a.e.) if the set of points where it fails to hold is a set of measure zero.
- Two functions f and g defined on the same domain E are said to be equivalent on E, written as $f \sim g$ on E, if f = g a.e. on E, i.e. $f(x) = g(x)$ for all $x \in E - E_1$, where $E_1 \subset E$ with $m(E_1) = 0$.
- $f^+ = \max(f, 0)$ and $f^{-1} = \max(-f, 0)$

  $|f| = f^+ + f^{-1}$
- Let A be subset of real numbers. We define the characteristic function χ_A of the set A as follows:

  $$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$
- A real valued function ϕ is called simple if it is measurable and assumes only a finite number of values.

10.3 Keywords

Almost Everywhere (a.e.): A property is said to hold almost everywhere (a.e.) if the set of points where it fails to hold is a set of measure zero.

Characteristic Function: Let A be subset of real numbers. We define the characteristic function χ_A of the set A as follows:

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Egoroff's Theorem: Let E be a measurable set with $m(E) < \infty$ and $\{f_n\}$ a sequence of measurable functions which converge to f a.e. on E. Then given n > 0 there is a set $A \subset E$ with $m(A) < n$ such that the sequence $\{f_n\}$ converges to f uniformly on E – A.

Equivalent Functions: Two functions f and g defined on the same domain E are said to be equivalent on E, written as $f \sim g$ on E, if f = g a.e. on E, i.e. $f(x) = g(x)$ for all $x \in E - E_1$, where $E_1 \subset E$ with $m(E_1) = 0$.

Haracteristic Function: Let A be subset of real numbers. We define the characteristic function χ_A of the set A as follows:

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Lebesgue Measurable Function: A function $f : E \to R^*$ is said to be a Lebesgue measurable function on E or a measurable function on E iff the set

$E(f > \alpha) = \{x \in E : f(x) > \alpha\} = f^{-1}\{\alpha, \infty)\}$ is a measurable subset of E $\forall\ \alpha \in R$.

Measurable Set: A set E is said to be measurable if for each set T, we have

$$m^*(T) = m^*(T \cap E) + m^*\{T \cap E^c)$$

Non-negative Functions: Let f be a function, then its positive part, written f^+ and its negative part, written f^{-1}, are defined to be the non-negative functions given by

$$f^+ = \max(f, 0) \text{ and } f^{-1} = \max(-f, 0) \text{ respectively.}$$

Riesz Theorem: Let $\{f_n\}$ be a sequence of measurable functions which converges in measure to f. Then there is a subsequence $\{f_{n_k}\}$ which converges to f a.e.

Simple Function: A real valued function ϕ is called simple if it is measurable and assumes only a finite number of values.

If ϕ is simple and has the values $\alpha_1, \alpha_2, \ldots \alpha_n$, then

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $$A_i = \{x : \phi(x) = \alpha_i\}$$

and $A_i \cap A_j$ is a null set.

Step Function: A real valued function S defined on an interval [a, b] is said to be a step function if these is a partition $a = x_o < x_1 \ldots < x_n = b$ such that the function assumes one and only one value in each interval.

Subsequence: If (x_n) is a given sequence in X and (n_k) is an strictly increasing sequence of positive integers, then $\{x_{n_k}\}$ is called a subsequence of (x_n).

10.4 Review Questions

1. If f is a measurable function and c is a real number, then is it true to say that cf is measurable?
2. A non-zero constant function is measurable if and only if X is measurable comment.
3. Let Q be the set of rational number and let f be an extended real-valued function such that $\{x : f(x) > \alpha\}$ is measurable for each $\alpha \in Q$. Then show that f is measurable.
4. Show that if f is measurable then the set $\{x : f(x) = \alpha\}$ is measurable for each extended real number α.
5. If f is a continuous function and g is a measurable function, then prove that the composite function fog is measurable.

Notes

6. Show that

(i) $\chi_{A \cap B} = \chi_A \cdot \chi_B$

(ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

(iii) $\chi_{A^c} = 1 - \chi_A$

10.5 Further Readings

Books

Dudley, R.M. (2002). *Real Analysis and Probability (2 ed.)*. Cambridge University Press

Strichartz, Robert (2000). *The Way of Analysis*. Jones and Bortlett.

Online links

mathworld.wolfram.com>calculus and Analysis > Measure theory

planetmath.org/Measurable functions.html

zeta.math.utsa.edu/~ mqr 328/class/real2/mfunct.pdf

Unit 11: Integration

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define the Riemann integral and Lebesgue integral of bounded function over a set of finite measure.
- Understand the Lebesgue integral of a non-negative function.
- Solve problems on integration.

Introduction

We now come to the main use of measure theory: to define a general theory of integration. The particular case of the integral with respect to the Lebesgue measure is not, in any way, simpler the general case, which will give us a tool of much wider applicability.

11.1 Integration

11.1.1 The Riemann Integral

Let f be a bounded real valued function defined on the interval [a, b] and let $a = x_0 < x_1 < \dots < x_n = b$ be a sub-division of [a, b].

Then for each sub-division we can define the sums

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

and

$$s = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$$

Notes

where $$M_i = \sup_{x_{i-1}<x\le x_i} f(x),$$

$$m_i = \inf_{x_{i-1}<x\le x_i} f(x)$$

We then define the *upper Riemann integral* of f by

$$R\int_a^{\bar{b}} f(x)\,dx = \inf S$$

where the infimum is taken over all possible sub-divisions of [a, b].

Similarly, we define the *lower Riemann integral*

$$R\int_{\underline{a}}^{b} f(x)\,dx = \sup S$$

The upper integral is always at least as large as the lower integral, and if the two are equal, we say that f is *Riemann integrable* and we call this common value the Riemann integral of f.

It will be denoted by

$$R\int_a^b f(x)\,dx$$

Note By a *step function* we mean function Ψ s.t.

$$\Psi(x) = \alpha_i \ \forall \ x \in [x_{i-1}, x_i]$$

for some sub-division of [a, b] and some set of constant α_i then

$$\int_a^b \Psi(x)\,dx = \int_{x_o}^{x_1} \Psi(x)\,dx + \ldots + \int_{x_{n-1}}^{x_n} \Psi(x)\,dx$$

$$= \int_{x_o}^{x_1} \alpha_1\,dx + \int_{x_1}^{x_2} \alpha_2\,dx + \ldots + \int_{x_{n-1}}^{x_n} \alpha_n\,dx$$

$$= \alpha_1 (x_1 - x_0) + \alpha_2 (x_2 - x_1) + \ldots + \alpha_n (x_{n-1} - x_n)$$

$$= \sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}) \qquad \ldots (1)$$

with this in mind, we see that

$$R\int_a^{\bar{b}} f(x)\,dx = \inf \cup_p (f)$$

$$= \inf \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$= \inf \int_a^b \Psi(x)\, dx \quad \text{for all step functions}$$

$$\Psi(x) \geq f(x)$$

Similarly $$R \int_a^b f(x)\, dx = \sup L_p(f)$$

$$= \sup \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$= \sup \int_a^b \phi(x)\, dx \quad \text{for all step function}$$

$$\phi(x) \leq f(x).$$

11.1.2 Lebesgue Integral of a Bounded Function over a Set of Finite Measure

Characteristic Function

The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

is called the *characteristic function* of E.

Simple Function

A linear combination $\phi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$ is called a *simple function* if the sets E_i are measurable.

This representation of ϕ is not unique.

However, a function ϕ is simple if and only if it is measurable and assumes only a finite number of values.

Canonical Representation

If ϕ is simple function and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ the set of non-zero values of ϕ, then

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where $E_i = \{x : \phi(x) = \alpha_i\}$.

Notes

This representation of ϕ is called the *canonical representation*. Here E_i's are disjoint and α_i's are finite in number, distinct and non-zero.

Elementary Integral

Definition: If ϕ vanishes outside a set of finite measure, we define the elementary integral of ϕ by $\int \phi(x)\,dx = \sum_{i=1}^{n} \alpha_i\, mE_i$ when ϕ has the canonical representation.

$$\phi = \sum_{i=1}^{n} \alpha_i\, \chi_{E_i}.$$

We sometimes abbreviate the expression for this integral $\int \phi$. If E is any measurable set, we define the elementary integral of ϕ over E by $\int_E \phi = \int \phi \cdot \chi_E$.

If E = [a, b], then the integral $\int_{[a,b]} \phi$ will be denoted by $\int_a^b \phi$.

Theorem 1: Let ϕ and Ψ be simple functions which vanish outside a set of finite measure, then

$$\int (a\phi + b\Psi) = a\int \phi + b\int \Psi \text{ and if } \phi \geq \Psi \text{ a.e., then } \int \phi \geq \int \Psi$$

Proof: Since ϕ and Ψ are simple functions.

Therefore these can be written in the canonical form

$$\phi = \sum^{m} \alpha_i\, \chi_{A_i}$$

and

$$\Psi = \sum_{j=1}^{m'} B_j\, \chi_{B_j}$$

where $\{A_i\}$ and $\{B_j\}$ are disjoint sequences of measurable sets and

$$A_i = \{x : \phi(x) = \alpha_i\}$$

and

$$B_j = \{x : \Psi(x) = \beta_j\}$$

The set E_k obtained by taking all intersections $A_i \cap B_j$ form a finite disjoint collection of measurable sets. We may write

$$\phi = \sum_{k=1}^{N} a_k\, \chi_{E_k} \text{ and}$$

$$\Psi = \sum_{k=1}^{N} b_k\, \chi_{E_k} \text{ (where N = mm')}$$

Now $$a\phi + b\Psi = a\sum_{k=1}^{N} a_k\, \chi_{E_k} + b\sum_{k=1}^{N} b_k\, \chi_{E_k}$$

$$= \sum_{k=1}^{N} (aa_k + bb_k)\, \chi_{E_k}$$

which is again a simple function.

Since $$\int \phi = \sum_i a_i\, m\, E_i$$

$\therefore$ $$\int (a\phi + b\Psi) = \sum_{k=1}^{N} (aa_k + bb_k)\, m\, E_k \quad \text{(by definition)}$$

$$= a\sum_{k=1}^{N} a_k\, mE_k + b\sum_{k=1}^{N} b_k\, mE_k$$

$$= a\int \phi + b\int \Psi$$

Now since $\phi \geq \Psi$ a.e.

$\Rightarrow$ $\phi - \Psi \geq 0$ a.e.

We have proved that

$$\int (a\phi + b\Psi) = a\int \phi + b\int \Psi$$

Put a = 1, b = –1 in the first part, we get

$$\int (\phi - \Psi) = \int \phi - \int \Psi$$

Since $\phi - \Psi \geq 0$ a.e. is a simple function, by the definition of the elementary integral, we have

$$\int (\phi - \Psi) \geq 0$$

$\Rightarrow$ $$\int \phi - \int \Psi \geq 0$$

$\Rightarrow$ $$\int \phi \geq \int \Psi$$

Theorem 2: Riemann integrable is Lebesgue integrable.

Proof: Since f is Riemann integrable over [a, b], we have

$$\inf_{\Psi_1 \geq f} \int_a^b \Psi_1(x)\, dx = \sup_{\phi \leq f} \int_a^b \phi_1(x)\, dx = R\int_a^b f(x)\, dx$$

where ϕ_1 and Ψ_1 vary over all step functions defined on [a, b].

Notes

Since we know that every step function is a simple function,

$$\therefore \qquad \sup_{\phi_1 \le f} \int_a^b \phi_1(x)\, dx \;\le\; \sup_{\phi \le f} \int_a^b \phi(x)\, dx$$

and

$$\inf_{\Psi_1 \ge f} \int_a^b \Psi_1(x)\, dx \;\ge\; \inf_{\Psi \ge f} \int_a^b \Psi(x)\, dx$$

where ϕ and Ψ vary over all the simple functions defined on [a, b]. Thus from the above relation, we have

$$R\int_a^b f(x)\, dx \le \sup_{\phi \le f} \int_a^b \phi(x)\, dx \le \inf_{\Psi \ge f} \int_a^b \Psi(x)\, dx \le R\int_a^b f(x)\, dx$$

$$\Rightarrow \qquad \sup_{\phi \le f} \int_a^b \phi(x)\, dx = \inf_{\Psi \ge f} \int_a^b \Psi(x)\, dx$$

$$\Rightarrow \qquad \int_a^b f(x)\, dx = R\int_a^b f(x)\, dx$$

Note The converse of this theorem is not true i.e.

A Lebesgue integrable function may not be Riemann integrable

e.g. Let f be a function defined on the interval [0, 1] as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Let us consider a partition p of an interval [0, 1].

$$U(p, f) = \sum_{i=1}^{N} M_i \,\Delta x_i$$

$$= 1\Delta x_1 + 1\Delta x_2 + \ldots\ldots + 1\Delta x_n = 1 - 0$$

$$= 1.$$

$$\therefore \qquad \overline{\int_0^1} f\, dx = \inf U(p, f) = 1 - 0 = 1.$$

$$\underline{\int_0^1} f\, dx = \sup L(p, f)$$

$$= \sup \{0\Delta x_1 + 0\Delta x_2 + \ldots\ldots + 0\Delta x_n\}$$

$$= 0$$

Notes

Thus $$\overline{\int} f\,dx \neq \underline{\int} f\,dx$$

∴ The function is not Riemann integrable.

Now for Lebesgue Integrability

Let A_1 be the set of all irrational numbers and A_2 be the set of all rational numbers in [0, 1].

The partition P = $\{A_1, A_2\}$ is a measurable partition of [0, 1] and $mA_1 = 0$, $mA_2 = 1$.

$$L(p, f) = \inf_{A_1} f(x) \cdot mA_1 + \inf_{A_2} f(x) \cdot mA_2$$

$$= 0 \cdot mA_1 + 1 \cdot mA_2 = 1.$$

$$U(p, f) = \sup_{A_1} f(x) \cdot mA_1 + \sup_{A_2} f(x) \cdot mA_2$$

$$= 0 \cdot mA_1 + 1 \cdot mA_2 = 1.$$

$$\sup_{P} L(p, f) = 1 = \inf_{P} U(p, f)$$

⇒ f is Lebesgue integrable over [0, 1].

Theorem 3: If f and g are bounded measurable functions defined on the set E of finite measure, then

(1) $\int_E (af + bg) = a\int_E f + b\int_E g$

(2) If f = g a.e., then $\int_E f = \int_E g$

(3) If f ≤ g a.e., then $\int_E f \le \int_E g$

Hence $\left|\int f\right| \le \int |f|$

(4) If A and B are disjoint measurable set of finite measure, then

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof of 1: Result is true if a = 0

Let a ≠ 0.

If Ψ is a simple function then so is a Ψ and conversely.

Hence for a > 0

$$\int_E af = \inf_{a\Psi \ge af} \int_E a\Psi$$

$$= \inf_{\Psi \geq f} \int_E a\Psi \qquad (\because a > 0)$$

$$= \inf_{\Psi \geq f} a \int_E \Psi$$

$$= a \inf_{\Psi \geq f} \int_E \Psi$$

$$= a \int_E f$$

Again if $a < 0$,

$$\int_E af = \inf_{a\Psi \geq af} \int_E a\Psi$$

$$= \sup_{\Psi \leq f} \int_E a\Psi \qquad (\because a < 0)$$

$$= \sup_{\Psi \leq f} a \int_E \Psi$$

$$= a \sup_{\Psi \leq f} \int_E \Psi$$

$$= a \int_E f$$

Therefore in each case

$$\int_E af = a \int_E f \qquad \ldots \text{(i)}$$

Now we prove that

$$\int_E (f+g) = \int_E f + \int_E g$$

Let Ψ_1 and Ψ_2 be two simple functions such that $\Psi_1 > f$ and $\Psi_2 \geq g$, then $\Psi_1 + \Psi_2$ is a simple function and $\Psi_1 + \Psi_2 \geq f + g$.

or $f + g = \Psi_1 + \Psi_2$

$$\therefore \quad \int_E (f+g) \leq \int_E (\Psi_1 + \Psi_2)$$

But $$\int_E \Psi_1 + \Psi_2 = \int_E \Psi_1 + \int_E \Psi_2$$

Notes

$\Rightarrow \qquad \int_E (f+g) \leq \int_E \Psi_1 + \int_E \Psi_2$

Since $\qquad \inf_{f \leq \Psi_1} \int_E \Psi_1 = \int_E f$

and $\qquad \inf_{g \leq \Psi_2} \int_E \Psi_2 = \int_E g$

$\therefore \qquad \int_E (f+g) \leq \int_E f + \int_E g \qquad \ldots (2)$

On the other hand if ϕ_1 and ϕ_2 are two simple functions such that $\phi_1 < f$ and $\phi_2 \leq g$. Then $\phi_1 + \phi_2$ is simple function and

$$\phi_1 + \phi_2 \leq f + g,$$

or $\qquad f + g \geq \phi_1 + \phi_2$

$\therefore \qquad \int_E (f+g) \geq \int_E (\phi_1 + \phi_2)$

But $\qquad \int_E (\phi_1 + \phi_2) = \int_E \phi_1 + \int_E \phi_2$

$\Rightarrow \qquad \int_E (f+g) \geq \int_E \phi_1 + \int_E \phi_2$

Since $\qquad \sup_{f \geq \phi_1} \int_E \phi_1 = \int_E f$

and $\qquad \sup_{f \geq \phi_2} \int_E \phi_2 = \int_E g$

$\therefore \qquad \int_E (f+g) \geq \int_E f + \int_E g \qquad \ldots (3)$

From (2) and (3), we get

$$\int_E (f+g) = \int_E f + \int_E g$$

$\therefore \qquad \int_E (af+bg) = \int_E af + \int_E bg$

$$= a\int_E f + b\int_E g \text{ from (i)}$$

Proof of 2: Since f = g a.e.

$\Rightarrow \qquad f - g = 0$ a.e.

Let $F = \{x : f(x) \neq g(x)\}$

Then by definition of a.e., we have mF = 0 and $F \subset E$.

$$\therefore \quad \int_E (f-g) = \int_{F \cup (E-F)} (f-g) = \int_F (f-g) + \int_{E-F} (f-g)$$

$$= (f - g)\, mF + (f - g)\, m(E - F)$$

$$= (f - g) \,.\, 0 + 0 \,.\, m(E - F) \; [\because mF = 0 \text{ and } f - g = 0 \text{ over } E - F]$$

$$= 0$$

$$\therefore \quad \int_E (f-g) = 0$$

$$\Rightarrow \quad \int_E f - \int_E g = 0 \Rightarrow \int_E f = \int_E g$$

Note Converse need not be true

e.g. Let the functions $f : [-1, 1] \to R$ and $g : [-1, 1] \to R$ be defined by

$$f(x) = \begin{cases} 2 \text{ if } x \leq 0 \\ 0 \text{ if } x > 0 \end{cases}$$

and $g(x) = 1 \;\forall\; x.$

Then $$\int_{-1}^{1} f(x)\,dx = 2 = \int_{-1}^{1} g(x)\,dx$$

But $f \neq g$ a.e.

In other words, they are not equal even for a single point in [–1, 1].

Proof of 3: $f \leq g$ a.e.

$\Rightarrow \quad f - g \leq 0$ a.e.

Let ϕ be simple function,

$$\phi = f - g$$

$$\Rightarrow \quad \phi \leq 0 \qquad [\because f - g = 0 \text{ a.e.}]$$

$$\Rightarrow \quad \int_E \phi \leq 0$$

$$\Rightarrow \quad \int_E (f-g) \leq 0$$

$$\Rightarrow \quad \int_E f - \int_E g \leq 0$$

$$\Rightarrow \quad \int_E f \leq \int_E g$$

Since $f \leq |f|$

$$\Rightarrow \quad \int_E f \leq \int_E |f| \quad \ldots (1)$$

Again $-f \leq |f|$

$$\Rightarrow \quad \int_E -f \leq \int_E |f|$$

or

$$-\int_E |f| \leq \int_E f \quad \ldots (2)$$

From (1) and (2) we get

$$-\int_E |f| \leq \int_E f \leq \int_E |f|$$

$$\Rightarrow \quad \left|\int_E f\right| \leq \int_E |f|.$$

Proof of 4: It follows from (3) and the fact that $\int_E 1 = mE$.

Proof of 5:

$$\int_{A\cup B} f = \int f\chi_{A\cup B}$$

Now

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$$

where A and B are disjoint measurable sets i.e.

$$A \cap B = \phi$$

$$\therefore \quad \int_{A\cup B} f = \int f(\chi_A + \chi_B) - \int f\chi_{A\cap B}$$

$$= \int f\chi_A + \int f\chi_B - 0 \qquad [\because A \cap B = \phi \text{ and } m(\phi) = 0]$$

$$= \int_A f + \int_B f$$

11.1.3 The Lebesgue Integral of a Non-negative Function

Definition: If f is a non-negative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h \leq f} \int_E h,$$

Notes

where h is a bounded measurable function such that

$$m\{x : h(x) \neq 0\} < \infty$$

Theorem 4: Let f be a non-negative measurable function. Show that $\int f = 0$ implies f = 0 a.e.

Proof: Let ϕ be any measurable simple function such that

$$\phi \leq f.$$

Since f = 0 a.e. on E

$$\Rightarrow \qquad \phi \leq 0 \text{ a.e.}$$

$$\therefore \qquad \int_E \phi(x)\, dx \leq 0$$

Taking supremum over all those measurable simple functions $\phi \leq f$, we get

$$\int_E f\, dx \leq 0 \qquad \ldots (1)$$

Similarly let Ψ be any measurable simple function such that $\Psi \geq f$

Since $\qquad f = 0$ a.e.

$$\therefore \qquad \Psi \geq 0 \text{ a.e.}$$

$$\Rightarrow \qquad \int_E \Psi(x)\, dx \geq 0$$

Taking infimum over all those measurable simple functions $\Psi \geq f$, we get

$$\int_E f\, dx \geq 0 \qquad \ldots (2)$$

From (1) and (2), we get

$$\int_E f\, dx = 0.$$

Conversely, let $\qquad \int_E f\, dx = 0$

If $\qquad E_n = \left\{x : f(x) > \frac{1}{n}\right\}$, then

$$\int_E f\, dx > \int_E \frac{1}{n}\chi_{E_n}(x)\, dx$$

But $\qquad \int_E \frac{1}{n}\chi_{E_n}(x)\, dx = \frac{1}{n} m E_n \qquad$ (By definition)

$\therefore \qquad \int_E f\,dx > \frac{1}{n} m E_n$

Or $\qquad \frac{1}{n} m E_n < \int_E f\,dx$

But $\qquad \int_E f\,dx < 0$

$\therefore \qquad \frac{1}{n} m E_n < 0$

$\Rightarrow \qquad m\,E_n < 0$

But $m\,E_n \geq 0$ is always true

$\therefore \qquad m\,E_n = 0$

But $\qquad \{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$

and $\qquad m\,E_n = 0$

$\Rightarrow \qquad m\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$

$\Rightarrow \qquad m\,\{x : f(x) > 0\} = 0$

$\therefore \qquad f = 0$ a.e. on E

Theorem 5: Let f and g be two non-negative measurable functions. If f is integrable over E and g (x) < f (x) on E, then g is also integrable over E, and

$$\int_E (f - g) = \int_E f - \int_E g.$$

Proof: Since we know that if f and g are non-negative measurable functions defined on a set E, then

$$\int_E (f + g) = \int_E f + \int_E g$$

Since $\qquad f = (f - g) + g,$

therefore we have

$$\int_E f = \int_E (f - g + g) = \int_E (f - g) + \int_E g \qquad \ldots (1)$$

Since the functions f – g and g are non-negative and measurable. Further, f being integrable over E, $\int_E f < \infty$ (by definition)

Notes

Therefore, each integral on the right of (1) is finite.

In particular, $\int_E g < \infty$,

which shows that g is an integrable function over E.

Since $$\int_E f = \int_E (f-g) + \int_E g$$

$$\Rightarrow \quad \int_E f - \int_E g = \int_E (f-g).$$

11.1.4 The General Lebesgue Integral

For the positive part f^+ of a function f, we define

$$f^+ = \max(f, 0)$$

and negative part f^- by f^-

$$f^- = \max(-f, 0)$$

and that f is measurable if and only if both f^+ and f^- are measurable.

Note

$$f = f^+ - f^-$$

and $$|f| = f^+ + f^-$$

Definition: A measurable function f is said to be *Lebesgue integrable* over E if f^+ and f^- are both Lebesgue integrable over E. In this case, we define $\int_E f = \int_E f^+ - \int_E f^-$.

Theorem 6: Let f and g be integrable over E, then

(a) The function of f is integrable over E, and $\int_E cf = c\int_E f$.

(b) Sum of two integrable functions is integrable i.e. the function f + g is integral over E, and

$$\int_E (f+g) = \int_E f + \int_E g$$

(c) If $f \le g$ a.e., then $\int_E f \le \int_E g$.

(d) If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f \le \int_B f.$$

Proof: (a) If $c \geq 0$, then

$$(cf)^+ = cf^+$$

$$(cf)^- = cf^-$$

and if $c < 0$, then

$$(cf)^* = (-c) \cdot f^-$$

$$(cf)^- = (-c) \cdot f^+$$

Since f is integrable so f^+ and f^- are also integrable and conversely. Hence the result follows.

(b) In order to prove the required result first of all we show that if f_1 and f_2 are non-negative integrable functions such that $f = f_1 - f_2$, then

$$\int_E f = \int_E f_1 - \int_E f_2 \qquad \dots (1)$$

Since $f = f^+ - f^-$,

Also $f = f_1 - f_2$,

then $f^+ - f^- = f_1 - f_2$

$\Rightarrow$ $f^+ + f_2 = f_1 + f^- \qquad \dots (2)$

Also we know that if f and g are non-negative measurable functions defined on a set E, then

$$\int_E (f+g) = \int_E f + \int_E g$$

Then from (2), we get

$$\int_E f^+ + \int_E f_2 = \int_E f_1 + \int_E f^-$$

$$\Rightarrow \qquad \int_E f^+ - \int_E f^- = \int_E f_1 - \int_E f_2 \qquad \dots (3)$$

But f is integrable so f^+ and f^- are integrable i.e.

$$\int f = \int f^+ - \int f^-$$

Therefore (3) becomes

Hence $$\int_E f = \int_E f_1 - \int_E f_2$$

which proves (1).

Now, if f and g are integrable functions over E, then

$f^+ + g^+$, $f^- + g^-$ and $f + g = (f^+ + g^+) - (f^- + g^-)$

and also integrable functions over E.

Notes

Furthermore f and g are integrable, it implies that |f| and |g| are integrable.

($\because$ A measurable function f is integrable over E if and only if |f| is integrable over E.)

Thus |f| + |g| is integrable over E.

($\because$ $\int_E (f+g) = \int_E f + \int_E g$ and by the definition of integrable)

Since |f + g| ≤ |+|f| + |g|

which shows that f + g is integrable.

Hence sum of two integrable functions is integrable.

Thus
$$\int_E (f+g) = \int_E (f^+ + g^+) - \int_E (f^- + g^+)$$
$$= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-$$
$$= \left(\int_E f^+ - \int_E f^-\right) + \left(\int_E g^+ - \int_E g^-\right)$$
$$= \int_E f + \int_E g$$

(c)
$$f \le g \text{ a.e.}$$
$$\Rightarrow \quad f - g \le 0 \text{ a.e.}$$
$$\Rightarrow \quad g - f \ge 0 \text{ a.e.}$$
$$\therefore \quad \int_E (g-f) \ge 0$$

Since g = f + (g – f) and f, g – f are integrable over E.

Then by the given hypothesis $(g - f)^- = 0$ a.e.

then $\int_E (g-f)^- = 0$,

(Since we know that if f = 0 a.e. then $\int_E f = 0$)

$$\therefore \int_E g = \int_E f + \int_E (g-f) = \int_E f + \int_E (g-f)^+ - \int_E (g-f)^-$$

becomes

$$\int_E g = \int_E f + \int_E (g-f)^+ - 0 = \int_E f + \int_E (g-f)^+ \qquad (\because (g-f) \ge 0)$$

Notes

$$\geq \int_E f$$

$$\therefore \qquad \int_E g \geq \int_E f \Rightarrow \int_E f \leq \int_E g$$

(d)
$$\int_{A \cup B} f = \int f \, \chi_{A \cup B}$$

$$= \int f \cdot \chi_A + \int f \cdot \chi_B$$

$$= \int_A f + \int_B f .$$

Example: Let f be a non-negative integrable function. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f(t)\, dt \text{ is continuous on R.}$$

Solution: Since f is a non-negative integrable function, then given ε > 0 there is a δ > 0 such that for every set A ⊂ R with m (A) < δ, we have

$$\left| \int_A f \right| < \varepsilon$$

If $x_o \in R$, then $\forall\ x \in R$ with $|x - x_o| < \delta$, we have

$$\left| \int_{x_o}^{x} f(t) \right| dt < \varepsilon$$

$$\Rightarrow \qquad \left| \int_{x_o}^{-\infty} f(t)\, dt \right| + \left| \int_{-\infty}^{x} f(t)\, dt \right| < \varepsilon$$

$$\Rightarrow \qquad \left| \int_{-\infty}^{x} f(t)\, dt \right| + \left| \int_{x_o}^{-\infty} f(t)\, dt \right| < \varepsilon$$

$$\Rightarrow \qquad \left| \int_{-\infty}^{x} f(t)\, dt \right| - \left| \int_{-\infty}^{x_o} f(t)\, dt \right| < \varepsilon$$

$$\Rightarrow \qquad |F(x) - F(x_o)| < \varepsilon$$

Hence F is continuous at x_o. Since $x_o \in R$ is arbitrary, F is continuous on R.

Notes

11.2 Summary

- Let ϕ and Ψ be simple functions which vanish outside a set of finite measure, then

$$\int (a\phi + b\Psi) = a\int \phi + b\int \Psi \text{ and if } \phi \geq \Psi \text{ a.e., then } \int \phi \geq \int \Psi$$

- A Lebesgue integrable function may not be Riemann integrable.
- Let A_1 be the set of all irrational numbers and A_2 be the set of all rational numbers in [0, 1].
- If f is a non-negative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h \leq f} \int_E h,$$

where h is a bounded measurable function such that

$$m\{x : h(x) \neq 0\} < \infty$$

- Let f and g be two non-negative measurable functions. If f is integrable over E and g (x) < f (x) on E, then g is also integrable over E, and

$$\int_E (f - g) = \int_E f - \int_E g.$$

11.3 Keywords

Canonical Representation: If ϕ is simple function and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ the set of non-zero values of ϕ, then

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where $E_i = \{x : \phi(x) = \alpha_i\}$.

Characteristic Function: The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

is called the *characteristic function* of E.

Elementary Integral: If ϕ vanishes outside a set of finite measure, we define the elementary integral of ϕ by $\int \phi(x)\,dx = \sum_{i=1}^{n} \alpha_i \, mE_i$ when ϕ has the canonical representation.

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}.$$

Lebesgue Integrable: A measurable function f is said to be Lebesgue integrable over E if f^+ and f^- are both Lebesgue integrable over E. In this case, we define $\int_E f = \int_E f^+ - \int_E f^-$.

Notes

Simple Function: A linear combination $\phi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$ is called a simple function if the sets E_i are measurable.

Simple Function: A linear combination $\phi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$ is called a *simple function* if the sets E_i are measurable.

This representation of ϕ is not unique.

However, a function ϕ is simple if and only if it is measurable and assumes only a finite number of values.

The Lebesgue Integral of a Non-negative Function: If f is a non-negative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h \le f} \int_E h,$$

where h is a bounded measurable function such that

$$m\{x : h(x) \neq 0\} < \infty$$

The Riemann Integral: Let f be a bounded real valued function defined on the interval [a, b] and let $a = x_0 < x_1 < \ldots < x_n = b$ be a sub-division of [a, b].

Then for each sub-division we can define the sums

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

and $$s = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$$

where $$M_i = \sup_{x_{i-1} < x \le x_i} f(x),$$

$$m_i = \inf_{x_{i-1} < x \le x_i} f(x)$$

11.4 Review Questions

1. Prove that $\int_E af = a\int_E f \; \forall$ real number a.

2. If f is bounded real valued measurable function defined on a measurable set E of finite measure such that $a \le f(x) \le b$, then show that $amE \le \int_E f \le bmE$.

3. If f and g are non-negative measurable functions defined on $E \in \mathcal{M}$ then prove that

   (a) $\int_E cf = c\int_E f, c > 0$

Notes

(b) $\int_E (f+g) = \int_E f + \int_E g$

(c) $\int f = 0 \Rightarrow f = 0$ a.e.

(d) If $f \leq g$ a.e. then $\int_E f \leq \int_E g$

4. If f is integrable over E, then show that $|f|$ is integrable over E, and $\left|\int_E f\right| \leq \int_E |f|$.

5. Show that if f is a non-negative measurable function then f = 0 a.e. on E iff $\int_E f = 0$.

6. If $\int_E f = 0$ and $f(x) \geq 0$ on E, then f = 0 a.e.

11.5 Further Readings

Books

Erwin Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons Inc., New York, 1989

Walter Rudin, *Real and Complex Analysis*, Third McGraw Hill Book Co., New York, 1987

R.G. Bartle, *The Elements of Integration and Lebesgue Measure,* Wiley Interscience, 1995

Online links

www.maths.manchester.ac.uk

www.uir.ac.za

Unit 12: General Convergence Theorems

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand bounded convergence theorem.
- State and prove monotone convergence theorem and Lebesgue dominated convergence theorem.
- Solve related problems on these theorems.

Introduction

Convergence of a sequence of functions can be defined in various ways and there are situations in which each of these definitions is natural and useful. In this unit, we shall study about convergence almost everywhere, pointwise and uniform convergence. We shall also prove bounded convergence theorem and monotone convergence theorem which are so useful in solving problems on convergence. The dominated convergence theorem is one of the most important results of Lebesgue's integration theory. It gives a general sufficient condition for the validity of proceeding to the limit of a sequence of functions under the integral sign. It is an invaluable tool to study functions defined by integrals.

12.1 General Convergence Theorems

12.1.1 Convergence almost Everywhere

Let $<f_n>$ be a sequence of measurable functions defined over a measurable set E. Then $<f_n>$ is said to converge almost everywhere in E if there exists a subset E_0 of E s.t.

Notes

(i) $f_n(x) \to f(x), \ \forall \ x \in E - E_0$,

and (ii) $m(E_0) = 0$.

12.1.2 Pointwise Convergence

Let $<f_n>$ be a sequence of measurable functions on a measurable set E. Then $<f_n>$ is said to converge "pointwise" in E, if $\exists$ a measurable function f on E such that

$$f_n(x) \to f(x) \ \forall \ x \in E \text{ or}$$

$$\underset{n\to\infty}{\text{lt}} f_n(x) = f(x)$$

12.1.3 Uniform Convergence, Almost Everywhere (a.e.)

Let $<f_n>$ be a sequence of measurable functions defined over a measurable set E. Then the sequence $<f_n>$ is said to converge uniformly a.e. to f, if $\exists$ a set $E_0 \subset E$ s.t.

(i) $m(E_0) = 0$ and

(ii) $<f_n>$ converges uniformly to f on the set $E - E_0$.

12.1.4 Bounded Convergence Theorem

Theorem 1: State and Prove: Bounded Convergence Theorem

Statement: Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \le M \ \forall$ n and x. If $f(x) = \lim_{n\to\infty} f_n(x)$ for each x in E, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Proof: Since $f(x) = \lim_{n\to\infty} \int_E f_n(x)$ and f_n is measurable on E

$\therefore$ f is also measurable on E

Let $\varepsilon > 0$ be given

Then $\exists$ measurable set $A \subset E$ with $mA < \dfrac{\varepsilon}{4M}$ and a positive integer N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2mE} \ \forall \ n \ge N \text{ and } x \in E - A$$

$$\therefore \quad \left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right|$$

$$\le \int_E |(f_n - f)|$$

Notes

$$= \int_{E-A} |(f_n - f)| + \int_A |(f_n - f)| \text{ as } (E - A) \cap A = \phi$$

$$\leq \frac{\varepsilon}{2mE} \int_{E-A} 1 + \int_A (|f_n| + |f|)$$

$$\leq \frac{\varepsilon}{2mE} m(E - A) + 2M \int_A 1$$

$$\leq \frac{\varepsilon}{2mE} mE + 2M\, mA \text{ as } m(E - A) \leq mE$$

$$< \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Thus $\left| \int_E f_n - \int_E f \right| < \varepsilon$

But ε was arbitrary

∴ $\lim_{n \to \infty} \int_E f_n = \int_E f$

12.1.5 Fatou's Lemma

If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \leq \underline{\lim}_{n \to \infty} \int_E f_n$$

i.e. $\int_E f \leq \lim_{n \to \infty} \inf \int_E f_n$

Proof: Since integrals over sets of measure zero are zero.

∴ Without loss of generality, we may assume that the convergence is everywhere. Let h be a bounded measurable function with $h \leq f$ and $h(x) = 0$ outside a set $E' \subset E$ of finite measure.

Define a function h_n by

$h_n(x) = \text{Min.} \{h(x), f_n(x)\}$

then $h_n(x) \leq h(x)$ and $h_n(x) \leq f_n(x)$

∴ h_n is bounded by the boundedness of h and vanishes outside E′ as $x \in E - E' \Rightarrow h(x) = 0 \Rightarrow h_n(x) = 0$ because

Notes

Since $h_n = h$ or $h_n = f_n$

$\therefore$ h_n is measurable function on E′

If $h_n = h$, then $h_n \to h$

If $h_n = f_n < h \le f$

then $f_n \to h$ as $f_n \to f$

$\Rightarrow$ $h_n \to h$

Thus $h_n \to h$

Since $h_n(x) \to h(x)$ for each $x \in E'$ and $\{h_n\}$ is a sequence of bounded measurable functions on E′

$\therefore$ By Bounded Convergence Theorem

$$\int_E h = \int_{E'} h + \int_{E-E'} h = \int_E h = \lim_{n\to\infty} \int_{E''} h_n$$

as $E = (E - E') \cup E'$ & $(E - E') \cap E' = \phi$

$$= \underline{\lim}_{n\to\infty} \int_{E'} h_n$$

$$\le \underline{\lim}_{n\to\infty} \int_{E'} f_n \text{ as } h_n \le f_n$$

$$\le \underline{\lim}_{n\to\infty} \int_{E'} f_n \text{ as } E' \subset E$$

$$\Rightarrow \quad \le \int_E h \le \underline{\lim}_{n\to\infty} \int_{E'} f_n$$

Taking supremum over all $h \le f$, we get

$$\sup_{n \le f} \int_{E'} h \le \int_E h = \underline{\lim}_{n\to\infty} \int_{E'} f_n$$

$$\Rightarrow \quad \int_E f \le \int_E f \le \underline{\lim}_{n\to\infty} \int_E f_n$$

Remarks:

(1) If in Fatou's Lemma, we take

$$f_n(x) = \begin{cases} 1, n \le x < n+1 \\ 0, \text{otherwise} \end{cases}$$

with E = R

then $\int_E f \le \underline{\lim}_{n\to\infty} \int_E f_n$

Thus in Fatou's Lemma, strict inequality is possible.

(2) $f_n \geq 0 \;\forall\; x \in E$ is essential for Fatou's Lemma

However, if we take

$$f_n(x) = \begin{cases} -n, \frac{1}{n} \leq x < \frac{2}{n} \\ 0, \text{otherwise} \end{cases}$$

with E = [0, 2]

Then $\int_E f \leq \underline{\lim}_{n\to\infty} \int_E f_n$.

12.1.6 Monotone Convergence Theorem

Statement: Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and let $f = \lim_{n\to\infty} \int f_n$. Then

$$\int f = \lim_{n\to\infty} \int f_n$$

Proof: Let h be a bounded measurable function with $h \leq f$ and $h(x) = 0$ outside a set $E' \subset E$ of finite measure

Define a function h_n by

$$h_n(x) = \text{Min. } \{h(x), f_n(x)\}$$

then $h_n(x) \leq h(x)$ and $h_n(x) \leq f_n(x)$

∴ h_n is bounded by the boundedness of h and vanishes outside E′ as

$x \in E - E' \Rightarrow h(x) = 0 \Rightarrow h_n(x) = 0$ because $f_n(x) \geq 0$

Since $h_n = h$ or $h_n = f_n$

∴ h_n is measurable function on E′

If $h_n = h$, then $h_n \to h$

then $f_n \to h$ as $f_n \to f$

⇒ $h_n \to h$

Thus $h_n \to h$

Since $h_n(x) \to h(x)$ for each $x \in E'$ and $\{h_n\}$ is a sequence of measurable functions on E′

∴ By Bounded Convergence Theorem

$$\int_E h = \int_{E'} h + \int_{E-E'} h = \int_{E'} h = \lim_{n\to\infty} \int_{E'} h_n$$

as $E = (E - E') \cup E'$ & $(E - E') \cap E' = \phi$.

$$= \underline{\lim}_{n\to\infty} \int_{E'} h_n$$

Notes

$$= \varliminf_{n\to\infty} \int_{E'} f_n$$

$$\le \varliminf_{n\to\infty} \int_{E} f_n \text{ as } E' \subset E$$

$$\Rightarrow \quad \int_E h \le \varliminf_{n\to\infty} \int_E f_n$$

Taking supremum over all h ≤ f, we get

$$\sup \int_E h \le \varliminf_{n\to\infty} \int_E f_n$$

$$\Rightarrow \quad \int_E f \le \varliminf_{n\to\infty} \int_E f_n \quad \ldots (1)$$

Since $\{f_n\}$ is monotonically increasing sequence and $f_n \to f$

$$\therefore \quad f_n \le f$$

$$\Rightarrow \quad \int f_n \le \int f$$

$$\Rightarrow \quad \varlimsup_{n\to\infty} \int f_n \le \int f \quad \ldots (2)$$

∴ From (1) and (2), we have

$$\int f \le \varliminf_{n\to\infty} \int f_n \le \varlimsup_{n\to\infty} \int f_n \le \int f$$

$$\int f \le \lim_{n\to\infty} \int f_n$$

Theorem 2: Let $\{u_n\}$ be a sequence of non-negative measurable functions, and let $f = \sum_{h=1}^{\infty} \int u_n$.

Then $\int f = \sum_{h=1}^{\infty} \int u_n$

Proof: Let $f_n = u_1 + u_2 + \ldots + u_n = \sum_{j=1}^{n} \int u_j$

then $f_n \to f$

i.e. $\lim_{n\to\infty} f_n = f$

Let h be a bounded measurable function with h ≤ f and h(x) = 0 outside a set E′ ⊂ E of finite measure.

Define a function h_n by

$$h_n(x) = \text{Min. } \{h(x), f_n(x)\}$$

then $h_n(x) \le h(x)$ and $h_n(x) \le f_n(x)$

$\therefore$ h_n is bounded by the boundedness of h and vanishes outside E′ as

$x \in E - E' \Rightarrow h(x) = 0 \Rightarrow h_n(x) = 0$ because $f_n(x) \ge 0$.

Since $h_n = h$ or $h_n = f_n$

$\therefore$ h_n is measurable function on E′

If $h_n = h$, then $h_n \to h$

If $h_n = f_n < h < f$

then $f_n \to h$ as $f_n \to f$

$\Rightarrow$ $h_n \to h$

Thus $h_n \to h$

Since $h_n(x) \to h(x)$ for each $x \in E'$ and $\{h_n\}$ is a sequence of measurable function on E′

$\therefore$ By Bounded Convergence Theorem

$$\int_E h = \int_{E'} h + \int_{E-E'} h = \int_{E'} h = \lim_{n\to\infty} \int_{E'} h_n$$

as $E = (E - E') \cup E'$ & $(E - E') \cap E' = \phi$

$$= \underline{\lim}_{n\to\infty} \int_{E'} h_n$$

$$= \underline{\lim}_{n\to\infty} \int_{E'} f_n$$

$$\le \underline{\lim}_{n\to\infty} \int_{E} f_n \quad \text{as } E' \subset E$$

$$\Rightarrow \quad \int_E h \le \underline{\lim}_{n\to\infty} \int_E f_n$$

Taking supremum over all $h \le f$, we get

$$\sup_{h\le f} \int_E h \le \underline{\lim}_{n\to\infty} \int_E f_n \quad \ldots (1)$$

$$\Rightarrow \quad \int_E f \le \underline{\lim}_{n\to\infty} \int_E f_n$$

Since $\{f_n\}$ is monotonically increasing sequence and $f_n \to f$

$$\therefore \quad f_n \le f$$

$$\Rightarrow \quad \int f_n \le \int f$$

$$\Rightarrow \quad \overline{\lim}_{n\to\infty} \int f_n \le \int f \quad \ldots (2)$$

∴ From (1) and (2), we have

$$\int f \le \underline{\lim}_{n\to\infty} \int f_n \le \overline{\lim}_{n\to\infty} \int f_n \le \int f$$

$$\Rightarrow \quad \int f \le \lim_{n\to\infty} \int f_n$$

$$= \lim_{n\to\infty} \int f_n$$

$$= \lim_{n\to\infty} \int \sum_{j=1}^{n} u_j$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \int u_j$$

$$= \sum_{j=1}^{n} \int u_j$$

Hence $$\int f = \sum_{n=1}^{\infty} \int u_n$$

Theorem 3: Let f be a non-negative function which is integrable over a set E. Then given ε > 0 there is a δ > 0 such that for every set A ⊂ E with mA < δ, we have

$$\int_A f < \varepsilon$$

Proof: If f is bounded function on E

Then ∃ positive real number M such that

$$|f(x)| \le M \;\; \forall \;\; x \in E$$

∴ For given ε > 0 ∃ $\delta = \frac{\varepsilon}{M}$ such that for every set A ⊂ E with mA < δ, we have

$$\int_A f \le \int_A M = M\, mA < M.\delta = M\frac{\varepsilon}{M} = \varepsilon$$

i.e. $$\int_A f < \varepsilon$$

Thus the result is true if f is a bounded function. So assume that f is not a bounded function on E.

Define a function f_n on E by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{otherwise} \end{cases}$$

Then each f_n is bounded and

$f_n \to f$ at each point

Since $\{f_n\}$ is an increasing sequence of bounded functions such that $f_n \to f$ on E

$\therefore$ By the monotone convergence theorem

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

$\therefore$ For given $\varepsilon > 0$ $\exists$ a positive integer N such that

$$\left| \int_E f_n - \int_E f \right| < \frac{\varepsilon}{2} \text{ for } n \geq N$$

$$\Rightarrow \quad \left| \int_E f - \int_E f_N \right| < \frac{\varepsilon}{2}$$

$$\Rightarrow \quad -\frac{\varepsilon}{2} < \int_E f - \int_E f_N < \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \int_E (f - f_N) < \frac{\varepsilon}{2}$$

Choose $\delta < \dfrac{\varepsilon}{2N}$

$\therefore$ If $mA < \delta$, then we have

$$\begin{aligned}
\int_A f &= \int_A \left[(f - f_N) + f_N\right] \\
&= \int_A (f - f_N) + \int_A f_N \\
&\leq \int_E (f - f_N) + \int_A N \quad \text{as } f_N \leq N \\
&< \frac{\varepsilon}{2} + N\, mA \\
&< \frac{\varepsilon}{2} + N \cdot \delta \\
&< \frac{\varepsilon}{2} + N \cdot \frac{\varepsilon}{2N} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Notes

$$\Rightarrow \qquad \int_A f < \varepsilon$$

12.1.7 Lebesgue Dominated Convergence Theorem

Theorem 4: State and prove Lebesgue dominated convergence theorem

Statement: Let g be an integrable function on E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and $\lim\limits_{n\to\infty} f_n = f$ a.e. on E. Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n .$$

Proof: Since we know that if f is a measurable function over a set E and there is an integrable function g such that $|f| \leq g$, then f is integrable over E. So clearly, each f_n is integrable over E.

Also $\lim\limits_{n\to\infty} f_n = f$ a.e. on E.

and $|f_n| \leq g$ a.e. on E

$\Rightarrow$ $|f| \leq g$ a.e. on E.

Hence f is integrable over E.

Let $\{\phi_n\}$ be a sequence of functions defined by $\phi_n = f_{n+g}$. Clearly, ϕ_n is a non-negative and integrable function for each n.

Therefore, by Fatou's Lemma, we have

$$\int_E (f+g) \leq \underline{\lim}_{n\to\infty} \int_E (f_n + g)$$

$$\Rightarrow \qquad \int_E f \leq \underline{\lim}_{n\to\infty} \int_E f_n \qquad \ldots (1)$$

Similarly, let $\{\Psi_n\}$ be a sequence of functions defined by $\Psi_n = g - f_n$. Clearly Ψ_n is a non-negative and integrable function for each n. So, given by Fatou's Lemma, we have

$$\int_E (g-f) \leq \underline{\lim}_{n\to\infty} \int_E (g - f_n)$$

$$\Rightarrow \qquad \int_E g - \int_E f \leq \int_E g - \underline{\lim}_{n\to\infty} \int_E f_n$$

$$\Rightarrow \qquad -\int_E f \leq -\underline{\lim}_{n\to\infty} \int_E f_n$$

$$\Rightarrow \qquad \int_E f \geq \overline{\lim} \int_E f_n \qquad \ldots (2)$$

Hence from (1) & (2), we get

$$\int_E f = \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n$$

Notes

But $$\underline{\lim}\int_E f_n = \overline{\lim}\int_E f_n = \lim\int_E f_n$$

Hence $$\int_E f = \lim\int_E f_n .$$

Corollary: Let $\{u_n\}$ be a sequence of integrable functions on E such that $\sum_{n=1}^{\infty} u_n$ converges a.e. on E. Let g be a function which is integrable on E and satisfy $\left|\sum_{i=1}^{n} u_i\right| \le g$ a.e. on E for each n. Then $\left|\sum_{n=1}^{\infty} u_n\right|$ is integrable on E and $\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\int_E u_n$.

Proof: Let $\sum_{i=1}^{n} u_i = f_n$.

Applying Lebesgue Dominated Convergence Theorem for the sequence $\{f_n\}$, we get

$$\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\int_E u_n$$

Corollary: If f is integrable over E and $\{E_i\}$ is a sequence of disjoint measurable sets such that $\sum_{i=1}^{\infty} E_i = E$, then

$$\int_E f = \sum_{i=1}^{\infty}\int_{E_i} f$$

Proof: Since $\{E_i\}$ is a sequence of disjoint measurable sets, we may write.

$$f = \sum_{i=1}^{\infty} f\cdot\chi_{E_i}$$

The function f. χ_{E_i} is integrable over E since $\left|f\,\chi_{E_i}\right| \le |f|$ and $|f|$ is integrable over E. Moreover

$$\left|\sum_{i=1}^{n} f\cdot\chi_{E_i}\right| \le |f|, \forall n \in N$$

Thus the conditions of above corollary are satisfied and hence

$$\int_E f = \int_E \sum_{i=1}^{\infty} f\cdot\chi_{E_i}$$

Notes

$$= \sum_{i=1}^{\infty} \int_E f \cdot \chi_{E_i}$$

$$= \sum_{i=1}^{\infty} \int_{E_i} f$$

Example: Show that the theorem of bounded convergence applies to $f_n(x) = \frac{nx}{1+n^2x^2}$, $0 \le x \le 1$.

Sol:

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

$$= \frac{1}{\frac{1}{nx}+nx}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{nx}}-\sqrt{nx}\right)^2+2}$$

$$\le \frac{1}{2}$$

Thus $\exists$ a number $\frac{1}{2}$ such that $|f_n(x)| \le \frac{1}{2}$.

Hence it satisfies the conditions of bounded convergence theorem.

Now

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim\int_0^1 \frac{nx}{1+n^2x^2}\,dx$$

$$= \lim_{n\to\infty}\frac{1}{2n}\log(1+n^2x^2) \qquad \left(\because \frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty}\frac{[1/(1+n^2x^2)]\,2nx^2}{2} \qquad \text{[Using L'Hospital Rule]}$$

$$= \lim_{n\to\infty}\frac{nx^2}{1+n^2x^2}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n}x^2}{\frac{1}{n^2}+x^2} = 0$$

and

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 \lim_{n\to\infty}\left(\frac{nx}{1+n^2x^2}\right)dx$$

$$= \int_0^1 (0)\,dx = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

This verifies the result of bounded convergence theorem.

Example: Use Lebesgue dominated convergence theorem to evaluate $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx$,

where

$$f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}, \; n = 1, 2, 3, \ldots \; 0 \le x \le 1.$$

Solution:

$$f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$$

$$= \frac{1}{x}\cdot\frac{n^{3/2}x^2}{1+n^2x^2}$$

$$\le \frac{1}{x} = g(x), \text{(say)}$$

$\Rightarrow$ $\qquad f_n(x) \le g(x)$

and $\qquad g(x) \in L\,(0, 1]$,

Hence by Lebesgue Dominated Convergence Theorem.

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

$$= \int_0^1 \lim_{n\to\infty}\left(\frac{n^{3/2}x}{1+n^2x^2}\right)dx$$

$$= \int_0^1 \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)\left(\frac{x}{\frac{1}{n^2}+x^2}\right)dx$$

$$= \int_0^1 0\,dx = 0.$$

Example: If (f_n) is a sequence of non-negative function s.t. $f_n \to f$ and $f_n \le f$ for each n, show that

$$\int f = \lim \int f_n$$

Solution: From the given hypothesis it follows that

$$\overline{\lim}\int f_n \leq \int f \qquad \dots (1)$$

Also by Fatou's Lemma, we have

$$\int f \leq \underline{\lim}\int f_n \qquad \dots (2)$$

Then from (1) and (2), we get

$$\int f \leq \underline{\lim}\int f_n \leq \overline{\lim}\int f_n \leq \int f .$$

Hence
$$\int f = \underline{\lim}\int f_n \leq \overline{\lim}\int f_n = \lim\int f_n .$$

Example: If $\alpha > 0$, prove that $\lim\limits_{n\to\infty}\int_o^n \left(1-\frac{x}{n}\right)^n x^{\alpha-1}dx = \int_o^\infty e^{-x}.x^{\alpha-1}dx$, where the integrals are taken in the Lebesgue sense.

Solution: If $f_n(x) = \left(1-\frac{x}{n}\right)^n .x^{\alpha-1} > 0$, then $f_n(x) \leq g(x)$, where $g(x) = e^{-x}.x^{\alpha-1}$ $\left[\text{recall } \lim\limits_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}\right]$

Also $g(x) \in L[0, \infty]$, hence by Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_o^n f_n(x)\,dx = \int_o^\infty \lim_{n\to\infty} f_n(x)\,dx$$

$$= \int_o^\infty \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n .x^{\alpha-1}\,dx$$

$$= \int_o^\infty e^{-x}.x^{\alpha-1}\,dx$$

Example: Show that if $\alpha > 1$,

$$\int_o^1 \frac{x\sin x}{1+(nx)^\alpha}dx = 0(n^{-1}) \text{ as } n \to \infty .$$

Solution: Consider the sequence $<f_n(x)>$ s.t.

$$f_n(x) = \frac{nx\sin x}{1+(nx)^\alpha},\ n = 1, 2, \dots\dots\dots$$

Obviously since $\alpha > 1$, and $x \in [0, 1]$

Notes

$$\left|\frac{nx\sin x}{1+(nx)^{\alpha}}\right| \le 1$$

If $\Psi(x) = 1, \ \forall x$, then $|\, fn\ (x)\,| \le \Psi\ (x), \ \forall x$.

Hence by dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_0^1 \frac{nx\sin x}{1+(nx)^{\alpha}}dx = \int_0^1 \lim_{n\to\infty}\frac{nx\sin x}{1+(nx)^{\alpha}}dx = \int_0^1 (0)\,dx = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} n\times\left[\int_0^1 \frac{x\sin x}{1+(nx)^{\alpha}}dx\right] = 0$$

$$\Rightarrow \qquad \int_0^1 \frac{x\sin x}{1+(nx)^{\alpha}}dx = 0\ (n^{-1}).$$

Example: Show that $\lim_{n\to\infty}\int_a^{\infty}\frac{n^2xe^{-n^2x^2}}{1+x^2}dx = 0$, if a > 0, but not for a = 0.

Solution: If a > 0, $\left.\begin{array}{l}\text{putting } nx = u\\ \text{and } \ du = ndx\end{array}\right\}$, we get

$$\int_a^{\infty}\frac{n^2xe^{-n^2}x^2}{1+x^2}dx = \int_{na}^{\infty}\frac{ue^{-u^2}du}{1+u^2/n^2} = \int_o^{\infty}\phi_{(na,\infty)}\frac{ue^{-u^2}}{1+u^2/n^2}du$$

Also $\left|\frac{u.e^{-u^2}}{1+(u^2/n^2)}\cdot\phi_{(na,\infty)}\right| < u.e^{-u^2} \in L\,[0,\infty]$

and $\lim_{n\to\infty}\phi_{(ne,\infty)}\frac{u.e^{-u^2}}{1+u^2/n^2} = 0$ as $\phi_{(\infty,\infty)} = 0$.

Hence by Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_a^{\infty}\frac{n^2xe^{-n^2x^2}}{1+x^2}dx = \lim_{n\to\infty}\int_a^{\infty}\phi_{(na,\infty)}\frac{u.e^{-u^2}}{1+u^2/n^2}du$$

$$= \int_a^{\infty}\lim_{n\to\infty}\phi_{(na,\infty)}\frac{u.e^{-u^2}}{1+u^2/n^2}du = \int_o^{\infty}0\,dx = 0\,.$$

Now when a = 0,

$$\int_o^{\infty}\frac{n^2xe^{-n^2x^2}}{1+x^2}dx > \int_0^1\frac{n^2xe^{-n^2x^2}}{1+x^2}dx$$

$$> \frac{1}{2}\int_0^1 n^2 x.e^{-n^2x^2}dx \text{ (putting 1 in place of } x^2)$$

$$= -\frac{1}{4}\left[e^{-n^2x^2}\right]_0^1 > \frac{1}{4}$$

12.2 Summary

- ***Bounded Convergence Theorem:*** Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| < M$ $\forall$ n and all x. If $f(x) = \lim_{n\to\infty} f_n(x)$ for each x in E, then

$$\int_E f = \lim_{n\to\infty}\int_E f_n$$

- ***Monotone Convergence Theorem:*** Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and let $f = \lim_{n\to\infty} f_n$. Then

$$\int f = \lim_{n\to\infty}\int f_n$$

- ***Lebesgue Dominated Convergence Theorem:*** Let g be an integrable function on E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and $\lim_{n\to\infty} f_n = f$ a.e. on E. Then

$$\int_E f = \lim_{n\to\infty}\int_E f_n$$

12.3 Keywords

Convergence almost Everywhere: Let $<f_n>$ be a sequence of measurable functions defined over a measurable set E. Then $<f_n>$ is said to converge almost everywhere in E if there exists a subset E_0 of E s.t.

(i) $f_n(x) \to f(x)$, $\forall\ x \in E - E_0$,

and (ii) $m(E_0) = 0$.

Convergence: Refers to the notion that some functions and sequence approach a limit under certain conditions.

Fatou's Lemma: If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \leq \lim_{n\to\infty}\inf \int_E f_n$$

Pointwise Convergence: Let $<f_n>$ be a sequence of measurable functions on a measurable set E. Then $<f_n>$ is said to converge "pointwise" in E, if $\exists$ a measurable function f on E such that

$$f_n(x) \to f(x)\ \forall\ x \in E \text{ or}$$

$$\operatorname*{lt}_{n\to\infty} f_n(x) = f(x)$$

Uniform Convergence, Almost Everywhere (a.e.): Let $<f_n>$ be a sequence of measurable functions defined over a measurable set E. Then the sequence $<f_n>$ is said to converge uniformly a.e. to f, if $\exists$ a set $E_0 \subset E$ s.t.

(i) $m(E_0) = 0$ and

(ii) $<f_n>$ converges uniformly to f on the set $E - E_0$.

12.4 Review Questions

1. Show that we may have strict inequality in Fatou's Lemma.

2. Let $<f_n>$ be an increasing sequence of non-negative measurable functions, and let $f = \lim f_n$.

   Show that $\int f = \lim \int f_n$.

   Deduce that $\int f = \sum_{n=1}^{\infty} \int u_n$, if u_n is a sequence of non-negative measurable functions and

   $f = \sum_{n=1}^{\infty} u_n$.

3. State the Monotone Convergence theorem. Show that it need not hold for decreasing sequences of functions.

4. Let $\{g_n\}$ be a sequence of integrable functions which converge a.e. to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g_n$ and $\{f_n\}$ converges to f a.e.

   If $\int g = \lim_{n\to\infty} \int g_n$

   then prove that $\int f = \lim_{n\to\infty} \int f_n$.

5. State and prove monotone convergence theorem.

12.5 Further Readings

Books

G.F. Simmons, *Introduction to Topology and Modern Analysis*, New York: McGraw Hill, 1963.

H.L. Royden, *Real analysis*, Prentice Hall, 1988.

Online links

dl.acm.org

math.stanford.edu

www.springerlink.com

Unit 13: Signed Measures

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define signed measure.
- Describe positive and negative and null sets.
- Solve problems on signed measure.

Introduction

We have seen that a measure is a non-negative set function. Now we shall assume that it takes both positive and negative values. Such assumption leads us to a new type of measure known as signed measure. In this unit, we shall start with definition of signed measure and we shall prove some important theorems on it.

13.1 Signed Measures

13.1.1 Signed Measure: Definition

Definition: Let the couple $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ represents a σ-algebra of subsets of X. An extended real valued set function

$$\gamma : \mathcal{A} \to [-\infty, \infty]$$

defined on $\mathcal{A}$ is called a signed measure, if it satisfies the following postulates:

(i) γ assumes at most one of the values $-\infty$ or $+\infty$.

(ii) $\gamma(\phi) = 0$.

Notes

(iii) If <A_n> is any sequence of disjoint measurable sets, then

$$\gamma\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \gamma(A_n),$$

i.e., γ is countably additive.

From this definition, it follows that a measure is a special case of a signed measure. Thus, every measure on $\mathcal{A}$ is a signed measure but the converse is not true in general, i.e. every signed measure is not a measure in general.

If $-\infty < \gamma(A) < \infty$, for very $A \in \mathcal{A}$, then we say that signed measure γ is finite.

13.1.2 Positive Set, Negative Set and Null Set

Definition

(a) ***Positive Set:*** Let (X, $\mathcal{A}$) be a measurable space and let A be any subset of X. Then $A \subset X$ is said to be a positive set relative to a signed measure γ defined on (X, $\mathcal{A}$), if

(i) $A \in \mathcal{A}$, i.e. A is measurable.

(ii) $\gamma(E) \geq 0, \ \forall E \subset A$ s.t. E is measurable.

Obviously, it follows from the above definition that:

(i) every measurable subset of a positive set is a positive set,

(ii) ϕ is a positive set w.r.t. every signed measure.

Also for A to be positive. $\gamma(A) \geq 0$ is the necessary condition, but not in general sufficient for A to be positive.

(b) ***Negative Set:*** Let (X, $\mathcal{A}$) be a measurable space. Then a subset A of X is said to be a negative set relative to a signed measure γ defined on measurable space (X, $\mathcal{A}$) if

(i) $A \in \mathcal{A}$ i.e., A is measurable.

(ii) $\gamma(E) \leq 0, \ \forall E \subset A$ s.t. E is measurable.

⇒ set A is negative w.r.t. γ, provided it is positive w.r.t. – γ.

(c) ***Null Set:*** A set $A \subset X$ is said to be a null set relative to a signed measure γ defined on measurable space (X, $\mathcal{A}$) is, A is both positive and negative relative to γ.

Thus, measure of every null set is zero.

Now, we know that a measurable set is a set of measure zero, iff every measurable subset of it has γ measure zero. Thus, if $A \subset X$ is a null set relative to γ then $\gamma(E) = 0, \ \forall$ measurable subsets $E \subset A$. In other words.

$$A \text{ is a null set} \Leftrightarrow \gamma(E) = 0, \ \forall \text{ measurable subsets } E \subset A.$$

Theorem 1: Countable union of positive sets w.r.t. a signed measure is positive.

Proof: Let (X, $\mathcal{A}$) be a measurable space and let γ be a signed measure defined on (X, $\mathcal{A}$). Let <A_n> be a sequence of positive subsets of X, let $A = \bigcup_{i=1}^{n} A_i$ and let B be any measurable subset of A.

Set $\mathcal{B}_n = B \cap A_n \cap A_{n-1}^C \cap \ldots\ldots \cap A_1^C, \forall n \in N.$

Notes

where $A_n^C(n = 1, 2, 3 \ldots n - 1)$ denotes complement of $A_n(n = 1, 2, 3, \ldots n - 1)$ with respect to X.

Now, we know that complement of a measurable set is also measurable so that each $A_n^C(n = 1, 2, 3 \ldots n - 1)$ is measurable relative to γ. Again, intersection of countable collection of measurable sets is also measurable. Hence B_n is a measurable subset of the positive set A_n. Thus

$$\gamma(B_n) \geq 0 \qquad \text{(by the definition of positive set)} \ldots \text{(i)}$$

Obviously, the set B_n are disjoint and

if $$B = \bigcup_{n=1}^{\infty} B_n \text{, we get} \qquad \ldots \text{(ii)}$$

$$\gamma(B) = \sum_{n=1}^{\infty} \gamma(B_n) \qquad \ldots \text{(iii)}$$

In view of (i).

$$\gamma(B) \geq 0.$$

Thus, we have

(1) A is measurable for

A_n is a positive set $\Rightarrow A_n$ is a measurable set

$\Rightarrow$ countable union $\bigcup_{n=1}^{\infty} A_n$ is measurable,

$\Rightarrow A = \bigcup_{n=1}^{\infty} A_n$ is measurable

(2) $\gamma(B) \geq 0, \ \forall \ B \subset A$ s.t. B is a measurable set.

Hence A is a positive set, by definition.

Theorem 2: Let $(X, \mathcal{A})$ be a measurable space and let γ be a signed measure defined on (X, A). If B is a measurable set with finite negative measure i.e., $-\infty < \gamma(B) < 0$, then prove that B contains a negative set $A \subset B$ with the property $\gamma(A) < 0$.

Proof: If B is itself a negative set, then we may take A = B and theorem is done. Therefore consider the case when B is not a negative set. Then there must exist a measurable subset $E_1 \subset B$ and a smallest positive integer n_1, s.t.

$$\gamma(E_1) > \frac{1}{n_1}$$

$\because$ $$B = (B - E_1) \cup E_1 \text{ and } (B - E_1) \cap E_1 = \phi,$$

$\therefore$ $$\gamma(B) = \gamma(B - E_1) + \gamma(E_1) \qquad \ldots \text{(i)}$$

or $$\gamma(B - E_1) = \gamma(B) - \gamma(E_1) \qquad \ldots \text{(ii)}$$

Since $\gamma(B)$ is finite, (i) implies that $\gamma(B - E_1)$ and $\gamma(E_1)$ are finite. Again $\gamma(B) < 0$, (ii) implies that $\gamma(B - E_1) < 0$.

Now, the set $B - E_1$ is either negative or contains a subset of positive measure. If the set $B - E_1$ is a negative set, then we may take $A = B - E_1$ and the theorem is done. So, suppose that $B - E_1$ is not a negative set. Then there must exist a measurable subset E_2 of $B - E_1$ and a smallest positive number n_2 with a property

$$\gamma(E_2) > \frac{1}{n_2}.$$

Since $\quad B = (B - E_1 \cup E_2) \cup (E_2 \cup E_2)$,

and $\quad (B - E_1 \cup E_2) \cap (E_1 \cup E_2) = \phi$,

we have $\quad \gamma(B) = \gamma(B - E_1 \cup E_2) + \gamma(E_2 \cup E_2)$

or $\quad \gamma(B - E_1 \cup E_2) = \gamma(B) - \gamma(E_2 \cup E_2)$

$$= \gamma(B) - \gamma(E_1) - \gamma(E_2).$$

As before, $\quad \gamma(B - E_1 \cup E_2) > 0 \; [\because \gamma(B) < 0, \gamma(E_r) > 0 \text{ for } r = 1, 2]$

Thus, $B - E_1 \cup E_2$ is a set of negative measure, which is either a negative set or contains a subset of positive measure. If $B - E_1 \cup E_2$ is a negative set, then the theorem is done by taking $B = A - E_1 \cup E_2$. Otherwise we repeat the above process.

On repeating this process, at some stage we shall get either a negative subset $A \subset B$ s.t. $\gamma(A) < 0$ or a sequence $<E_r>$ of disjoint measurable sets and a sequence $<n_r : r \in N>$ of positive integers s.t.

$$E_r \subset B - \left[\bigcup_{n=1}^{r-1} E_n\right] \text{ and } \frac{1}{n_r} < \gamma(E_r) < \infty$$

In first case, we have nothing to do. In the latter case, let

$$A = B - \left[\bigcup_{n=1}^{\infty} E_n\right] \text{ or } B = A \cup \left[\bigcup_{n=1}^{\infty} E_n\right] \qquad \ldots \text{(iii)}$$

Then as before, it follows that

$$\gamma(B) = \gamma(A) + \sum_{n=1}^{\infty} \gamma(E_n).$$

$$> \gamma(A) + \sum_{k=1}^{\infty} \frac{1}{n_k} \qquad \ldots \text{(iv)}$$

[∵ change of suffix is in material]

Since $\gamma(B)$ is finite and γ assumes at most one of the values $-\infty$ and ∞, it follows from (iv) that $\gamma(A)$ is finite and the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent.

Then $\quad \gamma(A) < \gamma(B) - \sum_{k=1}^{\infty} \frac{1}{n_k}$

= a finite negative number

(∵ $\gamma(B)$ is a finite negative number)

Notes

or $$\gamma(A) < 0.$$

Again, we know that difference of two measurable sets is measurable and enumerable union of measurable sets is measurable therefore it follows from (iii) that A is a measurable set.

Now we shall prove that A is a negative set. Let $E \subset A$ be an arbitrary measurable set.

Since $$A = B - \left[\bigcup_{n=1}^{\infty} E_n\right],$$

$$E = B - \bigcup_{n=1}^{\infty} E_n .$$

Since $n_k \to \infty$, we can choose k so large that

$$\gamma(E) \le \frac{1}{n_{k-1}}$$

Letting $n_k \to \infty$, we obtain

$$\gamma(E) \le 0.$$

Thus we have

(1) A is measurable.

(2) $\gamma(E) \le 0, \ \forall \ E \subset A$ s.t. E is measurable.

Hence A is a negative set.

13.1.3 Hahn Decomposition Theorem

Theorem 3: Let γ be a signed measure on a measurable space $(X, \mathcal{A})$. Then there exists a positive set P and a negative set Q s.t.

$$P \cap Q = \phi \text{ and } P \cup Q = X.$$

Proof: Let $(X, \mathcal{A})$ be a measurable space and let γ be a signed measure defined on a measurable space $(X, \mathcal{A})$. Since, by definition, γ assumes at most one of the values $+\infty$ or collection of all negative subsets of X w.r.t. γ and let $\mathcal{B}$ be a collection of all negative subsets of X w.r.t. γ and let

$$k = \inf \{ \gamma(E) : E \in \mathcal{B})$$

(i) $\Rightarrow$ that there exists a sequence $<E_n>$ in $\mathcal{B}$ such that

$$\lim_{n\to\infty} \gamma(E_n) = k.$$

Let $$Q = \bigcup_{n=1}^{\infty} E_n .$$

Since $\mathcal{B}$ is a family of negative sets, $< E_n>$ is a sequence of negative sets. Again, we know by remark of theorem 1 that countable union of negative sets is negative, it follows that Q is a negative subset of X so that

$$\gamma(Q) \ge K.$$

Now, $Q - E_n$ is a subset of Q, it follows that

$$\gamma (Q - E_n) \leq 0.$$

Since $(Q - E_n) \cap E_n = \phi$

and $Q = (Q - E_n) \cup E_n$,

we have $\gamma (Q) = \gamma (Q - E_n) + \gamma (E_n)$

$\Rightarrow$ $\gamma (Q) \leq \gamma (E_n),\ \forall\, n \in N$ and $E_n \in B$.

Therefore $\gamma (Q) \leq K$. … (iii)

(ii) and (iii) $\Rightarrow \gamma (Q) = K \Rightarrow -\infty < k$. … (iv)

Now we shall show that $p = Q^C$, the complement of Q w.r.t. γ is a positive subset of X. Suppose not, i.e. P is negative. Then $\forall\, E \subset P$ s.t. E is measurable and $\gamma (E) < 0$. Now we know that if $-\infty < \gamma (E) < 0$, we get a negative set $A \subset E$ s.t. $\gamma (A) < 0$.

A, Q are distinct negative subsets of X

$\Rightarrow$ $A \cup Q$ is negative set

$\Rightarrow$ $\gamma (A \cup Q) \geq K$ [using (i)]

$\Rightarrow$ $\gamma (A) + \gamma (Q) \geq K$,

$\Rightarrow$ $\gamma (A) + K \geq K$, [using (iv)]

$\Rightarrow$ $\gamma (A) \geq 0$,

$\Rightarrow$ a contradiction, for $\gamma (A) < 0$

$\Rightarrow$ $P = Q^C$ is a positive subset of X

$\Rightarrow$ Q is a negative subset of X.

Thus $X = P \cup Q$, $P \cap Q = \phi$.

13.1.4 Hahn Decomposition: Definition

A decomposition of a measurable space X into two subsets s.t. $X = P \cup Q$, $P \cap Q = \phi$,

where P and Q are positive and negative sets respectively relative to the signal measure γ, is called as Hahn decomposition for the signed measure γ. P and Q are respectively called positive and negative components of X.

Notice that Hahn decomposition is not unique.

13.2 Summary

- Let the couple $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ represents a σ-algebra of subsets of X. An extended real-valued set function

  $$\gamma : \mathcal{A} \to [-\infty, \infty]$$

  defined on $\mathcal{A}$ is called a signed measure, if it satisfies the following postulates:

  (i) γ assumes at most one of the values $-\infty$ or $+\infty$.

  (ii) $\gamma (\phi) = 0$.

  (iii) If $<A_n>$ is any sequence of disjoint measurable sets, then γ is countably additive.

- Let $(X, \mathcal{A})$ be a measurable space and then $A \subset X$ is said to be a positive set relative to a signed measure γ defined on $(X, \mathcal{A})$ if

  (i) A is measurable

  (ii) $\gamma(E) \geq 0,\ \forall\ E \subset A$ s.t. E is measurable.

- Let $(X, \mathcal{A})$ be a measurable space. Then $A \subset X$ is said to be negative set relative to a signed measure γ if

  (i) A is measurable

  (ii) $\gamma(E) \leq 0,\ \forall\ E \subset A$ s.t. E is measurable.

- $A \subset X$ is said to be a null set relative to a signed measure γ defined on measurable space $(X, \mathcal{A})$ is: A is both positive and negative relative to γ.

13.3 Keywords

Hahn Decomposition: Definition: A decomposition of a measurable space X into two subsets s.t. $X = P \cup Q$, $P \cap Q = \phi$.

Negative Set: Let $(X, \mathcal{A})$ be a measurable space. Then a subset A of X is said to be a negative set relative to a signed measure γ defined on measurable space $(X, \mathcal{A})$ if

(i) $A \in \mathcal{A}$ i.e., A is measurable.

(ii) $\gamma(E) \leq 0,\ \forall E \subset A$ s.t. E is measurable.

Null Set: A set $A \subset X$ is said to be a null set relative to a signed measure γ defined on measurable space $(X, \mathcal{A})$ is, A is both positive and negative relative to γ.

Positive Set: Let $(X, \mathcal{A})$ be a measurable space and let A be any subset of X. Then $A \subset X$ is said to be a positive set relative to a signed measure γ defined on $(X, \mathcal{A})$, if

(i) $A \in \mathcal{A}$, i.e. A is measurable.

(ii) $\gamma(E) \geq 0,\ \forall E \subset A$ s.t. E is measurable.

Signed Measure: Let the couple $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ represents a σ-algebra of subsets of X. An extended real valued set function

$$\gamma : \mathcal{A} \to [-\infty, \infty]$$

defined on $\mathcal{A}$ is called a signed measure, if it satisfies the following postulates:

(i) γ assumes at most one of the values $-\infty$ or $+\infty$.

(ii) $\gamma(\phi) = 0$.

13.4 Review Questions

1. If $\gamma(E) = \int_E x e^{-x^2} dx$, then find positive, negative and null sets w.r.t. γ. Also give a Hahn decomposition of R w.r.t. γ.
2. State and prove Hahn decomposition theorem for signed measures.
3. If μ is a measure and γ_1, γ_2 are the signed measures given by $\gamma_1(E) = \mu(A \cap E)$, $\gamma_2(E) = \mu(B \cap E)$, where $\mu(A \cap B) = 0$, show that $\gamma_1 \perp \gamma_2$.

Notes

4. Show that if γ_1 and γ_2 are two finite signed measures, then so is $a\gamma_1 + b\gamma_2$ where a, b are real numbers.

13.5 Further Readings

Books

Bartle, Robert G., *The Elements of Integration*, New York – London – Sydney: John Wiley and Sons

Cohn, Donald L. (1997) [1980], *Measure Theory (reprint ed.)*, Boston – Based – Stuttgart: Birkhauser Verlag

Online links

www.maths.bris.ac.uk

www.planetmath.org/signedmeasure.html

Unit 14: Radon-Nikodym Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Absolutely continuous measure function
- State Radon-Nikodym theorem
- Understand the proof of Radon-Nikodym theorem
- Solve problems on this theorem

Introduction

In mathematics, the Radon-Nikodym theorem is a result in measure theory that states that given a measurable space (X, Σ), if a σ-finite measure on (X, Σ) is absolutely continuous with respect to a σ-finite measure on (X, Σ), then there is a measurable function f on X and taking values in [0, ∞], such that for any measurable set A.

The theorem is named after Johann Radon, who proved the theorem for the special case where the underlying space is R^N in 1913, and for Otto Nikodym who proved the general case in 1930. In 1936 Hans Freudenthal further generalised the Radon-Nikodym theorem by proving the Freudenthal spectral theorem, a result in Riesz space theory, which contains the Radon-Nikodym theorem as a special case.

If Y is a Banach space and the generalisation of the Radon-Nikodym theorem also holds for functions with values in Y, then Y is said to have the Radon-Nikodym property. All Hibert spaces have the Radon-Nikodym property.

14.1 Radon-Nikodym Theorem

14.1.1 Absolutely Continuous Measure Function

Let $(X, \mathcal{A})$ be a measurable space and let γ and μ be measure functions defined on the space (X, A). The measure γ is said to be absolutely continuous w.r.t. μ if

$$\mu(A) = 0 \text{ or } |\mu|(A) = 0,\ A \in \mathcal{A} \Rightarrow \gamma(A) = 0, \text{ and is denoted by } \gamma \ll \mu.$$

Notes

Notes

- If μ is σ-finite, the converse is also true.
- If γ and μ are signed measures on $(X, \mathcal{A})$, then $\gamma \ll \mu$ if $|\gamma| \ll |\mu|$.

Radon-Nikodym Theorem

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. If γ be a measure defined on A s.t. γ is absolutely continuous w.r.t. μ, then there exists a non-negative measurable function f on s.t.

$$\gamma(A) = \int_A f \, d\mu, \forall A \in \mathcal{A}.$$

The function f is unique in the sense that if g is any measurable function with the property defined as above, then f = g almost everywhere with respect to μ.

Proof: To establish the existence of the function f, we shall use the following two Lemmas:

Lemma 1: Let E be a countable set of real numbers. Let for each $a \in E$ there is a set $F_a \in \mathcal{A}$ s.t. $F_a \subset F_b$, whenever b < a i.e. $<F_n>$ is a monotonic decreasing sequence of subsets of $\mathcal{A}$ corresponding to the sequence $<a_n>$ of real numbers in E. Then $\exists$ a measurable extended real valued function f on X s.t.

$$f(x) \leq a, x \in F_a,$$

and $$f(x) \geq a, x \in (X - F_a).$$

Proof: Let $f(x) = \inf \{a : x \in F_a\}$ $x \in X$ and let, conventionally

$$\inf \{\text{empty collection of real numbers}\} = \infty$$

Now, $x \in F_a \Rightarrow f(x) \leq a$

$x \notin F_a \Rightarrow x \in F_a$ for every b < a

$\Rightarrow f(x) \geq a$

Now, $f(x) < a \Rightarrow x \in F_b$ for some b < a

or $$\{x : f(x) < a\} = \bigcup_{b<a} [F_b].$$

Also $x \in F_b \Rightarrow f(x) \leq b < a$ for some b < a.

Hence f is measurable.

Again, by definition of f, we observe that

$$f(x) \leq a, x \in F_a,$$

and $$f(x) \geq a, x \notin F_a.$$

Thus f is the required function.

Lemma 2: Let E be a countable set of real numbers. Let corresponding to each $a \in E$, there is a set $F_a \in \mathcal{A}$ s.t.

$$\gamma(F_a - F_b) = 0 \text{ whenever } b > a.$$

Then there exists a measurable function f with the property

$$x \in F_a \Rightarrow f(x) \leq a \text{ a.e.}$$

and $$x \in (X - F_a) \Rightarrow f(x) > a \text{ a.e.}$$

Notes

Proof: Let $P = \bigcup_{b<a} \{F_a - F_b\}$.

Evidently γ (P) = 0.

Let $F'_a = F_a \cup P$.

This $\Rightarrow F'_a - F'_b = (F_a - F_b) - P = \phi$ for a < b.

In view of Lemma 1, it follows that $\exists$ a measurable function f s.t.

$f(x) \leq a, x \in F'_a$

and $f(x) \leq a, x \in X - F'_a$

Thus we have

$$\left.\begin{array}{ll} x \in F_a \Rightarrow f(x) \leq a & \text{a.e.,} \\ x \in X - F_a \Rightarrow f(x) > a & \text{a.e.,} \end{array}\right\} \text{except for } x \in P.$$

Proof of the main theorem

At first, suppose that μ is finite.

$\Rightarrow$ (γ – a μ) is a signed measure on $\mathcal{A}$ for each rational number a.

Let (P_a, Q_a) be a Hahn decomposition for the measure (γ – a μ).

Let P_O = X and $Q_O = \phi$.

By the definition of Hahn decomposition theorem,

$P_a \cup Q_a = X$,

and $P_b \cup Q_b = X$.

Therefore, $Q_a - Q_b = Q_a - (X - P_b)$

$= Q_a \cap P_b$.

Thus, $(\gamma - a\mu)(Q_a - Q_b) \leq 0$ … (i)

Similarly, we can prove that

$(\gamma - b\mu)(Q_a - Q_b) \geq 0$ … (ii)

Let a < b, then from (i) and (ii), we have

$\mu(Q_a - Q_b) = 0$.

Therefore, by Lemma (ii)

$f(x) \geq a$, a.e. $x \in P_a$

and $f(x) \leq a$, a.e. $x \in Q_a$,

where f is measurable

Since $Q_O = \phi$, it follows that f is non-negative

Again, let $A \in \mathcal{A}$ be arbitrary.

Define $A_r = A \cap \left(\frac{Q_{r+1}}{n_o} - \frac{Q_r}{n_o}\right)$

Notes

$$A_\infty = A - \cup\left(\frac{Q_r}{n_o}\right).$$

Evidently, $A = A_\infty \cup \left(\bigcup_{r=o}^{\infty} A_r\right)$,

where A is disjoint union of measurable sets.

$\therefore \qquad \gamma(A) = \gamma(A_\infty) + \sum_{r=o}^{\infty}\gamma(A_r)$.

Obviously $A_r \subset \left(\frac{Q_{r+1}}{n_o} - \frac{Q_r}{n_o}\right)$

$\Rightarrow \qquad \frac{r}{n_o} \le f(x) \le \frac{r+1}{n_o}, \forall x \in A_r$

$\Rightarrow \qquad \frac{r}{n_o}\mu(A_r) \le \int_{A_r} fd\mu \le \frac{r+1}{n_o}\mu(A_r)$ [by first mean value theorem]

Again $\quad \frac{r}{n_o}\mu(A_r) \le \gamma(A_r) \le \frac{r+1}{n_o}\mu(A_r)$, we have

$$\left[\gamma(A_r) - \frac{1}{n_o}\mu(A_r)\right] \le \int_{A_r} fd\mu \le \gamma(A_r) + \frac{1}{n_o}\mu(A_r) \qquad \dots \text{(iii)}$$

Now, if $\mu(A_\infty) > 0$, then $g(A_\infty) = 0$, [$\because$ $(\gamma - a\mu)\, A_\infty$ is positive, $\forall$ a]

and $\qquad \gamma(A_\infty) = 0$ if $\mu(A_\infty) = 0 \qquad [\because \gamma \ll \mu)]$

In either case, $\gamma(A_\infty) = \int_{A_r} f\, d\, \mu$.

Adding the inequalities (iii) over r, we get

$$\gamma(A) - \frac{1}{n_o}\mu(A) \le \int_A f\, d\, \mu \le \gamma(A) + \frac{1}{n_o}\mu(A_r).$$

Since n_o is arbitrary and μ (A) is assumed to be finite, it follows that

$$\gamma(A) = \int_A f\, d\, \mu \forall A \in \mathcal{A}.$$

To show that the theorem is true for σ-finite measure μ, decompose X into a countable union of X_i of finite μ-measure. Applying the same argument as above for each X_i, we get the required function.

To show the second part, let g be any measurable function satisfying the condition,

$$\gamma(A) = \int_A f\, d\, \mu \forall A \in \mathcal{A}.$$

Notes

For each n ∈ N, define

$$A_n = \left\{x \in X : f(x) - g(x) \geq \frac{1}{n}\right\} \in \mathcal{A}$$

and $$B_n = \left\{x \in X : g(x) - f(x) \geq \frac{1}{n}\right\} \in \mathcal{A}.$$

Since f (x) – g (x) ≥ $\frac{1}{n}$, $\forall \mathcal{A} x \in A_n$, we have by first mean value theorem

$$\int_{A_n} (f-g)d\mu \geq \frac{1}{n}\mu(A_n)$$

$$\int_{A_n} fd\mu - \int_{A_n} gd\mu \geq \frac{1}{n}\mu(A_n)$$

$$\gamma(A_n) - \gamma(A_n) \geq \frac{1}{n}\mu(A_n) \text{ or } 0 \geq \frac{1}{n}\mu(A_n)$$

⇒ $\mu(A_n) \leq 0.$

Since $\mu(A_n)$ is always greater than equal to zero, we have $\mu(A_n) = 0$.

Similarly, we can show that

⇒ $\mu(B_n) \leq 0.$

If $C = \{x \in X : f(x) \neq g(x)\}$

$$= \bigcup_{n=1}^{\infty}(A_n \cup B_n),$$

then $\mu(C) = 0 \Rightarrow f = g$ a.e. on X w.r.t. μ.

Hence the theorem.

Theorem 1: If γ_1, γ_2 are σ-finite signed measures on (X, A) and $\gamma_1 \ll \mu, \gamma_2\ \mu$, then

$$\frac{d(\gamma_1+\gamma_2)}{d\mu} = \frac{d\gamma_1}{d\mu} + \frac{d\gamma_2}{d\mu} \text{ and } \frac{d\gamma_1}{d\mu} = \frac{d(-\gamma_1)}{d\mu}$$

Proof: Since γ_1, γ_2 are σ-finite and $\gamma_1 \ll \mu$, $\gamma_2 \ll \mu$, we have that $\gamma_1 + \gamma_2$ is also σ-finite and $\gamma_1 + \gamma_2 \ll \mu$.

Now for any $A \in \mathcal{A}$,

$$(\gamma_1 + \gamma_2)(A) = \gamma_1(A) + \gamma_2(A)$$

$$= \int_A \frac{d\gamma_1}{d\mu}d\mu + \int_A \frac{d\gamma_2}{d\mu}d\mu = \int_A \left[\frac{d\gamma_1}{d\mu} + \frac{d\gamma_2}{d\mu}\right]d\mu$$

$$\Rightarrow \int_A \left[\frac{d\gamma_1+\gamma_2}{d\mu}\right]d\mu = \int_A \left[\frac{d\gamma_1}{d\mu} + \frac{d\gamma_2}{d\mu}\right]d\mu$$

Notes

$$\Rightarrow \qquad \frac{d(\gamma_1+\gamma_2)}{d\mu} = \frac{d\gamma_1}{d\mu} + \frac{d\gamma_2}{d\mu}$$

Prove the other result yourself.

Theorem 2: If γ is a σ-finite signed measures and μ is a σ-finite measure s.t. $\gamma \ll \mu$, show that

$$\frac{d|\gamma|}{d\mu} = \left|\frac{d\gamma}{d\mu}\right|.$$

Proof: Let $\gamma = \gamma^+ - \gamma^-$ with Hahn decomposition A, B.

Then on A, $\left|\frac{d\gamma}{d\mu}\right| = \frac{d\gamma^+}{d\mu}$ and on B, $\left|\frac{d\gamma}{d\mu}\right| = \frac{d\gamma^-}{d\mu}$

$$\Rightarrow \qquad \left|\frac{d\gamma}{d\mu}\right| = \frac{d\gamma^+}{d\mu} + \frac{d\gamma^-}{d\mu} = \frac{d(\gamma^+ + \gamma^-)}{d\mu} = \frac{d|\gamma|}{d\mu}.$$

Theorem 3: If γ be a σ-finite signed measure and μ, λ be σ-finite measures on (X, A) s.t. $\gamma \ll \mu$, $\mu \ll \lambda$: then show that

$$\frac{d\gamma}{d\lambda} = \frac{d\gamma}{d\mu}\cdot\frac{d\mu}{d\lambda}$$

Proof: Since we may write $\gamma = \gamma^+ - \gamma^-$ and

$$\frac{-d\gamma^-}{d\mu} = \frac{d(-\gamma^-)}{d\mu}, \frac{-d\gamma^-}{d\lambda} = \frac{d(-\gamma^-)}{d\lambda}.$$

we need to prove the above result for measures only.

If $\frac{d\gamma}{d\mu} = f$ and $\frac{d\mu}{d\lambda} = g$, (f, g are non-negative functions as obtained in Radon-Nikodym Theorem),

then we need to prove that

$$\gamma(F) = \int_F fg\, d\lambda.$$

Let Ψ be a measurable simple function s.t.

$$\Psi = \sum_{i=1}^{n} a_i\, \phi_{E_i},$$

then $\int_F \Psi d\mu = \sum_{i=1}^{n} a_i\, \mu(E_i \cap F)$

$$= \sum_{i=1}^{n} a_i \int_{E_i \cap F} g d\lambda = \int_F \Psi_g d\lambda.$$

Let $<\Psi_n>$ be a sequence of measurable simple function which converges to f, then

$$\gamma(F) = \int_F f d\mu = \lim \int_F \Psi_n d\mu.$$

$$= \lim \int_F \Psi_n g d\lambda = \int_F f_g d\lambda \text{ as } \Psi_n g \to fg$$

$\Rightarrow$

$$= \frac{d\gamma}{d\lambda} = fg = \frac{d\gamma}{d\mu} \cdot \frac{du}{d\lambda}.$$

14.2 Summary

- Let (X, $\mathcal{A}$) be a measurable space and let r and m be measure functions, defined on the space (X, A). The measure γ is said to be absolutely continuous w.r.t. μ if

  μ (A) = 0 or |μ| (A) = 0, A ∈ $\mathcal{A}$ ⇒ γ (A) = 0, and is denoted by γ ≪ μ.

- Let (X, $\mathcal{A}$, μ) be a σ-finite measure space. If Y be a measure defined on A s.t. is absolutely continuous w.r.t. μ, then there exists a non-negative measurable function f s.t.

  $$\gamma (A) = \int_A f d\mu, \forall A \in \mathcal{A}$$

  The function f is unique in the sense that if g is any measurable function with the property defined as above, then f = g almost everywhere with respect to μ.

14.3 Keywords

Absolutely Continuous Measure Function: Let (X, $\mathcal{A}$) be a measurable space and let γ and μ be measure functions defined on the space (X, A). The measure γ is said to be absolutely continuous w.r.t. μ if

μ (A) = 0 or |μ| (A) = 0, A ∈ $\mathcal{A}$ ⇒ γ (A) = 0, and is denoted by γ ≪ μ.

Radon-Nikodym Theorem: Let (X, $\mathcal{A}$, μ) be a σ-finite measure space. If γ be a measure defined on A s.t. γ is absolutely continuous w.r.t. μ, then there exists a non-negative measurable function f on s.t.

$$\gamma (A) = \int_A f \, d\mu, \forall A \in \mathcal{A}.$$

The function f is unique in the sense that if g is any measurable function with the property defined as above, then f = g almost everywhere with respect to μ.

14.4 Review Questions

1. Show that $\frac{d\gamma}{d\mu} = \left(\frac{d\mu}{d\gamma}\right)^{-1}$,

   where μ and γ are σ-finite signed measures and μ ≪ γ, γ ≪ μ.

2. If γ (E) = $\int_E f d\mu$, , where $\int_E f d\mu$ exists, then find |γ| (E).

3. State and prove Radon-Nikodym theorem.

14.5 Further Readings

Notes

Book

G.E. Shilov and B.L. Gurevich, *Integral*, *Measure and Derivative: A Unified Approach*, Richard A. Silverman, trans. Dover Publications, 1978.

Online links

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