LOVELY PROFESSIONAL UNIVERSITY

# TOPOLOGY I

Edited by

Dr. Sachin Kaushal

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## SYLLABUS

## Topology I

***Objectives:*** For some time now, topology has been firmly established as one of basic disciplines of pure mathematics. It's ideas and methods have transformed large parts of geometry and analysis almost beyond recognition. In this course we will study not only introduce to new concept and the theorem but also put into old ones like continuous functions. Its influence is evident in almost every other branch of mathematics.In this course we study an axiomatic development of point set topology, connectivity, compactness, separability, metrizability and function spaces.

| Sr. No. | Content |
|---|---|
| 1 | Topological Spaces, Basis for Topology, The order Topology, The Product Topology on X * Y, The Subspace Topology. |
| 2 | Closed Sets and Limit Points, Continuous Functions, The Product Topology, The Metric Topology, The Quotient Topology. |
| 3 | Connected Spaces, Connected Subspaces of Real Line, Components and Local Connectedness, |
| 4 | Compact Spaces, Compact Subspaces of Real Line, Limit Point Compactness, Local Compactness |
| 5 | The Count ability Axioms, The Separation Axioms, Normal Spaces, Regular Spaces, Completely Regular Spaces |

CONTENT

# Unit 1: Topological Spaces

**Notes**

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Describe the concept of topological spaces;

Notes

- Explain the different kinds of topologies
- Solve the problems on intersection and union of topologies;
- Define open set and closed set;
- Describe the neighborhood of a point and solve related problems;
- Explain the dense set, separable space and related theorems and problems;
- Know the concept of limit point and derived set;
- Define interior and exterior of a set.

## Introduction

Topology is that branch of mathematics which deals with the study of those properties of certain objects that remain invariant under certain kind of transformations as bending or stretching. In simple words, topology is the study of continuity and connectivity.

Topology, like other branches of pure mathematics, is an axiomatic subject. In this, we use a set of axioms to prove propositions and theorems.

This unit starts with the definition of a topology and moves on to the topics like stronger and weaker topologies, discrete and indiscrete topologies, cofinite topology, intersection and union of topologies, open set and closed set, neighborhood, dense set, etc.

## 1.1 Topology and Different Kinds of Topologies

### 1.1.1 Topology

***Definition 1:*** Let X be a non-empty set. A collection T of subsets of X is said to be a topology on X if

(i) $X \in T, \phi \in T$

(ii) the intersection of any two sets in T belongs to T i.e. $A \in T, B \in T \Rightarrow A \cap B \in T$

(iii) the union of any (finite or infinite) no. of sets in T belongs to T.

i.e. $A_\alpha \in T \ \forall \ \alpha \in \Lambda \Rightarrow UA_\alpha \in T$ where $\Lambda$ is an arbitrary set.

The pair (X, T) is called a Topological space.

*Example 1:* Let X = {p, q, r, s, t, u} and $T_1$ = {X, $\phi$, {p}, {r, s}, {p, r, s}, {q, r, s, t, u}}

Then $T_1$ is a topology on X as it satisfies conditions (i), (ii) and (iii) of definition 1.

*Example 2:* Let X = {a, b, c, d, e} and $T_2$ = {X, $\phi$, {a}, {c, d}, {a, c, e}, {b, c, d}}

Then $T_2$ is not a topology on X as the union of two members of $T_2$ does not belong to $T_2$.

$$\{c, d\} \cup \{a, c, e\} = \{a, c, d, e\}$$

So, $T_2$ does not satisfy condition (iii) of definition 1.

Notes

## 1.1.2 Different Kinds of Topologies

### Stronger and Weaker Topologies

Let X be a set and let $T_1$ and $T_2$ be two topologies defined on X. If $T_1 \subset T_2$, then $T_1$ is called smaller or weaker topology than $T_2$.

If $T_1 \subset T_2$, then we also say that $T_2$ is longer or stronger topology than $T_1$.

### Comparable and Non-comparable Topologies

***Definitions:*** The topologies $T_1$ and $T_2$ are said to be comparable if $T_1 \subset T_2$ or $T_2 \subset T_1$.

The topologies $T_1$ and $T_2$ are said to be non-comparable if $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$.

*Example 3:* If X = {s, t} then $T_1 = \{\phi, \{s, X\}\}$ and $T_2 = \{\phi, \{t\}, X\}$ are non-comparable as $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$.

### Discrete and Indiscrete Topology

Let X be any non-empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

It may be noted that T in above definition satisfy the conditions of definition 1 and so is a topology.

Let X be any non-empty set and T = {X, $\phi$}. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

Again, it may be checked that T satisfies the conditions of definition 1 and so is also a topology.

*Example 4:* If X = {a, b, c} and T is a topology on X with $\{a\} \in T$, $\{b\} \in T$, $\{c\} \in T$, prove that T is the discrete topology.

*Solution:* The subsets of X are:

$$X_1 = \phi, X_2 = \{a\}, X_3 = \{b\}, X_4 = \{c\}, X_5 = \{a, b\}, X_6 = \{a, c\}, X_7 = \{b, c\}, X_8 = \{a, b, c\} = X$$

In order to prove that T is the discrete topology, we need to prove that each of these subsets belongs to T. As T is a topology, so X and $\phi$ belongs to T.

i.e. $X_1 \in T$, $X_8 \in T$.

Clearly, $X_2 \in T$, $X_3 \in T$, $X_4 \in T$

Now $X_5 = \{a, b\} = \{a\} \cup \{b\}$

since $\{a\} \in T$, $\{b\} \in T$ (Given)

and T is a topology and so by definition 1, their union is also in T i.e. $X_5 = \{a, b\} \in T$

similarly, $X_6 = \{a, c\} = \{a\} \cup \{c\} \in T$ and $X_7 = \{b, c\} = \{b\} \cup \{c\} \in T$

Hence, T is the discrete topology.

Notes

## Cofinite Topology

Let X be a non-empty set, and let T be a collection of subsets of X whose complements are finite along with $\phi$, forms a topology on X and is called cofinite topology.

*Example 5:* Let X = {l, m, n} with topology

$$T = \{\phi, \{l\}, \{m\}, \{n\}, \{l, m\}, \{m, n\}, \{l, n\}, X\}$$

is a cofinite topology since the compliments of all the subsets of X are finite.

> *Note* If X is finite, then topology T is discrete.

***Theorem 1:*** Let X be an infinite set and T be the collection of subsets of X consisting of empty set $\phi$ and all those whose complements are finite. Show that T is a topology on X.

***Proof:***

(i) Since $X' = \phi$, which is finite, so $X \in T$.

Also $\phi \in T$ (by definition of T)

(ii) Let $G_1, G_2, \in T$

$\Rightarrow$ $G'_1, G'_2$ are finite

$\Rightarrow$ $G'_1 \cup G'_2$ is finite

$\Rightarrow$ $(G_1 \cap G_2)'$ is finite (by De-Morgan's law $(G'_1 \cup G'_2 = (G_1 \cap G_2)')$

$\Rightarrow$ $G_1 \cap G_2 \in T$

(iii) If $\{G_\alpha : \alpha \in \Lambda\}$ is an arbitrary collection of sets in T, then

$G'_\alpha$ is finite $\forall \alpha \in \Lambda$

$\Rightarrow$ $\cap \{G'_\alpha : \alpha \in \Lambda\}$ is finite

$\Rightarrow$ $[\cup \{G_\alpha : \alpha \in \Lambda\}]'$ is finite (by De-Morgan's law)

$\Rightarrow$ $\cup \{G_\alpha : \alpha \in \Lambda\} \in T$

Hence T is a topology for X.

## Co-countable Topology

Let X be a non-empty set. Let T be the collection of subsets of X whose complements are countable along with $\phi$, forms a topology on X and is called co-countable topology.

***Theorem 2:*** Let X be a non-empty set. Let T be the collection of all subsets of X, whose complements are countable together with empty set $\phi$. Show that T is a topology on X.

***Proof:***

(i) Since $X' = \phi$, which is countable

so, $X \in T$

Also, by definition, $\phi \in T$

(ii) Let $G_1, G_2 \in T$

$\Rightarrow$ $G'_1, G'_2$ are countable

$\Rightarrow$ $G'_1, \cup G'_2$ is countable

$\Rightarrow$ $(G_1 \cap G_2)'$ is countable (by De-Morgan's law)

$\Rightarrow$ $G_1 \cap G_2 \in T$

(iii) Let $\{G_\alpha : \alpha \in \Lambda\}$ be an arbitrary collection of members of sets in T.

$\Rightarrow$ $G'_\alpha$ is countable $\forall\ \alpha \in \Lambda$

$\Rightarrow$ $\cap \{G'_\alpha : \alpha \in \Lambda\}$ is countable

$\Rightarrow$ $[\cup \{G_\alpha : \alpha \in \Lambda\}'$ is countable (by De-Morgan's law)

$\Rightarrow$ $\cup \{G_\alpha : \alpha \in \Lambda\} \in T$

Hence, T is a topology for X.

### Self Assessment

1. Construct three topologies $T_1, T_2, T_3$ on a set X = {a, b, c} s.t. $T_1 \subset T_2 \subset T_3$.
2. Let X = {a, b, c} and T = {ϕ, X, {b}, {a, b}. Is T is a topology for X?

## 1.2 Intersection and Union of Topologies

Intersection of any two topologies on a non-empty set is always topology on that set. While the union of two topologies may not be a topology on that set.

*Example 6:* Let X = {1, 2, 3, 4}

$T_1$ = {ϕ, X, {1}, {2}, {1, 2}}

$T_2$ = {ϕ, X, {1}, {3}, {1, 3}}

$T_1 \cap T_2$ = {ϕ, X, {1}} is a topology on X.

$T_1 \cup T_2$ = {ϕ, X, {1}, {2}, {3}, {1, 2}, {1, 3}} is not a topology on X.

*Example 7:* If $T_1$ and $T_2$ are two topologies defined on the same set X, then $T_1 \cap T_2$ is also a topology on X but $T_1 \cup T_2$ is not a topology on X.

*Solution: Part I:* Let $T_1, T_2$ be two topologies on the same set X.

We are to prove that $T_1 \cap T_2$ is a topology on X.

By assumption,

(i) $X \in T_1, X \in T_2$

$\phi \in T_1, \phi \in T_2$

(ii) $A, B \in T_1 \Rightarrow A \cap B \in T_1$

$A, B \in T_2 \Rightarrow A \cap B \in T_2$

Notes

(iii) $G_\alpha \in T_1 \ \forall \ \alpha \in \Delta \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T_1$

$G_\alpha \in T_2 \ \forall \ \alpha \in \Delta \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T_2$

Then (I) $X \in T_1 \cap T_2$, $\phi \in T_1 \cap T_2$ by (i)

(II) $A \in T_1 \cap T_2$, $B \in T_1 \cap T_2 \Rightarrow A \cap B \in T_1 \cap T_2$

For $A \in T_1 \cap T_2$, $B \in T_1 \cap T_2$

$\Rightarrow \quad A \in T_1, A \in T_2$ and $B \in T_1, B \in T_2$

$\Rightarrow \quad A \cap B \in T_1, A \cap B \in T_2$ by (ii)

$\Rightarrow \quad A \cap B \in T_1 \cap T_2$

(iv) $G_\alpha \in T_1 \cap T_2 \ \forall \ \alpha \in \Delta$

$\Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T_1 \cap T_2$

For $G_\alpha \in T_1 \cap T_2 \ \forall \ \alpha \in \Delta$

$\Rightarrow \quad G_\alpha \in T_1 \ \forall \ \alpha \in \Delta$ and $G_\alpha \in T_2 \ \forall \ \alpha \in \Delta$

$\Rightarrow \quad \cup G_\alpha \in T_1$ and $\cup G_\alpha \in T_2$ by (iii)

Thus, $T_1 \cap T_2$ is topology on X.

*Part II:* Let X = {a, b, c}. Then $T_1 = \{X, \phi \ \{a\}\}$ and $T_2 = \{X, \phi, \{b\}\}$ are topologies on X.

Let $G_1 = \{a\} \in T_1$, $G_2 = \{b\} \in T_2$.

Then $G_1 \cup G_2 = \{a, b\} \notin T_1 \cup T_2$.

Consequently $T_1 \cup T_2$ is not a topology on X.

## Self Assessment

3. Prove that the intersection of an arbitrary collection of topologies for a set X is a topology for X.

4. Let $T_n$ be a topology on a set X $\forall \ n \in \Delta$, $\Delta$ being an index set. Then $\cap \{T_n : r \in \Delta\}$ is a topology on X.

## 1.3 Open Set, Closed Set and Closure of a Set

### 1.3.1 Definition of Open Set and Closed Set

Let (X, T) be a topological space. Any set $A \in T$ is called an open set and X-A is a closed set.

*Example 8:* If T = {$\phi$, {a}, X} be a topology on X = {a, b} then $\phi$, X and {a} are T-open sets.

*Example 9:* Let X = {a, b, c} and T = {$\phi$, {a}, {b, c}, X} be a topology on X.

Since X – {a} = {b, c}

X – {b, c} = {a}

Therefore, T-closed sets are $\phi$, {b, c} and X, which are the complements of T-open sets X, {b, c}, {a} and $\phi$ respectively.

Notes

*Note* In every topological space, X and $\phi$ are open as well as closed.

### 1.3.2 Door Space

A topological space (X, T) is said to be a door space if every subset of X is either T-open or T-closed.

*Example 10:* Let X = {1, 2, 3) and T = {$\phi$, {1, 2}, {2, 3}, {2}, X}

Then, T-closed sets are X, {3}, {1}, {1, 3}, $\phi$.

This shows that every subset of X is either T-open or T-closed.

### 1.3.3 Closure of a Set

Let (X, T) be a topological space and A is a subset of X, then the closure of A is denoted by $\bar{A}$ or Cl (A) is the intersection of all closed sets containing A or all closed superset of A.

*Example 11:* If T = {$\phi$, {a}, {a, b}, {a, c, d}, {a, b, c}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then find the closure of the sets {a}, {b}

*Solution:* Closed subset of X are

$\phi'$, {a}′, {a, b}′, {a, c, d}′, (a, b, e}′, {a, b, c, d}′, X′ = X, {b, c, d, e}, {c, d, e}, {b, e}, {c, d}, {e}, $\phi$

then $\{\bar{a}\} = X$

$\{\bar{b}\} = X \cap \{b, c, d, e\} \cap \{b, e\} = \{b, e\}$

***Theorem 3:*** A is closed iff $A = \bar{A}$

***Proof:*** Let us suppose that A is closed

$\therefore$ $A \subseteq \bar{A}$ (by definition of closure)

Now also $\bar{A} \subseteq A$ (A is common in all supersets of A)

$\therefore$ $\bar{A} = A$

Conversely, let us suppose that $A = \bar{A}$

Since we know that $\bar{A}$ is closed. (by definition of closure of A)

$\therefore$ $A = \bar{A}$ is closed

$\Rightarrow$ A is closed

### 1.3.4 Properties of Closure of Sets

***Theorem 4:*** Let (X, T) be a topological space and let A, B be any two subsets of X. Then

(i) $\bar{\phi} = \phi$, $\bar{X} = X$;

(ii) $A \subset \bar{A}$

(iii) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

Notes

(iv) $\left(\overline{A \cup B}\right) = \bar{A} \cup \bar{B}$

(v) $\left(\overline{A \cup B}\right) \subset \bar{A} \cap \bar{B}$

(vi) $\bar{\bar{A}} = \bar{A}$

***Proof:***

(i) Since $\phi$ and X are open as well as closed.

So, $\phi$, X being closed, we have

$\bar{\phi} = \phi, \; \bar{X} = X$

(ii) Since we know that $\bar{A}$ is the smallest T-closed set containing A so $A \subset \bar{A}$

(iii) Let $A \subseteq B$

Then $A \subseteq B \subseteq \bar{B}$

i.e. $\bar{B}$ is a closed superset of A. $(\because B \subseteq \bar{B})$

But $\bar{A}$ is the smallest closed superset of A.

$\therefore \quad \bar{A} \subseteq \bar{B}$

Thus, $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$.

(iv) We have $A \subseteq A \cup B \Rightarrow \bar{A} \subseteq \overline{A \cup B}$ by (iii)

and $B \subseteq A \cup B \Rightarrow \bar{B} \subseteq \overline{A \cup B}$ by (iii)

Hence $\bar{A} \cup \bar{B} \subseteq \left(\overline{A \cup B}\right)$ … (I)

Since $\bar{A}$, $\bar{B}$ are closed sets, $\bar{A} \cup \bar{B}$ is also closed.

$\Rightarrow$ $A \cup B \subseteq \bar{A} \cup \bar{B}$ … (II)

From (1) & (2), we have $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(v) We have

$(A \cap B) \subseteq A \Rightarrow \overline{A \cap B} \subseteq \bar{A}$ by (iii)

and $(A \cap B) \subseteq B \Rightarrow \overline{A \cap B} \subseteq \bar{B}$ by (iii)

Hence $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

(vi) We know that if A is a T-closed subset then $\bar{A}$ = A by the theorem: In a topological space (X, T) if A is subset of X then A is closed iff $\bar{A}$ = A.

But $\bar{A}$ is also a T-closed subset.

$\therefore \quad \bar{\bar{A}} = A.$

***Theorem 5:*** In a topological space, an arbitrary union of open sets is open and a finite intersection of open sets is open. Prove it.

***Proof:*** Let (X, T) be a topological space

Let $G_i \in T \;\forall\; i \in N$

Let $G = \bigcup_{i=1}^{\infty} G_i$, $\quad H = \bigcap_{i=1}^{n} G_i$

We are to prove that G and H are open subsets of X. By definition of topology,

(i) $G_i \in T \;\forall\; i \in N \Rightarrow \bigcup_{i=1}^{\infty} G_i \in T \Rightarrow G \in T$

(ii) $G_i \in T \;\forall\; i \in N \Rightarrow G_1 \cap G_2 \in T$

$G_1 \cap G_2 \in T, G_3 \in T$

$\Rightarrow G_1 \cap G_2 \cap G_3 \in T$

By induction, it follows that

$\bigcap_{i=1}^{n} G_i = H \in T$

Hence proved.

***Theorem 6:*** In a topological space (X, T), prove that an arbitrary intersection of closed sets is closed and finite union of closed sets is closed.

***Proof:*** Let (X, T) be a topological space,

Let $F_i \subset X$ be closed $\forall\; i \in N$

Let $H = \bigcap_{i=1}^{\infty} F_i$, $F = \bigcup_{i=1}^{n} F_i$

We are to prove that F and H are closed sets $F_i$ is closed $\forall\; i \in N$

$\Rightarrow$ $X - F_i$ is open $\forall\; i \in N$

Also, we know, $\bigcup_{i=1}^{\infty}(X - F_i)$ and $\bigcap_{i=1}^{n}(X - F_i)$ are open sets

[$\because$ An arbitrary union of open sets is open and a finite intersection of open sets is open]

$\Rightarrow$ $X - \bigcap_{i=1}^{\infty} F_i$ and $X - \bigcup_{i=1}^{\infty} F_i$ are open sets (by De Morgan's Law)

$\Rightarrow$ $\bigcap_{i=1}^{\infty} F_i$, $\bigcup_{i=1}^{n} F_i$ are closed sets (by definition of closed sets)

i.e. H, F are closed sets.

Hence, proved.

## Self Assessment

5. Give two examples of a proper non-empty subset of a topological space such that it is both open and closed and prove your assertion.

6. On the real line show that every open interval is an open but every open set need not be an open interval.
7. Let (Y, U) be a subspace of a topological space (X, T). Then every U-open set is also T-open iff Y is T-open.

## 1.4 Neighborhood

Let (X, T) be a topological space. $A \subset X$ is called a neighbourhood of a point $x \in X$ if $\exists$ $G \in T$ with $x \in G$ such that $G \subset A$. The word neighborhood is, in short, written as 'nhd'.

Let G be any open set such that $G \subset X$ with $x \in G$ is also nhd of a point $x \in X$.

*Example 12:* Let T = {ϕ, X, {b}, {a, b}, {a, b, d}}, be a topology on X = {a, b, c, d}. Find T-nhds of (i) a, (ii) b and (iii) c.

*Solution:*

(i) T-open sets containing 'a' are X, {a, b}, {a, b, d}.

super set of X is X

supersets of {a, b} are {a, b}, {a, b, c}, {a, b, d}, X

supersets of {a, b, d} are {a, b, d}, X.

T-nhds of 'a' are {a, b}, {a, b, c}, {a, b, d}, X

(ii) T-open sets containing b are

{b}, {a, b}, {a, b, d}, X

supersets of {a, b} are {a, b}, {a, b, c}, {a, b, d}, X

supersets of {a, b, d} are {a, b, d}, X

supersets of {b} are {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {b, c, d}, {a, b, d}, X

T-nhds of 'b' are {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, X

(iii) T-open set containing 'c' is X.

Hence T-nhd of 'c' is X.

***Theorem 7:*** Let (X, T) be a topological space and $A \subset X$. Then A is T-open $\Leftrightarrow$ A contains T-nhds of each of its points.

***Proof:*** Let (X, T) be a topological space and $A \subset X$.

*Step I:* Given A is an open set.

To show: A contains T-nhd of each of its points. Clearly $x \in A \subset A$ $\forall$ $x \in A$ and A is an open set. This shows that A contains T-nhd of each of its points.

*Step II:* Given A contains T-nhd of each of its point, then any $x \in A \Rightarrow \exists$ nhd $N_x \subset X$ such that

$$x \in N_x \subset A \qquad \ldots(1)$$

To show: A is an open set

By definition of nhd, $\exists$ open set $G_x$ s.t.

$$x \in G_x \subset N_x \qquad \ldots(2)$$

**Notes**

From (1) and (2), we get

$$x \in G_x \subset N_x \subset A \qquad \text{...(3)}$$

$\Rightarrow \quad x \in G_x \subset A$

which is true $\forall$ x $\in$ A

$$\therefore \quad \bigcup_{x \in A} G_x \subset A \qquad \text{...(4)}$$

Let G = $\bigcup_{x \in A} G_x$ and an arbitrary union of open sets is open and so G is an open set.

$$\therefore \quad G \subset A \qquad \text{...(5) [Using (4)]}$$

for any $\quad x \in A \Rightarrow x \in G_x \subset G \Rightarrow x \in G \Rightarrow A \subset G \qquad$ ...(6)

from (5) & (6), we get

$$A = G$$

$\Rightarrow$ A is an open set.

***Theorem 8:*** Let X be a topological space. Then the intersection of two nhds of x $\in$ X is also a nhd of x.

***Proof:*** Let $N_1$ and $N_2$ be two nhds of x $\in$ X then $\exists$ open sets $G_1$ and $G_2$ such that

$$x \in G_1 \subseteq N_1 \text{ and}$$

$$x \in G_2 \subseteq N_2$$

$$\therefore \qquad x \in G_1 \cap G_2 \subseteq N_1 \cap N_2$$

$\therefore$ $G_1 \cap G_2$ is an open set containing x and contained in $N_1 \cap N_2$.

This shows that $N_1 \cap N_2$ is also a nhd of x.

***Theorem 9:*** Let (y, $\bigcup$) be a subspace of a topological space (X, T). A subset of Y is $\bigcup$-nhd of a point y $\in$ Y iff it is the intersection of Y with a T-nhd of the point y $\in$ Y.

***Proof:*** Let (y, $\bigcup$) $\subset$ (X, T) and y $\in$ Y be arbitrary, then y $\in$ X.

*Step I:* Let $N_1$ be a $\bigcup$-nhd of y, then

$$\exists\, V \in \bigcup \text{ s.t. } y \in V \subset N_1 \qquad \text{...(1)}$$

To show: $N_1 = N_2 \cap Y$ for some T-nhd $N_2$ of y.

$$y \in V \in \bigcup \Rightarrow G \in T \text{ s.t. } V = G \cap Y$$

$$\Rightarrow y \in G \cap Y \Rightarrow y \in G, y \in Y \qquad \text{... (2)}$$

Let $\qquad N_2 = N_1 \bigcup G$

Then $\qquad N_1 \subset N_2, G \subset N_2 \qquad$ ...(3)

From (2) and (3), y $\in$ G $\subset$ $N_2$ where G $\in$ T

This shows that $N_2$ is a T-nhd of y.

$$N_2 \cap Y = (N_1 \cup G) \cap Y = (N_1 \cap Y) \cup (G \cap Y)$$

$$= (N_1 \cap Y) \cup V = N_1 \cup V = N_1 \qquad \text{[by (1)]}$$

$$\because \qquad N_1 \subset Y \text{ and } V \cup N_1$$

so, $N_2$ has the following properties

$N_1 = N_2 \cap Y$ and $N_2$ is a $\bigcup$-nhd of y.

This completes the proof.

*Step II:* Conversely Let $N_2$ be a T-nhd of y so that

$$\exists\, A \in T \text{ s.t. } y \in A \subset N_2 \qquad \ldots(4)$$

To show: $N_2 \cap Y$ is a U-nhd of y.

$\therefore \quad y \in Y, y \in A \Rightarrow y \in Y \cap A$ [by (4)]

$\Rightarrow y \in A \cap Y \subset N_2 \cap Y$ [by (3)]

$A \in T \Rightarrow A \cap Y \in \bigcup$

Thus, we have $y \in A \cap Y \subset N_2 \cap Y$, where $A \cap Y \in \bigcup$.

This shows that $N_2 \cap Y$ is a $\bigcup$-nhd of y.

### Self Assessment

8. Let $T = \{X, \phi, \{p\}, \{p, q\}, \{p, q, t\}, \{p, q, r, s\}, \{p, r, s\}\}$ be the topology on

   $X = \{p, q, r, s, t\}$

   List the nhds of the points r, t.

9. Prove that a set G in a topological space X is open iff G is a nhd of each of its points.

## 1.5 Dense Set and Boundary Set

### 1.5.1 Dense Set and No where Dense

Let (X, T) be a topological space and $A \subset X$ then A is said to be dense or everywhere dense in X if $\bar{A} = X$.

*Example 13:* Consider the set of rational number $Q \subseteq R$, then only closed set containing Q in R, which shows that $\bar{Q} = R$.

Hence, Q is dense in R.

*Note* Rational are dense in R and countable but irrational numbers are also dense in R but not countable.

*Example 14:* Prove that A set is always dense in its subset

*Solution:* Let $A \subset B$ then $A \subset B \subset \bar{B}$

$\Rightarrow A \subset \bar{B}$

$\Rightarrow \bar{B} \supset A$

$\Rightarrow$ B is dense in A.

**Notes**

*Example 15:* If $T = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, e\}, \{a, b, c, d\}, X\}$ be a topology on $X = \{a, b, c, d, e\}$ then which of the set $\{a\}$, $\{b\}$, $\{c, e\}$ are dense in X.

*Solution:* A is called dense in X if $\bar{A} = X$ (By definition)

$$\{\bar{a}\} = \cap \{F : F \text{ is closed subset s.t. } F \supset \{a\}\} = X.$$

$$\{\bar{b}\} = X \cap \{b, c, d, e\} \cap \{b, e\} = \{b, e\}$$

$$\{\overline{c, e}\} = X \cap \{b, c, d, e\} \cap \{c, d, e\} = \{c, d, e\}$$

This shows that $\{a\}$ is the only dense set in X.

***Definition***

- A is said to be dense in itself if $A \subset D(A)$.
- A is said to be nowhere dense set in X if $\text{int}(\bar{A}) = \phi$ i.e., if the interior of the closure of A is an empty set.

## 1.5.2 Boundary Set

The Boundary set of A is the set of all those points which belong neither to the interior of A nor to the interior of its complement and is denoted by b(A).

Symbolically, $b(A) = X - A^\circ \cup (X - A)^\circ$.

Elements of b(A) are called bounding points of A. Boundary points are, sometimes called frontier points.

*Example 16:* Define nowhere dense set and give an example of it.

*Solution:* $\because D(N) = \phi$

For if a is any real number, then consider a real number $\in > 0$, so small that open set $(a - \in, a + \in)$ does not contain any point of N.

$$\therefore \quad Z = \{n : n \in N\} \cup \{0\} \cup \{-n : n \in N\}$$

$$D(Z) = D\{n : n \in N\} \cup D\{0\} \cup D\{-n : n \in N\}$$

$$= \phi \cup \phi \cup \phi$$

$$= \phi \subset Z$$

$$\therefore \quad D(Z) \subset Z \Rightarrow Z \text{ is closed.}$$

$$\Rightarrow \quad Z = \bar{Z}$$

$$\text{Int}(\bar{Z}) = \text{Int}(Z) = \cup \{G \subset R : G \text{ is open}, G \subset Z\}$$

$$= \phi$$

$\therefore$ An open subset of R will be an open interval, say $G = (a_1, a_2)$. This open interval contains all real numbers (rationals and irrationals) x s.t. $a_1 < x < a_2$ and therefore $G \not\subset Z$.

$$\therefore \quad \text{Int}(Z) = \phi$$

This proves that Z is nowhere dense set in R.

*Example 17:* Prove that every non-empty subset of an indiscrete space is dense in X.

*Solution:* Let (X, T) be an indiscrete space.

Let $A \subset X$ be non-empty set.

To show: A is dense in X.

For this, we are to prove $\bar{A} = X$

By definition of an indiscrete topology,

$T = \{\phi, X\}$

T-open sets are $\phi$, X

T-closed sets are $X - \phi$, X i.e. X, $\phi$.

Since $A \neq \phi$ by assumption.

$\therefore$ The only closed superset of A is X,

so that $\bar{A} = X$.

*Example 18:* Let T = {X, $\phi$, {p}, {p, q}, {p, q, t}, {p, q, r, s}, {p, r, s}} be the topology on X = {p, q, r, s, t}

Determine boundary of the following sets

(i) $B = \{q\}$

$B^\circ = \cup \{\phi\} = \phi$

$(X - B)^\circ = \{p, r, s, t\}^\circ = \cup \{f, \{p\}, \{p, r, s\}\}$

$= \{p, r, s\}$

$b(B) = X - B^\circ \cup (X - B)^\circ$

$= X - \phi \cup \{p, r, s\}$

$= \{q, t\}$

### Self Assessment

10. In a topological space, prove that:

    (i) A is dense $\Leftrightarrow$ it intersects every non-empty open set.

    (ii) A is closed $\Leftrightarrow$ A contains its boundary.

11. In any topological space, prove that

    $b(A) = \phi \Leftrightarrow$ A is open as well as closed.

## 1.6 Separable Space, Limit Point and Derived Set

### 1.6.1 Separable Space

Let X be a topological space and A be subset of X, then X is said to be separable if

(i) $\bar{A} = X$

(ii) A is countable

*Example 19:* Let X = {1, 2, 3, 4, 5} be a non-empty set and T = {$\phi$, X, {3}, {3, 4}, {2, 3}, {2, 3, 4}} is a topology defined on X. Suppose a subset A = {1, 3, 5} $\subseteq$ X. The closed set are:

X, $\phi$, {1, 2, 4, 5}, {1, 2, 5}, {1, 4, 5}, {1, 5}.

So, we have $\bar{A}$ = X. Since A is finite and dense in X. So X is a separable space.

***Theorem 10:*** Show that the cofinite topological space (X, T) is separable.

***Solution:*** Let (X, T) be a cofinite topological space.

(i) When X is countable.

Then X $\subset$ X and $\bar{X} = X$

This shows that X is separable.

(ii) Let A $\subset$ X s.t. A is finite.

By definition of cofinite topological space A′ = X – A is open so that A is closed.

$\Rightarrow$ every finite set A is T-closed and so $\bar{A}$ = X.

Now $\bar{A}$ = X, A is countable.

This shows that (X, T) is separable.

*Example 20:* A discrete space X is separable iff X is countable.

*Solution:* As we know that every subset of a discrete space (X, T) is both open and closed. Also, A is said to be everywhere dense in X if $\bar{A}$ = X.

Also, X is separable if $\exists$ A $\subset$ X s.t. $\bar{A}$ = X and A is countable.

So, the only everywhere dense subset of X is X itself.

$\Rightarrow$ X can have a countable dense subset iff X is countable.

Hence, X is separable iff X is countable.

### 1.6.2 Limit Point or Accumulation Point or Cluster Point

Let (X, T) be a topological space and A $\subset$ X. A point x $\in$ X is said to be the *limit point* or *accumulation point* or *cluster point* of A if each open set containing 'x' contains at least one point of A different from x.

Thus, it is clear from the above definition that the limit point of a set A may or may not be the point of A.

*Example 21:* Let X = {a, b, c} with topology

T = {$\phi$, {a, b}, {c}, X} and A = {a}, then b is the only limit point of A, because the open sets containing b namely {a, b} and X also contains a point of A.

Whereas, 'a' and 'b' are not limit point of C = {c}, because the open set {a, b} containing these points do not contain any point of C. The point c is also not a limit point of C, since then open set {c} containing 'c' does not contain any other point of C different from c. Thus, the set C = {c} has no limit points.

*Example 22:* Prove that every real number is a limit point of R.

*Solution:* Let $x \in R$ then every nhd of x contains at least one point of R other than x

$\therefore$ x is a limit point of R.

But x was arbitrary.

$\therefore$ every real number is a limit point of R.

*Example 23:* Prove that every real number is a limit point of R – Q.

*Solution:* Let x be any real number, then every nhd of X contains at least one point of R – Q other than x

$\therefore$ x is a limit point of R – Q.

But x was arbitrary.

$\therefore$ every real number is a limit point of R – Q.

### 1.6.3 Derived Set

***Definition:*** The set of all limit points of A is called the derived set of A and is represented by D(A).

*Example 24:* Every derived set in a topological space is a closed.

*Solution:* Let (X, T) be a topological space and $A \subset X$.

To show: D(A) is a closed set.

As we know that B is a closed set if $D(B) \subset B$.

Hence, D(A) is closed iff $D[D(A)] \subset D(A)$.

Let $x \in D[D(A)]$ be arbitrary, then x is a limit point of D(A) so that

$$(G - \{x\}) \cap D(A) \neq \phi \ \forall\ G \in T \text{ with } x \in G$$

$$\Rightarrow \quad (G - \{x\}) \cap A \neq \phi$$

$$\Rightarrow \quad x \in D(A)$$

Hence proved.

[For every nhd of an element of D(A) has at least one point of A].

*Example 25:* In any topological space, prove that $A \cup D(A)$ is closed.

*Solution:* Let (X, T) be a topological space and $A \subset X$.

To prove: $A \cup D(A)$ is a closed set.

Let $x \in X - A \cup D(A)$ be arbitrary then $x \notin A \cup D(A)$ so that $x \notin A$, $x \notin D(A)$

$$x \notin D(A) \Rightarrow \exists\ G \in T \text{ with } x \in G \quad \text{s.t.}$$

$$(G - \{x\}) \cap A = \phi$$

$$\Rightarrow G \cap A = \phi \qquad (\because \quad x \notin A) \qquad \ldots(1)$$

**Notes**

For this G, we also claim

$$G \cap D(A) = \phi$$

Let $y \in G$ be arbitrary.

Now G is an open set containing 'y' s.t.

$G \cap A = \phi$, showing that $y \notin D(A)$.

$\therefore$ any $y \in G \Rightarrow y \notin D(A)$

This shows $G \cap D(A) = \phi$

$\therefore$ $G \cap A = \phi, G \cap D(A) = \phi$

Now, $G \cap [A \cup D(A)] = (G \cap A) \cup [G \cap D(A)]$

$= \phi \cup \phi = \phi$

$\Rightarrow$ $G \subset X - A \cup D(A)$

$\therefore$ any $x \in X - A \cup D(A)$

$\Rightarrow$ $G \in T$ with $x \in G$ s.t. $G \subset X - A \cup D(A)$

This proves that x is an interior point of $X - A \cup D(A)$.

Since x is arbitrary point of $X - A \cup D(A)$.

Hence, every point of $X - A \cup D(A)$ is an interior point of $X - A \cup D(A)$.

$\therefore$ $X - A \cup D(A)$ is open.

i.e., $A \cup D(A)$ is closed.

***Theorem 11:*** Let (X, T) be a topological space and $A \subseteq X$, then A is closed iff $A' \subseteq A$ or $A \supseteq D(A)$. A subset A of X in a topological space (X, T) is closed iff A contains each of its limit points.

***Proof:*** Let A be closed $\Rightarrow A^C$ is open.

Let $x \in A^C$

then $A^C$ is open set containing x but containing no point of A other than x. This shows that x is not a limit point of A.

Thus, no point of $A^C$ is a limit point of A. Consequently, every limit point of A is in A and therefore $A' \subseteq A$.

Conversely, Let $A' \subseteq A$.

To show: A is closed.

Let x be an arbitrary point of $A^C$.

Then $x \in A^C \Rightarrow x \notin A \quad \Rightarrow x \notin A$ and $x \notin A' \quad (\because A' \subseteq A)$

$\Rightarrow$ $x \notin A$ and x is not a limit point of A.

$\Rightarrow$ $\exists$ an open set G such that $x \in G$ and $G \cap A = \phi \quad \Rightarrow \quad G \subseteq A^C$

$\Rightarrow$ $x \in G \subseteq A^C$

$\Rightarrow$ $A^C$ is the nhd of each of its points and therefore $A^C$ is open.

Hence A is closed.

Notes

*Theorem 12:* In any topological space, prove that

$$\bar{A} = A \cup D(A)$$

***Proof:*** Let (X, T) be a topological space and $A \subset X$.

To show: $\bar{A} = A \cup D(A)$

Since $A \cup D(A)$ is closed and hence

$$\overline{A \cup D(A)} = A \cup D(A) \quad \ldots(1)$$

$\because \quad A \subset A \cup D(A)$

$\therefore \quad \bar{A} \subset \overline{A \cup D(A)} = A \cup D(A)$ [Using (1)]

$$\bar{A} \subset A \cup D(A) \quad \ldots(2)$$

Now, We are to prove that

$$A \cup D(A) \subset \bar{A} \quad \ldots(3)$$

But, $A \subset \bar{A}$ …(4)

To prove (3), we are to prove

$$D(A) \subset \bar{A} \quad \ldots(5)$$

i.e., to show that

$$D(A) \subset \cap_i \{F_i \subset X : F_i \text{ is closed } F_i \supset A\} \quad \ldots(6)$$

Let $x \in D(A)$ be arbitrary.

$x \in D(A) \Rightarrow x$ is a limit point of A

$\Rightarrow$ x is a limit point of all those sets which contain A.

$\Rightarrow$ x is a limit point of all those $F_i$ appearing on R.H.S. of (6).

$\Rightarrow \quad x \in D(F_i) \subset F_i \quad (\because \; F_i$ is closed)

$\Rightarrow \quad x \in F_i$ for each i

$\Rightarrow \quad x \in \cap_i \{F_i \subset X : F_i \text{ is closed}\}$

$\Rightarrow \quad x \in \bar{A}$

Thus any $x \in D(A) \Rightarrow x \in \bar{A}$

$$D(A) \subset \bar{A}$$

Hence the result (5) proved.

From (4) & (5), we get

$$A \cup D(A) \subset \bar{A} \cup \bar{A} = \bar{A}$$

i.e., $A \cup D(A) \subset \bar{A}$

Hence the result (3) proved.

Combining (2) & (3), we get the required result.

## Self Assessment

12. Let X = {a, b, c} and let T = {ϕ, X, {b}, {c}}, find the set of all cluster points of set {a, b}.

Notes

13. Let $X = \{a, b, c\}$ and let $T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, show that $D(\{a\}) = \{c\}$, $D(\{c\}) = \phi$ and find derived sets of other subsets of X.

## 1.7 Interior and Exterior

### 1.7.1 Interior Point and Exterior Point

***Interior Point:*** Let X be a topological space and let $A \subset X$.

A point $x \in A$ is called an interior point of A iff $\exists$ an open set G such that $x \in G \subseteq A$.

The set of all interior points of A is known as the interior of A and is denoted by Int (A) or $A°$. Symbolically,

$$A° = \text{Int}(A) = \cup \{G \in T : G \subset A\}.$$

*Example 26:* Let $T = \{\phi, \{a\}, \{b, c\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology on $X = \{a, b, c, d\}$ then

Int (A) = Union of all open subsets of X which are contained in A.

$$\text{Int}[\{a\}] = \phi \cup \{a\} = \{a\}$$

$$\text{Int}[\{a, b\}] = \phi \cup \{a\} \cup \{a, b\} = \{a, b\}$$

***Exterior Point:*** Let X be a topological space and let $A \subset X$.

A point $x \in A$ is called an exterior point of A iff it is an interior point of $A^C$ or $X - A$.

The set of all exterior points of A is called the exterior of A and is denoted by ext (A).

Symbolically,

$$\text{ext}(A) = (X - A)° \text{ or } (A^C)°.$$

*Example 27:* Let $T = \{X, \phi, \{p\}, \{p, q\}, \{p, q, t\}, \{p, q, r, s\}, \{p, r, s\}\}$ be the topology on $X = \{p, q, r, s, t\}$

Determine exterior of (i)

$$B = \{q\}$$

*Solution:*

$$\text{ext}(B) = (X - B)° = \{p, r, s, t\}°$$

$$= \cup \{Q, \{p\}, \{p, r, s\}\}$$

$$= \{p, r, s\}$$

### 1.7.2 Interior Operator and Exterior Operator

***Interior Operator:*** Let X be a non-empty set and P(X) be its power set. Then, an interior operator 'i' on X is a mapping $i : P(X) \to P(X)$ which satisfies the following four axioms:

(i) $i(X) = X$

(ii) $i(A) \subseteq A$

(iii) $i(A \cap B) = i(A) \cap i(B)$

(iv) $i(i(A) = i(A)$, where A and B are subsets of X.

Notes

***Exterior Operator:*** Let X be a topological space. Then, an exterior operator on X is a mapping $e : P(X) \to P(X)$ satisfying the following postulates:

(i) $e(\phi) = X$, $e(X) = \phi$

(ii) $e(A) \subseteq A'$

(iii) $e[\{e(A)\}'] = e(A)$

(iv) $e(A \cup B) = e(A) \cap e(B)$ where A and B are subsets of X.

***Theorem 13:*** Prove that $\text{int}(A) = \cup \{G : G \text{ is open}, G \subseteq A\}$.

or

Let X be a topological space and let $A \subseteq X$. Then, $A°$ is the union of all open subsets of A.

***Proof:*** Let $x \in A° \leftrightarrow x$ is an interior point of A.

$\leftrightarrow$ A is a nhd of x.

Then $\exists$ an open set G such that $x \in G \subset A$ and hence $x \in \cup \{G : G \text{ is an open subset of } A\}$

Now let $\quad x \in \cup \{G : G \text{ is open}, G \subseteq A\}$ …(1)

$\Rightarrow$ $x \in$ some T-open set G which is contained in A

$\Rightarrow$ $x \in A°$ by definition of $A°$

$\therefore \quad \cup \{G : G \text{ is open}, G \subseteq A\} \subseteq A°$ …(2)

Thus from (1) and (2), we get

$$A° = \cup \{G : G \text{ is open}, G \subseteq A\}$$

***Theorem 14:*** Let X be a topological space and let A be a subset of X. Then int (A) is an open set.

***Proof:*** Let x be any arbitrary point of int (A). Then x is an interior point of A.

This implies that A is a nhd. of x i.e., $\exists$ an open G such that $x \in G \subset A$.

Since G contains a nhd of each of its points, it follows that A is a nhd of each of the point of G.

Thus, each point of G is a interior point of A.

Therefore, $x \in G \subset \text{int }(A)$.

Thus, it is shown that to each $x \in A°$, these exists an open set G such that $x \in G \subset \text{int }(a)$.

Hence $A°$ is a nhd of each of its point and consequently int (A) is open.

***Theorem 15:*** Let X be a topological space and let $A \subseteq X$. Then $A°$ is the largest open set contained in A.

***Proof:*** Let G be any open subset of A and let x be an arbitrary element of G i.e. $x \in G \subset A$.

Thus A is a nhd of x i.e., x is an interior point of A.

Hence $\quad x \in A°$

$\therefore \quad x \in G \Rightarrow x \in A°$.

Thus $G \subset A° \subset A$.

Hence $A°$ contains every open subset of A and it is, therefore, the largest open subset of A.

***Theorem 16:*** Let X be a topological space and let $A \subseteq X$. Then A is open iff $A° = A$.

***Proof:*** Let A be a T-open set.

Since every T-open set is a T-nhd of each of its point, therefore every point of A is a T-interior point of A. Consequently $A \subset A^\circ$,

Again, since each T-interior point of A belongs to A therefore $A^\circ \subset A$.

Hence, $A = A^\circ$

Consequently, if $A = A^\circ$, then A must be a T-open set for $A^\circ$ is a T-open set.

### 1.7.3 Properties of Interior

***Theorem 17:*** Let (X, T) be a topological space and $A, B \subset X$. Then

(i) $\phi^\circ = \phi$

(ii) $X^\circ = X$

(iii) $A \subset B \Rightarrow A^\circ \subset B^\circ$

(iv) $(A^\circ)^\circ = A^\circ$ or $A^{\circ\circ} = A^\circ$.

***Proof:*** Let (X, T) be a topological space and $A, B \subset X$.

(i) & (ii), By definition of T, $\phi, X \in T$, consequently.

$$\phi^\circ = \phi, \quad X^\circ = X$$

For A is open $\Leftrightarrow A^\circ = A$.

(iii) Suppose $A \subset B$

any $x \in A^\circ \Rightarrow x$ is an interior point of A.

$\Rightarrow \exists$ open set G s.t. $x \in G \subset A$

$\Rightarrow x \in G \subset A \subset B \Rightarrow x \in G \subset B$ & G is open.

$\Rightarrow x \in B^\circ$

$\therefore \quad A^\circ \in B^\circ$.

(iv) We Know that $A^\circ$ is open

Also G is open $\Leftrightarrow G^\circ = G$ …(1)

In view of this, we get

$$(A^\circ)^\circ = A^\circ \quad \text{or} \quad A^{\circ\circ} = A^\circ \qquad \text{(on putting } G = A^\circ \text{ in (1))}$$

***Theorem 18:*** Let i be an interior operator defined on a set X. Then these exists a unique topology T on X s.t. for each $A \subset X$.

$$i(A) = \text{T-interior of } A.$$

***Proof:*** Let i be an interior operator on X. Then a map

$$i : P(X) \to P(X) \text{ s.t.}$$

(i) $i(X) = X$

(ii) $i(A) \subset A$

(iii) $i(A \cap B) = i(A) \cap i(B)$

(iv) $i[i(A)] = i(A)$, where $A, B \subset X$

P(X) being power set of X.

Notes

To prove that $\exists$ unique topology T on X s.t. $i(A) = A°$, where $A°$ = T-interior of A.

Write $T = \{A \subset X : i(A) = A\}$

(1) $X \in T$, for $i(X) = X$

(2) To prove $\phi \in T$

$i(\phi) \subset \phi$, by (ii)

But $\phi \subset i(\phi)$

$i(\phi) \subset \phi$ So that $\phi \in T$

(3) $G_1, G_2, \in T \Rightarrow G_1 \cap G_2 \in T$

For $G_1, G_2, \in T \Rightarrow i(G_1) = G_1, i(G_2) = G_2$

$\Rightarrow i(G_1 \cap G_2) = i(G_1) \cap i(G_2)$ by (iii)

$= G_1 \cap G_2$

$\Rightarrow i(G_1 \cap G_2) = G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2 \in T$

(4) To prove $G_\alpha \in T \ \forall \ \alpha \in \Delta \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T$

Firstly we shall prove that

$A \subset B \Rightarrow i(A) \subset i(B)$,

where $A, B, \subset X$ …(1)

$A \subset B \Rightarrow A \cap B = A$

$\Rightarrow i(A) = i(A \cap B)$

$= i(A) \cap i(B)$, by (iii)

$\subset I (B)$

$\Rightarrow i(A) \subset i(B)$. Hence the result (1).

Let $G_\alpha \in T \ \forall \ \alpha \in \Delta$ so that

$i(G_\alpha) = G_\alpha$ …(2)

Also let $\cup \{G_\alpha : \alpha \in \Delta\} = G$.

Then $G_\alpha \subset G \Rightarrow i(G_\alpha) \subset i(G)$, by (1)

$\Rightarrow G_\alpha \subset i(G)$, by (2)

$\Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \subset i(G)$

$\Rightarrow G \subset i(G)$

But $i(G) \subset G$, by (ii).

Consequently $i(G) = G$ so that $G \in T$. Hence the result (4). From (1), (2), (3) and (4), it follows that T is a topology on X.

Remains to prove that

$i(A) = A°$.

By (iv), $i[i(A)] = i(A)$

By construction of T, $\Rightarrow$ i(A) $\in$ T.

Thus, i(A) is T-open set s.t. i(A) $\subset$ A.

Let B be an open set s.t. B $\subset$ A.

$$B \in T, B \subset A \Rightarrow i(B) = (B), i(B) \subset i(A)$$

$$\Rightarrow B \subset i(A)$$

Thus i(A) contains any open set B s.t. B $\subset$ A. It follows that i(A) is the largest open subset of A. Consequently i(A) = $A^\circ$.

### 1.7.4 Properties of Exterior

***Theorem 19:*** Let (X, T) be a topological space and A, B $\subset$ X. Then

(i) $\text{ext}(X) = \phi$

(ii) $\text{ext}(\phi) = X$

(iii) $\text{ext}(A) \subset A'$

(iv) $\text{ext}(A) = \text{ext}[(\text{ext}(A))']$

(v) $A \subset B \Rightarrow \text{ext}(B) \subset \text{ext}(A)$

(vi) $A^\circ \subset \text{ext}[\text{ext}(A)]$

(vii) $\text{ext}(A \cup B) = \text{ext}(A) \cap \text{ext}(B)$.

***Proof:***

(i) $\text{ext}(X) = (X - X)^\circ = \phi$ as we know that $\text{ext}(A) = (X - A)^\circ$

(ii) $\text{ext}(\phi) = (X - \phi)^\circ = X^\circ = X$

(iii) $\text{ext}(A) = (X - A)^\circ \subset X - A = A'$ or $\text{ext}(A) \subset A'$ for $B^\circ \subset B$

(iv) $[\text{ext}(A)]' = [(X - A)^\circ]' = X - (X - A)^\circ$

or $\quad \text{ext}[\{\text{ext}(A)\}'] = \text{ext}[X - (X - A)^\circ]$

$$= [X - \{X - (X - A)^\circ\}]^\circ$$

$$= [(X - A)^\circ]^\circ = (X - A)^{\circ\circ}$$

$$= (X - A)^\circ \qquad [\text{As } B^{\circ\circ} = B^\circ \ \forall\ B]$$

$$= \text{ext}(A)$$

$\Rightarrow \quad \text{ext}(A) = \text{ext}[(\text{ext}(A))']$

(v) $A \subset B \Rightarrow X - B \subset X - A$

$$\Rightarrow (X - B)^\circ \subset (X - A)^\circ$$

$$\Rightarrow \text{ext}(B) \subset \text{ext}(A)$$

(vi) $\text{ext}(A) = (X - A)^\circ \subset X - A$

$$\Rightarrow \text{ext}(A) \subset X - A$$

As $A \subset B \Rightarrow \text{ext}(B) \subset \text{ext}(A)$, we get

$$\text{ext}(X - A) \subset \text{ext}[\text{ext}(A)] \qquad \ldots(1)$$

Notes

But ext (X – A) = ext (A′) = (X – A′)° = [X – (X – A)]°

$= A°$

Now (1) becomes $A° \subset \text{ext}\,[\text{ext}\,(A)]$

(vii) $\text{ext}\,(A \cup B) = [X - (A \cup B)]° = [(X - A) \cap (X - B)]°$

$= (A' \cap B')°$

$= (A')° \cap (B')°$

$= \text{ext}\,(A) \cap \text{ext}\,(B).$

***Theorem 20: Exterior Operator:*** The exterior, by definition of interior function 'e' on X is a function

$$e : P(X) \to P(X) \quad \text{s.t.}$$

(i) $e\,(X) = \phi$

(ii) $e\,(\phi) = X$

(iii) $e\,(A) \subset A'$

(iv) $e\,(A) = e\,[(e\,(A))']$

(v) $e\,(A \cup B) = e\,(A) \cap e\,(B)$

For any sets A, B ⊂ X. Then there exists a unique topology T on X s.t. e (A) = T-exterior of A.

***Proof:*** Write $T = \{G \subset X : e\,(G') = G\}$

We are to show that T is a topology on X.

(i) $e\,(\phi') = e\,(X) = \phi$ by (i)

$e\,(X') = e\,(\phi) = X$ by (ii)

Now $e\,(\phi') = \phi, \quad e\,(X') = X \quad \Rightarrow \quad \phi, X \in T$

(ii) Let $G_1, G_2 \in T$

Then $e\,(G_1') = G_1, \quad e\,(G_2') = G_2$

But $(G_1 \cap G_2)' = G_1' \cup G_2'$

$e\,[(G_1 \cap G_2)'] = e\,(G_1' \cup G_2')$

$= e\,(G_1') \cap e\,(G_2')$ by (v)

$= G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2 \in T$

(iii) Firstly, we shall show that

$$A \subset B \Rightarrow e\,(B) \subset e\,(A) \quad \ldots(1)$$

$A \subset B \Rightarrow A \cup B = B \Rightarrow e\,(B) = e\,(A \cup B)$

$= e\,(A) \cap e\,(B) \subset e\,(A)$

$\Rightarrow \quad e\,(B) \subset e\,(A)$

Let $G_\alpha \in T\ \forall\ \alpha \in \Delta$

Then $e\,(G_\alpha') = G_\alpha \quad \ldots(2)$

**Notes**

Let $G = \cup \{G_\alpha : \alpha \in \Delta\}$

Then $G' = \cap \{G'_\alpha : \alpha \in \Delta\}$, (By De Morgan's law)

By (iii), $e(G') \subset G'' = G$ or $e(G') \subset G$ …(3)

$G_a \subset G \Rightarrow G' \subset G'_a \Rightarrow e(G'_a) \subset e(G')$ by (1)

$\Rightarrow G_\alpha \subset e(G')$ by (2)

$\Rightarrow \cup G_\alpha \subset e(G')$

$\Rightarrow G \subset e(G')$ …(4)

From (3) & (4),

$e(G') = G$ so that $G \in T$

So, $G_\alpha \in T \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T$

This shows that T is a topology on X.

It remains to prove that

$e(A) =$ T-exterior of A.

By (iv), $e(A) = e[(e(A))']$

$\Rightarrow e(A) \in T$ [By (iii)],

$e(A) \subset A'$

Thus, e (A) is an open set contained in A′.

Also, e (A) is the largest open set contained in A′.

$\therefore$ T-interior of $A' = e(A)$

or T-exterior of $A = e(A)$

### Self Assessment

14. Let X = {a, b, c} and let T = {ϕ, X, {b}, {a, c}}, find the interior of the set {a, b}.
15. If T = {ϕ, {a}, {a, b}, {a, c, d}, {a, b, e}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then find the interior points of the subset A = {a, b, c} on X.

## 1.8 Summary

- Topology deals with the study of those properties of certain objects that remain invariant by stretching or bending.
- Let X be any non-empty set and T be the collection of all subsets of X. Then T is called discrete topology.
- Let X be any non-empty set and T = {X, ϕ}, then T is called indiscrete topology.
- Let T be a collection of subset of X where complements are finite along with ϕ, forms a topology on X is called cofinite topology.
- Let (X, T) be a topological space. Any set A ∈ T is called an open set and X – A is called closed set.
- Closure of a set is the intersection of all closed sets containing A where A is subset of X.

- Let (X, T) be a topological space. $A \subset X$ is called a neighbourhood of a point $x \in X$ if $\exists\, G \in T$ with $x \in G$ such that $G \subset A$.
- Let (X, T) be a topological space. $A \subset X$ is said to be dense or everywhere dense in X if $\bar{A} = X$.
- If A said to be nowhere dense set in X if int $(\bar{A}) = \phi$.
- Let {X, T} be a topological space and $A \subset X$ then X is said to be separable if

  (i) $\bar{A} = X$ (ii) A is countable
- Let (X, T) be a topological space and $A \subset X$.

  A point $x \in X$ is said to be the limit point if each open set containing x contains at least one point of A different form x.
- The set of all limit point of A is called derived set of A.
- Let (X, T) be a topological space and $A \subset X$. A point $x \in A$ is called a interior point of A iff there exists an open set G such that $x \in G \subseteq A$. It is denoted by Int (A) or A°.
- A point $x \in X$ is called an exterior point of A iff it is an interior point of $A^C$ or X – A. It is denoted by ext (A).

## 1.9 Keywords

***Complement:*** The complement of a set A w.r.t. the universal set X is defined as the set X-A and is denoted by A°.

symbolically, $A° = X - A = \{x \in X : x \notin A\}$.

***Intersection:*** The intersection of two sets A and B, denoted by $A \cap B$, is

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

***Subset:*** If every element of set A is also an element of set B, then A is called a subset of B. It is denoted by the symbol $A \subset B$.

***Superset:*** $A \subset B$ is also expressed by writing, $B \supset A$.

***Union:*** The union of two sets A and B, denoted by $A \cup B$, is

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

## 1.10 Review Questions

1. Let X = {a, b, c, d, e, f}, which of the following collections of subsets of X is a topology on X? (Justify your answers).

   (a) $T_1 = \{X, \phi, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$;

   (b) $T_2 = \{X, \phi, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$;

   (c) $T_3 = \{X, \phi, \{b\}, \{a, b, c\}, \{d, c, f\}, \{b, d, e, f\}\}$.

2. If X = {a, b, c, d, e, f} and T is the discrete topology on X, which of the following statements are true?

   (a) $X \in T$ (b) $\{X\} \in T$

   (c) $\{\phi\} \in T$ (d) $\phi \in T$

   (e) $\{\phi\} \in X$ (f) $a \in T$

**Notes**

3. Let (X, T) be any topological space. Verify that the intersection of any finite number of member of T is a member of T.

4. List all possible topologies on the following sets:

   (a) X = {a, b}; (b) Y = {a, b, c}

5. Let X be an infinite set and T a topology on X. If every infinite subset of X is in T, prove that T is the discrete topology.

6. Let (X, T) be a topological space with the property that every subset is closed. Prove that it is a discrete space.

7. Consider the topological space (X, T) where the set X = {a, b, c, d, e}, the topology T = {X, $\phi$, {a}, {c, d}, {a, c, d}, {b, c, d, e}}, and A = {a, b, c}. Then b, d, and e are limit points of A but a and c are not limit points of A.

8. Let X = {a, b, c, d, e} and T = {X, $\phi$, {a}, {c, d}, {a, c, d}, {b, c, d, e}} show that $\{\overline{b}\}$ = {b, c}, $\{\overline{a, c}\}$ = X, and $\{\overline{b, d}\}$ = {b, c, d, e}.

9. Let X = {a, b, c, d, e, f} and

   $T_1$ = {X, $\phi$, {a}, {c, d}, {a, c, d}, {b, c, d, e, f}},

   (a) Find all the limit points of the following set:

      (i) {a},

      (ii) {b, c},

      (iii) {a, c, d},

      (iv) {b, d, e, f},

   (b) Hence, find the closure of each of the above sets.

10. (a) Let A and B be subsets of a topological space (X, T). Prove carefully that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

    (b) Give an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

11. Let S be a dense subset of a topological space (X, T). Prove that for every open subset $\cup$ of X, $\overline{S \cap \cup} = \overline{U}$.

12. Let E be a non-empty subset of a topological space (X, T). Show that $\overline{E} = E \cup d(E)$, where d (E) is derived set of E.

13. Define interior operator. Explain how can this operator be used to define a topology on a set X.

14. Prove that A subset of topological space is open iff it is nhd of each of its points.

15. (a) Show that A° is the largest open set contained in A.

    (b) Show that the set of all cluster points of set in a topological space is closed.

16. The union of two topologies for a set X is not necessarily a topology for X. Prove it.

17. Let X be a topological space. Let $A \subseteq X$. Then prove that $A \cup A'$ is closed set.

18. Show that $A \cup D(A)$ is a closed set. Also show that $A \cup D(A)$ is the smallest closed subset of X containing A.

19. In a topological space, prove that $(X - A)^\circ = X - \overline{A}$. Int $A' = (\overline{A})'$. Hence deduce, that $A^\circ = (\overline{A'})'$.

Notes

20. Let (X, T) be a topological space and $A \subset X$. A point x of A is an interior point of A iff it is not a limit point of X – A.

21. Let T = {X, $\phi$, {p}, {p, q}, {p, q, t}, {p, q, r, s}, {p, r, s}} be the topology on X = {p, q, r, s, t}. Determine limit points, closure, interior, exterior and boundary of the following sets:

    (a) A = {r, s, t} (b) B = {p}

22. It T = {$\phi$, {a}, {a, b}, {a, c, d}, {a, b, e}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then

    (a) Point out T-open subsets of X.

    (b) Point out T-closed subsets of X.

    (c) Find the closure of the sets {a}, {b}, {c}.

    (d) Find the interior points of the subset A = {a, b, c} on X.

    (e) Which of the sets {a}, {b}, {c, e} are dense in X?

### Answers: Self Assessment

1. $T_1$ = {$\phi$, X}, $T_2$ = {$\phi$, X, {b}}, {a, b}, $T_3$ = {$\phi$, X, {a}, {b}, {a, b}}.
2. Yes
8. nhd of *r* are {p, r, s}, {p, q, r, s}

   nhd of t is {p, q, t}
12. D(A) = {c}
13. D ({b}) = D ({a, b}) = D({b, c}) = D({c, a}) = {c}

## 1.11 Further Readings

*Books*

J. L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison–Wesley, Mass. 1970.

# Unit 2: Basis for Topology

**Notes**

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Define the term basis for topology;
- Solve the questions related to basis for topology;
- Describe the sub-base and related theorems;
- State the standard topology.

## Introduction

In mathematics, a base or basis $\mathcal{B}$ for a topological space X with topology T is a collection of open sets in T such that every open set in T can be written as a union of elements of $\mathcal{B}$. We say that the base generates the topology T. Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology.

In this unit, we shall study about basis, sub-base, standard topology and lower limit topology.

## 2.1 Basis for a Topology

***Definition:*** Basis

A collection of subsets $\mathcal{B}$ of X is called a basis or a base for a topology if:

1. The union of the elements of B is X.
2. If $x \in B_1 \cap B_2$, $B_1$, $B_2$, $\in \mathcal{B}$, then there exists a B of $\mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Notes

***Another Definition:***

$\mathcal{B}$ is said to be a base for the topology T on X if $x \in G \in T \Rightarrow \exists\, B \in \mathcal{B}$ s.t. $x \in B \subset G$.

The elements of $\mathcal{B}$ are referred to as basic open sets.

*Example 1:*

(1) $\mathcal{S}$, the standard topology on $\mathcal{R}$, is generated by the basis of open intervals (a,b) where a < b.

(2) A basis for another topology on $\mathcal{R}$ is given by half open intervals [a,b), a < b. It generated the lower limit topology $\mathcal{L}$.

(3) The Open intervals (a,b), a < b with a & b rational is a *countable* basis. It generates the same topology as $\mathcal{S}$.

*Example 2:* Let X = {1, 2, 3, 4}. Let A = {{1, 2}, {2, 4}, {3}}. Determine the topology on X generated by the elements of A and hence determine the base for this topology.

*Solution:*

Let X = {1, 2, 3, 4} and

A = {{1, 2}, {2, 4}, {3}}.

Finite intersections of the members of A form the class $\mathcal{B}$ given by

$\mathcal{B}$ = {{1, 2}, {3}, {2, 4}, $\phi$, {2}, X}.

The unions of the members of $\mathcal{B}$ form the class T given by

T = {{1, 2}, {3}, {2, 4}, $\phi$, {2}, X, {1, 2, 3}, {1, 2, 4}, {3, 2, 4}, {3, 2}}.

It can be easily verified that $\mathcal{B}$ is a base for the topology T on X.

### 2.1.1 Topology Generated by Basis

***Lemma 1:*** Let $\mathcal{B}$ be a basis for a topology T on a set X. Then T equals the collection of all unions of elements of $\mathcal{B}$.

***Proof:*** Each element of $\mathcal{B}$ is open, so arbitrary unions of elements in $\mathcal{B}$ are open i.e., in T. We must show any $\cup \in T$ equals a union of basis elements. For each $x \in \cup$, choose a set $\mathcal{B}_x \subset \cup$ that contains x.

What does the union $\cup_x\ \mathcal{B}_x$ of these basis elements equal? All of $\cup$ i.e. $\cup_{is}$ a union of basis elements. How to find a basis for your topology.

***Lemma 2:*** Let (X, T) be a topological space. Suppose $\mathcal{B}$ is a collection of open sets of X s.t. $\forall$ open sets $\cup$ and $\forall\, x \in \cup$, there exists an element $B \in \mathcal{B}$ s.t. $x \in B \subset \cup$. Then $\mathcal{B}$ is a basis for T.

***Proof:*** We show the two basis conditions:

1. Since X itself is open in the topology, our hypothesis tells us that $\forall\, x \in X$, there exists $B \in \mathcal{B}$ containing x.

2. Let $x \in B_1 \cap B_2$. Since $B_1$, $B_2$ are open, so is $B_1 \cap B_2$; by our hypothesis, there exists $B \in \mathcal{B}$ containing x with $B \subset B_1 \cap B_2$.

So, $\mathcal{B}$ is a basis and generates a topology T′; we must show T′ = T.

Take $\cup \in T$; by hypothesis, there is a set $B \in \mathcal{B}$ with $x \in B \subset \cup$; this is the definition of $\cup$ being an open set in topology T′.

Conversely, take V open in topology T′. Then by the previous lemma, V equals a union of elements of sets in $\mathcal{B}$.

By hypothesis, each set in B is open in topology T; thus V is a union of open sets from T, so it is open in T.

***Lemma 3:*** Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies T and T′, respectively, on X. Then the following are equivalent:

1. T′ is finer than T.
2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

***Proof:*** (2) $\Rightarrow$ (1)

Given any element $\cup$ of T,

We are to show that $\cup \in T'$.

Let $x \in \cup$.

Since $\mathcal{B}$ generates T, there is an element $B \in \mathcal{B}$ such that $x \in B \subset \cup$.

Condition (2) tells us $\exists$ an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then

$x \in B', \subset \cup,$

so, $\cup \in T'$, by definition

(1) $\Rightarrow$ (2)

Given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$.

Now B belongs to T by definition

and $T \subset T'$ by condition (1)

$\therefore$ $B \in T'$.

Since T′ is generated by $\mathcal{B}'$,

there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

### 2.1.2 A Characterisation of a Base for a Topology

***Theorem 1:*** Let (X, T) be a topological space. A sub-collection $\mathcal{B}$ of T is a base for T iff every T-open set can be expressed as union of members of $\mathcal{B}$.

or

If T be a topology on X and $\mathcal{B} \subset T$, show that following conditions are equivalent:

(i) Each $G \in T$ is the union of members of $\mathcal{B}$.

(ii) For any x belonging to an open set G, $\exists\ B \in \mathcal{B}$ with $x \in B \subset G$.

***Proof:*** Let $\mathcal{B}$ be a base for the topological space (X, T) so that $x \in G \in T$.

$\Rightarrow$ $\quad \exists\ B \in \mathcal{B}$ s.t. $x \in B \in \mathcal{B}$ s.t. $x \in B \subset G$ …(1)

To show: $G = \cup \{B : B \in \mathcal{B} \text{ and } B \subset G\}$ …(2)

From (1), the statement (2) at once follows.

Conversely, suppose that $\mathcal{B} \in T$ s.t. (2) holds.

Also, suppose that (X, T) is a topological space.

To prove: statement (1).

Let $x \in X$ be arbitrary and G be an open set s.t. $x \in G$.

Then $x \in G \in T$.

Now (2) suggests that

$$\exists\, B \in \mathcal{B} \text{ s.t. } x \in B \subset G.$$

Hence the result (1).

### Self Assessment

1. Let X = {a, b, c, d} and A = {{a, b}, {b, c}, {d}}. Determine a base $\mathcal{B}$ (generated by A) for a unique topology T on X.
2. Let $\mathcal{B}$ be a base for the topology T on X. Let $\mathcal{B}^* \subset T$ s.t. $\mathcal{B} \subset \mathcal{B}^*$. Show that $\mathcal{B}^*$ is a base for the topology T on X.
3. What is necessary and sufficient condition for a family to become a base for a topology?
4. Let B be a base for X and let Y be a subspace of X. Then if we intersect each element of B with Y, the resulting collection of sets is a base for the subspace Y. Prove it.

## 2.2 Sub-base

***Definition:*** Let (X, T) be a topological space. Let $\mathcal{S} \subset T$ s.t. $S \neq \phi$.

$\mathcal{S}$ is said to be sub, base or open sub-base or semi bases for the topology T on X if finite intersections of the members of $\mathcal{S}$ form a base for the topology T on X i.e. the unions of the members of $\mathcal{S}$ give all the members of T. The elements of $\mathcal{S}$ are referred to as sub-basic open sets.

*Example 3:* Let $a, b \in \mathcal{R}$ be arbitrary s.t. $a < b$. Clearly $(-\infty, b) \cap (a, \infty) = (a, b)$

The open intervals (a, b) form a base for the usual topology on $\mathcal{R}$. Hence, by definition, the family of infinite open intervals forms a sub-base for the usual topology on $\mathcal{R}$.

***Theorem 2:*** Let $\mathcal{S}$ be a non-empty collection of subsets of a non empty set X. Then $\mathcal{S}$ is a sub-base for a unique topology T for X, i.e., finite intersections of members of $\mathcal{S}$ form a base for T.

***Proof:*** Let $\mathcal{B}$ be the collection of all finite intersections of members of $\mathcal{S}$. Then we have to show that $\mathcal{B}$ is a base for a unique topology on X.

For this, we have to show that $\mathcal{B}$ satisfies conditions (1) and (2).

(1) Since X is the intersection of empty collection of members of $\mathcal{S}$, it follows that

$X \in \mathcal{B}$ and so $X = \bigcup \{B : B \in \mathcal{B}\}$.

(2) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then $B_1, B_2$ are finite intersections of members of $\mathcal{S}$. Hence, $B_1 \cap B_2$ is also a finite intersection of members of S and so $B_1 \cap B_2 \in \mathcal{B}$.

Hence, $\mathcal{B}$ is a base for a unique topology on X for which $\mathcal{S}$ is sub-base.

*Example 4:* Find out a sub-base $\mathcal{S}$ for the discrete topology T on X = {a, b, c} s.t. $\mathcal{S}$ does not contain any singleton set.

*Solution:* Let X = {a, b, c}. Let T be the discrete topology on X.

If we write $\mathcal{B} = \{\{x\} : x \in X\}$, then by the theorem:

"Let X be an arbitrary set and $\mathcal{B}$ a non empty subset of the power set P(X) of X. $\mathcal{B}$ is a base for some topology on X iff

(i) $\bigcup \{B : B \in \mathcal{B}\} = X$

(ii) $x \in B_1, B_2$ and $B_1, B_2, \in \mathcal{B} \Rightarrow \exists\, B \in \mathcal{B}$ s.t. $x \in B \subset B_1 \cap B_2$.

$\mathcal{B}$ is a base for the topology T on X."

Any family B* of subsets of X. $\mathcal{S}$ does not contain any singleton set. Hence, $\mathcal{S}$ is the required sub-base.

## Self Assessment

5. Let $\mathcal{S}$ be a sub-base for the topologies T and $T_1$ on X. Show that $T = T_1$.
6. Let (Y, $\cup$) be a sub-base of (X, T) and $\mathcal{S}$ a sub-base for T on X. Show that the family $\{Y \cap S : S \in \mathcal{S}\}$ is a sub-base for $\cup$ on Y.
7. Given a non empty family $\mathcal{S}$ of subsets of a set X, show that $\exists$ weakest topology T on X in which all the members of $\mathcal{S}$ are open sets and $\mathcal{S}$ is a sub-base for T.
8. Let X = {a, b, c, d, e}. Find a sub-base $\mathcal{S}$ for the discrete topology T on X which does not contain any singleton set.

## 2.3 Standard Topology and Lower Limit Topology

### 2.3.1 Standard Topology

If $\mathcal{B} = \{(a, b) : a, b \in R \text{ s.t. } a < b\}$ i.e. $\mathcal{B}$ is a collection of open intervals on real line, the topology generated by $\mathcal{B}$ is called standard topology on $\mathcal{R}$.

### 2.3.2 Lower Limit Topology

If $\mathcal{B}_1 = \{(a, b] : a, b \in R \text{ and } a < b\}$ i.e. $\mathcal{B}_1$ is a collection of semi-open intervals, the topology generated by $\mathcal{B}_1$ is called lower limit topology on $\mathcal{R}$.

When $\mathcal{R}$ is given the lower limit topology, we denote it by $\mathcal{R}_\ell$.

Finally let K denote the set of all numbers of the form $\frac{1}{n}$, for $n \in Z_+$ and let $\mathcal{B}_2$ be the collection of all open intervals (a, b) along with all sets of the form (a, b) –K. The topology generated by $\mathcal{B}_2$ will be called the K-topology on $\mathcal{R}$. When $\mathcal{R}$ is given this topology, we denote it by $\mathcal{R}_K$.

***Lemma:*** The topologies of $\mathcal{R}_l$ and $\mathcal{R}_k$ are strictly finer than the standard topology on $\mathcal{R}$, but are not comparable with one another.

***Proof:*** Let T, T′ and T″ be the topologies or $\mathcal{R}_1$ , $\mathcal{R}_\ell$ and $\mathcal{R}_K$, respectively. Given a basis elements (a, b) for T and a point x of (a, b), the basis element [x, b) for T′ contains x and lies in (a, b). On the other hand, given the basis element [x, b) for T′, there is no open interval (a, b) that contains x and lies in [x, d). Thus T′ is strictly finer than T.

Notes

A similar argument applies to $\mathcal{R}_K$. Given a basis element (a, b) for T and a point x of (a, b), this same interval is a basis element for T″ that contains x. On the other hand, given the basis element B = (–1, 1) –K for T″ and the point O of B, there is no open interval that contains O and lies in B.

Now, it can be easily shown that the topologies of $\mathcal{R}_\ell$ and $\mathcal{R}_K$ are not comparable.

### Self Assessment

9. Consider the following topologies on $\mathcal{R}$:

   $T_1$ = the standard topology,

   $T_2$ = the topology of $\mathcal{R}_K$,

   $T_3$ = the finite complement topology,

   $T_4$ = the upper limit topology, having all sets (a, b) as basis,

   $T_5$ = the topology having all sets $(-\infty, a) = \{x : x < a\}$ as basis

   Determine, for each of these topologies, which of the others it contains.

## 2.4 Summary

- A base (or basis) $\mathcal{B}$ for a topological space X with topology $\top$ is a collection of open sets in T such that every open set in T can be written as a union of elements of $\mathcal{B}$.
- ***Sub-base:*** Let X be any set and $\mathcal{S}$ a collection of subsets of X. Then $\mathcal{S}$ is a sub-base if a base of X can be formed by a finite intersection of elements of $\mathcal{S}$.
- ***Standard Topology:*** If $\mathcal{B}$ is the collection of all open intervals in the real line $(a, b) = \{x : a < x < b\}$, the topology generated by $\mathcal{B}$ is called standard topology on the real line.
- ***Lower Limit Topology:*** If $\mathcal{B}'$ is the collection of all half-open intervals of the form

  $[a, b) = \{X : a \leq x < b\}$,

  where a< b, the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathcal{R}$.

## 2.5 Keywords

***Finer:*** If $T_1 \subset T_2$, then we say that $T_2$ is longer or finer than $T_1$.

***Subset:*** If A and B are sets and every element of A is also an element of B then, A is subset of B denoted by $A \subseteq B$.

***Topological Space:*** It is a set X together with T, a collection of subsets of X, satisfying the following axioms.

(1) The empty set and X are in T.

(2) T is closed under arbitrary union.

(3) T is closed under finite intersection.

## 2.6 Review Questions

1. Let $\mathcal{B}$ be a basis for a topology on a non empty set X. It $\mathcal{B}_1$ is a collection of subsets of X such that $T \supseteq \mathcal{B}_1 \supset \mathcal{B}$, prove that $\mathcal{B}_1$ is also a basis for T.

Notes

2. Show that the collection $\mathcal{B} = \{(a, b) : a, b \in \mathcal{R}, a < b\}$ of all open intervals in $\mathcal{R}$ is a base for a topology on $\mathcal{R}$.
3. Show that the collection $\mathcal{C} = \{[a, b] : a, b \in \mathcal{R}, a < b\}$ of all closed intervals in $\mathcal{R}$ is not a base for a topology on $\mathcal{R}$.
4. Show that the collection $\mathcal{L} = \{(a, b] : a, b \in \mathcal{R}, a < b\}$ of half-open intervals is a base for a topology on $\mathcal{R}$.
5. Show that the collection $\mathcal{S} = \{[a, b) : a, b \in \mathcal{R}, a < b\}$ of half-open intervals is a base for a topology on $\mathcal{R}$.
6. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a sub-basis.
7. If $\mathcal{S}$ is a sub-base for the topology T on X, then $\mathcal{S} = \{X, \phi\}$ is also a sub-base for T on X.

## Answers: Self Assessment

1. $\mathcal{B} = \{\{a, b\}, \{b, c\}, \{d\}, \{b\}, \phi, X\}$

   $T = \{\mathcal{B}, \{a, b, d\}, \{b, c, d\}, \{b, d\}, \{a, b, c\}\}$.

8. $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, a\}\}$.

## 2.7 Further Readings

*Books*

Engelking, Ryszard (1977), *General Topology*, PWN, Warsaw.

Willard, Stephen (1970), *General Topology*, Addison-Wesley. Reprinted 2004, Dover Publications.

*Richa Nandra, Lovely Professional University*



# Unit 3: The Order Topology

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Understand the order topology;
- Solve the problems on order topology;
- Describe the open intervals, closed intervals and half-open intervals.

## Introduction

If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the order topology; in this unit, we consider it and study some of its properties.

## 3.1 The Order Topology

### 3.1.1 Intervals

Suppose that X is a set having a simple order relation <. Given elements a and b of X such that a < b, there are four subsets of X that are called the intervals determined by a and b. They are the following:

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$[a, b] = \{x \mid a \leq x \leq b\}$$

The notation used here is familiar to you already in the case where X is the real line, but these are intervals in an arbitrary ordered set.

- A set of the first type is called an ***open interval*** in X.
- A set of the last type is called a ***closed interval*** in X.
- Sets of the second and third types are called ***half-open intervals***.

> *Note* The use of the term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X and so they will.

### 3.1.2 Order Topology

***Definition:*** Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_o, b)$, where $a_o$ is the smallest element (if any) of X.

(3) All intervals of the form $(a_o, b_o]$, where $b_o$ is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the ***order topology***. If X has no smallest element, there are no sets of type (2), and if X has no largest element, there are no sets of type (3).

> *Notes* One has to check that $\mathcal{B}$ satisfies the requirements for a basis.
>
> (A) First, note that every element x of X lies in at least one element of $\mathcal{B}$ : The smallest element (if any) lies in all sets of type (2), the largest element (if any) lies in all sets of type (3), and every other element lies in a set of type (1).
>
> (B) Second, note that the intersection of any two sets of the preceding types is again a set of one of these types, or is empty.

*Example 1:* The standard topology on $\mathcal{R}$ is just the order topology derived from the usual order on $\mathcal{R}$.

*Example 2:* Consider the set $\mathcal{R} \times \mathcal{R}$ in the dictionary order; we shall denote the general element of $\mathcal{R} \times \mathcal{R}$ by $x \times y$, to avoid difficulty with notation. The set $\mathcal{R} \times \mathcal{R}$ has neither a largest nor a smallest element, so the order topology on $\mathcal{R} \times \mathcal{R}$ has as basis the collection of all open intervals of the form $(a \times b, c \times d)$ for $a < c$, and for $a = c$ and $b < d$. The subcollection consisting of only intervals of the second type is also a basis for the order topology on $\mathcal{R} \times \mathcal{R}$, as you can check.

*Example 3:* The positive integers $Z_+$ form an ordered set with a smallest element. The order topology on $Z_+$ is the discrete topology, for every one-point set is open : If $n > 1$, then the one-point set $\{n\} = \{n-1, n+1\}$ is a basis element; and if $n=1$, the one-point set $\{1\} = [1, 2)$ is a basis element.

*Example 4:* The set $X = \{1, 2\} \times Z_+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting $1 \times n$ by $a_n$ and $2 \times n$ by $b_n$, we can represent X by

$$a_1, a_2, \ldots ; \quad b_1, b_2, \ldots$$

The order topology on X is not the discrete topology. Most one-point sets are open, but there is an exception the one-point set $\{b_1\}$. Any open set containing $b_1$ must contain a basis element about $b_1$ (by definition), and any basis element containing $b_1$ contains points of the $a_i$ sequence.

### 3.1.3 Rays

***Definition:*** If X is an ordered set, and a is an element of X, there are four subsets of X that are called the rays determined by a. They are the following:

$$(a, +\infty) = \{x \mid x > a\}$$

$$(-\infty, a) = \{x \mid x < a\},$$

$$[a, +\infty) = \{x \mid x \geq a\},$$

$$(-\infty, a] = \{x \mid x \leq a\}.$$

sets of first two types are called open rays; and sets of the last two types are called closed rays.

*Notes*

(1) The use of the term "open" suggests that open rays in X are open sets in the order topology. And so they are (consider, for example, the ray $(a, +\infty)$. If X has a largest element $b_o$, then $(a, +\infty)$ equals the basis element $(a, b_o]$. If X has no largest element, then $(a, +\infty)$ equals the union of all basis elements of the form $(a, x)$, for $x > a$. In either case, $(a, +\infty)$ is open. A similar argument applies to the ray $(-\infty, a)$.

(2) The open rays, in fact, form a sub-basis for the order topology on X, as we now show. Because the open rays are open in the order topology, the topology they generate is contained in the order topology. On the other hand, every basis element for the order topology equals a finite intersection of open rays; the interval (a, b) equals the intersection of $(-\infty, b)$ and $(a, +\infty)$, while $[a_o, b)$ and $(a, b_o]$, if they exist, are themselves open rays. Hence the topology generated by the open rays contains the order topology.

### 3.1.4 Order Topology on the Linearly Ordered Set

The order topology $T_<$ on the linearly ordered set X is the topology generated by all open rays. A linearly ordered space is a linearly ordered set with the order topology.

### 3.1.5 Lemma (Basis for the Order Topology)

Let (X, <) be a linearly ordered set.

(1) The union of all open rays and all open intervals is a basis for the order topology $T_<$.

(2) If X has no smallest and no largest element, then the set $\{(a, b) \mid a, b \in X, a < b\}$ of all open intervals is a basis for the order topology.

***Proof:*** As we know

$$B_{S<} = \{\text{Finite intersections of S-sets}\}$$

$$= S \cup \{(a, b) \mid a, b \in X, a < b\}$$ is a basis for the topology generated by the sub-basis $S_<$.

If X has a smallest element $a_o$ then $(-\infty, b) = [a_o, b)$ is open. If X has no smallest element, then the open ray $(-\infty, b) = \cup_{a<c} (a, c)$ is a union of open intervals and we do not need this open ray in the basis. Similar remarks apply to the greatest element when it exists.

## 3.2 Summary

- Open interval : $(a, b) = \{x \mid a < x < b\}$

  Closed interval : $[a, b] = \{x \mid a \leq x \leq b\}$

- Half open intervals : $(a, b] = \{x \mid a < x \leq b\}$

  $[a, b) = \{x \mid a \leq x < b\}$

- The order topology $T_<$ on the linearly ordered set X is the topology generated by all open rays. A linearly ordered space is a linearly ordered set with the order topology.

- Open rays : $(a, +\infty) = \{x \mid x > a\}$

  $(-\infty, a) = \{x \mid x < a\}$

- Closed rays : $(-\infty, a] = \{x \mid x \leq a\}$

  $[a, +\infty) = \{x \mid x \geq a\}$

## 3.3 Keywords

***Basis:*** A basis $\mathcal{B}$ for a topological space X with topology T is a collection of open sets in T such that every open set in T can be written as a union of elements of $\mathcal{B}$.

***Discrete Space:*** Let X be any non empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

***Open and Closed Set:*** Any set $A \in T$ is called an open subset of X or simply a open set and X – A is a closed subset of X.

## 3.4 Review Questions

1. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

2. Show that the dictionary order topology on the set R × R is the same as the product topology $R_d \times R$, where $R_d$ denotes R in the discrete topology. Compare this topology with the standard topology on $R^2$.

Notes

## 3.5 Further Readings

*Books*

Baker, *Introduction to Topology* (1991).

Dixmier, *General Topology* (1984).

*Online links*

http://mathforum.org/isaac/problems/bridgesl.html

http://www.britannica.com

mathworld.wolfram.com/ordertopology.html

# Unit 4: The Product Topology on X × Y

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Describe the product topology;
- Solve the problems on product topology;
- Define projection mappings;
- Discuss the problems on projection mappings.

## Introduction

A product space is the Cartesian product of a family of topological space equipped with a natural topology called the product topology. This topology differs from another, perhaps more obvious, topology called the box topology, which can also be given to a product space and which agrees with the product topology when the product is over only finitely many spaces. However the product topology is 'correct' in that it makes the product space a categorical product of its factors, whereas the box topology is too fine, this is the sense in which the product topology is natural.

## 4.1 Product Topology

Given two sets X and Y, their product is the set $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.

For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and more generally $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

If X and Y are topological spaces, we can define a topology on X × Y by saying that a basis consists of the subsets U × V as U ranges over open sets in X and V ranges over open sets in Y.

The criterion for a collection of subsets to be a basis for a topology is satisfied since

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

This is called the product topology on X × Y.

*Example 1:* A basis for the product topology on $\mathbb{R} \times \mathbb{R}$ consists of the open rectangles $(a_1, b_1) \times (a_2, b_2)$. This is also a basis for the usual topology on $\mathbb{R}^2$, so the product topology coincides with the usual topology.

*Example 2:* Take the topology $T = \{\phi, \{a, b\}, \{a\}\}$ on $X = \{a, b\}$.

Then the product topology on $X \times X$ is

$\{\phi, X \times X, \{(a, a)\}, \{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(a, a), (a, b), (b, a)\}\}$ where the last open set in the list is not in the basis.

***Theorem 1:*** If $(X_1, T_1)$ and $(X_2, T_2)$ are any two topological spaces, then the collection

$$\mathcal{B} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$$

is a base for some topology on $X = X_1 \times X_2$.

***Proof:*** Suppose, $(X_1, T_1)$ and $(X_2, T_2)$ be any two topological spaces.

Write $X = X_1 \times X_2$,

$\mathcal{B} = \{U_1 \times U_2 : U_1 \in T_1, U_2 \in T_2\}$.

To show: $\mathcal{B}$ is a base for some topology on X.

(i) To prove: $U \{B : B \in \mathcal{B}\} = X$.

$X_1 \in T_1, X_2 \in T_2 \quad \Rightarrow X_1 \times X_2 \in \mathcal{B}$

$\Rightarrow X \in \mathcal{B}$

$\Rightarrow X = U \{B : B \in \mathcal{B}\}$

(ii) Let $U_1 \times U_2, V_1 \times V_2 \in \mathcal{B}$ and let

$(x_1, x_2) \in (U_1 \times U_2) \bigcap (V_1 \times V_2)$

To prove: $\exists\, W_1 \times W_2 \in \mathcal{B}$ s.t.

$(x_1, x_2) \in W_1 \times W_2 \subset (U_1 \times U_2) \bigcap (V_1 \times V_2)$

$(x_1, x_2) \in (U_1 \times U_2) \bigcap (V_1 \times V_2)$

$\Rightarrow (x_1, x_2) \in U_1 \times U_2$ and $(x_1, x_2) \in V_1 \times V_2$

$\Rightarrow x_1 \in U_1, x_2 \in U_2 ; x_1 \in V_1, x_2 \in V_2$

$\Rightarrow x_1 \in U_1 \bigcap V_1 ; x_2 \in U_2 \bigcap V_2$

$\Rightarrow x_1 \in W_1 ; x_2 \in W_2$

On taking $W_1 = U_1 \bigcap V_1$,

$W_2 = U_2 \bigcap V_2$

$\Rightarrow (x_1, x_2) \in W_1 \times W_2$

$U_1 \times U_2 \in \mathcal{B}, V_1 \times V_2 \in \mathcal{B}$

$\Rightarrow U_1 \in T_1, U_2 \in T_2 ; V_1 \in T_1, V_2 \in T_2$

$\Rightarrow U_1 \bigcap V_1 \in T_1, U_2 \bigcap V_2 \in T_2$

$\Rightarrow W_1 \in T_1, W_2 \in T_2$

$\Rightarrow W_1 \times W_2 \in \mathcal{B}$

So, we have proved that

$\exists\ W_1 \times W_2 \in B$ s.t $(x_1, x_2) \in W_1 \times W_2$

Now, it remains to prove that

$W_1 \times W_2 \subset (U_1 \times U_2) \cap (V_1 \times V_2)$

Let $(y_1, y_2) \in W_1 \times W_2$ be arbitrary.

$(y_1, y_2) \in W_1 \times W_2 \quad \Rightarrow y_1 \in W_1, y_2 \in W_2$

$\Rightarrow y_1 \in U_1 \cap V_1, y_2 \in U_2 \cap V_2$

$\Rightarrow y_1 \in U_1, y_1 \in V_1$ and $y_2 \in U_2, y_2 \in V_2$

$\Rightarrow (y_1, y_2) \in U_1 \times U_2$ and $(y_1, y_2) \in V_1 \times V_2$

$\Rightarrow (y_1, y_2) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

Finally, any $(y_1, y_2) \in W_1 \times W_2$

$\Rightarrow \quad (y_1, y_2) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

This proves that

$W_1 \times W_2 \subset (U_1 \times U_2) \cap (V_1 \times V_2)$

It immediately follows from (i) and (ii) that B is a base for some topology, say, T on X.

***Theorem 2:*** Let $(X_1, T_1)$ and $(X_2, T_2)$ be two topological spaces and let $\mathcal{B}_1, \mathcal{B}_2$ be bases for $T_1$ and $T_2$ respectively.

Let $X = X_1 \times X_2$

Then $\mathcal{B} = \{\mathcal{B}_1 \times \mathcal{B}_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a base for the product topology T on X.

***Proof:*** Let $\mathcal{C} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$.

Then $\mathcal{C}$ is a base for the topology T on X (refer theorem 1)

We are to prove that $\mathcal{B}$ is a base for T on X.

By definition of base,

for $(x_1, x_2) \in G \in T$

$\Rightarrow \exists\ G_1 \times G_2 \in \mathcal{C}$ s.t. $(x_1, x_2) \in G_1 \times G_2 \subset G$ ...(1)

Again $(x_1, x_2) \in G_1 \times G_2 \in C$

$\Rightarrow x_1 \in G_1 \in T_1, \quad x_2 \in G_2 \in T_2.$

Applying definition of base,

$x_1 \in G_1 \in T_1 \Rightarrow \exists\ B_1 \in \mathcal{B}_1$, s.t. $x_1 \in B_1 \subset G_1$ ...(2)

$x_2 \in G_2 \in T_2 \Rightarrow \exists\ B_2 \in \mathcal{B}_2$, s.t. $x_2 \in B_2 \subset G_2$ ...(3)

$B_1 \in \mathcal{B}, B_2 \in \mathcal{B}_2 \Rightarrow B_1 \times B_2 \in \mathcal{B}.$

Now (2) and (3) $\Rightarrow \exists\ B_1 \times B_2 \in B$ s.t.

$(x_1, x_2) \in B_1 \times B_2 \subset G_1 \times G_2 \subset G$

or $\quad (x_1, x_2) \in B_1 \times B_2 \subset G$

Notes

Thus, we have shown that

$(x_1, x_2) \in G \in T$

$\Rightarrow \quad \exists\, B_1 \times B_2 \in \mathcal{B} \qquad \text{s.t.} \qquad (x_1, x_2) \in B_1 \times B_2 \subset G$

By definition,

This proves that $\mathcal{B}$ is base for T on X.

***Remark:*** From the theorems (1) and (2), it is clear that

$\mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$,

$\mathcal{C} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$

both are bases for the same topology T on X.

***Theorem 3:*** Let (X, T) and (Y, $\mathcal{V}$) be any two topological spaces and let L and M be sub-bases for $T_1$ and $\mathcal{V}$ respectively. Then the collection $\mathcal{A}$ of all subsets of the form L × Y and X × M, is a sub-base for the product topology T on X × Y, where $L \in \mathcal{L}$, $M \in \mathcal{M}$.

***Proof:*** Now in order to prove that $\mathcal{A}$ is a sub-base for T on X × Y, we are to prove that: the collection $\mathcal{G}$ of finite intersections of members of $\mathcal{A}$ form a base for T on X × Y.

Since the intersection of empty sub collection of $\mathcal{A}$ is X × Y and so X × Y ∈ G.

Next let $\{L_1 \times Y, L_2 \times Y, ..., L_p \times Y\} \cup \{X \times M_1, X \times M_2, ..., X \times M_q\}$ be a non empty finite sub-collection of $\mathcal{A}$. This intersection of these elements belong to $\mathcal{G}$, by construction of $\mathcal{G}$. This element of $\mathcal{G}$ is

$(L_1 \times Y) \cap (L_2 \times Y) \cap ... (L_p \times Y) \cap (X \times M_1) \cap (X \cap M_2) \cap ... \cap (X \times M_q)$

$= [(L_1 \cap L_2 \cap ... \cap L_p) \times Y] \cap [X \times (M_1 \cap M_2 \cap ... \cap M_q)]$

[For $A \times (B \cap C) = (A \times B) \cap (A \cap C)$]

$= [(L_1 \cap L_2 \cap ... \cap L_p) \cap X] \times [(M_1 \cap M_2 \cap ... \cap M_q) \cap Y]$

[For $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$]

$= (L_1 \cap L_2 \cap ... \cap L_p) \times (M_1 \cap M_2 \cap ... \cap M_q)$

[For $L_n \subset X$ and $M_n \subset Y \;\; \forall\; n$]

$$= \left[\bigcap_{n=1}^{p} L_r\right] \times \left[\bigcap_{r=1}^{q} M_r\right] \qquad ...(1)$$

We suppose that $\mathcal{B}$ is base for $T_1$ on X generated by the elements of $\mathcal{L}$ and $\mathcal{C}$ is a base for $\mathcal{V}$ on Y generated by the elements of $\mathcal{M}$.

As we know that the finite intersections of sub-base form the base for that topology.

In view of the above statements,

$$\bigcap_{r=1}^{p} L_r \in \mathcal{B} \qquad \bigcap_{r=1}^{q} M_r \in \mathcal{C}$$

From (i), it follows that $\mathcal{G}$ is expressible as

$$\mathcal{G} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

Then $\mathcal{G}$ is a base for the product topology T on X × Y. (Refer Theorem 2).

**Notes**

But $\mathcal{G}$ is obtained from the finite intersections of members of $\mathcal{A}$.

It follows that $\mathcal{A}$ is a sub-base for the product topology T on X × Y.

***Theorem 4:*** The product of two second axiom spaces is a second axiom space.

***Proof:*** Let $(X, T_1)$ and $(Y, T_2)$ be two second countable spaces.

Let (X × Y × T) be the product topological space.

To prove that (X × Y × T) is second countable.

Our assumption implies that ∃ countable bases.

$$B_1 = \{B_i : i \in N\} \text{ and } \{C_i : i \in N\}$$

for X and Y respectively. Recall that

$$B = \{G_1 \times G_2; G_1 \in T_1, G_2 \in T_2\}$$

is a base for the topology T on X × Y.

Write

$$C = \{B_i \times C_{j;}\ i, j \in N_1\} = B_1 \times B_2$$

$B_1$ and $B_2$ are countable = $B_1 \times B_2$ are countable

$$\Rightarrow C \text{ is countable}$$

By definition of base B

any $(x, y) \in N \in T \Rightarrow \exists\, G \times H \in B$ s.t. $(x, y) \in G \times H \subset N$ …(1)

$$\Rightarrow x \in G \in T_1, y \in H \in T_2$$

$$\Rightarrow \exists\, B_i \in B_1, C_j \in B_2 \text{ s.t. } x \in B_i \subset G, y \in C_j \subset H$$

This $\Rightarrow (x, y) \in B_i, \times C_j \subset G \times H \subset N.$

Thus any $(x, y) \in N \in T \Rightarrow \exists\, B_i \times C_j \in C$ s.t. $(x, y) \in B_i \times B_j \subset N$. By definition this proves that C is a base for the topology T on X × Y. Also C has been shown to be countable. Hence (X × Y, T) is second countable.

***Theorem 5:*** The product space of two Hausdorff space is Hausdorff space.

***Proof:*** Let (X, T) be a product topological space of two Hausdorff space $(X_1, T_1)$ and $(X_2, T_2)$.

To prove that (X, T) is Hausdorff space.

Consider a pair of distinct elements $(x_1, x_2)$ and $(y_1, y_2)$ in X.

***Case I.*** When $x_1 = y_1$

then $x_2 \neq y_2$, $\therefore (x_1, x_2) \neq (y_1, y_2)$

By the Hausdorff space property, given a pair of elements

$x_2, y_2 \in X_2$ s.t. $x_2 \neq y_2$, there are disjoint open sets

$$G_2, H_2 \neq X_2 \text{ s.t. } x_2 \in G_2, y_2 \in H_2$$

Then $X_1 \times G_2$ and $X \times H_2$ are disjoint open sets in X. for

$$x_1 \in X_1, x_2 \in G_2 \Rightarrow (x_1, x_2) \in X_1 \times G_2.$$

$$y_1 \in X_1, y_2 \in H_2 \Rightarrow (y_1, y_2) \in X_1 \times H_2.$$

$\therefore$ Given a pair of distinct elements $(x_1, x_2)$, $(y_1, y_2) \in X$ there are disjoint open subsets $X_1 \times G_2$, $X_1 \times H_2$ of X s.t. $(x_1, x_2) \in X_1 \times G_2$, $(y_1, y_2) \in X_1 \times H_2$.

The leads to the conclusion that (X, T) is a Hausdorff space.

*Example 3:* Let $T_1 = \{\phi, \{1\}, X_1\}$ be a topology on $X_1 = \{1, 2, 3\}$ and $T_2 = \{\phi , X_2, \{a\}, \{b\} \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ be a topology for $X_2 = \{a, b, c, d\}$.

Find a base for the product topology T.

*Solution:* Let $B_1$ be a base for $T_1$ and $B_2$ be a base for $T_2$. Then $B = \{B_1 \times B_2 : B_1 \in B_1, B_2 \in B_2\}$ is a base for the product topology T.

We can take $B_1 = \{\{1\}, X_1\}$

$B_2 = \{\{a\}, \{b\}, \{c, d\}\}$.

The elements of B are

$\{1\} \times\{a\}, \{1\} \times\{b\}, \{1\} \times\{c, d\}, \{1, 2, 3\} \times\{a\}, \{1, 2, 3\} \times\{b\}, \{1, 2, 3\} \times\{c, d\}$.

That is to say

$$B = \begin{Bmatrix} \{(1,a)\}, \{(1,b)\}, \{(1,c),(1,d)\} \\ \{(1,a),(2,a),(3,a)\}, \{(1,b),(2,d),(3,b)\} \\ \{(1,c),(2,c),(3,c),(1,d),(2,d),(3,d)\} \end{Bmatrix}$$

is a base for T.

## Self Assessment

1. Let X and X′ denote a single set in the topologies T and T′ respectively let Y and Y′ denote a single set in the topologies U and U′ respectively. Assume these sets are non-empty.

   (a) Show that if $T' \supset T$ and $U' \supset U$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

   (b) Does the converse of (a) hold? Justify your answer.

## 4.2 Projection Mappings

***Definition:***

The mappings,

$\pi_x : X \times Y \to X$ s.t. $\pi_x(x, y) = x \ \forall \ (x, y) \in X \times Y$

$\pi_x : X \times Y \to Y$ s.t. $\pi_y(x, y) = y \ \forall \ (x, y) \in X \times Y$

are called projection maps of X × Y onto X and Y spaces respectively.

***Theorem 6:*** If (X, T) is the product space of topological spaces $(X_1, T_1)$ and $(X_2, T_2)$, then the projection maps $\pi_1$ and $\pi_2$ are continuous and open.

***Proof:*** Let (X, T) be a product topological space of topological spaces $(X_1, T_1)$ and $(X_2, T_2)$. Then $X = X_1 \times X_2$.

Define maps

$\pi_1 : X \to X_1$ s.t. $\pi_1(x_1, x_2) = x_1 \ \forall \ (x_1, x_2) \in X$

$\pi_2 : X \to X_2$ s.t. $\pi_2(x_1, x_2) = x_2 \ \forall \ (x_1, x_2) \in X$.

Notes

Then $\pi_1$ and $\pi_2$ both are called projection maps on the first and second coordinate spaces respectively.

*Step (i):* To prove: projection maps are continuous maps.

Firstly, we shall show that $\pi_1$ is continuous.

Let $G \subset X_1$ be an arbitrary open set.

$$\begin{aligned}\pi_1^{-1}(G) &= \{(x_1, x_2) \in X : \pi_1(x_1, x_2) \in G\} \\ &= \{(x_1, x_2) \in X : x_1 \in G\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 : x_1 \in G\} \\ &= G \times X_2 \\ &= \text{An open set in } X.\end{aligned}$$

For G is open in $X_1$, $X_2$ is open in $X_2$

$\Rightarrow G \times X_2$ is open in X.

$\Rightarrow \pi_1^{-1}(G)$ is open in X.

Thus, we have prove that

any open set $G \subset X_1 \Rightarrow \pi_1^{-1}[G]$ is open in X.

$\Rightarrow \pi_1$ is continuous.

Similarly, we can prove that $\pi_2$ is continuous map. Consequently, projection maps are continuous maps.

*Step (ii):* To prove that projection maps are open maps. We shall first show that $\pi_2$ is an open map.

Let $G \subset X$ be an arbitrary open set.

Let $x_2 \in \pi_2[G]$ be arbitrary.

$x_2 \in \pi_2[G] \Rightarrow \exists (u_1, u_2) \in G$ s.t. $\pi_2(u_1, u_2) = x_2$

$\Rightarrow u_2 = x_2$ $\quad [\because \pi_2(u_1, u_2) = u_2]$

Now $(u_1, x_2) \in G$

Let $\mathcal{B}$ be the base for the topology T on X.

By definition of base,

$(u_1, x_2) \in G \in T \Rightarrow \exists\, U_1 \times U_2 \in \mathcal{B}$ s.t. $(u_1, x_2) \in U_1 \times U_2 \subset G$

$\Rightarrow \pi_2(u_1, x_2) \in \pi_2(U_1 \times U_2) \subset \pi_2(G)$

$\Rightarrow x_2 \in \pi_2(U_1 \times U_2) \subset \pi_2(G)$

$\Rightarrow x_2 \in U_2 \subset \pi_2(G)$.

For $\pi_2(U_1 \times U_2) = \{\pi_2(x_1, x_2) : (x_1, x_2) \in U_1 \times U_2\}$

$= \{x_2 : x_1 \in U_1, x_2 \in U_2\} = U_2$.

$\therefore$ Given any $x_2 \in \pi_2[G] \Rightarrow \exists$ open set $U_2 \subset X_2$ s.t. $x_2 \in U_2 \subset \pi_2[G]$.

This proves that $x_2$ is an interior point of $\pi_2[G]$. But $x_2$ is an arbitrary point of $\pi_2[G]$.

Notes

$\therefore$ Every point of $\pi_2[G]$ is an interior point.

This proves that $\pi_2[G]$ is open in $X_2$.

$\therefore$ Any open set $G \subset X$

$\Rightarrow$ $\pi_2(G)$ is open in $X_2$.

This proves that the map $\pi_2 : X \to X_2$ is an open map. Similarly, we can show that $\pi_1$ is an open map. Consequently, projection maps are open maps.

This completes the proof of the theorem.

***Theorem 7:*** Let (X, T) be the product topological space of $(X_1, T_1)$ and $(X_2, T_2)$.

Let $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$

be the projection maps on the first and second co-ordinate spaces respectively.

Let $f : Y \to X$ be another map, where Y is another topological space. Show that f is continuous iff $\pi_1$ o f and $\pi_2$ o f are continuous maps.

***Proof:*** Let $X = X_1 \times X_2$

Let (X, T) be the product topological space of $(X_1, T_1)$ and $(X_2, T_2)$.

Let (Y, U) be another topological space.

Let $\mathcal{B}$ be the base for the topology T on X.

Let $\pi_1 : X \to X_1$,

$\pi_2 : X \to X_2$ be projection maps.

Let $f : Y \to X$ be another map.

Then $\pi_1 \text{ o } f : Y \to X_1$

$\pi_2 \text{ o } f : Y \to X_2$ are also maps.

Let f be continuous.

To prove that $\pi_1$ o f and $\pi_2$ o f are continuous maps.

By theorem 6, projection maps are continuous, i.e. $\pi_1$ and $\pi_2$ are continuous maps.

Also f is given to be continuous.

This means that $\pi_1$ o f, $\pi_2$ o f are continuous maps. Conversely, suppose that $\pi_1$ o f, $\pi_2$ o f are continuous maps.

To show that f is continuous.

Let $G \subset X$ be an arbitrary open set.

If we prove that $f^{-1}(G)$ is open in Y, the result will follow.

Let $y \in f^{-1}(G)$ be an arbitrary, then $f(y) \in G$.

$\therefore$ f(y) is an element of $X = X_1 \times X_2$ and hence it can be taken as $f(y) = (x_1, x_2) \in G$

By definition of base,

$(x_1, x_2) \in G \in T \Rightarrow \exists\, U_1 \times U_2 \in \mathcal{B}$ s.t. $(x_1, x_2) \in U_1 \times U_2 \subset G$

$\Rightarrow$ $\pi_1(x_1, x_2) \in \pi_1(U_1 \times U_2) \subset \pi_1(G)$ and

$\pi_2(x_1, x_2) \in \pi_2(U_1 \times U_2) \subset \pi_2(G)$

$\Rightarrow \quad x_1 \in U_1 \subset \pi_1(G)$ and $x_2 \in U_2 \subset \pi_2(G)$ ...(1)

For $\pi_1(U_1 \times U_2) = \{\pi_1(x_1, x_2) : (x_1, x_2) \in U_1 \times U_2\}$

$= \{x_1 : x_1 \in U_1, x_2 \in U_2\}$

$= U_1$

Similarly, $\pi_2(U_1 \times U_2) = U_2$

$(\pi_1 \text{ o } f)(y) = \pi_1(f(y))$

$= \pi_1(x_1, x_2)$

$= x_1$

Similarly, $(\pi_2 \text{ o } f)(y) = x_2$

Thus, $(\pi_1 \text{ o } f)(y) = x_1$, $(\pi_2 \text{ o } f)(y) = x_2$

In this event (1) takes the form

$$\left.\begin{matrix}(\pi_1 \text{ o } f)(y) \in U_1 \subset \pi_1(G) \\ (\pi_1 \text{ o } f)(y) \in U_2 \subset \pi_2(G)\end{matrix}\right\} \qquad ...(2)$$

This $y \in (\pi_1 \text{ o } f)^{-1}(U_1)$ and

$y \in (\pi_2 \text{ o } f)^{-1}(U_2)$

$\Rightarrow \quad y \in [(\pi_1 \text{ o } f)^{-1}(U_1)] \cap [(\pi_2 \text{ o } f)^{-1}(U_2)]$ ...(3)

$\because \quad \pi_1$ o f, $\pi_2$ o f are given to be continuous and hence $(\pi_1 \text{ o } f)^{-1}(U_1)$ and $(\pi_2 \text{ o } f)^{-1}(U_2)$ are open in Y.

$\Rightarrow \quad [(\pi_1 \text{ o } f)^{-1}(U_1) \cap [(\pi_2 \text{ o } f)^{-1}(U_2)]$ is open in Y.

On taking $(\pi_1 \text{ o } f)^{-1}(U_1) = V_1$, $[\pi_2 \text{ o } f)^{-1}(U_2) = V_2$.

We have $V_1 \cap V_2$ as an open set in Y.

According to (3), $y \in V_1 \cap V_2 = V$ (say)

any $v \in V \Rightarrow v \in V_1$ and $v \in V_2$

$\Rightarrow \quad v \in (\pi_1 \text{ o } f)^{-1}(U_1)$, $v = (\pi_2 \text{ o } f)^{-1}(U_2)$

$\Rightarrow \quad (\pi_1 \text{ o } f)(v) \in U_1$, $(\pi_2 \text{ o } f)(v) \in U_2$

$\Rightarrow \quad (\pi_1 \text{ o } f)(v) \in U_1 \subset \pi_1(G)$ and

$(\pi_2 \text{ o } f)(v) \in U_2 \subset \pi_2(G)$ [from (2)]

$\Rightarrow \quad v \in (\pi_1 \text{ o } f)^{-1}[\pi_1(G)]$ and $v \in (\pi_2 \text{ o } f)^{-1}[\pi_2(G)]$

$\Rightarrow \quad v \in (f^{-1} \text{ o } \pi_1^{-1})[\pi_1(G)]$ and

$v \in (f^{-1} \text{ o } \pi_2^{-1})[\pi_2(G)]$

$\Rightarrow \quad v \in f^{-1}(G)$ and $V \in f^{-1}(G)$

$\therefore \quad$ any $v \in V \Rightarrow v \in f^{-1}(G)$

$\Rightarrow V \subset f^{-1}(G)$

Notes

Thus we have shown that

any $y \in f^{-1}(G) \Rightarrow \exists$ an open set $V \subset Y$ s.t. $y \in V \subset f^{-1}(G)$.

$\Rightarrow$ y is an interior point of $f^{-1}(G)$ and hence every point of $f^{-1}(G)$ is an interior point, showing thereby $f^{-1}(G)$ is open in Y.

***Theorem 8:*** The product topology is the coarser (weak) topology for which projections are continuous.

***Proof:*** Let $(X \times Y, T)$ be product topological space of $(X, T_1)$ and $(Y, T_2)$.

Let $\mathcal{B}$ be a base for T. Then

$\mathcal{B} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$

The mappings, $\pi_x : X \times Y \to X$ s.t. $\pi_x(x, y) = x$

and $\pi_y : X \times Y \to Y$ s.t. $\pi_y(x, y) = y$

are called projection maps.

These maps are continuous. [Refer theorem (4)]

Let T* be any topology on $X \times Y$ for which $\pi_x$ and $\pi_y$ are continuous.

To prove: T is the coarest (weakest) topology for which projections are continuous, we have to show that $T \subset T^*$.

For this, we have to show that

any $G \in T \Rightarrow G \in T^*$

Let $G \in T$, by definition of base,

$G \in T \Rightarrow B_1 \subset B$ s.t. $G = \bigcup\{B : B \in \mathcal{B}_1\}$

$\Rightarrow G = \bigcup\{G_1 \times G_2 : G_1 \times G_2 \in \mathcal{B}_1\}$

$G_1 \subset X \Rightarrow G_1 \cap X = G_1$

$G_2 \subset X \Rightarrow G_2 \cap X = G_2$

Then $G = \bigcup\{(G_1 \cap X) \times (G_2 \cap Y) : G_1 \times G_2 \in \mathcal{B}_1\}$

$= \bigcup\{(G_1 \times G_2) \cap (X \times Y) : G_1 \times G_2 \in \mathcal{B}_1\}$

[For $(a \times b) \cap (c \times d) = (a \cap c) \times (b \cap d)$]

or $G = \{\pi_x^{-1}(G_1) \cap \pi_y^{-1}(G_2) : G_1 \times G_2 \in \mathcal{B}_1\}$ … (1)

$\pi_x : X \times Y \to X$, $G_1 \in T$, $\pi_x$ is continuous

$\Rightarrow \pi_x^{-1}(G_1) \in T^*$

Similarly, $\pi_y^{-1}(G_2) \in T^*$

This implies $\pi_x^{-1}(G_1) \cap \pi_y^{-1}(G_2) \in T^*$, be definition of topology.

In this event (1) declares that G is an arbitrary union of T* open sets and hence G is T* open set and so $G \in T^*$.

any $G \in T \Rightarrow G \in T^*$

Notes

*Example 4:* Let B be a member of the defining base for the product space $X = x_i X_i$, show that the projection of B into any coordinate space is open.

or

Each projection is a continuous map.

*Solution:* Let B be a member of the defining base for the product space $X = x_i X_i$ so that B is expressible as

$B = x\{X_i : i \neq j_1, j_2, ..., j_m\} \times G_{j_1} \times ... \times G_{j_m}$

where $G_{i_k}$ is an open subset of $X_{i_k}$.

The projection map $\pi_\alpha$ is defined as

$$\pi_\alpha : X \to X_\alpha$$

$$\pi_\alpha(B) = \begin{cases} X_\alpha & \text{if } \alpha \neq j_1, j_2, ...j_m \\ G_\alpha & \text{if } \alpha \in \{j_1, j_2, ...j_m\} \end{cases}$$

In either case, $\pi_\alpha(B)$ is an open set.

***Theorem 9:*** Let $y_o$ be a fixed element of Y and let $A = X \times \{y_o\}$. Then the restriction $f_x$ or $\pi_x$ to A is a homeomorphism of the subspace A of X × Y onto X. Also the restriction $f_y$ of $\pi_y$ to $B = \{x_o\} \times Y$ into Y is a homeomorphism, where $x_o \in X$.

***Proof:*** Let (X × Y, T) be the product topological space of $(X, T_1)$ and $(X, T_2)$. Let $x \in X$ and $y \in Y$ be arbitrary. Then the projection maps are defined as

$\pi_x : X \times Y \to X$ s.t. $\pi_x (x, y) = x$

and $\pi_y : X \times Y \to X$ s.t. $\pi_y (x, y) = y$.

Let $x_o \in X$ and $y_o \in Y$ be fixed elements.

Let $f_x$ be the restriction of $\pi_x$ to A so that $f_x$ is a map s.t. $f_x : A \to X$

s.t. $f_x(x, y_o) = x$.

To prove that $f_x$ is a homeomorphism, we have to prove that

(i) $f_x$ is one-one onto

(ii) $f_x$ is continuous

(iii) $f_x^{-1}$ is continuous

$f_x (x_1, y_o) = f_x (x_2, y_o) \Rightarrow x_1 = x_2$, by definition of $f_x$

$\Rightarrow (x_1, y_o) = (x_2, y_o)$.

Hence $f_x$ is one-one.

Given any $x \in X$, $\exists (x, y_o) \in A$ s.t. $f_x (x, y_o) = x$.

This proves that $f_x$ is onto. Hence the result (i).

$\pi_x$ is a projection map $\Rightarrow \pi_x$ is continuous.

Also $f_x$ is its restriction $\Rightarrow f_x$ is continuous. Hence (ii).

Notes

(iv) To prove $f_x^{-1} : X \rightarrow A$ is continuous. We have to prove: given any V open subset of A.

$[f_x^{-1}]^{-1}$ (V) = $f_x$ (V) is open in X.

Now V is expressible as $V = A \cap B$, where $B \in T$.

Let B be a base for T. Then

$$B = \{G \times H : G \in T_1, H \in T_2\}$$

By definition of base,

$$B \in T \Rightarrow \exists B_1 \subset B \text{ s.t.}$$

$$B = \bigcup \{G \times H \in T_1 \times T_2 : G \times H \in B_1\}$$

Then $A \cap B = \bigcup \{A \cap (G \times H) : G \times H \in B_1\}$

$$= \bigcup \{(X \times \{y_o\}) \cap (G \times H) : G \times H \in B_1\}$$

$$= \begin{cases} \bigcup \{G \times \{y_o\}) : G \times H \in B_1\} & \text{if } y_o \in H \\ \text{or} \bigcup \{G \times \phi \quad : G \times H \in B_1\} & \text{if } y_o \notin H \end{cases}$$

$$= \begin{cases} \bigcup \{G \times y_o\} : G \times H \in B_1\} & \text{if } y_o \in H \\ \text{or } \phi \quad : & \text{if } y_o \notin H \end{cases}$$

Moreover $\phi$ is an open set and an arbitrary union of open sets is open.

In either case, $f_x$ $(A \cap B)$ is open in X, i.e., $f_x$ (V) is open in X.

## Self Assessment

2. Prove that the collection

$$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a sub basis for the product topology on X × Y.

3. A map $f : X \rightarrow Y$ is said to be an open map if for every open set U of X, the set f(U) is open in Y, show that $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are open maps.

## 4.3 Summary

- If X and Y are topological spaces, the product topology on X × Y is the topology whose basis is $\{A \times B : A \in T_X, B \in T_Y\}$.
- Given any product of sets X × Y, there are projections maps $\pi_x$ and $\pi_y$ from X × Y to X and to Y given by $(x, y) \rightarrow x$ and $(x, y) \rightarrow y$.
- If (X, Y) is the product space if topological spaces $(X_1, T_1)$ and $(X_2, T_2)$, then the projection maps $\pi_1$ and $\pi_2$ are continuous and open.

## 4.4 Keywords

***Basis:*** A collection $\mathcal{B}$ of open sets in a topological space X is called a basis for the topology if every open set in X is a union of sets in $\mathcal{B}$.

***Coarser:*** Let T and T′ are two topologies on a given set X. If T′ ⊃ T, we say that T is coarser than T′.

***Hausdorff space:*** A topological space (X, T) is called a Hausdorff space if a given pair of distinct points x, y ∈ X, ∃ G, H ∈ T s.t. x ∈ G, Y ∈ H, G ⋂ H = ϕ.

***Interior point:*** Let (X, T) be a topological space and $\mathcal{A} \subset X$. A point $x \in \mathcal{A}$ is called an interior point of $\mathcal{A}$ iff ∃ an open set G such that x ∈ G ⊆ A.

## 4.5 Review Questions

1. Let $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ be the bases for topological spaces $(X_1, T_1), (X_2, T_2), ..., (X_n, T_n)$ respectively. Then prove that the family $\{O_1 \times O_2 \times ... \times O_n : O_i \in \mathcal{B}_i, i = 1, 2, ... , n\}$ is a basis for the product topology on $X_1 \times X_2 \times ... \times X_n$.
2. Prove that the product of any finite number of indiscrete spaces is an indiscrete space.
3. Let $X_1$ and $X_2$ be infinite sets and $T_1$ and $T_2$ the finite-closed topology on $X_1$ and $X_2$, respectively. Show that the product topology, T on $X_1 \times X_2$ is not the finite-closed topology.
4. Let $(X_1, T_1)$, $(X_2, T_2)$ and $(X_3, T_3)$ be topological spaces. Prove that

$$[(X_1, T_1) \times (X_2, T_2)] \times (X_3, T_3) \cong (X_1, T_1) \times (X_2, T_2) \times (X_3, T_3)$$

5. (a) Let $(X_1, T_1)$ and $(X_2, T_2)$ be topological spaces. Prove that

$$(X_1, T_1) \times (X_2, T_2) \cong (X_2, T_2) \times (X_1, T_1)$$

   (b) Generalise the above result to products of any finite number of topological spaces.

## 4.6 Further Readings

*Books*

H.F. Cullen, *Introduction to General Topology,* Boston, MA: Heath.

K.D. Joshi, *Introduction to General Topology,* New Delhi, Wiley.

S. Willard, *General Topology,* MA: Addison-Wesley.

# Unit 5: The Subspace Topology

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Describe the concept of subspace of topological space;
- Explain the problems related to subspace topology;
- Derive the theorems on subspace topology.

## Introduction

We shall describe a method of constructing new topologies from the given ones. If (X, T) is a topological space and $Y \subseteq X$ is any subset, there is a natural way in which Y can "inherit" a topology from parent set X. It is easy to verify that the set $U \cap Y$, as U runs through T, is a topology on Y. This prompts the definition of subspace or relative topology.

## 5.1 Subspace of a Topological Space

***Definition:*** Let (X, T) be a topological space, V be a non empty subset of X and $T_Y$ be the class of all intersections of Y with open subsets of X i.e.

$$T_Y = \{Y \cap U : U \in T\}$$

Then $T_Y$ is a topology on Y is called the subspace topology (or the relative topology induced on Y by T. The topological space $(Y, T_Y)$ is said to be a subspace of (X, T).

*Note* Let $A \subset Y \subset X$

(1) It A is open in Y, Y is open in X, then A is open in X.

(2) It A is closed in Y, Y is closed in X, then A is closed in X.

***Remark:*** Consider the usual topology T on R and the relative topology $\cup$ on Y = [0, 1]. Then $\left(0, \frac{1}{2}\right)$ is $\cup$-open as well as T-open $\left(\frac{1}{2}, 1\right] = \left(\frac{1}{2}, 2\right) \cap [0, 1] = G \cap [0, 1]$

where $$G = \left(\frac{1}{2}, 2\right) \in T$$

$$\therefore \quad \left(\frac{1}{2}, 1\right] = G \cap Y.$$

This shows that $\left(\frac{1}{2}, 1\right]$ is $\cup$-open but not T-open

$$\left(\frac{1}{2}, \frac{2}{3}\right) = \left(\frac{1}{2}, \frac{2}{3}\right) \cap [0, 1]$$

$$= G \cap Y$$

where $$G = \left(\frac{1}{2}, \frac{2}{3}\right) \in T$$

$\left(\frac{1}{2}, \frac{2}{3}\right) \in \cup$ and also $\left(\frac{1}{2}, \frac{2}{3}\right) \in T$.

Similarly, $\left(0, \frac{1}{2}\right]$ is not $\cup$-open as well as it is not T-open.

### 5.1.1 Solved Examples on Subspace Topology

*Example 1:* Let X = {a, b, c, d, e, f}

T = {X, $\phi$, {a}, {c, d}, {a, c, d}, {b, c, d, e, f}}

and Y = {b, c, e}.

Then the subspace topology on Y is

$T_Y$ = {Y, $\phi$, {c}}.

*Example 2:* Consider the topology

T = {$\phi$, {1}, {2, 3}, X} on

X = {1, 2, 3} and a subset Y = {1, 2} of X.

Then $Y \cap \phi = \phi$

$Y \cap \{1\} = \{1\}$,

$Y \cap \{2, 3\} = \{2\}$ and

$Y \cap X = V$.

Hence, the relative topology on Y is

$T_Y$ = {$\phi$, {1}, {2}, V}.

***Theorem 1:*** A subspace of a topological space is itself a topological space.

***Proof:***

(i) $\phi \in T$ and $\phi \cap Y = \phi \Rightarrow \phi \in T_Y$,

$X \in T$ and $X \cap Y = Y \Rightarrow Y \in T_Y$,

(ii) Let $\{H_\alpha : \alpha \in \Lambda\}$ be any family of sets in $T_Y$.

Then $\forall\ \alpha \in \Lambda\ \exists$ a set $G_\alpha \in T$ such that $H_\alpha = G_\alpha \cap Y$

$$\therefore \quad \bigcup \{H_\alpha : \alpha \in \Lambda\} = \bigcup \{G_\alpha \cap Y : \alpha \in \Lambda\}$$

$$= [\bigcup \{G_\alpha : \alpha \in \Lambda\}] \cap Y \in T_Y$$

since $\bigcup \{G_\alpha : \alpha \in \Lambda\} \in T$

(iii) Let $H_1$ and $H_2$ be any two sets in $T_Y$.

Then $H_1 = G_1 \cap Y$ and $H_2 = G_2 \cap Y$ for some $G_1, G_2 \in T$.

$$\therefore \quad H_1 \cap H_2 = (G_1 \cap Y) \bigcap (G_2 \bigcap Y)$$

$$= (G_1 \cap G_2) \cap Y \in T_Y, \text{ since } G_1 \cap G_2 \in T$$

Hence, $T_Y$ is a topology for Y.

*Example 3:* Let (Y, V) be a subspace of a topological space (X, T) and let (Z, W) be a subspace of (Y, V). Then prove that (Z, W) is a subspace of (X, T).

*Solution:* Given that $(Y, V) \subset (X, T)$ …(1)

and $(Z, W) \subset (Y, V)$ …(2)

We are to prove that $(Z, W) \subset (X, T)$

From (1) and (2), we get

$$Z \subset Y \subset X \quad \text{…(3)}$$

From (1), $V = \{G \bigcap Y : G \in T\}$ …(4)

and (2), $W = \{H \cap Z : H \in V\}$ …(5)

From (4) and (5), we get $H = G \cap Y$

$$\Rightarrow \quad H \cap Z = (G \cap Y) \cap Z$$

$$= G \cap (Y \cap Z)$$

$$= G \cap Z \quad \text{[Using (3)]}$$

so, $H \cap Z = G \cap Z$ …(6)

Using (6) in (5), we get

$$W = \{G \cap Z : G \in T\}$$

$$\Rightarrow \quad (Z, W) \subset (X, T)$$

Hence, (Z, W) is a subspace of (X, T).

*Example 4:* If T is usual topology on $\mathcal{R}$, then find relative topology $\cup$ on $\mathcal{N} \subset \mathcal{R}$.

*Solution:* Every open interval on $\mathcal{R}$ is T-open set.

Let $$G = \left(n - \frac{1}{2}, n + \frac{1}{2}\right), n \in \mathcal{N}.$$

Then $G \in T$. Now $\mathcal{U} = \{G \cap \mathcal{N} : G \in T\}$

If $G = \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$

then $G \cap \mathcal{N} = \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \cap \mathcal{N}$

$= \{n\}$

Or $\mathcal{U} = \{\{n\} : n \in \mathcal{N}\}$

Every singleton set of $\mathcal{N}$ is $\mathcal{U}$-open set.

As an arbitrary subset of $\mathcal{N}$ is an arbitrary union of singleton sets and so every subset of $\mathcal{N}$ is $\mathcal{U}$-open.

Consequently, $\mathcal{U}$ is a discrete topology on $\mathcal{N}$.

*Example 5:* Define relative topology. Consider the topology : $T = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ on $X = \{a, b, c, d\}$. If $Y = \{b, c, d\}$ is a subset of X, then find relative topology on Y.

*Solution:* If $\mathcal{U}$ is relative topology on $\mathcal{Y}$, then

$$\mathcal{U} = \{G \cap Y, G \in T\}$$

$$\Rightarrow \quad \mathcal{U} = \{\phi \cap Y, \{a\} \cap Y, \{b, c\} \cap Y, \{a, b, c\} \cap Y, X \cap Y\}$$

$$\Rightarrow \quad \mathcal{U} = \{\phi , \phi, \{b, c\}, \{b, c\}, Y\}$$

$$\Rightarrow \quad \mathcal{U} = \{\phi, Y, \{b, c\}\}$$

*Example 6:* Let X be a topological space and let Y and Z be subspaces of X such that $Y \subset Z$. Show that the topology which Y has a subspace of X is the same as that which it has as a subspace of Z.

*Solution:* Let (X, T) be a topological space and Y, Z be subspaces of X such that

$$Y \subset Z \subset X.$$

Further assume $(Y, T_1) \subset (Z, T_2) \subset (X, T)$ …(1)

$(Y, T_3) \subset (X, T)$ …(2)

We are to show that $T_1 = T_3$

By definition (1) declares that

$T_1 = \{G \cap Y : G \in T_2\}$ …(3)

$T_2 = \{H \cap Z : H \in T\}$ …(4)

$T_3 = \{P \cap Z : P \in T\}$ …(5)

Using (4) in (3), we get

$$G \cap Y = (H \cap Z) \cap Y = H \cap (Y \cap Z) = H \cap Y$$

Now, (3) becomes

$T_1 = \{H \cap Y : H \in T\}$ …(6)

From (5) and (6), we get $T_1 = T_3$.

***Theorem 2:*** Let $(Y, \mathcal{U})$ be a subspace of a topological space $(X, T)$. A subset of Y is $\mathcal{U}$-nhd. of a point $y \in Y$ iff it is the intersection of Y with a T-nhd. of the point $y \in Y$.

***Proof:*** Let $(Y, \mathcal{U}) \subset (X, T)$ and $y \in Y$ be arbitrary, then $y \in X$.

(I) Let $N_1$ be a $\mathcal{U}$-nhd of y, then

$$\exists \quad V \in \mathcal{U} \quad \text{s.t.} \quad y \in V \subset N_1 \qquad \ldots(1)$$

To prove : $N_1 = N_2 \cap Y$ for some T-nhd $N_2$ of y.

$$y \in V \in \mathcal{U} \Rightarrow \exists\, G \in T \text{ s.t. } V = G \cap Y$$

$$\Rightarrow y \in G \cap Y \Rightarrow y \in G, y \in Y \qquad \ldots(2)$$

Write $N_2 = N_1 \cup G$.

Then $N_1 \subset N_2$, $G \subset N_2$.

so, (2) implies $y \in G \subset N_2$, where $G \in T$

This shows that $N_2$ is a T-nhd of y.

$$\begin{aligned} N_2 \cap Y &= (N_1 \cup G) \cap Y \\ &= (N_1 \cap Y) \cup (G \cap Y) \\ &= (N_1 \cap Y) \cup V \\ &= N_1 \cup V && \text{[by (1)]} \\ &= N_1 && \because \quad N_1 \subset Y \quad \text{and} \quad V \subset N_1 \end{aligned}$$

Finally, $N_2$ has the following properties

$$N_1 = N_2 \cap Y \quad \text{and} \quad N_2 \text{ is a T-nhd of y.}$$

This completes the proof.

(II) Conversely, Let $N_2$ be a T-nhd. of y so that

$$\exists \quad A \in T \quad \text{s.t.} \quad y \in A \subset N_2 \qquad \ldots(3)$$

We are to prove that $N_2 \cap Y$ is a $\mathcal{U}$-nhd of y.

$$\therefore \qquad y \in Y, y \in A \Rightarrow y \in Y \cap A \qquad \text{[by (3)]}$$

$$\Rightarrow y \in A \cap Y \subset N_2 \cap Y \qquad \text{[by (3)]}$$

$$A \in T \Rightarrow A \cap Y \in \mathcal{U}$$

Thus, we have $y \in A \cap Y \subset N_2 \cap Y$, where $A \cap Y \in \mathcal{U}$.

$\Rightarrow N_2 \cup Y$ is a $\mathcal{U}$-nhd of y.

*Example 7:* Let $(Y, \mathcal{U})$ be a subspace of a topological space $(X, T)$. Then every $\mathcal{U}$-open set is also T-open iff Y is T-open.

*Solution:* Let $(Y, \mathcal{U}) \subset (X, T)$ and let

any $G \in \mathcal{U} \Rightarrow G \in T$ $\qquad \ldots(1)$

i.e. every $\mathcal{U}$-open set is also T-open set.

*To show:* Y is T-open, it is enough to prove that $y \in T$.

Let $G \in \bigcup$ be arbitrary, then $G \in T$, by (1).

We can write $G = H \cap Y$ for some T-open set H.

Now, $G = H \cap Y, G \in T \Rightarrow H \cap Y \in T$

Again $H \cap Y \in T, H \in T$ and $Y \subset X \Rightarrow Y \in T$

Conversely, let $(Y, \bigcup) \subset (X, T)$ and let $Y \in T$ for any $G \in \bigcup \Rightarrow G \in T$

$\therefore \quad G \in \bigcup \Rightarrow \exists\, A \in T$ s.t. $G = A \cap Y$

Again $\quad A \in T, Y \in T \Rightarrow A \cap Y \in T \Rightarrow G \in T$

Finally, any $G \in \bigcup \Rightarrow G \in T$.

### 5.1.2 Basis for the Subspace Topology

*Example 8:* Consider the subset Y = [0, 1] of the real line $\mathbb{R}$, in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$, where (a, b) is an open interval in $\mathbb{R}$, such a set is of one of the following types:

$$(a, b) \cap Y = \begin{cases} (a,b) & \text{if a and b are in Y,} \\ [0,b) & \text{if only b is in Y,} \\ (a,1] & \text{if only a is in Y,} \\ Y \text{ or } \phi & \text{if neither a nor b is in Y.} \end{cases}$$

By definition, each of these sets is open in Y. But sets of the second and third types are not open in the larger space $\mathbb{R}$.

Note that these sets form a basis for the order topology on Y. Thus, we see that in the case of the set Y = [0, 1], its subspace topology (as a subspace of $\mathbb{R}$) and its order topology are the same.

*Example 9:* Let Y be the subset $[0, 1) \cup \{2\}$ of $\mathbb{R}$. In the subspace topology on Y the one-point set {2} is open, because it is the intersection of the open set $\left(\frac{3}{2}, \frac{5}{2}\right)$ with Y. But in the order topology on Y, the set {2} is not open. Any basis element for the order topology on Y that contains 2 is of the form

$$\{x \mid x \in Y \text{ and } a < x \leq 2\}$$

for some $a \in Y$; such a set necessarily contains points of Y less than 2.

***Lemma 1:*** If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

***Proof:*** Given $\bigcup$-open in X and given $y \in \bigcup \cap Y$, we can choose an element B of $\mathcal{B}$ such that

$y \in B \subset \bigcup$. Then $y \in B \cap Y \subset \bigcup \cap Y$

Now as we know

"If X is a topological space and $\mathcal{C}$ is a collection of open sets of X such that for each open set $\bigcup$ of X and each x in $\bigcup$, there is an element c of $\mathcal{C}$ such that $x \in \mathcal{C} \subset \bigcup$. The $\mathcal{C}$ is a basis for the topology of X."

Thus, we can say that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Notes

*Lemma 2:* Let Y be a subspace of X. If $\cup$ is open in Y and Y is open in X, then $\cup$ is open in X.

***Proof:*** Since $\cup$ is open in Y,

$$\cup = Y \cap V \quad \text{for some set V open in X.}$$

Since Y and V are both open in X,

so is $Y \cap V$.

### 5.1.3 Subspace of Product Topology

***Theorem 3:*** If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

***Proof:*** The set $\cup \times V$ is the general basis element for $X \times Y$, where $\cup$ is open in X and V is open in Y.

$\therefore$, $(\cup \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$.

Now, $(\cup \times V) \cap (A \times B) = (\cup \cap A) \times (V \cap B)$.

Since $\cup \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(\cup \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

So, we can say that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same.

Hence, the topologies are the same.

## 5.2 Summary

- A subspace of a topological space is itself a topology space.
- If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.
- Let Y be a subspace of X. If $\cup$ is open in Y and Y is open in X, then $\cup$ is open in X.
- If A is a subspace of X and B is a subspace of Y then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

## 5.3 Keywords

***Basis:*** Let X be a topological space A set $\mathcal{B}$ of open set is called a basis for the topology if every open set is a union of sets in $\mathcal{B}$.

***Closed Set:*** Let (X, T) be a topological space. Let set $A \in T$. Then X–A is a closed set.

***Intersection:*** The intersection of A and B is written $A \cap B$. $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.

***Neighborhood:*** Let (X, T) be a topological space. $A \subset X$ is called a neighborhood of a point $x \in X$ if $\exists\, G \in T$ with $x \in G$ s.t. $G \subset A$.

***Open set:*** Let (X, T) be a topological space. Any set $A \in T$ is called an open set.

***Product Topology:*** Let X and Y be topological space. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $\cup \times V$, where $\cup$ is an open subset of X and V is an open subset of Y.

Notes

***Subset:*** If A and B are sets and every element of A is also an element of B, then A is subset of B denoted by $A \subseteq B$.

***Subspace:*** Given a topological space (X, T) and a subset S of X, the subspace topology on S is defined by

$$T = \{S \cap \cup : \cup \in T\}$$

***Topological Space:*** It is a set X together with T, a collection of subsets of X, satisfying the following axioms. (1) The empty set and X are in T; (2) T is closed under arbitrary union and (3) T is closed under finite intersection. Then collection T is called a topology on X.

## 5.4 Review Questions

1. Let $X = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\} \subset X$ and

   $T = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{1, 4, 5\}, \{1, 2, 4, 5\}\}$.

   Find relative topology T, on A.

2. Let (X, T) be a topological space and $X^* \subset X$. Let $T^*$ be the collection of all sets which are intersections of $X^*$ with members of T. Prove that $T^*$ is a topology on $X^*$.

3. Show that if Y is a subspace of X, and $A \subset Y$, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

4. If T and T′ are topologies on X and T′ is strictly finer than T, what do you say about the corresponding subspace topologies on the subset Y of X?

5. Let A be a subset of X. If $\mathcal{B}$ is a base for the topology of X, then the collection

   $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$

   is a base for the subspace topology on A.

6. Let $(Y, \cup)$ be a subspace of (X, T). If F and $F_1$ are the collections of all closed subsets of (X, T) and $(Y, \cup)$ respectively, then $F_1 \subset F \Leftrightarrow Y \in F$.

## 5.5 Further Readings

*Books*


Willard, Stephen. *General Topology*, Dover Publication (2004).

Bourbaki, Nicolas, *Elements of Mathematics: General Topology*, Addison-Wesley (1966).

Simmons. *Introduction to Topology and Modern Analysis*.

James & James. *Mathematics Dictionary*.

# Unit 6: Closed Sets and Limit Point

CONTENTS



## Objectives

After studying this unit, you will be able to:

- Define closed sets;
- Solve the problems related to closed sets;
- Understand the limit points and derived set;
- Solve the problems on limit points.

## Introduction

On the real number line we have a notion of 'closeness'. For example each point in the sequence 1..01..001..0001..00001.. is closer to 0 than the previous one. Indeed, in some sense 0 is a limit point of this sequence. So the interval (0, 1] is not closed as it does not contain the limit point 0. In a general topological space me do not have a 'distance function', so we must proceed differently. We shall define the notion of limit point without resorting to distance. Even with our new definition of limit point, the point 0 will still be a limit point of (0, 1]. The introduction of the notion of limit point will lead us to a much better understanding of the notion of closed set.

## 6.1 Closed Sets

A subset A of a topological space X is said to be closed if the set X-A is open.

*Example 1:* The subset [a, b] of R is closed because its complement

$R - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.

Similarly, $[a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open. These facts justify our use of the terms "closed interval" and "closed ray". The subset [a, b) of R is neither open nor closed.

*Example 2:* In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

***Theorem 1:*** Let X be a topological space. Then the following conditions hold:

(a) $\phi$ and X are closed.

(b) Arbitrary intersections of closed sets are closed.

(c) Finite unions of closed sets are closed.

***Proof:***

(a) $\phi$ and X are closed because they are the complements of the open sets X and $\phi$, respectively.

(b) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha).$$

Since the sets $X - A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\cap A_\alpha$ is closed.

(c) Similarly, if $A_i$ is closed for i = 1, …, n, consider the equation

$$X - \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i).$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open. Hence $\cup A_i$ is closed.

***Theorem 2:*** Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

***Proof:*** Assume that A = C $\cap$ Y, where C is closed in X. Then X – C is open in X, so that (X – C) $\cap$ Y is open in Y, by definition of the subspace topology. But (X – C) $\cap$ Y = Y – A. Hence Y – A is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. Then Y – A is open in Y, so that by definition it equals the intersection of an open set U of X with Y. The set X – U is closed in X and A = Y $\cap$ (X – U), so that A equals the intersection of a closed set of X with Y, as desired.

*Example 3:* Let (Y, U) $\subset$ (X, T) and A $\subset$ Y.

Then A is U-closed iff A = F $\cap$ Y for some T closed set F.

or

A is U-closed iff A is the intersection of Y and a T-closed F.

*Solution:* Let (Y, U) $\subset$ (X, T) and A $\subset$ Y, i.e. (Y, U) is subspace of (X, T).

To prove that A is U-closed iff

A = F $\cap$ Y for some T-closed set F.

A is U-closed $\Leftrightarrow$ Y – A is U-open.

Then Y – A can be expressed as:

Y – A = G $\cap$ Y for some T-open set G.

From which A = Y – G $\cap$ Y = X $\cap$ Y – G $\cap$ Y

= (X – G) $\cap$ Y

= F $\cap$ Y, where F = X – G is a T-closed set.

This completes the proof.

## Self Assessment

1. Show that if A is closed in Y and Y is closed in X, then A is closed in X.
2. Show that if A is closed in X and B is closed in Y, then A × B is closed in X × Y.

## 6.2 Limit Point

Let (X, T) be a topological space and $A \subset X$. A point $x \in X$ is said to be the limit point or accumulation point of A if each open set containing x contains at least one point of A different from x.

Thus it is clear from the above definition that the limit point of a set A may or may not be the point of A.

*Note* Limit point is also known as accumulation point or cluster point.

*Example 4:* Let X = {a, b, c} with topology $T = \{\phi, \{a, b\}, \{c\}, X\}$ and A = {a}, then b is the only limit point of A, because the open sets containing b namely {a, b} and X also contains a point of A.

Where as 'a' and 'b' are not limit point of C = {c}, because the open set {a, b} containing these points do not contain any point of C. The point 'c' is also not a limit point of C, since then open set {c} containing 'c' does not contain any other point of C different from C. Thus, the set C = {c} has no limit points.

*Example 5:* Prove that every real number is a limit point of R.

*Solution:* Let $x \in R$

then every nhd of x contains at least one point of R other than x.

$\therefore$ x is a limit point of R.

But x was arbitrary.

$\therefore$ every real number is a limit point of R.

*Example 6:* Prove that every real number is a limit point of R – Q.

*Solution:* Let x be any real number, the every nhd of x contains at least one point of R – Q other than x.

$\therefore$ x is a limit point of R – Q

But x was arbitrary

$\therefore$ every real number is a limit point of R – Q.

### 6.2.1 Derived Set

The set of all limit points of A is called the derived set of A and is denoted by D(A).

*Notes*

1. In terms of derived set, the closure of a set $A \subset X$ is defined as $\bar{A} = A + D(A) = A \cup D(A)$.
2. If every point of A is an isolated point of A, then A is known as isolated set.

*Example 7:* Every derived set in a topological space is a closed.

*Solution:* Let (X, T) be a topological space and $A \subset X$.

*Aim:* D(A) is a closed set.

Recall that B is a closed set if $D(B) \subset B$.

Hence D(A) is closed iff $D[D(A)] \subset D(A)$.

Let $x \in D[D(A)]$ be arbitrary, then x is a limit point of D(A) so that $(G - \{x\}) \cap D(A) \neq \phi \forall G \in T$ with $x \in G$.

$\Rightarrow (G - \{x\}) \cap A \neq \phi \Rightarrow x \in D(A)$.

Hence the result.

[For every nhd of an element of D(T) has at least one point of A].

*Example 8:* Let (X, T) be a topological space and $A \subseteq X$, then A is closed iff $A' \subseteq A$ or $A \supseteq D(A)$.

*Solution:* Let A be closed.

$\Rightarrow$ $A^c$ is open.

Let $x \in A^c$.

Then $A^c$ is an open set containing x but containing no point of A other than x.

This shows that x is not a limit point of A.

Thus, no point of $A^c$ is a limit point of A.

Consequently, every limit point of A is in A and therefore

$A' \subseteq A$

Conversely, Let $A' \subseteq A$

we have to show that A is closed.

Let x be arbitrary point of $A^c$.

Then $x \in A^c$

$\Rightarrow$ $x \notin A$

$\Rightarrow$ $x \notin A$ and $x \notin A'$

$\Rightarrow$ $x \notin A$ and x not a limit point of A.

$\Rightarrow$ $\exists$ an open set G such that $x \in G$ and $G \cap A = \phi$

$\Rightarrow$ $x \in G \subseteq A^c$.

$\Rightarrow$ $A^c$ is the nhd of each of its point and therefore $A^c$ is open.

Hence A is closed.

*Example 9:* Let (X, T) be a topological space and $A \subset X$. A point x of A is an interior point of A iff it is not a limit point of X – A.

*Solution:* Let (X, T) be a topological space and $A \subset X$. Suppose a point x of A is an interior point of A so that $x \in A$, $x \in A°$.

To prove that x is not a limit point of X – A i.e., $x \notin D(X - A)$

$$x \in A \Rightarrow \exists\, G \in T \text{ with } x \in G \text{ s.t. } G \subset A$$

$$\Rightarrow G \cap (X - A) = \phi$$

$$\Rightarrow (G - \{x\}) \cap (X - A) = \phi \qquad [\because\ x \notin (X - A)\ ]$$

$\therefore$ G is an open set containing set.

$$(G - \{x\}) \cap (X - A) = \phi$$

This immediately shows that $x \notin D(X - A)$.

Conversely suppose that (X, T) is topological space and $A \subset X$ s.t. a point x of A is not a limit point of (X – A).

To prove that $x \in A°$.

By hypothesis $x \in A$, $x \notin D(X - A)$

$x \notin D(X - A) \Rightarrow \exists\, G \in T$ with $x \in G$ s.t. $(G - \{x\}) \cap (X - A) = \phi$

$\Rightarrow$ $G \cap (X - A) = \phi$ $\qquad [\because\ x \notin X - A]$

$\Rightarrow$ $G \subset A$.

$\therefore$ $x \in A \Rightarrow \exists\, G \in T$ with $x \in G$ s.t. $G \subset A$. This proves that $x \in A°$.

## Self Assessment

3. Let x be a topological space and let A, B be subset of x. Then.

   (a) $\phi' = \phi$ or $D(\phi) = \phi$

   (b) $A \subseteq B \Rightarrow A' \subseteq B'$ or $A \subset B \Rightarrow D(A) \subset D(B)$;

   (c) $x \in A' \Rightarrow x \in (G - \{x\})'$ ;

## 6.3 Summary

- A subset A of a topological space X is said to be closed if the set X - A is open.
- Let (X, T) be a topological space and $A \subset X$. A point $x \in X$ is said to be the limit point of A if each open set containing x contains at least one point of A different from x.
- The set of all limit points of A is called the derived set of A and is denoted by D(A).

## 6.4 Keywords

***Discrete Topology:*** Let X be any non-empty set and T be the collection of all subsets of X. Then T is called discrete topology on the set X.

***Open and Closed Set:*** Let (X, T) be a topological space. Any set A $\in$ T is called an open set and X – A is a closed set.

***Subspace:*** Let (X, T) be a topological space and a subset S of X, the subspace topology on S is defined by $T_S = \{S \cap U \mid U \in T\}$.

## 6.5 Review Questions

1. Let X be a topological space and A be a subset of X. Then prove that $\bar{A}$ is the smallest closed set containing A.
2. Prove that A is closed iff $A = \bar{A}$.
3. Let (Y, U) $\subset$ (X, S) and A $\subset$ Y. Prove that A point y $\in$ Y is U-limit point of A iff y is a T-limit point of A.
4. Show that every closed set in a topological space is the disjoint union of its set of isolated points and its set of limit points, in the sense that it contains these sets.
5. Show that if U is open in X and A is closed in X, then U – A is open in X, and A – U is closed in X.

## 6.6 Further Readings

*Books*

J. L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

# Unit 7: Continuous Functions

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Understand the concept of continuity;
- Define Homeomorphism;
- Define open and closed map;
- Understand the theorems and problems on continuity.

## Introduction

The concept of continuous functions is basic to much of mathematics. Continuous functions on the real line appear in the first pages of any calculus look, and continuous functions in the plane and in space follow not far behind. More general kinds of continuous functions arise as one goes further in mathematics. In this unit, we shall formulate a definition of continuity that will include all these as special cases and we shall study various properties of continuous functions.

## 7.1 Continuity

### 7.1.1 Continuous Map and Continuity on a Set

***Definition:*** Let (X, T) and (Y, U) be any two topological spaces.

Let $f : (X, T) \to (Y, U)$ be a map.

The map f of said to be continuous at $x_0 \in X$ is given any U-open set H containing $f(x_0)$, $\exists$a T-open set G containing $x_0$ s.t. $f(G) \subset H$.

If the map in continuous at each $x \in X$ then the map is called a continuous map.

**Notes**

***Definition:*** Continuity on a set. A function

$$f:(X,T) \Rightarrow (Y,U)$$

is said to be continuous on a set $A \subset X$ if it is continuous at each point of A.

*Notes* The following have the same meaning:

(a) f is a continuous map.

(b) f is a continuous relative to T and U

(c) f is T – U continuous map.

*Example 1:* Let $\mathcal{R}$ denote the set of real numbers in its usual topology, and let $\mathcal{R}_\ell$ denote the same set in the lower limit topology. Let

$$f: \mathcal{R} \to \mathcal{R}_\ell$$

be the identity function;

$$f(x) = x \text{ for every real number } x.$$

then f is not a continuous function; the inverse image of the open set [a, b) of $\mathcal{R}_\ell$ equals itself, which is not open in $\mathcal{R}$. On the other hand, the identity function.

$$g: \mathcal{R}_\ell \to \mathcal{R}$$

is continuous, because the inverse image of (a, b) is itself, which is open in $\mathcal{R}_\ell$.

## 7.1.2 Homeomorphism

***Definition:*** A map $f : (X, T) \to (Y, U)$ is said to be homeomorphism or topological mapping if

(a) f is one-one onto.

(b) f and $f^{-1}$ are continuous.

In this case, the spaces X and Y are said to be homeomorphic or topological equivalent to one another and Y is called the homeomorphic image of X.

*Example 2:* Let T denote the usual topology on R and a any non-zero real number. Then each of the following maps is a homeomorphism

(a) $f : (R, T) \to (R, T)$ s.t. $f(x) = a + x$

(b) $f : (R, T) \to (R, T)$ s.t. $f(x) = ax$

(c) $f : (R, T) \to (R, T)$ s.t. $f(x) = x^3$ where $x \in R$.

*Example 3:* Show that (R, U) and (R, D) are not homeomorphic.

*Solution:* Every singleton is D-open and image of a singleton is again singleton which is not U-open. Consequently no one-one D – U continuous map of R onto R can be homeomorphism. From this the required result follows.

### 7.1.3 Open and Closed Map

***Definition:*** Open Map

A map f : (X, T) → (Y, U) is called an open or interior map if it maps open sets onto open sets i.e. if

$$\text{any } G \in T \Rightarrow f(G) \in U.$$

***Definition:*** Closed Map

A map f : (X, T) → (Y, U) is called a closed map if

any T-closed set f ⇒ f(F) is U-closed set.

*Example 4:* (i) Let T denote the usual topology on R. Let a be any non-zero real number, Then each of the following map is open as well as closed.

(a) f : (R, T) → (R, T) s.t. f(x) = a + x

(b) f : (R, T) → (R, T) s.t. f(x) = ax

In this case if a = 0, then this map is closed but not open.

(ii) The identity map f : (X, T) → (X, T) is open and as well as closed.

(iii) A map from an indiscrete space into a topological space is open as well as closed.

(iv) A map from a topological space into a discrete space is open as well as closed.

*Note* Proof of (i) b,

Let a ≠ 0 and A = (b, c) ∈ T arbitrary.

Then f(b) = ab, f(c) = ac.

$\therefore$ $f(A) = (ab, ac) \in J$

i.e., image of an open set is an open set under the map f(x) = ax, a ≠ 0. Hence this map is open.

Similarly $f([b,c]) = [ab, bc]$, i.e. image of a closed set is closed.

$\therefore$ f is a closed map

Consider the case in which a = 0

Then $f(x) = ax = 0, \forall x \in R$

$\therefore$ $f(x) = 0 \ \forall x \in R.$

Now $f([b,c]) = \{0\} =$ A Finite set $=$ A closed set for a finite set is a T-closed set.

Now the image of a closed set is closed and hence f is a closed map.

Again f (5, 6) = {0} ≠ an open set.

$\therefore$ image of an open set is not open.

Consequently, f is not open.

### 7.1.4 Theorems and Solved Examples

**Notes**

***Theorem 1:*** The function f: (X, J) $\to$ (Y, U) is continuous iff $f^{-1}$ (V) is open in X for every open set V in Y.

***Proof:*** Let f: (X, J) $\to$ (Y, U) be a map.

(i) Suppose f is continuous. Let G be an open subset of Y.

To prove that $f^{-1}(G)$ is open in X.

If $f^{-1}(G) = \phi$, then $f^{-1}(G) \in J$.

If $f^{-1}(G) \neq \phi$, then $\exists x \in f^{-1}(G)$ so that $f(x) \in G$.

Continuity of f $\Rightarrow$ f is continuous at x.

$\Rightarrow \exists\, H \in J$ s.t. $x \in H$ and $f(H) \subset G$.

$\Rightarrow x \in H \subset f^{-1}(G)$, $H \subset J$.

Thus we have shown that $f^{-1}(G)$ is a nhd of each of its points and so $f^{-1}(G)$ is J-open.

Conversely, suppose that f : (X, J) $\to$ (Y, U) is a map such that $f^{-1}(V)$ is open in X for each open set $V \subset Y$.

To prove that f is continuous.

Let $V \in U$ be arbitrary.

Then, by assumption, $f^{-1}$ (V) is open in X.

Take $U = f^{-1}(V)$, so that $U \in J$.

i.e $F(U) = f(f^{-1}(V)) \subset V$, or $f(U) \subset V$.

given any $V \in U$, $\exists\, U \in J$ s.t. $f(U) \subset V$.

This proves that f is a continuous map.

***Theorem 2:*** A map f : X $\to$ Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

A map f : (X, d) $\to$ (Y, p) be continuous iff $f^{-1}(F)$ is closed in X $\forall F \subset Y$ is closed where (X, d) and (Y, p) are metric spaces.

***Proof:*** Let f : X $\to$ Y be a continuous map.

To prove that $f^{-1}(c)$ is closed in X for each closed set $C \subset Y$.

Let $C \subset Y$ be an arbitrary closed set.

Continuity of f implies that $f^{-1}(Y - C)$ is open in X. (Refer theorem (1))

i.e. $f^{-1}(Y)$-$f^{-1}(C)$ is open in X.

i.e. $X - f^{-1}(C)$ is open in X.

or $f^{-1}(C)$ is closed in X.

Conversely, suppose that f : (X, T) $\to$ (Y, U) is a map such that $f^{-1}(C)$ is closed for each closed set $C \subset Y$.

To prove that f is continuous.

Let $G \subset Y$ be an arbitrary open set, then Y – G is closed in Y.

By hypothesis, $f^{-1}(Y - G)$ is closed in X.

Notes

i.e., $f^{-1}(Y) - f^{-1}(G)$ is closed in X,

i.e., $X - f^{-1}(G)$ is closed in X,

i.e. $f^{-1}(G)$ is open in X,

$\therefore$ any $G \subset Y$ is open $\Rightarrow f^{-1}(G)$ is open in X

This proves that f is continuous map.

***Theorem 3:*** Let f : (X, T) → (Y, U) be a map, Let $\mathcal{S}$ be a sub-base for the topology U on Y. Then f is continuous iff $f^{-1}(S)$ is open in X whenever $S \in \mathcal{S}$

or

f is continuous ⇔ the inverse image of each sub-basic open set is open.

***Proof:*** Let f : (X, T) → (Y, U) be continuous map. Let $\mathcal{S}$ be a sub-base for the topology U on Y. Let $S \in \mathcal{S}$ be arbitrary.

To prove that $f^{-1}(S)$ is open in X.

$S \in \mathcal{S} \Rightarrow S \in U$ ($\because$ $\mathcal{S} \subset U \Rightarrow S$ is open in Y)

$\Rightarrow f^{-1}(S)$ is open in X, (by Theorem 1).

Conversely, suppose that f : (X, T) → (Y, U) is a map such that $f^{-1}(S)$ is open in X whenever $S \in \mathcal{S}$, $\mathcal{S}$ being a sub-base for the topology U on Y. Let B be a base for U on Y.

To prove that f is continuous.

Let $G \subset Y$ be an open set, then $G \in U$.

By definition of base,

$$G \in U \Rightarrow \exists\, \mathcal{B}_1 \subset \mathcal{B} \text{ s.t. } G = \cup \{B : B \in \mathcal{B}_1\} \quad \ldots(1)$$

By the definition of sub-base, any $B \in \mathcal{B}$ can be expressed as

$B = \bigcap_{i=1}^{n} S_i$ for same choice of $S_1, S_2, \ldots S_n \in \mathcal{S}$

$$f^{-1}(B) = f^{-1}\left[\bigcap_{i-=1}^{n} S_i\right] = \bigcap_{i=1}^{n} f^{-1}(S_i) \quad \ldots(2)$$

By hypothesis, $f^{-1}(S_i)$ is open in X, Being a finite intersection of open sets in X, $\bigcap_{i=1}^{n} f^{-1}(S_i)$ is open in X, i.e. $f^{-1}(B)$ is open in X

i.e. $f^{-1}(G) = f^{-1}[\cup\{B : B \in \mathcal{B}_1\}]$

$= \cup\left[f^{-1}(B) : B \in \mathcal{B}_1\right]$

= An arbitrary union subsets of X

= open subset of X.

$\therefore$ $f^{-1}(G)$ is open in X.

Thus we have shown that

any $G \subset Y \Rightarrow f^{-1}(G)$ is open in X.

This proves that f is continuous.

**Notes**

***Theorem 4:*** Let (X, T) and (Y, U) be topological spaces.

Let f : (X, T) $\rightarrow$ (Y, U) be a map. Then f is continuous iff $f^{-1}(B)$ is open for every B $\in \mathcal{B}$, $\mathcal{B}$ being a base for U on Y.

or

f is continuous iff the inverse image of each basic open set is open.

***Proof:*** Let (X, T) and (Y, U) be topological spaces.

Let $\mathcal{B}$ be a base for U on Y. Let f : X $\rightarrow$ Y be a continuous map.

To prove that $f^{-1}(B)$ is open in X for every B $\in \mathcal{B}$

$$B \in B \Rightarrow B \in U$$

($\because$ $B \subset U \Rightarrow B$ is open in Y.)

$\Rightarrow$ $f^{-1}(B)$ is open in X. Then f is continuous.

Conversely, suppose that f : X $\rightarrow$ Y is map such that $f^{-1}(B)$ is open in X for each B $\in \mathcal{B}$, $\mathcal{B}$ being a base for the topology U on Y. Let G $\in$ U be arbitrary. Then, by definition of base,

$\exists \mathcal{B}_1 \subset \mathcal{B}$ s.t. $G = \cup\{B : B \in \mathcal{B}_1\}$

$\therefore$ $f^{-1}(G) = f^{-1} \cup \{B : B \in \mathcal{B}_1\}$

$= \cup\{f^{-1}(B) : B \in B_1\}$

= An arbitrary union of open subsets of X

[$\because$ $f^{-1}(B)$ is open in X, by assumption]

$\Rightarrow$ An open subset of X.

$\therefore$ $f^{-1}(G)$ is open in X

Starting from an arbitrary open subset G of Y we are able to show that $f^{-1}(G)$ is open in X, showing thereby f is continuous.

***Theorem 5:*** To show that a one-one onto continuous map f : X $\rightarrow$ X′ is a homeomorphism if f is either open or closed.

***Proof:*** For the sake of convenience, we take X′ = Y.

Suppose f : (X, T) $\rightarrow$ (Y, V) is one-one onto and continuous map. Also suppose that f is either open or closed.

To prove that f is a homeomorphism, it is enough to show that $f^{-1}$ is continuous. For this we have to show that.

$f^{-1}\overline{(B)} \subset \overline{f^{-1}(B)}$. For any set $B \subset Y$.

$B \subset Y \Rightarrow \overline{f^{-1}(B)} \subset X$ is closed set

Also f is a closed map.

$\Rightarrow$ $\overline{f\left[f^{-1}(B)\right]} = \overline{\left[f\left(\overline{f^{-1}(B)}\right)\right]}$

Notes

Evidently

$$f^{-1}(B) \subset \overline{f^{-1}(B)} \qquad \text{...(1)}$$

$$\Rightarrow \quad f\{f^{-1}(B)\} \subset f\overline{\{f^{-1}(B)\}}$$

$$\Rightarrow \quad \overline{\left[f\{f^{-1}(B)\}\right]} \subset \overline{\left[f\overline{\{f^{-1}(B)\}}\right]}$$

$$\Rightarrow \quad \overline{\left[f\{f^{-1}(B)\}\right]} \subset f\overline{\{f^{-1}(B)\}} \qquad \text{(on using (1))}$$

$$\Rightarrow \quad ff^{-1}\overline{(B)} \subset f\overline{\left[f^{-1}(B)\right]}$$

$$\Rightarrow \quad f^{-1}\overline{(B)} \subset \overline{f^{-1}(B)}$$

$\Rightarrow$ $f^{-1}$ is continuous.

Similarly we can show that if f is open, that $f^{-1}$ is continuous

***Theorem 6:*** A map $f : (X, T) \to (Y, V)$ is closed iff

$\overline{f(A)} \subset f(\overline{A})$ for every $A \subset X$.

***Proof:*** Let $(X, T) \to (Y, V)$ be closed map and $A \subset X$ arbitrary.

To prove $\overline{f(A)} \subset f(\overline{A})$

$\overline{A}$ is closed subset of X, f is closed.

$\Rightarrow$ $f(\overline{A})$ is closed subset of Y.

$$\Rightarrow \quad \overline{f(\overline{A})} = f(\overline{A}) \qquad \text{...(1)}$$

But $A \subset \overline{A}$

$\Rightarrow$ $f(A) \subset f(A)$

$\Rightarrow$ $\overline{f(A)} \subset \overline{f(\overline{A})}$

$\Rightarrow$ $\overline{f(A)} \subset \overline{f(\overline{A})} = f(\overline{A})$, By (1)

$\Rightarrow$ $\overline{f(A)} \subset f(\overline{A})$,

Conversely, suppose $\overline{f(A)} \subset f(\overline{A}) \ \forall \ A \subset X$. ...(2)

To prove that f is closed.

Let F be a closed subset of X so that $\overline{F} = F$

$$\overline{F} = F \Rightarrow f(\overline{F}) = f(F) \qquad \text{...(3)}$$

Also, by (2), $\overline{\overline{f(F)}} \subset \overline{f(F)}$

Combining this with (3),

$\overline{f(F)} \subset f(F)$

But $f(F) \subset \overline{f(F)}$ [For $C \subset \overline{C}$ is true for any set C]

Combining the last two.

$\overline{f(F)} = f(F).$

$\Rightarrow$ f(F) is closed.

Thus F is closed. $\Rightarrow$ f(F) is closed.

$\therefore$ f is closed map.

***Theorem 7:*** A function f : (X, T) → (Y, V) is continuous iff

$\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}(B^{\circ}), B \subset Y.$

or $f^{-1}(B^{\circ}) \subset \left[f^{-1}(B)\right]^{\circ}$

***Proof:*** Let f : (X, T) → (Y, V) be a topological map. Let B ⊂ Y be arbitrary.

(i) Suppose f is continuous.

To prove that $\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}(B^{\circ})$

$B \subset Y \Rightarrow B^{\circ}$ is open in Y.

$\Rightarrow$ $f^{-1}(B^{\circ})$ is open in X. For f is continuous.

$\Rightarrow$ $\left[f^{-1}(B^{\circ})\right]^{\circ} = f^{-1}(B^{\circ})$ …(1)

$B^{\circ} \subset B \Rightarrow f^{-1}(B^{\circ}) \subset f^{-1}(B)$

$\Rightarrow f^{-1}(B) \supset f^{-1}(B^{\circ})$

$\Rightarrow \left[f^{-1}(B)^{\circ}\right] \supset \left[f^{-1}(B^{\circ})\right]^{\circ} = f^{-1}(B^{\circ}),$ [by (1)]

$\Rightarrow \left[f^{-1}(B)^{\circ}\right] \supset f^{-1}(B^{\circ})$

Proved.

(ii) Suppose $\left[f^{-1}(B)^{\circ}\right] \supset f^{-1}(B^{\circ})$ …(2)

To prove f is continuous.

Let G be an open subset of Y and hence G = G°

If we show that $f^{-1}(G)$ in open in X, the result will follow:

$\left[f^{-1}(G)\right]^{\circ} \supset f^{-1}(G^{\circ}),$ [by (2)]

$= f^{-1}(G)$

Notes

$\therefore \quad \left[f^{-1}(G)\right]^{\circ} \supset f^{-1}(G)$

But $\left[f^{-1}(G)\right]^{\circ} \subset f^{-1}(G)$ is always $\left[\text{for } C^{\circ} \subset C \ \forall \ C\right]$

Combining the last two, $[f^{-1}(G)]^{\circ} = f^{-1}(G)$

$\therefore \quad f^{-1}(G)$ is open in X.

*Example 5:* Let f : R → R be a constant map.

Prove that f is continuous.

*Solution:* Let f : R → R be a map given by

$$f(x) = c \ \forall \ x \in R. \quad \ldots(1)$$

Then evidently f is a constant map.

To show that f is continuous.

Let $G \subset R$ be an arbitrary open set.

By definition, $f^{-1}(G) = \left[x \in R : f(x) \in G\right] \quad \ldots(2)$

From (1) and (2), $f^{-1}(G) = \begin{cases} R \text{ if } c \in G, \\ \phi \text{ if } c \notin G, \end{cases}$

$\phi$ and R both are open sets in R and hence $f^{-1}(G)$ is open in R.

Given any open set G in R, we are able to show that $f^{-1}(G)$ is open in R. This proves that f is a continuous map.

*Example 6:* Let T and U be any two topologies on R. Let

$$f : (R, T) \to (R, U)$$

be a map given by $f(x) = 1 \ \forall \ x \in R$.

Then show that f is continuous.

*Hint:* take C = 1. Instead of writing

"Let $G \subset R$ be an open set", write

"$G \in U$ and $f^{-1}(G) \in T$".

Do these changes in the preceding solution.

## 7.2 Summary

- Let f : (X, T) → (Y, U) be a map.

  The map f is said to be continuous at $x_0 \in X$ is given any U open set H containing $f(x_0)$, $\exists$a T-open set G containing $x_0$ s.t. $f(G) \subset H$.

  If map is continuous at each $x \in X$, then the map is called a continuous map.

- A function f : (X, T) → (Y, U) is said to be continuous on a set $A \subset X$ if it is continuous at each point of A.

- A map $f : (X, T) \to (Y, U)$ is said to be homeomorphism or topological mapping if
  - (a) f is one-one onto.
  - (b) f and $f^{-1}$ are continuous.
- A map $f : (X, T) \to (Y, U)$ is called an open map if it maps open sets onto open sets i.e. if any $G \in T \Rightarrow f(G) \in U$.
- A map $f : (X, T) \to (Y, U)$ is called a closed map if any T-closed set F

  $\Rightarrow$ f(F) is U-closed set.

## 7.3 Keywords

***Discrete Space:*** Let X be any non empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

***Indiscrete Space:*** Let X be any non empty set and $T = \{X, \phi\}$. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

***Open and Closed set:*** Let (X, T) be a topological space. Any set $A \in T$ is called an open set and $X - A$ is a closed set.

## 7.4 Review Questions

1. In any topological space, prove that f and g are continuous maps $\Rightarrow$ gof is continuous map. Let A, B, C be metric spaces if $f : A \to B$ is continuous and $g : B \to C$ is continuous, then $gof : A \to C$ is continuous.
2. Show that characteristic function of $A \subset X$ is continuous on X iff A is both open and closed in X.
3. Suppose (X, T) is a discrete topological space and (Y, U) is any topological space. Then show that any map

$$f : (X, T) \to (Y, U)$$

   is continuous.
4. Let T be the cofinite topology on R. Let U denote the usual topology on R. Show that the identity map

$$f : (R, T) \to (R, U)$$

   is discontinuous, where as the identity map

$$g : (R, U) \to (R, T)$$

   is a continuous map.
5. Show that the map

$$f : (R, U) \to (R, U) \text{ given by}$$

$$f(x) = x^2 \ \forall \ x \in R \text{ is not open}$$

   U-denotes usual topology.

## 7.5 Further Readings

*Books*


J. L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Notes

# Unit 8: The Product Topology

CONTENTS



## Objectives

After studying this unit, you will be able to:

- Understand the product topology;
- Define Cartesian product and box topology;
- Solve the problems on the product topology.

## Introduction

There are two main techniques for making new topological spaces out of old ones. The first of these, and the simplest, is to form subspaces of some given space. The second is to multiply together a number of given spaces. Our purpose in this unit is to describe the way in which the latter process is carried out.

Previously, we defined a topology on the product X × Y of two topological spaces. In present unit, we generalize this definition to more general cartesian products. So, let us consider the cartesian products

$$X_1 \times \ldots \times X_n \quad \text{and} \quad X_1 \times X_2 \times \ldots,$$

where each $X_i$ is a topological space. There are two possible ways to proceed. One way is to take as basis all sets of the form $\cup_1 \times \ldots \times \cup_n$ in the first case, and of the form $\cup_1 \times \cup_2 \times \ldots$ in the second case, where $\cup_i$ is an open set of $X_i$ for each i.

## 8.1 The Product Topology

### 8.1.1 The Product Topology: Finite Products

***Definition:*** Let $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ be topological spaces. Then the product topology T on the set $X_1 \times X_2 \times \ldots \times X_n$ is the topology having the family $\{O_1 \times O_2 \times \ldots \times O_n, O_i \in T_i, i = 1, \ldots, n\}$

as a basis. The set $X_1 \times X_2 \times \ldots \times X_n$ with the topology T is said to be the product of the spaces $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ and is denoted by $(X_1 \times X_2 \times \ldots, X_n, T)$ or $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$.

***Proposition:*** Let $B_1, B_2, \ldots, B_n$ be bases for topological spaces $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$, respectively. Then the family $\{O_1 \times O_2 \times \ldots \times O_n : O_i \in B_i, i=1, ..., n\}$ is a basis for the product topology on $X_1 \times X_2 \times \ldots \times X_n$.

*Example 1:* Let $C_1, C_2, \ldots, C_n$ be closed subsets of the topological spaces $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$, respectively. Then $C_1 \times C_2 \times \ldots \times C_n$ is a closed subset of the product space $(X_1 \times X_2 \times \ldots \times X_n, T)$.

*Solution:* Observe that

$$(X_1 \times X_2 \times \ldots \times X_n)\setminus(C_1 \times C_2 \times \ldots \times C_n)$$
$$= [\,(X_1\setminus C_1) \times X_2 \times \ldots \times X_n\,] \cup [\,X_1 \times (X_2\setminus C_2) \times X_3 \times \ldots \times X_n\,] \cup \ldots \cup [\,X_1 \times X_2 \times \ldots \times X_{n-1} \times (X_n\setminus C_n)\,]$$

which is a union of open sets (as a product of open sets is open) and so is an open set in $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$. Therefore, its complement, $C_1 \times C_2 \times \ldots \times C_n$, is a closed set, as required.

*Notes*

(i) We now see that the euclidean topology on $R^n$, $n \geq 2$, is just the product topology on the set $R \times R \times \ldots R = R^n$.

(ii) Any product of open sets is an open set or more precisely: if $O_1, O_2, \ldots, O_n$ are open subsets of topological spaces $(X_1, T_1), (X_2, T_2) \ldots, (X_n, T_n)$, respectively, then $O_1 \times O_2 \times \ldots O_n$ is an open subset of $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$.

(iii) Any product of closed sets is a closed set.

### 8.1.2 The Product Topology: Infinite Products

Let $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n), \ldots$ be a countably infinite family of topological spaces. Then the product, $\prod_{i=1}^{\infty} X_i$, of the sets $X_i$, $i \in N$ consists of all the infinite sequences $\langle x_1, x_2, x_3, \ldots, x_n, \ldots\rangle$, where $x_i \in X_i$ for all i. (The infinite sequence $\langle x_1, x_2, \ldots, x_n, \ldots\rangle$ is sometimes written as $\prod_{i=1}^{\infty} x_i$ ). The product space, $\prod_{i=1}^{\infty}(X_i, T_i)$, consists of the product $\prod_{i=1}^{\infty} X_i$ with the topology T having as its basis the family

$$B = \left\{\prod_{i=1}^{\infty} O_i : O_i \in T_i \quad \text{and} \quad O_i = X_i \text{ for all but a finite number of i.}\right\}$$

The topology T is called the product topology. So a basic open set is of the form

$$O_1 \times O_2 \times \ldots \times O_n \times X_{n+1} \times X_{n+2} \times \ldots$$

*Note* It should be obvious that a product of open sets need not be open in the product topology T. In particular, if $O_1, O_2, O_3, \ldots, O_n, \ldots$ are such that $O_i \in T_i$ and $O_i \neq X_i$ for all i, then $\prod_{i=1}^{\infty} O_i$ cannot be expressed as a union of members of B and so is not open in the product space $\left(\prod_{i=1}^{\infty} X_i, T\right)$.

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*Example 2:* Let $(X_1, T_1), \dots, (X_n, T_n), \dots$ be a countably infinite family of topological spaces. Then the box topology T′ on the product $\prod_{i=1}^{\infty} X_i$ is that topology having as its basis the family

$$B' = \left\{\prod_{i=1}^{\infty} O_i : O_i \in T_i\right\}$$

It is readily seen that if each $(X_i, T_i)$ is a discrete space, then the box product $\left(\prod_{i=1}^{\infty} X_i, T'\right)$ is a discrete space. So if each $(X_i, T)$ is a finite set with the discrete topology, then $\left(\prod_{i=1}^{\infty} X_i, T'\right)$ is an infinite discrete space, which is certainly not compact. So, we have a box product of the compact spaces $(X_i, T_i)$ being a non-compact space.

*Example 3:* Let $(X_i, T_i), \dots, (Y_i, T_i')$, $i \in N$, be countably infinite families of topological spaces having product spaces $\left(\prod_{i=1}^{\infty} X_i, T\right)$ and $\left(\prod_{i=1}^{\infty} Y_i, T'\right)$ respectively. If the mapping $h_i: (X_i, T_i) \to (Y_i, T_i)$ is continuous for each $i \in N$, then so is the mapping $h: \left(\prod_{i=1}^{\infty} X_i, T\right) \to \left(\prod_{i=1}^{\infty} Y_i, T'\right)$ given by $h : \left(\prod_{i=1}^{\infty} x_i\right) = \prod_{i=1}^{\infty} h_i(x_i)$; that is, $h(\langle x_1, x_2, \dots, x_n, \dots\rangle) = \langle h_1(x_1), h_2(x_2), \dots, h_n(x_n), \dots\rangle$.

*Solution:* It suffices to show that if O is a basic open set in $\left(\prod_{i=1}^{\infty} Y_i, T'\right)$, then $h^{-1}(O)$ is open in $\left(\prod_{i=1}^{\infty} X_i, T\right)$. Consider the basic open set $\cup_1 \times \cup_2 \times \dots \cup_n \times Y_{n+1} \times Y_{n+2} \times \dots$ where $\cup_i \in T'$, for $i = 1, \dots, n$. Then

$$h^{-1}(\cup_1 \times \dots \times \cup_n \times Y_{n+1} \times Y_{n+2} \times \dots)$$
$$= h_1^{-1}(\cup_1) \times \dots \times h_n^{-1}(\cup_n) \times h^{-1}(Y_{n+1}) \times h^{-1}(X_{n+2}) \times \dots$$

and the set on the right hand side is in T, since the continuity of each $h_i$ implies $h_1^{-1}(\cup_i) \in T_i$, for $i = 1, \dots, n$. So h is continuous.

### 8.1.3 Cartesian Product

***Definition:*** Let $\{A_\alpha\}_{\alpha\in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha\in J} A_\alpha$. The cartesian product of this index family, denoted by $\prod_{\alpha\in J} A_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha\in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha\in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

### 8.1.4 Box Topology

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha\in J} X_\alpha$ the collection of all sets of the for $\prod_{\alpha\in J} \cup_\alpha$, where $\cup_\alpha$ is open in $X_\alpha$, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

*Example 4:* Consider euclidean n-space $\mathbb{R}^n$. A basis for $\mathbb{R}$ consists of all open intervals in $\mathbb{R}$; hence a basis for the topology of $\mathbb{R}^n$ consists of all products of the form

$$(a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n)$$

Since $\mathbb{R}^n$ is a finite product, the box and product topologies agree. Whenever we consider $\mathbb{R}^n$, we will assume that it is given this topology, unless we specifically state otherwise.

*Example 5:* Consider $\mathbb{R}^w$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^w = \prod_{n\in\mathbb{Z}_+} X_n,$$

where $X_n = \mathbb{R}$ for each n. Let us define a function $f : \mathbb{R} \to \mathbb{R}^w$ by the equation

$$f(t) = (t, t, t, \ldots);$$

the $n^{th}$ coordinate function of f is the function $f_n(t) = t$. Each of the coordinate functions $f^n : \mathbb{R} \to \mathbb{R}$ is continuous; therefore, the function f is continuous if $\mathbb{R}^w$ is given the product topology. But f is not continuous if $\mathbb{R}^w$ is given the box topology. Consider, for example, the basic element.

$$B = (-1, 1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \ldots$$

for the box topology. We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. This would mean that $f((-\delta, \delta)) \subset B$ so that, applying $\pi_n$ to both sides of the inclusion.

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset \left(-\tfrac{1}{n}, \tfrac{1}{n}\right)$$

for all n, a contradiction.

***Theorem 1:*** Let $\{X_\alpha\}$ be an indexed family of spaces; Let $A_\alpha \subset X_\alpha$ for each α. If $\Pi X_\alpha$ is given either the product or the box topology, then

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$

***Proof:*** Let $x = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$; we show that $x \in \overline{\prod A_\alpha}$.

Let $\cup = \Pi \cup_\alpha$ be a basis element for either the box or product topology that contains x. Since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in \cup_\alpha \cap A_\alpha$ for each α. Then $y = (y_\alpha)$ belongs to both $\cup$ and $\Pi A_\alpha$. Since $\cup$ is arbitrary, it follows that x belongs to the closure of $\Pi A_\alpha$.

Conversely, suppose $x = (x_\alpha)$ lies in the closure of $\Pi A_\alpha$, in either topology. We show that for any given index β, we have $x_\beta \in \bar{A}_\beta$. Let $V_\beta$ be an arbitrary open set of $X_\beta$ containing $x_\beta$. Since $\prod_\beta^{-1}(V_\beta)$ is open in $\Pi X_\alpha$ in either topology, it contains a point $y = (y_\alpha)$ of $\Pi A_\alpha$. Then $y_\beta$ belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \bar{A}_\beta$.

***Theorem 2:*** Let $f : A \to \Pi_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\Pi X_\alpha$ have the product topology. Then the function f is continuous if and only if each function $f_\alpha$ is continuous.

***Proof:*** Let $\pi_\beta$ be the projection of the product onto its βth factor. The function $\pi_\beta$ is continuous, for if $\cup_\beta$ is open in $X_\beta$, the set $\pi_\beta^{-1}(\cup_\beta)$ is a sub basis element for the product topology on $X_\alpha$. Now suppose that $f : A \to \Pi X_\alpha$ is continuous. The function $f_\beta$ equals the composite $\pi_\beta$ of; being the composite of two continuous functions, it is continuous.

Conversely suppose that each co-ordinate function $f_\alpha$ is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each sub-basis element is open in A; we remarked on this fact when we defined continuous functions. A typical sub-basis element for the product topology on $\Pi X_\alpha$ is a set of the form $\pi_\beta^{-1}(\cup_\beta)$, where β is some index and $\cup_\beta$ is open $X_\beta$. Now

$$f^{-1}\left(\pi_\beta^{-1}(\cup_\beta)\right) = f_\beta^{-1}(\cup_\beta),$$

because $f_\beta = \pi_\beta$ of. Since $f_\beta$ is continuous, this set is open in A, as desired.

## 8.2 Summary

- Let $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ be topological spaces. Then the product topology T on the set $X_1 \times X_2 \times \ldots \times X_n$ is the topology having the family $\{O_1 \times O_2 \times \ldots \times O_n, O_i \in T_i, i = 1, \ldots, n\}$ as a basis. The set $X_1 \times X_2 \times \ldots \times X_n$ with the topology T is said to be the product of the space $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ and is denoted by $(X_1 \times X_2 \times \ldots, X_n, T)$.
- The product space, $\prod_{i=1}^{\infty} (X_i, T_i)$, consists of the product $\prod_{i=1}^{\infty} X_i$ with the topology T having as its basis the family

$$B = \left\{\prod_{i=1}^{\infty} O_i : O_i \in T_i \quad \text{and} \quad O_i \in X_i \text{ for all but a finite number of i.}\right\}$$

  The topology T is called the product topology.
- The cartesian product of this index family, denoted by $\prod_{\alpha \in J} A_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.
- Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the for $\prod_{\alpha \in J} \cup_\alpha$, where $\cup_\alpha$ is open in $X_\alpha$, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

## 8.3 Keywords

***Discrete Space:*** Let X be any non empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

***Indiscrete Space:*** Let X be any non empty set and T = {X, ϕ}. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

***Open & Closed Set:*** Any set $A \in T$ is called an open subset of X or simply a open set and X – A is a closed subset of X.

***Topological Space:*** Let X be a non empty set. A collection T of subsets of X is said to be a topology on X if

(i) $X \in T, \phi \in T$

(ii) $A \in T, B \in T \Rightarrow A \cap B \in T$

(iii) $A_\alpha \in T \ \forall \ \alpha \in \Lambda \Rightarrow \cup A_\alpha \in T$ where Λ is an arbitrary set.

## 8.4 Review Questions

1. If $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ are discrete spaces, prove that the product space $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$ is also a discrete space.
2. Let $X_1$ and $X_2$ be infinite sets and $T_1$ and $T_2$ the finite-closed topology on $X_1$ and $X_2$, respectively. Show that the product topology, T, on $X_1 \times X_2$ is not the finite-closed topology.

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3. Prove that the product of any finite number of indiscrete spaces is an indiscrete space.

4. For each $i \in N$, let $C_i$ be a closed subset of a topological space $(X_i, T_i)$. Prove that $\prod_{i=1}^{\infty} C_i$ is a closed subset of $\prod_{i=1}^{\infty}(X_i, T_i)$.

5. Let $(X_i, T_i)$, $i \in N$, be a countably infinite family of topological spaces. Prove that each $(X_i, T_i)$ is homeomorphic to a subspace of $\prod_{i=1}^{\infty}(X_i, T_i)$.

## 8.5 Further Readings

*Books*

Dixmier, *General Topology* (1984).

James R. Munkres, *Topology*, Second Edition, Pearson Prentice Hall.

*Online links*

mathworld.wolfram.com/product topology.html

www.history.mcs.st-and.ac.uk/~john/MT4522/Lectures/L1.5.html

# Unit 9: The Metric Topology

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Define metric space and pseudo metric space;
- Understand the definitions of open and closed spheres, boundary set, open and closed set;
- Define convergence of a sequence in a metric space and interior, closure and boundary of a point;
- Define neighborhood and limit point;
- Solve the problems on metric topology.

## Introduction

The most important class of topological spaces is the class of metric spaces. Metric spaces provide a rich source of examples in topology. But more than this, most of the applications of topology to analysis are via metric spaces. The notion of metric space was introduced in 1906 by Maurice Frechet and developed and named by Felix Hausdorff in 1914.

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set. Topologies given in this way lie at the heart of

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modern analysis. For example, In this section, we shall define the metric topology and shall give a number of examples. In the next section, we shall consider some of the properties that metric topologies satisfy.

## 9.1 The Metric Topology

### 9.1.1 Metric Space

Let $X \neq \phi$ be any given space.

Let $x, y, z \in X$ be arbitrary.

A function $d : X \times X \to R$ having the properties listed below:

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality)

is called a **distance function** or **a metric** for X. Instead of saying, "Let X be a non-empty set with a metric d defined on it". We always say, "Let (X, d) be a metric space".

Evidently, d is a real valued map and d denotes the distance between x and y. A set X, together with a metric defined on it, is called metric space.

*Example 1:*

(1) Let $X = R$ and $\rho(x, y) = |x - y|\ \forall x, y \in X$. Then $\rho$ is a metric on X. This metric is defined as usual metric on R.

(2) Let $x, y \in R$ be arbitrary

Let $$\rho(x, y) = \begin{cases} 0 & \text{iff} \quad x = y \\ 1 & \text{iff} \quad x \neq y \end{cases}$$

Then $\rho$ is a metric on R.

This metric is defined as trivial metric or discrete metric on R.

### 9.1.2 Pseudo Metric Space

Let $X \neq \phi$ be any given space. Let $x, y, z \in X$ be arbitrary. A function $d : X \times X \to R$ having the properties listed below:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, y) + d(y, z) \geq d(x, z)$,

Where $x, y, z \in X$

is called pseudo metric on x. The set X together with the pseudo metric d is called pseudo metric space. Pseudo metric differs from metric in the sense that.

$$d(x, y) = 0 \text{ even if } x \neq y$$

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Thus for a pseudo metric

$$x = y \Rightarrow d(x, y) = 0$$

but not conversely.

***Remark:*** Thus every metric space is a pseudo metric space but every pseudo metric space is not necessarily metric space.

*Example 2:* Consider a map $d : R \times R \to R$ s.t.

$$d(x, y) = |x^2 - y^2| \quad \forall\ x, y \in R$$

Evidently $\quad d(x, y) = 0 \Rightarrow x = +y.$

It can be shown that d is a pseudo metric but not metric.

### 9.1.3 Open and Closed Sphere

Let $(X, \rho)$ be a metric space.

Let $x_o \in X$ and $r \in R^+$. Then set $\{ x \in X : \rho(x_o, x) < r \}$ is defined as open sphere (or simply sphere) with centre $x_o$ and radius r.

The following have the same meaning:

Open sphere, closed sphere, open ball and open disc.

We denote this **open sphere** by the symbol $S(x_o, r)$ or by $S_r(x_o)$ or by $B_r(x_o, d)$ or $B(x_o, r)$. This open sphere is also called as Spherical neighborhood of the point $x_o$ or r-nhd of the point $x_o$.

We denote closed sphere by $S_r[x_o]$ and is defined as

$$S_r[x_o] = \{ x \in X : \rho(x, x_o) \le r \}.$$

The following have the same meaning:

Closed sphere, closed ball, closed cell and disc.

#### Examples on Open Sphere

In case of usual metric, we see that

(i) If $X = R$, then $S_r(x_o) = (x_o - r, x_o + r)$ = open interval with $x_o$ as centre.

(ii) If $X = R^2$, then $S_r(x_o)$ = open circle with centre $x_o$ and radius r.

(iii) If $X = R^3$, then $S_r(x_o)$ = open sphere with centre $x_o$ and radius r.

### 9.1.4 Boundary Set, Open Set, Limit Point and Closed Set

#### Boundary Set

Let (X, d) be a metric space and $A \subset X$. A point x in X is called a boundary point of A if each open sphere centered at x intersects A and A′. The boundary of A is the set of all its boundary points and is denoted by b(A). It has following properties.

(1) b(A) is a closed set

(2) $b(A) = A \cap A'$

(3) A is closed $\Leftrightarrow$ A contains its boundary.

### Open Set

Let $(X, \rho)$ be a metric space.

A non-empty set $G \subset X$ is called an open set if any $x \in G \Rightarrow \exists\, r \in R^+$ s.t. $S_r(x) \subset G$.

### Limit Point

Let $(X, \rho)$ be a metric space and $A \subset X$. A point $x \in X$ is called a limit point or limiting point or accumulation point or cluster point if every open sphere centered on x contains a point of A other than x, i.e., $x \in X$ is called limit point of A if $(S_{r(x)} - \{x\}) \cap A \neq \phi$, $r \in R^+$.

The set of all limiting points of a set A is called **derived set of A** and is denoted by D(A).

### Closed Set

Let $(X, \rho)$ be a metric space and $A \subset X$. A is called a closed set if the derived set of A i.e., $D(A) \subset A$ i.e., if every limit point of A belongs to the set itself.

## 9.1.5 Convergence of a Sequence in a Metric Space

Let $<x_n>$ be a sequence in a metric space $(X, \rho)$. This sequence is said to coverage to $x_0 \in X$, if given any $\varepsilon > 0$, $\exists\, n_o \in N$ s.t. $n \geq n_0 \Rightarrow \rho(x_n, x_0) < \varepsilon$ or equivalently, given any $\varepsilon > 0$, $\exists\, n_0 \in N$ s.t. $n \geq n_0 \Rightarrow x_n \in S_\varepsilon(x_0)$.

## 9.1.6 Theorems on Closed Sets and Open Sets

***Theorem 1:*** In a metric space $(X, \rho)$, $\phi$ and X are closed sets.

***Proof:*** Let $(X, \rho)$ be a metric space.

To prove that $\phi$ and X are closed sets.

$\because \qquad D(\phi) = \phi \subset \phi$

$\therefore \qquad D(\phi) \subset \phi.$

$\Rightarrow \phi$ is a closed set.

All the limiting points of X belong to X. For X is the universal set.

i.e., any $\qquad x \in D(x) \Rightarrow x \in X$

$\because \qquad D(X) \subset X$

$\Rightarrow$ X is a closed set.

***Theorem 2:*** Let (X, d) be a metric space. Show that $F \subset X$, F is closed $\Leftrightarrow F'$ is open.

***Proof:*** Let (X, d) be a metric space.

Let F be a closed subset of X, so that $D(F) \subset F$.

To prove that $F'$ is open in X.

Let $x \in F'$ be arbitrary. Then $x \notin F$.

$$D(F) \subset F,\ x \notin F \Rightarrow x \notin D(F)$$

$$\Rightarrow (S_{r(x)} - \{x\}) \cap F = \phi \text{ for some } r > 0$$

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$$\Rightarrow S_{r(x)} \cap F = \phi \quad [\because x \notin F]$$
$$\Rightarrow S_{r(x)} \subset X - F$$
$$\Rightarrow S_{r(x)} \subset F'.$$

$\therefore$ Given $x \in F'$, $\exists$ any open sphere $S_{r(x)}$ s.t.

$$S_{r(x)} \subset F'$$

By definition, this proves that F′ is open.

Conversely suppose that F′ is open in X.

To prove that F is closed in X.

Let $x \in F'$ be arbitrary, then $x \notin F$.

$\because$ F′ is open, $\exists_r \subset R^+$ s.t., $S_{r(x)} \subset F'$

$$\Rightarrow S_{r(x)} \cap F = \phi$$
$$\Rightarrow (S_{r(x)} - \{x\}) \cap F = \phi$$
$$\Rightarrow x \notin D(F).$$

Thus, any $x \in F' \Rightarrow x \notin D(F)$

i.e. any $x \in X - F \Rightarrow x \in X - D(F)$

$$\Rightarrow X - F \subset X - D(F) \text{ or } D(F) \subset F$$
$$\Rightarrow F \text{ is closed.}$$

***Theorem 3:*** In any metric space (X, d), each open sphere is an open set.

***Proof:*** Let (X, d) be a metric space. Let $S_{r_0(x_0)}$ be an open sphere in X.

To prove that $S_{r_0(x_0)}$ is an open set.

Let $x \in S_{r_0(x_0)}$ be arbitrary, then $d(x, x_0) < y_0$

Write $\quad r = r_0 - d(x, x_0) \quad$ ...(1)

By definition $\quad S_{r_0(x_0)} = \{y \in X : d(y, x_0) < r_0\}$

$$S_{r(x)} = \{y \in X : d(y, x) < r\}.$$

We claim $\quad S_{r(x)} \subset S_{r_{(0)}(x_0)}$

Let $y \in S_{r(x)}$ be arbitrary

Then $\quad d(x, y) < r$

$$d(y, x_0) \leq d(y, x) + d(x, x_0)$$
$$< r + d(x, x_0) = r_0. \quad \text{[on using (1)]}$$

$\therefore \quad d(y, x_0) < r_0$

$$\Rightarrow y \in S_{r_0(x_0))}$$

and $\quad y \in S_{r(x)} \Rightarrow y \in S_{r_0(x_0)}$

$$\Rightarrow S_{r(x)} \subset S_{r_0(x_0)}$$

Thus we have shown that for given any $x \in S_{r_0(x_0)}$, $\exists\, r > 0$ s.t. $S_{r(x)} \subset S_{r_0(x_0)}$.

By definition, this proves that $S_{r_0(x_0)}$ is an open set.

***Theorem 4:*** In any metric space, any closed sphere is a closed set.

***Proof:*** Let $S_{r_0[x_0]}$ denote a closed sphere in a metric space (X, d).

To prove that $S_{r_0[x_0]}$ is a closed set.

For this we must show that $S'_{r_0[x_0]}$ is open in X.

Let $x \in S'_{r_0[x_0]}$ be arbitrary,

$$x \in S'_{r_0[x_0]} \Rightarrow x \notin S_{r_0[x_0]}$$

$$\Rightarrow d(x, x_0) > r_0, \qquad \left[\because S_{r_0[x_0]} = \{y \in X : d(y, x_0) \le r_0\}\right]$$

$$\Rightarrow d(x, x_0) - r_0 > 0$$

$$\Rightarrow r > 0, \text{ on taking } r = d(x, x_0) - r_0 \qquad \ldots(1)$$

We claim $S_{r(x)} \subset S'_{r_0[x_0]}$.

Let $y \in S_{r(x)}$ be arbitrary, so that, d (y, x) < r.

$\therefore \quad d(x, x_0) \le d(x, y) + d(y, x_0)$.

$\therefore \quad d(y, x_0) \ge d(x, x_0) - d(x, y) > d(x, x_0) - r = r_0 \qquad$ [on using (1)]

$\therefore \quad d(y, x_0) > r_0 \Rightarrow y \notin S_{r_0[x_0]}$

Thus, any $\quad y \in S_{r(x)} \Rightarrow y \in S'_{r_0[x_0]}$

$$\Rightarrow S_{r(x)} \subset S'_{r_0[x_0]}$$

$\therefore$ Given any $x \in S'_{r_0[x_0]}, \exists r > 0$ s.t. $S_{r(x)} \subset S'_{r_0[x_0]}$

This prove that $S'_{r_0[x_0]}$ is open in x.

*Example 3:* Give an example to show that the union of an infinite collection of closed sets in a metric space is not necessarily closed.

*Solution:* Let $\{[\frac{1}{n}, 1] : n \in N\}$ be the infinite collection for the usual metric space ($\mathcal{R}$, d).

Now each member of this collection is a closed set, being a closed interval.

But $\cup\{[\frac{1}{n}, 1] : n \in N\} = \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup \ldots = ]0, 1]$.

Since ]0, 1] is not closed, it follows that the union of an infinite collection of closed sets is not closed.

*Example 4:* Show that every closed interval is a closed set for the usual metric on $\mathcal{R}$.

*Solution:* Let x, y $\in \mathbb{R}$ where x < y. We shall show that [x, y] is closed.

Now $\quad \mathcal{R} - [x, y] = \{a \in \mathcal{R} : a < x \text{ or } a > y\}$

$$= \{a \in \mathcal{R} : a < x\} \cup \{a \in \mathcal{R} : a > y\}$$

$$= ]-\infty, x[ \cup ]y, \infty[$$

which is open, being a union of two open sets.

Hence [x, y] is closed.

*Example 5:* Give an example of two closed subsets A and B of the real line $\mathcal{R}$ such that d(A, B) = 0 but $A \cap B = \phi$.

*Solution:* Let $A = \{2, 3, 4, 5, \ldots\}$

$$B = \left\{2\tfrac{1}{2}, 3\tfrac{1}{3}, 4\tfrac{1}{4}, \ldots\right\},$$

Clearly $A \cap B = \phi$.

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

If $n \in A$ and $n + \frac{1}{n} \in B$

$$d(A, B) = \lim_{n\to\infty} d\left(n, n + \tfrac{1}{n}\right)$$

$$= \lim_{n\to\infty} \tfrac{1}{n} \qquad [\because \text{ d is usual metric for } \mathcal{R}]$$

$$= 0$$

### 9.1.7 Interior, Closure and Boundary of a Point

#### Interior

Let (X, d) be a metric space and $A \subset X$.

A point $x \in A$ is called an interior point of A if $\exists\, r \in R^+$ s.t. $S_{r(x)} \subset A$.

The set of all interior point of A is called the interior of A and is denoted by A°, or by Int. (A).

Thus $A° = \text{int.}(A) = \{x \in A : S_{r(x)} \subset A \text{ for some r})$

Alternatively, we define

$$A° = \cup (S_{r(x)} : S_{r(x)} \subset A).$$

Evidently

(i) A° is an open set.

For an arbitrary union of open sets is open.

(ii) A° is the largest open subset of A.

#### Closure

Let (X, d) be a metric space and $A \subset X$.

The closure of A, denoted by $\bar{A}$, is defined as the intersection of all closed sets that contain A. Symbolically

$$\bar{A} = \cap \{F \subset X : F \text{ is closed}, F \supset A\} \qquad \ldots(1)$$

Evidently

(i) $\bar{A}$ is closed set

For an arbitrary intersection of closed sets is closed.

(ii) $\bar{A} \supset A$.

(iii) $\bar{A}$ is the smallest closed set which contain A.

Alternatively we define

$$\bar{A} = A \cup D(A) \qquad ...(2)$$

A point $x \in \bar{A}$ is called a point of closure of A.

Alternatively, a point $x \in X$ is called a point of closure of A iff $x \in A$ or $x \in D(A)$.

## Boundary of a Point

Let (X, d) be a metric space. Let $A \subset X$

(i) ***Boundary or Frontier*** of a set A is denoted by b(A) or $F_r(A)$ and is defined as

$$b(A) = F_r(A) = X - A° \cup (X - A)°.$$

Elements of b(A) are called boundary points of A.

(ii) ***The exterior of A*** is defied as the set $(X - A)°$ and is denoted by ext (A).

Symbolically ext (A) = $(X - A)°$.

(iii) A is said to be *dense or everywhere dense* in X if $\bar{A} = X$.

(iv) A is said to be *somewhere dense* if $(\bar{A})° \neq \phi$ i.e., if closure of A contains some open set.

(v) A is said to be *nowhere dense* (or non where dense set) if $(\bar{A})° = \phi$.

(vi) A metric space (X, d) is said to be separable if $\exists A \subset X$ s.t. A is countable and $\bar{A} = X$.

(vii) A is said to be *dense* in itself of $A \subset D(A)$.

*Example 6:*

(1) To find the boundary of set of integers Z and set of rationals Q.

$$Z° = \cup \{G \subset R : G \text{ is open ad } G \subset Z\} = \phi$$

For every sub set of R contains fractions also.

$$\text{Similarly } (R - Z)° = \phi$$

$$b(Z) = R - Z° \cup (R - Z)° = R - \phi \cup \phi = R$$

$$b(Z) = R.$$

$$\text{Similarly } b(Q) = R.$$

(2) Give two examples of limit points

(i) If $A = \left\{1 + \frac{1}{n} : n \in N\right\}$,

i.e. $A = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, ....\right\}$, then

1 is limit point of A. For $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$.

Notes

(ii) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ....\right\} = \left\{\frac{1}{n} : n \in N\right\}$

then 0 is the limit point of A

For $\lim_{n\to\infty} \frac{1}{n} = 0$

### 9.1.8 Neighborhood

Let (X, d) be a metric space and $x \in X$, $A \subset X$

A subset A of X is called a neighborhood (nbd) of x if $\exists$ open sphere $S_{r(x)}$ s.t. $S_{r(x)} \subset A$

This means that A is nbd of a point x iff x is an interior point of A.

From the definition of nbd, it is clear that:

(1) Every superset of a nbd of a point is also a nbd.

(2) Every open sphere $S_{r(x)}$ is a nbd of x.

(3) Every closed sphere $S_{r(x)}$ is a nbd of x.

(4) Intersection of two nbds of the same point is given a nbd of that point.

(5) A set is open if it contains a nbd of each of its points.

(6) Nbd of a point need not be an open set.

### 9.1.9 Theorems and Solved Examples

***Theorem 5:*** A subset of a metric space is open iff it is a nbd of each of its point.

***Proof:*** Let A be a subset of a metric space (X, d).

*Step I:* Given A is a nbd of each of its points.

*Aim:* A is an open set

Recall that a set N is called nbd of a point $x \in X$ if $\exists$ open set $G \subset X$ s.t. $x \in G \subset N$.

Let $p \in A$ be arbitrary, then by assumption, A is a nbd of p. By definition of nbd, $\exists$ open set $G_p \subset X$ s.t. $p \in G_p \subset A$.

It is true $\forall p \in A$

Take $A = \cup \{G_p : p \in G_p, G_p \text{ is an open set}, G_p \subset A\}$

= An arbitrary union of open sets

= open set

$\therefore$ A is an open set.

*Step II:* Let A be an open subset of X.

*Aim:* A is a nbd of each of its points. By assumption, we can write $p \in A \subset A \ \forall p \in A$ .

$\Rightarrow$ A is a nbd of each of its points.

*Problem:* Every set of discrete metric space is open.

*Solution:* Let (X, d) be a discrete metric space. Let x, y ∈ X be arbitrary. By definition of discrete metric,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Let r be any positive real number s.t. r ≤ 1.

Then $S_{r\,(x)} = \{y \in X : d\,(y, x) < r \leq 1\}$

$= \{y \in X : d\,(y, x) < 1\}$

$= \{y \in X : d\,(y, x) = 0\}$ (by definition of d)

$= \{y \in X : y = x\} = \{0\}$

or $S_{r\,(x)} = \{0\}$

But every open sphere is an open set.

∴ {x} is an open set is X ∀ x ∈ X.

If $A = \{x_1, x_2, ..., x_n\}$ = finite set ⊂ X, then

$A = \bigcup_{r=1}^{n} \{x_r\}$ = finite union of open sets.

= open set.

Hence every finite subset of X is open set. ...(1)

If $B = \{x_1, x_2, x_3, ....\} \subset X$, then

B is an infinite subset of X.

Now $B = \bigcup_{r=1}^{\infty} \{x_r\}$

= Arbitrary union of open sets

= Open set,

∴ B is an open set. ...(2)

From (1) and (2), it follows that every subset (finite or infinite) is an open set in X.

*Problem:* A finite set in any metric space has no limit point.

*Solution:* Let A be a finite subset of a metric space (X, d). We know that "x ∈ X is a limit point of any set B if every open sphere $S_{r\,(x)}$ contains an infinite number of points of B other than *x*."

This condition can not be satisfied here as A is finite set.

Hence A has no limit point.

***Theorem 6:*** Let (X, d) be a metric space. A subset A of X is closed if given any x ∈ X – A, d (x, A) ≠ 0.

***Proof:*** Let (X, d) be a metric space and A ⊂ X be an arbitrary closed set.

To prove that

Given any x ∈ X – A, d (x, A) ≠ 0

Notes

A is closed ⇒ X – A is open.

By definition of open set,

any $x \in X - A \Rightarrow \exists_r \in R^+$ s.t. $S_{r_{(x)}} \subset X - A \Rightarrow S_{r_{(x)}} \cap A = \phi$

$$\Rightarrow d(X, A) \geq r \Rightarrow d(x, A) \neq 0.$$

Conversely let A be any subset of a metric space (X, d).

Let any $x \in X - A \Rightarrow d(x, A) \neq 0.$

To prove that A is closed.

Let $x \in X - A$ be arbitrary so that, by assumption

$$d(x, A) = r \neq 0 \Rightarrow S_{r_{(x)}} \cap A = \phi \Rightarrow S_{r_{(x)}} \subset X - A$$

$\therefore$ $x \in X - A \Rightarrow \exists\, r \in R^+$ s.t. $S_{r_{(x)}} \in X - A.$

By definition, this implies X – A is open

⇒ A is closed

Proved.

*Problem:* In any metric space, show that

$$X - \bar{A} = (X - A)^\circ$$

or $(\bar{A})' = (A')^\circ.$

*Solution:* $(\bar{A})' = X - \bar{A}$

= X – Intersection of all closed super sets of A

$= X - \bigcap_i F_i$ where $F_i$ is closed and $F_i \supset A$

$= \bigcup_i (X - F_i)$ where $X - F_i$ is open and $X - F_i \subset X - A$

= Union of open subsets of X – A = A′

$= (A')^\circ.$

Proved.

*Problem:* In any metric space (X, d), prove that A is open ⇔ A° = A.

***Solution:*** Let A be a subset of a metric space (X, d). By definition of interior,

$$A^\circ = \cup\{S_{r_{(x)}} : S_{r_{(x)}} \subset A\} \quad ...(1)$$

since every open sphere is an open set and arbitrary union of open sets is open.

Consequently,

A° is an open set. ...(2)

By (1), it is clear that A° ⊂ A ...(3)

and A° is largest open subset of A. ...(4)

Notes

(i) Given $A = A°$ ...(5)

*Aim:* A is an open set

(2) and (5) ⇒ A is an open set.

(ii) Given A is an open set. ...(6)

*Aim:* $A = A°$

(4) and (6) ⇒ $A = A°$.

## 9.1.10 Uniform Convergence

A sequence defined on a metric space (X, d) is said to be uniformly convergent if given $\varepsilon > 0$, $\exists$ $n_0 \in N$ s.t. $n \geq n_0 \Rightarrow d(f_n(x), f(x)) < \varepsilon \ \forall \ \in X$.

***Theorem 7:*** Let $<f_n(x)>$ be a sequence of continuous functions defined on a metric space (X, d). Let this sequence converge uniformly to f on X. Then f(x) is continuous on X.

OR

Uniform limit of a sequence of continuous function is continuous.

***Proof:*** Since $< f_n(x)>$ converges uniformly to f on (X, d). Hence given $\varepsilon > 0$, $\exists n_0 \in N$ independent of $x \in N$ s.t. $n \geq n_0$.

$$\Rightarrow d(f_n(x), f(x)) < \varepsilon/3 \quad ...(1)$$

Let $a \in X$ be arbitrary. To prove that f is continuous on X, we have to prove that f is continuous at x = a, for this we have to show that given $\varepsilon > 0$, $\exists \ \delta > 0$ s.t. $d(x, a) < \delta$

$$\Rightarrow d(f(x), f(a)) < \varepsilon. \quad ...(2)$$

Continuity of $f_n$ at $a \in X$

$$\Rightarrow d(f_n(x), f_n(a)) < \frac{\varepsilon}{3} \text{ for } d(x, a) < \delta \quad ...(3)$$

By (1), $d(f_n(a), f(a)) < \frac{\varepsilon}{3} \forall \ n \geq n_0$ ...(4)

If $d(x, a) < \delta$, then

$$d(f(x), f(a)) \leq d[f(x), f_n(x)] + d[f_n(x), f_n(a)] + d[f_n(a), f(a)]$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ by (1), (3) and (4)}$$

or $d(f(x), f(a) < \varepsilon$ for $d(x, a) < \delta$. Hence the result (2).

***Theorem 8:*** **Frechet space.** Let F be the set of infinite sequences of real numbers.

Let $x, y, z \in F$, then

$$x = <x_n> = <x_1, x_2 ...>, y = <y_n>, z = <z_n>$$

where $x_n, y_n, z_n \in R$

we define a map

$$d : F \times F : \to R \text{ s.t.}$$

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{[1 + |x_n - y_n|]}$$

Notes

To show that d is metric on F.

(i) $d(x, y) \geq 0$. For $|x_n - y_n| \geq 0 \ \forall n$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

For $$d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} = 0$$

$$\Leftrightarrow \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1+|x_n - y_n|} = 0 \ \forall n$$

$$\Leftrightarrow |x_n - y_n| = 0 \ \forall_n \Leftrightarrow x_n = y_n \forall n$$

$$\Leftrightarrow x = y$$

(iii) $$d(x, y) = d(y, x)$$

For $$|x_n - y_n| = |y_n - x_n|$$

(iv) $$d(x, y) \geq d(x, z) + d(z, y)$$

Here we use the fact that

$$\frac{|\alpha+\beta|}{1+|\alpha+\beta|} \geq \frac{|\alpha|}{1+|\alpha|} + \frac{|\beta|}{1+|\beta|}$$

In view of this, we have

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{1+|x_n - y_n|} \cdot \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|(x_n - z_n)+(z_n - y_n)|}{1+|(x_n - z_n)+(z_n - y_n)|}$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - z_n|}{1+|x_n - z_n|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|z_n - y_n|}{1+|z_n - y_n|}$$

$$= d(x, z) + d(z, y)$$

Thus d is metric on F. The fair (F, d) is a metric space and this metric space is called **Frechet space**.

*Example 7:* In a metric space (X, d), prove that

$$F \text{ is closed} \Leftrightarrow D(F) \subset F.$$

Prove that a subset F of a metric space X contains all its limit points iff X – F is open.

*Solution:* Let (X, d) be a metric space and $F \subset X$.

We know that F is closed $\Leftrightarrow$ X – F is open.

*Step I:* Let X – F be open so that F is closed,

*Aim:* $D(F) \subset F$.

Let $x \in X - F$ be arbitrary. Then $X - F$ is an open set containing x s.t. $(X - F) \cap F = \phi$.

$\Rightarrow$ x is not a limit point of F

$$\Rightarrow x \notin D(F) \Rightarrow x \in X - D(F)$$

Thus, $\forall\ x \in X - F \Rightarrow x \in X - D(F)$

$\therefore$ $X - F \subset X - D(F)$

or, $D(F) \subset F$

*Step II:* Given $D(F) \subset F$. ...(1)

To prove F is closed.

Let $y \in X - F$, then $y \notin F$

$$y \notin F, D(F) \subset F \Rightarrow y \notin D(F)$$

$$\Rightarrow \exists \text{ open set } G \text{ with } y \in G \text{ s.t.}$$

$$(G - \{y\}) \cap F = \phi$$

$$\Rightarrow G \cap F = \phi \text{ as } y \notin F$$

$$\Rightarrow G \subset X - F$$

Thus we have show that

$$\text{any } y \in X - F \Rightarrow \exists \text{ open set } G \text{ with } y \in G \text{ s.t. } G \subset X - F$$

$$\Rightarrow X - F \text{ is open} \Rightarrow F \text{ is closed.}$$

## 9.2 Summary


- Let $X \neq \phi$ be any given space. Let $x, y, z, \in X$ be arbitrary. A function $d : X \times X \to R$ having the properties listed below:

  (i) $d(x, y) \geq 0$

  (ii) $d(x, y) = 0$ iff $x = y$

  (iii) $d(x, y) = d(y, x)$

  (iv) $d(x, y) + d(y, z) \geq d(x, z)$

  is called a distance function or a metric for X.

- Let $X \neq \phi$ be any given space. Let $x, y, z \in X$ be arbitrary. A function $d : X \times X \to R$ having the properties listed below:

  (i) $d(x, y) \geq 0$

  (ii) $d(x, y) = 0$ if $x = y$

  (iii) $d(x, y) = d(y, x)$

  (iv) $d(x, y) + d(y, z) \geq d(x, z)$, where $x, y, z \in X$ is called pseudo metric on X.

- Let $(X, \rho)$ be a metric space. Let $x_0 \in X$ and $r \in R^+$. Then set $\{x \in X : \rho(x_0, x) < r\}$ is defined as open sphere with centre $x_0$ and radius r.

- Closed sphere:

  $S_r[x_0] = \{x \in X : \rho(x, x_0) \leq x\}$

Notes

- Let $(X, \rho)$ be a metric space. A non empty set $G \subset X$ is called an open set if any $x \in G \Rightarrow \exists$ $r \in R^+$ s.t. $S_{r(x)} \subset G$.
- Let $(X, \rho)$ be a metric space and $A \subset X$. A point $x \in X$ is called a limit point if every open sphere centered on x contains a point of A other than x, i.e. $x \in X$ is called the limit point of A if $\left(S_{r_{(x)}} - \{x\}\right) \cap A \neq \phi$, $r \in R^+$.
- Let $(X, \rho)$ be a metric space and $A \subset X$. A is called a closed set if the derived set of A i.e. $D(A) \subset A$ i.e. if every limit point of A belongs to the set itself.
- Set $<x_n>$ be a sequence in a metric space $(X, \rho)$. This sequence is said to converge to $x_0 \in X$, if given any $\in > 0$, $\exists\, n_o \in N$ s.t. $n \geq n_0 \Rightarrow \rho(x_n, x_0) < \in$.
- A point $x \in A$ is called an interior point of A if $\exists\, r \in R^+$ s.t. $S_{r_{(x)}} \subset A$.
- The closure of A, denoted by $\bar{A}$, is defined as the intersection of all closed sets that contain A.
- Boundary of a set A is denoted by b(A) is defined as $b(A) = X - A^\circ \cup (X - A)^\circ$.
- The exterior of A is defined as the set $(X - A)^\circ$ and is denoted by ext (A).
- A is said to be dense or everywhere dense in X if $\bar{A} = X$.
- A is said to be nowhere dense if $(\bar{A})^\circ = \phi$.
- A metric space (X, d) is said to be separable if $\exists\, A \subset X$ s.t. A is countable and $\bar{A} = X$.
- A sequence defined on a metric space (X, d) is said to be uniformly convergent if given $\in > 0$, $\exists\, n_0 \in N$ s.t. $n \geq n_0$

  $\Rightarrow d(f_n(x), f(x)) < \in \quad \forall\, x \in X$.

## 9.3 Keywords

***Frechet Space:*** A topology space (X, T) is said to satisfy the $T_1$ – axiom of separation if given a pair of distinct point $x, y \in X$.

$\exists\, G, H \in T$ s.t. $x \in G, y \notin G; y \in H, x \notin H$.

In this case the space (X, T) is called Frechet Space.

***Intersection:*** The intersection of two sets A and B, denoted by $A \cap B$, is defined as the set containing those elements which belong to A and B both. Symbolically

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

***Union:*** The union of two sets A ad B, denoted by $A \cup B$, is defined as the set of those elements which either belong to A or to B. Symbolically

$A \cup B = \{x : x \in A \text{ or } x : B\}$

## 9.4 Review Questions

1. In any metric space (X, d), show that

   (a) an arbitrary intersection of closed sets is closed.

   (b) any finite union of closed sets is closed.

2. Let R be the set of all real numbers and let

$$d(x, y) = \frac{|x - y|}{1 + |x - y|} \text{ for all } x, y \in R.$$

   Prove that d is a metric for R.

3. Every derived set in a metric space is a closed set.

4. Let A and B is disjoint closed set in a metric space (X, d). Then ∃ disjoint open sets G, H s.t. $A \subset G$, $B \subset H$.

5. Let $X \neq \phi$ and let d be a real function of ordered pairs of X which satisfies the following two conditions:

   $d(x, y) = 0 \Leftrightarrow x = y$

   and $d(x, y) \leq d(x, z) + d(z, y)$.

   Show that d is a metric on X.

6. Give an example of a pseudo metric which is not metric.

7. Let X be a metric space. Show that every subset of X is open ⇔ each subset of X which consists of single point is open.

8. In a metric space prove that

   (a) $(\bar{A}) = \text{Int}(A')$,

   (b) $\bar{A} = \{x : d(x, A) = 0\}$.

## 9.5 Further Readings

*Books*

B. Mendelson, *Introduction to Topology*, Dover Publication.

J. L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

Notes

# Unit 10: The Quotient Topology

**CONTENTS**



## Objectives

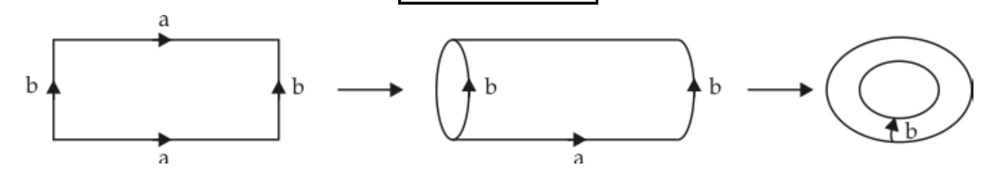
After studying this unit, you will be able to:

- Understand the quotient map, open map and closed map;
- Explain the quotient topology;
- Solve the theorems and questions on quotient topology.

## Introduction

The quotient topology is not a natural generalization of something. You have already studied in analysis. Nevertheless, it is easy enough to motivate. One motivation comes from geometry, where one often has occasion to use 'cut-and-paste' techniques to construct such geometric objects as surfaces. The torus (surface of a doughnut), for example can be constructed by taking a rectangle and 'pasting' its edges together appropriately in Figure 10.1.

**Figure 10.1**




Formalizing these constructions involves the concept of quotient topology.

## 10.1 The Quotient Topology

Notes

### 10.1.1 Quotient Map, Open and Closed Map

**Quotient Map**

Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

The condition is stronger than continuity, some mathematicians call it 'strong continuity'. An equivalent condition is to require that a subset A of Y be closed in Y if and only if $p^{-1}(A)$ is closed in X. Equivalence of the two conditions follow from equation

$$f^{-1}(Y - B) = X - f^{-1}(B).$$

***Open map:*** A map $f : X \to Y$ is said to be an open map if for each open set U of X, the set f(U) is open in Y.

***Closed Map:*** A map $f : X \to Y$ is said to be a closed map if for each closed set A of X, the set f(A) is closed in Y.

*Example 1:* Let X be the subspace $[0, 1] \cup [2, 3]$ of $\mathbb{R}$ and let Y be the subspace [0, 2] of $\mathbb{R}$. The map $p : X \to Y$ defined by

$$p(x) = \begin{cases} x & \text{for} \quad x \in [0,1], \\ x-1 & \text{for} \quad x \in [2,3] \end{cases}$$

is readily seen to be surjective, continuous and closed. Therefore, it is a quotient map. It is not, however, an open map; the image of the open set [0, 1] of X is not open in Y.

*Note* If A is the subspace $[0, 1] \cup [2, 3]$ of X, then the map $q : A \to Y$ obtained by restricting p is continuous with surjective but it is not a quotient map. For the set [2, 3] is open in A and is saturated w.r.t q, but its image is not open in Y.

*Example 2:* Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate, then $\pi_1$ is continuous and surjective. Furthermore, $\pi_1$ is an open map. For if U × V is a non-empty basis element for $\mathbb{R} \times \mathbb{R}$, then $\pi_1(U \times V) = U$ is open in $\mathbb{R}$; it follows that $\pi_1$ carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. However, $\pi_1$ is not a closed map. The subset

$$C = \{x \times y \mid xy = 1\}$$

of $\mathbb{R} \times \mathbb{R}$ is closed, but $\pi_1(C) = \mathbb{R} - \{0\}$, which is not closed in $\mathbb{R}$.

*Note* If A is the subspace of $\mathbb{R} \times \mathbb{R}$ that is the union of C and the origin {0}, then the map $q : A \to \mathbb{R}$ obtained by restricting $\pi_1$ is continuous and surjective, but it is not a quotient map. For the one-point set {0} is open in A and is saturated with respect to q. But its image is not open in $\mathbb{R}$.

## 10.1.2 Quotient Topology

If X is a space and A is a set and if p : X → A is a surjective map, than there exists exactly one topology T on A relative to which p is a quotient map; it is called the quotient topology induced by p.

The topology T is of course defined by letting it consists of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that T is a topology. The sets $\phi$ and A are open because $p^{-1}(\phi) = \phi$ and $p^{-1}(A) = X$. The other two conditions follow from the equations

$$p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha),$$

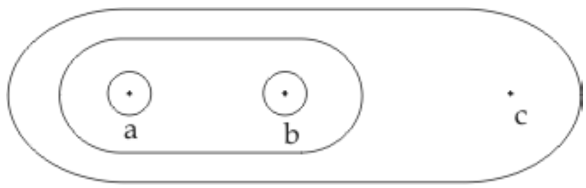
$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

*Example 3:* Let p be the map of the real line $\mathbb{R}$ onto the three point set A = {a, b, c} defined by

$$p(x) = \begin{cases} a & \text{if} \quad x > 0 \\ b & \text{if} \quad x < 0 \\ c & \text{if} \quad x = 0 \end{cases}$$

You can check that the quotient topology on A induced by p is the one indicated in figure (10.2) below

**Figure 10.2**




## 10.1.3 Quotient Space

Let X be a topological space and let X* be a partition of X into disjoint subsets whose union is X. Let p : X → X* be the surjective map that carries each point of X to the element of X* containing it. In the quotient topology induced by p, the space X* is called a quotient space of X.

Given X*, there is an equivalence relation on X of which the elements of X* are the equivalence classes. One can think of X* as having been obtained by 'identifying' each pair of equivalent points. For this reason, the quotient space X* is often called an identification space, or a decomposition space of the space X.

We can describe the topology of X* in another way. A subset U of X* is a collection of equivalence classes, and the set $p^{-1}(U)$ is just the union of the equivalence classes belonging to U. Thus the typical open set of X* is a collection of equivalence classes whose union is an open set of X.

Notes

*Example 4:* Let X be the closed unit ball $\{x \times y \mid x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$, and let X* be the partition of X consisting of all the one-point sets $\{x \times y\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{x \times y\} \mid x^2 + y^2 = 1\}$. One can show that X* is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere, defined by

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

***Theorem 1:*** Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.

(1) If A is either open or closed in X, then q is a quotient map.

(2) If p is either an open map or a closed map, then q is a quotient map.

***Proof:*** *Step (1):* We verify first the following two equations:

$$q^{-1}(V) = p^{-1}(V) \qquad \text{if } V \subset p(A);$$

$$p(U \cap A) = p(U) \cap p(A) \qquad \text{if } U \subset X$$

To check the first equation, we note that since $V \subset p(A)$ and A is saturated, $p^{-1}(V)$ is contained in A. It follows that both $p^{-1}(V)$ and $q^{-1}(V)$ equal all points of A that are mapped by p into V. To check the second equation, we note that for any two subsets U and A of X, we have the inclusion

$$p(U \cap A) \subset p(U) \cap p(A)$$

To prove the reverse inclusion, suppose $y = p(u) = p(a)$, for $u \in U$ and $a \in A$. Since A is saturated, A contains the set $p^{-1}(p(a))$, so that in particular A contains u. They $y = p(u)$, where $u \in U \cap A$.

*Step (2):* Now suppose A is open or p is open. Given the subset V of p(A), we assume that $q^{-1}(V)$ is open in A and show that V is open in p(A).

Suppose first that A is open. Since $q^{-1}(V)$ is open in A and A is open in X, the set $q^{-1}(V)$ is open in X. Since $q^{-1}(V) = p^{-1}(V)$, the latter set is open in X, so that V is open in Y because p is a quotient map. In particular, V is open in p(A).

Now suppose p is open. Since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, we have $p^{-1}(V) = U \cap A$ for some set U open in X.

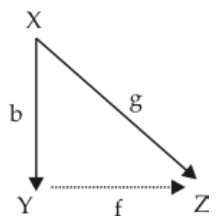
Now $p(p^{-1}(V) = V$ because p is surjective, then

$$V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$$

The set p(U) is open in Y because p is an open map; hence V is open in p(A).

*Step (3):* The proof when A or p is closed is obtained by replacing the word ‘open’ by the word ‘closed’ throughout step 2.

***Theorem 2:*** Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.



***Proof:*** For each $y \in Y$, the set $g(p^{-1}(\{y\}))$ is a one-point set in Z (since g is constant on $p^{-1}(\{y\})$). If we let f(y) denote this point, then we have defined a map $f : Y \to Z$ such that for each $x \in X$, $f(p(x)) = g(x)$. If f is continuous, then $g = f \circ p$ is continuous. Conversely, suppose g is continuous. Given an open set V of $\overline{Z}$, $g^{-1}(V)$ is open in X. But $g^{-1}(V) = p^{-1}(F^{-1}(V))$; because p is a quotient map, it follows that $f^{-1}(V)$ is open in Y. Hence f is continuous. If f is a quotient map, then g is the composite of two quotient maps and is thus a quotient map. Conversely, suppose that g is a quotient map. Since g is subjective, so is f.

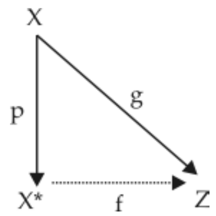
Let V be a subset of Z; we show that U is open in Z if $f^{-1}(V)$ is open in Y. Now the set $p^{-1}(f^{-1}(V))$ is open in X because p is continuous. Since this set equals $g^{-1}(V)$, the latter is open in X. Then because g is a quotient map, V is open in Z.

***Corollary (1):*** Let $g : X \to Z$ be a surjective continuous map. Let X* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$$

Give X* the quotient topology.

(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.




(b) If Z is Hausdorff, so is X*.

***Proof:*** By the preceding theorem, g induces a continuous map $f : X^* \to Z$; it is clear that f is bijective. Suppose that f is a homeomorphism. Then both f and the projection map $p : X \to X^*$ are quotient map. So that their composite q is a quotient map. Conversely, suppose that g is a quotient map. Then it follows from the preceding theorem that f is a quotient map. Being bijective, f is thus a homeomorphism.

Suppose Z is Hausdorff. Given distinct points of X*, their images under f are distinct and thus possess disjoint neighbourhoods U and V. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of the two given points of X*.

## 10.2 Summary

- Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in y if and only if $p^{-1}(U)$ is open in X.
- A map $f : X \to Y$ is said to be an open map if for each open set U of X, the set f(U) is open in Y.
- A map $f : X \to Y$ is said to be a closed map if for each closed set A of X, the set f(A) is closed in Y.
- If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology T on A relative to which p is a quotient map; it is called the quotient topology induced by p.

**Notes**

- Let X be a topological space and let X* be a partition of X into disjoint subsets whose union is X. Let p : X → X* be the surjective map that carries each point of X to the element of X* containing it. In the quotient topology induced by p, the space X* is called a quotient space of X.

## 10.3 Keywords

***Equivalence relation:*** A relation R in set A is an equivalence relation iff it is reflexive, symmetric and transitive.

***Homeomorphism:*** A map f : (X, T) → (Y, U) is said to be homeomorphism if (i) f is one-one onto (ii) f and $f^{-1}$ are continuous.

## 10.4 Review Questions

1. Prove that the product of two quotient maps needs not be a quotient map.
2. Let p : X → Y be a continuous map. Show that if there is a continuous map f : Y → X such that p o f equals the identify map of Y, then p is a quotient map.
3. Show that a subset G of Y is open in the quotient topology (relative to f : X → Y) iff $f^{-1}(G)$ is an open subset of X.
4. Show that if f is a continuous, open mapping of the topological space X onto the topological space Y, then the topology for Y must be the quotient topology.
5. Show that Y, with the quotient topology, is a $T_1$-space iff $f^{-1}(y)$ is closed in X for every $y \in Y$.
6. Show that if X is a countably compact $T_1$-space, then Y is countably compact with the quotient topology.
7. Show that if f is a continuous, closed mapping of X onto Y, then the topology for Y must be the quotient topology.
8. Show that a subset F of Y is closed in the quotient topology (relative to f : X → Y) iff $f^{-1}(F)$ is a closed subset of X.

## 10.5 Further Readings

*Books*

J.L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

# Unit 11: Connected Spaces, Connected Subspaces of Real Line

CONTENTS



## Objectives

After studying this unit, you will be able to:

- Define connected spaces;
- Solve the questions on connected spaces;
- Understand the theorems and problems on connected subspaces of the real line.

## Introduction

The definition of connectedness for a topological space is a quite natural one. One says that a space can be "separated" if it can be broken up into two "globs" – disjoint open sets. Otherwise, one says that it is connected. Connectedness is obviously a topological property, since it is formulated entirely in terms of the collection of open sets of X. Said differently, if X is connected, so is any space homeomorphic to X.

Now how to construct new connected spaces out of given ones. But where can we find some connected spaces to start with? The best place to begin is the real line. We shall prove that R is connected, and so are the intervals.

## 11.1 Connected Spaces

***Definition:*** A topological space X is said to be disconnected iff there exists two non-empty separated sets A and B such that $E = A \cup B$.

In this case, we say that A and B form a partition or separation of E and we write, $E = A \mid B$.

A topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.

A subspace Y of a topological space X is said to be connected if it is connected as a topological space it its own right.

Notes

*Note* A set is said to be connected iff it has no separation

*Example 1:*

(1) Let X be an indiscrete topological space. Then X is connected since the indiscrete topology consists of the empty set $\phi$ and the whole space X only.

(2) Let X be a discrete topological space with at least two elements. Then X is disconnected since if A is any non-empty proper subset of X, then A and $A^c$ are disjoint non-empty open subsets of X such that $X = A \cup A^c$.

(3) $\phi$ is connected. Since $\phi$ cannot be expressed as the union of two non-empty separated sets. So $\phi$ has no separation and is therefore connected.

***Theorem 1:*** In a topological space X the following statements are equivalent:

(i) X is connected;

(ii) The empty set $\phi$ and the whose space X are the only subsets of X that are both open and closed in X i.e. X has no non-trivial subset that is both open and closed in X;

(iii) X cannot be represented as the union of two non-empty disjoint closed sets.

(iv) X cannot be represented as the union of two non-empty separated sets.

***Proof:*** We shall prove the theorem by showing that

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)

Let X be connected.

Suppose A is a non-trivial subset of X that is simultaneously open and closed in X. Then $B = A^c$ is non-empty, open and $X = A \cup B$, $A \cap B = \phi$

This is contrary to the given hypothesis that X is connected and accordingly (ii) must be true

(ii) $\Rightarrow$ (iii)

Let (ii) be true.

Suppose X = A U B, where A and B are two disjoint non-empty closed sets.

Then $A = B^c$ is a non-trivial subset of X that is open as well as closed in X. This contradicts the given hypothesis (ii) and thus (ii) must be true.

(iii) $\Rightarrow$ (iv)

Let (iii) be true.

Suppose $X = A \cup B$

where $A \neq \phi$, $B \neq \phi$, $A \cap \overline{B} = \phi = \overline{A} \cap B$.

Then clearly $X = \overline{A} \cup \overline{B}$

where $\overline{A}$ and $\overline{B}$ are non-empty closed sets.

Also $A \cap \overline{B} = \phi$

Notes

$\Rightarrow \overline{B} \subseteq A^C$

$\overline{A} \cap B = \phi \Rightarrow \overline{A} \subseteq B^C$

$\Rightarrow \overline{A} \cap \overline{B} \subseteq B^C \cap A^C = (B \cup A)^C = X^C = \phi$

i.e., $\overline{A} \cap \overline{B} = \phi$

Thus X can be represented as the union of two disjoint non-empty closed sets.

This contradicts the given hypothesis (iii) and thus (iv) must be true.

(iv) $\Rightarrow$ (i)

Let (iv) be true

Suppose that X is disconnected.

Then there exist disjoint non-empty open sets G and H such that $X = G \cup H$.

Since G and H are open and $G \cap H = \phi$, is follows $G \cap \overline{H} = \phi$ and $\overline{G} \cap H = \phi$.

This contradicts the given hypothesis (iv) and thus (i) must be true.

Hence the proof of the theorem.

***Theorem 2:*** The closure of a connected set is connected

OR

If A is connected subset then show that A is also connected.

***Proof:*** Let (X, T) be a topological space and A be a subset of X.

If A is connected, then we have to show that $\overline{A}$ is also connected.

If $\overline{A}$ is not connected then it has a separation.

Let $\overline{A} = G \mid H$

So by theorem, Let (X, T) be a topological space and let E be a connected subset of (X, T). If E has a separation $X = A \mid B$, then either $E \subseteq A$ or $E \subseteq B$, we have

$$\overline{A} \subseteq G \text{ or } \overline{A} \subseteq H$$

If $\overline{A} \subseteq G$

$\Rightarrow \overline{\overline{A}} \subseteq \overline{G}$

$\Rightarrow \overline{A} \subseteq \overline{G}$

$\Rightarrow \overline{A} \cap H \subseteq \overline{G} \cap H$

$\Rightarrow \overline{A} \cap H = \phi$ (∵ G and H are separated.) ...(1)

Also $\overline{A} = G \cup H$ ...(2)

$\Rightarrow H \subset \overline{A}$

**Notes**

Now from (1) and (2), we get

$H = \phi$,

which contradicts the given fact that H is non-empty.

Hence $\overline{A}$ is also a connected set.

***Theorem 3:*** If every two points of a set E are contained in same connected subset of E, then E is connected.

***Proof:*** Let us suppose that E is not connected.

Then, it must a separation $E = A \mid B$

i.e. E is the union of non-empty separated sets A and B.

Since A and B are non-empty, let $a \in A$ and $b \in B$.

Then, A and B being disjoint

$\Rightarrow$ a, b are two distinct points of E.

So, by given hypothesis there exists a connected subset C of E such that $a, b \in c$

But, C being a connected subset of a disconnected set E with the separation $E = A \mid B$,

we have $C \subseteq A$ or $C \subseteq B$.

This is not possible, since A and B are disjoint and C contains at least one point of A and one that of B, which leads to a contradiction.

Hence E is connected.

***Theorem 4:*** A topological space (X, T) is connected iff the only non-empty subset of X which is open and closed is X itself.

***Proof:*** Let (X, T) be a connected space.

Let A be a non-empty subset of X that is both open and closed. Then $A^c$ is both open and closed.

$\therefore \quad \overline{A} = A$ and $\overline{A}^C = A^C$

Thus $A \cap A^C = \phi$

$\Rightarrow \overline{A} \cap A^C = \phi$ and $A \cap \overline{A}^C = \phi$

Also $X = A \cup A^C$

Therefore A and $A^C$ are two separated sets whose union in X.

Now if $A \neq \phi$ and $A^C \neq \phi$, then we have separation $X = A \mid A^C$, which leads to the contradiction as X is connected.

So either $A = \phi$ or $A^C = \phi$

But $A = \phi$ or $A^C = \phi$

But $A \neq \phi$

So $A^C = \phi$

$\therefore \quad X = A \cup A^C = A \cup \phi = A$

This shows that the only non-empty subset of X that is both open and closed is X itself. Conversely, let the only subset of X which is both open and closed be X itself.

Then, there exists no non-empty proper subset of X which is both open and closed.

Hence (X, T) is not disconnected and therefore, it is connected.

***Theorem 5:*** A continuous image of connected space is connected.

***Proof:*** Let $f : X \to Y$ be a continuous mapping of a connected space X into an arbitrary topological space Y.

We shall show that f[X] is connected as a subspace of Y.

Let us suppose f[X] is disconnected.

Then there exists G and H both open in Y such that

$G \cap f[X] \neq \phi$, $H \cap f[X] \neq \phi$

$(G \cap f[X]) \cap (H \cap f[X]) = \phi$

and $(G \cap f[X]) \cup (H \cap f[X]) = f[X]$

It follows that

$$\begin{aligned} \phi &= f^{-1}[\phi] \\ &= f^{-1}[(G \cap f[X]) \cap (H \cap f[X])] \\ &= f^{-1}((G \cap H) \cap f[X]) \\ &= f^{-1}[G] \cap f^{-1}[H] \cap f^{-1}(f[X]) \\ &= f^{-1}[G] \cap f^{-1}[H] \cap X \\ &= f^{-1}[G] \cap f^{-1}[H] \end{aligned}$$

and

$$\begin{aligned} X &= f^{-1}(f[X]) \\ &= f^{-1}[(G \cap f[X]) \cup (H \cap f[X])] \\ &= f^{-1}[(G \cup H) \cap f[X])] \\ &= f^{-1}[G \cup H] \cap f^{-1}(f[X]) \\ &= f^{-1}[G] \cup f^{-1}[H] \cap X \\ &= f^{-1}[G] \cup f^{-1}[H] \end{aligned}$$

Since f is continuous and G and H are open in Y both intersecting f[X].

If follows that $f^{-1}[G]$ and $f^{-1}[H]$ are both non-empty open subsets of X.

Thus X has been expressed as union of two disjoint open subsets of X and consequently X is disconnected, which is a contradiction.

Hence f[X] must be connected.

*Example 2:* Show that (X, T) is connected space if X = {a, b, c, d} and T = {X,$\phi$, {a, b}}.

*Solution:* T-open sets are X, $\phi$, {a, b}.

T-closed sets are $\phi$, X, {c, d}

For X-{a, b} = {c, d}

Thus $\exists$ non-proper subset of X which is both open and closed. Consequently (X, T) is not disconnected. It follows that (X, T) is connected.

**Notes**

*Example 3:* Show that every indiscrete space is connected.

*Solution:* Let (X, T) be an indiscrete space so that T = {ϕ, X}. Then T-open sets are ϕ, X. T-closed sets are X, ϕ. Hence the only non-empty subset of X which is both open and closed is X.

∴ X is connected, by theorem (4).

### Self Assessment

1. Prove that the closure of connected set is connected.
2. Prove that a continuous image of a connected space is a connected set.
3. Prove that connectedness is preserved under continuous map.

## 11.2 Connected Subspaces of Real Line

***Theorem 6:*** The set of real numbers with the usual metric is a connected space.

***Proof:*** Let if possible (R, U) be a disconnected space. Then there most exist non-empty closed subsets A and B of R such that

$$A \cup B = R \text{ and } A \cap B = \phi$$

Since A and B are non-empty, ∃ a point $a \in A$ and $b \in B$

Since $A \cap B = \phi$

∴ $a \neq b$

Thus $a < b$ or $a > b$

Let $a < b$

We have $[a, b] \subseteq p$

$\Rightarrow [a, b] \subseteq A \cup B$

Thus $x \in [a, b] \Rightarrow x \in A$ or $x \in B$

Let $p = \sup([a, b] \cap A)$

Then $a \leq p \leq b$

Since A is closed, $p \in A$

Again $A \cap B = \phi$ and $p \in B$

$\Rightarrow p < b$

Also by definition of p

$p + \varepsilon \in B \ \forall \in > 0$

∴ $p + \varepsilon \leq b$

Again since B is closed, $p \in B$.

Thus, we get

$p \in A$ and $p \in B \Rightarrow p \in A \cap B$

But $A \cap B = \phi$

Thus we get a contradiction. Hence R is connected.

Notes

***Theorem 7:*** A subspace of the real line R is connected iff it is an interval. In particular, R is connected.

***Proof:*** Let E be a subspace of R.

We first prove that if E is connected, then it is an interval. Let us suppose that E is not an interval. Then there exists real numbers a, b, c with $a < c < b$ such that $a, b \in E$ but $c \notin E$.

Let A = $]-\infty, C[$ and B=$]c, \infty[$.

Then A and B are open subset of R such that $a \in A$ and $b \in B$.

Now, $E \cap A \neq \phi$ and $E \cap B \neq \phi$, since $a \in E \cap A$ and $b \in E \cap B$.

Also, $(E \cap A) \cap (E \cap B) = E \cap (A \cap B) = \phi$ ($\because A \cap B = \phi$)

and $(E \cap A) \cup (E \cap B) = E \cap (A \cup B) = E \cap R - \{c\} = E$

Thus, $A \cup B$ forms a disconnection of E i.e., E is disconnected, a contradiction.

Hence E must be an interval.

Conversely, Let E be an interval and if possible let E is disconnected.

Then E is the union of two non-empty disjoint sets G and H, both closed in E, i.e. $E = G \cup H$.

Let $a \in G$ and $b \in H$

Since $G \cap H = \phi$, we have $a \neq b$

So either $a < b$ or $b < a$

Without any loss of generality we may assume that $a < b$.

Since $a, b \in E$ and E is an interval, we have $[a, b] \subset E = G \cup H$.

Let $p = \sup\{G \cap [a, b]\}$, then clearly $a \leq p \leq b$

Consequently, $p \in E$.

But, G being closed in E, the definition of p shows that $p \in G$ and therefore, $p \neq b$.

Consequently, $p < b$

Moreover, the definition of p shows that $p + \varepsilon \in H$ for each $\varepsilon > 0$ for which $p + \varepsilon \leq b$.

This shows that every nhd. of p contains at least one point of H, other than p. So, p is a limit point of H. But H being closed, we have $p \in H$.

Thus, $p \in G \cap H$ and therefore $G \cap H \neq \phi$, which is a contradiction.

Hence E must be connected

***Theorem 8:*** Prove that the real line is connected.

***Proof:*** Let R be an interval and if possible let R is disconnected. Then R is the union of two non-empty disjoint sets G and H, both closed in R, i.e. $R = G \cup H$.

Let $a \in G$ and $b \in H$.

Since $G \cap H = \phi$, we have $a \neq b$

So either $a < b$ or $b < a$

Without any loss of generality, we may assume that $a < b$.

Since $a, b \in R$ and R is an interval, we have $[a, b] \subset R = G \cup H$

**Notes**

Let $p = \sup\{G \cap [a, b]\}$, then clearly $a \le p \le b$

Consequently $p \in R$

But G being closed in R, the definition of p shows that $p \in G$ and therefore $p \neq b$.

Consequently, $p < b$

Moreover the definition of p shows that $p + \varepsilon \in H$ for each $\in > 0$ for which $p + \in \le b$.

This shows that every nhd. of p contains at least one point of H other than p. So p is a limit point of H.

But H being closed, we have $p \in H$

Thus $p \in G \cap H$ and therefore $G \cap H \neq \phi$, which is a contradiction.

Hence R must be connected.

*Example 4:* Show that if X is a connected topological space and f is a non-constant continuous real function defined on X then X is uncountably infinite.

*Solution:* $f : X \to \mathcal{R}$ is continuous and X is connected, so f (X) is a connected subspace of $\mathcal{R}$.

Suppose that f (X) is not connected, there exists a non-empty proper subset E of f (X) such that E is both open and closed in f (X).

As f is continuous

$\Rightarrow f^{-1}(E)$ is non-empty proper subset of X which is both open and closed in X.

This contradicts the fact that X is connected. Hence f (X) must be a connected subspace of $\mathcal{R}$.

Also f is non-constant, there exist $x, y \in X$ such that $f(x) \neq f(y)$

Let $a = f(x)$ and $b = f(y)$.

Without any loss of generality we may suppose that $a < b$. Now $a, b \in f(X)$, f (x) is a connected subspace of $\mathcal{R}$

$\Rightarrow [a, b] \subseteq f(X)$.

[$\because$ a subspace E of real line $\mathcal{R}$ is connected iff E is an interval
i.e. if $a, b \in E$ and $a < c < b$ then $c \in E$. In particular $\mathcal{R}$ is connected.]

Since [a, b] is uncountably infinite, it follows that f (X) is uncountably infinite and consequently X must be uncountably infinite.

*Example 5:* Show that the graph of a continuous real function defined on an interval is a connected subspace of the Euclidean plane.

*Solution:* Let $f : 1 \to \mathcal{R}$ be continuous and let G be the graph of f.

Then $G = I \times f(I) \subseteq \mathcal{R}^2$.

Now since I is connected by the theorem "A subspace E of the real line $\mathcal{R}$ is connected iff E is an interval."

Also, f is continuous, it follows that f (I) is a connected subspace of R since continuous image of a connected space is connected. Also we know that connectedness is a product invariant property, hence G is connected.

Notes

*Example 6:* The spaces $\mathcal{R}^n$ and $C^n$ are connected.

*Solution:* We know that $\mathcal{R}^n$ is a topological space can be regarded as the product of n replicas of the real line $\mathcal{R}$. But $\mathcal{R}$ is connected therefore $\mathcal{R}^n$ is connected since the product of any non-empty class of connected spaces is connected.

We next prove that $C^n$ and $\mathcal{R}^{2n}$ are essentially the same as topological spaces by taking a homomorphism f of $C^n$ onto $\mathcal{R}^{2n}$.

Let $z = (z_1, z_2, \ldots, z_n)$ be an arbitrary element in $C^n$.

Let us suppose that each coordinate $z_K$ is of the form

$$z_K = a_K + ib_K$$

where $a_K$ and $b_K$ are its real and imaginary parts.

Let us define f by

$$f(z) = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n).$$

f is clearly a one-to-one mapping of $C^n$ onto $\mathcal{R}^{2n}$ and if we observe that $||f(z)|| = ||z||$, then f is a homeomorphism which shows that $\mathcal{R}^{2n}$ is connected. Hence $C^n$ is also connected.

### Self Assessment

4. Show that if f is continuous map of a connected space X into R, then f(X) is an interval.
5. Show that a subset $\mathcal{A}$ of the real line that contains at least two distinct points is connected if and only if it is an interval.

## 11.3 Summary

- A topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.
- The closure of a connected set is connected.
- If every two points of a set E are contained in some connected subset of E, then E is connected.
- A continuous image of connected space is connected.
- The set of real numbers with the usual metric is a connected space.
- A subspace of the real line R is connected iff it is an interval. In particular, R is connected.

## 11.4 Keyword

***Separated set:*** Let A, B be subsets of a topological space (X, T). Then the set A and B are said to be separated iff

(i) $A \neq \phi, B \neq \phi$

(ii) $A \cap \overline{B} = \phi, \overline{A} \cap B = \phi$

Notes

## 11.5 Review Questions

1. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \phi$ for all n. Show that $UA_n$ is connected.
2. Let $p : X \rightarrow Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and if Y is connected, then X is connected.
3. Let $Y \subset X$ ; let X and Y be connected. Show that if A and B form a separation of X – Y, then $Y \cup A$ and $Y \cup B$ are connected.
4. Let (X, T) be a topological space and let E be a connected subset of (X, T). If E has a separation $X = A | B$, then either $E \subseteq A$ or $E \subseteq B$.
5. Prove that if a connected space has a non-constant continuous real map defined on it, then it is uncountably infinite.
6. Show that a set is connected iff A is not the union of two separated sets.
7. Let $f : S' \rightarrow R$ be a continuous map. Show there exists a point x of S′ such that $f(x) = f(-x)$.
8. Prove that connectedness is a topological property.
9. Prove that the space $R^n$ and $C^n$ are connected.

## 11.6 Further Readings

*Books*

William W. Fairchild, Cassius Ionescu Tulcea, *Topology*, W.B. Saunders Company.

B. Mendelson, *Introduction to Topology*, Dover Publication.

*Online links*

www.mathsforum.org

www.history.mcs.st/andrews.ac.uk/HistTopics/topology/in/mathematics.htm

*Richa Nandra, Lovely Professional University*



# Unit 12: Components and Local Connectedness

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Understand the term components of a topological space;
- Solve the problems on components of a topological space;
- Define locally connectedness;
- Solve the problems on locally connectedness.

## Introduction

Given an arbitrary space X, there is a natural way to break it up into piece that are connected. We consider that process now. Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components or the "connected components" of X.

Connectedness is a useful property for a space to possess. But for some purposes, it is more important that the space satisfy a connectedness condition locally. Roughly speaking, local connectedness means that each point has "arbitrary small" neighbourhoods that are connected. So, in this unit, we shall deal with two important topics components and local connectedness.

## 12.1 Components of a Topological Space

***Definition:*** A subset E of a topological space X is said to be a component of X if

1. E is a connected set and
2. E is not a proper subset of any connected subspace of X i.e. if E is a maximal connected subspace of X.

Notes

*Notes*

(i) Let (X, J) be a topological space and E be a subset of X. If $x \in E$, then the union of all connected sets containing x and contained in E, is called component of E with respect to x and is denoted by C (E, x).

(ii) Since the union of any family of connected sets having a non-empty intersection is a connected set, therefore the component of E with respect of x i.e. C (E, x) is a connected set.

(iii) If E is a component of X, the $E \neq \phi$.

*Example 1:*

(i) If X is a connected topological space, then X has only one component, namely X itself.

(ii) If X is a discrete topological space, then each singleton subset of X is its component.

***Theorem 1:*** In a topological space (X, T) each point in X is contained in exactly one component of X.

***Proof:*** Let x be any point of X

Let $A_x = \{A_i\}$ be the class of all connected subspaces of X which contains x

$$A_x \neq \phi \text{ as } \{x\} \in A_x$$

Also (i) $$A_i \neq \phi \text{ since } x \in \bigcap_i A_i$$

Therefore by theorem, Let X be a topological space and $\{A_i\}$ be a non-empty class of connected subspaces of X such that $\bigcap_i A_i \neq \phi$ then $A = U_i\ A_i$ is connected subspace of $X, \bigcup_i A_i = C_x$ (say) is connected subspace of X.

Further, $x \in C_x$ and if B is any connected subspace of X containing x, then $B \in A_x$ and so $B \subseteq C_x$.

Therefore $C_x$ is a maximal connected subspace i.e. a component of X containing x.

Now we shall prove that $C_x$ is the only component which contains x.

Let $C_x^*$ be any other component of X which contain x. The $C_x^*$ is one of the $A_i's$ and is therefore contained in $C_x$. But $C_x^*$ is maximal as a connected sub-space of X, therefore we must have $C_x^* = C_x$ i.e. $C_x$ is unique in the sense that each point $x \in X$ is contained in exactly one component $C_x$ of X.

***Theorem 2:*** In a topological space each components is closed.

***Proof:*** Let (X, T) be a topological space and let C be a component of X.

By the definition of component, C is the largest connected set containing x. Then, $\bar{C}$ is also a connected set containing x.

Thus $\bar{C} \subset C$

Also $C \subseteq \bar{C}$

Therefore $C = \bar{C}$

Hence C is closed.

Notes

***Theorem 3:*** In a topological space X each connected sub space of X is contained in a component of X.

***Proof:*** Let E be any connected subspace of X.

If $E = \phi$, then E is contained in every component of X.

Let $E \neq \phi$, and let $x \in E$

Then $x \in X$

Let $E_x$ be the union of all connected subsets of X containing x. Then, $E_x$ is a component of X containing x.

Now, E is a connected set containing x and $E_x$ is the largest connected set containing x. So $E \subseteq E_x$.

***Theorem 4:*** In a topological space (X, T), a connected subspace of X which is both open and closed, in a component of X.

***Proof:*** Let G be a connected subspace of X which is both open and closed.

If G = $\phi$, then G is contained in every component.

If G $\neq \phi$, then G contains a point $x_1 \in X$ and so

$$G \subset C\,(X, x_i) = C$$

We shall show that G = C

In order to show that G = C, let us assume that G is a proper subset of C, so that

$G \cap C \neq \phi$ and $G' \cap C \neq \phi$ where $G' = X - C$.

Since G is both open and closed, $G'$ is also both open and closed.

Also $$(G \cap C) \cap (G' \cap C) = (G \cap G') \cap$$

$$= \phi \cap C = \phi$$

and $$(G \cap C) \cup (G' \cap C) = (G \cup G') \cap C = X \cap C = X$$

which shows that C is disconnected, which is a contradiction of the given fact that C is connected

Hence G = C.

***Theorem 5:*** The product of any non-empty class of connected topological spaces is connected i.e. connectedness is a product invariant property.

***Proof:*** Let {$x_i$} be a non-empty connected topological spaces and $X = \Pi_i \; X_i$ be the product space.

Let $a =< a_i > \in X$ and E be a component of a.

We claim that $X \subseteq \overline{E} = E$ ($\because$ E is closed)

Let $x = < x_i >$ be any point of X and let $G = \Pi\{X_i : i \neq i_1, ...i_m\} X G_1 \; X....X G_i$

be any basic open set containing x.

Now $H = \Pi\left\{\{a_i\}; i \neq i_1, i_2, ...., i_m\right\} \times X_{i_1} \times X_{i_a} \times .... \times X_{i_m}$ is homeomorphic to $X_{i_1} \times X_2 \times ... X_{i_m}$ and is therefore connected

($\because$ connectedness is a topological property)

**Notes**

Further $a \in H$, H connected and E a component of a implies that H is a subset of E. But $G \cap H \neq \phi$, so that G contains a point of H and hence of E.

Thus we have shown that every basic nhd of x contains a point of E.

Consequently every nhd of x will contain a point of E and therefore $x \in \bar{E}$.

Thus $x \in X \Rightarrow x \in \bar{E} = E$, so that $X \subseteq E$ But $E \subseteq X$.

Hence X = E and is therefore connected.

***Theorem 6:*** The component of a topological space X form a position of X i.e. any two components of X are either disjoint or identical and the union of all the components is X.

***Proof:*** For each $x \in X$, let C (X, x) the union of all connected sets containing x.

Then C (X, x) is a component of X.

Clearly, the family $\{C_x : x \in X\}$ consists of all components of X and $X = \cup \{C_x : x \in X\}$. Now let $C(X, x_1)$ and $C(X, x_2)$ be the components of X with respect of $x_1$ and $x_2$ respectively, $x_1 \neq x_2$

If $C(X, x_1) \cap C(X, x_2) = \phi$, we are done

so, let $C(X, x_1) \cap C(X, x_2) = \phi$

Let $x \in C(X, x_1) \cap C(X, x_2)$

them $x \in C(X, x_1)$ and $x \in C(X, x_2)$

Now $C(X, x_1)$ and $C(X, x_2)$ are connected sets containing x and C (X, x) is a component containing x, therefore

$$C(X, x_1) \subseteq C(X, x)$$

and $$C(X, x_2) \subseteq (X, x)$$

But $C(X, x_1)$ and $C(X, x_2)$ being components, they cannot be contained in a larger connected subset of X.

Therefore $C(X, x_1) = C(X, x_2) = C(X, x)$

Thus, any two components of X one either disjoint or identical.

Hence, the components of X form a partition of X.

## Self Assessment

1. Prove that the components of E corresponding to different points of E are either equal or disjoint.

## 12.2 Local Connectedness

### 12.2.1 Locally Connected Spaces

A topological space X is said to locally connected at a point $x \in X$ if every nhd. of x contains a connected nhd. of x i.e. if N is any open set containing x then there exists a connected open set G containing x such that $G \subseteq N$

or

A topological space (X, T) is said to be locally connected iff for every point $x \in X$ and every nhd. G of x, there exists a connected nhd. H such that $x \in H \subset G$. Thus the space (X, T) is locally connected iff the family of all open connected sets is a base for T.

Notes

*Example 2:* Each interval and each ray in the real line in both connected and locally connected. The subspace [–1, 0) ∪ (0, 1] of ℝ is not connected, but it is locally connected.

### 12.2.2 Locally Connected Subset

Let (X, T) be a topological space and let (Y, $T_y$) be a sub-space of (X, T)

The subset Y ⊆ X is said to be locally connected if (Y, $T_y$) is a locally connected space.

### 12.2.3 Theorems and Solved Examples

***Theorem 7:*** Every discrete space is locally connected.

***Solution:*** Let x be an arbitrary point of a discrete space X. We know that every subset of a discrete space is open and that every singleton set is connected. Hence {x} is a connected open nhd. of x. Also every open nhd. of x must contain {x}.

Hence X is locally connected.

*Example 3:* Give two examples of locally connected space which are not connected.

Or

Is locally connected space always connected? Justify.

*Solution:*

1. Let X be a discrete space containing more than one point.

   Let x ∈ X. Then {x} is an open connected set and is obtained in every open set containing x. So, X is locally connected at each point of X. Also, every singleton subset of X is a non-empty proper subset of X which is both open and closed. So X is disconnected.

2. Consider the usually topological space (R, U)

   Let A ⊂ R, which is the union of two disjoint open intervals.

   Then A is not a interval and therefore it is not connected.

   To show that A is locally connected.

   Let x be an arbitrary point of A and $G_x$ be a set open in A such that x ∈ $G_x$. Then there exists an open interval $I_x$ such that x ∈ $I_x \subseteq G_x$. But $I_x$ being an interval, it is connected in R and therefore in A.

   Thus every open nhd. of x in A contains a connected open nhd. of x in A.

   Hence A is locally connected.

*Example 4:* Give example of a space which is connected but not locally connected.

*Solution:* Consider the subspace A ∪ B of the Euclidean Plane $R^2$, where

$$A = \{(0, y) : -1 \le y \le 1\}$$

and

$$B = \left\{(x, y) : y = \sin\left(\frac{1}{x}\right), 0 < x \le 1\right\}$$

The A ∩ B = ϕ and each point of A is a limit point of B and so A and B are not separated. Consequently, A ∪ B is connected.

**Notes**

But $A \cup B$ is not locally connected at (0, 1), since the open disc with centre (0, 1) and radius $\left(\frac{1}{4}\right)$ does not contain any connected open subset of $R^2$ containing (0, 1).

Hence $A \cup B$ is connected but not locally connected.

***Theorem 8:*** Every component of a locally connected space is open.

***Proof:*** Let (X, T) be a locally connected space and E be a component of X.

We shall show that E is an open set.

Let x be any element of E.

Since X is locally connected, there exists a connected space set $G_x$ which contains x. Since E is a component, we have $x \in G_x \subset E$ clearly, $E = \cup \{G_x : x \in E\}$.

Therefore E, being a union of open sets, is an open set.

***Theorem 9:*** A topological space X is locally connected iff the components of every open subspace of X are open in X.

***Proof:*** Let X be locally connected and Y be an open subspace of X.

Let E be a component of Y.

We are to show that E is open in X i.e. if x is any element of E then there exists a nhd. G of x such that $G \subseteq E$.

Now $E \subseteq$, Y, Y open in X, $x \in Y$ and X is locally connected implies that there exists a connected open set G containing x such that $G \subseteq Y$.

Since the topology which G has as a subspace of Y is the same as that it has as a subspace of X, therefore G is also connected as a subspace of Y and consequently $G \subseteq E$ as E is a component of Y.

Conversely, let the components of every open subspace of X be open in X, Let $x \in X$ and Y an open subset of X containing x. Let $E_x$ be a component of Y containing x. Then by hypothesis, $E_x$ is open and connected in Y and therefore in X.

*Example 5:* Give an example of locally connected space which is totally disconnected.

*Solution:* Every discrete space is locally connected as well as totally disconnected.

Let x be an arbitrary point of a discrete space X.

We know that every subset of a discrete space is open and that every singleton set is connected.

Hence {x} is a connected open nhd. of x. Also every open nhd. of x must contain {x}.

Hence X is locally connected.

To prove X is totally disconnected.

Let x, y be any two distinct points of a discrete space X.

The $G = \{x\}$ and $H = X - \{x\}$ are both non-empty open disjoint sets whose union is X such that $x \in G$ and $y \in H$. It follows that X is totally disconnected.

***Theorem 10:*** Local connectedness neither implies nor is implied by connectedness.

***Proof:*** The union of two disjoint open intervals on the real line forms a space which is locally connected but not connected. Example of a space which is connected but not locally connected.

Let X be the subspace of Euclidean plane defined by

$X = A \cup B$ where

$A = \{(x, y) : x = 0, y \in [-1, 1]\}$

and $B = \{(x, y) : 0 \leq x \leq 1 \text{ and } y = \sin \frac{1}{x}\}$.

Since B is the image of (0, 1] under a continuous mapping f give by

$$f(x) = \left(x, \sin\frac{1}{x}\right)$$

So B is connected.

($\because$ Continuous image of a connected space is connected).

Since $X = \overline{B}$, therefore X is connected. But it is not locally connected because each point $x \in A$ has a nhd. which does not contain any connected nhd. of x.

***Theorem 11:*** The image of a locally connected space under a mapping which is both open and continuous is locally connected. Hence locally connectedness is a topological property.

***Proof:*** Let X be a locally connected space and Y be an arbitrary topological space.

Let $f : X \to Y$ be a map which is both open and continuous. Without any loss of generality we may assume that f is onto. We shall show that Y = f (X) is locally connected.

Let y = f (x), $x \in X$, be any point of Y and G be any nhd. of y. Since f is continuous.

$\Rightarrow f^{-1}(G)$ is open in X containing $f^{-1}(y) = x$.

Thus, $f^{-1}(G)$ is open, nhd. of x.

Now X being locally connected, these exists a connected open set H such that $x \in H \subseteq f^{-1}(G)$.

$\therefore y = f(x) \in f(H) \subseteq f[f^{-1}(G)] \subseteq G$,

where f (H) is open, since f is open.

Moreover, the continuous image of a connected set is connected, it follows that f (H) is connected.

This shows that f (X) is locally connected at each point.

Hence, f (X) is locally connected.

## Self Assessment

2. Show that a connected subspace of a locally connected space has a finite number of components.

3 Show that the product X × Y of locally connected sets X and Y is locally connected.

## 12.3 Summary

- A subset of E of a topological space X is said to be a component of X if
  - (i) E is a connected set &
  - (ii) E is not a proper subset of any connected subspace of X i.e. if E is a maximal connected subspace of X.
- A topological space X is said to locally connected at a point $x \in X$ if every nhd of x contains a connected nhd. of x i.e. if N is any open set containing x then there exists a connected open set G containing x such that $G \subseteq N$.
- Let (X, T) be a topological space and let $(Y, T_y)$ be a subspace of (X, T). The subset $y \subseteq X$ is said to be locally connected if $(y, T_y)$ is a locally connected space.

## 12.4 Keywords

Notes

***Connected:*** A topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.

***Discrete Space:*** Let X be any non empty set of T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

***Open Set:*** Let (X, T) be a topological space. Any set $A \in T$ is called an open set.

***Partition:*** A topological space X is said to be disconnected if there exists two non-empty separated sets A and B such that $E = A \cup B$. In this case, we say that A and B form a partition of E and we write $E = A/B$.

## 12.5 Review Questions

1. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected.
2. A space X is said to be weakly locally connected at x if for every neighbourhood U of x, there is a connected subspace of X contained in U that contains a neighbourhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected.
3. Prove that a space X is locally connected if and only if for every open set U of X and each component of U is open in X.
4. Prove that the components of X are connected disjoint subspaces of X whose union is X, such that each non-empty connected subspace of X intersects only one of them.

## 12.6 Further Readings

*Books*

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

# Unit 13: Compact Spaces and Compact Subspace of Real Line

CONTENTS



## Objectives

After studying this unit, you will be able to:

- Define open covering of a topological space;
- Understand the definition of a compact space;
- Solve the problems on compact spaces and compact subspace on real line.

## Introduction

The notion of compactness is not nearly so natural as that of connectedness. From the beginning of topology, it was clear that the closed interval [a, b] of the real line had a certain property that was crucial for proving such theorems as the maximum value theorem and the uniform continuity theorem. But for a long time, it was not clear how this property should be formulated for an arbitrary topological space. It used to be thought that the crucial property of [a, b] was the fact that every infinite subset of [a, b] has a limit point, and this property was the one dignified with the name of compactness. Later, mathematicians realized that this formulation does not lie at the heart of the matter, but rather that a stranger formulation, in terms of open coverings of the space, is more central. The latter formulation is what we now call compactness. It is not as natural of intuitive as the former; some familiarity with it is needed before its usefulness becomes apparent.

## 13.1 Compact Spaces

***Definition:*** A collection $\mathcal{A}$ of subsets of a space X is said to *cover X*, or to be a covering of X, if the union of the elements of A is equal to X. It is called an *open covering* of X if its elements are open subsets of X.

***Definition:*** A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite sub-collection that also covers X.

*Example 1:* The real line ℝ is not compact, for the covering of ℝ by open intervals

$$\mathcal{A} = \{(n, n+2) / n \in \mathbb{Z}\}$$

contains no finite sub-collection that covers ℝ.

*Example 2:* The following subspace of ℝ is compact

$$X = \{0\} \cup \{1/n \in \mathbb{Z}_+\}.$$

Given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing O. The set U contains all but finitely many of the point 1/n; choose, for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite sub-collection of $\mathcal{A}$ that covers X.

***Lemma (i):*** Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite sub-collection covering Y.

***Proof:*** Suppose that Y is compact and $\mathcal{A} = \{A_x\}_{\alpha \in T}$ is a covering of Y by sets open in X. Then the collection

$$\{A_x \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite sub-collection

$$\{A_{\alpha_1} \cap Y, ...., A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, ..., A_{\alpha_n}\}$ is a sub-collection of $\mathcal{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set $A_\alpha$ open in X such that

$$A'_\alpha = A_\alpha \cap Y$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite sub-collection $\{A_{\alpha_1},...., A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, ..., A'_{\alpha_n}\}$ is a sub-collection of $\mathcal{A}'$ that covers Y.

***Theorem 1:*** Every closed subspace of a compact space is compact.

***Proof:*** Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X by $\mathcal{A}$ joining to $\mathcal{A}$ the single open set X – Y that is

$$B = \mathcal{A} \cup \{X - Y\}$$

Some finite sub-collection of $\mathcal{B}$ covers X. If this sub-collection contains the set X – Y, discard X – Y; otherwise, leave the sub-collection alone. The resulting collection is a finite sub-collection of $\mathcal{A}$ that cover Y.

***Theorem 2:*** Every compact subspace of a Hausdorff space is closed.

***Proof:*** Let Y be a compact subspace of the Hausdorff space X. We shall prove that X – Y is open. So that Y is closed. Let $x_0$ be a point of X – Y. We show there is a neighborhood of $x_0$ that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods $U_y$ and $V_y$ of the points $x_0$ and y, respectively (using the Hausdorff condition). The collection $\{V_y / y \in Y\}$ is a covering of Y by sets in X; therefore, finitely many of them $V_{y1}, ...., V_{yn}$ cover Y. The open set

$$V = V_{y1} \cup .... \cup V_{yn}$$

contains Y, and it is disjoint from the open set

$$U = U_{y1} \cap \dots \cap U_{yn}$$

formed by taking the intersection of the corresponding neighborhoods of $x_0$. For if z is a point of V, then $z \in V_{yi}$ for some i, hence $z \notin U_{yi}$ and so $z \notin U$. Then U is a neighbourhood of $x_0$, disjoint from Y, as desired.

***Theorem 3:*** The image of a compact space under a continuous map is compact.

***Proof:*** Let $f : X \to Y$ be continuous; let X be compact. Let $\mathcal{A}$ be a covering of the set f(X) by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them. Say

$$f^{-1}(A_1), \dots, f^{-1}(A_n), \text{ cover X, then the sets } A_1 \dots, A_n \text{ cover } f(X)$$

*Note* Use of the proceeding theorem is as a tool for verifying that a map is a homeomorphism

***Theorem 4:*** Let $f : X \to Y$ be a bijective function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

***Proof:*** We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map $f^{-1}$. If A is closed in X, then A is compact by theorem (1). Therefore by the theorem just proved f(A) is compact. Since Y is Hausdorff, f(A) is closed in Y by theorem (2)

*Example 3:* Show by means of an example that a compact subset of a topological space need not be closed.

*Solution:* Suppose (X, I) is an indiscrete topological space such that X contains more than one element. Let A be a proper subset of X and let $(A, I_1)$ be a subspace of (X, I). Here, we have $I_1 = \{\phi, A\}$. For $I = \{\phi, X\}$. Hence, the only $I_1$ – open cover of A is {A} which is finite. Hence A is compact. But A is not I-closed. For the only I-closed sets are $\phi$, X. Thus A is compact but not closed.

***Theorem 5:*** A closed subset of a countably compact space is countably compact.

***Proof:*** Let Y be a closed subset of a countably compact space (X, T).

Let $\{G_n : n \in N\}$ be a countable T-open cover of Y, then

$$Y \subset \bigcup_n G_n.$$

But $\quad X = Y' \cup Y$

Hence $\quad X = Y' \cup \{G_n : n \in N\}$

This shows that the family consisting of open sets $Y', G_1, G_2, G_3, \dots$ forms an open countable cover of X which is known to be countably compact. Hence this cover must be reducible to a finite subcover, say

$Y', G_1, G_2, \dots, G_n$ so that $X = Y' \cup \left[\bigcup_{i=1}^{n} G_i\right]$

$\Rightarrow \quad Y \subset \bigcup_{i=1}^{n} G_i$

Notes

It means that $\{G_i : 1 \le i \le n\}$ is finite subcover of the countable cover

$$\{G_n : n \in N\}$$

Hence Y is countably compact

## Self Assessment

1. Prove that a topological space is compact if every basic open cover has a finite sub-cover.
2. Show that every cofinite topological space (X, T) is compact.
3. Show that if $(Y, T_1)$ is a compact subspace of a Hausdorff space (X, T), then Y is T-closed.

## 13.2 Compact Subspaces of the Real Line

The theorems of the preceding section enable is to construct new compact spaces from existing ones, but in order to get very far we have to find some compact spaces to start with. The natural place to begin is the real line.

Application include the extreme value theorem and the uniform continuity theorem of calculus, suitably generalised.

***Theorem 6:*** *Extreme Value Theorem*

Let f: $X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

***Proof:*** Since f is continuous and X is compact, the set A = f(X) is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have m = f(c) and M = F(d) for some points c and d of X.

If A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), ..., (-\infty, a_n)\}$$

covers A. If $a_i$ is the largest of the elements $a_1,..., a_n$, then $a_i$ belongs to none of these sets, contrary to the fact that they cover A.

A similar argument shows that A has a smallest element.

***Definition:*** Let (X, d) be a metric space; let A be a non-empty subset of X. For each $x \in X$, we define the *distance from x to A* by the equation

$$d(x, A) = \inf \{d (x, a) \mid a \in A\}.$$

It is easy to show that for fixed A, the function d (x, A) is continuous function of x.

Given $x, y \in X$, one has the inequalities

$$d(x, A) \le d(x, a) \le d(x, y) + d(y, a),$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \le \inf d (y, a) = d(y, A),$$

Notes

so that

$$d(x, A) - d(y, A) \leq d(x, y).$$

The same inequality holds with x and y interchanged, continuity of the function d(x, A) follows.

Now we introduce the notion of Lebesgue number. Recall that the diameter of a bounded subset A of a metric space (X, d) is the number

$$\sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

***Lemma (1) (The Lebesgue number Lemma):*** Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than $\delta$, there exists an element of $\mathcal{A}$ containing it.

The number $\delta$ is called a Lebesgue number for the covering $\mathcal{A}$.

***Proof:*** Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number of $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$.

Choose a finite subcollection $\{A_1, ..., A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - A_i$, and define f: $X \rightarrow \mathbb{R}$ be letting f(x) be the average of the numbers $d(x, C_i)$. That is,

$$f(x) = \frac{1}{n}\sum_{i=1}^{n}(x, c_i)$$

We show that f(x) > 0 for all x. Given $x \in X$, choose i so that $x \in A_i$. Then choose $\in$ so $\in$-neighborhood of x lies in $A_i$. Then $d(x, c_i) \geq \in$, so that $f(x) \geq \in/n$.

Since f is continuous, it has a minimum value $\delta_i$ we show that $\delta$ is our required Lebesgue number. Let B be a subset of X of diameter less that $\delta$. Choose a point $x_0$ of B; then B lies in the $\delta$-neighborhood of $x_0$. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest of the number $d(x_0, C_i)$. Then the $\delta$-neighborhood of $x_0$ is contained in the element $A_m - X - C_m$ of one covering $\mathcal{A}$.

***Definition:*** *Uniformly Continuous*

A function F from the metric space $(X, d_X)$ to the metric $(Y, d_Y)$ is said to be uniformly continuous if given $\in > 0$, there is a $\delta > 0$ such that for every pair of points $x_0, x_1$ of X,

$$d_x(x_0, x_1) < d \Rightarrow d_Y(f(x_0), f(x_1)) < \in.$$

***Theorem 7:*** *Uniform Continuity Theorem*

Let f: $X \rightarrow Y$ be a continuous map of the compact metric space $(X, d_x)$ to be metric space $(Y, d_y)$. Then f is uniformly continuous.

***Proof:*** Given $\in > 0$, take the open covering of Y by balls B $(y, \in/2)$ of radius $\in/2$. Let A be the open covering of X by the inverse images of these balls under f. Choose $\delta$ to be a Lebesgue number for the covering A. Then if $x_1$ and $x_2$ are two points of X such that $dx(x_1, x_2) < \delta$, the two point set $\{x_1, x_2\}$ has diameter less than $\delta$. So that its image $\{f(x_1), f(x_2)\}$ lies in some ball B $(y, \in/2)$. Then dy $(f(x_1), f(x_2) < \in$, as desired.

Finally, we prove that the real numbers are uncountable. The interesting thing about this proof is that it involves no algebra at all-no decimal or binary expansions of real numbers or the like-just the other properties of $\mathbb{R}$.

**Notes**

***Theorem 8:*** Every closed and bounded interval on the real line is compact.

***Proof:*** Let $I_1$ = [a, b] be a closed and bounded interval on $\mathcal{R}$. If possible, let $I_1$ be not compact. Then there exists an open covering $\mathcal{C} = \{G_i\}$ of $I_1$, having no finite sub covering.

Let us write $$I_1 = [a, b] = \left[a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right] \quad \ldots(1)$$

Since $I_1$ is not covered by a finite sub-class of $\mathcal{C}$ and therefore at least one of the intervals of the union in (1) cannot be covered by any finite sub-class of $\mathcal{C}$.

Let us denote such an interval by $I_2 = [a_1, b_1]$.

Now writing $$I_2 = [a_1, b_1] = \left[a_1, \frac{a_1+b_1}{2}\right] \cup \left[\frac{a_1+b_1}{2}, b_1\right] \quad \ldots(2)$$

As argued before, at least one of the intervals in the union of (2) cannot be covered by a finite sub-class of $\mathcal{C}$.

Let us denote such an interval by $I_3 = [a_2, b_2]$.

On continuing this process we obtain a nested sequence $\langle I_n \rangle$ of closed intervals such that none of these intervals $I_n$ can be covered by a finite sub-class of $\mathcal{C}$.

Clearly the length of the inverval.

$$I_n = \frac{a-b}{2^n}$$

Thus lim $|I_n| = 0$.

Hence, by the nested closed interval property, $\cap I_n \neq \phi$.

Let $p \in \cap I_n$, then $p \in I_n \; \forall \; n \in N$.

In particular $p \in I_1$.

Now since $\mathcal{C}$ is an open covering of $I_1$, there exists some $A_{\alpha_0}$ in C such that $p \in A_{\alpha_0}$.

Since $A_{\alpha_0}$ is open there exists an open interval $(p - \varepsilon, p + \varepsilon)$ such that $p \in (p - \varepsilon, p + \varepsilon) \subseteq A_{\alpha_0}$.

Since $\ell(I_n) \to 0$ as $n \to \infty$, there exists some

$$I_{n_0} \subseteq (p - \varepsilon, p + \varepsilon) \subseteq A_{\alpha_0}.$$

This contradicts our assumption that no $I_n$ is covered by a finite number of members of $\mathcal{C}$.

Hence [a, b] is compact.

*Example 4:* The real line is not compact.

*Solution:* Let $\mathcal{C} = \{\, ]-n, n[ \; : n \in N\}$.

Then each member of $\mathcal{C}$ is clearly an open interval and therefore, a U-open set.

Also if p is any real number, then there exists a positive integer $n_p$ such that $n_p > |p|$.

Then clearly $p \in \, ]-n_p, n_p[ \; \in \mathcal{C}$.

Thus each point of $\mathcal{R}$ is contained in some member of $\mathcal{C}$ and therefore $\mathcal{C}$ is an open covering of $\mathcal{R}$.

Notes

Now if $\mathcal{C}^*$ is a fanily of finite numbr of sets in $\mathcal{C}$, say

$$\mathcal{C}^* = \{\ ]-n_1, n_1[,\ ]-n_2, n_2[, \ldots,\ ]-n_K, n_K[\ \}$$

and if $n^* = \max \{n_1, n_2, \ldots, n_K\}$, then

$$n^* \notin \bigcup_{i=1}^{K} (]-n_i, n_i[)$$

Thus it follows that no finite sub-family of $\mathcal{C}$ cover $\mathcal{R}$.

Hence ($\mathcal{R}$, U) is not compact.

***Theorem 9:*** A closed and bounded subset (subspace) of $\mathcal{R}$ is compact.

***Proof:*** Let $I_1 = [a_1, b_1]$ be a closed and bounded subset of $\mathcal{R}$. Let $G = \{(c_i, d_i) : i \in \Delta\}$ be an open covering of $I_1$.

To prove that ∃ finite subcover of the original cover G.

Suppose the contrary.

Then ∃ no finite subcover of the cover G.

Divide $I_1$ into two equal closed intervals.

$$\left[a_1, \frac{a_1+b_1}{2}\right] \text{ and } \left[\frac{a_1+b_1}{2}, b_1\right].$$

Then, by assumption, at least one of these two intervals will not be covered by any finite subclass of the cover G. Call that interval by the name $I_2$.

Write $I_2 = [a_2, b_2]$

Then $[a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$.

Divide $I_2$ into two equal closed intervals $\left[a_2, \frac{a_2+b_2}{2}\right]$ and $\left[\frac{a_2+b_2}{2}, b_2\right]$. Again by assumption, at least one of these two intervals will not be covered by any finite sub-family of the cover G. Call that interval by the name $I_3$.

Write $I_3 = [a_3, b_3]$.

Repeating this process an infinite number of times, we get a sequence of intervals $I_1, I_2, I_3, \ldots$ with the properties.

(i) $I_n \supset I_{n+1}\ \forall\ n \in N$.

(ii) $I_n$ is closed $\forall\ n \in N$.

(iii) $I_n$ is not covered by any finite sub-family of G.

(iv) $\underset{n\to\infty}{\text{lt}} [I_n] = 0$, where $|I_n|$ denotes the length of the interval $I_n$ and similar is the meaning of $|[a_r, b_r]|$.

Evidently the sequence of intervals $\langle I_n \rangle$ satisfies all the conditions of nested closed interval property.

This $\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$

So that ∃ a number $p_0 \in \bigcap_{n=1}^{\infty} I_n$.

## Self Assessment

Notes

4. Prove that if X is an ordered set in which every closed interval is compact, the X has the least upper bound property.

5. Let X be a metric space with metric d; let $A \subset X$ be non-empty. Show that d(x, A) = 0 if and only if $x \in \overline{A}$.

## 13.3 Summary

- A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its element are ope subsets of X.
- A space X is said to be compact if every open covering A of X contains a finite subcollection that also cover X.
- Let A be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than $\delta$, there exists an element of A continuing it. The number $\delta$ is called a Lebesgue number for the covering A.

## 13.4 Keywords

***Closed Open Set:*** Let (X, T) be a topological space. Any set $A \in T$ is called an open set and X – A is a closed set.

***Countably Compact:*** A topological space (X, T) is said to be countably compact iff every countable T-open cover of X has a finite subcover.

***Homeomorphism:*** A map $f : (X, T) \to (Y, U)$ is said to be homeomorphism if (i) f is one-one onto (ii) f and $f^{-1}$ are continuous.

***Indiscrete Topology:*** Let X be any non-empty set ad T = {X, $\phi$}. Then T is called the indiscrete topology.

## 13.5 Review Questions

1. Let T and T′ be two topologies on the set X; suppose that $T' \supset T$. What does compactness of X under one of these topologies imply about compactness under the other?

2. Show that if X is compact Hausdorff under both T and T′, then either T and T′ are equal or they are not comparable.

3. Show that a finite union of compact subspaces of X is compact.

4. Let A and B be disjoint compact subspace of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

5. Let Y be a subspace of X. If $Z \subset Y$, then show that Z is compact as a subspace of Y $\Leftrightarrow$ it is compact as a subspace of X.

6. Prove that a closed subset of a compact space is compact.

## 13.6 Further Readings

*Books*

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York

S. Willard, *General Topology*, Addison-Wesley Mass. 1970.

# Unit 14: Limit Point Compactness



**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Define limit-point compactness and solve related problems;
- Define the term sequentially compact and solve questions on it.

## Introduction

In this unit, we introduce limit point compactness. In some ways, this property is more natural and intuitive than that of compactness. In the early days of topology, it was given the name "compactness", while the open covering formulation was called "bicompactness". Later, the word "compact" was shifted to apply the open covering definition, leaving this one to search for a new name. It still has not found a name on which everyone agrees. On historical grounds, some call it "Frechet compactness" others call it the "Bolzano–Weierstrass property". We have invented the term "limit point compactness". It seems as good a term as any at least it describes what the property is about.

## 14.1 Limit Point Compactness and Sequentially Compact

### 14.1.1 Limit Point Compactness

A space X is said to be limit point compact if every infinite subset of X has a limit point.

***Theorem 1:*** Compactness implies limit point compactness, but not conversely.

***Proof:*** Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contra positive – if A has no limit point, then A must be finite.

So suppose A has no limit point. Then A contains all its limit points, so that A is closed. Further more, for each $a \in A$, we can choose a neighborhood $U_a$ of a such that $U_a$ intersects A in the point a alone. The space of X is covered by the open set X – A and the open sets $U_a$; being compact, it can be covered by finitely many of these sets. Since X – A does not intersect A, and each set $U_a$ contains only one point of A, the set A must be finite.

*Example 1:* Let Y consist of two points; give Y the topology consisting of Y and the empty set. Then the space $X = Z_+ \times Y$ is limit point compact, for every non-empty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X.

### 14.1.2 Sequentially Compact

Let X be a topological space. If $(x_n)$ is a sequence of points of X, and if

$$n_1 < n_2 < ... < n_i < ....$$

is an increasing sequence of positive integers, then the sequence $(y_i)$ defined by setting $y_i = x_{ni}$ is called a subsequence of the sequence $(x_n)$. The space X is said $t_o$ be sequentially compact if every sequence of points of X has a convergent subsequence.

***Theorem 2:*** Let X be a metrizable space. Then the following are equivalent:

1. X is compact
2. X is limit point compact
3. X is sequentially compact

***Proof:*** We have already proved that (1) ⇒ (2). To show that (2) ⇒ (3), assume that X is limit point compact. Given a sequence $(x_n)$ of points of X, consider the set $A = \{x_n n \in Z_+\}$. If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence $(x_n)$ has a subsequence that is constant, and therefore converges trivially. On the other hand, if A is infinite, then A has a limit point of x. We define a subsequence of $(x_n)$ converging to x as follows.

First choose $n_1$ so that

$$x_{n_1} \in B(x, 1)$$

Then suppose that the positive integer $n_{i-1}$ is given. Because the ball B (x, 1/i) intersects A in infinitely many points, we an choose an index $n_i > n_{i-1}$ such that

$$x_{ni} \in B(x, 1/i)$$

Then the subsequence $x_{n1}, x_{n2}, ...,$ converges to x.

Finally, we show that (3) ⇒ (1). This is the hardest part of the proof.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X. (This would follow from compactness, but compactness is what we are trying to prove.) Let $\mathcal{A}$ be an open covering of X. We assume that there is no $\delta > 0$ such that each set of diameter less than $\delta$ has an element of $\mathcal{A}$ containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than 1/n that is not contained in any element of $\mathcal{A}$; let $C_n$ be such a set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence $(x_{ni})$ of the sequence $(x_n)$ converges, say to the point a. Now a belongs to some element A of the collection $\mathcal{A}$; because A is open, we may choose an $\in > 0$ such that $B(a, \in) \subset A$. If i is large enough that $1/n_i < \in/2$, then the set $C_{ni}$ lies in the $\frac{\in}{2}$ neighborhood of $x_{ni}$; if i is also chosen large enough that $d(x_{ni}, a) < \in/2$, then $C_{ni}$ lies in the $\in$-neighborhood of a. But this means that $C_{ni} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\in > 0$, there exists a finite covering of X by open $\in$-balls. Once again, we proceed by contradiction. Assume that there exists an $\in > 0$ such that X cannot be covered by finitely many $\in$-balls. Construct a sequence of points $x_n$ of X as follows: First, choose $x_1$ to be any point of X. Noting that the ball $B(x_1, \in)$ is not all of X

(Otherwise X could be covered by a single $\in$-ball), choose $x_2$ to be a point of X not in $B(x_1, \in)$. In general, given $x_1, ...., x_n$, choose $x_{n+1}$ to be a point in the union

$$B(x_1, \in) \cup .....\cup B(x_n, \in)$$

using the fact that these balls do not cover X. Note that by construction $d(x_{n+1}, x_i) \geq \in$ for i = 1, ..., n. Therefore, the sequence $(x_n)$ can have no convergent subsequence; in fact, any ball of radius $\in/2$ can contain $x_n$ for at most one value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let $\mathcal{A}$ be an open covering of X. Because X is sequentially compact, the open covering $\mathcal{A}$ has a Lebesgue number $\delta$. Let $\in = \delta/3$; use sequential compactness of X to find a finite covering of X by open $\in$- balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of $\mathcal{A}$. Choosing one such element of $\mathcal{A}$ for each of these $\in$-balls, we obtain a finite subcollection of $\mathcal{A}$ that covers X.

*Example 2:* Prove that a continuous image of a sequentially compact set is sequentially compact.

*Solution:* Let (X, T) be a sequentially compact topological space so that every sequence $\langle x_n \rangle$ in X has a convergent subsequence $\langle x_{i_k} : K \in N \rangle$ and let this subsequence converge to $x_{i_0}$, i.e.,

$$x_{i_K} \to x_{i_0} \in X.$$

Let $f : (X, T) \to (Y, U)$ be a continuous map.

To prove that f (X) is sequentially compact set.

f is continuous map $\Rightarrow$ f is sequentially continuous

Furthermore $x_{i_K} \to x_{i_0}$.

This implies that $f(x_{i_K}) \to f(x_{i_0})$.

Showing thereby f (X) is sequentially compact.

*Example 3:* A finite subset of a topological space is necessarily sequentially compact.

*Solution:* Let (X, T) be a topological space and $A \subset X$ be finite and $\langle x_n \rangle$ be a sequene in A so that $x_n \in A \; \forall$ n. Also $\langle x_n \rangle$ contains infinite number of terms. It follows that at least one element of A, say $x_0$ must appear infinite number of times in $\langle x_n \rangle$. Thus $\langle x_0, x_0, x_0, \ldots \rangle$ is a subsequence of $\langle x_n \rangle$ and this subsequence converges to $x_0 \in A$, showing thereby A is a sequentially compact.

***Theorem 3:*** A metric space is sequentially compact iff it has the Bolzano Weierstrass Property.

***Proof I:*** Let (X, d) be a sequentially compact metric space. To prove that (X, d) has Bolzano Weierstrass Property,

Let $A \subset X$ be an infinite set.

If we show that A has a limit point in X, the result will follow.

A is an infinite set $\Rightarrow$ A contains an enumerable set, say $\{x_n : n \in N\}$

$\Rightarrow \langle x_n \in A, n \in N \rangle$ is a sequence with infinitely many distinct points.

By the assumption of sequential compactness, the sequence $\langle x_n \rangle$ has a convergent subsequence $\langle x_{in} : n \in N \rangle$ (say). Let this convergent sequence $\langle x_{in} : n \in N \rangle$ converge to $x_0$. Then $x_0 \in X$ and $\langle x_n \rangle$ also converges to $x_0$, i.e., $x_n \to x_0$. Consequently $x_0$ is a limit point of the set $\{x_n : n \in N\}$.

Evidently $\{x_n : n \in N\} \subset A$.

So that $D(\{x_n : n \in N\}) \subset D(A)$.

But $x_0 \in D\{x_n : n \in N\}$ and hence $x_0 \in D(A)$, i.e., A has a limit point $x_0 \in X$.

***Proof II:*** Conversely, suppose that the metric space (X, d) has Bolzano Weierstrass property.

To prove that X is sequentially compact.

By the assumption of Bolzano Weierstrass property, every infinite subset of X has a limit point in X. Let $\langle x_n \rangle$ be an asbitrary sequence in X.

***Case (i):*** If the sequence $\langle x_n \rangle$ has an element x which is infinitely repeated, then it has a constant subsequence $\langle x, x, \ldots, x, \ldots \rangle$ which certainly converges to x.

***Case (ii):*** If the sequence $\langle x_n \rangle$ has infinitely many distinct points then by assumption, the set $\{x_n : x \in N\}$ has a limit point, say $x_0 \in X$. Consequently $x_0$ is a limit of the sequence $\langle x_n : n \in N \rangle$ with infinitely many distinct points so that this sequence contains a subsequence $\langle x_{in} : n \in N\}$ which also converges to $X_0$.

$\therefore$ In either case, we have shown that every sequence in X contains a convergent subsequence so that X is sequentially compact.

Hence the result.

## 14.2 Summary

- A space X is said to be limit point compact if every infinite subset of X has a limit point.
- Compactness implies limit point compactness, but not conversely.
- A topological space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

## 14.3 Keywords

***BWP:*** A topological space (X, T) is said to have Bolzano Weierstrass Property denoted by BWP if every infinite subset has a limit point.

***Compact Space:*** A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X.

***Lebesgue Covering Lemma:*** Every open covering of a sequentially compact space has a Lebesgue number.

***Lebesgue Number:*** Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d), a real number $\delta > 0$ is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < \delta \Rightarrow A \subset G_{i0}$ for at least one index $i_0 \in \Delta$.

***Metrizable:*** Any topological space (X, T), if it is possible to find a metric on $\rho$ on X which induces the topology T i.e. the open sets determined by the metric $\rho$ are precisely the members of $\delta$, then X is said to the metrizable.

***Open Cover:*** Let (X, T) be a topological space and $A \subset X$. Let G denote a family of subsets of X. G is called a cover of A if $A \subset U \{G : G \in G\}$. If every member of G is a open set, then the cover G is called an open cover.

## 14.4 Review Questions

1. Show that [0, 1] is not limit point compact as a subspace of $\mathbb{R}_l$.
2. Let X be limit point compact.
   (a) If $f : X \to Y$ is continuous, does it follow that f(X) is limit point compact?
   (b) If A is a closed subset of X, does it follow that A is limit point compact?
   (c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?
3. A space X is said to be countably compact if every countable open covering of X contains a finite subcollection that covers X. Show that for a $T_1$ space X, countable compactness is equivalent to limit point compactness.

   [***Hint:*** If no finite subcollection of $U_n$ covers X, choose $x_n \notin U_1 \cup ... \cup U_n$, for each n.]

## 14.5 Further Readings

*Books*

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley Mass. 1970.

# Unit 15: Local Compactness

CONTENTS



## Objectives

After studying this unit, you will be able to:

- Describe the local compactness;
- Solve the problems on local compactness;
- Explain the theorems on local compactness.

## Introduction

In this unit, we study the notion of local compactness and we prove the theorems that every continuous image of a locally compact space is locally compact and many other theorems.

## 15.1 Locally Compact

Let (X, T) be a topological space and let $x \in X$ be arbitrary. Then X is said to be locally compact at x if the closure of any neighborhood of x is compact.

X is called locally compact if it is compact at each of its points, but need not be compact as whole.

***Alternative definition:*** A topological space (X, T) is locally compact if each element $x \in X$ has a compact neighborhood.

*Example 1:* Show that $\mathbb{R}$ is locally compact.

*Solution:* Let $x \in R$ be arbitrary.

Evidently, $\overline{S_r(x)} = S_r[x]$

$S_r[x]$ is compact, being closed and bounded subset of $\mathbb{R}$. Thus the closure of the neighborhood $S_r(x)$ of x is compact and hence the result.

*Example 2:* Show that compactness ⇒ locally compact.

*Solution:* Let (X, T) be a compact topological space. To prove that X is locally compact.

For this, we must show that the closure of any neighborhood of any point $x \in X$ is compact. This follow from the fact that X is the neighborhood of each of its points and $X = \overline{X}$, X is compact.

Notes

***Theorem 1:*** Let (X, T) and (Y, $\mathcal{U}$) be topological spaces and f : (X, T) $\xrightarrow{\text{onto}}$ (Y, U) be a continuous open map. Then if X is locally compact, then Y is also.

Or

Every open continuous image of a locally compact space is locally compact.

***Proof:*** Let f : (X, T) → (Y, U) be a continuous open map and X a locally compact space.

We claim Y is locally compact.

Let $y \in Y$ be arbitrary and $U \subset Y$ a nbd of y.

$y \in Y$, $f : X \to Y$ is onto $\Rightarrow \exists\, x \in X$ s.t. $f(x) = y$

∵ f is continuous

∴ Given any nbd U of y, ∃ a nbd $V \subset X$ of x s.t. $f(V) \subset U$. X is locally compact.

⇒ X is locally compact at x and V is a nbd of x.

⇒ ∃ is compact set A s.t. $x \in A^o \subset A \subset V$

⇒ $f(x) \in f(A^o) \subset f(A) \subset f(V) \subset U$

⇒ $y \in f(A^o) \subset f(A) \subset U$ ...(1)

Now, f is open, $A^o \subset X$ is open.

⇒ $f(A^o) \subset Y$ is open

⇒ $f(A^o) = [f(A^o)]^o$ ...(2)

From (1), $f(A^o) \subset f(A)$

Thus $[f(A^o)]^o \subset [f(A)]^o$

⇒ $f(A^o) \subset [f(A)]^o$, (on using (2))

⇒ $f(A^o) \subset [f(A)]^o \subset f(A)$

Using this in (1),

$y \in f(A^o) \subset [f(A)]^o \subset f(A) \subset U$

or $y \in [f(A)]^o \subset f(A) \subset U$

Taking B = f(A) = continuous image of compact set A

= compact set

We obtain $y \subset B^o \subset B \subset U$, B is compact.

Finally, we have shown that given any $y \in Y$ and a nbd U of y, ∃ a compact set $B \subset Y$, s.t. $y \subset B^o \subset B \subset U$.

Hence Y is locally compact at y so that Y is locally compact.

***Theorem 2:*** Every locally compact $T_2$-space is a regular space.

***Proof:*** Let (X, T) be a locally compact $T_2$-space. To prove that (X, T) is a regular space. Let $x \in X$ be arbitrary and G a nbd of x.

By definition of locally compact space,

∃ a compact set $A \subset X$ s.t. $x \in A^o \subset A \subset G$.

A is compact, X is $T_2$-space ⇒ A is closed.

Notes

$\Rightarrow \quad (\overline{A^o}) \subset \overline{A} = A$

$\Rightarrow \quad (\overline{A^o}) \subset A$ ...(1)

$\therefore \quad x \in A^o \subset A \subset G$

$\therefore \quad x \in A^o \subset (\overline{A^o}) \subset A \subset G,$ [by (1)]

Taking $A^o = U$

$x \in U \subset \overline{U} \subset G$

Thus we have shown that given any nbd G of x, $\exists$ a nbd U of x s.t.

$x \in U \subset \overline{U} \subset G$

Consequently X is regular.

***Theorem 3:*** Any open subspace of a locally compact space is a locally compact.

***Proof:*** Let (Y, U) be an open subspace of a locally compact space (X, T) so that Y is open in X.

To prove that Y is locally compact.

Let $x \in Y \subset X$ be arbitrary and G a U-nbd of x in Y, then $x \in X, G \subset Y$.

X is locally compact $\Rightarrow$ X is locally compact at x.

G is a U-nbd of x in Y $\Rightarrow \exists\, G_1 \in U$ s.t. $x \in G_1 \subset G$

$\Rightarrow \quad G_1 \in T$ s.t. $x \in G_1 \subset G$. For Y is open in X.

$\Rightarrow$ G is a T-nbd of x in X.

Also X is locally compact $\Rightarrow \exists$ a compact set $A \subset X$ s.t. $x \in A^o \subset A \subset G$. But $G \subset Y$.

$\Rightarrow \quad x \in A^o \subset A \subset G \subset Y$

Thus (i) $A \subset Y$, A is U-compact.

For A is T-compact $\Rightarrow$ A is U-compact.

(ii) G is a nbd of x in Y s.t. $x \in A^o \subset A \subset G$.

This proves that Y is locally compact at any $y \in Y$ and hence the result follows. Proved.

***Theorem 4:*** Every closed subspace of a locally compact space is locally compact.

***Proof:*** Let (Y, U) be a closed subspace of a locally compact space (X, T), then Y is T-closed set. Let $y \in Y \subset X$ be arbitrary.

To prove that Y is locally compact, we have to prove that Y is locally compact at y.

X is locally compact $\Rightarrow$ X is locally compact at y

$\Rightarrow \quad \exists$ T-open nbd N of x s.t. $\overline{N}$ is T-compact.

$\Rightarrow \quad N \cap Y$ is U-open nbd of y.

$N \cap Y \subset N \Rightarrow \overline{N \cap Y} \subset \overline{N}$.

Notes

Thus $\overline{N \cap Y}$ is a closed subset of a compact set $\overline{N}$. Hence $\overline{N \cap Y}$ is compact.

Y is T-closed ⇒ T-closure of N ∩ Y = U-closure of N ∩ Y.

Thus N ∩ Y is U-open nbd of y s.t. $\overline{N \cap Y}$ is compact, showing thereby Y is locally compact at y.

## 15.2 Summary

- A topological space (X, T) is locally compact if each element x ∈ X has a compact neighborhood.
- Any open subspace of a locally compact space is a locally compact.
- Every locally compact $T_2$-space is a regular space.
- Every closed subspace of a locally compact space is locally compact.

## 15.3 Keywords

***Closure:*** Let (X, T) be a topological space and A ⊂ X. The closure of $\mathcal{A}$ is defined as the intersection of all closed sets which contain A and is denoted by the symbol $\overline{A}$.

***Compact set:*** Let (X, T) be a topological space and A ⊂ X. A is said to be a compact set if every open covering of A is reducible to finite sub-covering.

***Interior point:*** A point x ∈ A is called an interior point of A if ∃ r ∈ $R^+$ s.t. $S_r(x)$ ⊂ A.

***Neighborhood:*** Let ε > 0 be any real number. Let $x_0$ be any point on the real line. Then the set {x ∈ R : | x – $x_0$ | < ε} is defined as the ∈-neighborhood of the point $x_0$.

***Regular space:*** A regular space is a topological space in which every nbd of a point contains a closed neighborhood of the same point.

***$T_2$-space:*** A $T_2$-space is a topological space (X, T) fulfilling the $T_2$-axiom: every two points x, y ∈ X have disjoint neighborhoods.

## 15.4 Review Questions

1. Show that the rationals Q are not locally compact.
2. Let X be a locally compact space. If f : X → Y is continuous, does it follow that f(x) is locally compact? What if f is both continuous and open? Justify your answer.
3. If f : $X_1$ → $X_2$ is a homeomorphism of locally compact Hausdorff spaces, show f extends to a homeomorphism of their one-point compactifications.
4. Is every open subspace of a locally compact space is locally compact? Give reasons in support of your answer.
5. Show by means of an example that locally compact space need not be compact.
6. Show that local compactness is a closed hereditary property.
7. $X_1$, $X_2$ are L-compact if and only if $X_1 \times X_2$ is L-compact.

## 15.5 Further Readings

*Books*

Kelley, John (1975), *General Topology,* Springer.

Sbeen, Lynn Arthur, Seebach, J. Arthur Jr. (1995), *Counter examples in Topology* (Dover reprint of 1978 ed.) Berlin, New York.

Willard, Stephan (1970), *General Topology,* Addison-Wesley (Dover Edition).

*Online links*

www.math.stanford.edu

# Unit 16: The Countability Axioms

**Notes**

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Define countability axioms;
- Understand and describe the theorems on countability axioms;
- Discuss the theorems on countability axioms related to the metric spaces.

## Introduction

The concept we are going to introduce now, unlike compactness and connectedness, do not arise naturally from the study of calculus and analysis. They arise instead from a deeper study of topology itself. Such problems as imbedding a given space in a metric space or in a compact Hausdorff space are basically problems of topology rather than analysis. These particular problems have solutions that involve the countability and separation axioms. In this unit, we shall introduce countability axioms and explore some of their consequences.

## 16.1 Countability Axioms

### 16.1.1 First Axiom of Countability

Let (X, T) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local base at each $x \in X$. The space X, in this case, is called first countable or first axiom space.

*Example 1:* Consider $x \in R$

$$A_n = \left(x-\frac{1}{n}, x+\frac{1}{n}\right) \forall\ x \in N$$

Notes

Take $B_x = \{A_n : n \in N\}$.

Evidently, $B_x$ is a local base at $x \in X$ for the usually topology on R.

Clearly, $B_x \sim N$ under the map $A_n \to n$.

Therefore, $B_x$ is a countable local base at $x \in X$. But $x \in X$ is arbitrary.

Hence R with usual topology is first countable.

### 16.1.2 Second Axiom of Countability

Let (X, T) be a topological space. The space X is said to satisfy the second axiom of countability if $\exists$ a countable base for T on X.

In this case, the space X is called second countable or second axiom space.

*Note* A second countable space is also called completely separable space.

*Example 2:* The set of all open intervals (r, s) and r with s as rational numbers forms a base, say B for the usual topology $\cup$ of R. Since Q, Q × Q are countable sets and so B is a countable base for $\cup$ on R.

$\therefore$ (R, $\cup$) is second countable.

### 16.1.3 Hereditary Property

Let (X, T) be a topological space. A property P of X is said to be hereditary if the property is possessed by every subspace of X.

E.g. first countable, second countable are hereditary properties, where as closed sets, open sets, are not hereditary properties.

### 16.1.4 Theorems and Solved Examples on Countability Axioms

***Theorem 1:*** Let (X, T) be a second axiom space and let C be any collection of disjoint open subsets of X. Then C is a countable collection.

***Proof:*** Let (X, T) be a second countable space, then $\exists$ a countable base

$$\mathcal{B} = \{B_n : n \in \Delta\} \text{ for topology T on X.}$$

Let C be a collection of disjoint open subsets of X.

Let $A \in C$ be arbitrary, then $A \in T$.

By definition of base, $\exists\ B_n \in \mathcal{B}$ st. $B_n \subset A$.

We associate with A, a least positive integer n s.t. $B_n \subset A$.

Members of C are disjoint

$\Rightarrow$ distinct integers will be associated with distinct member of C.

If we now order the members of C according to the order of associated integers, then we shall get a sequence containing all the members of C. Hence, C is a countable collection.

***Theorem 2:*** Let (X, T) be a first axiom space. Then $\exists$ is a nested (monotone decreasing) local base at every point of X.

Notes

***Proof:*** Let (X, T) be first axiom space, then ∃ is a countable local base

$$B(x) = \{B_n : n \in N\} \text{ at every point } x \in X.$$

Write $C_1 = B_1, B_2 = B_1 \cap B_2, C_3 = B_1 \cap B_2 \cap B_3, \ldots,$

$$C_n = \bigcap_{i=1}^{n} B_i.$$

Then $C_1 \supset C_2 \supset C_3 \supset \ldots \supset C_n$.

$$x \in B_n \in \mathcal{B} \quad \forall\, n \quad \Rightarrow x \in C_n \in T \quad \forall\, n.$$

It follows that $C(x) = \{C_n : n \in N\}$ is a nested local base at x.

***Theorem 3:*** A second countable space is always first countable space.

Or

Prove that second axiom of countability ⇒ first axiom of countability.

***Proof:*** Let (X, T) be a topological space which satisfies the second axiom of countability so that (X, T) is second countable.

To prove that (X, T) also satisfies the first axiom of countability.

i.e., to prove that (X, T) is first countable.

By hypothesis, ∃ a countable base $\mathcal{B}$ for topology T on X.

$\mathcal{B}$ is countable ⇒ $\mathcal{B} \sim N$

This show that $\mathcal{B}$ can be expressed as

$$\mathcal{B} = \{B_n : n \in N\}$$

Let $x \in X$ be arbitrary.

Write $L_x = \{B_n \in \mathcal{B} : x \in B_n\}$

(i) $L_x$, being a subset of a countable set $\mathcal{B}$, is countable.

(ii) Since members of $\mathcal{B}$ are T open sets and hence the members of $L_x$. For $L_x \subset \mathcal{B}$.

(iii) Any $G \in L_x \Rightarrow x \in G$, according to the construction of $L_x$.

(iv) Let $G \in T$ for arbitrary s.t. $x \in G$.

Then, by definition of base,

$$x \in G \in J \Rightarrow B_r \in \mathcal{B} \quad \text{s.t. } x \in B_r \subset G,$$

$$\Rightarrow \exists\, B_r \in L_x \quad \text{s.t. } B_r \subset G,$$

For $B_r \in \mathcal{B}$ with $x \in B_r \Rightarrow B_r \in L_x$.

Finally $x \in G \in T \Rightarrow \exists\, B_r \in L_x$ s.t. $B_r \subset G$. ...(1)

From (i), (ii), (iii), (iv) and (1), it follows that $L_x$ is a countable local base at $x \in X$. Hence, by definition, X is first countable.

***Theorem 4:*** To prove that first countable space does not imply second countable space. Give an example of a first countable space which does not imply second countable space.

***Proof:*** We need only give an example of a space which does satisfy the first axiom of countability but not the second axiom of countability.

Notes

Let T be a discrete topology on an infinite set X so that every subset of X is open in X and hence in, particular, each singleton set {x} is open in X for each $x \in X$.

Write $\mathcal{B} = \{\{x\} : x \in X\}$.

Then it is easy to verify that $\mathcal{B}$ is a base for the topology T on X and $\mathcal{B}$ is not countable. For X is not countable. Hence X is not second countable.

If we take $L_x = \{x\}$ then evidently $L_x$ is a countable local base at $x \in X$ as it has only one number.

For any $G \in T$ with $x \in G$, $\exists\{x\}$ s.t. $x \in \{x\} \subset G$. From what has been done, it follows that X is first countable but not second countable.

***Theorem 5:*** Show that the property of a space being first countable is hereditary.

***Proof:*** Let $(Y, \bigcup)$ be a subspace of a first countable space (X, T).

If we show that $(Y, \bigcup)$ is first countable, we can conclude the required result.

Let $y \in Y$ be arbitrary, then $y \in X$. $[\because Y \subset X]$

X is first countable $\Rightarrow \exists$ a countable local base at each $x \in X$ and hence, in particular, $\exists$ a countable local base $\mathcal{B}$ at $y \in X$.

Members of $\mathcal{B}$ can be enumerated as $B_1, B_2, B_3, B_4, \ldots$

i.e. $\mathcal{B}_n = \{B_n : n \in N\}$.

Evidently, $y \in B_n \ \forall \ n \in N$.

Write $B_1 = \{Y \cap B_n : n \in N\}$ ...(1)

$y \in Y, y \in B_n \ \forall \ n \in N \Rightarrow y \in Y \cap B_n \ \forall \ n \in N$ ...(2)

$B_n \in \mathcal{B} \ \forall \ n \in N \Rightarrow B_n \in T \Rightarrow Y \cap B_n \in \bigcup$ ...(3)

We claim $\mathcal{B}_1$ is a countable local base at y for $\bigcup$ on Y.

(i) Evidently $N \sim B_1$ under the map $n \to Y \cap B_n$. Hence $B_1$ is countable. ...(4)

(ii) any $G \in \mathcal{B}_1 \Rightarrow y \in G$ ...(5)

(iii) $\mathcal{B}_1$ is family of all $\cup$ - open sets. ...(6)

(iv) let $G \in \bigcup$ be arbitrary s.t.

$y \in G$, then $\exists H \in T$ s.t. $G = H \cap Y$.

$y \in H$. For $y \in G = H \cap Y$.

By definition of local base.

$$y \in H \in T \Rightarrow \exists B_r \in \mathcal{B} \text{ s.t. } y \in B_r \subset H$$

or $$y \in H \cap Y \in \bigcup \Rightarrow \exists B_r \cap Y \in \mathcal{B}_1 \text{ s.t. } B_r \cap Y \subset H \cap Y$$

or $$y \in G \in \bigcup \Rightarrow \exists B_r \cap Y \in \mathcal{B}_1 \text{ s.t. } y \in B_r \cap Y \subset G \quad \text{...(7)}$$

The result (1), (4), (5), (6) and (7) taken together imply that $\mathcal{B}_1$ is a local base at $y \in Y$ for the topology $\cup$ on Y and hence $(Y, \bigcup)$ is first countable.

***Theorem 6:*** Show that the property of a space being second countable is hereditary.

or

Prove that every subspace of a second countable space is second countable.

***Proof:*** Let (Y, $\bigcup$) be a sub-space of a topological space (X, T) which is second countable so that there exists a countable base $\mathcal{B}$ for the topology T.

If we show that (Y, $\bigcup$) is second countable, the result will follow

$$\mathcal{B} \text{ is countable} \Rightarrow \mathcal{B} \sim N$$

$$\Rightarrow \mathcal{B} \text{ is expressible as}$$

$$\mathcal{B} = \{B_n : N \in N\}$$

Write

(i) Evidently $\mathcal{B}_1 \sim N$ under the map $Y \cap B_n \to n$.

$\therefore$ $\mathcal{B}_1$ is countable.

(ii) $\mathcal{B}_1$ is a family of all $\bigcup$–open sets.

For $B_n \in \mathcal{B} \Rightarrow B_n \in T,$

$\therefore B \subset T \Rightarrow Y \cap B_n \in \bigcup$

(iii) any $y \in G \in \cup \Rightarrow \exists\, B_r \cap Y \in \mathcal{B}_1$

s.t. $y \in Y \cap B_r \subset G.$

For proving this let $G \in \bigcup$

s.t. $y \in G$, then $\exists\, H \in T$ s.t. $G = H \cap Y$.

$y \in G \Rightarrow y \in H \cap Y \Rightarrow y \in H$ and $y \in Y$.

By definition of base,

any $y \in H \in T \Rightarrow \exists\, B_r \in \mathcal{B}$ s.t. $y \in B_r \subset H$

from which any $y \in H \cap Y \in \bigcup$

$\Rightarrow \exists\, Y \cap B_r \subset \mathcal{B}_1$ s.t. $y \in Y \cap B_r \in G$

i.e. any $y \in G \in \cup \Rightarrow \exists\, Y \cap B_r \in B_1$

s.t. $y \in Y \cap B_r \subset G.$

Thus it follows that $\mathcal{B}_1$ is a countable base for the topology $\bigcup$ and Y. Consequently, (Y, $\bigcup$) is second countable.

***Theorem 7:*** A second countable space is always separable.

***Proof:*** Let (X, T) be a second countable space.

*To prove:* (X, T) is separable.

Since X is second countable and hence $\exists$ a countable base $\mathcal{B}$ for the topology T on X. Members of $\mathcal{B}$ may be enumerated as $B_1, B_2, B_3, \ldots$ .

Choose an element $x_i$ from each $B_i$ and take A as the collection of all these $x_i$'s.

That is to say, $x_i \in B_c \in \mathcal{B} \;\; \forall \;\; i \in N$ ...(1)

and $A = \{x_i : i \in N\}$ ...(2)

Evidently $N \sim A$ under the map $i \to x_i$

Therefore, A is enumerable.

Notes

Clearly, $A \subset X$

We claim $\bar{A} = X.$

Suppose not, then $X - \bar{A} \neq \phi$ ...(3)

Let $y \in X - \bar{A}$ be arbitrary. $\bar{A}$ is closed and hence $X - \bar{A}$ is open. It amounts to saying that

$$y \in X - \bar{A} \in T.$$

By definition of base

$$y \in X - \bar{A} \in T \Rightarrow \exists B_y \in \mathcal{B} \text{ s.t. } y \in B_y \subset X - \bar{A}.$$

In particular $\Rightarrow$

$$x_{n_o} \in X - \bar{A} \in T$$

$\Rightarrow$ $\exists B_{n_o} \in \mathcal{B}$ s.t. $x_{n_o} \subset X - \bar{A}.$

Now $x_{n_o} \in X - \bar{A} \Rightarrow x_{n_o} \notin \bar{A} \supset A$

$\Rightarrow x_{n_o} \notin A$ ...(4)

$x_{n_o} \in B_n \Rightarrow x_{n_o} \in A$, according to (1) and (2), Contrary to (4).

Hence our assumption $X - \bar{A} \neq \phi$ is wrong.

Consequently $X - \bar{A} = \phi$ i.e. $X = \bar{A}$

Thus, we have shown that

$\exists A \subset X$ s.t. $\bar{A} = X$ and X is enumerable set. By definition, this proves that X is separable.

***Theorem 8:*** Every second axion space is hereditarily separable.

***Proof:*** Let (Y, ∪) be a subspace of second axion, i.e. second countable space (X, T).

To prove the required result, we have to show that (Y, ∪) is second countable and separable since every second countable space is separable. [Refer theorem (7)].

Now it remains to show that (Y, ∪) is second countable. Now write the proof of Theorem (6).

*Example 3:* Prove that (R, ∪) is a second axiom space (Second countable.).

*Solution:* We know that Q is a countable subset of R. If we write

$\mathcal{B} = \{(a, b) : a < b \text{ and } a, b \in Q\}$

Then $\mathcal{B}$ forms a countable base for the usual topology ∪ and R so that R is second countable.

*Example 4:* Prove that ($R^2$, ∪) is second countable.

*Solution:* If we write

$\mathcal{B} = \{S_r(x) : x, r \in Q\}$

then $\mathcal{B}$ forms a countable base for the usual topology ∪ on $R^2$. Hence ($R^2$, ∪) is second countable space.

### 16.1.5 Theorems Related to Metric Spaces

**Notes**

***Theorem 9:*** A metric space is second countable iff it is separable.

***Proof:***

(i) Let (x, ρ) be a metric space. Let T be the metric topology on X corresponding to the metric ρ. Let (X, T) be second countable. To prove that X is separable.

Here write the complete proof of the theorem (6).

(ii) Conversely, suppose that (x, ρ) is a metric space and T is a metric topology on X corresponding to the metric ρ. Also, suppose that X is separable, so that

$\exists\, A \subset X$ s.t. $\bar{A} = X$ and A is countable.

A is countable $\Rightarrow$ A is expressible as

$A = \{a_n : n \in N\}$

To prove that X is second countable.

We know that each open sphere forms an open set.

Let $a_n \in A$ be arbitrary.

Write $\mathcal{B} = \{S_r(a_n) : r \in Q^+, n \in N\}$.

Q is an enumerable set

$\Rightarrow$ $Q^+$ is an enumerable set

$\therefore$ $Q^+ \subset Q$

Then $\mathcal{B}$ is a countable base for the topology T on X.

$\therefore$ X is second countable.

Let $G \in T$ be arbitrary s.t. $x \in G$.

x being an arbitrary point of X.

By definition of open set in a metric space,

$\exists$ a positive real number $\in$ s.t. $S_{(x, \in)} \subset G$ ...(1)

Since A is dense in X and so there will exist a point $a \in A$ s.t.

$$\rho(a, x) < \frac{\in}{3} \qquad ...(2)$$

Since Q is dense in R for the usual topology on R and hence its subset $Q^+$ is also dense in R with usual topology so that $\exists\, r \in Q^+$ s.t.

$$\frac{\in}{3} < r < \frac{2\in}{3}$$

*Aim:* $S_{r(a)} \subset S_{\in(x)} \subset G$.

Also let $y \in S_{(a,r)}$ be arbitrary so that $\rho(y, a) < r$ ...(3)

$$\rho(x,y) \le \rho(x,a) + \rho(a,y)$$

$$< \frac{\in}{3} + r < \frac{\in}{3} + \frac{2\in}{3}, \qquad \text{from (3)}$$

$$\Rightarrow \qquad \rho(x,y) < \in \;\Rightarrow\; y \in S_{(x,\in)}$$

Finally, any $y \in S_{(x,r)} \Rightarrow y \in S_{(x,\in)}$

$$\therefore \quad S_{(a,\,r)} \subset S_{(x,\,\in)}$$

From (2) and (3), $\rho(a,x) < r$, so that $x \in S_{(a,r)}$.

Thus, we have shown that

$$x \in S_{(a,r)} \subset S_{(x,\in)} \subset G.$$

from which $x \in S_{(a,r)} \subset G$.

Thus, $\quad x \in G \in T \Rightarrow \exists\, r \in Q^+$ s.t. $x \in S_{(a,r)}$

i.e. $\quad x \in G \in T \Rightarrow \exists S_{(a,r)} \in \mathcal{B}$ s.t. $x \in S_{(a,r)} \subset G$.

This proves that $\mathcal{B}$ is a base for the topology T on X. From what has been done, it follows that $\mathcal{B}$ is enumerable base for the topology T on X and hence X is second countable.

*Example 5:* Every separable metric space is second countable.

*Solution:* Refer second part of the above theorem.

***Theorem 10:*** A metric space is first countable.

***Proof:*** Let (X, ρ) be a metric space. Let T be metric topology on X, corresponding to the metric ρ on X. Let ρ ∈ X be arbitrary.

To prove that (X, T) is first countable, it suffices to show that ∃ a countable local base at p for the topology T on X.

Write $L_p = \{S_{(p,\,r)} : r \in Q^+\}$.

Q is enumerable and hence its subset $Q^+$,

$Q^+$ is enumerable ⇒ $L_p$ is enumerable.

Let G ∈ T be arbitrary s.t. ρ ∈ G.

Then, by definition of an open set.

$$\exists\, s \in R^+ \text{ s.t. } S_{(p,\,s)} \subset G.$$

Choose a positive rational number r s.t. r < s.

Then $\quad S_{(p,r)} \subset S_{(p,s)} \subset G$

or $\quad S_{(p,r)} \subset G$.

Given any G ∈ T with ρ ∈ G.

$$\exists\, r \in Q^+ \text{ s.t. } S_{(p,\,r)} \subset G.$$

Now $L_p$ has the following properties:

(i) every member of $L_p$ is an open set containing p.

∵ each open sphere forms an open set.

(ii) $L_p$ is enumerable set.

**Notes**

(iii) Given any $G \in T$ with $p \in G$, $\exists\, r \in Q^+$ s.t.

$S_{(p, r)} \subset G$.

From what has been done, it follows that $L_p$ is an enumerable local base at p of the topology T on X.

## 16.2 Summary

- Let (X, T) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local base at each $x \in X$.
- Let (X, T) be a topological space. The space X is said to satisfy the second axiom of countability if $\exists$ a countable base for T on X.
- Let (X, T) be a topological space. A property P of X is said to be hereditary if the property is possessed by every subspace of X.

## 16.3 Keywords

***Base:*** $\mathcal{B}$ is said to be a base for the topology T on X if $x \in G \in T \Rightarrow \exists\, B \in \mathcal{B}$ s.t. $x \in B \subset G$.

***Local Base:*** A family $\mathcal{B}_x$ of open subsets of X is said to be a local base at $x \in X$ for the topology T on X if

(i) any $B \in \mathcal{B}_x \Rightarrow x \in B$

(ii) any $G \in T$ with $y \in G \Rightarrow \exists\, B \in \mathcal{B}_x$ s.t. $y \in B \subset G$.

***Open Sphere:*** Let (X, ρ) be a metric space. Let $x_0 \in X$ and $r \in R^+$. Then set $\{x \in X : \rho(x_0, x) < r\}$ is defined as open sphere with centre $x_0$ and radius r.

***Separable:*** Let X be a topological space and A be a subset of X, then X is said to be separable if

(i) $\bar{A} = X$

(ii) A is countable.

## 16.4 Review Questions

1. Prove that the property of being a first axiom space is a topological property.
2. For each point x in a first axiom $T_1$ – space,

   $\{x\} = \bigcap_{n \in N} B_n(x)$
3. Prove that the property of being a second axiom space is a topological property.
4. In a second axiom $T_1$ – space, a set is compact iff it is countable compact.
5. Show that in a second axiom space, every collection of non empty disjoint open sets is countable.
6. Give an example of a separable space which is not second countable.
7. Show that every separable metric space is second countable. Is a separable topological space is second countable? Justify your answer.
8. Every sub-space of a second countable space is second countable and hence show that it is also separable.

## 16.5 Further Readings

Books

G.F. Simmons, *Introduction to Topology and Modern Analysis,* McGraw Hill International Book Company, New York, 1963.

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

# Unit 17: The Separation Axioms

**Notes**

**CONTENTS**



## Objectives

After studying this unit, you will be able to:

- Define $T_0$-axiom and solve related problems;
- Explain the $T_1$-axiom and related theorems;
- Describe the $T_2$-axiom and discuss problems and theorems related to it.

## Introduction

The topological spaces we have been studying thus far have been generalizations of the real number system. We have obtained some interesting results, yet because of the degree of generalization many intuitive properties of the real numbers have been lost. We will now consider topological spaces which satisfy additional axioms that are motivated by elementary properties of the real numbers.

## 17.1 $T_0$-Axiom or Kolmogorov Spaces

A topological space X is said to be a $T_0$-space if for any pair of distinct points of X, there exist at least one open set which contains one of them but not the other.

In other words, a topological space X is said to be a $T_0$-space if it satisfy following axiom for any $x, y \in X$, $x \neq y$, there exist an open set $\mathcal{U}$ such that $x \in U$ but $y \notin U$.

*Example 1:* Let X = {a, b, c} with topology T = {$\phi$, X, {a}, {b}, {a, b}} defined on X, then (X, T) is a $T_0$-space because

(i) for a and b, there exist an open set {a} such that $a \in \{a\}$ and $b \notin \{a\}$

(ii) for a and c, there exist an open set {b} and $b \in \{b\}$ and $c \notin \{b\}$

### Examples of $T_0$-space

(i) Every metric space is $T_0$-space.

(ii) If (X, T) is cofinite topological space, then it is $T_0$-space.

(iii) Every discrete space is $T_0$-space.

(iv) An indiscrete space containing only one point is a $T_0$-space.

## 17.1.1 $T_1$-Axiom of Separation or Frechet Space

A topological space (X, T) is said to satisfy the $T_1$-Axiom of separation if given a pair of distinct points $x, y \in x$

$$\exists G, H \in T \text{ s.t. } x \in G, y \notin G, y \in H, x \notin H$$

In this case the space (X, T) is called $T_1$-space or Frechet space.

*Example 2:* Let X = {a, b, c} with topology T = {$\phi$, X, {a}, {b}, {a, b}} defined on X is not a $T_1$-space because for $a, c \in X$, we have open sets {a} and X such that $a \in \{a\}$, $c \notin \{a\}$. This shows that we cannot find an open set which contains c but not a, so (X, T) is not a $T_1$-space. But we have already showed that (X, T) is a $T_0$-space. This shows that a $T_0$-space may not be a $T_1$-space. But the converse is always true.

***Theorem 1:*** A topological space (X, T) is a $T_1$-space if f(x) is closed for each $x \in X$. In a topological space, show that $T_1$-space $\Leftrightarrow$ each point is a closed set.

***Proof:*** (i) Let (X, T) be a topological space s.t. {x} is closed $\forall\ x \in X$.

To prove that X is $T_1$-space.

Consider $x, y \in X$ s.t. $x \neq y$.

Then, by hypothesis, {x} and {y} are disjoint closed sets. This means that X-{x} and X-{y} are T-open sets.

Write G = X-{y}, H = X – {x},

Then $G, H \in T$ s.t. $x \in G, y \in G, y \in H, x \notin H$.

This proves that (X, T) is a $T_1$-space.

(ii) Conversely, suppose that (X, T) is a $T_1$-space.

To prove that {x} is closed $\forall\ x \in X$.

Since X is a $T_1$-space.

$\therefore$ Given a pair of distinct points $x, y \in X$, $\exists\ G, H \in T$.

s.t. $x \in G, y \notin G$ and $y \in H, x \notin H$.

Evidently, $G \subset X - \{y\}$, $H \subset X - \{x\}$.

Given any $x \in X - \{y\} \Rightarrow \exists\ G \in T$ s.t. $x \in G \subset x - \{y\}$.

This proves that every point x of X – {y} is an interior point of X – {y}, meaning thereby X – {y} is open, i.e., {y} is closed. Furthermore, given any $y \in X - \{x\} \Rightarrow \exists\ H \in T$ s.t. $y \in H \subset X - \{x\}$.

This implies that every point y of X – {x} is an interior point of X – {x}. Hence X – {x} is open , i.e.,

{x} is closed.

**Notes**

Finally {x}, {y} are closed sets in X.

Generalising this result.

{x} is closed $\forall$ x $\in$ X.

*Example 3:* Prove that in a $T_1$-space all finite sets are closed.

*Solution:* Let (X, T) be a $T_1$-space.

To prove that {x} is closed $\forall$ x $\in$ X.

Now write (ii) part of the proof of the theorem 1

Let A be an arbitrary finite subset of X.

Then A = $\cup$ {{x}} : x $\in$ A}

= finite union of closed sets = closed set.

$\therefore$ A is a closed set.

*Example 4:* A topological space (X, T) is a $T_1$-space iff T contains the cofinite topology on X.

*Solution:* Let (X, T) be a $T_1$-space.

To prove that T contains cofinite topology on X, we have to show that T contains subsets A of X s.t. X – A is finite.

Here we shall make use of the fact that

X is $T_1$-space $\Rightarrow$ {x} is closed $\forall$ x $\in$ X

$\Rightarrow$ X – {x} is open subset of X $\Rightarrow$ X – {x} $\in$ T

Thus X – {x} $\in$ T $\Rightarrow$ X – (X – {x}) = {x} = finite set.

This is true $\forall$ x $\in$ X.

Hence by definition T contains cofinite topology on X.

Conversely, suppose that T contains cofinite topology on X.

To prove that (X, T) is $T_1$-space.

{x} is a finite subset of X.

Also T contains cofinite topology.

Consequently X – {x} $\in$ T so that

{x} is closed $\forall$ x $\in$ X

$\Rightarrow$ (X, T) is $T_1$-space.

***Theorem 2:*** A topological space X is a $T_1$-space of X iff every singleton subset {x} of X is closed.

***Proof:*** Let X be a $T_1$-space and x $\in$ X.

By the $T_1$-axiom, we know that if y $\neq$ x $\in$ X, than there exists an open set $G_y$ which contain y but not x i.e.

$y = G_y \subseteq \{x\}^c$

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Then $\{x\}^c = \cup \{y : y \neq x\} \subseteq \{G_y : y \neq x\} \subseteq \{x\}^c$.

Therefore $\{x\}^c = \cup \{G_y : y \neq x\}$.

Thus $\{x\}^c$ being the union of open sets is an open set. Hence {x} is a closed set.

Conversely, let us suppose that {x} is closed.

We have to prove that X is a $T_1$-space.

Let x and y be two distinct points of X.

Since {x} is a closed set, $\{x\}^c$ is an open set which contains y but not x.

Similarly $\{y\}^c$ is an open set which contains x but not y.

Hence X is a $T_1$-space.

***Theorem 3:*** The property of being a $T_1$-space is preserved by one-to-one onto, open mappings and hence is a topological property.

***Proof:*** Let (X, T) be a $T_1$-space and let (Y, V) be a space homomorphic to the topological space (X, T).

Let f be a one-one open mapping of (X, T) onto (Y, V).

We shall prove that (Y, V) is also a $T_1$-space.

Let $y_1$, $y_2$ be any two distinct points of y.

Since the mapping f is one-one onto, there exist, points $x_1$ and $x_2$ in X such that

$$x_1 \neq x_2 \text{ and } f(x_1) = y_1 \text{ and } f(x_2) = y_2$$

Since (X, T) is a $T_1$-space, there exists T-open sets G and H such that

$$x_1 \in G \text{ but } x_2 \in G$$

$$x_2 \in H \text{ but } x_1 \in H$$

Again, since f is an open mapping, f[G] and f[H] are V-open subsets such that

$$f(x) \in f[G] \text{ but } f(x_2) \notin f[G]$$

$$\text{and } f(x_2) \in f[H] \text{ but } f(x_1) \in f[x]$$

Hence (Y, V) is also a $T_1$-space.

Thus, the property of being a $T_1$-space is preserved under one-one onto, open mappings.

Hence it is a topological property.

***Theorem 4:*** Every subspace of $T_1$-space is a $T_1$-space i.e. the property being a $T_1$-space is hereditary.

***Proof:*** Let (X, T) be a $T_1$-space and let (X*, T*) be a subspace of (X, T).

Let $x_1$ and $x_2$ be two distinct point of X*. Since $X^* \subset X$, $x_1$ and $x_2$ are also distinct points of X. But (X, T) is a $T_1$-space, therefore there exist T-open sets G and H such that

$x_1 \in G$ but $x_2 \notin G$

and $x_2 \in H$ but $x_1 \notin H$

Then $G_1 = G \cap X^*$

and $H_1 = H \cap X^*$ are T*-open sets such that

$X_1 \in G_1$ but $x_2 \notin G_1$

**Notes**

and $x_2 \in H_2$ but $x_1 \notin H_1$

Hence (X*, T*) is a $T_1$-space.

## Self Assessment

1. Show that any finite $T_1$-space is a discrete space. Is a discrete space $T_1$ space? Justify your answer.
2. If (X, T) is a $T_0$-space and $T_1$ is finer than T, then (X, $T_1$) is also $T_0$-space.
3. A finite subset of a $T_1$-space has no cluster point.
4. If (X, T) is a $T_1$-space and $T^* \geq T$, then (X, T*) is also a $T_1$-space.

## 17.2 $T_2$-Axiom of Separation or Hausdorff Space

A topological space (X, T) is said to satisfy the $T_2$-axiom or separation if given a pair of distinct points $x, y \in X$.

$\exists\, G, H \in T$ s.t. $x \in G$, $y \in H$, $G \cap x = \phi$

In this case the space (X, T) is called a $T_2$-space or Hausdorff space or separated space.

*Example 5:* Let X = {1, 2, 3} be a non-empty set with topology T = P(X) (all the subsets of X, powers set or discrete topology). Hence

For 1, 2 $\quad 1 \in \{1\}, 2 \notin \{1\}$

For 2, 3 $\quad 2 \in \{2\}, 3 \notin \{2\}$

For 3, 1 $\quad 3 \in \{3\}, 1 \notin \{3\}$ and (X, T) is a $T_2$-space

For 1, 2 $\quad 1 \in \{1\}, 2 \in \{2\} \Rightarrow \{1\} \cap \{2\} = \phi$

For 2, 3 $\quad 2 \in \{2\}, 3 \in \{3\} \Rightarrow \{2\} \cap \{3\} = \phi$

For 3, 1 $\quad 3 \in \{3\}, 1 \in \{1\} \Rightarrow \{3\} \cap \{1\} = \phi$

*Example 6:* Show that every $T_2$-space is a $T_1$-space.

*Solution:* Let (X, T) be a $T_2$-space.

Let x, y be any two distinct points of X. Since the space is $T_2$, then there exist open nhd. G and H of x and y respectively such that $G \cap H = \phi$.

Thus G and H are open sets such that

$x \in G$ but $y \notin G$

and $y \in H$ but $x \notin H$

Hence the space is $T_1$.

*Example 7:* Prove that every $T_2$-space is a $T_1$-space but converse is not true. Justify.

*Solution:* Let (X, T) be a $T_2$-space.

Let x, y be any two distinct points of X.

Since the space is $T_2$, $\exists$ open nhds G and H of x and y respectively such that $G \cap H = \phi$

Notes

Thus, G and H are open sets such that

$x \in G$ but $y \notin G$

and $y \in H$ but $x \notin H$

Hence, the space (X, T) is a $T_1$-space.

Conversely, let us consider the cofinite topology T on an infinite set X.

Let x be an arbitrary point of X.

by definition of T,

X – {x} is open, for {x} is finite set and so {x} is T-closed.

Thus, every singleton subset of X is closed.

It follows that the space (X, T) is a $T_1$-space. Now we shall show that the space (X, T) is not a $T_2$-space.

For this topology, no two open subsets of X can be disjoint.

Let if possible G and H be two open disjoint subsets of X, then

$$G \cap H = \phi$$

$$\Rightarrow \quad (G \cap H)' = \phi'$$

$$\Rightarrow \quad G' \cup H' = X \qquad \text{(by De-Morgan's law)}$$

Here $G' \cup H'$ being the union of two finite sets is finite, where as X is infinite.

Hence for this topology no two open sets can be disjoint i.e. no two distinct points can be separated by open sets.

Hence, (X, T) is not $T_2$-space.

***Theorem 5:*** Every subspace of a $T_2$-space is a $T_2$-space

or

Prove that every subspace of a Hausdorff space is also Hausdorff.

***Proof:*** Let (X, T) be a Hausdorff space and $(Y, T_y)$ be a subspace of it.

Let x and y be any two distinct points of Y.

Then x and y are distinct points of X.

But (X, T) is a Hausdorff space, $\exists$ T-open nhds. G and H of x and y respectively such that

$$G \cap H = \phi$$

Consequently, $Y \cap G$ and $Y \cap H$ are $T_y$-open nhds of x and y respectively.

Also $x \in G, x \in Y \Rightarrow x \in Y \cap G$

and $y \in H, y \in y \Rightarrow y \in Y \cap H$

and since $G \cap H = \phi$, we have

$(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \phi = \phi$

This shows that $(Y, T_y)$ is also a $T_2$-space. Hence, every subspace of a Hausdorff space is also a Hausdorff space.

**Notes**

***Theorem 6:*** The property of being a Hausdorff space is a topological invariant.

or

The property of being a Hausdorff space is preserved by one-one onto open mapping and hence is a topological property.

***Proof:*** Let (X, T) be a $T_2$-space and let (Y, $T_y$) be any topological space.

Let f be a one-one open mapping of X onto Y. Let $y_1$, $y_2$ be two distinct elements of Y. Since f is one-one onto map, there exists distinct elements $x_1$ and $x_2$ of X such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Since (X, T) is a $T_2$-space, ∃ T-open nhds. G and H of $x_1$ and $x_2$ such that $G \cap H = \phi$

Now, f being open, it follows that f(G) and f(H) are open subsets of Y such that

$$y_1 = f(x_1) \in f(G)$$

$$y_2 = f(x_2) \in f(H)$$

and $f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$

This shows that (Y, $T_y$) is also a $T_2$-space.

Since a property being a $T_2$-space is preserved under one-one, onto, open maps, it is preserved under homeomorphism.

Hence, it is a topological property.

***Theorem 7:*** Prove that every compact subset of Hausdorff space is closed.

***Proof:*** Let (Y, T*) be a compact subset of Hausdorff space (X, T).

In order to prove that Y is T-closed, we have to show that X – Y is T-open.

Let x be an arbitrary element of X – Y.

Since (X, T) is a $T_2$-space, then for each $y \in Y$, ∃ T-open sets $G_y$ and $H_y$ such that

$$x \in G_y,\ y \in H_y \text{ and } G_y \cap H_y = \phi$$

Now consider the class

$\mathcal{C} = \{H_y \cap Y : y \in Y\}$

Clearly, $\mathcal{C}$ is T*-open cover of Y.

Since (Y, T*) is a compact subset of (X, T), there must exist a finite sub cover of $\mathcal{C}$ i.e. ∃ n points $y_1, y_2, ..., y_n$ in Y such that

$\{H_{yi} \cap Y : i \in T_n\}$ is a finite sub cover of $\mathcal{C}$.

Thus $Y \subset \bigcup_{i=1}^{n} \{H_{y_i}\}$

Let $N = \bigcap_{i=1}^{n} \{G_{yi}\}$, then N is T-nhd of x, and $N \cap \left[\bigcup_{i=1}^{n} \{H_{y_i}\}\right] = \phi$.

Thus, $N \cap Y = \phi \Rightarrow N \subset X - Y$

i.e. X – Y contains a T-nhd of each of its points.

Hence, X – Y is T-open i.e. Y is T-closed.

*Example 8:* Show that every convergent sequence in Hausdorff space has a unique limit.

*Solution:* Let (X, T) be a Hausdorff space.

Let $<x_n>$ be a sequence of points of Hausdorff space X.

Let $\underset{n\to\infty}{Lt}\ x_n = x$

Suppose, if possible,

$\underset{n\to\infty}{Lt}\ x_n = y$, where $x \neq y$.

Since X is a Hausdorff space, $\exists$ open sets G and H such that $x \in G$, $y \in H$

and $G \cap H = \phi$ ...(1)

Since $x_n \to x$ and $x_n \to y$

and G, H are nhds of x and y respectively, $\exists$ positive integers $n_1$ and $n_2$ such that

$x_n \in G\ \forall\ n \geq n_1$ and

$x_n \in H\ \forall\ n \geq n_2$

Let $n_0 = \max(n_1, n_2)$, then $x_n \in G \cap H\ \forall\, n \geq n_0$

This contradicts (1).

Hence, the limit of the sequence must be unique.

*Note* Converse of the above theorem is not true.

*Example 9:* Show that each singleton subset of a Hausdorff space is closed.

*Solution:* Let X be a Hausdorff space and let $x \in X$.

Let $y \in X$ be any arbitrary point of X other than x i.e. $x \neq y$.

Since X is a $T_2$-space, $\exists$ a nhd of y which does not contain x.

It follows that y is not a limit point of {x} and consequently $D(\{x\}) = \phi$

Hence $\overline{\{x\}} = x$.

This shows that {x} is T-closed.

*Example 10:* Show that every finite $T_2$-space is discrete.

*Solution:* Let (X, T) be a finite $T_2$-space. We know that every singleton subset of X is T-closed. Also a finite union of closed sets is closed. It follows that every finite subset of X is closed.

Hence, the space is discrete.

***Theorem 8:*** A first countable space in which every convergent sequence has a unique limit is a Hausdorff space.

Notes

***Proof:*** Let (X, T) be a first countable space in which every convergent sequence has a unique limit. If possible, let (X, T) be not a Hausdorff space.

Then given $x, y \in X$, $x \neq y$, $\exists$ open sets G and H

$$\text{such that } x \in G,\ y \in H,\ G \cap H \neq \phi$$

Now (X, T) being first countable, there exists monotone decreasing local bases

$\mathcal{B}_x = \{B_n(x) : x \in N\}$ and

$\mathcal{B}_y = \{B_n(y) : n \in N\}$ at x and y respectively.

Clearly, $B_n(x) \cap B_n(y) \neq \phi \quad \forall\ n \in N$

[$\because$ $B_n(x)$ and $B_n(y)$ are open nhds. of x and y respectively]

Let $\quad x_n \in B_n(x) \cap B_n(y) \quad \forall\ n \in N$

But $B_n(x)$ and $B_n(y)$ being monotone decreasing local bases at x and y respectively, $\exists$ a positive integer $n_0$ such that

$$n > n_0 \Rightarrow B_n(x) \subset G \quad \text{and}$$
$$B_n(y) \subseteq H$$

$$\Rightarrow \quad x_n \in B_n(x) \subseteq G \text{ and}$$
$$x_n \in B_n(y) \subseteq H$$

$\Rightarrow \quad x_n \in G$ and $x_n \in H$

$\therefore \quad x_n \to x$ and $x_n \to y$

But, this contradicts the fact that every convergent sequence in X has a unique limit.

Hence, (X, T) must be a Hausdorff space.

***Theorem 9:*** The product space of two Hausdorff spaces is Hausdorff.

***Proof:*** Let X and Y be two Hausdorff spaces. We shall prove that X × Y is also a Hausdorff spaces.

Let $(x_1, y_1)$ and $(x_2, y_2)$ be any two distinct points of X × Y.

Then either $x_1 \neq x_2$ or $y_1 \neq y_2$

Let us take $x_1 \neq x_2$

Since X is a Hausdorff space, $\exists$ T open nhds. G and H of $x_1$ and $x_2$ respectively such that $x_1 \in G$, $x_2 \in H$ and $G \cap H = \phi$

Then G × Y and H × Y are open subsets of X × Y such that

$$(x_1, y_1) \in G \times Y,$$
$$(x_2, y_2) \in H \times Y \text{ and}$$
$$(G \times Y) \cap (H \times Y) = (G \cap H) \times Y$$
$$= \phi \times Y = \phi$$

Thus, in this case, distinct points $(x_1, y_1)$ and $(x_2, y_2)$ of X × Y have disjoint open nhds.

Similarly, when $y_1 \neq y_2$ $\exists$ disjoint open nhds of $(x_1, y_1)$ and $(x_2, y_2)$

Hence X × Y is Hausdorff.

Notes

## Self Assessment

5. Show that one-to-one continuous mapping of a compact topological space onto a Hausdorff space is a homeomorphism.
6. The product of any non-empty class of Hausdorff spaces is a Hausdorff space. Prove it.
7. Show that if (X, T) is a Hausdorff space and T* is finer than T, then (X, T*) is a $T_2$-space.
8. Show that every finite Hausdorff space is discrete.

## 17.3 Summary


- $T_0$-axiom of separation:

  A topological space (X, T) is said to satisfy the $T_0$-axiom

  If for $x, y \in X$, either $\exists\, G \in T$ s.t. $x \in G, y \notin G$

  or $\exists\, H \in T$ s.t. $y \in H, x \notin H$
- $T_1$-axiom:

  A topological space (X, T) is said to satisfy the $T_1$-axiom if

  for $x, y \in X\ \exists\, G, H \in T$

  s.t. $x \in G, y \notin G; y \in H, x \notin H$
- $T_2$-axiom:

  A topological space (X, T) is said to satisfy the $T_2$-axiom if for $x, y \in X$

  $\exists\, G, H \in T$ s.t. $x \in G, y \in H, G \cap H = \phi$


## 17.4 Keywords

***Cofinite topology:*** Let X be a non-empty set, and let T be a collection of subsets of X whose complements are finite along with $\phi$, forms a topology on X and is called cofinite topology.

***Compact:*** A compact space is a topological space in which every open cover has a finite sub cover.

***Discrete:*** Let X be any non-empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X.

***Indiscrete space:*** Let X be any non-empty set and T = {X, $\phi$}. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

***Limit point:*** A point $x \in X$ is said to be the limit point of $A \subset X$ if each open set containing x contains at least one point of A different from x.

## 17.5 Review Questions

1. Show that A finite subset of a $T_1$-space has no limit point.
2. Prove that for any set X there exists a unique smallest T such that (X, T) is a $T_1$-space.
3. (X, T) is a $T_1$-space iff the intersection of the nhds of an arbitrary point of X is a singleton.
4. Show that a topological space X is a $T_1$-space iff each point of X is the intersection of all open sets containing it.

Notes

5. For any set X, there exists a unique smallest topology T such that (X, T) is a $T_1$-space.
6. A $T_1$-space is countably compact iff every infinite open covering has a proper subcover.
7. If (X, T) is a $T_1$-space and $T^* \geq T$, then $(X, T^*)$ is also a $T_1$-space.
8. If $(X, T_1)$ is a Hausdorff space, $(X, T_2)$ is compact and $T_1 \leq T_2$ than $T_1 = T_2$.
9. If f and g are continuous mappings of a topological space X into a Hausdorff space, then the set of points at which f and g are equal is a closed subset of X.
10. If f is a continuous mapping of a Hausdorff space X into itself, show that the set of fixed points; i.e. {x : f(x) = x}, is closed.
11. Show that every infinite Hausdorff space contains an infinite isolated set.
12. If (X, T) is a $T_2$-space and $T^* \geq T$, then prove that $(X, T^*)$ is also a $T_2$-space.

## 17.6 Further Readings

*Books*

Eric Schechter (1997), *Handbook of Analysis and its Foundations,* Academic Press.

Stephen Willard, *General Topology,* Addison Wesley, 1970 reprinted by Dover Publications, New York, 2004.

# Unit 18: Normal Spaces, Regular Spaces and Completely Regular Spaces

CONTENTS



## Objectives

After studying this unit, you will be able to:

- Define normal space;
- Solve the problems on normal space;
- Discuss the regular space;
- Describe the completely regular space;
- Solve the problems on regular and completely regular space.

## Introduction

Now we turn to a more through study of spaces satisfying the normality axiom. In one sense, the term "normal" is something of a misnomer, for normal spaces are not as well-behaved as one might wish. On the other hand, most of the spaces with which we are familiar do satisfy this axiom, as we shall see. Its importance comes from the fact that the results one can prove under the hypothesis of normality are central to much of topology. The Urysohn metrization theorem and the Tietze extension theorem are two such results; we shall deal with them later. We shall study about regular spaces and completely regular spaces.

## 18.1 Normal Space

A topological space (X, T) is said to be normal space if given a pair of disjoint closed sets $C_1$, $C_2 \subset X$.

$\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $C_1 \subset G_1$, $C_2 \subset G_2$.

*Example 1:* Metric spaces are normal.

*Solution:* Before proving this, we need a preliminary fact. Let X be a metric space with metric d. Given a subset $A \subset X$ define the distance d(x, A) from a point $x \in X$ to A to the greatest lower

**Notes**

bound of the set of distances d(x, a) from x to points $a \in A$. Note that $d(x, A) \geq 0$, and $d(x, A) = 0$ iff x is in the closure of A since $d(X, A) = 0$ is equivalent to saying that every ball $B_r(x)$ contains points of A.

*Example 2:* A compact Hausdorff space is normal.

*Solution:* Let A and B be disjoint closed sets in a compact Hausdorff space X. In particular, this implies that A and B are compact since they are closed subsets of a compact space. By the argument in the proof of the preceding example we know that for each $x \in A$, $\exists$ disjoint open sets $U_x$ and $V_x$ with $x \in U_x$ and $B \subset V_x$. Letting x very over A, we have an open cover of A by the sets $U_x$.

So, there is a finite subcover. Let U be the union of the sets $U_x$ in this finite subcover and let V be the intersection of the corresponding sets $V_x$. Then U and V are disjoint open nhds. of A and B.

*Example 3:* A closed sub-space of a normal space is a normal space.

*Solution:* Let (X, T) be a topological space which is normal and (Y, U) a closed sub-space of (X, T) so that Y is closed in X. To prove that Y is a normal space.

Let $F_1, F_2 \subset Y$ be disjoint sets which are closed in Y. Y is closed in X, a subset F of Y is closed in Y iff F is closed in X.

$\therefore$ $F_1$ and $F_2$ are disjoint closed sets in X.

By the property of normal space (X, T).

$$\exists\, G_1, G_2 \in T \text{ s.t. } F_1 \subset G_1, F_2 \subset G_2, G_1 \cap G_2 = \phi$$

$$F_1 \subset G_1 \Rightarrow F_1 \cap Y \subset G_1 \cap Y \Rightarrow F_1 = F_1 \cap Y \subset G_1 \subset Y$$

$$\Rightarrow F_1 \subset G_1 \cap Y.$$

Similarly $F_2 \subset G_2 \Rightarrow F_2 \subset G_2 \subset Y$.

By definition of relative topology,

$$G_1, G_2 \in T \Rightarrow Y \cap G_1, Y \cap G_2 \in U$$

Also $(G_1 \cap Y) \cap (G_2 \cap Y) = (Y \cap Y) \cap (G_1 \cap G_2) = Y \cap \phi = \phi$.

Finally given a pair of disjoint closed sets $F_1, F_2$ in Y, $\exists$ disjoint sets.

$$G_1 \cap Y, G_2 \cap Y \in U \text{ s.t. } F_1 \subset G_1 \cap Y, F_2 \subset G_2 \cap Y.$$

This proves that (Y, U) is a normal space.

### Self Assessment

1. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.
2. Give an example of a normal space with a subspace that is not normal.
3. Show that paracompact space (X, T) is normal.

## 18.2 Regular Space

A topological space (X, T) is said to be regular space if: given an element $x \in X$ and closed set $F \subset X$ s.t. $x \notin F$, $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $x \in G_1$, $F \subset G_2$.

*Notes* A regular $T_1$-space is called a $T_3$-space.

A normal $T_1$-space is called a $T_4$-space.

**Examples of Regular Space**

1. Every discrete space is regular.
2. Every indiscrete space is regular.

*Example 4:* Give an example to prove that a regular space is not necessarily a $T_1$-space.

*Solution:* Let X = {a, b, c} and let T = $\{T^{\phi}, X, \{c\}, \{a, b\}\}$ be a topology on X.

The closed subsets of X are $\phi$, X, {c}, {a, b). Clearly this space (X, T) satisfies the R-axiom and it is a regular space. But it is not a $T_1$-space, for the singleton subset {b} is not a closed set.

Thus, this space (X, T) is a regular but not a $T_1$-space.

*Example 5:* Give an example of $T_2$-space which is not a $T_3$-space.

*Solution:* Consider a topology T on the set $\mathcal{R}$ of all real numbers such that the T-nhd. of every non-zero real number is the same as its $\cup$-nhd but T-nhd. of 0 are of the form

$$G - \left\{\frac{1}{n} : n \in N\right\}$$

where G is a $\cup$-nhd. of 0.

Then T is finer than $\cup$.

Now, (R, $\cup$) is Hausdorff and $\cup \subset T$, so (R, T) is Hausdorff.

But $\left\{\frac{1}{n} : n \in N\right\}$ being T-closed, cannot be separated from 0 by disjoint open sets.

Hence, (R, T) is not a regular space.

Thus, (R, T) is $T_2$ but not $T_3$.

***Theorem 1:*** A topological space (X, T) is a regular space iff each nhd. of an element $x \in X$ contains the closure of another nhd. of x.

***Proof:*** Let (X, T) be a regular space.

Then for a given closed set F and $x \in X$ such that $x \notin F$ there exist disjoint open sets G, H such that

$x \in G$ and $F \subset H$.

Now $x \in G \Rightarrow G$ is a nhd. of x ($\because$ G is open)

Again, $G \cap H = \phi$

$\Rightarrow G \subset X - H$

$\Rightarrow \overline{G} \subset (\overline{X - H}) = X - H$ (Since H is open and so X–H is closed)

**Notes**

$\Rightarrow \bar{G} \subset X - H$

$\Rightarrow \bar{G} \subset X - F$

$\Rightarrow \bar{G} \subset X - F = M$ (say)

$\Rightarrow \bar{G} \subset M$.

Since F is a closed set, M is an open set and

$x \notin F \Rightarrow x \in X - F$.

$\Rightarrow x \in M$, thus M is a nhd. of x.

Hence, if M is a nhd. of x, there exists a nhd. G of x such that

$x \in G \subset \bar{G} \subset M$.

Conversely, Let $N_1$ and $N_2$ be the nhds. of $x \in X$.

If $\overline{N_2} \subset N_1$, then we have to show that (X, T) is a regular space.

Let F be a closed subset of X and let x be an element of X such that $x \notin F$.

Now F is closed and $x \notin F$.

$\Rightarrow x \in X - F$ and $X - F$ is open.

$\Rightarrow X - F$ is a nhd. of x.

Let $X - F = N_1$, then by hypothesis

$$x \in N_2 \subset \overline{N_2} \subset X - F \qquad (\because \overline{N_2} \subset N_1)$$

Let us write $N_2 = G_1$ and

$X - \overline{N_2} = G_2$

Then
$$\begin{aligned} G_1 \cap G_2 &= N_2 \cap (X - \overline{N_2}) \\ &= (N_2 \cap X) - (N_2 \cap \overline{N_2}) \\ &= N_2 - N_2 \\ &= \phi. \end{aligned}$$

$\therefore G_1 \cap G_2 = \phi$.

Also $x \in N_2 \Rightarrow x \in G_1$

and $\overline{N_2} \subset X - F \Rightarrow F \subset X - \overline{N_2}$

or $F \subset G_2$

Since $\overline{N_2}$ is a closed set, therefore $G_2$ is open.

Thus, we have proved that for a given closed subset F of X and $x \in X$ such that $x \notin F$ there exist disjoint open subsets $G_1$, $G_2$ such that

$x \in G$, and $F \subset G_2$

Hence X is a regular space.

***Theorem 2:*** Prove that a normal space is a regular space i.e. to say, X is a $T_4$-space $\Rightarrow$ X is a $T_3$-space.

***Proof:*** Let (X, T) be a $T_4$-space so that

(i) X is a $T_1$-space

(ii) X is a regular space

To prove that X is a $T_3$-space. For this we must show that

(iii) X is a $T_1$-space

(iv) X is a regular space

Evidently (i) $\Rightarrow$ (iii)

If we show that (ii) $\Rightarrow$ (iv), the result will follow. Let $F \subset X$ be a closed set and $x \in X$ s.t. $x \notin F$. X is a $T_1$-space $\Rightarrow$ {x} is closed in X.

By normality, given a pair of disjoint closed sets {x} and F in X, $\exists$ disjoint open sets G, H in X s.t. $\{x\} \subset G$, $F \subset X$, i.e. given a closed set $F \subset X$ and $x \in X$ s.t. $x \notin F$. $\exists$ disjoint open sets G, H in X s.t. $\{x\} \subset G$, $F \subset H$. This proves that (X, T) is a regular space.

*Example 6:* Show that the property of a space being regular is hereditary property.

*Solution:* Let (Y, $\bigcup$) be a subspace of a regular space (X, T). We claim that the property of regularity is hereditary property. If we show that (Y, $\bigcup$) is regular, the result will follow.

Let F be a $\bigcup$-closed set and $p \in Y$ s.t. $p \notin F$.

Let $\overline{F}^T$ = closure of F w.r.t. the topology T. and $\overline{F}^{\cup}$ = closure of F w.r.t. the topology $\bigcup$ we know that $\overline{F}^{\cup} = \overline{F}^T \cap Y$.

Since F is a $\bigcup$-closed set $\Rightarrow F = \overline{F}^{\cup} \Rightarrow F = \overline{F}^T \cap Y$.

$p \notin F \Rightarrow p \notin \overline{F}^T \cap Y \Rightarrow p \notin \overline{F}^T$ or $p \notin Y$

$\Rightarrow p \notin \overline{F}^T$ for $p \in Y$.

$\overline{F}^T$ is a T-closed set.

$\because$ closure of any set is always closed.

By the regularity of (X, T), given a closed set $\overline{F}^T$ and a point $p \in X$. s.t. $p \notin \overline{F}^T$; $\exists$ disjoint sets G, $H \in T$ with $p \in G$, $\overline{F}^T \subset H$.

Consequently, $F = \overline{F}^T \cap Y \subset H \cap Y$, $p \in G \cap Y$

$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap (Y \cap Y) = \phi \cap Y = \phi$

Thus, we have shown that given a $\bigcup$-closed set F and a point $p \in Y$ s.t. $p \notin F$, we are able to find out the disjoint open sets $G \cap Y$, $H \cap Y$ in Y s.t. $p \in G \cap Y$, $F \subset H \cap Y$.

This proves that (Y, $\bigcup$) is regular. Hence proved.

## Self Assessment

4. Show that the usual topological space (R, $\cup$) is regular.

5. Show that every $T_3$-space is a $T_2$-space.

**Notes**

6. Give an example to show that a normal space need not be a regular.
7. Prove that regularity is a topological property.

## 18.3 Completely Regular Space

A topological space (X, T) is called a completely regular space if given a closed set $F \subset X$ and a point $x \in X$ s.t. $x \notin F$, $\exists$ a continuous map $f : X \to [0, 1]$ with the property,

$$f(x) = 0,\ f(F) = \{1\}$$

*Example 7:* Every metric space is a completely regular space.

*Solution:* Let (X, d) be a metric space.

Let $a \in X$ and F be a closed set in X not containing a.

Define $F : X \to R$ by

$$f(x) = \frac{d(x,a)}{d(x,a) + d(x,F)}\ \forall\ x \in X,$$

where $$d(x,F) = \inf\{d(x, y) : y \in F\},$$

$$d(x, F) = 0 \Leftrightarrow x \in \overline{F} = F,$$

Consequently $d(x, a) + d(x, F) \neq 0$ as $a \notin F$.

Thus we see that $f \in \mathcal{C}(X, R)$, $0 \leq f(x) \leq 1$ for every $x \in X$, $f(a) = 0$ and $f(F) = \{1\}$.

***Theorem 3:*** Every subspace of a completely regular space is completely regular i.e. complete regularity is hereditary property.

***Proof:*** Let $(Y, T_Y)$ be a subspace of a completely regular space (X, T).

Let F be a $T_Y$-closed subset of Y and $y \in Y - F$. Since F is a $T_Y$-closed, there exists a T-closed subset F* of X such that

$$F = Y \cap F^*$$

Also $y \notin F \Rightarrow y \notin Y \cap F^*$

$\Rightarrow \quad y \notin F^* \qquad (\because y \in Y)$

and $y \in Y \Rightarrow y \in X$.

It follows that F* is a T-closed subset of X and $y \in X - F^*$.

Since X is completely regular, there exists a continuous real valved function $f : X \to [0, 1]$, such that

$$f(y) = 0 \text{ and } f(F^*) = \{1\}.$$

Let g denote the restriction of f to Y. Then g is a continuous mapping of Y into [0, 1].

Now by the definition of g.

$$g(x) = f(x)\ \forall\ x \in Y.$$

Hence $f(y) = 0 \Rightarrow g(y) = 0$

and $f(x) = 1\ \forall\ x \in F^*$

and $F \subset F^* \Rightarrow g(x) = f(x) = 1 \ \forall\ x \in F$

$\therefore\ g(F) = \{1\}$.

Hence for every $T_Y$-closed subset F of Y and for each point $y \in Y - F$, there exists a continuous mapping g of Y into [0, 1] such that

$$g(y) = 0 \quad \text{and} \quad g(F) = \{1\}.$$

Hence $(Y, T_Y)$ is also completely regular.

***Theorem 4:*** A completely regular space is regular.

***Proof:*** Let (X, T) be a completely regular space, then given any closed set $F \subset X$ and $p \in X$ s.t. $p \notin F$; $\exists$ continuous map $f : X \to [0, 1]$ with the property that

$$f(p) = 0, \quad f(F) = \{1\}.$$

To prove that (X, T) is a regular space.

Consider the set [0, 1] with usual topology. It is easy to verity that [0, 1] is a $T_2$-space, then we can find out disjoint open sets G, H in [0, 1] s.t. $0 \in G$, $1 \in H$.

By hypothesis, f is continuous, hence $f^{-1}(G)$, $f^{-1}(H)$ are open in X.

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(H \cap G) = f^{-1}(\phi), = \phi$$

$$f^{-1}(G) = \{x \in X : f(x) \in G\}.$$

Furthermore, $\quad f(p) = 0 \in G \Rightarrow f(p) \in G \Rightarrow p \in f^{-1}(G)$

$$f(F) = 1 \in H \Rightarrow f(F) = \{1\} \subset H$$

$$\Rightarrow f(F) \subset H$$

$$\Rightarrow F \subset f^{-1}(H).$$

Given any closed set $F \subset X$ and $p \in X$ s.t. $p \notin F$; $\exists$ disjoint open sets $f^{-1}(G)$, $f^{-1}(H)$ in X s.t. $p \in f^{-1}(G)$, $F \subset f^{-1}(H)$, in X s.t. $p \in f^{-1}(G)$, $F \subset f^{-1}(H)$, showing thereby X is regular.

***Theorem 5:*** A Tychonoff space is a $T_3$-space. Or Completely regular space $\Rightarrow$ regular space.

***Proof:*** Let (X, T) be a Tychonoff space, then

(i) X is a $T_1$-space

(ii) X is a completely regular space.

To prove that (X, T) is a $T_3$-space, it suffices to show that

(iii) X is a $T_1$-space.

(iv) X is a regular space

Evidently (i) $\Rightarrow$ (iii)

Prove as in Theorem (1)

Hence the result.

*Example 8:* Prove that a topological space (X, T) is completely regular iff for every $x \in X$ and every open set G containing x there exists a continuous mapping f of X into [0, 1] such that

$$f(x) = 0 \quad \text{and} \quad f(Y) = 1 \ \forall\ y \in X - G.$$

**Notes**

*Solution:* Let (X, T) be a topological space for which the given conditions hold. Let F be a T-closed subset of X and let x be a point of X such that $x \notin F$. Then X – F is a T-open set containing x. By the given condition there exits a continuous mapping $f : X \to [0, 1]$ such that

$$f(x) = 0 \quad \text{and} \quad f(Y) = 1\ \forall\ y \in X - (X - F) \text{ i.e. } y \in F.$$

Hence the space is completely regular.

Conversely, Let (X, T) be a completely regular space and let G be an open subset of X containing x.

Then X – G is a closed subset of X such that $x \notin X - G$. Since X is completely regular there exists a continuous mapping $f : X \to [0, 1]$ such that

$$f(x) = 0 \quad \text{and} \quad f(X - G) = \{1\}$$

### Self Assessment

8. Let F be a closed subset of a completely regular space (X, T) and $x_0 \in F'$, then prove that there exists a continuous map $f : X \to [0, 1]$ s.t. $f(x_0) = 1$, $f(F) = \{0\}$.
9. Prove that a normal space is completely regular iff it is regular.

## 18.4 Summary

- A topological space (X, T) is said to be normal space if: given a pair of disjoint closed sets $C_1, C_2 \subset X$. $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $C_1 \subset G_1$, $C_2 \subset G_2$.
- Matric spaces are normal.
- A closed subspace of a normal space is a normal space.
- A topological space (X, T) is said to be regular space if: given an element $x \in X$ and closed set $F \subset X$ s.t. $x \notin F$, $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $x \in G_1$, $F \subset G_2$.
- A regular $T_1$-space is called a $T_3$-space.
- A normal $T_1$-space is called a $T_4$-space.
- A normal space is a regular.
- A topological space (X, T) is called a completely regular space if : given a closed set $F \subset X$ and a point $x \in X$ s.t. $x \notin F$, $\exists$ a continuous map $f : X \subset [0, 1]$ with the property, $f(x) = 0$, $f(F) = \{1\}$.
- Every metric space is a completely regular space.
- Complete regularity is hereditary property.
- A completely regular space is regular.

## 18.5 Keywords

***Compact:*** A topological space (X, T) is called compact if every open cover of X has a finite sub cover.

***Hausdorff Space:*** It is a topological space in which each pair of distinct points can be separated by disjoint neighbourhoods.

***Metric Space:*** Any metric space is a topological space, the topology being the set of all open sets.

***Tychonoff Space:*** Tychonoff space is a Hausdorff space (X, T) in which any closed set A and any $x \notin A$ are functionally separated.

## 18.6 Review Questions

1. Prove that regularity is a hereditary property.
2. Prove that normality is a topological property.
3. Prove that complete regularity is a topological property.
4. Show that if X is completely regular, then every pair of disjoint subsets A and B such that A is compact and B is closed, there exists a real valued continuous mapping F of X such that f(A) = {0} and f(B) = {1}.
5. Show that a closed subspace of a normal space is normal.
6. Show that a completely regular space is regular and hence a Tychonoff space is a $T_3$-space.
7. Give an example of Hausdorff space which is not normal.
8. Show that a topological space X is normal iff for any closed set F and an open set G containing F there exists an open set H such that

$$F \subset H,\ \bar{H} \subset G \text{ i.e. } F \subset H \subset \bar{H} \subset G.$$

## 18.7 Further Readings

*Books*


A.V. Arkhangel'skii, V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, Reidel (1984).

J.L. Kelly, *General Topology*, Springer (1975).

Stephen Willard (1970), *General Topology*.