COMPLEX ANALYSIS

Edited By

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SYLLABUS

Complex Analysis

Objectives:

- To emphasize the role of the theory of functions of a complex variable, their geometric properties and indicating some applications. Introduction covers complex numbers; complex functions; sequences and continuity; and differentiation of complex functions. Representation formulas cover integration of complex functions; Cauchy's theorem and Cauchy's integral formula; Taylor series; and Laurent series. Calculus of residues covers residue calculus; winding number and the location of zeros of complex functions; analytic continuation.
- To understand classical concepts in the local theory of curves and surfaces including normal, principal, mean, and Gaussian curvature, parallel transports and geodesics, Gauss's theorem Egregium and Gauss-Bonnet theorem and Joachimsthal's theorem, Tissot's theorem.

Sr. No.	Content
1	Set Theory Finite, Countable and Uncountable Sets, Metric spaces; Definition and examples
2	Compactness of k-cells and Compact Subsets ofEuclidean, Space kR , Perfect sets and Cantor's set,Connected sets in a metric space, Connected subset of Real
3	Sequences in Metric Spaces, Convergent sequences and Subsequences, Cauchy sequence, complete metric space, Cantor's intersection theorem and Baire's Theorem, Branch contraction Principle.
4	Limit of functions, continuous functions, Continuity and compactness, continuity and connectedness, Discontinuities and Monotonic functions
5	Sequences and series: Uniform convergence, Uniform convergence and continuity, Uniform convergence and integration, Uniform convergence and differentiation

CONTENT

Notes

Unit 1: Complex Numbers

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the meaning of geometry
- Explain the polar coordinates

Introduction

Let us hark back to the first grade when the only numbers you knew were the ordinary everyday integers. You had no trouble solving problems in which you were, for instance, asked to find a number x such that $3x = 6$. You were quick to answer "2". Then, in the second grade, Miss Holt asked you to find a number x such that $3x = 8$. You were stumped—there was no such "number"! You perhaps explained to Miss Holt that 3 (2) = 6 and 3 (3) = 9, and since 8 is between 6 and 9, you would somehow need a number between 2 and 3, but there isn't any such number. Thus, you were introduced to "fractions."

These fractions, or rational numbers, were defined by Miss Holt to be ordered pairs of integers—thus, for instance, (8, 3) is a rational number. Two rational numbers (n, m) and (p, q) were defined to be equal whenever $nq = pm$. (More precisely, in other words, a rational number is an equivalence class of ordered pairs, etc.) Recall that the arithmetic of these pairs was then introduced: the sum of (n, m) and (p, q) was defined by

$$(n, m) + (p, q) = (nq + pm, mq),$$

and the product by

$$(n, m)\,(p, q) = (np, mq).$$

Subtraction and division were defined, as usual, simply as the inverses of the two operations.

In the second grade, you probably felt at first like you had thrown away the familiar integers and were starting over. But no. You noticed that $(n, 1) + (p, 1) = (n + p, 1)$ and also $(n, 1)(p, 1) = (np, 1)$. Thus, the set of all rational numbers whose second coordinate is one behave just like the integers.

Notes

If we simply abbreviate the rational number $(n, 1)$ by n, there is absolutely no danger of confusion: 2 + 3 = 5 stands for (2, 1) + (3, 1) = (5, 1). The equation 3x = 8 that started this all may then be interpreted as shorthand for the equation (3, 1) (u, v) = (8,1), and one easily verifies that $x = (u, v)$ = (8, 3) is a solution. Now, if someone runs at you in the night and hands you a note with 5 written on it, you do not know whether this is simply the integer 5 or whether it is shorthand for the rational number (5, 1). What we see is that it really doesn't matter. What we have "really" done is expanded the collection of integers to the collection of rational numbers. In other words, we can think of the set of all rational numbers as including the integers–they are simply the rationals with second coordinate 1.

One last observation about rational numbers. It is, as everyone must know, traditional to write the ordered pair (n, m) as $\frac{n}{m}$. Thus, n stands simply for the rational number $\frac{n}{1}$, etc.

Now why have we spent this time on something everyone learned in the second grade? Because this is almost a paradigm for what we do in constructing or defining the so-called complex numbers. Watch.

Euclid showed us there is no rational solution to the equation $x^2 = 2$. We were thus led to defining even more new numbers, the so-called real numbers, which, of course, include the rationals. This is hard, and you likely did not see it done in elementary school, but we shall assume you know all about it and move along to the equation $x^2 = -1$. Now we define complex numbers. These are simply ordered pairs (x, y) of real numbers, just as the rationals are ordered pairs of integers. Two complex numbers are equal only when there are actually the same–that is $(x, y) = (u, v)$ precisely when $x = u$ and $y = v$. We define the sum and product of two complex numbers:

$$(x, y) + (u, v) = (x + u, y + v)$$

and

$$(x, y)\,(u, v) = (xu - yv, xv + yu)$$

As always, subtraction and division are the inverses of these operations.

Now let's consider the arithmetic of the complex numbers with second coordinate 0:

$$(x, 0) + (u, 0) = (x + u, 0),$$

and

$$(x, 0)\,(u, 0) = (xu, 0).$$

Note that what happens is completely analogous to what happens with rationals with second coordinate 1. We simply use x as an abbreviation for (x, 0) and there is no danger of confusion: x + u is short-hand for (x, 0) + (u, 0) = (x + u, 0) and xu is short-hand for (x, 0) (u, 0). We see that our new complex numbers include a copy of the real numbers, just as the rational numbers include a copy of the integers.

Next, notice that x(u, v) = (u, v)x = (x, 0) (u, v) = (xu, xv). Now any complex number z = (x, y) may be written

$$\begin{aligned} z = (x, y) &= (x, 0) + (0, y) \\ &= x + y(0, 1) \end{aligned}$$

When we let $\alpha = (0,1)$, then we have

$$z = (x, y) = x + \alpha y$$

Notes

Now, suppose $z = (x, y) = x + \alpha y$ and $w = (u, v) = u + \alpha v$. Then we have

$$zw = (x + \alpha y)(u + \alpha v)$$
$$= xu + \alpha(xv + yu) \pm \alpha^2 yv$$

We need only see what α^2 is: $\alpha^2 = (0, 1)(0, 1) = (-1, 0)$, and we have agreed that we can safely abbreviate $(-1, 0)$ as -1. Thus, $\alpha^2 = -1$, and so

$$zw = (xu - yv) + \alpha(xv + yu)$$

and we have reduced the fairly complicated definition of complex arithmetic simply to ordinary real arithmetic together with the fact that $\alpha^2 = -1$.

Let's take a look at division–the inverse of multiplication. Thus, $\frac{z}{w}$ stands for that complex number you must multiply w by in order to get z . An example:

$$\frac{z}{w} = \frac{x+\alpha y}{u+\alpha v} = \frac{x+\alpha y}{u+\alpha v}.\frac{u-\alpha v}{u-\alpha v}$$
$$= \frac{(xu+yv)+\alpha(yu-xv)}{u^2+v^2}$$
$$= \frac{xu+yv}{u^2+v^2} + \alpha\frac{yu-xv}{u^2+v^2}$$

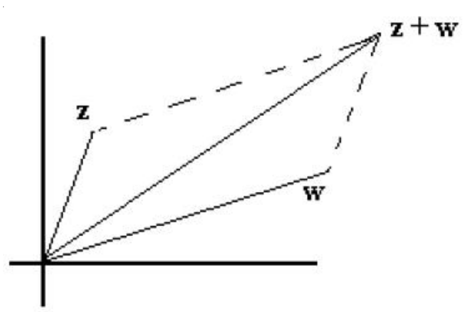
Notes This is just fine except when $u^2 + v^2 = 0$; that is, when $u = v = 0$. We may, thus, divide by any complex number except $0 = (0, 0)$.

One final note in all this. Almost everyone in the world except an electrical engineer uses the letter *i* to denote the complex number we have called α. We shall accordingly use *i* rather than α to stand for the number (0, 1).

1.1 Geometry

We now have this collection of all ordered pairs of real numbers, and so there is an uncontrollable urge to plot them on the usual coordinate axes. We see at once that there is a one-to-one correspondence between the complex numbers and the points in the plane. In the usual way, we can think of the sum of two complex numbers, the point in the plane corresponding to z + w is the diagonal of the parallelogram having z and w as sides:

Figure 1.1



Notes

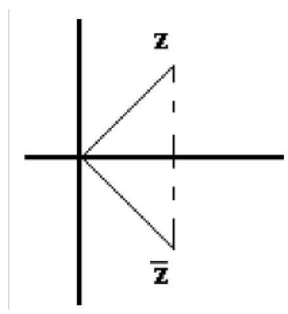
We shall postpone until the next section the geometric interpretation of the product of two complex numbers.

The **modulus** of a complex number z = x + iy is defined to be the non-negative real number $\sqrt{x^2 + y^2}$, which is, of course, the length of the vector interpretation of z. This modulus is traditionally denoted $|z|$, and is sometimes called the length of z.

Notes $|(x,0)| = \sqrt{x^2} = |x|$, and so $|\bullet|$ is an excellent choice of notation for the modulus.

The conjugate $\bar{z}$ of a complex number z = x + iy is defined by $\bar{z}$ = x – iy. Thus, $|z|^2 = z\bar{z}$. Geometrically, the conjugate of *z* is simply the reflection of *z* in the horizontal axis:

Figure 1.2



Observe that if z = x + iy and w = u + iv, then

$$\overline{(z+w)} = (x + u) - i(y + v)$$

$$= (x - iy) + (u - iv)$$

$$= \bar{z} + \bar{w}$$

In other words, the conjugate of the sum is the sum of the conjugates. It is also true that $\overline{zw} = \bar{z}\,\bar{w}$. If z = x + iy, then x is called the real part of z, and y is called the imaginary part of z. These are usually denoted Re z and Im z, respectively. Observe then that $z + \bar{z} = 2$ Rez and $z - \bar{z} = 2$ Imz.

Now, for any two complex numbers z and w consider

$$|z + w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + (w\bar{z} + \bar{w}z) + w\bar{w}$$

$$= |z|^2 + 2\,\mathrm{Re}(w\bar{z}) + |w|^2$$

$$\leq |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2$$

In other words,

$$|z + w| \leq |z| + |w|$$

Notes

the so-called **triangle inequality**. (This inequality is an obvious geometric fact–can you guess why it is called the triangle inequality?)

1.2 Polar coordinates

Now let's look at polar coordinates (r, θ) of complex numbers. Then we may write z = r(cos θ + i sin θ). In complex analysis, we do not allow r to be negative; thus, r is simply the modulus of z. The number θ is called an argument of z, and there are, of course, many different possibilities for θ. Thus, a complex numbers has an infinite number of arguments, any two of which differ by an integral multiple of 2π. We usually write θ = arg z. The principal argument of z is the unique argument that lies on the interval (–π, π].

Example: For 1 – i, we have

$$1 - i = \sqrt{2}(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right))$$

$$= \sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right))$$

$$= \sqrt{2}(\cos\left(\frac{399\pi}{4}\right) + i\sin\left(\frac{399\pi}{4}\right))$$

Each of the numbers $\frac{7\pi}{4}, \frac{\pi}{4},$ and $\frac{399\pi}{4}$ is an argument of 1 – i, but the principal argument is $-\frac{\pi}{4}$.

Suppose z = r(cos θ + i sin θ) and w = s(cos ξ + i sin ξ). Then

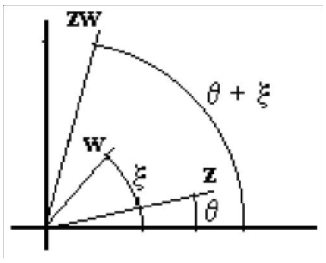
$$zw = r(\cos\theta + i\sin\theta)\, s(\cos\xi + i\sin\xi)$$

$$= rs[(\cos\theta\cos x - \sin\theta\sin x) + i(\sin\theta\cos\xi + \sin\xi\cos\theta)]$$

$$= rs(\cos(\theta + \xi) + i\sin(\theta + \xi))$$

We have the nice result that the product of two complex numbers is the complex number whose modulus is the product of the moduli of the two factors and an argument is the sum of arguments of the factors. A picture:

Figure 1.3



We now define exp(iθ), or $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Notes

We shall see later as the drama of the term unfolds that this very suggestive notation is an excellent choice. Now, we have in polar form

$$z = re^{i\theta},$$

where $r = |z|$ and θ is any argument of z. Observe we have just shown that

$$e^{i\theta}\, e^{i\xi} = e^{i(\theta+\xi)}.$$

It follows from this that $e^{i\theta}\, e^{-i\theta} = 1$. Thus,

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

It is easy to see that

$$\frac{z}{w} = \frac{re^{i\theta}}{se^{i\xi}} = \frac{r}{s}(\cos(\theta-\xi) + i\sin(\theta-\xi))$$

1.3 Summary

- The **modulus** of a complex number $z = x + iy$ is defined to be the nonnegative real number $\sqrt{x^2+y^2}$, which is, of course, the length of the vector interpretation of z.
- The conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by $\bar{z} = x - iy$.
- In other words, the conjugate of the sum is the sum of the conjugates. It is also true that $\overline{zw} = \bar{z}\,\bar{w}$. If $z = x + iy$, then x is called the real part of z, and y is called the imaginary part of z. These are usually denoted Re z and Im z, respectively. Observe then that $z + \bar{z} = 2\,\mathrm{Re}z$ and $z - \bar{z} = 2\,\mathrm{Im}z$.

  Now, for any two complex numbers z and w consider

  $$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + (w\bar{z} + \bar{w}z) + w\bar{w} \\ &= |z|^2 + 2\mathrm{Re}(w\bar{z}) + |w|^2 \\ &\le |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2 \end{aligned}$$

  In other words,

  $$|z+w| \le |z| + |w|$$

  the so-called **triangle inequality**. (This inequality is an obvious geometric fact–can you guess why it is called the triangle inequality?)
- We shall see later as the drama of the term unfolds that this very suggestive notation is an excellent choice. Now, we have in polar form

  $$z = re^{i\theta},$$

  where $r = |z|$ and θ is any argument of z. Observe we have just shown that

  $$e^{i\theta}\, e^{i\xi} = e^{i(\theta+\xi)}.$$

  It follows from this that $e^{i\theta}\, e^{-i\theta} = 1$. Thus

  $$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

Notes

It is easy to see that

$$\frac{z}{w} = \frac{re^{i\theta}}{se^{i\xi}} = \frac{r}{s}(\cos(\theta - \xi) + i\sin(\theta - \xi))$$

1.4 Keywords

Modulus: The **modulus** of a complex number z = x + iy is defined to be the non-negative real number $\sqrt{x^2 + y^2}$, which is, of course, the length of the vector interpretation of z.

Argument: Polar coordinates (r, θ) of complex numbers. Then we may write z = r(cos θ + i sin θ). In complex analysis, we do not allow r to be negative; thus, r is simply the modulus of z. The number θ is called an argument of z, and there are, of course, many different possibilities for θ.

1.5 Self Assessment

1. The .... of a complex number z = x + iy is defined to be the nonnegative real number $\sqrt{x^2 + y^2}$, which is, of course, the length of the vector interpretation of z.
2. The conjugate $\bar{z}$ of a complex number z = x + iy is defined by ....
3. It is also true that $\overline{zw} = \bar{z}\,\bar{w}$. If z = x + iy, then x is called the real part of z, and y is called the .... of z.
4. a .... has an infinite number of arguments, any two of which differ by an integral multiple of 2π.

1.6 Review Questions

1. Find the following complex numbers in the form x + iy:

   (a) (4 – 7i) (–2 + 3i) (b) $(1 - i)^3$

   (c) $\frac{(5+2i)}{(1+i)}$ (d) $\frac{1}{i}$

2. Find all complex z = (x, y) such that

$$z^2 + z + 1 = 0$$

3. Prove that if wz = 0, then w = 0 or z = 0.
4. (a) Prove that for any two complex numbers, $\overline{zw} = \bar{z}\,\bar{w}$.

   (b) Prove that $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.

   (c) Prove that $||z| - |w|| \le |z - w|$.

5. Prove that $|zw| = |z||w|$ and that $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

Notes

6. Sketch the set of points satisfying

   (a) $|z - 2 + 3i| = 2$ (b) $|z + 2i| \le 1$

   (c) $\text{Re}(\bar{z} + i) = 4$ (d) $|z - 1 + 2i| = |z + 3 + i|$

   (e) $|z + 1| + |z - 1| = 4$ (f) $|z + 1| - |z - 1| = 4$

7. Write in polar form $re^{i\theta}$:

   (a) i (b) $1 + i$

   (c) -2 (d) $-3i$

   (e) $\sqrt{3} + 3i$

8. Write in rectangular form—no decimal approximations, no trig functions:

   (a) $2e^{i3\pi}$ (b) $e^{i100\pi}$

   (c) $10e^{i\pi/6}$ (d) $\sqrt{2}\, e^{i5\pi/4}$

9. (a) Find a polar form of $(1 + i)\ (1 + i\sqrt{3})$.

   (b) Use the result of a) to find $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$.

10. Find the rectangular form of $(1 + i)^{100}$.

11. Find all z such that $z^3 = 1$. (Again, rectangular form, no trig functions.)

12. Find all z such that $z^4 = 16i$. (Rectangular form, etc.)

Answers: Self Assessment

1. modulus
2. $\bar{z} = x - iy$
3. imaginary part
4. complex numbers

1.7 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 2: Complex Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Explain the function of a complex variable
- Describe the functions of a complex variable
- Define derivatives

Introduction

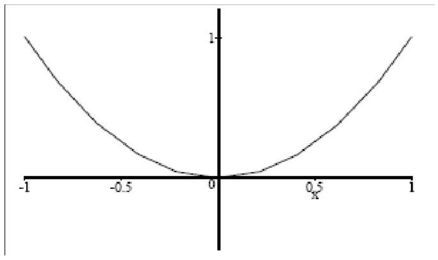
There are equations such as $x^2 + 3 = 0$, $x^2 - 10x + 40 = 0$ which do not have a root in the real number system R . There does not exist any real number whose square is a negative real number. If the roots of such equations are to be determined then we need to introduce another number system called complex number system. Complex numbers can be defined as ordered pairs (x, y) of real numbers and represented as points in the complex plane, with rectangular coordinates x and y. In this unit, we shall review the function of the complex variable.

2.1 Functions of a Real Variable

A function $\gamma : I \to C$ from a set I of reals into the complex numbers C is actually a familiar concept from elementary calculus. It is simply a function from a subset of the reals into the plane, what we sometimes call a vector-valued function. Assuming the function γ is nice, it provides a vector, or parametric, description of a curve. Thus, the set of all $\{\gamma(t) : \gamma(t) = e^{it} = \cos t + i \sin t = (\cos t, \sin t), 0 \le t \le 2\pi\}$ is the circle of radius one, centered at the origin.

We also already know about the derivatives of such functions. If $\gamma(t) = x(t) + iy(t)$, then the derivative of γ is simply $\gamma'(t) = x'(t) + iy'(t)$, interpreted as a vector in the plane, it is tangent to the curve described by γ at the point $\gamma(t)$.

Example 1: Let $\gamma(t) = t + it^2$, $-1 \le t \le 1$. One easily sees that this function describes that part of the curve $y = x^2$ between $x = -1$ and $x = 1$:



Another example. Suppose there is a body of mass M "fixed" at the origin–perhaps the sun–and there is a body of mass m which is free to move–perhaps a planet. Let the location of this second body at time t be given by the complex-valued function z(t). We assume the only force on this mass is the gravitational force of the fixed body. This force f is thus,

$$f = \frac{GMm}{|z(t)|^2}\left(-\frac{z(t)}{|z(t)|}\right)$$

where G is the universal gravitational constant. Sir Isaac Newton tells us that

$$mz''(t) = f = \frac{GMm}{|z(t)|^2}\left(-\frac{z(t)}{|z(t)|}\right)$$

Hence,

$$z'' = -\frac{GM}{|z|^3}z$$

Next, let's write this in polar form, $z = re^{i\theta}$:

$$\frac{d^2}{dt^2}(re^{i\theta}) = -\frac{k}{r^2}e^{i\theta}$$

where we have written GM = k. Now, let's see what we have.

$$\frac{d}{dt}(re^{i\theta}) = r\frac{d}{dt}(e^{i\theta}) + \frac{d}{dt}e^{i\theta}$$

Now,

$$\begin{aligned}\frac{d}{dt}(re^{i\theta}) &= \frac{d}{dt}(\cos\theta + i\sin\theta)\\ &= (-\sin\theta + i\cos\theta)\frac{d\theta}{dt}\\ &= i(\cos\theta + i\sin\theta)\frac{d\theta}{dt}\\ &= i\frac{d\theta}{dt}e^{i\theta}.\end{aligned}$$

(Additional evidence that our notation $e^{i\theta} = \cos\theta + i\sin\theta$ is reasonable.)

Notes

Thus,

$$\frac{d}{dt}(re^{i\theta}) = r\frac{d}{dt}(e^{i\theta}) + \frac{d}{dt}e^{i\theta}$$

$$= r\left(i\frac{d\theta}{dt}e^{i\theta}\right) + \frac{dr}{dt}e^{i\theta}$$

$$= \left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)e^{i\theta}.$$

Now,

$$\frac{d^2}{dt^2}(re^{i\theta}) = \left(\frac{d^2r}{dt^2} + i\frac{dr}{dt}\frac{d\theta}{dt} + ir\frac{d^2\theta}{dt^2}\right)e^{i\theta} + \left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)i\frac{d\theta}{dt}e^{i\theta}$$

$$= \left[\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\right]e^{i\theta}$$

Now, the equation $\frac{d^2}{dt^2}(re^{i\theta}) = -\frac{k}{r^2}e^{i\theta}$ becomes

$$\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = -\frac{k}{r^2}.$$

This gives us the two equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2},$$

and,

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

Multiply by r and this second equation becomes

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

This tells us that

$$\alpha = r^2\frac{d\theta}{dt}$$

is a constant. (This constant α is called the angular momentum.) This result allows us to get rid of $\frac{d\theta}{dt}$ in the first of the two differential equations above:

$$\frac{d^2r}{dt^2} - r\left(\frac{\alpha}{r}\right)^2 = -\frac{k}{r^2}$$

or

Notes

$$\frac{d^2r}{dt^2} - \frac{\alpha^2}{r^3} = -\frac{k}{r^2}.$$

Although this now involves only the one unknown function r, as it stands it is tough to solve. Let's change variables and think of r as a function of θ. Let's also write things in terms of the function $s = \frac{1}{r}$. Then,

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\alpha}{r^2}\frac{d}{d\theta}.$$

Hence,

$$\frac{dr}{dt} = \frac{a}{r^2}\frac{dr}{d\theta} = -\alpha\frac{ds}{d\theta},$$

and our differential equation looks like

$$\frac{d^2r}{dt^2} - \frac{\alpha^2}{r^3} = -\alpha^2 s^2 \frac{d^2 s}{d\theta^2} - \alpha^2 s^3 = -ks^2,$$

or,

$$\frac{d^2s}{d\theta^2} + s = \frac{k}{\alpha^2}.$$

This one is easy. From high school differential equations class, we remember that

$$s = \frac{1}{r} = A\cos(\theta + \varphi) + \frac{k}{\alpha^2},$$

where A and φ are constants which depend on the initial conditions. At long last,

$$r = \frac{\alpha^2/k}{1 + \varepsilon\cos(\theta + \varphi)},$$

where we have set $\varepsilon = A\alpha^2/k$. The graph of this equation is, of course, a conic section of eccentricity ε.

2.2 Functions of a Complex Variable

The real excitement begins when we consider function $f : D \to C$ in which the domain D is a subset of the complex numbers. In some sense, these too are familiar to us from elementary calculus—they are simply functions from a subset of the plane into the plane:

$$f(z) = f(x, y) = u(x, y) + iv(x, y) = (u(x, y), v(x, y))$$

Thus, $f(z) = z^2$ looks like $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$. In other words, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The complex perspective, as we shall see, generally provides richer and more profitable insights into these functions.

The definition of the limit of a function *f* at a point $z = z_0$ is essentially the same as that which we learned in elementary calculus:

$$\lim_{z \to z_0} f(z) = L$$

means that given an $\varepsilon > 0$, there is a δ so that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. As you could guess, we say that f is continuous at z_0 if it is true that $\lim_{z \to z_0} f(z) = f(z_0)$. If f is continuous at each point of its domain, we say simply that f is continuous.

Suppose both $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z)$ exist. Then the following properties are easy to establish:

$$\lim_{z \to z_0} [f(z) \pm g(z)] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z)$$

$$\lim_{z \to z_0} [f(z)g(z)] = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z)$$

and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$$

provided, of course, that $\lim_{z \to z_0} g(z) \neq 0$.

It now follows at once from these properties that the sum, difference, product, and quotient of two functions continuous at z_0 are also continuous at z_0. (We must, as usual, except the dreaded 0 in the denominator.)

It should not be too difficult to convince yourself that if $z = (x, y)$, $z_0 = (x_0, y_0)$, and $f(z) = u(x, y) + iv(x, y)$, then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0, y_0)} u(x, y) + i \lim_{(x,y) \to (x_0 y_0)} v(x, y)$$

Thus, f is continuous at $z_0 = (x_0, y_0)$ precisely when u and v are.

Our next step is the definition of the derivative of a complex function f. It is the obvious thing. Suppose *f* is a function and z_0 is an interior point of the domain of *f*. The derivative $f'(z_0)$ of *f* is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example 2:

Suppose $f(z) = z^2$. Then, letting $\Delta z = z - z_0$, we have

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{2z_0 \Delta z + (\Delta z)^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} (2z_0 + \Delta z) \\
&= 2z_0
\end{aligned}$$

No surprise here–the function $f(z) = z^2$ has a derivative at every z, and it's simply 2z.

Notes

Another Example

Let f(z) = $z\bar{z}$. Then,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\overline{(z_0 + \Delta z)} - z_0 \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z}) + \bar{z}_0 \Delta z + \Delta z \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right)$$

Suppose this limit exists, and choose Δz = (Δx, 0). Then,

$$\lim_{\Delta z \to 0} \left(\bar{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right) = \lim_{\Delta x \to 0} \left(\bar{z}_0 + \Delta x + z_0 \frac{\Delta x}{\Delta z} \right)$$

$$= \bar{z}_0 + z_0$$

Now, choose Δz = (0, Δy). Then,

$$\lim_{\Delta z \to 0} \left(\bar{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right) = \lim_{\Delta x \to 0} \left(\bar{z}_0 + i\Delta x - z_0 \frac{i\Delta y}{i\Delta y} \right)$$

$$= \bar{z}_0 - z_0$$

Thus, we must have $\bar{z}_0 + z_0 = \bar{z}_0 - z_0$, or z_0 = 0. In other words, there is no chance of this limit's existing, except possibly at z_0 = 0. So, this function does not have a derivative at most places.

Now, take another look at the first of these two examples. It looks exactly like what you did in Mrs. Turner's 3rd grade calculus class for plain old real-valued functions. Meditate on this and you will be convinced that all the "usual" results for real-valued functions also hold for these new complex functions: the derivative of a constant is zero, the derivative of the sum of two functions is the sum of the derivatives, the "product" and "quotient" rules for derivatives are valid, the chain rule for the composition of functions holds, etc., For proofs, you need only go back to your elementary calculus book and change x's to z's.

A bit of jargon is in order. If f has a derivative at z_0, we say that f is differentiable at z_0. If f is differentiable at every point of a neighborhood of z_0, we say that f is analytic at z_0. (A set S is a neighborhood of z_0 if there is a disk D = {z : $|z - z_0| < r, r > 0$} so that $D \subset S$. If f is analytic at every point of some set S, we say that f is analytic on S. A function that is analytic on the set of all complex numbers is said to be an entire function.

2.3 Derivatives

Suppose the function f given by f(z) = u(x, y) + iv(x, y) has a derivative at $z = z_0 = (x_0, y_0)$. We know this means there is a number $f'(z_0)$ so that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Notes

Choose Δz = (Δx, 0) = Δx. Then,

$$f'z(_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta z \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0 + y_0)}{\Delta x} + i \frac{v(x_0, \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right]$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

Next, choose Δz = (0, Δy) = iΔy. Then,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} - i \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \right]$$

$$= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{dy}(x_0, y_0)$$

We have two different expressions for the derivative $f'(z_0)$, and so

$$\frac{\partial v}{\partial x}(x_0, y_0) + i \frac{\partial u}{dy}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0)$$

or,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0),$$

$$\frac{\partial u}{\partial x}(x_0, y_0) = -i \frac{\partial v}{\partial x}(x_0, y_0)$$

These equations are called the **Cauchy-Riemann Equations.**

We have shown that if f has a derivative at a point z_0, then its real and imaginary parts satisfy these equations. Even more exciting is the fact that if the real and imaginary parts of f satisfy these equations and if in addition, they have continuous first partial derivatives, then the function f has a derivative. Specifically, suppose u(x, y) and v(x, y) have partial derivatives in a neighborhood of $z_0 = (x_0, y_0)$, suppose these derivatives are continuous at z_0, and suppose

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0),$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)$$

Notes

We shall see that f is differentiable at z_0.

$$= \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \frac{[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y}$$

Observe that

$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)] + [u(x_0, y_0 + \Delta y) - u(x_0,y_0].$

Thus,

$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \frac{\partial u}{\partial x}(\xi, y_0+\Delta y),$

and,

$$\frac{\partial u}{\partial x}(\xi, y_0+\Delta y) = \frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1,$$

where

$$\lim_{\Delta z\to 0}\varepsilon_1 = 0.$$

Thus,

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x\left[\frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1\right].$$

Proceeding similarly, we get

$$= \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \frac{[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y}$$

$$= \frac{\Delta x\left[\frac{du}{dx}(x_0,y_0)+\varepsilon_1+i\frac{dv}{dx}(x_0,y_0)+i\varepsilon_2\right]+\Delta y\left[\frac{du}{dy}(x_0,y_0)+\varepsilon_3+i\frac{dv}{dy}(x_0,y_0)+i\varepsilon_4\right]}{\Delta x+i\Delta y},.$$

where $\varepsilon_i \to 0$ as $\Delta z \to 0$. Now, unleash the Cauchy-Riemann equations on this quotient and obtain,

$$= \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \frac{\Delta x\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+i\Delta y\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]}{\Delta x+i\Delta y}+\frac{\text{stuff}}{\Delta x+i\Delta y}$$

Notes

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \frac{\text{stuff}}{\Delta x + i\Delta y}.$$

Here,

$$\text{stuff} = \Delta x(\varepsilon_1 + i\varepsilon_2) + \Delta y(\varepsilon_3 + i\varepsilon_4).$$

It's easy to show that

$$\lim_{\Delta z \to 0} \frac{\text{stuff}}{\Delta z} = 0,$$

and

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

In particular we have, as promised, shown that f is differentiable at z_0.

Example 3:

Let's find all points at which the function f given by $f(z) = x^3 - i(1 - y)^3$ is differentiable. Here we have $u = x^3$ and $v = -(1 - y)^3$. The Cauchy-Riemann equations, thus, look like

$$3x^2 = 3(1 - y)^2, \text{ and}$$

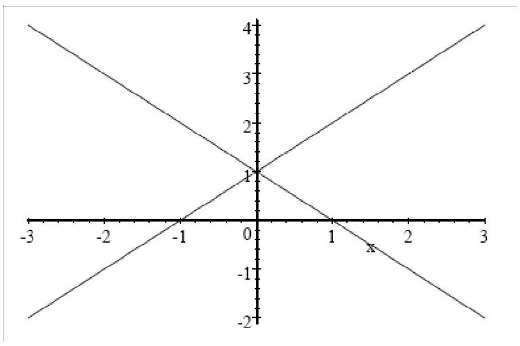
$$0 = 0.$$

The partial derivatives of u and v are nice and continuous everywhere, so f will be differentiable everywhere the C-R equations are satisfied. That is, everywhere

$$x^2 = (1 - y)^2; \text{ that is, where}$$

$$x = 1 - y, \text{ or } x = -1 + y.$$

This is simply the set of all points on the cross formed by the two straight lines



Notes

2.4 Summary

- A function $\gamma : I \to C$ from a set I of reals into the complex numbers C is actually a familiar concept from elementary calculus. It is simply a function from a subset of the reals into the plane, what we sometimes call a vector-valued function. Assuming the function γ is nice, it provides a vector, or parametric, description of a curve. Thus, the set of all $\{\gamma(t) : \gamma(t) = e^{it} = \cos t + i \sin t = (\cos t, \sin t), 0 \leq t \leq 2\pi\}$ is the circle of radius one, centered at the origin.

  We also already know about the derivatives of such functions. If $\gamma(t) = x(t) + iy(t)$, then the derivative of γ is simply $\gamma'(t) = x'(t) + iy'(t)$, interpreted as a vector in the plane, it is tangent to the curve described by γ at the point $\gamma(t)$.

- The real excitement begins when we consider function $f : D \to C$ in which the domain D is a subset of the complex numbers. In some sense, these too are familiar to us from elementary calculus—they are simply functions from a subset of the plane into the plane:

  $$f(z) = f(x, y) = u(x, y) + iv(x, y) = (u(x, y), v(x, y))$$

  Thus $f(z) = z^2$ looks like $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$. In other words, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The complex perspective, as we shall see, generally provides richer and more profitable insights into these functions.

  The definition of the limit of a function f at a point $z = z_0$ is essentially the same as that which we learned in elementary calculus:

  $$\lim_{z \to z_0} f(z) = L$$

2.5 Keywords

Elementary calculus: A function $\gamma : I \to C$ from a set I of reals into the complex numbers C is actually a familiar concept from elementary calculus.

Limit of a function: The definition of the limit of a function f at a point $z = z_0$ is essentially the same as that which we learned in elementary calculus.

Derivatives: Suppose the function f given by $f(z) = u(x, y) + iv(x, y)$ has a derivative at $z = z_0 = (x_0, y_0)$. We know this means there is a number $f'(z_0)$ so that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

2.6 Self Assessment

1. A function $\gamma : I \to C$ from a set I of reals into the complex numbers C is actually a familiar concept from ....
2. The real excitement begins when we consider function .... in which the domain D is a subset of the complex numbers.
3. The definition of the .... f at a point $z = z_0$ is essentially the same as that which we learned in elementary calculus.
4. If f has a derivative at z_0, we say that f is .... at z_0.
5. Suppose the function f given by $f(z) = u(x, y) + iv(x, y)$ has a derivative at $z = z_0 = (x_0, y_0)$. We know this means there is a number $f'(z_0)$ so that ....

Answers: Self Assessment

Notes

1. Elementary calculus
2. $f : D \to C$
3. Limit of a function
4. Differentiable
5. $f'(z_0) = \lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

2.7 Review Questions

1. (a) What curve is described by the function $\gamma(t) = (3t + 4) + i(t - 6),\ 0 \le t \le 1$?

   (b) Suppose z and w are complex numbers. What is the curve described by

   $$\gamma(t) = (1 - t)w + tz,\ 0 \le t \le 1?$$

2. Find a function γ that describes that part of the curve $y = 4x^3 + 1$ between $x = 0$ and $x = 10$.
3. Find a function γ that describes the circle of radius 2 centered at $z = 3 - 2i$.
4. Note that in the discussion of the motion of a body in a central gravitational force field, it was assumed that the angular momentum α is nonzero. Explain what happens in case $\alpha = 0$.
5. Suppose $f(z) = 3xy + i(x - y^2)$. Find $\lim_{z \to 3+2i} f(z)$, or explain carefully why it does not exist.
6. Prove that if f has a derivative at z, then f is continuous at z.
7. Find all points at which the valued function f defined by $f(z) = \bar{z}$ has a derivative.
8. Find all points at which the valued function f defined by

   $$f(z) = (2 \pm i)z^3 - iz^2 + 4z - (1 + 7i)$$

   has a derivative.
9. Is the function f given by

   $$f(z) = \begin{cases} \dfrac{(z)^2}{2}, & z \ne 0 \\ 0, & z = 0 \end{cases}$$

   differentiable at z = 0? Explain.
10. At what points is the function f given by $f(z) = x^3 + i(1 - y)^3$ analytic? Explain.
11. Find all points at which $f(z) = 2y - ix$ is differentiable.
12. Suppose f is analytic on a connected open set D, and $f'(z) = 0$ for all $z \in D$. Prove that f is constant.

Notes

13. Find all points at which

$$f(z) = \frac{x}{x^2+y^2} - i\frac{x}{x^2+y^2}$$

is differentiable. At what points is f analytic? Explain.

14. Suppose f is analytic on the set D, and suppose Re f is constant on D. Is f necessarily constant on D? Explain.

15. Suppose f is analytic on the set D, and suppose $|f(z)|$ is constant on D. Is f necessarily constant on D? Explain.

2.8 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 3: Elementary Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define exponential function
- Discuss the trigonometric functions
- Describe the logarithms and complex exponents

Introduction

As we know, Complex functions are, of course, quite easy to come by — they are simply ordered pairs of real-valued functions of two variables. We have, however, already seen enough to realize that it is those complex functions that are differentiable are the most interesting. It was important in our invention of the complex numbers that these new numbers in some sense included the old real numbers — in other words, we extended the reals. We shall find it most useful and profitable to do a similar thing with many of the familiar real functions. That is, we seek complex functions such that when restricted to the reals are familiar real functions. As we have seen, the extension of polynomials and rational functions to complex functions is easy; we simply change x's to z's. Thus, for instance, the function f defined by :

$$f(z) = \frac{z^2 + z + 1}{z + 1}$$

has a derivative at each point of its domain, and for $z = x + 0i$, becomes a familiar real rational function :

$$f(x) = \frac{x^2 + x + 1}{x + 1}.$$

What happens with the trigonometric functions, exponentials, logarithms, etc., is not so obvious. Let us begin.

3.1 The Exponential Function

Let the so-called exponential function exp be defined by

$$\exp(z) = ex(\cos y + i \sin y),$$

where, as usual, z = x + iy. From the Cauchy-Riemann equations, we see at once that this function has a derivative every where—it is an entire function. Moreover,

$$\frac{d}{dz}\exp(z) = \exp(z).$$

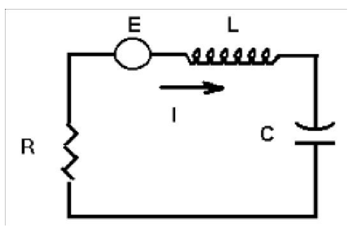
Note next that if z = x + iy and w = u + iv, then

$$\begin{aligned}\exp(z + w) &= e^{x+u}[\cos(y + v) + i \sin(y + v)]\\ &= e^x e^u[\cos y \cos v - \sin y \sin v + i(\sin y \cos v + \cos y \sin v)]\\ &= e^x e^u(\cos y + i \sin y)(\cos v + i \sin v)\\ &= \exp(z)\exp(w).\end{aligned}$$

We, thus, use the quite reasonable notation $e^z = \exp(z)$ and observe that we have extended the real exponential e^x to the complex numbers.

Example: Recall from elementary circuit analysis that the relation between the voltage drop V and the current flow I through a resistor is $V = RI$, where R is the resistance. For an inductor, the relation is $V = L\frac{dl}{dt}$, where L is the inductance; and for a capacitor, $C\frac{dV}{dt} = I$, where C is the capacitance. (The variable t is, of course, time.) Note that if V is sinusoidal with a frequency ω, then so also is I. Suppose then that $V = A\sin(\omega t + \varphi)$. We can write this as $V = Im(Ae^{i\varphi}e^{i\omega t}) = Im(Be^{i\omega t})$, where B is complex. We know the current I will have this same form: $I = Im\,(Ce^{i\omega t})$. The relations between the voltage and the current are linear, and so we can consider complex voltages and currents and use the fact that $e^{i\omega t} = \cos\omega t + i\sin\omega t$. We, thus, assume a more or less fictional complex voltage V, the imaginary part of which is the actual voltage, and then the actual current will be the imaginary part of the resulting complex current.

What makes this a good idea is the fact that differentiation with respect to time t becomes simply multiplication by $i\omega$: $\frac{d}{dt}Ae^{i\omega t} = iwtAe^{i\omega t}$. If $I = be^{i\omega t}$, the above relations between current and voltage become $V = i\omega LI$ for an inductor, and $i\omega VC = I$, or $V = \frac{1}{i\omega C}$ for a capacitor. Calculus is thereby turned into algebra. To illustrate, suppose we have a simple RLC circuit with a voltage source $V = \alpha\sin\omega t$. We let $E = ae^{i\omega t}$.



Then the fact that the voltage drop around a closed circuit must be zero (one of Kirchoff's celebrated laws) looks like

Notes

$$i\omega LI + \frac{I}{i\omega C} + RI = ae^{i\omega t}, \text{or}$$

$$i\omega Lb + \frac{b}{i\omega C} + Rb = a$$

Thus,

$$b = \frac{a}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$

In polar form,

$$b = \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\varphi},$$

where

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R} \; (R \neq 0)$$

Hence,

$$I = \text{Im}(be^{i\omega t}) = \text{Im}\left(\frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i(\omega t + \varphi)}\right)$$

$$= \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \varphi)$$

This result is well-known to all, but it is hoped that you are convinced that this algebraic approach afforded us by the use of complex numbers is far easier than solving the differential equation. You should note that this method yields the steady state solution—the transient solution is not necessarily sinusoidal.

3.2 Trigonometric Functions

Define the functions cosine and sine as follows:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

where we are using $e^z = \exp(z)$.

First, let's verify that these are honest-to-goodness extensions of the familiar real functions, cosine and sine–otherwise we have chosen very bad names for these complex functions.

Notes

So, suppose $z = x + 0i = x$. Then,

$$e^{ix} = \cos x + i \sin x, \text{ and}$$

$$e^{-ix} = \cos x - i \sin x.$$

Thus,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Next, observe that the sine and cosine functions are entire–they are simply linear combinations of the entire functions e^{iz} and e^{-iz}. Moreover, we see that

$$\frac{d}{dz}\sin z = \cos z, \text{and} \frac{d}{dz}\cos z = -\sin z,$$

just as we would hope.

It may not have been clear to you back in elementary calculus what the so-called hyperbolic sine and cosine functions had to do with the ordinary sine and cosine functions.

Now perhaps it will be evident. Recall that for real t,

$$\sin h\, t = \frac{e^t - e^{-t}}{2}, \text{ and } \cos h\, t = \frac{e^t + e^{-t}}{2}$$

Thus,

$$\sin (it) = \frac{e^{i(it)} - e^{-i(it)}}{2i} = \frac{e^t - e^{-t}}{2} = i \sin h\, t$$

Similarly,

$$\cos (it) = \cos ht.$$

Most of the identities you learned in the 3rd grade for the real sine and cosine functions are also valid in the general complex case. Let's look at some.

$$\sin^2 z + \cos^2 z = \frac{1}{4}\left[-(e^{iz} - e^{-iz})^2 + (e^{iz} + e^{-iz})^2\right]$$

$$= \frac{1}{4}\left[-e^{2iz} + 2e^{iz}e^{-iz} - e^{-2iz} + e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}\right]$$

$$= \frac{1}{4}(2+2) = 1$$

It is also relative straight-forward and easy to show that:

$$\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w, \text{ and}$$

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w$$

Other familiar ones follow from these in the usual elementary school trigonometry fashion. Let's find the real and imaginary parts of these functions:

$$\sin z = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cos hy + i \cos x \sin hy.$$

In the same way, we get $\cos z = \cos x \cos h\,y - i \sin x \sin hy$.

3.3 Logarithms and Complex Exponents

In the case of real functions, the logarithm function was simply the inverse of the exponential function. Life is more complicated in the complex case—as we have seen, the complex exponential function is not invertible.

There are many solutions to the equation $e^z = w$.

If $z \neq 0$, we define log z by

$$\log z = \ln|z| + i \arg z.$$

There are thus many log z's; one for each argument of z. The difference between any two of these is, thus, an integral multiple of $2\pi i$. First, for any value of log z we have

$$e^{\log z} = e^{\ln|z| + i \arg z} = e^{\ln|z|}e^{i \arg z} = z.$$

This is familiar. But next there is a slight complication:

$$\log(e^z) = \ln e^x + i \arg e^z = x + y(y + 2k\pi)i$$

$$= z + 2k\pi i,$$

where k is an integer. We also have

$$\log(zw) = \ln(|z|\,|w|) + i \arg(zw)$$

$$= \ln|z| + i \arg z + \ln|w| + i \arg w + 2k\pi i$$

$$= \log z + \log w + 2k\pi i$$

for some integer k.

There is defined a function, called the **principal logarithm,** or **principal branch** of the logarithm, function, given by

$$\text{Log } z = \ln|z| + i\text{Arg } z,$$

where Arg z is the principal argument of z. Observe that for any log z, it is true that $\log z = \text{Log } z + 2k\pi i$ for some integer k which depends on z. This new function is an extension of the real logarithm function:

$$\text{Log } x = \ln x + i\text{Arg } x = \ln x.$$

This function is analytic at a lot of places. First, note that it is not defined at $z = 0$, and is not continuous anywhere on the negative real axis ($z = x + 0i$, where $x < 0$). So, let's suppose $z_0 = x_0 + iy_0$, where z_0 is not zero or on the negative real axis, and see about a derivative of Log z :

$$\lim_{z \to z_0} \frac{\text{Log } z - \text{Log } z_0}{z - z_0} = \lim_{z \to z_0} \frac{\text{Log } z - \text{Log } z_0}{e^{\log z} - e^{\log z_0}}$$

Notes

Now if we let w = Log z and w_0 = Log z_0, and notice that $w \to w_0$ as $z \to z_0$, this becomes

$$\lim_{z \to z_0} \frac{\text{Log } z - \text{Log } z_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}}$$

$$= \frac{1}{e^{w_0}} = \frac{1}{z_0}$$

Thus, Log is differentiable at z_0, and its derivative is $\frac{1}{z_0}$.

We are now ready to give meaning to z^c, where c is a complex number. We do the obvious and define

$$z^c = e^{c \log z}.$$

There are many values of log z, and so there can be many values of z^c. As one might guess, $e^{c\text{Log } z}$ is called the principal value of z^c.

Note that we are faced with two different definitions of z^c in case c is an integer. Let's see, if we have anything to unlearn. Suppose c is simply an integer, c = n. Then

$$z^n = e^{n \log z} = e^{n(\text{Log } z + 2k\pi i)}$$

$$= e^{n\text{Log } z} e^{2kn\pi i} = e^{n\text{Log } z}$$

There is, thus, just one value of z^n, and it is exactly what it should be: $e^{n\text{Log } z} = |z|^n e^{in \arg z}$. It is easy to verify that in case c is a rational number, z^c is also exactly what it should be.

Far more serious is the fact that we are faced with conflicting definitions of z^c in case z = e. In the above discussion, we have assumed that e^z stands for exp(z). Now we have a definition for e^z that implies that e^z can have many values. For instance, if someone runs at you in the night and hands you a note with $e^{1/2}$ written on it, how do you know whether this means exp(1/2) or the two values $\sqrt{e}$ and $-\sqrt{e}$? Strictly speaking, you do not know. This ambiguity could be avoided, of course, by always using the notation exp(z) for $e^x e^{iy}$, but almost everybody in the world uses e^z with the understanding that this is exp(z), or equivalently, the principal value of e^z. This will be our practice.

3.4 Summary

- Let the so-called exponential function exp be defined by

  $$\exp(z) = ex(\cos y + i \sin y),$$

  where, as usual, z = x + iy. From the Cauchy-Riemann equations, we see at once that this function has a derivative every where—it is an entire function. Moreover,

  $$\frac{d}{dz}\exp(z) = \exp(z).$$

  Note next that if z = x + iy and w = u + iv, then

  $$\begin{aligned} \exp(z + w) &= e^{x+u}[\cos(y + v) + i \sin(y + v)] \\ &= e^x e^u[\cos y \cos v - \sin y \sin v + i(\sin y \cos v + \cos y \sin v)] \\ &= e^x e^u(\cos y + i \sin y)(\cos v + i \sin v) \\ &= \exp(z) \exp(w). \end{aligned}$$

Notes

We, thus, use the quite reasonable notation $e^z = \exp(z)$ and observe that we have extended the real exponential e^x to the complex numbers.

- First, let's verify that these are honest-to-goodness extensions of the familiar real functions, cosine and sine–otherwise we have chosen very bad names for these complex functions.

  So, suppose z = x + 0i = x. Then,

$$e^{ix} = \cos x + i \sin x, \text{ and}$$

$$e^{-ix} = \cos x - i \sin x.$$

  Thus,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- In the case of real functions, the logarithm function was simply the inverse of the exponential function. Life is more complicated in the complex case—as we have seen, the complex exponential function is not invertible.

  There are many solutions to the equation $e^z = w$.

  If $z \neq 0$, we define log z by

$$\log z = \ln|z| + i \arg z.$$

- There are many values of log z, and so there can be many values of z^c. As one might guess, $e^{c\text{Log } z}$ is called the principal value of z^c.

  Note that we are faced with two different definitions of z^c in case c is an integer. Let's see if we have anything to unlearn. Suppose c is simply an integer, c = n. Then

$$z^n = e^{n \log z} = e^{n(\text{Log } z + 2k\pi i)}$$

$$= e^{n\text{Log } z} e^{2kn\pi i} = e^{n\text{Log } z}$$

  There is, thus, just one value of z^n, and it is exactly what it should be: $e^{n\text{Log } z} = |z|^n e^{in \arg z}$. It is easy to verify that in case c is a rational number, z^c is also exactly what it should be.

3.5 Keywords

Exponential function: Let the so-called exponential function exp be defined by exp(z) = ex(cos y + i sin y),

Logarithm function: The logarithm function was simply the inverse of the exponential function.

Principal value: There are many values of log z, and so there can be many values of z^c. As one might guess, $e^{c\text{Log } z}$ is called the principal value of z^c.

3.6 Self Assessment

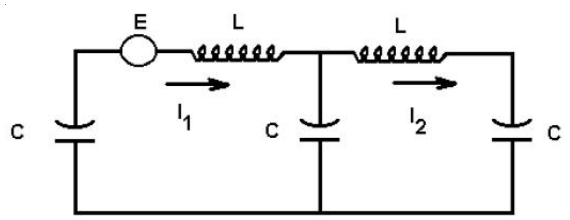
1. Let the so-called exponential function exp be defined by ....
2. If $z \neq 0$, we define log z by ....

Notes

3. New function is an extension of the real logarithm function: ....
4. There are many values of log z, and so there can be many values of z^c. As one might guess, $e^{c\text{Log } z}$ is called the .... of z^c.

3.7 Review Questions

1. Show that exp(z + 2πi) = exp(z)
2. Show that $\dfrac{\exp(z)}{\exp(w)} = \exp(z - w)$.
3. Show that |exp(z)| = e^x, and arg (exp(z) = y + 2kπ for any arg (exp(z)) and some integer k.
4. Find all z such that exp(z) = –1, or explain why there are none.
5. Find all z such that exp(z) = 1 + i, or explain why there are none.
6. For what complex numbers w does the equation exp(z) = w have solutions? Explain.
7. Find the indicated mesh currents in the network:



8. Show that for all z,

   (a) sin(z + 2p) = sin z; (b) cos(z + 2π) = cos z;

   (c) $\sin\left(z + \dfrac{\pi}{2}\right) = \cos z$.

9. Show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$.
10. Find all z such that sin z = 0.
11. Find all z such that cos z = 2, or explain why there are none.
12. Is the collection of all values of $\log(i^{1/2})$ the same as the collection of all values of $\dfrac{1}{2}\log i$? Explain.
13. Is the collection of all values of $\log(i^2)$ the same as the collection of all values of 2log i ? Explain.
14. Find all values of $\log(z^{1/2})$. (in rectangular form)
15. At what points is the function given by Log $(z^2 + 1)$ analytic? Explain.
16. Find the principal value of

    (a) i^i. (b) $(1 - i)^{4i}$

17. Find all values of $|i^i|$.

Notes

Answers: Self Assessment

1. $\exp(z) = ex(\cos y + i \sin y)$
2. $\log z = \ln|z| + i \arg z$
3. $\text{Log } x = \ln x + i\text{Arg } x = \ln x.$
4. principal value

3.8 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati,T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Notes

Unit 4: Integration

CONTENTS

Objectives

After studying this unit, you will be able to:

- Explain the evaluation of integrals
- Discuss the anti derivatives

Introduction

If $\gamma : D \rightarrow C$ is simply a function on a real interval $D = [\alpha, \beta]$, then the integral $\int_{\alpha}^{\beta} \gamma(t)dt$ of course, simply an ordered pair of everyday 3rd grade calculus integrals:

$$\int_{\alpha}^{\beta} \gamma(t)dt = \int_{\alpha}^{\beta} x(t)dt + i\int_{\alpha}^{\beta} y(t)dt,$$

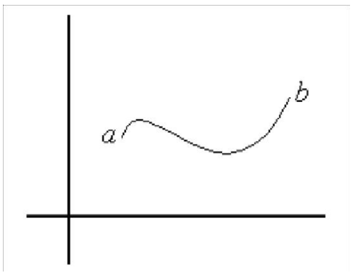
where g(t) = x(t) + iy(t). Thus, for example,

Nothing really new here. The excitement begins when we consider the idea of an integral of an honest-to-goodness complex function $f : D \rightarrow C$, where D is a subset of the complex plane. Let's define the integral of such things; it is pretty much a straightforward extension to two dimensions of what we did in one dimension back in Mrs. Turner's class.

4.1 Integral

Suppose f is a complex-valued function on a subset of the complex plane and suppose a and b are complex numbers in the domain of f. In one dimension, there is just one way to get from one number to the other; here we must also specify a path from a to b. Let C be a path from a to b, and we must also require that C be a subset of the domain of f.

Figure 4.1



Note we do not even require that a ≠ b; but in case a = b, we must specify an *orientation* for the closed path C. We call a path, or curve, **closed** in case the initial and terminal points are the same, and a **simple closed** path is one in which no other points coincide. Next, let P be a **partition** of the curve; that is, $P = \{z_0, z_1, z_2, \ldots, z_n\}$ is a finite subset of C, such that $a = z_0$, $b = z_n$, and such that z_j comes immediately after z_{j-1} as we travel along C from a to b.

A Riemann sum associated with the partition P is just what it is in the real case:

$$S(P) = \sum_{j=1}^{n} f(z_j^*)\Delta z_j,$$

where z_j^* is a point on the arc between z_{j-1} and z_j, and $\Delta zj = z_j - z_{j-1}$.

Notes For a given partition P, there are many S(P)—depending on how the points z_j^* are chosen.)

there is a number L so that given any ε > 0, there is a partition Pε of C such that

$$|S(P) - L| < \varepsilon$$

whenever P ⊃ Pε, then f is said to be integrable on C and the number L is called the **integral of** f on C. This number L is usually written $\int_C f(z)dz$.

Some properties of integrals are more or less evident from looking at Riemann sums:

$$\int_C cf(z)dz = c\int_C f(z)dz$$

for any complex constant c.

$$\int_C (f(z) + g(z))dz = \int_C f(z)dz + \int_C g(z)dz$$

4.2 Evaluating Integrals

Now, how on Earth do we ever find such an integral? Let γ : [α, β] → C be a complex description of the curve C. We partition C by partitioning the interval [α, β] in the usual way: $\alpha = t_0 < t_1 < t_2$

Notes

$< \ldots < t_n = \beta$. Then $\{a = \gamma(\alpha), \gamma(t_1), \gamma(t_2), \ldots, \gamma(\beta) = b\}$ is partition of C. (Recall we assume that $g'(t) \neq 0$ for a complex description of a curve C.) A corresponding Riemann sum looks like

$$S(P) = \sum_{j=1}^{n} f(\gamma(t_j^*))(\gamma(t_j) - \gamma(t_{j-1})).$$

We have chosen the points $z_j^* = \gamma(t_j^*)$, where $t_{j-1} \leq t_j^* \leq t_j$. Next, multiply each term in the sum by 1 in disguise:

$$S(P) = \sum_{j=1}^{n} f(\gamma(t_j^*))(\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}})(t_j - t_{j-1}).$$

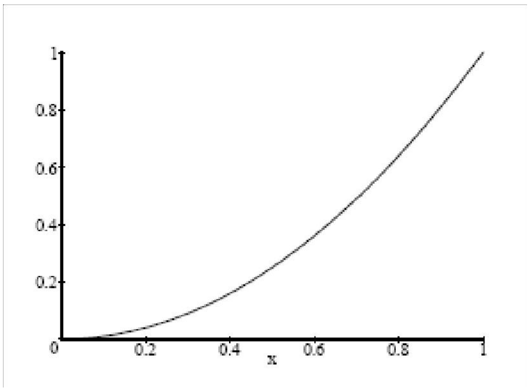
Hope it is now reasonably convincing that "in the limit", we have

$$\int_C f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

(We are, of course, assuming that the derivative γ' exists.)

Example 1: We shall find the integral of $f(z) = (x^2 + y) + i(xy)$ from $a = 0$ to $b = 1 + i$ along three different paths, or contours, as some call them.

First, let C_1 be the part of the parabola $y = x^2$ connecting the two points. A complex description of C_1 is $\gamma_1(t) = t + it^2$, $0 \leq t \leq 1$:

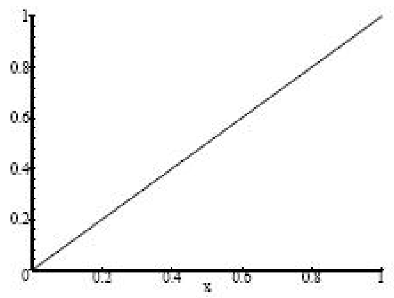


Now, $\gamma_1'(t) = 1 + 2ti$, and $f(\gamma_1(t)) = (t^2 + t^2) = itt^2 = 2t^2 + it^3$. Hence,

$$\begin{aligned}\int_{C_1} f(z)dz &= \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt \\ &= \int_0^1 (2t^2 + it^3)(1 + 2ti)\,dt \\ &= \int_0^1 (2t^2 - 2t^4 + 5t^3 i)\,dt \\ &= \frac{4}{15} + \frac{5}{4}i\end{aligned}$$

Notes

Next, let's integrate along the straight line segment C_2 joining 0 and 1 + i.



Here we have $\gamma_2(t) = t + it$, $0 \le t \le 1$. Thus, $\gamma_2'(t) = 1 + i$, and our integral looks like

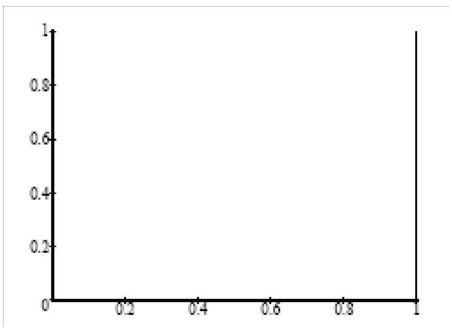
$$\int_{C_2} f(z)dz = \int_0^1 f(\gamma_2(t))\gamma_2'(t)dt$$

$$= \int_0^1 [(t^2 + t) + it^2](1+i)\,dt$$

$$= \int_0^1 [t + i(t + 2t^2)]\,dt$$

$$= \frac{1}{2} + \frac{7}{6}i$$

Finally, let's integrate along C_3, the path consisting of the line segment from 0 to 1 together with the segment from 1 to 1 + i.



We shall do this in two parts: C_{31}, the line from 0 to 1 ; and C_{32}, the line from 1 to 1 + i. Then we have

$$\int_{C_3} f(z)dz = \int_{C_{31}} f(z)dz + \int_{C_{32}} f(z)dz.$$

Notes

For C_{31}, we have $\gamma(t) = t$, $0 \le t \le 1$. Hence,

$$\int_{C_{31}} f(z)dz = \int_0^1 t^2 dt = \frac{1}{3}.$$

For C_{32}, we have $\gamma(t) = 1 + it$, $0 \le t \le 1$. Hence,

$$\int_{C_{32}} f(z)dz = \int_0^1 (1 + t + it)idt = -\frac{1}{2} + \frac{3}{2}i.$$

Thus,

$$\int_{C_3} f(z)dz = \int_{C_{31}} f(z)dz + \int_{C_{32}} f(z)dz.$$

$$= -\frac{1}{6} + \frac{3}{2}i.$$

Suppose there is a number M so that $|f(z)| \le M$ for all $z \in C$. Then,

$$\left|\int_C f(z)dz\right| = \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt\right|$$

$$\le \int_\alpha^\beta |f(\gamma(t))\gamma'(t)|dt$$

$$\le M\int_\alpha^\beta |\gamma'(t)|dt = ML,$$

where $L = \int_\alpha^\beta |\gamma'(t)|dt$ is the length of C.

4.3 Antiderivatives

Suppose D is a subset of the reals and $\gamma : D \to C$ is differentiable at t. Suppose further that g is differentiable at $\gamma(t)$. Then let's see about the derivative of the composition $g(\gamma(t)$. It is, in fact, exactly what one would guess. First,

$$g(\gamma(t)) = u(x(t), y(t)) + iv(x(t), y(t)),$$

where $g(z) = u(x, y) + iv(x, y)$ and $\gamma(t) = x(t) + iy(t)$. Then,

$$\frac{d}{dt}g(\gamma(t)) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right).$$

The places at which the functions on the right-hand side of the equation are evaluated are obvious. Now, apply the Cauchy-Riemann equations:

Notes

$$\frac{d}{dt}g(\gamma(t)) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial x}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial x}\frac{dy}{dt}\right)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)$$

$$= g'(\gamma(t))\gamma'(t).$$

Now, back to integrals. Let $F : D \to C$ and suppose $F'(z) = f(z)$ in D. Suppose that a and b are in D and that $C \subset D$ is a contour from a to b. Then

$$\int_C f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt,$$

where $\gamma : [\alpha, \beta] \to C$ describes C. From our introductory discussion, we know that $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$. Hence,

$$\int_C f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt$$

$$= \int_\alpha^\beta \frac{d}{dt}F(\gamma(t))dt = F(\gamma(\beta)) - F(\gamma(\alpha))$$

$$= F(b) - F(a)$$

This is very pleasing.

Notes Integral depends only on the points a and b and not at all on the path C. We say the integral is path independent. Observe that this is equivalent to saying that the integral of f around any closed path is 0. We have, thus, shown that if in D the integrand f is the derivative of a function F, then any integral $\int_C f(z)dz$ for $C \subset D$ is path independent.

Example:

Let C be the curve $y = \frac{1}{x^2}$ from the point z = 1 + i to the point z = $3 + \frac{i}{9}$. Let's find

$$\int_C z^2 dz$$

This is easy—we know that $F'(z) = z^2$, where $F(z) = \frac{1}{3}z^3$. Thus,

$$\int_C z^2 dz = \frac{1}{3}\left[(1+i)^3 - \left(3 + \frac{i}{9}\right)^3\right]$$

$$= -\frac{260}{27} - \frac{728}{2187}i$$

Now, instead of assuming f has an antiderivative, let us suppose that the integral of f between any two points in the domain is independent of path and that f is continuous. Assume also that every point in the domain D is an interior point of D and that D is connected. We shall see that in this case, f has an antiderivative. To do so, let z_0 be any point in D, and define the function F by

$$F(z) = \int_{C_z} f(z)dz,$$

where C_z is any path in D from z_0 to z. Here is important that the integral is path independent, otherwise F(z) would not be well-defined.

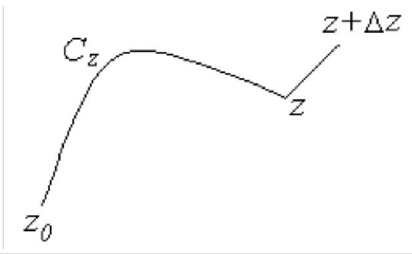
Notes Also we need the assumption that D is connected in order to be sure there always is at least one such path.

Now, for the computation of the derivative of F:

$$F(z + \Delta z) - F(z) = \int_{L_{\Delta z}} f(s)ds$$

where $L_{\Delta z}$ is the line segment from z to z + Δz.

Figure 4.2



Next, observe that $\int_{L_{\Delta z}} ds = \Delta z$. Thus, $f(z) = \frac{1}{\Delta z}\int_{L_{\Delta z}} (f(s) - f(z))ds$.

Now then,

$$\left|\frac{1}{\Delta z}\int_{L_{\Delta z}} (f(s) - f(z))ds\right| \le \left|\frac{1}{\Delta z}\right||\Delta z|\max\{|f(s) - f(z)| : s \in L_{\Delta z}\}$$

$$\le \max\{|f(s) - f(z)| : s \in L_{\Delta z}\}.$$

We know f is continuous at z, and so $\lim_{\Delta z \to 0} \max\{|f(s) - f(z)| : s \in L_{\Delta z}\} = 0$. Hence,

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \lim_{\Delta z \to 0}\left(\frac{1}{\Delta z}\int_{L_{\Delta z}} (f(s) - f(z))ds\right)$$

$$= 0$$

In other words, F′(z) = f(z), and so, just as promised, f has an antiderivative! Let's summarize what we have shown in this section:

Suppose $f : D \to C$ is continuous, where D is connected and every point of D is an interior point. Then f has an antiderivative if and only if the integral between any two points of D is path independent.

4.4 Summary

- If $\gamma : D \to C$ is simply a function on a real interval D = [α, β], then the integral $\int_{\alpha}^{\beta}\gamma(t)dt$ of course, simply an ordered pair of everyday 3rd grade calculus integrals:

$$\int_{\alpha}^{\beta}\gamma(t)dt = \int_{\alpha}^{\beta}x(t)dt + i\int_{\alpha}^{\beta}y(t)dt,$$

where g(t) = x(t) + iy(t).

- A Riemann sum associated with the partition P is just what it is in the real case:

$$S(P) = \sum_{j=1}^{n} f(z_j^*)\Delta z_j,$$

where z_j^* is a point on the arc between z_{j-1} and z_j, and $\Delta zj = z_j - z_{j-1}$.

- Suppose D is a subset of the reals and $\gamma : D \to C$ is differentiable at t. Suppose further that g is differentiable at γ(t). Then let's see about the derivative of the composition g(γ(t). It is, in fact, exactly what one would guess. First,

$$g(\gamma(t)) = u(x(t), y(t)) + iv(x(t), y(t)),$$

where g(z) = u(x, y) + iv(x, y) and γ(t) = x(t) + iy(t).

- f is continuous at z, and so $\lim_{\Delta z\to 0}\max\{|f(s)-f(z)| : s \in L_{\Delta z}\} = 0.$ Hence,

$$\lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \lim_{\Delta z\to 0}\left(\frac{1}{\Delta z}\int_{L_{\Delta z}}(f(s)-f(z))ds\right)$$

$$= 0$$

In other words, F′(z) = f(z), and so, just as promised, f has an antiderivative! Let's summarize what we have shown in this section:

Suppose $f : D \to C$ is continuous, where D is connected and every point of D is an interior point. Then f has an antiderivative if and only if the integral between any two points of D is path independent.

Notes

4.5 Keywords

Calculus integrals: If $\gamma : D \to C$ is simply a function on a real interval $D = [\alpha, \beta]$, then the integral $\int_{\alpha}^{\beta}\gamma(t)dt$ of course, simply an ordered pair of everyday 3rd grade calculus integrals: $\int_{\alpha}^{\beta}\gamma(t)dt = \int_{\alpha}^{\beta}x(t)dt + i\int_{\alpha}^{\beta}y(t)dt.$

Antiderivative: Suppose $f : D \to C$ is continuous, where D is connected and every point of D is an interior point. Then f has an antiderivative if and only if the integral between any two points of D is path independent.

4.6 Self Assessment

1. If $\gamma : D \to C$ is simply a function on a real interval $D = [\alpha, \beta]$, then the integral $\int_{\alpha}^{\beta}\gamma(t)dt$ of course, simply an ordered pair of everyday 3rd grade .... $\int_{\alpha}^{\beta}\gamma(t)dt = \int_{\alpha}^{\beta}x(t)dt + i\int_{\alpha}^{\beta}y(t)dt.$

2. A Riemann sum associated with the partition P is just what it is in the real case: .... where z_j^* is a point on the arc between z_{j-1} and z_j, and $\Delta zj = z_j - z_{j-1}$.

3. Integral depends only on the points a and b and not at all on the path C. We say the integral is ....

4. If in D the integrand f is the derivative of a function F, then any integral .... for $C \subset D$ is path independent.

5. f is continuous at z, and so .... Hence,

$$\lim_{\Delta z \to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \lim_{\Delta z \to 0}\left(\frac{1}{\Delta z}\int_{L_{\Delta z}}(f(s)-f(z))ds\right)$$
$$= 0$$

6. Suppose $f : D \to C$ is continuous, where D is connected and every point of D is an interior point. Then f has an .... if and only if the integral between any two points of D is path independent.

4.7 Review Questions

1. Evaluate the integral $\int_C \bar{z}dz,$ where C is the parabola $y = x^2$ from 0 to 1 + i.

2. Evaluate $\int_C \frac{1}{z}dz,$ where C is the circle of radius 2 centered at 0 oriented counter clockwise.

3. Evaluate $\int_C f(z)dz,$ where C is the curve $y = x^3$ from $-1 - i$ to $1 + i$, and $f(z) = \begin{cases} 1 & \text{for} \quad y < 0 \\ 4y & \text{for} \quad y \geq 0 \end{cases}$

Notes

4. Let C be the part of the circle $\gamma(t) = e^{it}$ in the first quadrant from a = 1 to b = i. Find as small an upper bound as you can for $\left|\int_C (z^2 - \bar{z}^4 + 5)dz\right|$.

5. Evaluate $\int_C f(z)dz$ where f(z) = $z + 2\bar{z}$ and C is the path from z = 0 to z = 1 + 2i consisting of the line segment from 0 to 1 together with the segment from 1 to 1 + 2i.

6. Suppose C is any curve from 0 to π + 2i. Evaluate the integral

$$\int_C \cos\left(\frac{z}{2}\right)dz.$$

7. (a) Let F(z) = log z, $-\frac{3}{4}\pi < \arg z < \frac{5}{4}\pi$. Show that the derivative F′(z) = $\frac{1}{z}$.

   (b) Let G(z) = $\log z, -\frac{\pi}{4} < \arg z < \frac{7\pi}{4}$. Show that the derivative G′(z) = $\frac{1}{z}$.

   (c) Let C_1 be a curve in the right-half plane $D_1 = \{z : \text{Re}z \geq 0\}$ from –i to i that does not pass through the origin. Find the integral

$$\int_{C_1} \frac{1}{z}dz.$$

   (d) Let C_2 be a curve in the left-half plane $D_2 = \{z : \text{Re}z \leq 0\}$ from –i to i that does not pass through the origin. Find the integral.

$$\int_{C_2} \frac{1}{z}dz.$$

8. Let C be the circle of radius 1 centered at 0 with the clockwise orientation. Find

$$\int_C \frac{1}{z}dz.$$

9. (a) et H(z) = z^c, $-\pi < \arg z < \pi$. Find the derivative H′(z).

   (b) Let K(z) = z^c, $-\frac{\pi}{4} < \arg z < \frac{7\pi}{4}$. Find the derivative K′(z).

   (c) Let C be any path from –1 to 1 that lies completely in the upper half-plane and does not pass through the origin. (Upper half-plane $\{z : \text{Im}z \geq 0\}$.) Find

$$\int_C F(z)dz,$$

   where F(z) = z^i, $-\pi < \arg z \leq \pi$.

10. Suppose P is a polynomial and C is a closed curve. Explain how you know that $\int_C P(z)dz = 0$.

Notes

Answers: Self Assessment

1. calculus integrals
2. $S(P) = \sum_{j=1}^{n} f(z_j^*)\Delta z_j,$
3. path independent
4. $\int_C f(z)dz$
5. $\lim_{\Delta z \to 0} \max\{|f(s) - f(z)| : s \in L_{\Delta z}\} = 0$
6. antiderivative

4.8 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati,T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 5: Cauchy's Theorem

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define homotopy
- Discuss the Cauchy's theorem
- Describe examples of Cauchy's theorem

Introduction

In earlier unit, you have studied about complex functions and complex number. Cauchy-Riemann equations which under certain conditions provide the necessary and sufficient condition for the differentiability of a function of a complex variable at a point. A very important concept of analytic functions which is useful in many application of the complex variable theory. Let's discuss the concept of Cauchy's theorem.

5.1 Homotopy

Suppose D is a connected subset of the plane such that every point of D is an interior point—we call such a set a region—and let C_1 and C_2 be oriented closed curves in D. We say C_1 is **homotopic** to C_2 in D if there is a continuous function $H : S \to D$, where S is the square $S = \{(t, s) : 0 \leq s, t \leq 1\}$, such that H(t,0) describes C_1 and H(t,1) describes C_2, and for each fixed s, the function H(t, s) describes a closed curve C_s in D.

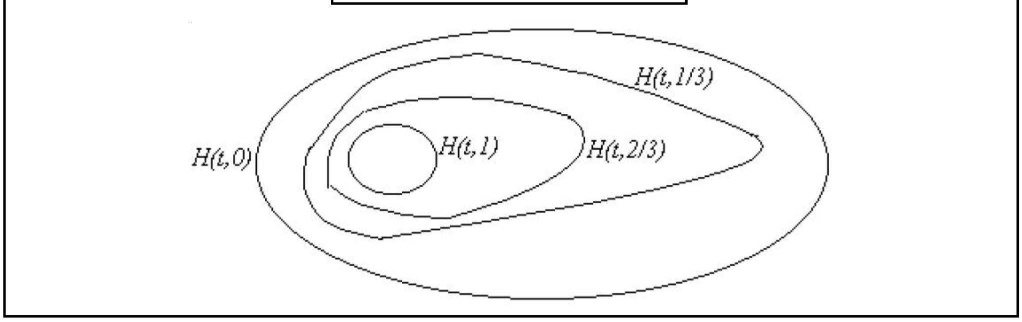
The function H is called a **homotopy** between C_1 and C_2. Note that if C_1 is homotopic to C_2 in D, then C_2 is homotopic to C_1 in D. Just observe that the function $K(t, s) = H(t,1 - s)$ is a homotopy.

It is convenient to consider a point to be a closed curve. The point c is a described by a constant function $\gamma(t) = c$. We thus speak of a closed curve C being homotopic to a constant—we sometimes say *C* is **contractible** to a point.

Notes

Emotionally, the fact that two closed curves are homotopic in D means that one can be continuously deformed into the other in D.

Figure 5.1

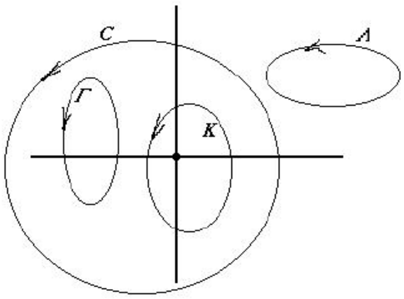


Example 1:

Let D be the annular region D ={z : 1 < | z | < 5}. Suppose C_1 is the circle described by $\gamma_1(t) = 2e^{i2\pi t}$, $0 \le t \le 1$; and C_2 is the circle described by $\gamma_2(t) = 4e^{i2\pi t}$, $0 \le t \le 1$. Then $H(t, s) = (2 + 2s)e^{i2\pi t}$ is a homotopy in D between C_1 and C_2. Suppose C_3 is the same circle as C_2 but with the opposite orientation; that is, a description is given by $\gamma_3(t) = 4e^{-i2\pi t}$, $0 \le t \le 1$. A homotopy between C_1 and C_3 is not too easy to construct—in fact, it is not possible! The moral: orientation counts. From now on, the term "closed curve" will mean an oriented closed curve.

Another Example

Let D be the set obtained by removing the point z = 0 from the plane. Take a look at the picture. Meditate on it and convince yourself that C and K are homotopic in D, but Γ and Λ are homotopic in D, while K and Γ are not homotopic in D.



5.2 Cauchy's Theorem

Suppose C_1 and C_2 are closed curves in a region D that are homotopic in D, and suppose f is a function analytic on D. Let H(t, s) be a homotopy between C_1 and C_2. For each s, the function $\gamma_s(t)$ describes a closed curve C_s in D. Let I(s) be given by

$$I(s) = \int_{C_s} f(z)dz.$$

Then,

$$I(s) = \int_0^1 f(H(t,s))\frac{\partial H(t,s)}{\partial t}dt.$$

Now, let's look at the derivative of I(s). We assume everything is nice enough to allow us to differentiate under the integral:

$$I'(s) = \frac{d}{ds}\left[\int_0^1 f(H(t,s))\frac{\partial H(t,s)}{\partial t}\,dt\right]$$

$$= \int_0^1\left[f'(H(t,s))\frac{\partial H(t,s)}{\partial t}\frac{\partial H(t,s)}{\partial t} + f(H(t,s))\frac{\partial^2 H(t,s)}{\partial s\partial t}\right]dt$$

$$= \int_0^1\left[f'(H(t,s))\frac{\partial H(t,s)}{\partial t}\frac{\partial H(t,s)}{\partial t} + f(H(t,s))\frac{\partial^2 H(t,s)}{\partial t\partial s}\right]dt$$

$$= \int_0^1 \frac{\partial}{\partial t}\left[f(H(t,s))\frac{\partial H(t,s)}{\partial s}\right]dt$$

$$= f(H(1, s))\frac{\partial H(1,s)}{\partial s} - f(H(0, s))\frac{\partial H(0,s)}{\partial s}.$$

But we know each H(t, s) describes a closed curve, and so H(0, s) = H(1, s) for all s. Thus,

$$I'(s) = f(H(1,s))\frac{\partial H(1,s)}{\partial s} - f(H(0,s))\frac{\partial H(0,s)}{\partial s} = 0.$$

which means I(s) is constant! In particular, I(0) = I(1), or

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

This is a big deal. We have shown that if C_1 and C_2 are closed curves in a region *D* that are homotopic in *D*, and *f* is analytic on *D*, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

An easy corollary of this result is the celebrated Cauchy's Theorem, which says that if *f* is analytic on a simply connected region D, then for any closed curve C in D,

$$\int_C f(z)dz = 0.$$

In court testimony, one is admonished to tell the truth, the whole truth, and nothing but the truth. Well, so far in this chapter, we have told the truth, but we have not quite told the whole truth. We assumed all sorts of continuous derivatives in the preceding discussion. These are not always necessary—specifically, the results can be proved true without all our smoothness assumptions—think about approximation.

Example 2:

Look at the picture below and convince your self that the path C is homotopic to the closed path consisting of the two curves C_1 and C_2 together with the line L. We traverse the line twice, once from C_1 to C_2 and once from C_2 to C_1.

Observe then that an integral over this closed path is simply the sum of the integrals over C_1 and C_2, since the two integrals along L, being in opposite directions, would sum to zero. Thus, if f is analytic in the region bounded by these curves (the region with two holes in it), then we know that

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

Notes

5.3 Summary

- Suppose D is a connected subset of the plane such that every point of D is an interior point—we call such a set a region—and let C_1 and C_2 be oriented closed curves in D. We say C_1 is **homotopic** to C_2 in D if there is a continuous function H : S → D, where S is the square S = {(t, s) : 0 ≤ s, t ≤ 1}, such that H(t,0) describes C_1 and H(t,1) describes C_2, and for each fixed s, the function H(t, s) describes a closed curve C_s in D.
- The function H is called a **homotopy** between C_1 and C_2. Note that if C_1 is homotopic to C_2 in D, then C_2 is homotopic to C_1 in D. Just observe that the function K(t, s) = H(t,1 – s) is a homotopy.

  It is convenient to consider a point to be a closed curve. The point c is a described by a constant function $\gamma(t) = c$. We, thus, speak of a closed curve C being homotopic to a constant—we sometimes say *C* is **contractible** to a point.

  Emotionally, the fact that two closed curves are homotopic in D means that one can be continuously deformed into the other in D.
- Suppose C_1 and C_2 are closed curves in a region D that are homotopic in D, and suppose f is a function analytic on D. Let H(t, s) be a homotopy between C_1 and C_2. For each s, the function $\gamma_s(t)$ describes a closed curve C_s in D. Let I(s) be given by $I(s) = \int_{C_s} f(z)dz$.

5.4 Keywords

Homotopy: The function H is called a homotopy between C_1 and C_2. Note that if C_1 is homotopic to C_2 in D, then C_2 is homotopic to C_1 in D. Just observe that the function K(t, s) = H(t,1 – s) is a homotopy.

Contractible: It is convenient to consider a point to be a closed curve. The point c is a described by a constant function $\gamma(t) = c$. We thus speak of a closed curve C being homotopic to a constant—we sometimes say *C* is contractible to a point.

Cauchy's Theorem: Suppose C_1 and C_2 are closed curves in a region D that are homotopic in D, and suppose f is a function analytic on D. Let H(t, s) be a homotopy between C_1 and C_2. For each s, the function $\gamma_s(t)$ describes a closed curve C_s in D. Let I(s) be given by $I(s) = \int_{C_s} f(z)dz$.

5.5 Self Assessment

1. Suppose D is a connected subset of the plane such that every point of D is an interior point—we call such a set a region—and let C_1 and C_2 be oriented closed ....
2. It is convenient to consider a point to be a closed curve. The point c is a described by a constant function $\gamma(t) = c$. We thus speak of a closed curve C being homotopic to a constant—we sometimes say C is .... to a point.
3. Emotionally, the fact that two closed curves are .... in D means that one can be continuously deformed into the other in D.
4. If f is analytic in the region bounded by these curves (the region with two holes in it), then we know that ....

5.6 Review Questions

Notes

1. Suppose C_1 is homotopic to C_2 in D, and C_2 is homotopic to C_3 in D. Prove that C_1 is homotopic to C_3 in D.
2. Explain how you know that any two closed curves in the plane C are homotopic in C.
3. A region D is said to be simply connected if every closed curve in D is contractible to a point in D. Prove that any two closed curves in a simply connected region are homotopic in D.
4. Prove Cauchy's Theorem.
5. Let S be the square with sides x = ± 100, and y = ± 100 with the counterclockwise orientation. Find $\int_S \frac{1}{z} dz$.
6. (a) Find $\int_C \frac{1}{z-1} dz$, where C is any circle centered at z = 1 with the usual counterclockwise orientation: $\gamma(t) = 1 + Ae^{2\pi it}$, $0 \le t \le 1$.

   (b) Find $\int_C \frac{1}{z+1} dz$, where C is any circle centered at z = –1 with the usual counterclockwise orientation.

   (c) Find $\int_C \frac{1}{z^2-1} dz$, where C is the ellipse $4x^2 + y^2 = 100$ with the counterclockwise orientation. [Hint: partial fractions]

   (d) Find $\int_C \frac{1}{z^2-1} dz$, where C is the circle $x^2 - 10x + y^2 = 0$ with the counterclockwise orientation.
7. Evaluate $\int_C \text{Log}(z+3)dz$, where C is the circle $|z| = 2$ oriented counterclockwise.
8. Evaluate $\int_C \frac{1}{z^n} dz$ where C is the circle described by $\gamma(t) = e^{2\pi it}$, $0 \le t \le 1$, and n is an integer ≠ 1.
9. (a) Does the function $f(z) = \frac{1}{z}$ have an antiderivative on the set of all $z \neq 0$? Explain.

   (b) How about $f(z) = \frac{1}{z^n}$, n an integer ≠ 1?
10. Find as large a set D as you can so that the function e^{z^2} have an antiderivative on D.
11. Explain how you know that every function analytic in a simply connected region D is the derivative of a function analytic in D.

Notes

Answers: Self Assessment

1. curves in D
2. contractible
3. homotopic
4. $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$

5.7 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 6: Cauchy's Integral Formula

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Cauchy's integral formula
- Discuss functions defined by integrals
- Describe liouville's theorem
- Explain maximum moduli

Introduction

In last unit, you have studied about concept of Cauchy's theorem. A very important concept of analytic functions which is useful in many application of the complex variable theory. This unit provides you information related to Cauchy's integral formula, functions defined by integrals and maximum moduli.

6.1 Cauchy's Integral Formula

Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its inside, and suppose z_0 is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz.$$

This is the famous Cauchy Integral Formula. Let's see why it's true.

Notes

Let ε > 0 be any positive number. We know that f is continuous at z_0 and so there is a number δ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Now let ρ > 0 be a number such that ρ < δ and the circle $C_0 = \{z : |z - z_0| = \rho\}$ is also inside C. Now, the function $\frac{f(z)}{z - z_0}$ is analytic in the region between C and C_0; thus,

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_0} \frac{f(z)}{z - z_0} dz.$$

We know that $\int_C \frac{1}{z - z_0} dz = 2\pi i$, so we can write

$$\int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_0} \frac{1}{z - z_0} dz$$

$$= \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

For $z \in C_0$ we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

$$\leq \frac{\varepsilon}{\rho}.$$

Thus,

$$\left| \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| = \left| \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz \right|$$

$$\leq \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon.$$

which is exactly what we set out to show.

Look at this result. It says that if *f* is analytic on and inside a simple closed curve and we know the values *f* (*z*) for every *z* on the simple closed curve, then we know the value for the function at every point inside the curve—quite remarkable indeed.

Example:

Let C be the circle $|z| = 4$ traversed once in the counterclockwise direction. Let's evaluate the integral

$$\int_C \frac{\cos z}{z^2 - 6z + 5} dz.$$

We simply write the integrand as

$$\frac{\cos z}{z^2 - 6z + 5} = \frac{\cos z}{(z - 5)(z - 1)} = \frac{f(z)}{z - 1},$$

where **Notes**

$$f(z) = \frac{\cos z}{z-5}.$$

Observe that f is analytic on and inside C, and so,

$$\int_C \frac{\cos z}{z^2 - 6z + 5} dz = \int_C \frac{f(z)}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i \frac{\cos 1}{1-5} = -\frac{i\pi}{2}\cos 1$$

6.2 Functions defined by Integrals

Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the function G be defined by :

$$G(z) = \int_C \frac{g(s)}{s-z} ds$$

for all $z \notin C$. We shall show that G is analytic. Here we go.

Consider,

$$\frac{G(z+\Delta z) - G(z)}{\Delta z} = \frac{1}{\Delta z}\int_C \left[\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right] g(s) ds$$

$$= \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)} ds.$$

Next,

$$\frac{G(z+\Delta z) - G(z)}{\Delta z} - \int_C \frac{g(s)}{(s-z)^2} ds = \int_C \left[\frac{1}{(s-z-Dz)(s-z)} - \frac{1}{(s-z)^2}\right] g(s) ds$$

$$= \int_C \left[\frac{(s-z)-(s-z-\Delta z)}{(s-z-\Delta z)(s-z)^2}\right] g(s) ds$$

$$= \Delta z \int_C \frac{g(s0}{(s-z-\Delta z)(s-z)^2} ds.$$

Now we want to show that

$$\lim_{\Delta z \to 0}\left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds\right] = 0.$$

To that end, let M = max $\{|g(s)| : s \in C\}$, and let d be the shortest distance from z to C. Thus, for $s \in C$, we have $|s - z| \geq d > 0$ and also

$$|s - z - \Delta z| \geq |s - z| - |\Delta z| \geq d - |\Delta z|.$$

Notes

Putting this all together, we can estimate the integrand above:

$$\left|\frac{g(s)}{(s-z-\Delta z)(s-z)^2}\right| \le \frac{M}{(d-|\Delta z|)d^2}$$

for all $s \in C$. Finally,

$$\left|\Delta z\int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2}ds\right| \le |\Delta z|\frac{M}{(d-|\Delta z|)d^2}\text{length}(C),$$

and it is clear that

$$\lim_{\Delta z\to 0}\left[\Delta z\int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2}ds\right] = 0,$$

just as we set out to show. Hence G has a derivative at z, and

$$G'(z) = \int_C \frac{g(s)}{(s-z)^2}ds$$

Next we see that G′ has a derivative and it is just what you think it should be. Consider

$$\begin{aligned}
\frac{G'(z+\Delta z)-G'(z)}{\Delta z} &= \frac{1}{\Delta z}\int_C\left[\frac{1}{(s-z-\Delta z)^2}-\frac{1}{(s-z)^2}\right]g(s)ds \\
&= \frac{1}{\Delta z}\int_C\left[\frac{(s-z)^2-(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds \\
&= \frac{1}{\Delta z}\int_C\left[\frac{2(s-z)\Delta z-(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds \\
&= \int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds
\end{aligned}$$

Next,

$$\begin{aligned}
&= \frac{G'(z+\Delta z)-G'(z)}{\Delta z} - 2\int_C \frac{g(s)}{(s-z)^3}ds \\
&= \int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right]g(s)ds \\
&= \int_C\left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds \\
&= \int_C\left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z)^2+4\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds
\end{aligned}$$

Notes

$$= \int_C \left[\frac{3\Delta z(s-z) - 2(\Delta z)^2}{(s-z-\Delta z)^2 (s-z)^3} \right] g(s)ds$$

Hence,

$$\left| \frac{G'(z+\Delta z) - G'(z)}{\Delta z} - 2\int_C \frac{g(s)}{(s-z)^3} ds \right| = \left| \int_C \left[\frac{3\Delta z(s-z) - 2(\Delta z)^2}{(s-z-\Delta z)^2 (s-z)^3} \right] g(s)ds \right|$$

$$\leq |\Delta z| \frac{(|3m| + 2|\Delta z|)M}{(d-\Delta z)^2 d^3},$$

where m = max $\{|s - z| : s \in C\}$. It should be clear then that

$$\lim_{\Delta z \to 0} \left| \frac{G'(z+\Delta z) - G'(z)}{\Delta z} - 2\int_C \frac{g(s)}{(s-z)^3} ds \right| = 0,$$

or in other words,

$$G''(z) = 2\int_C \frac{g(s)}{(s-z)^3} ds.$$

Suppose f is analytic in a region D and suppose C is a positively oriented simple closed curve in D. Suppose also the inside of C is in D. Then from the Cauchy Integral formula, we know that

$$2\pi i f(z) = \int_C \frac{f(s)}{s-z} ds$$

and so with g = f in the formulas just derived, we have

$$f'(z) = \frac{1}{2pi} \int_C \frac{f(s)}{(s-z)^2} ds, \text{ and } f''(z) = \frac{2}{2\pi i} \int_C \frac{f(s)}{(s-z)^3} ds$$

for all z inside the closed curve C. Meditate on these results. They say that the derivative of an analytic function is also analytic. Now suppose f is continuous on a domain D in which every point of D is an interior point and suppose that $\int_C f(z)dz = 0$ for every closed curve in D. Then we know that f has an antiderivative in D—in other words f is the derivative of an analytic function. We now know this means that f is itself analytic. We thus have the celebrated Morera's Theorem:

If $f : D \to C$ is continuous and such that $\int_C f(z)dz = 0$ for every closed curve in D, then f is analytic in D.

Example:

Let's evaluate the integral

$$\int_C \frac{e^z}{z^3} dz,$$

where C is any positively oriented closed curve around the origin. We simply use the equation

$$f''(z) = \frac{2}{2\pi i} \int_C \frac{f(s)}{(s-z)^3} ds$$

Notes

with z = 0 and f(s) = e^s. Thus,

$$\pi i e^0 = \pi i = \int_C \frac{e^z}{z^3} dz.$$

6.3 Liouville's Theorem

Suppose f is entire and bounded; that is, f is analytic in the entire plane and there is a constant M such that $|f(z)| \leq M$ for all z. Then it must be true that $f'(z) = 0$ identically. To see this, suppose that $f'(w) \neq 0$ for some w. Choose R large enough to insure that $\frac{M}{R} < |f'(w)|$. Now let C be a circle centered at 0 and with radius $\rho > \max\{R, |w|\}$. Then we have :

$$\frac{M}{R} < |f'(w)| \leq \left| \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-w)^2} ds \right|$$

$$\leq \frac{1}{2\pi} \frac{M}{\rho^2} 2\pi\rho = \frac{M}{\rho},$$

a contradiction. It must, therefore, be true that there is no w for which $f'(w) \neq 0$; or, in other words, $f'(z) = 0$ for all z. This, of course, means that f is a constant function. What we have shown has a name, Liouville's Theorem:

The only bounded entire functions are the constant functions.

Let's put this theorem to some good use. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a polynomial. Then

$$p(z) = \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + ... + \frac{a_0}{z^n} \right) z^n.$$

Now choose R large enough to insure that for each j = 1, 2,...,n, we have $\left| \frac{a_{n-j}}{z^j} \right| < \frac{|a_n|}{2n}$ whenever $|z| > R$. (We are assuming that $a_n \neq 0$.) Hence, for $|z| > R$, we know that

$$|p(z)| \geq \left| |a_n| - \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + ... + \frac{a_0}{z^n} \right| \right| |z|^n$$

$$\geq \left| |a_n| - \left| \frac{a_{n-1}}{z} \right| - \left| \frac{a_{n-2}}{z^2} \right| - ... - \left| \frac{a_0}{z^n} \right| \right| |z|^n$$

$$> \left| |a_n| - \frac{|a_n|}{2n} - \frac{|a_n|}{2n} - ... - \frac{|a_n|}{2n} \right| |z|^n$$

$$> \frac{|a_n|}{2} |z|^n.$$

Hence, for $|z| > R$,

$$\frac{1}{|p(z)|} < \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n| R^n}.$$

Now suppose $p(z) \neq 0$ for all z. Then $\frac{1}{p(z)}$ is also bounded on the disk $|z| \leq R$. Thus, $\frac{1}{p(z)}$ is a bounded entire function, and hence, by Liouville's Theorem, constant! Hence the polynomial is constant if it has no zeros. In other words, if p(z) is of degree at least one, there must be at least one z_0 for which $p(z_0) = 0$. This is, of course, the celebrated **fundamental theorem of algebra.**

6.4 Maximum Moduli

Suppose f is analytic on a closed domain D. Then, being continuous, $|f(z)|$ must attain its maximum value somewhere in this domain. Suppose this happens at an interior point. That is, suppose $|f(z)| \leq M$ for all $z \in D$ and suppose that $|f(z_0)| = M$ for some z_0 in the interior of D. Now z_0 is an interior point of D, so there is a number R such that the disk Λ centered at z_0 having radius R is included in D. Let C be a positively oriented circle of radius $\rho \leq R$ centered at z_0. From Cauchy's formula, we know

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s - z_0} ds.$$

Hence,

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + \rho e^{it})dt,$$

and so,

$$M = |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{it})| dt \leq M.$$

since $|f(z_0 + \rho e^{it})| \leq M$. This means

$$M = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{it})| dt.$$

Thus,

This integrand is continuous and non-negative, and so must be zero. In other words, $|f(z)| = M$ for all $z \in C$. There was nothing special about C except its radius $\rho \leq R$, and so we have shown that f must be constant on the disk Λ.

It is easy to see that if D is a region (i.e. connected and open), then the only way in which the modulus $|f(z)|$ of the analytic function *f* can attain a maximum on D is for *f* to be constant.

6.5 Summary

- Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its inside, and suppose z_0 is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz.$$

This is the famous Cauchy Integral Formula.

Notes

- Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the function G be defined by

$$G(z) = \int_C \frac{g(s)}{s-z} ds,$$

for all $z \notin C$. We shall show that G is analytic.

- Suppose f is analytic in a region D and suppose C is a positively oriented simple closed curve in D. Suppose also the inside of C is in D. Then from the Cauchy Integral formula, we know that

$$2\pi i f(z) = \int_C \frac{f(s)}{s-z} ds$$

and so with g = f in the formulas just derived, we have

$$f'(z) = \frac{1}{2pi}\int_C \frac{f(s)}{(s-z)^2} ds, \text{ and } f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3} ds$$

for all z inside the closed curve C. Meditate on these results. They say that the derivative of an analytic function is also analytic. Now suppose f is continuous on a domain D in which every point of D is an interior point and suppose that $\int_C f(z)dz = 0$, for every closed curve in D.

- Suppose f is entire and bounded; that is, f is analytic in the entire plane and there is a constant M such that $|f(z)| \le M$ for all z. Then it must be true that $f'(z) = 0$, identically. To see this, suppose that $f'(w) \neq 0$, for some w. Choose R large enough to insure that $\frac{M}{R} < |f'(w)|$. Now let C be a circle centered at 0 and with radius $\rho > \max\{R, |w|\}$. Then we have :

$$\frac{M}{R} < |f'(w)| \le \left|\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-w)^2} ds\right|$$

$$\le \frac{1}{2\pi}\frac{M}{\rho^2} 2\pi\rho = \frac{M}{\rho},$$

a contradiction. It must therefore be true that there is no w for which $f'(w) \neq 0$; or, in other words, $f'(z) = 0$ for all z. This, of course, means that f is a constant function. What we have shown has a name, Liouville's Theorem:

The only bounded entire functions are the constant functions.

6.6 Keywords

Cauchy Integral Formula: Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its inside, and suppose z_0 is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0} dz.$$

This is the famous Cauchy Integral Formula.

Function: Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the function G be defined by

$$G(z) = \int_C \frac{g(s)}{s-z} ds$$

for all $z \notin C$. We shall show that G is analytic.

Fundamental theorem of algebra: In other words, if p(z) is of degree at least one, there must be at least one z_0 for which $p(z_0) = 0$. This is, of course, the celebrated fundamental theorem of algebra.

6.7 Self Assessment

1. Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its inside, and suppose z_0 is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

   This is the famous ....

2. It says that if f is analytic on and inside a simple closed curve and we know the values f(z) for every z on the ...., then we know the value for the function at every point inside the curve—quite remarkable indeed.

3. Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the .... G be defined by

$$G(z) = \int_C \frac{g(s)}{s-z} ds,$$

   for all $z \notin C$. We shall show that G is analytic.

4. If $f : D \to C$ is .... such that $\int_C f(z)dz = 0$ for every closed curve in D, then f is analytic in D.

5. Now suppose $p(z) \neq 0$ for all z. Then $\frac{1}{p(z)}$ is also bounded on the disk $|z| \leq R$. Thus, $\frac{1}{p(z)}$ is a bounded entire function, and hence, by .... , constant!

6. Suppose f is analytic on a closed domain D. Then, being continuous, $|f(z)|$ must attain its .... somewhere in this domain.

Notes

6.8 Review Questions

1. Suppose f and g are analytic on and inside the simple closed curve C, and suppose moreover that f(z) = g(z) for all z on C. Prove that f(z) = g(z) for all z inside C.

2. Let C be the ellipse $9x^2 + 4y^2 = 36$ traversed once in the counterclockwise direction. Define the function g by :

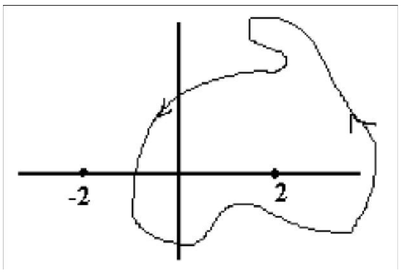
$$g(z) = \int_C \frac{s^2 + s + 1}{s - z}\, ds.$$

Find : (a) g(i) (b) g(4i)

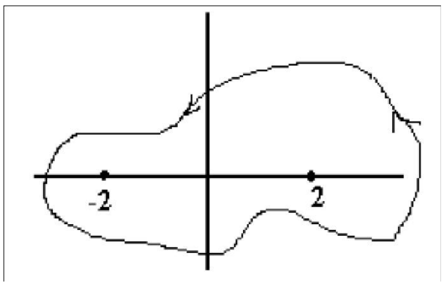
3. Find :

$$\int_C \frac{e^{2z}}{z^2 - 4} dz$$

where, C is the closed curve in the picture:



4. Find $\int_\Gamma \frac{e^{2z}}{z^2 - 4} dz,$ where Γ is the contour in the picture:



5. Evaluate :

$$\int_C \frac{\sin z}{z^2} dz$$

where, C is a positively oriented closed curve around the origin.

Notes

6. Let C be the circle $|z - i| = 2$ with the positive orientation. Evaluate :

(a) $\int_C \frac{1}{z^2+4}dz$ (b) $\int_C \frac{1}{(z^2+4)^2}dz$

7. Suppose f is analytic inside and on the simple closed curve C. Show that :

$$\int_C \frac{f'(z)}{z-w}dz = \int_C \frac{f(z)}{(z-w)^2}dz$$

for every $w \notin C$.

8. (a) Let α be a real constant, and let C be the circle $\gamma(t) = eit$, $-\pi \le t \le \pi$. Evaluate :

$$\int_C \frac{e^{az}}{z}dz.$$

(b) Use your answer in part a) to show that :

$$\int_0^\pi e^{\alpha\cos t}\cos(\alpha \sin t)dt = \pi.$$

9. Suppose f is an entire function, and suppose there is an M such that $\text{Re}f(z) \le M$ for all z. Prove that f is a constant function.

10. Suppose w is a solution of $5z^4 + z^3 + z^2 - 7z + 14 = 0$. Prove that $|w| \le 3$.

11. Prove that, if p is a polynomial of degree n, and if $p(a) = 0$, then $p(z) = (z - a)q(z)$, where q is a polynomial of degree $n - 1$.

12. Prove that, if p is a polynomial of degree $n \ge 1$, then

$$p(z) = c(z - z_1)^{k_1} (z - z_2)k^2 \dots (z - zj)^{k_j},$$

where $k_1, k_2, \dots, k_j$ are positive integers such that n ƒ- k1 ƒy k2 ƒyƒ∑ ƒykj.

13. Suppose p is a polynomial with real coefficients. Prove that p can be expressed as a product of linear and quadratic factors, each with real coefficients.

14. Suppose f is analytic and not constant on a region D and suppose $f(z) \ne 0$ for all $z \in D$. Explain why $|f(z)|$ does not have a minimum in D.

15. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic on a region D. Prove that if $u(x, y)$ attains a maximum value in D, then u must be constant.

Answers: Self Assessment

1.	Cauchy Integral Formula.	2.	simple closed curve
3.	function	4.	continuous
5.	Liouville's Theorem	6.	maximum value

Notes

6.9 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 7: Transformations and Conformal Mappings

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define transformation
- Discuss bilinear transformation
- Describe transformation of a circle
- Explain conformal mapping

Introduction

In earlier unit you have studied about concept of Cauchy's theorem, Cauchy's integral formula, functions defined by integrals and maximum moduli. A conformal map is a function which preserve angles. In the most common case, the function between domain is in the complex plane. This unit will explain you the concept of transformation and conformal mapping.

7.1 Transformations

Here, we shall study how various curves and regions are mapped by elementary analytic function. We shall work in $\not\subset_\infty$ i.e. the extended complex plane. We start with the linear function.

$$w = Az \qquad \ldots(1)$$

Notes

where A is non zero complex constant and $z \neq 0$. We write A and z in exponential form as

$$A = ae^{i\alpha}, \qquad z = re^{i\theta}$$

Then
$$w = (ar)\, e^{i(\alpha + \theta)} \qquad \ldots(2)$$

Thus we observe from (2) that transformation (1) expands (or contracts) the radius vector representing z by the factor $a = | A |$ and rotates it through an angle $\alpha = \arg A$ about the origin. The image of a given region is, therefore, geometrically similar to that region. The general linear transformation,

$$w = Az + B \qquad \ldots(3)$$

is evidently an expansion or contraction and a rotation, followed by a translation. The image region mapped by (3) is geometrically congruent to the original one.

Now we consider the function,

$$w = \frac{1}{z} \qquad \ldots(4)$$

which establishes a one to one correspondence between the non zero points of the z–plane and the w–plane. Since $z\bar{z} = | z |^2$, the mapping can be described by means of the successive transformations

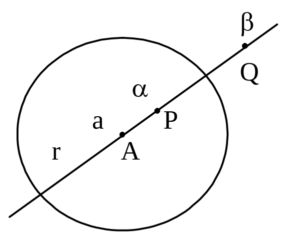
$$Z = \frac{1}{|z|^2} z, \qquad w = \bar{Z} \qquad \ldots(5)$$

Geometrically, we know that if P and Q are inverse points w.r.t. a circle of radius r with centre A, then

$$(AP)\,(AQ) = r^2$$

Thus a and b are inverse points w.r.t. the circle $| z - a | = r$ if

$$(\alpha - a)\,\overline{(\beta - a)} = r^2$$

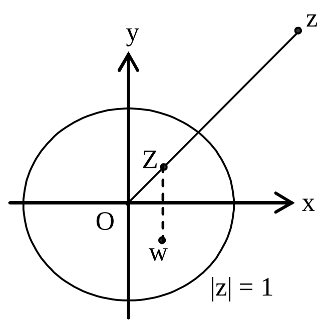


where the pair $\alpha = a$, $\beta = \infty$ is also included. We note that α, β, a are collinear. Also points α and β are inverse w.r.t a straight line *l* if β is the reflection of a in *l* and conversely. Thus, the first of the transformation in (5) is an inversion w.r.t the unit circle $| z | = 1$ i.e. the image of a non-zero point z is the point Z with the properties

$$z\bar{Z} = 1, \qquad | Z | = \frac{1}{|z|} \text{ and } \arg Z = \arg z$$

Thus, the point exterior to the circle $| z | = 1$ are mapped onto the non-zero points interior to it and conversely. Any point on the circle is mapped onto itself. The second of the transformation in (5) is simply a reflection in the real axis.

Notes



Since $\lim_{z\to 0} \frac{1}{z} = \infty$ and $\lim_{z\to\infty} \frac{1}{z} = 0$,

it is natural to define a one–one transformation w = T(z) from the extended z plane onto the extended w plane by writing

$$T(0) = \infty, \quad T(\infty) = 0$$

and $$T(z) = \frac{1}{z}$$

for the remaining values of z. It is observed that T is continuous throughout the extended z plane.

When a point w = u + iv is the image of a non-zero point z = x + iy under the transformation $w = \frac{1}{z}$, writing $w = \frac{\bar{z}}{|z|^2}$ results in

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2} \qquad \ldots(6)$$

Also, since $z = \frac{1}{w} = \frac{\bar{w}}{|w|^2}$, we get

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2} \qquad \ldots(7)$$

The following argument, based on these relations (6) and (7) between co–ordinates shows the important result that the mapping $w = \frac{1}{z}$ transforms circles and lines into circles and lines.

When a, b, c, d are real numbers satisfying the condition $b^2 + c^2 > 4ad$, then the equation

$$a(x^2 + y^2) + bx + cy + d = 0 \qquad \ldots(8)$$

represents an arbitrary circle or line, where $a \neq 0$ for a circle and a = 0 for a line.

If x and y satisfy equation (8), we can use relations (7) to substitute for these variables. Thus, using (7) in (8), we obtain

$$d(u^2 + v^2) + bu - cv + a = 0 \qquad \ldots(9)$$

Notes

which also represents a circle or a line. Conversely, if u and v satisfy (9), it follows from (6) that x and y satisfy (8). From (8) and (9), it is clear that

(i) a circle ($a \neq 0$) not passing through the origin ($d \neq 0$) in the z plane is transformed into a circle not passing through the origin in the w plane.

(ii) a circle ($a \neq 0$) through the origin ($d = 0$) in the z plane is transformed into a line which does not pass through the origin in the w plane.

(iii) a line ($a = 0$) not passing through the origin ($d \neq 0$) in the z plane is transformed into a circle through the origin in the w plane.

(iv) a line ($a = 0$) through the origin ($d = 0$) in the z plane is transformed into a line through the origin in the w plane.

Hence, we conclude that $w = \frac{1}{z}$ transforms circles and lines into circles and lines respectively.

Remark : In the extended complex plane, a line may be treated as a circle with infinite radius.

7.1.1 Bilinear Transformation

The transformation

$$w = \frac{az+b}{cz+d}, \ ad - bc \neq 0 \qquad \ldots(1)$$

where a, b, c, d are complex constants, is called bilinear transformation or a linear fractional transformation or Möbius transformation. We observe that the condition $ad - bc \neq 0$ is necessary for (1) to be a bilinear transformation, since if

$$ad - bc = 0, \text{ then } \frac{b}{a} = \frac{d}{c} \text{ and we get}$$

$$w = \frac{a(z+b/a)}{c(z+d/c)} = \frac{a}{c} \text{ i.e. we get a constant function which is not linear.}$$

Equation (1) can be written in the form

$$cwz + dw - az - b = 0 \qquad \ldots(2)$$

Since (2) is linear in z and linear in w or bilinear in z and w, therefore, (1) is termed as bilinear transformation.

When $c = 0$, the condition $ad - bc \neq 0$ becomes $ad \neq 0$ and we see that the transformation reduces to general linear transformation. When $c \neq 0$, equation (1) can be written as

$$w = \frac{a\,(z+b/a)}{c\,(z+d/c)} = \frac{a}{c}\left[1 + \frac{b/a - d/c}{z+d/c}\right]$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2} = \frac{1}{z+d/c} \qquad \ldots(3)$$

We note that (3) is a composition of the mappings

$$z_1 = z + \frac{d}{c}, \quad z_2 = \frac{1}{z_1}, \quad z_3 = \frac{bc-ad}{c^2}\, z_2$$

and thus, we get $w = \frac{a}{c} + z_3$.

The above three auxiliary transformations are of the form

$$w = z + \alpha, \quad w = \frac{1}{z}, \quad w = \beta z \qquad \ldots(4)$$

Hence, every bilinear transformation is the resultant of the transformations in (4).

But we have already discussed these transformations and thus, we conclude that a bilinear transformation always transforms circles and lines into circles and lines respectively, because the transformations in (4) do so.

From (1), we observe that if c = 0, a, d ≠ 0, each point in the w plane is the image of one and only one point in the z-plane. The same is true if c ≠ 0, except when $z = -\frac{d}{c}$ which makes the denominator zero. Since we work in extended complex plane, so in case $z = -\frac{d}{c}$, $w = \infty$ and thus, we may regard the point at infinity in the w–plane as corresponding to the point $z = -\frac{d}{c}$ in the z–plane.

Thus, if we write

$$T(z) = w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \qquad \ldots(5)$$

Then, $T(\infty) = \infty, \quad \text{if } c = 0$

and $T(\infty) = \frac{a}{c}, \quad T\left(-\frac{d}{c}\right) = \infty, \quad \text{if } c \neq 0$

Thus, T is continuous on the extended z-plane. When the domain of definition is enlarged in this way, the bilinear transformation (5) is one–one mapping of the extended z-plane onto the extended w-plane.

Hence, associated with the transformation T, there is an inverse transformation T^{-1} which is defined on the extended w-plane as

$$T^{-1}(w) = z \text{ if and only if } T(z) = w.$$

Thus, when we solve equation (1) for z, then

$$z = \frac{-dw+b}{cw-a}, \text{ ad bc} \neq 0 \qquad \ldots(6)$$

and thus,

$$T^{-1}(w) = z = \frac{-dw+b}{cw-a}, \text{ ad bc} \neq 0$$

Evidently, T^{-1} is itself a bilinear transformation, where

$$T^{-1}(\infty) = \infty \quad \text{if } c = 0$$

Notes

and $$T^{-1}\left(\frac{a}{c}\right) = \infty, \quad T^{-1}(\infty) = -\frac{d}{c}, \text{ if } c \neq 0$$

From the above discussion, we conclude that inverse of a bilinear transformation is bilinear. The points $z = -\frac{d}{c}$ $(w = \infty)$ and $z = \infty(w = \frac{a}{c})$ are called critical points.

Theorem

Composition (or resultant or product) of two bilinear transformations is a bilinear transformation.

Proof. We consider the bilinear transformations

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \qquad \ldots(1)$$

and $$w_1 = \frac{a_1 w + b_1}{c_1 w + d_1}, \quad a_1 d_1 - b_1 c_1 \neq 0 \qquad \ldots(2)$$

Putting the value of w from (1) in (2), we get

$$w_1 = \frac{a_1\left(\frac{az+b}{cz+d}\right) + b_1}{c_1\left(\frac{az+b}{cz+d}\right) + d_1} = \frac{(a_1 a + b_1 c)z + (b_1 d + a_1 b)}{(c_1 a + d_1 c)z + (d_1 d + c_1 b)}$$

Taking $A = a_1 a + b_1 c, \quad B = b_1 d + a_1 b,$

$C = c_1 a + d_1 c, \quad D = d_1 d + c_1 b$, we get

$$w_1 = \frac{Az+B}{Cz+D}$$

Also
$$\begin{aligned}
AD - BC &= (a_1 a + b_1 c)(d_1 d + c_1 b) - (b_1 d + a_1 b)(c_1 a + d_1 c) \\
&= (a_1 a d_1 d + a_1 a c_1 b + b_1 c d_1 d + b_1 c c_1 b) - (b_1 d c_1 a + b_1 d\, d_1 c + a_1 b c_1 a + a_1 b d_1 c) \\
&= a_1 a d_1 d + b_1 b c_1 c - b_1 d c_1 a - a_1 b d_1 c \\
&= ad(a_1 d_1 - b_1 c_1) - bc(a_1 d_1 - b_1 c_1) \\
&= (ad - bc)(a_1 d_1 - b_1 c_1) \neq 0
\end{aligned}$$

Thus $$w_1 = \frac{Az+B}{Cz+D}, \quad AD - BC \neq 0$$

is a bilinear transformation.

This bilinear transformation is called the resultant (or product or composition) of the bilinear transformations (1) and (2).

The above property is also expressed by saying that bilinear transformations form a group.

7.1.2 Definitions

Notes

(i) The points which coincide with their transforms under bilinear transformation are called its fixed points. For the bilinear transformation $w = \frac{az+b}{cz+d}$, fixed points are given by

$$w = z \qquad \text{i.e. } z = \frac{az+b}{cz+d} \qquad \ldots(1)$$

Since (1) is a quadratic in z and has in general two different roots, therefore, there are generally two invariant points for a bilinear transformation.

(ii) If z_1, z_2, z_3, z_4 are any distinct points in the z-plane, then the ratio

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is called cross ratio of the four points z_1, z_2, z_3, z_4. This ratio is invariant under a bilinear transformation i.e.

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

7.1.3 Transformation of a Circle

First we show that if p and q are two given points and K is a constant, then the equation

$$\left|\frac{z-p}{z-q}\right| = K, \qquad (1)$$

represents a circle. For this, we have

$$|z-p|^2 = K^2|z-q|^2$$

$$\Rightarrow \quad (z-p)\left(\overline{z-p}\right) = K^2(z-q)\left(\overline{z-q}\right)$$

$$\Rightarrow \quad (z-p)(\bar{z}-\bar{p}) = K^2(z-q)(\bar{z}-\bar{q})$$

$$\Rightarrow \quad z\bar{z} - \bar{p}z - p\bar{z} + p\bar{p} = K^2(z\bar{z} - \bar{q}z - q\bar{z} + q\bar{q})$$

$$\Rightarrow \quad (1-K^2)z\bar{z} - (p-qK^2)\bar{z} - (\bar{p}-\bar{q}K^2)z = K^2q\bar{q} - p\bar{p}$$

$$\Rightarrow \quad z\bar{z} - \left(\frac{p-qK^2}{1-K^2}\right)\bar{z} - \left(\frac{\bar{p}-\bar{q}K^2}{1-K^2}\right)z + \frac{|p|^2 - K^2|q|^2}{1-K^2} = 0 \qquad \ldots(2)$$

Equation (2) is of the form

$$z\bar{z} + b\bar{z} + \bar{b}z + c = 0 \qquad \text{(c is being a real constant)}$$

which always represents a circle.

Thus equation (2) represents a circle if $K \neq 1$.

If K = 1, then it represents a straight line

$$|z-p| = |z-q|$$

Notes

Further, we observe that in the form (1), p and q are inverse points w.r.t. the circle. For this, if the circle is $|z - z_0| = r$ and p and q are inverse points w.r.t. it, then

$$z - z_0 = re^{i\theta}, \quad p - z_0 = qe^{i\lambda},$$

$$q - z_0 = \frac{\rho^2}{a} e^{i\lambda}$$

Therefore,

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{\rho e^{i\theta} - ae^{i\lambda}}{\rho e^{i\theta} - \dfrac{\rho^2}{a} e^{i\lambda}}\right| = \frac{a}{\rho}\left|\frac{\rho e^{i\theta} - ae^{i\lambda}}{ae^{i\theta} - \rho e^{i\lambda}}\right|$$

$$= K \left|\frac{\rho(\cos\theta + i\sin\theta) - a(\cos\lambda + i\sin\lambda)}{a(\cos\theta + i\sin\theta) - \rho(\cos\lambda + i\sin\lambda)}\right|, K = \frac{a}{\rho}$$

$$= K \left|\frac{(\rho\cos\theta - a\cos\lambda) + i(\rho\sin\theta - a\sin\lambda)}{(a\cos\theta - \rho\cos\lambda) + i(a\sin\theta - \rho\sin\lambda)}\right|$$

$$= K \left[\frac{(\rho\cos\theta - a\cos\lambda)^2 + (\rho\sin\theta - a\sin\lambda)^2}{(a\cos\theta - \rho\cos\lambda)^2 + (a\sin\theta - \rho\sin\lambda)^2}\right]^{1/2}$$

$= K$, where $K \neq 1$, since $a \neq r$

Thus, if p and q are inverse points w.r.t. a circle, then its equation can be written as

$$\left|\frac{z-p}{z-q}\right| = K, \quad K \neq 1, \text{ K being a real constant.}$$

Theorem

In a bilinear transformation, a circle transforms into a circle and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetric about the line.

Proof : We know that $\left|\frac{z-p}{z-q}\right| = K$ represents a circle in the z-plane with p and q as inverse points, where $K \neq 1$. Let the bilinear transformation be

$$w = \frac{az+b}{cz+d} \quad \text{so that} \quad z = \frac{dw-b}{-cw+a}$$

Then under this bilinear transformation, the circle transforms into

$$\left|\frac{\dfrac{dw-b}{-cw+a} - p}{\dfrac{dw-b}{-cw+a} - q}\right| = K \quad \Rightarrow \quad \left|\frac{dw-b-p(q-cw)}{dw-b-q(a-cw)}\right| = K$$

Notes

$$\Rightarrow \quad \left|\frac{w(d+cp)-(ap+b)}{w(d+cq)-(aq+b)}\right| = K \quad \Rightarrow \quad \left|\frac{w-\dfrac{ap+b}{cp+d}}{w-\dfrac{aq+b}{cq+d}}\right| = K\left|\frac{cq+d}{cp+d}\right| \qquad \ldots(1)$$

The form of equation (1) shows that it represents a circle in the w-plane whose inverse points are $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$. Thus, a circle in the z-plane transforms into a circle in the w-plane and the inverse points transform into the inverse points.

Also if $K\left|\frac{cq+d}{cp+d}\right| = 1$, then equation (1) represents a straight line bisecting at right angle the join of the points $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$ so that these points are symmetric about this line. Thus, in a particular case, a circle in the z-plane transforms into a straight line in the w-plane and the inverse points transform into points symmetrical about the line.

Example: Find all bilinear transformations of the half plane Im $z \geq 0$ into the unit circle $|w| \leq 1$.

Solution. We know that two points z, $\bar{z}$, symmetrical about the real z-axis(Im z = 0) correspond to points w, $\frac{1}{\bar{w}}$, inverse w.r.t. the unit w–circle. ($|w|\ \frac{1}{|\bar{w}|} = 1$). In particular, the origin and the point at infinity in the w-plane correspond to conjugate values of z.

Let

$$w = \frac{az+b}{cz+d} = \frac{a(z+b/a)}{c\,(z+d/c)} \qquad \ldots(1)$$

be the required transformation.

Clearly $c \neq 0$, otherwise points at ∞ in the two planes would correspond.

Also, $w = 0$ and $w = \infty$ are the inverse points w.r.t. $|w| = 1$. Since in (1), $w = 0$, $w = \infty$ correspond respectively to $z = -\frac{b}{a}$, $z = -\frac{d}{c}$, therefore, these two values of z–plane must be conjugate to each other. Hence, we may write

$$-\frac{b}{a} = \alpha,\ -\frac{d}{c} = \bar{\alpha} \qquad \text{so that}$$

$$w = \frac{a}{c}\,\frac{z-\alpha}{z-\bar{\alpha}} \qquad (2)$$

The point $z = 0$ on the boundary of the half plane Im $z \geq 0$ must correspond to a point on the boundary of the circle $|w| = 1$, so that

$$1 = |w| = \left|\frac{a}{c}\right|\left|\frac{0-\alpha}{0-\bar{\alpha}}\right| = \left|\frac{a}{c}\right|$$

Notes

$\Rightarrow \qquad \frac{a}{c} = e^{i\lambda} \quad \Rightarrow \quad a = ce^{i\lambda}$, where λ is real.

Thus, we get

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \qquad \ldots(3)$$

Since $z = \alpha$ gives $w = 0$, α must be a point of the upper half plane i.e. Im $\alpha > 0$. With this condition, (3) gives the required transformation. In (3), if z is real, obviously $|w| = 1$ and if Im $z > 0$, then z is nearer to α than to $\bar{\alpha}$ and so $|w| < 1$. Hence, the general linear transformation of the half plane Im $z \geq 0$ on the circle $|w| \leq 1$ is

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right), \quad \text{Im } \alpha > 0.$$

Example: Find all bilinear transformations of the unit $|z| \leq 1$ into the unit circle $|w| \leq 1$.

OR

Find the general homographic transformations which leaves the unit circle invariant.

Solution. Let the required transformation be

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\frac{(z+b/a)}{(z+d/c)} \qquad \ldots(1)$$

Here, $w = 0$ and $w = \infty$, correspond to inverse points

$$z = -\frac{b}{a}, \quad z = -\frac{d}{c}, \quad \text{so we may write}$$

$$-\frac{b}{a} = \alpha, \quad -\frac{d}{c} = \frac{1}{\bar{\alpha}} \quad \text{such that } |a| < 1.$$

So,
$$w = \frac{a}{c}\left(\frac{z-\alpha}{z-1/\alpha}\right) = \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right) \qquad \ldots(2)$$

The point $z = 1$ on the boundary of the unit circle in z-plane must correspond to a point on the boundary of the unit circle in w-plane so that

$$1 = |w| = \left|\frac{a\bar{\alpha}}{c}\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{a\bar{\alpha}}{c}\right|$$

or $a\,\bar{\alpha} = c\,e^{i\lambda}$, where λ is real.

Hence (2) becomes,

$$w = e^{il}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right), \quad |a| < 1 \qquad \ldots(3)$$

This is the required transformation, for if $z = e^{i\theta}$, $a = be^{i\beta}$, then

$$|w| = \left|\frac{e^{i\theta} - be^{i\beta}}{be^{i(\theta-\beta)} - 1}\right| = 1.$$

If $z = re^{i\theta}$, where $r < 1$, then

$$|z - \alpha|^2 - |\bar{\alpha} z - 1|^2 = r^2 - 2rb\cos(\theta - \beta) + b^2 - \{b^2r^2 - 2br\cos(\theta - \beta) + 1\}$$

$$= (r^2 - 1)(1 - b^2) < 0$$

and so

$$|z - a|^2 < |z - 1|^2 \quad \Rightarrow \quad \frac{|z-\alpha|^2}{|\bar{\alpha}z - 1|^2} < 1$$

i.e. $\qquad |w| < 1$

Hence, the result.

Example: Show that the general transformation of the circle $|z| \leq \rho$ into the circle $|w| \leq \rho'$ is

$$w = \rho\rho' e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z - \rho^2}\right), \quad |a| < \rho$$

Solution. Let the transformation be

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\left(\frac{z+b/a}{z+d/c}\right) \qquad \ldots(1)$$

The points $w = 0$ and $w = \infty$, inverse points of $|w| = \rho'$ correspond to inverse point $z = -b/a$, $z = -d/c$ respectively of $|z| = r$, so we may write

$$-\frac{b}{a} = \alpha, \quad -\frac{d}{c} = \frac{\rho^2}{\bar{\alpha}}, \quad |a| < \rho$$

Thus, from (1), we get

$$w = \frac{a}{c}\left(\frac{z-\alpha}{z - \dfrac{\rho^2}{\bar{\alpha}}}\right) = \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z - \rho^2}\right) \qquad \ldots(2)$$

Equation (2) satisfied the condition $|z| \leq \rho$ and $|w| \leq \rho'$. Hence, for $|z| = r$, we must have $|w| = \rho'$ so that (2) becomes

$$\rho' = |w| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{z-\alpha}{\bar{\alpha}z - z\bar{z}}\right|, \quad z\bar{z} = r^2$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1}{z}\right|\left|\frac{z-\alpha}{\bar{z}-\bar{\alpha}}\right| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1}{z}\right|\left|\frac{z-\alpha}{\overline{z-\alpha}}\right|$$

Notes

$$= \left|\frac{a\bar{\alpha}}{c}\right|\frac{1}{\rho}, \quad |z-a| = |\overline{z-\alpha}|$$

$\Rightarrow$ $$\rho\rho' = \left|\frac{a\bar{\alpha}}{c}\right| \quad \Rightarrow \quad \frac{a\bar{\alpha}}{c} = \rho\rho' e^{i\lambda}, \ \lambda \text{ being real.}$$

Thus, the required transformation becomes

$$w = \rho\rho' e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-\rho^2}\right), \quad |a| < \rho.$$

Example: Find the bilinear transformation which maps the point 2, i, –2 onto the points 1, i, –1.

Solution. Under the concept of cross–ratio, the required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Using the values of z_i and w_i, we get

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

or $$\frac{w-1}{w+1} = \left(\frac{z-2}{z+2}\right)\left(\frac{2+i}{2-i}\right)\left(\frac{1-i}{1+i}\right)$$

or $$\frac{w-1}{w+1} = \frac{4-3i}{5}\frac{z-2}{z+2}$$

or $$\frac{w-1+w+1}{w-1-(w+1)} = \frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

or $$-w = \frac{3z(3-i)+2i(3-i)}{-iz(z-i)-6(3-i)} = \frac{3z+2i}{-(iz+6)}$$

or $$w = \frac{3z+2i}{iz+6}$$

which is the required transformation.

7.2 Conformal Mappings

Let S be a domain in a plane in which x and y are taken as rectangular Cartesian co-ordinates. Let us suppose that the functions u(x, y) and v(x, y) are continuous and possess continuous partial derivatives of the first order at each point of the domain S. The equations

$$u = u(x, y), \qquad v = v(x, y)$$

set up a correspondence between the points of S and the points of a set T in the (u, v) plane. The set T is evidently a domain and is called a map of S. Moreover, since the first order partial derivatives of u and v are continuous, a curve in S which has a continuously turning tangent is

Notes

mapped on a curve with the same property in T. The correspondence between the two domains is not, however, necessarily a one–one correspondence.

For example, if we take $u = x^2$, $v = y^2$, then the domain $x^2 + y^2 < 1$ is mapped on the triangle bounded by $u = 0$, $v = 0$, $u + v = 1$, but there are four points of the circle corresponding to each point of the triangle.

7.2.1 Definition

A mapping from S to T is said to be isogonal if it has a one–one transformation which maps any two intersecting curves of S into two curves of T which cut at the same angle. Thus, in an isogonal mapping, only the magnitude of angle is preserved.

An isogonal transformation which also conserves the sense of rotation is called conformal mapping. Thus, in a conformal transformation, the sense of rotation as well as the magnitude of the angle is preserved.

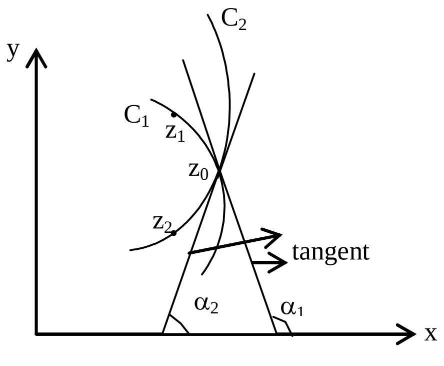
The following theorem provides the necessary condition of conformity which briefly states that if f(z) is analytic, mapping is conformal.

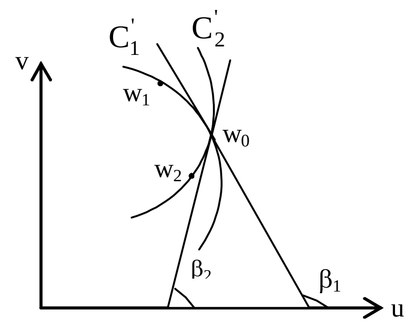
Theorem: Prove that at each point z of a domain D where f(z) is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

Proof. Let $w = f(z)$ be an analytic function of z, regular and one valued in a region D of the z-plane. Let z_0 be an interior point of D and let C_1 and C_2 be two continuous curves passing through z_0 and having definite tangents at this point, making angles a_1, a_2, say, with the real axis.

We have to discover what is the representation of this figure in the w-plane. Let z_1 and z_2 be points on the curves C_1 and C_2 near to z_0. We shall suppose that they are at the same distance r from z_0, so we can write $z_1 - z_0 = re^{i\theta 1}$, $z_2 - z_0 = re^{i\theta 2}$.

Then as $r \to 0$, $\theta_1 \to \alpha_1$, $q_2 \to \alpha_2$. The point z_0 corresponds to a point w_0 in the w-plane and z_1 and z_2 correspond to point w_1 and w_2 which describe curves C'_1 and C_2', making angles b_1 and b_2 with the real axis.





Let $w_1 - w_0 = \rho_1\ e^{i\phi_1}$, $w_2 - w_0 = \rho_2\ e^{i\phi_2}$,

where $\rho_1, \rho_2 \to 0 \quad \Rightarrow \quad \phi_1, \phi_2 \to \beta_1, \beta_2$, respectively.

Now, by the definition of an analytic function,

$$\lim_{z \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = f'(z_0)$$

Notes

Since $f'(z_0) \neq 0$, we may write it in the form $Re^{i\lambda}$ and thus,

$$\lim \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}} = Re^{i\lambda} \quad \text{i.e. } \lim \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)} = Re^{i\lambda}$$

$\Rightarrow$ $$\lim \frac{\rho_1}{r} = R = |f'(z_0)|$$

and $$\lim (\phi_1 - \theta_1) = \lambda$$

i.e. $$\lim \phi_1 - \lim \theta_1 = \lambda$$

i.e. $$\beta_1 - \alpha_1 = \lambda \quad \Rightarrow \quad \beta_1 = \alpha_1 + \lambda$$

Similarly, $\beta_2 = \alpha_2 + \lambda$.

Hence, the curves C'_1 and C'_2 have definite tangents at w_0 making angles $\alpha_1 + \lambda$ and $\alpha_2 + \lambda$ respectively with the real axis. The angle between C'_1 and C'_2 is

$$\beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 - \lambda) = \alpha_1 - \alpha_2$$

which is the same as the angle between C_1 and C_2. Hence the curve C_1' and C_2' intersect at the same angle as the curves C_1 and C_2. Also the angle between the curves has the same sense in the two figures. So the mapping is conformal.

Special Case : When $f'(z_0) = 0$, we suppose that $f'(z)$ has a zero of order n at the point z_0. Then in the neighbourhood of this point (by Taylor's theorem)

$$f(z) = f(z_0) + a(z - z_0)^{n+1} + \ldots, \text{ where } a \neq 0$$

Hence, $$w_1 - w_0 = a(z - z_0)^{n+1} + \ldots.$$

i.e. $$\rho_1 \, e^{i\phi_1} = |a| \, r^{n+1} \, e^{i[d + (n+1)q1]} + \ldots$$

where, $$\delta = \arg a$$

Hence, $$\lim \phi_1 = [d + (n + 1)\, \theta_1] = \delta + (n + 1)\, \alpha_1 \qquad | \ \delta \text{ is constant}$$

Similarly, $$\lim \phi_2 = d + (n + 1)\, \alpha_2$$

Thus, the curves C'_1 and C'_2 still have definite tangent at w_0, but the angle between the tangents is

$$\lim(\phi_2 - \phi_1) = (n + 1)\, (\alpha_2 - \alpha_1)$$

Thus, the angle is magnified by (n + 1).

Also the linear magnification, $$R = \lim \frac{\rho_1}{r} = 0 \qquad \left| \because \lim \frac{\rho_1}{r} = R = |f'(z_0)| = 0\right.$$

Therefore, the conformal property does not hold at such points where $f'(z) = 0$

A point z_0 at which $f'(z_0) = 0$ is called a critical point of the mapping. The following theorem is the converse of the above theorem and is sufficient condition for the mapping to be conformal.

Theorem: If the mapping $w = f(z)$ is conformal then show that $f(z)$ is an analytic function of z.

Proof. Let $w = f(z) = u(x, y) + iv(x, y)$

Here, $u = u(x, y)$ and $v = v(x, y)$ are continuously differentiable equations defining conformal transformation from z-plane to w-plane. Let ds and $d\sigma$ be the length elements in z-plane and w-plane respectively so that

$$ds^2 = dx^2 + dy^2, \quad d\sigma^2 = du^2 + dv^2 \qquad \ldots(1)$$

Notes

Since u, v are functions of x and y, therefore

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\therefore \qquad du^2 + dv^2 = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2$$

i.e.

$$d\sigma^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]dx^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dy^2$$

$$+ 2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)dxdy \qquad \ldots(2)$$

Since the mapping is given to be conformal, therefore, the ratio $d\sigma^2 : d\sigma^2$ is independent of direction, so that from (1) and (2), comparing the coefficients, we get

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0}$$

$$\Rightarrow \qquad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \qquad \ldots(3)$$

and

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \qquad \ldots(4)$$

Equations (3) and (4) are satisfied if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \ldots(5)$$

or

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \ldots(6)$$

Equation (6) reduces to (5) if we replace v by –v i.e. by taking as image figure obtained by the reflection in the real axis of the w-plane.

Thus, the four partial derivatives u_x, u_y, $v_{x-,}$ v_y exist, are continuous and they satisfy C–R equations (5). Hence, f(z) is analytic.

Remarks

(i) The mapping w = f(z) is conformal in a domain D if it is conformal at each point of the domain.

(ii) The conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shapes. Smaller portions of these domains and curves are conformally mapped by analytic function to well-known domains and curves.

Notes

Example: Discuss the mapping $w = \bar{z}$.

Solution. We observe that the given mapping replaces every point by its reflection in the real axis. Hence, angles are conserved but their signs are changed and thus, the mapping is isogonal but not conformal. If the mapping $w = \bar{z}$ is followed by a conformal transformation, then resulting transformation of the form $w = f(\bar{z})$ is also isogonal but not conformal, where f(z) is analytic function of z.

Example: Discuss the nature of the mapping $w = z^2$ at the point $z = 1 + i$ and examine its effect on the lines Im z = Re z and Re z = 1 passing through that point.

Solution. We note that the argument of the derivative of $f(z) = z^2$ at $z = 1 + i$ is

$$[\arg 2z]_{z=1+i} = \arg(2 + 2i) = \pi/4$$

Hence, the tangent to each curve through $z = 1 + i$ will be turned by the angle $\pi/4$. The co-efficient of linear magnification is $|f'(z)|$ at $z = 1 + i$, i.e. $|2 + 2i| = 2\sqrt{2}$. The mapping is

$$w = z^2 = x^2 - y^2 + 2ixy = u(x, y) + iv(x, y)$$

We observe that mapping is conformal at the point $z = 1 + i$, where the half lines $y = x(y \geq 0)$ and $x = 1(y \geq 0)$ intersect. We denote these half lines by C_1 and C_2, with positive sense upwards and observe that the angle from C_1 to C_2 is $\pi/4$ at their point of intersection. We have

$$u = x^2 - y^2, \quad v = 2xy$$

The half line C_1 is transformed into the curve C'_1 given by

$$u = 0, \quad v = 2y^2 \,(y \geq 0)$$

Thus, C'_1 is the upper half $v \geq 0$ of the v–axis.

The half line C_2 is transformed into the curve C'_2 represented by

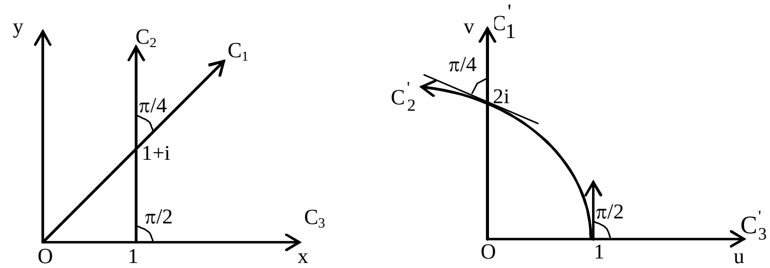
$$u = 1 - y^2, \quad v = 2y \ (y \geq 0)$$

Hence, C'_2 is the upper half of the parabola $v^2 = -4(u - 1)$. We note that, in each case, the positive sense of the image curve is upward.

For the image curve C'_2,

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v}$$

In particular, $\frac{dv}{du} = -1$ when $v = 2$. Consequently, the angle from the image curve C'_1 to the image curve C'_2 at the point $w = f(1 + i) = 2i$ is $\frac{\pi}{4}$, as required by the conformality of the mapping there.



Notes The angle of rotation and the scalar factor (linear magnification) can change from point to point. We note that they are 0 and 2 respectively, at the point z = 1, since f ′(1) = 2, where the curves C_2 and C'_2 are the same as above and the non-negative x-axis (C_3) is transformed into the non-negative u-axis (C'_3).

Example: Discuss the mapping $w = z^a$, where a is a positive real number.

Solution. Denoting z and w in polar as $z = re^{i\theta}$, $w = re^{i\phi}$, the mapping gives $r = r^a$, $\phi = a\theta$.

Thus the radii vectors are raised to the power a and the angles with vertices at the origin are multiplied by the factor a. If a > 1, distinct lines through the origin in the z–plane are not mapped onto distinct lines through the origin in the w-plane, since, e.g. the straight line through the origin at an angle $\frac{2\pi}{a}$ to the real axis of the z-plane is mapped onto a line through the origin in the w–plane at an angle 2π to the real axis i.e. the positive real axis itself. Further, $\frac{dw}{dz} = az^{a-1}$, which vanishes at the origin if a > 1 and has a singularity at the origin if a < 1. Hence, the mapping is conformal and the angles are therefore preserved, excepting at the origin. Similarly the mapping $w = e^z$ is conformal.

Example: Prove that the quadrant | z | < 1, 0 < arg z < $\frac{\pi}{2}$ is mapped conformally onto a domain in the w-plane by the transformation $w = \frac{4}{(z+1)^2}$.

Solution. If $w = f(z) = \frac{4}{(z+1)^2}$, then f (z) is finite and does not vanish in the given quadrant. Hence, the mapping w = f(z) is conformal and the quadrant is mapped onto a domain in the w-plane provided w does not assume any value twice i.e. distinct points of the quadrant are mapped to distinct points of the w-plane. We show that this indeed is true. If possible, let $\frac{4}{(z_1+1)^2} = \frac{4}{(z_2+1)^2}$, where $z_1 \neq z_2$ and both z_1 and z_2 belong to the quadrant in the z-plane. Then, since $z_1 \neq z_2$, we have $(z_1 - z_2)(z_1 + z_2 + 2) = 0$

$\Rightarrow z_1 + z_2 + 2 = 0$ i.e. $z_1 = -z_2 - 2$. But since z_2 belongs to the quadrant, $-z_2 - 2$ does not, which contradicts the assumption that z_1 belongs to the quadrant. Hence w does not assume any value twice.

7.3 Summary

- Here, we shall study how various curves and regions are mapped by elementary analytic function. We shall work in $\mathbb{C}_\infty$ i.e. the extended complex plane. We start with the linear function.

$$w = Az \qquad \ldots(1)$$

where A is non-zero complex constant and $z \neq 0$. We write A and z in exponential form as

$$A = ae^{i\alpha}, \qquad z = re^{i\theta}$$

Notes

Then, $$w = (ar)\, e^{i(\alpha + \theta)} \quad \ldots(2)$$

Thus, we observe from (2) that transformation (1) expands (or contracts) the radius vector representing z by the factor a = | A | and rotates it through an angle α = arg A about the origin. The image of a given region is, therefore, geometrically similar to that region. The general linear transformation

$$w = Az + B \quad \ldots(3)$$

is evidently an expansion or contraction and a rotation, followed by a translation. The image region mapped by (3) is geometrically congruent to the original one.

- $$d(u^2 + v^2) + bu - cv + a = 0$$

  which also represents a circle or a line. Conversely, if u and v satisfy (9), it follows from (6) that x and y satisfy (8). From (8) and (9), it is clear that

  (i) a circle ($a \neq 0$) not passing through the origin ($d \neq 0$) in the z plane is transformed into a circle not passing through the origin in the w plane.

  (ii) a circle ($a \neq 0$) through the origin ($d = 0$) in the z plane is transformed into a line which does not pass through the origin in the w plane.

  (iii) a line ($a = 0$) not passing through the origin ($d \neq 0$) in the z plane is transformed into a circle through the origin in the w plane.

  (iv) a line ($a = 0$) through the origin ($d = 0$) in the z plane is transformed into a line through the origin in the w plane.

- The transformation

  $$w = \frac{az+b}{cz+d},\ ad - bc \neq 0 \quad \ldots(1)$$

  where a, b, c, d are complex constants, is called bilinear transformation or a linear fractional transformation or Möbius transformation. We observe that the condition $ad - bc \neq 0$ is necessary for (1) to be a bilinear transformation, since if

  $ad - bc = 0$, then $\frac{b}{a} = \frac{d}{c}$ and we get

- In a bilinear transformation, a circle transforms into a circle and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetric about the line.

- Let S be a domain in a plane in which x and y are taken as rectangular Cartesian co-ordinates. Let us suppose that the functions u(x, y) and v(x, y) are continuous and possess continuous partial derivatives of the first order at each point of the domain S. The equations

  $$u = u(x, y), \qquad v = v(x, y)$$

  set up a correspondence between the points of S and the points of a set T in the (u, v) plane. The set T is evidently a domain and is called a map of S. Moreover, since the first order partial derivatives of u and v are continuous, a curve in S which has a continuously turning tangent is mapped on a curve with the same property in T. The correspondence between the two domains is not, however, necessarily a one-one correspondence.

7.4 Keywords

Transformation: The linear function.

$$w = Az$$

where A is non-zero complex constant and $z \neq 0$. We write A and z in exponential form as

$$A = ae^{i\alpha}, \qquad z = re^{i\theta}$$

Then $$w = (ar)\, e^{i(\alpha + \theta)}$$

One–one transformation: It is natural to define a one–one transformation $w = T(z)$ from the extended z plane onto the extended w plane by writing

$$T(0) = \infty, \quad T(\infty) = 0$$

Bilinear transformation: The transformation

$$w = \frac{az+b}{cz+d}, \; ad - bc \neq 0$$

where a, b, c, d are complex constants, is called bilinear transformation or a linear fractional transformation or Möbius transformation.

Conformal Mappings: Let S be a domain in a plane in which x and y are taken as rectangular Cartesian co-ordinates. Let us suppose that the functions u(x, y) and v(x, y) are continuous and possess continuous partial derivatives of the first order at each point of the domain S. The equations

$$u = u(x, y), \qquad v = v(x, y)$$

7.5 Self Assessment

1. The general linear transformation .... is evidently an expansion or contraction and a rotation, followed by a translation.

2. Any point on the circle is mapped onto itself. The second of the transformation in (5) is simply a .... in the real axis.

3. It is natural to define a .... $w = T(z)$ from the extended z plane onto the extended w plane by writing

$$T(0) = \infty, \quad T(\infty) = 0$$

4. The transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b, c, d are complex constants, is called .... or a linear fractional transformation or Möbius transformation.

5. Composition (or resultant or product) of two bilinear transformations is a ....

6. The points which coincide with their transforms under bilinear transformation are called its fixed points. For the bilinear transformation ...., fixed points are given by $w = z$

   i.e. $z = \dfrac{az+b}{cz+d}$

   Since (1) is a quadratic in z and has in general two different roots, therefore, there are generally two invariant points for a bilinear transformation.

Notes

7. In a bilinear transformation, a circle transforms into a circle and inverse points transform into ....
8. An .... which also conserves the sense of rotation is called conformal mapping. Thus in a conformal transformation, the sense of rotation as well as the magnitude of the angle is preserved.

7.6 Review Questions

Find the bilinear transformation which maps

(i) 1, –i, 2 onto 0, 2, –i respectively.

(ii) 1, i, 0 onto 1, i, –1 respectively.

(iii) 0, 1, ∞ onto ∞, –i, 1 respectively.

(iv) –1, ∞, i into 0, ∞, 1 respectively.

(v) ∞, i, 0 onto 0, i, respectively.

(vi) 1, 0, –1 onto i, ∞, 1 respectively.

(vii) 1, i, –1 onto i, 0, –i respectively.

Answers: Self Assessment

1. $w = Az + B$
2. reflection
3. one-one transformation
4. bilinear transformation
5. bilinear transformation.
6. $w = \dfrac{az+b}{cz+d}$
7. inverse points.
8. isogonal transformation

7.7 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati,T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Notes

Unit 8: Series

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define sequences
- Discuss series and power series
- Describe integration by power series
- Explain differentiation by power series

Introduction

In earlier unit, you have studied about concept of transformation and conformal mapping. In this unit, we shall introduce you to series representation of a complex valued function f (z) . In order to obtain and analyze these series, we need to develop some concepts related to series. We shall start the unit by discussing basic facts regarding the convergence of sequences and series of complex numbers.

8.1 Sequences

The basic definitions for complex sequences and series are essentially the same as for the real case. A sequence of complex numbers is a function $g : Z_+ \to C$ from the positive integers into the complex numbers. It is traditional to use subscripts to indicate the values of the function. Thus, we write $g(n) \equiv z_n$ and an explicit name for the sequence is seldom used; we write simply (z_n) to

Notes

stand for the sequence g which is such that g(n) = z_n. For example, $\left(\frac{i}{n}\right)$ is the sequence g for which g(n) = $\frac{i}{n}$.

The number L is a limit of the sequence (z_n) if given an $\varepsilon > 0$, there is an integer N_ε such that $|zn - L| < \varepsilon$ for all $n \geq N_\varepsilon$. If L is a limit of (z_n), we sometimes say that (z_n) converges to L. We frequently write lim (z_n) = L. It is relatively easy to see that if the complex sequence (z_n) = ($u_n + iv_n$) converges to L, then the two real sequences (u_n) and (v_n) each have a limit: (u_n) converges to ReL and (v_n) converges to ImL. Conversely, if the two real sequences (u_n) and (v_n) each have a limit, then so also does the complex sequence ($u_n + iv_n$). All the usual nice properties of limits of sequences are thus true:

$\lim(z_n \pm w_n) = \lim(z_n) \pm \lim(w_n)$;

$\lim(z_n w_n) = \lim(z_n)\ \lim(wn)$; and

$$\lim\left(\frac{z_n}{w_n}\right) = \frac{\lim(z_n)}{\lim(w_n)}.$$

provided that $\lim(z_n)$ and $\lim(w_n)$ exist. (And in the last equation, we must, of course, insist that $\lim(w_n) \neq 0$.)

A necessary and sufficient condition for the convergence of a sequence (a_n) is the celebrated Cauchy criterion: given $\varepsilon > 0$, there is an integer N_ε so that $|a_n - a_m| < \varepsilon$ whenever $n,m > N_\varepsilon$.

A sequence (f_n) of functions on a domain D is the obvious thing: a function from the positive integers into the set of complex functions on D. Thus, for each $z \in D$, we have an ordinary sequence (fn(z)). If each of the sequences (fn(z)) converges, then we say the sequence of functions (f_n) converges to the function f defined by $f(z) = \lim(f_n(z))$. This pretty obvious stuff. The sequence (f_n) is said to converge to f uniformly on a set S if given an $\varepsilon > 0$, there is an integer N_ε so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N_\varepsilon$ and all $z \in S$.

Notes It is possible for a sequence of continuous functions to have a limit function that is not continuous. This cannot happen if the convergence is uniform.

To see this, suppose the sequence (f_n) of continuous functions converges uniformly to f on a domain D, let $z_0 \in D$, and let $\varepsilon > 0$. We need to show there is a δ so that $|f(z_0) - f(z)| < \varepsilon$ whenever $|z_0 - z| < \delta$. Let's do it. First, choose N so that $|f_N(z) - f(z)| < \frac{\varepsilon}{3}$. We can do this because of the uniform convergence of the sequence (f_n). Next, choose δ so that $|f_N(z_0) - f_N(z)| < \frac{\varepsilon}{3}$ whenever $|z_0 - z| < \delta$. This is possible because f_N is continuous.

Now then, when $|z_0 - z| < \delta$, we have

$$\begin{aligned}|f(z_0) - f(_z)| &= |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)| \\ &\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)|\end{aligned}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

Now suppose, we have a sequence (f_n) of continuous functions which converges uniformly on a contour C to the function f. Then the sequence $\left(\int_C f_n(z)dz\right)$ converges to $\int_C f_n(z)dz.$ This is easy to see. Let $\varepsilon > 0$. Now let N be so that $|f_n(z) - f(z)| < \frac{\varepsilon}{A}$ for $n > N$, where A is the length of C. Then,

$$\left|\int_C f_n(z)dz - \int_C f(z)dz\right| = \left|\int_C (f_n(z) - f(z))dz\right|$$

$$< \frac{\varepsilon}{A}A = \varepsilon$$

whenever $n > N$.

Now suppose (f_n) is a sequence of functions each analytic on some region D, and suppose the sequence converges uniformly on D to the function f. Then f is analytic. This result is in marked contrast to what happens with real functions—examples of uniformly convergent sequences of differentiable functions with a non-differentiable limit abound in the real case. To see that this uniform limit is analytic, let $z_0 \in D$, and let $S = \{z : |z - z_0| < r\} \subset D$. Now consider any simple closed curve $C \subset S$. Each f_n is analytic, and so $\int_C f_n(z)dz = 0$ for every n. From the uniform convergence of (f_n), we know that $\int_C f(z)dz$ is the limit of the sequence $\left(\int_C f_n(z)dz\right)$, and so $\int_C f(z)dz = 0.$ Morera's theorem now tells us that f is analytic on S, and hence at z_0. Truly a miracle.

8.2 Series

A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + ... + a_n$. In other words, there is sequence (a_n) so that $s_n = s_{n-1} + a_n$. The s_n are usually called the partial sums. Recall from Mrs. Turner's class that if the series $\left(\sum_{j=1}^{n} a_j\right)$ has a limit, then it must be true that $\lim_{n\to\infty}(a_n) = 0.$

Consider a series $\left(\sum_{j=1}^{n} fj(z)\right)$ of functions. Chances are this series will converge for some values of z and not converge for others. A useful result is the celebrated Weierstrass M-test: Suppose (M_j) is a sequence of real numbers such that $M_j \geq 0$ for all $j > J$, where J is some number., and suppose also that the series $\left(\sum_{j=1}^{n} M_j\right)$ converges. If for all $z \in D$, we have $|fj(z)| \leq M_j$ for all $j > J$, then the series $\left(\sum_{j=1}^{n} f_j(z)\right)$ converges uniformly on D.

Notes

To prove this, begin by letting ε > 0 and choosing N > J so that

$$\sum_{j=m}^{n} M_j < \varepsilon$$

for all n, m > N. (We can do this because of the famous Cauchy criterion.) Next, observe that

$$\left|\sum_{j=m}^{n} f_j(z)\right| \le \sum_{j=m}^{n} \left|f_j(z)\right| \le \sum_{j=m}^{n} M_j < \varepsilon.$$

This shows that $\left(\sum_{j=1}^{n} f_j(z)\right)$ converges. To see the uniform convergence, observe that

$$\left|\sum_{j=m}^{n} f_j(z)\right| = \left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| < \varepsilon$$

for all z ∈ D and n > m > N. Thus,

$$\lim_{n\to\infty}\left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| = \left|\sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| \le \varepsilon$$

for m > N. (The limit of a series $\left(\sum_{j=0}^{n} a_j\right)$ is almost always written as $\sum_{j=0}^{\infty} a_j$.)

8.3 Power Series

We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ... + c_n(z - z_0)^n.$$

(We start with n = 0 for esthetic reasons.) These are the so-called power series. Thus,

a power series is a series of functions of the form $\left(\sum_{j=0}^{n} c_j(z-z_0)^j\right)$.

Let's look first at a very special power series, the so-called Geometric series:

$$\left(\sum_{j=0}^{n} z^j\right).$$

Here,

$$s_n = 1 + z + z^2 + ... + z^n, \text{ and}$$

$$zs_n = z + z^2 + z^3 + ... + z^{n+1}.$$

Subtracting the second of these from the first gives us

$$(1 - z)s_n = 1 - z^{n+1}.$$

Notes

If z = 1, then we can't go any further with this, but I hope it's clear that the series does not have a limit in case z = 1. Suppose now $z \neq 1$. Then we have,

$$s_n = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}.$$

Now if $|z| < 1$, it should be clear that $\lim(z^{n+1}) = 0$, and so

$$\lim\left(\sum_{j=0}^{n} z^j\right) = \lim s_n = \frac{1}{1-z}.$$

Or,

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \text{ for } |z| < 1.$$

There is a bit more to the story. First, note that if $|z| > 1$, then the Geometric series does not have a limit (why?). Next, note that if $|z| \leq \rho < 1$, then the Geometric series converges uniformly to $\frac{1}{1-z}$.

Notes $\left(\sum_{j=0}^{n} \rho^j\right)$ has a limit and appeal to the Weierstrass M-test.

Clearly, a power series will have a limit for some values of z and perhaps not for others. First, note that any power series has a limit when $z = z_0$. Let's see what else we can say. Consider a power series $\left(\sum_{j=0}^{n} c_j (z - z_0)^j\right)$. Let

$$\lambda = \lim\sup\left(\sqrt[j]{|c_j|}\right).$$

(Recall from 6th grade that $\lim\sup(a_k) = \lim(\sup\{a_k : k \geq n\})$.) Now let $R = \frac{1}{\lambda}$. (We shall say R = 0 if $l = \infty$, and $R = \infty$ if $\lambda = 0$.) We are going to show that the series converges uniformly for all $|z - z_0| \leq \rho < R$ and diverges for all $|z - z_0| > R$.

First, let's show the series does not converge for $|z - z_0| > R$. To begin, let k be so that

$$\frac{1}{|z - z_0|} < k < \frac{1}{R} = \lambda.$$

There are an infinite number of c_j for which $\sqrt[j]{|c_j|} > k$, otherwise $\lim\sup\left(\sqrt[j]{|c_j|}\right) \leq k$. For each of these c_j, we have,

$$\left|c_j (z - z_0)^j\right| = \left(\sqrt[j]{|c_j|}\,|z - z_0|\right)^j > (k|z - z_0|^j > 1.$$

It is, thus, not possible for $\lim_{n\to\infty}\left|c_n (z - z_0)^n\right| = 0$, and so the series does not converge.

Notes

Next, we show that the series *does* converge uniformly for $|z - z_0| \le \rho < R$. Let k be so that

$$\lambda = \frac{1}{R} < k < \frac{1}{\rho}.$$

Now, for j large enough, we have $\left(\sqrt[j]{|c_j|}\right) < k$. Thus, for $|z - z_0| \le \rho$, we have

$$\left|c_j(z-z_0)^j\right| = \left(\sqrt[j]{|c_j|}\,|z-z_0|^j\right) < (k|z-z_0|^j < (k\rho)^j.$$

The geometric series $\left(\sum_{j=0}^{n}(k\rho)^j\right)$ converges because $k\rho < 1$ and the uniform convergence of $\left(\sum_{j=0}^{n}c_j(z-z_0)^j\right)$ follows from the M-test.

Example:

Consider the series $\left(\sum_{j=0}^{n}\frac{1}{j!}z^j\right)$. Let's compute $R = 1/\limsup\left(\sqrt[j]{|c_j|}\right) = \limsup\,(\sqrt[j]{j!})$. Let K be any positive integer and choose an integer m large enough to insure that $2m > \frac{K^{2K}}{(2K)!}$. Now consider $\frac{n!}{K^n}$, where $n = 2K + m$:

$$\frac{n!}{K^n} = \frac{(2K+m)!}{K^{2K+m}} = \frac{(2K+m)(2K+m-1)...(2K+1)(2K}{KmK^{2K}}$$
$$> 2m\frac{(2K)!}{K^{2K}} > 1$$

Thus $\sqrt[n]{n!} > K$. Reflect on what we have just shown: given any number K, there is a number n such that $\sqrt[n]{n!}$ is bigger than it. In other words, $R = \limsup\left(\sqrt[j]{j!}\right) = \infty$, and so the series $\left(\sum_{j=0}^{n}\frac{1}{j!}z^j\right)$ converges for all z.

Let's summarize what we have. For any power series $\left(\sum_{j=0}^{n}c_j(z-z_0)^j\right)$, there is a number $R = \frac{1}{\limsup\left(\sqrt[j]{|c_j|}\right)}$ such that the series converges uniformly for $|z - z_0| \le \rho < R$ and does not converge for $|z - z_0| > R$.

Notes We may have $R = 0$ or $R = \infty$.

The number R is called the radius of convergence of the series, and the set $|z - z_0| = R$ is called the circle of convergence. Observe also that the limit of a power series is a function analytic inside the circle of convergence (why?).

8.4 Integration of Power Series

Inside the circle of convergence, the limit

$$S(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$$

is an analytic function. We shall show that this series may be integrated "term-by-term"—that is, the integral of the limit is the limit of the integrals. Specifically, if *C* is any contour inside the circle of convergence, and the function g is continuous on *C*, then

$$\int_C g(z)S(z)dz = \sum_{j=0}^{\infty} c_j \int_C g(z)(z - z_0)^j dz.$$

Let's see why this. First, let ε > 0. Let M be the maximum of $|g(z)|$ on *C* and let *L* be the length of *C*. Then there is an integer *N* so that

$$\left| \sum_{j=n}^{\infty} c_j (z - z_0)^j \right| < \frac{\varepsilon}{ML}$$

for all *n* > *N*. Thus,

$$\left| \int_C \left(g(z) \sum_{j=n}^{\infty} c_j (z - z_0)^j \right) dz \right| < ML \frac{\varepsilon}{ML} = \varepsilon,$$

Hence,

$$\left| \int_C g(z)S(z)dz - \sum_{j=0}^{n-1} c_j \int_C g(z)(z - z_0)^j dz \right| = \left| \int_C \left(g(z) \sum_{j=n}^{\infty} c_j (z - z_0)^j \right) dz \right| < \varepsilon.$$

8.5 Differentiation of Power Series

Again, let

$$S(z) = \sum_{j=0}^{\infty} c_j (z - z_0)j.$$

Now we are ready to show that inside the circle of convergence,

$$S'(z) = \sum_{j=1}^{\infty} jc_j (z - z_0)^{j-1}.$$

Let z be a point inside the circle of convergence and let C be a positive oriented circle centered at z and inside the circle of convergence. Define

$$g(s) = \frac{1}{2\pi i (s - z)^2},$$

Notes

and apply the result of the previous section to conclude that

$$\int_C g(s)S(s)ds = \sum_{j=0}^{\infty} c_j \int_C g(s)(s - z_0)^j ds, \text{or}$$

$$\frac{1}{2\pi i}\int_C \frac{S(s)}{(s-z)^2} ds = \sum_{j=0}^{\infty} c_j \frac{1}{2\pi i}\int_C \frac{(s-z_0)^j}{(s-z)^2} ds. \text{ Thus}$$

$$S'(z) = \sum_{j=0}^{\infty} jc_j(z - z_0)^{j-1}.$$

8.6 Summary

- The basic definitions for complex sequences and series are essentially the same as for the real case. A sequence of complex numbers is a function $g : Z_+ \to C$ from the positive integers into the complex numbers. It is traditional to use subscripts to indicate the values of the function. Thus, we write $g(n) \equiv z_n$ and an explicit name for the sequence is seldom used; we write simply (z_n) to stand for the sequence g which is such that $g(n) = z_n$. For example, $\left(\frac{i}{n}\right)$ is the sequence g for which $g(n) = \frac{i}{n}$.

- The number L is a limit of the sequence (z_n) if given an $\varepsilon > 0$, there is an integer N_ε such that $|zn - L| < \varepsilon$ for all $n \geq N_\varepsilon$. If L is a limit of (z_n), we sometimes say that (z_n) converges to L. We frequently write $\lim (z_n) = L$. It is relatively easy to see that if the complex sequence $(z_n) = (u_n + iv_n)$ converges to L, then the two real sequences (u_n) and (v_n) each have a limit: (u_n) converges to ReL and (v_n) converges to ImL. Conversely, if the two real sequences (u_n) and (v_n) each have a limit, then so also does the complex sequence $(u_n + iv_n)$. All the usual nice properties of limits of sequences are, thus, true:

  $\lim(z_n \pm w_n) = \lim(z_n) \pm \lim(w_n)$;

  $\lim(z_n w_n) = \lim(z_n) \lim(wn)$; and

  $$\lim\left(\frac{z_n}{w_n}\right) = \frac{\lim(z_n)}{\lim(w_n)}.$$

  provided that $\lim(z_n)$ and $\lim(w_n)$ exist. (And in the last equation, we must, of course, insist that $\lim(w_n) \neq 0$.).

- A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + ... + a_n$. In other words, there is sequence (a_n) so that $s_n = s_{n-1} + a_n$. The s_n are usually called the partial sums. Recall from Mrs. Turner's class that if the series $\left(\sum_{j=1}^{n} a_j\right)$ has a limit, then it must be true that $\lim_{n\to\infty}(a_n) = 0$.

  Consider a series $\left(\sum_{j=1}^{n} fj(z)\right)$ of functions. Chances are this series will converge for some values of z and not converge for others. A useful result is the celebrated Weierstrass M-test: Suppose (M_j) is a sequence of real numbers such that $M_j \geq 0$ for all $j > J$, where J is some

Notes

number, and suppose also that the series $\left(\sum_{j=1}^{n} M_j\right)$ converges. If for all $z \in D$, we have $|fj(z)| \leq M_j$ for all $j > J$, then the series $\left(\sum_{j=1}^{n} f_j(z)\right)$ converges uniformly on D.

- We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ... + c_n(z - z_0)^n.$$

(We start with n = 0 for esthetic reasons.) These are the so-called power series. Thus, a power series is a series of functions of the form $\left(\sum_{j=0}^{n} c_j(z - z_0)^j\right)$.

8.7 Keywords

Sequence: A sequence of complex numbers is a function $g : Z_+ \rightarrow C$ from the positive integers into the complex numbers.

Partial sums: A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + ... + a_n$. In other words, there is sequence (a_n) so that $s_n = s_{n-1} + a_n$. The s_n are usually called the partial sums.

Power series: We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ... + c_n(z - z_0)^n.$$

(We start with n = 0 for esthetic reasons.) These are the so-called power series.

8.8 Self Assessment

1. A .... of complex numbers is a function $g : Z_+ \rightarrow C$ from the positive integers into the complex numbers.
2. A necessary and sufficient condition for the convergence of a sequence (a_n) is the celebrated Cauchy criterion: given $\varepsilon > 0$, there is an integer N_ε so that .... whenever $n, m > N_\varepsilon$.
3. A sequence (f_n) of functions on a domain D is the obvious thing: a function from the positive integers into the set of .... on D.
4. A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + ... + a_n$. In other words, there is sequence (a_n) so that $s_n = s_{n-1} + a_n$. The s_n are usually called the ....
5. We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ... + c_n(z - z_0)^n.$$

(We start with n = 0 for esthetic reasons.) These are the so-called ....

6. if C is any contour inside the circle of convergence, and the function g is continuous on C, then ....

Notes

8.9 Review Questions

1. Prove that a sequence cannot have more than one limit. (We, thus, speak of the limit of a sequence.)
2. Give an example of a sequence that does not have a limit, or explain carefully why there is no such sequence.
3. Give an example of a bounded sequence that does not have a limit, or explain carefully why there is no such sequence.
4. Give a sequence (f_n) of functions continuous on a set D with a limit that is not continuous.
5. Give a sequence of real functions differentiable on an interval which converges uniformly to a non-differentiable function.
6. Find the set D of all z for which the sequence $\left(\frac{z^n}{z^n - 3^n}\right)$ has a limit. Find the limit.
7. Prove that the series $\left(\sum_{j=1}^{n} a_j\right)$ converges if and only if both the series $\left(\sum_{j=1}^{n} \mathrm{Re}\, a_j\right)$ and $\left(\sum_{j=1}^{n} \mathrm{Im}\, a_j\right)$ converge.
8. Explain how you know that the series $\left(\sum_{j=1}^{n} \left(\frac{1}{z}\right)^j\right)$ converges uniformly on the set $|z| \geq 5$.
9. Suppose the sequence of real numbers (α_j) has a limit. Prove that

$$\lim \mathrm{sum}(\alpha_j) = \lim(\alpha_j).$$

10. For each of the following, find the set D of points at which the series converges:

(a) $\left(\sum_{j=0}^{n} j!z^j\right)$.

(b) $\left(\sum_{j=0}^{n} jz^j\right)$.

(c) $\left(\sum_{j=0}^{n} \frac{j^2}{3^j} z^j\right)$.

(d) $\left(\sum_{j=0}^{n} \frac{(-1)^j}{2^{2j}(j!)^2} z^{2j}\right)$.

11. Find the limit of

$$\left(\sum_{j=0}^{n} (j+1)z^j\right).$$

For what values of z does the series converge?

Notes

12. Find the limit of

$$\left(\sum_{j=1}^{n} \frac{z^j}{j}\right).$$

For what values of z does the series converge?

13. Find a power series $\left(\sum_{j=0}^{n} c_j (z-1)^j\right)$ such that

$$\frac{1}{z} = \sum_{j=0}^{\infty} c_j (z-1)^j, \text{for} \ |z-1| < 1.$$

14. Find a power series $\left(\sum_{j=0}^{n} c_j (z-1)^j\right)$ such that

$\log z = \sum_{j=0}^{\infty} c_j (z-1)j, \text{for} \ |z-1| < 1.$

Answers: Self Assessment

1. sequence
2. $|a_n - a_m| < \varepsilon$
3. complex functions
4. partial sums
5. power series.
6. $\int_C g(z)S(z)dz = \sum_{j=0}^{\infty} c_j \int_C g(z)(z-z_0)^j dz.$

8.10 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati,T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Richa Nandra, Lovely Professional University

Notes

Unit 9: Taylor and Laurent Series

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss Taylor series
- Describe the concept of Laurent series

Introduction

In last unit, you have studied about concept of power series and also discussed basic facts regarding the convergence of sequences and series of complex numbers. We shall show that if f (z) is analytic in some domain D then it can be represented as a power series at any point $z_0 \in D$ in powers of $(z - z_0)$ which is the Taylor series of f (z) . If f (z) fails to be analytic at a point z_0 , we cannot find Taylor series expansion of f (z) at that point. However, it is often possible to expand f (z) in an infinite series having both positive and negative powers of $(z - z_0)$. This series is called the Laurent series.

9.1 Taylor Series

Suppose f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius ρ, where $|z - z_0| < \rho < r$. Then for $s \in C$ we have

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{(s-z_0)}\left[\frac{1}{1-\dfrac{z-z_0}{s-z_0}}\right] = \sum_{j=0}^{\infty}\frac{(z-z_0)^j}{(s-z_0)^{j+1}}$$

since $\left|\dfrac{z-z_0}{s-z_0}\right| < 1$. The convergence is uniform, so we may integrate

Notes

$$\int_C \frac{f(s)}{s-z}dz = \sum_{j=0}^{\infty}\left(\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j, \text{ or}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j.$$

We have, thus, produced a power series having the given analytic function as a limit:

$$f(z) = \sum_{j=0}^{\infty} c_j (z-z_0)^j, |z-z_0| < r,$$

where

$$c_j = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds$$

This is the celebrated Taylor Series for f at $z = z_0$.

We know we may differentiate the series to get

$$f'(z) = \sum_{j=0}^{\infty} jc_j (z-z_0)^{j-1}$$

and this one converges uniformly where the series for f does. We can, thus, differentiate again and again to obtain

$$f^{(n)}(z) = \sum_{j=n}^{\infty} j(j-1)(j-2)...(j-n+1)c_j(z-z_0)^{j-n}.$$

Hence,

$$f^{(n)}(z_0) = n!c_n, \text{ or}$$

$$c_n = \frac{f(n)(z_0)}{n!}.$$

But we also know that,

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}ds.$$

This gives us,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}ds, \text{for } n = 0,1,2,.....$$

This is the famous **Generalized Cauchy Integral Formula.** Recall that we previously derived this formula for n = 0 and 1.

What does all this tell us about the radius of convergence of a power series? Suppose we have,

$$f(z) = \sum_{j=0}^{\infty} c_j (z-z_0)^j,$$

Notes

and the radius of convergence is R. Then we know, of course, that the limit function f is analytic for $|z - z_0| < R$. We showed that if f is analytic in $|z - z_0| < r$, then the series converges for $|z - z_0| > r$. Thus $r \le R$, and so f cannot be analytic at any point z for which $|z - z_0| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f is analytic.

Example:

Let $f(z) = \exp(z) = e^z$. Then $f(0) = f'(0) = ... = f^{(n)}(0) = ... = 1$, and the Taylor series for f at $z_0 = 0$ is

$$\sum_{j=0}^{\infty} \frac{1}{j!} z^j$$

and this is valid for all values of z since f is entire. (We also showed earlier that this particular series has an infinite radius of convergence.)

9.2 Laurent Series

Suppose f is analytic in the region $R_1 < |z - z_0| < R_2$, and let C be a positively oriented simple closed curve around z_0 in this region.

Notes We include the possibilities that R_1 can be 0, and $R_2 = \infty$.

We shall show that for $z \notin C$ in this region

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j},$$

where,

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \text{for } j = 0, 1, 2, ...$$

and

$$b_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{-j+1}} ds, \text{for } j = 1, 2, ...$$

The sum of the limits of these two series is frequently written

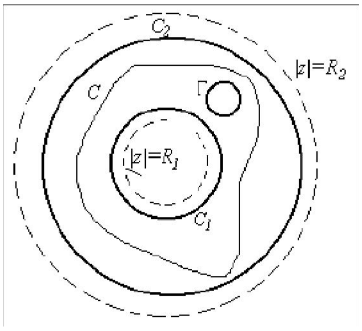
$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j,$$

where,

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \ j = 0, \pm 1, \pm 2, ...$$

This recipe for f(z) is called a Laurent series, although it is important to keep in mind that it is really two series.

Okay, now let's derive the above formula. First, let r_1 and r_2 be so that $R_1 < r_1 \le |z - z_0| \le r_2 < R_2$ and so that the point z and the curve C are included in the region $r_1 \le |z - z_0| \le r_2$. Also, let Γ be a circle centered at z and such that Γ is included in this region.



Then $\frac{f(z)}{s-z}$ is an analytic function (of s) on the region bounded by C_1, C_2, and Γ, where C_1 is the circle $|z| = r_1$ and C_2 is the circle $|z| = r^2$. Thus,

$$\int_{C_2} \frac{f(z)}{s-z}\,ds = \int_{C_1} \frac{f(z)}{s-z}\,ds + \int_{\Gamma} \frac{f(z)}{s-z}\,ds.$$

(All three circles are positively oriented, of course.) But $\int_{\Gamma} \frac{f(z)}{s-z}\,ds = 2\pi i f(z),$ and so we have

$$2\pi i f(z) = \int_{C_2} \frac{f(z)}{s-z}\,ds - \int_{C_1} \frac{f(z)}{s-z}\,ds$$

Look at the first of the two integrals on the right-hand side of this equation. For $s \in C_2$, we have $|z - z_0| < |s - z_0|$, and so

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)}$$

$$= \frac{1}{s-z_0}\left[\frac{1}{1-\left(\frac{z-z_0}{s-z_0}\right)}\right]$$

$$= \frac{1}{s-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^j$$

$$= \sum_{j=0}^{\infty}\frac{1}{(s-z_0)^{j+1}}(z-z_0)^j.$$

Notes

Hence,

$$\int_{C_2} \frac{f(z)}{s-z}\,ds = \sum_{j=0}^{\infty}\left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}}\,ds\right)(z-z_0)^j$$

$$= \sum_{j=0}^{\infty}\left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}}\,ds\right)(z-z_0)^j$$

For the second of these two integrals, note that for $s \in C_1$ we have $|s - z_0| < |z - z_0|$, and so

$$\frac{1}{s-z} = \frac{-1}{(z-z_0)-(s-z_0)} = \frac{-1}{z-z_0}\left[\frac{1}{1-\left(\frac{s-z_0}{z-z_0}\right)}\right]$$

$$= \frac{-1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^j = -\sum_{j=0}^{\infty}(s-z_0)^j\frac{1}{(z-z_0)^{j+1}}$$

$$= -\sum_{j=0}^{\infty}(s-z_0)^{j-1}\frac{1}{(z-z_0)^j} = -\sum_{j=1}^{\infty}\left(\frac{1}{(s-z_0)^{-j+1}}\right)\frac{1}{(z-z_0)^j}$$

As before,

$$\int_{C_1} \frac{f(s)}{s-z}\,ds = -\sum_{j=1}^{\infty}\left(\int_{C_1} \frac{f(s)}{(s-z_0)^{-j+1}}\,ds\right)\frac{1}{(z-z_0)^j}$$

$$= -\sum_{j=1}^{\infty}\left(\int_{C_1} \frac{f(s)}{(s-z_0)^{-j+1}}\,ds\right)\frac{1}{(z-z_0)^j}$$

Putting this altogether, we have the Laurent series:

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s-z}\,ds$$

$$= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z_0)^{j+1}}\,ds\right)(z-z_0)^j + \sum_{j=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z0)^{-j+1}}\,ds\right)\frac{1}{(z-z_0)^j}.$$

Let f be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

First, observe that f is analytic in the region $0 < |z| < 1$. Let's find the Laurent series for f valid in this region. First,

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

Notes

From our vast knowledge of the Geometric series, we have

$$f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j.$$

Now let's find another Laurent series for f, the one valid for the region $1 < |z| < \infty$.

First,

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right].$$

Now since $\left|\frac{1}{z}\right| < 1$, we have

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] = \frac{1}{z}\sum_{j=0}^{\infty} z^{-j} = \sum_{j=1}^{\infty} z^{-j},$$

and so

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j}$$

$$f(z) = \sum_{j=2}^{\infty} z^{-j}.$$

9.3 Summary

Suppose f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius ρ, where $|z - z_0| < \rho < r$. Then for $s \in C$, we have,

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{(s-z_0)}\left[\frac{1}{1-\frac{z-z_0}{s-z_0}}\right] = \sum_{j=0}^{\infty}\frac{(z-z_0)^j}{(s-z_0)^{j+1}}$$

since $\left|\frac{z-z_0}{s-z_0}\right| < 1$. The convergence is uniform, so we may integrate

$$\int_C \frac{f(s)}{s-z}dz = \sum_{j=0}^{\infty}\left(\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j, \text{ or}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j.$$

Notes

We have, thus, produced a power series having the given analytic function as a limit:

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j, |z - z_0| < r,$$

where,

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds$$

9.4 Keywords

Taylor series: f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius ρ, where $|z - z_0| < \rho < r$. Then for $s \in C$ we have

$$\frac{1}{s - z} = \frac{1}{(s - z_0) - (z - z_0)} = \frac{1}{(s - z_0)} \left[\frac{1}{1 - \frac{z - z_0}{s - z_0}} \right] = \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(s - z_0)^{j+1}}$$

Cauchy Integral Formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds, \text{for } n = 0, 1, 2, ...$$

This is the famous Generalized Cauchy Integral Formula. Recall that we previously derived this formula for n = 0 and 1.

9.5 Self Assessment

1. f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius ρ, where $|z - z_0| < \rho < r$. Then for $s \in C$ we have

$$\frac{1}{s - z} = \frac{1}{(s - z_0) - (z - z_0)} = \frac{1}{(s - z_0)} \left[\frac{1}{1 - \frac{z - z_0}{s - z_0}} \right] = ...$$

2. The circle of .... is the largest circle centered at z_0 inside of which the limit f is analytic.

3. Suppose f is analytic in the region ...., and let C be a positively oriented simple closed curve around z_0 in this region.

4. Laurent series for f, the one valid for the region $1 < |z| < \infty$. When, $f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j$ is equal to ....

9.6 Review Questions

Notes

1. Show that for all z,

$$e^z = e\sum_{j=0}^{\infty}\frac{1}{j!}(z-1)^j.$$

2. What is the radius of convergence of the Taylor series $\left(\sum_{j=0}^{n} c_j z^j\right)$ for tanh z ?

3. Show that

$$\frac{1}{1-z} = \sum_{j=0}^{\infty}\frac{(z-i)^j}{(1-i)^{j+1}}$$

for $|z - i| < \sqrt{2}$.

4. If $f(z) = \frac{1}{1-z}$, what is $f^{(10)}(i)$?

5. Suppose f is analytic at z = 0 and $f(0) = f'(0) = f''(0) = 0$. Prove there is a function g analytic at 0 such that $f(z) = z^3 g(z)$ in a neighborhood of 0.

6. Find the Taylor series for f(z) = sin z at $z_0 = 0$.

7. Show that the function f defined by

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is analytic at z = 0, and find f′(0).

8. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which the series converge to f(z).

9. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z(1+z^2)}$$

and specify the regions in which the series converge to f(z).

10. Find the Laurent series in powers of $z - 1$ for $f(z) = \frac{1}{z}$ in the region $1 < |z - 1| < \infty$.

Notes

Answers: Self Assessment

1. $\sum_{j=0}^{\infty} \frac{(z-z_0)^j}{(s-z_0)^{j+1}}$

2. convergence

3. $R_1 < |z - z_0| < R_2$

4. $f(z) = \sum_{j=2}^{\infty} z^{-j}$.

9.7 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati,T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H.Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 10: Residues and Singularities

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss residues
- Describe the concept of singularities

Introduction

In last unit, you have studied about the Taylor series. Taylor series representation of a complex valued function is discussed. In earlier unit, we have introduced the concept of absolute and uniform convergence of power series and defined its radius of convergence. This unit will explain zeros and singularities of complex valued functions and use the Laurent series to classify these singularities.

10.1 Residues

A point z_0 is a singular point of a function *f* if *f* is not analytic at z_0, but is analytic at some point of each neighborhood of z_0. A singular point z_0 of f is said to be isolated if there is a neighborhood of z_0 which contains no singular points of f save z_0. In other words, f is analytic on some region $0 < |z - z_0| < \varepsilon$.

Example:

The function f given by

$$f(z) = \frac{1}{z(z^2 + 4)}$$

has isolated singular points at z = 0, z = 2i, and z = –2i.

Notes

Every point on the negative real axis and the origin is a singular point of Log z, but there are no isolated singular points.

Suppose now that z_0 is an isolated singular point of f . Then there is a Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j$$

valid for $0 < |z - z_0| < R$, for some positive R. The coefficient c_{-1} of $(z - z_0)^{-1}$ is called the residue of f at z_0, and is frequently written

$$\operatorname*{Res}_{z=z_0} f.$$

Now, why do we care enough about c_{-1} to give it a special name? Well, observe that if C is any positively oriented simple closed curve in $0 < |z - z_0| < R$ and which contains z_0 inside, then

$$c_{-1} = \frac{1}{2\pi i} \int_C f(z)dz.$$

This provides the key to evaluating many complex integrals.

Example:

We shall evaluate the integral

$$\int_C f^{1/z} dz$$

where C is the circle $|z| = 1$ with the usual positive orientation. Observe that the integrand has an isolated singularity at z = 0. We know then that the value of the integral is simply $2\pi i$ times the residue of $e^{1/z}$ at 0. Let's find the Laurent series about 0. We already know that

$$e^z = \sum_{j=0}^{\infty} \frac{1}{z!} z^j$$

for all z. Thus,

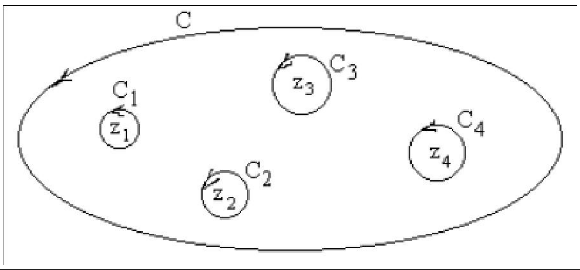
$$e^{1/z} = \sum_{j=0}^{\infty} \frac{1}{j!} z^{-j} = 1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \ldots$$

The residue $c_{-1} = 1$, and so the value of the integral is simply $2\pi i$.

Now suppose, we have a function *f* which is analytic everywhere except for isolated singularities, and let C be a simple closed curve (positively oriented) on which f is analytic. Then there will be only a finite number of singularities of f inside C (why?). Call them $z_1, z_2, \ldots, z_n$. For each $k = 1, 2, \ldots, n$, let C_k be a positively oriented circle centered at z_k and with radius small enough to insure that it is inside C and has no other singular points inside it.

Notes

Figure 10.1



Then,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C2} f(z)dz + ... + \int_{C_n} f(z)dz$$

$$= 2\pi i \operatorname*{Res}_{z=z_1} f + 2\pi i \operatorname*{Res}_{z=z_2} f + ... + 2\pi i \operatorname*{Res}_{z=z_n} f$$

$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f.$$

This is the celebrated Residue Theorem. It says that the integral of f is simply $2\pi i$ times the sum of the residues at the singular points enclosed by the contour C.

10.2 Poles and other Singularities

In order for the Residue Theorem to be of much help in evaluating integrals, there needs to be some better way of computing the residue—finding the Laurent expansion about each isolated singular point is a chore. We shall now see that in the case of a special but commonly occurring type of singularity the residue is easy to find. Suppose z_0 is an isolated singularity of f and suppose that the Laurent series of f at z_0 contains only a finite number of terms involving negative powers of $z - z_0$. Thus,

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} + ... + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + ...$$

Multiply this expression by $(z - z_0)^n$:

$$\phi(z) = (z - z_0)^n f(z) = c_{-n} + c_{-n+1}(z - z_0) + ... + c_{-1}(z - z_0)^{n-1} + ...$$

What we see is the Taylor series at z_0 for the function $\phi(z) = (z - z_0)^n f(z)$. The coefficient of $(z - z_0)^{n-1}$ is what we seek, and we know that this is

$$\frac{\phi^{(n-1)}(z_0)}{(n-1)!}$$

The sought after residue c_{-1} is thus,

$$c_{-1} = \operatorname*{Res}_{z=z_0} f = \frac{\phi^{(n-1)}(z_0)}{(n-1)!},$$

where $\phi(z) = (z - z_0)^n f(z)$.

Notes

Example:

We shall find all the residues of the function

$$f(z) = \frac{e^z}{z^2(z^2+1)}.$$

First, observe that f has isolated singularities at 0, and ± i. Let's see about the residue at 0. Here, we have,

$$\phi(z) = z^2 f(z) = \frac{e^z}{z^2(z^2+1)}.$$

The residue is simply $\phi'(0)$:

$$\phi'(z) = \frac{(z^2+1)e^z - 2ze^z}{(z^2+1)^2}.$$

Hence,

$$\operatorname*{Res}_{z=0} f = \phi'(0) = 1.$$

Next, let's see what we have at z = i:

$$f(z) = (z - i)f(z) = \frac{e^z}{z^2(z^2+1)},$$

and so

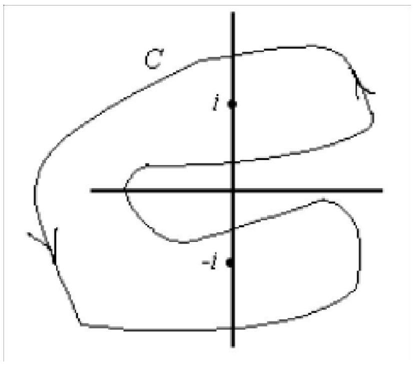
$$\operatorname*{Res}_{z=0} f(z) = \phi(i) = -\frac{e^i}{2i}.$$

In the same way, we see that

$$\operatorname*{Res}_{z=-i} f = \frac{e^{-i}}{2i}.$$

Let's find the integral $\int_C \frac{e^z}{z^2(z^2+1)} dz$, where C is the contour pictured:

Figure 10.2



Notes

This is now easy. The contour is positive oriented and encloses two singularities of f; viz, i and –i. Hence,

$$\int_C \frac{e^z}{z^2(z^2+1)}dz = 2\pi i\left[\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=-i} f\right]$$

$$= 2\pi i\left[-\frac{e^i}{2i} + \frac{e^{-i}}{2i}\right]$$

$$= -2\pi i \sin 1.$$

There is some jargon that goes with all this. An isolated singular point z_0 of f such that the Laurent series at z_0 includes only a finite number of terms involving negative powers of $z - z_0$ is called a pole. Thus, if z_0 is a pole, there is an integer n so that $\phi(z) = (z - z_0)^n f(z)$ is analytic at z_0, and $f(z_0) \neq 0$. The number n is called the order of the pole. Thus, in the preceding example, 0 is a pole of order 2, while i and –i are poles of order 1. (A pole of order 1 is frequently called a simple pole.) We must hedge just a bit here. If z_0 is an isolated singularity of f and there are no Laurent series terms involving negative powers of $z - z_0$, then we say z_0 is a removable singularity.

Example:

Let

$$f(z) = \frac{\sin z}{z};$$

then the singularity z = 0 is a removable singularity:

$$f(z) = \frac{1}{z}\sin z = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...$$

and we see that in some sense f is "really" analytic at z = 0 if we would just define it to be the right thing there.

A singularity that is neither a pole or removable is called an essential singularity.

Let's look at one more labor-saving trick—or technique, if you prefer. Suppose f is a function:

$$f(z) = \frac{p(z)}{q(z)},$$

where p and q are analytic at z0, and we have $q(z_0) = 0$, while $q'(z_0) \neq 0$, and $p(z_0) \neq 0$.

Then,

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z_0) + p'(z_0)(z - z_0) + ...}{q'(z_0)(z - z_0) + \frac{q^n(z_0)}{2}(z - z_0)^2 ...}$$

Notes

and so

$$\phi(z) = (z - z_0)f(z) = \frac{p(z_0) + p'(z_0)(z - z_0) + ...}{q'(z_0) + \frac{q^n(z_0)}{2}(z - z_0) + ...}$$

Thus, z_0 is a simple pole and

$$\operatorname*{Res}_{z=z_0} f = \phi(z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Example:

Find the integral :

$$\int_C \frac{\cos z}{(e^z - 1)} dz,$$

where C is the rectangle with sides $x = \pm 1$, $y = -\pi$, and $y = 3\pi$.

The singularities of the integrand are all the places at which $e^z = 1$, or in other words, the points $z = 0, \pm 2\pi i, \pm 4\pi i,....$ The singularities enclosed by C are 0 and $2\pi i$. Thus,

$$\int_C \frac{\cos z}{(e^z - 1)} dz \; = \; 2\pi i\left[\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=2\pi i} f\right],$$

where

$$f(z) = \frac{\cos z}{e^z - 1}.$$

Observe this is precisely the situation just discussed: $f(z) = \frac{p(z)}{q(z)}$, where p and q are analytic, etc., etc. Now,

$$\frac{p(z)}{q'(z)} = \frac{\cos z}{e^z}.$$

Thus,

$$\operatorname*{Res}_{z=0} f = \frac{\cos 0}{1} = 1, \text{and}$$

$$\operatorname*{Res}_{z=2\pi i} f = \frac{\cos 2\pi i}{e^{2\pi i}} = \frac{e^{-2p} + e^{2\pi}}{2} = \cosh 2\pi.$$

Finally,

$$\int_C \frac{\cos z}{e^z - 1} dz \; = \; 2\pi i\left[\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=2\pi i} f\right]$$

$$= 2\pi i(1 + \cosh 2\pi)$$

Notes

10.3 Summary

- A point z_0 is a singular point of a function f if f not analytic at z_0, but is analytic at some point of each neighborhood of z_0. A singular point z_0 of f is said to be isolated if there is a neighborhood of z_0 which contains no singular points of f save z_0. In other words, f is analytic on some region $0 < |z - z_0| < \varepsilon$.
- Now suppose we have a function f which is analytic everywhere except for isolated singularities, and let C be a simple closed curve (positively oriented) on which f is analytic. Then there will be only a finite number of singularities of f inside C (why?). Call them $z_1, z_2, \ldots, z_n$. For each $k = 1, 2, \ldots, n$, let C_k be a positively oriented circle centered at z_k and with radius small enough to insure that it is inside C and has no other singular points inside it.

  Then,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C2} f(z)dz + \ldots + \int_{C_n} f(z)dz$$

$$= 2\pi i \operatorname*{Res}_{z=z_1} f + 2\pi i \operatorname*{Res}_{z=z_2} f + \ldots + 2\pi i \operatorname*{Res}_{z=z_n} f$$

$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f.$$

  This is the celebrated Residue Theorem. It says that the integral of f is simply $2\pi i$ times the sum of the residues at the singular points enclosed by the contour C.
- In order for the Residue Theorem to be of much help in evaluating integrals, there needs to be some better way of computing the residue—finding the Laurent expansion about each isolated singular point is a chore. We shall now see that in the case of a special but commonly occurring type of singularity the residue is easy to find. Suppose z_0 is an isolated singularity of f and suppose that the Laurent series of f at z_0 contains only a finite number of terms involving negative powers of $z - z_0$. Thus,

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \ldots$$

  Multiply this expression by $(z - z_0)^n$:

$$\phi(z) = (z - z_0)^n f(z) = c_{-n} + c_{-n+1}(z - z_0) + \ldots + c_{-1}(z - z_0)^{n-1} + \ldots$$

  What we see is the Taylor series at z_0 for the function $\phi(z) = (z - z_0)^n f(z)$. The coefficient of $(z - z_0)^{n-1}$ is what we seek, and we know that this is

$$\frac{\phi^{(n-1)}(z_0)}{(n-1)!}$$

10.4 Keywords

Singular point: A singular point z_0 of f is said to be isolated if there is a neighborhood of z_0 which contains no singular points of f save z_0.

Residue Theorem. It says that the integral of f is simply $2\pi i$ times the sum of the residues at the singular points enclosed by the contour C.

Notes

10.5 Self Assessment

1. A .... z_0 of f is said to be isolated if there is a neighborhood of z_0 which contains no singular points of f save z_0.
2. .... says that the integral of f is simply $2\pi i$ times the sum of the residues at the singular points enclosed by the contour C.
3. In order for the .... to be of much help in evaluating integrals, there needs to be some better way of computing the residue—finding the Laurent expansion about each isolated singular point is a chore.
4. Suppose z_0 is an isolated singularity of f and suppose that the .... of f at z_0 contains only a finite number of terms involving negative powers of $z - z_0$. Thus,

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \ldots$$

10.6 Review Questions

1. Evaluate the integrals. In each case, C is the positively oriented circle $|z| = 2$.

   (a) $\int_C e^{1/z^2} dz.$

   (b) $\int_C \sin\left(\frac{1}{z}\right) dz.$

   (c) $\int_C \cos\left(\frac{1}{z}\right) dz.$

   (d) $\int_C \frac{1}{z}\sin\left(\frac{1}{z}\right) dz.$

   (e) $\int_C \frac{1}{z}\cos\left(\frac{1}{z}\right) dz.$

2. Suppose f has an isolated singularity at z_0. Then, of course, the derivative f′ also has an isolated singularity at z_0. Find the residue $\operatorname{Res}_{z=z_0} f'$.
3. Given an example of a function f with a simple pole at z_0 such that $\operatorname{Res}_{z=z_0} f = 0,$ or explain carefully, why there is no such function.
4. Given an example of a function f with a pole of order 2 at z_0 such that $\operatorname{Res}_{z=z_0} f = 0,$ explain carefully, why there is no such function.
5. Suppose g is analytic and has a zero of order n at z0 (That is, g(z) = (z – z_0)nh(z), where $h(z_0) \neq 0$.). Show that the function f given by

$$f(z) = \frac{1}{g(z)}$$

   has a pole of order n at z_0. What is $\operatorname{Res}_{z=z_0} f$?

Notes

6. Suppose g is analytic and has a zero of order n at z_0. Show that the function f given by

$$f(z) = \frac{g'(z)}{g(z)}$$

has a simple pole at z_0, and $\operatorname*{Res}_{z=z_0} f = n$.

7. Find :

$$\int_C \frac{\cos z}{z^2 - 4} dz,$$

where C is the positively oriented circle $|z| = 6$.

8. Find :

$$\int_C \tan z dz,$$

where C is the positively oriented circle $|z| = 2\pi$.

9. Find :

$$\int_C \frac{1}{z^2 + z + 1} dz,$$

where C is the positively oriented circle $|z| = 10$.

Answers: Self Assessment

1. Singular point
2. Residue Theorem
3. Residue Theorem
4. Laurent series

10.7 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H. Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Richa Nandra, Lovely Professional University

Unit 11: Rouche's Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the concept of argument principle
- Describe the Rouche's theorem

Introduction

In last unit, you have studied about the Taylor series, singularities of complex valued functions and use the Laurent series to classify these singularities. This unit will explain the concept related to argument principle and Rouche's theorem.

11.1 Argument Principle

Let C be a simple closed curve, and suppose f is analytic on C. Suppose moreover that the only singularities of f inside C are poles. If $f(z) \neq 0$ for all $z \in C$, then $\Gamma = (C)$ is a closed curve which does not pass through the origin. If

$$\gamma(t),\ a \leq t \leq b$$

is a complex description of C, then

$$\xi(t) = f(\gamma(t)),\ \alpha \leq t \leq \beta$$

is a complex description of Γ. Now, let's compute

$$\int_C \frac{f'(z)}{f(z)} dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)dt.$$

Notes

But notice that $\xi'(t) = f'(\gamma(t))\gamma'(t)$. Hence,

$$\int_C \frac{f'(z)}{f(z)}dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)dt = \int_\alpha^\beta \frac{\xi'(t)}{\xi(t)}dt$$

where $|n|$ is the number of times Γ "winds around" the origin. The integer n is positive in case Γ is traversed in the positive direction, and negative in case the traversal is in the negative direction.

Next, we shall use the Residue Theorem to evaluate the integral $\int_C \frac{f'(z)}{f(z)}dz.$ The singularities of the integrand $\frac{f'(z)}{f(z)}$ are the poles of f together with the zeros of f. Let's find the residues at these points. First, let $Z = \{z_1, z_2, ..., z_K\}$ be set of all zeros of f. Suppose the order of the zero z_j is n_j. Then $f(z) = (z - z_j)^{n_j} h(z)$ and $h(z_j) \neq 0$. Thus,

$$\frac{f'(z)}{f(z)} = \frac{(z-p_j)^{m_j}h'(z) - m_j(z-p_j)^{m_j-1}h(z)}{(z-pj)^{2m_j}}.\frac{(z-p_j)m_j}{h(z)}$$

$$= \frac{h'(z)}{h(z)} - \frac{m_j}{(z-p_j)^{m_j}}.$$

Now then,

$$\phi(z) = (z-p_j)^{m_j}\frac{f'(z)}{f(z)} = (z-p_j)^{m_j}\frac{h'(z)}{h(z)} - m_j,$$

and so

$$\operatorname*{Res}_{z=p_j}\frac{f'}{f} = \phi(p_j) = -m_j.$$

The sum of all these residues is

$$-P = -m_1 - m_2 - ... - m_j$$

Then,

$$\int_C \frac{f'(z)}{f(z)}dz = 2\pi i(N-P);$$

and we already found that

$$\int_C \frac{f'(z)}{f(z)}dz = n2\pi i,$$

where n is the "winding number", or the number of times Γ winds around the origin—$n > 0$ means Γ winds in the positive sense, and n negative means it winds in the negative sense. Finally, we have

$n = N - P,$

where $N = n_1 + n_2 + ... + n_K$ is the number of zeros inside C, counting multiplicity, or the order of the zeros, and $P = m_1 + m_2 + ... + m_j$ is the number of poles, counting the order. This result is the celebrated **argument principle.**

Notes

11.2 Rouche's Theorem

Suppose f and g are analytic on and inside a simple closed contour C. Suppose that $|f(z)| > |g(z)|$ for all $z \in C$. Then we shall see that f and f + g have the same number of zeros inside C. This result is Rouche's Theorem. To see why it is so, start by defining the function $\psi(t)$ on the interval $0 \leq t \leq 1$:

$$\psi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(t)}{f(z) + tg(z)} dz.$$

Observe that the denominator of the integrand is never zero:

$$|f(z) + tg(z)| \geq ||f(t) - t|g(t)|| \geq ||f(t)| - |g(t)|| > 0.$$

Observe that Ψ, is continuous on the interval [0,1] and is integer-valued—$\Psi(t)$ is the number of zeros of f + tg inside C. Being continuous and integer-valued on the connected set [0,1], it must be constant. In particular, $\Psi(0) = \Psi(1)$. This does the job!

$$\psi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$$

is the number of zeros of f inside C, and

$$\psi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + g'(t)}{f(z) + g(z)} dz.$$

is the number of zeros of f + g inside C.

Example:

How many solutions of the equation $z^6 - 5z^5 + z^3 - 2 = 0$ are inside the circle $|z| = 1$? Rouche's Theorem makes it quite easy to answer this. Simply let $f(z) = -5z^5$ and let $g(z) = z^6 + z^3 - 2$. Then $|f(z)| = 5$ and $|g(z)| \leq |z|^6 + |z|^3 + 2 = 4$ for all $|z| = 1$. Hence $|f(z)| > |g(z)|$ on the unit circle. From Rouche's Theorem we know then that f and f + g have the same number of zeros inside $|z| = 1$. Thus, there are 5 such solutions.

The following nice result follows easily from Rouche's Theorem. Suppose U is an open set (i.e., every point of U is an interior point) and suppose that a sequence (f_n) of functions analytic on U converges uniformly to the function f. Suppose further that f is not zero on the circle $C = \{z : |z : z_0| = R\} \subset U$. Then there is an integer N so that for all $n \geq N$, the functions f_n and f have the same number of zeros inside C.

This result, called Hurwitz's Theorem, is an easy consequence of Rouche's Theorem. Simply observe that for $z \in C$, we have $|f(z)| > \in > 0$ for some ε. Now let N be large enough to insure that $|f_n(z) - f(z)| < \varepsilon$ on C. It follows from Rouche's Theorem that f and $f + (f_n - f) = f_n$ have the same number of zeros inside C.

Example:

On any bounded set, the sequence (f_n), where $f_n(z) = 1 + z + \frac{z^2}{2} + ... + \frac{z_n}{n!}$, converges uniformly to $f(z) = e^z$, and $f(z) \neq 0$ for all z. Thus for any R, there is an N so that for n > N, every zero of

$1 + z + \frac{z^2}{2} + ... + \frac{z_n}{n!}$, has modulus > R. Or to put it another way, given an R there is an N so that for n > N no polynomial $1 + z + \frac{z^2}{2} + ... + \frac{z_n}{n!}$, has a zero inside the circle of radius R.

11.3 Summary

- Let C be a simple closed curve, and suppose f is analytic on C. Suppose moreover that the only singularities of f inside C are poles. If $f(z) \neq 0$ for all $z \in C$, then Γ = (C) is a closed curve which does not pass through the origin. If

  $$\gamma(t),\ \alpha \leq t \leq \beta$$

  is a complex description of Γ. Now, let's compute

  $$\int_C \frac{f'(z)}{f(z)} dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)dt.$$

  But notice that ξ′(t) = f′(γ(t))γ′(t). Hence,

  $$\int_C \frac{f'(z)}{f(z)} dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)dt = \int_\alpha^\beta \frac{\xi'(t)}{\xi(t)} dt$$

  where |n| is the number of times Γ "winds around" the origin. The integer n is positive in case Γ is traversed in the positive direction, and negative in case the traversal is in the negative direction.

- We shall use the Residue Theorem to evaluate the integral $\int_C \frac{f'(z)}{f(z)} dz.$ The singularities of the integrand $\frac{f'(z)}{f(z)}$ are the poles of f together with the zeros of f. Let's find the residues at these points. First, let Z = {$z_1, z_2, ..., z_K$} be set of all zeros of f. Suppose the order of the zero z_j is n_j. Then $f(z) = (z - z_j)^{n_j} h(z)$ and $h(z_j) \neq 0$. Thus,

  $$\frac{f'(z)}{f(z)} = \frac{(z - p_j)^{m_j} h'(z) - m_j (z - p_j)^{m_j - 1} h(z)}{(z - pj)^{2m_j}} . \frac{(z - p_j)m_j}{h(z)}$$

  $$= \frac{h'(z)}{h(z)} - \frac{m_j}{(z - p_j)^{m_j}}.$$

- Suppose f and g are analytic on and inside a simple closed contour C. Suppose that |f(z)| > |g(z)| for all $z \in C$. Then we shall see that f and f + g have the same number of zeros inside C. This result is Rouche's Theorem. To see why it is so, start by defining the function ψ(t) on the interval $0 \leq t \leq 1$:

  $$\psi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(t)}{f(z) + tg(z)} dz.$$

Notes

Observe that the denominator of the integrand is never zero:

$$|f(z) + tg(z)| \geq ||f(t) - t|g(t)|| \geq ||f(t)| - |g(t)|| > 0.$$

Observe that Ψ¸ is continuous on the interval [0,1] and is integer-valued—Ψ(t) is the number of zeros of f + tg inside C. Being continuous and integer-valued on the connected set [0,1], it must be constant. In particular, Ψ(0) = Ψ(1). This does the job!

$$\psi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

is the number of zeros of f inside C, and

$$\psi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z)+g'(t)}{f(z)+g(z)}dz.$$

is the number of zeros of f + g inside C.

11.4 Keyword

Rouche's Theorem: Suppose f and g are analytic on and inside a simple closed contour C. Suppose moreover that $|f(z)| > |g(z)|$ for all $z \in C$. Then we shall see that f and f + g have the same number of zeros inside C. This result is Rouche's Theorem.

11.5 Self Assessment

1. Let C be a simple closed curve, and suppose f is analytic on C. Suppose moreover that the only .... of f inside C are poles.
2. The integer n is positive in case G is traversed in the ...., and negative in case the traversal is in the negative direction.
3. Being continuous and integer-valued on the connected set [0,1], it must be constant. In particular, ...

11.6 Review Questions

1. Let C be the unit circle $|z| = 1$ positively oriented, and let f be given by $f(z) = z^3$. How many times does the curve f„°C„± wind around the origin? Explain.
2. Let C be the unit circle $|z| = 1$ positively oriented, and let f be given by

$$f(z) = \frac{z^2+2}{z^3}.$$

   How many times does the curve f(C) wind around the origin? Explain.

3. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a0$, with $a_n \neq 0$. Prove there is an R > 0 so that if C is the circle $|z| = R$ positively oriented, then

$$\int_C \frac{p'(z)}{p(z)}dz = 2n\pi i.$$

4. How many solutions of $3e^z - z = 0$ are in the disk $|z| < 1$? Explain.

5. Suppose f is entire and f(z) is real if and only if z is real. Explain how you know that f has at most one zero.

6. Show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros inside the circle $|z| = 1$.

7. How many solutions of $2z^4 - 2z^3 + 2z^2 - 2z + 9 = 0$ lie inside the circle $|z| = 1$?

8. Use Rouche's Theorem to prove that every polynomial of degree n has exactly n zeros (counting multiplicity, of course).

9. Let C be the closed unit disk $|z| \leq 1$. Suppose the function f analytic on C maps C into the open unit disk $|z| < 1$ — that is, $|f(z)| < 1$ for all $z \in C$. Prove there is exactly one $w \in C$ such that $f(w) = w$. (The point w is called a fixed point of f .)

Answers: Self Assessment

1. singularities
2. positive direction
3. $Y(0) = Y(1)$

11.7 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H. Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Unit 12: Fundamental Theorem of Algebra

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the concept of fundamental theorem on algebra
- Describe the calculus of residues
- Discuss the multivalued functions and its branches

Introduction

In last unit, you have studied about the Taylor series, singularities of complex valued functions and use the Laurent series to classify these singularities. Also you studied about the concept related to argument principle and Rouche's theorem. This unit will explain fundamental theorem on algebra.

12.1 Fundamental Theorem of Algebra

Every polynomial of degree *n* has exactly *n* zeros.

Proof. Let us consider the polynomial

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,\ a_n \neq 0$$

We take $f(z) = a_n z^n$, $g(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1}$

Let C be a circle $|z| = r$, where $r > 1$.

Now, $$|f(z)| = |a_n z^n| = |a_n|\, r^n$$

$$|g(z)| \leq |a_0| + |a_1|\, r + |a_2|\, r^2 + \ldots + |a_{n-1}|\, r^{n-1}$$

$$\leq (|a_0| + |a_1| + \ldots + |a_{n-1}|)\, r^{n-1}$$

Therefore,

$$\left|\frac{g(z)}{f(z)}\right| = \frac{(|a_0|+|a_1|+...+|a_{n-1}|)r^{n-1}}{|a_n|r^n}$$

$$= \frac{|a_0|+|a_1|+...+|a_{n-1}|}{|a_n|r}$$

Hence $|g(z) < |f(z)|$, provided that

$$\frac{|a_0|+|a_1|+...+|a_{n-1}|}{|a_n|r} < 1$$

i.e. $$r > \frac{|a_0|+|a_1|+...+|a_{n-1}|}{|a_n|} \qquad (1)$$

Since r is arbitrary, therefore, we can choose r large enough so that (1) is satisfied. Now, applying Rouche's theorem, we find that the given polynomial f(z + g(z) has the same number of zeros as f(z). But f(z) has exactly n zeros all located at z = 0. Hence, the given polynomial has exactly n zeros.

Example: Determine the number of roots of the equation

$$z^8 - 4z^5 + z^2 - 1 = 0$$

that lie inside the circle $|z| = 1$

Solution. Let C be the circle defined by $|z| = 1$

Let us take $f(z) = z^8 - 4z^5$, $g(z) = z^2 - 1$.

On the circle C,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^2-1}{z^8-4z^5}\right| \le \frac{|z|^2+1}{|z|^5|4-z^3|}$$

$$\le \frac{1+1}{4-|z|^3} = \frac{2}{4-1} = \frac{2}{3} < 1$$

Thus, $|g(z)| < |f(z)|$ and both f(z) and g(z) are analytic within and on C, Rouche's theorem implies that the required number of roots is the same as the number of roots of the equation $z^8 - 4z^5 = 0$ in the region $|z| < 1$. Since $z^3 - 4 \neq 0$ for $|z| < 1$, therefore, the required number of roots is found to be 5.

Inverse Function

If f(z) = w has a solution z = F(w), then we may write

f{F(w) } = w, F{ f(z)} = z. The function F defined in this way, is called inverse function of f.

Theorem. (Inverse Function Theorem)

Let a function w = f(z) be analytic at a point $z = z_0$ where $f'(z_0) \neq 0$ and $w_0 = f(z_0)$.

Notes

Then there exists a neighbourhood of w_0 in the w-plane in which the function $w = f(z)$ has a unique inverse $z = F(w)$ in the sense that the function F is single-valued and analytic in that neighbourhood such that $F(w_0) = z_0$ and

$$F'(w) = \frac{1}{f'(z)}.$$

Proof. Consider the function $f(z) - w_0$. By hypothesis, $f(z_0)\ w_0 = 0$. Since $f'(z_0) \neq 0$, f is not a constant function and therefore, neither $f(z) - w_0$ not $f'(z)$ is identically zero. Also $f(z) - w_0$ is analytic at $z = z_0$ and so it is analytic in some neighbourhood of z_0. Again, since zeros are isolated, neither $f(z) - w_0$ nor $f'(z)$ has any zero in some deleted neighbourhood of z_0. Hence, there exists $R > 0$ such that $f(z) - w_0$ is analytic for $|z - z_0| \leq R$ and $f(z) - w_0 \neq 0$, $f'(z) \neq 0$ for $0 < |z - z_0| \leq R$. Let D denote the open disc

$$\{z : |z - z_0| < R\}$$

and C denotes its boundary

$$\{z : |z - z_0| = R\}.$$

Since $f(z) - w_0$ for $|z - z_0| \leq R$, we conclude that $|f(z) - w_0|$ has a positive minimum on the circle C. Let

$$\min_{z \in C} |f(z) - w_0| = m$$

and choose d such that $0 < d < m$.

We now show that the function f(z) assumes exactly once in D every value w_1 in the open disc $T = \{w : |w - w_0| < d\}$. We apply Rouche's theorem to the functions $w_0 - w_1$ and $f(z)-w_0$. The condition of the theorem are satisfied, since

$$|w_0 - w_1| < d < m = |f(z) - w_0| \leq |f(z) - w_0| \text{ on C.}$$

Thus, we conclude that the functions.

$$f(z) - w_0 \text{ and } (f(z) - w_0) + (w_0 - w_1) = f(z) - w_1$$

have the same number of zeros in D. But the function $f(z) - w_0$ has only one zero in D i.e. a simple zeros at z_0, since $(f(z) - w_0)' = f'(z) \neq 0$ at z_0.

Hence, $f(z) - w_1$ must also have only one zero, say z_1 in D. This means that the function f(z) assumes the value w, exactly once in D. It follows that the function $w = f(z)$ has a unique inverse, say $z = F(w)$ in D such that F is single-valued and $w = f\{F(w)\}$. We now show that the function F is analytic in D. For fix w_1 in D, we have $f(z) = w_1$ for a unique z_1 in D. If w is in T and $F(w) = z$, then

$$\frac{F(w) - F(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} \tag{2}$$

It is noted that T is continuous. Hence, $z \to z_1$ whenever $w \to w_1$. Since $z_1 \in D$, as shown above $f'(z_1)$ exists and is zero. If we let $w \to w$, then (2) shows that

$$F'(w_1) = \frac{1}{f'(z_1)}.$$

Thus $F'(w)$ exists in the neighbourhood T of w_0 so that the function F is analytic there.

12.2 Calculus of Residues

The main result to be discussed here is Cauchy's residue theorem which does for meromorphic functions what Cauchy's theorem does for holomorphic functions. This theorem is extremely important theoretically and for practical applications.

The Residue at a Singularity

We know that in the neighbourhood of an isolated singularity z = a, a one valued analytic function *f*(z) may be expanded in a Laurent's series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

The co-efficient b_1 is called the residue of f(z) at z = a and is given by the formula

$$\text{Res}\,(z = a) = b_1 = \frac{1}{2\pi i}\int_{\gamma} f(z)\,dz \qquad \Big|\because b_n \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)dz}{(z-a)^{-n+1}}$$

Where g is any circle with centre z = a, which excludes all other singularities of f(z). In case, z = a is a simple pole, then we have

$$\text{Res}\,(z = a) = b_1 = \lim_{z\to a} (z\text{-}a)\, f(z) \qquad \Big|\because \sum_{0}^{\infty} a_n\,(z-a)^n + \frac{b_1}{z-a}$$

A more general definition of the residue of a function f(z) at a point z = a is as follows.

If the point z = a is the only singularity of an analytic function f(z) inside a closed contour C, then the value f(z) dz is called the residue of f(z) at a.

Residue at Infinity

If f(z) is analytic or has an isolated singularity at infinity and if C is a circle enclosing all its singularities in the finite parts of the z-plane, the residue of f(z) at infinity is defined by

$$\text{Res}\,(z = \infty) = \frac{1}{2\pi i}\int_{C} f(z)\,dz, \qquad \Big|\text{ or Res}\,(z = \infty) - \frac{1}{2\pi i}\int_{C} f(z)\,dz,$$

Integration taken in positive sense

the integration being taken round C in the negative sense w.r.t. the origin, provided that this integral has a definite value. By means of the substitution z = w^{-1}, the integral defining the residue at infinity takes the form

$$\frac{1}{2\pi i}\int[-f(w^{-1})]\frac{dw}{w^2},$$

taken in positive sense round a sufficiently small circle with centre at the origin.

Thus, we also say if

$$\lim_{w\to 0}\,[-f(w^{-1})\; w^{-1}] \qquad \text{or} \qquad \lim_{w\to 0}\,[-z\, f(z)]$$

has a definite value, that value is the residue of f(z) at infinity.

Notes

Remarks. (i) The function may be regular at infinity, yet has a residue there.

For example, consider the function $f(z) = \dfrac{b}{z-a}$ for this function

$$\text{Res } (z = \infty) = -\frac{1}{2\pi i}\int_C f(z)\, dz$$

$$= -\frac{1}{2\pi i}\int_C \frac{b}{z-a}\, dz$$

$$= -\frac{b}{2\pi i}\int_0^{2\pi} \frac{re^{i\theta} i d\theta}{re^{i\theta}}, \text{ C being the circle } |z-a| = r$$

$$= -\frac{b}{2\pi}\int_0^{2\pi} d\theta = -b$$

$\therefore$ Res $(z = \infty) = -b$

Also, $z = a$ is a simple pole of $f(z)$ and its residue there is $\dfrac{1}{2\pi i}\displaystyle\int_C f(z)\, dz = b$

$| \text{ or } \lim\limits_{z\to a} (z-a) f(z) = b$

Thus, Res $(z = a) = b$ – Res $(z = \infty)$

(ii) If the function is analytic at a point $z = a$, then its residue at $z = a$ is zero but not so at infinity.

(iii) In the definition of residue at infinity, C may be any closed contour enclosing all the singularities in the finite parts of the z-plane.

Calculation of Residues

Now, we discuss the method of calculation of residue in some special cases.

(i) If the function $f(z)$ has a simple pole at $z = a$, then, Res $(z = a) = \lim\limits_{z\to a} (z-a) f(z)$.

(ii) If $f(z)$ has a simple pole at $z = a$ and $f(z)$ is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$ i.e. a rational function, then

$$\text{Res } (z = a) = \lim_{z\to a} (z-a) f(z) = \lim_{z\to a} (z-a) \frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z\to a} \frac{\phi(z)}{\dfrac{\psi(z)-\psi(a)}{z-a}}$$

$$= \frac{\phi(a)}{\psi'(a)},$$

where $\psi(a) = 0$, $\psi'(a) \neq 0$, since $\psi(z)$ has a simple zero at $z = a$

Notes

(iii) If *f*(z) has a pole of order m at z = a then we can write

$$f(z) = \frac{\phi(z)}{(z-a)^m} \qquad (1)$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

Now, $\text{Res}\ (z = a) = b_1 = \frac{1}{2\pi i}\int_C f(z)\, dz = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^m} dz$

$$= \frac{1}{\lfloor m-1}\frac{\lfloor m-1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^{m-1+1}}\, dz$$

$$= \frac{1}{\lfloor m-1}\phi^{m-1}(a) \qquad \text{[By Cauchy's integral formula for derivatives]} \qquad (2)$$

Using (1), formula (2) take the form

$$\text{Res}\ (z = a) = \frac{1}{\lfloor m-1}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] \text{ as } z \to a$$

i.e. $$\text{Res}\ (z = a) = \lim_{z\to a}\frac{1}{\lfloor m-1}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] \qquad (3)$$

Thus, for a pole of order m, we can use either formula (2) or (3).

(iv) If z = a is a pole of any order for f(z), then the residue of f(z) at z = a is the co-efficient of $\frac{1}{z-a}$ in Laurent's expansion of f(z)

(v) Res ($z = \infty$) = Negative of the co-efficient of $\frac{1}{z}$ in the expansion of f(z) in the neighbourhood of $z = \infty$.

Example: (a) Find the residue of $\frac{z^4}{z^2+a^2}$ at z = –ia

Solution. Let $f(z) = \frac{z^4}{z^2+a^2}$.

Poles of f(z) are $z = \pm\, ia$

Thus z = –ia is a simple pole, so

$$\text{Res}\ (z = -ia) = \lim_{z\to -ia}(z + ia)\, f(z)$$

$$= \lim_{z\to -ia}(z+ia)\frac{z^4}{(z+ia)(z-ia)}$$

Notes

$$= \lim_{z \to -ia} \frac{z^4}{z - ia} = \frac{a^4}{-2ia} \qquad \left| \begin{array}{l} \text{or } \phi(z) = \dfrac{z^4}{z - ia} \\ \text{as } f(z) = \dfrac{z^4/(z - ia)}{(z + ia)} \end{array} \right.$$

$$= \frac{a^4 i}{2a} = \frac{ia^3}{2}$$

(b) Find the residues of $e^{iz}\, z^{-4}$ at its poles.

Solution. Let $f(z) = \dfrac{e^{iz}}{z^4}$

f(z) has pole of order 4 at z = 0, so

$$\text{Res}\,(z = 0) = \frac{1}{\lfloor 3}\left[\frac{d^3}{dz^3}(e^{iz})\right]_{z=0} = -\frac{i}{6} \qquad |\ \phi(z) = e^{iz}$$

Alternatively, by the Laurent's expansion

$$\frac{e^{iz}}{z^4} = \frac{1}{z^4} + \frac{i}{z^3} - \frac{1}{\lfloor 2 z^2} - \frac{i}{\lfloor 3 z} + \ldots$$

we find that

$$\text{Res}\,(z = 0) = \text{co-efficient of } \frac{1}{z}$$

$$= -\frac{i}{6}$$

(c) Find the residue of $\dfrac{z^3}{z^2 - 1}$ at $z = \infty$.

Solution. Let $f(z) = \dfrac{z^3}{z^2 - 1} = \dfrac{z^3}{z^2\left(1 - \dfrac{1}{z^2}\right)} = z\left(1 - \dfrac{1}{z^2}\right)^{-1}$

$$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \ldots.\right)$$

$$= z + \frac{1}{z} + \frac{1}{z^3} + \ldots$$

Therefore,

$$\text{Res}\,(z = \infty) = -\left(\text{co-efficient of } \frac{1}{z}\right) = -1$$

(d) Find the residues of at its poles.

Solution. Let $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$

Notes

Poles of f(z) are z = 1 (order four) and z = 2, 3 (simple)

Therefore,

$$\text{Res } (z = 2) = \lim_{z\to 2} (z-2)\, f(z) = \lim_{z\to 2} \frac{z^3}{(z-1)^4(z-3)} = -8$$

$$\text{Res } (z = 3) = \lim_{z\to 3} (z - 3)\, f(z) = \frac{27}{16}$$

For z = 1, we take $\phi(z) = \dfrac{z^3}{(z-2)(z-3)}$

where $\qquad f(z) = \dfrac{\phi(z)}{(z-1)^4}$ and thus, $\text{Res } (z = 1) = \dfrac{\phi^3(1)}{\underline{\lfloor 3}}$

$$\text{Now, } f(z) = z + 5 - \frac{8}{z-2} + \frac{27}{z-3}$$

$$\phi^3(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\phi^3(1) = \frac{303}{8}$$

Thus,

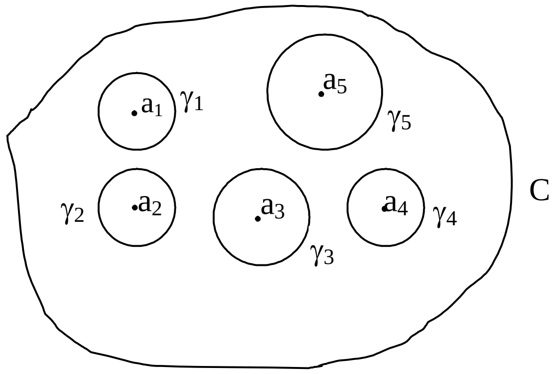
$$\text{Res } (z = 1) = \frac{303}{8\underline{\lfloor 3}} = \frac{101}{16}$$

Theorem. (Cauchy Residue Theorem)

Let f(z) be one-valued and analytic inside and on a simple closed contour C, except for a finite number of poles within C. Then

$$\int_C f(z)\, dz = 2\pi i \text{ [Sum of residues of f(z) at its poles within C]}$$

Proof. Let $a_1, a_2, \ldots, a_n$ be the poles of *f*(z) inside C. Draw a set of circles g_r of radii ∈ and centre a_r (r = 1, 2,…, n) which do not overlap and all lie within C. Then, *f*(z) is regular in the domain bounded externally by C and internally by the circles g_r.



Notes

Then by cor. to Cauchy's Theorem, we have

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz \tag{4}$$

Now, if a_r is a pole of order m, then by Laurent's theorem, f(z) can be expressed as

$$f(z) = \phi(z) + \sum_{s=1}^{m} \frac{b_s}{(z-a_r)^s}$$

where $\phi(z)$ is regular within and on γ_r.

Then

$$\int_{\gamma_r} f(z)\,dz = \sum_{s=1}^{m} \int_{\gamma_r} \frac{b_s}{(z-a_r)^s}\,dz$$

(5)

where $\int_{\gamma_r} f(z)\,dz = 0,$ by Cauchy's theorem

Now, on γ_r $|z-a_r| = \in$ i.e. $z = a_r + \in e^{i\theta}$

$\Rightarrow$ $dz = \in ie^{i\theta}\,d\theta$

where θ varies from 0 to 2π as the point z moves once round g_r.

Thus, $\int_{\gamma_r} f(z)\,dz = \sum_{s=1}^{m} b_s \in^{1-s} \int_0^{2\pi} e^{(1-s)i\theta}\,i d\theta$

= 2pi b_1

= 2pi [Residue of f(z) at a_r]

where $\int_0^{2\pi} e^{(1-s)i\theta}\,d\theta = \begin{cases} 0, \text{ if } s \neq 1 \\ 2\pi \text{ if } s = 1 \end{cases}$

Hence, from (4), we find

$$\int_C f(z)dz = \sum_{r=1}^{n} 2\pi i\,[\text{Residue of } f(z) \text{ at } a_r]$$

$$= 2\pi i \left[\sum_{r=1}^{n} \text{Residue of } f(z) \text{ at } a_r\right]$$

= $2\pi i$ [sum of Residues of f(z) at its poles inside C.]

which proves the theorem.

Remark. If f(z) can be expressed in the form $f(z) = \dfrac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$, then the pole z = a is a pole of type I or overt.

If f(z) is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$, where $\phi(z)$ and $\psi(z)$ are analytic and $\phi(a) \neq 0$ and $\psi(z)$ has a zero of order *m* at *z* = *a*, then *z* = *a* is a pole of type II or covert. Actually, whether a pole of f(z) is overt or covert, is a matter of how f(z) is written.

12.3 Jordan's Inequality

Notes

If $0 \le \theta \le \pi/2$, the $\frac{2\theta}{\pi} \le \sin\theta \le \theta$

This inequality is called Jordan inequality. We know that as θ increases from 0 to π/2, cosθ decreases steadily and consequently, the mean ordinate of the graph of y = cos x over the range $0 \le x \le \theta$ also decreases steadily. But this mean ordinate is given by

$$\frac{1}{\theta}\int_0^\theta \cos x \, dx = \frac{\sin\theta}{\theta}$$

It follows that when $0 \le \theta \le \pi/2$,

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$$

Jordan's Lemma

If f(z) is analytic except at a finite number of singularities and if $f(z) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to\infty}\int_T e^{imz} f(z)\, dz = 0 \text{ , } m > 0$$

where T denotes the semi-circle $|z| = R$, $I_m.\ z \ge 0$, R being taken so large that all the singularities of f(z) lie within T.

Proof. Since $f(z) \to 0$ uniformly as $|z| \to \infty$, there exists $\in > 0$ such that $|f(z)| < \in \ \forall\ z$ on T.

Also $|z| = R \quad \Rightarrow z = Re^{i\theta} \Rightarrow dz = Re^{i\theta}\, id\theta \quad \Rightarrow |dz| = Rd\theta$

$$|e^{imz}| = |e^{im\,Re^{i\theta}}| = |e^{imR\cos\theta}\, e^{-mR\sin\theta}|$$

$$= e^{-mR\sin\theta}$$

Hence, using Jordan inequality,

$$\left|\int_T e^{imz} f(z)\, dz\right| \le \left|\int_T e^{imz} f(z)\right| |dz|$$

$$< \int_0^\pi e^{-mR\sin a} \in R\, d\theta$$

$$= 2 \in R\int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

$$= 2 \in R\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta \qquad \left|\begin{array}{l} \because \frac{2a}{\pi} \le \sin\theta \\ \text{i.e.} -\sin\theta \le \frac{-2\theta}{\pi} \end{array}\right.$$

$$= 2 \in R\,\frac{(1-e^{-mR})}{2mR/\pi}$$

$$= \frac{\in\pi}{m}(1-e^{-mR}) < \frac{\in\pi}{m}$$

Notes

Hence $\lim_{R\to\infty} \int_T e^{imz}\ f(z)\ dz = 0$

Example: By method of contour integration prove that

$$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{2a} e^{-ma},\ \text{where } m \geq 0,\ a > 0$$

Solution. We consider the integral

$$\int_C f(z)\ dz,\ \text{ where } f(z) = \frac{e^{imz}}{z^2 + a^2}$$

and C is the closed contour consisting of T, the upper half of the large circle $|z| = R$ and real axis from $-R$ to R.

$$\text{Now, } \frac{1}{z^2 + a^2} \to 0 \text{ as } |z| = R\to\infty$$

Hence by Jordan lemma,

$$\lim_{R\to\infty} \int_T \frac{e^{im}z}{z^2 + a^2}\, dz = 0$$

i.e. $\quad \lim_{R\to\infty} \int_T f(z)dz = 0 \qquad (1)$

Now, poles of f(z) are given by $z = \pm ia$ (simple), out of which $z = ia$ lies within C.

$$\therefore \qquad \text{Res } (z = ia) = \frac{e^{-ma}}{2ia}$$

Hence by Cauchy's residue theorem,

$$\int_C f(z) = 2pi\, \frac{e^{-ma}}{2ia} = \frac{\pi}{a} e^{-ma}$$

or

$$\int_T f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{a} e^{-ma}$$

Making $R\to\infty$ and using (1), we get

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2}\, dx = \frac{\pi}{a} e^{-ma}$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{a} e^{-ma}$$

or $\quad \displaystyle\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{2a} e^{-ma}$

Hence the result.

Deduction. (i) Replacing m by a and a by 1 in the above example, we get

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}$$

Putting a = 1, we get

$$\int_0^\infty \frac{\cos x}{x^2+1} = \frac{\pi}{2}e^{-1} = \frac{\pi}{2e}$$

(ii) Taking m = 1, a = 2, we get

$$\int_0^\infty \frac{\cos x}{x^2+4}\,dx = \frac{\pi}{4e^2}$$

Example: Prove that $\int_{-\infty}^\infty \frac{x^3 \sin mx}{x^4+a^4} = \frac{\pi}{2}e^{-ma/\sqrt{2}}\cos\left(\frac{ma}{\sqrt{2}}\right)$ m > 0, a > 0

Solution. Consider the integral $\int_C f(z)\,dz$, where

$$f(z) = \frac{z^3 e^{imz}}{z^4+a^4}$$

and C is the closed contour….

Since $\frac{z^3}{z^4+a^4} \to 0$ as $|z| = R \to \infty$, so by

Jordan lemma,

$$\lim_{R\to\infty}\int_T \frac{z^3 e^{imz}}{z^4+a^4}\,dz = 0 \qquad (2)$$

Poles of f(z) are given by

$z^4 + a^4 = 0$

or $\quad z^4 = -a^4 = e^{2n\pi i}\, e^{\pi i}\, a^4$

or $\quad z = a\, e^{(2n+1)\pi i/4}$, n = 0, 1, 2, 3.

Out of these four simple poles, only

$z = ae^{i\pi/4}$, $a\, e^{i3\pi/4}$ lie within C.

If $f(z) = \frac{\phi(z)}{\psi(z)}$, then Res $(z = \alpha) = \lim_{z\to\alpha}\frac{\phi(z)}{\psi'(z)}$, a being simple pole.

$\therefore$ For the present case,

$$\text{Res}\,(z=\alpha) = \lim_{z\to\alpha}\frac{z^3 e^{imz}}{4z^3} = \lim_{z\to\alpha}\frac{e^{imz}}{4}.$$

Thus, $\quad$ Res $(z = ae^{i\pi/4})$ + Res $(z = a\, e^{iz\pi/4})$

Notes

$$= \frac{1}{4}[\exp(\text{ima } e^{i\pi/4}) + \exp(\text{ima } e^{iz\pi/4})]$$

$$= \frac{1}{4}\left[\exp\left\{\text{ima}\left(\frac{i+1}{\sqrt{2}}\right)\right\} + \exp\left\{\text{ima}\left(\frac{-1+i}{\sqrt{2}}\right)\right\}\right]$$

$$= \frac{1}{4}\exp\left(\frac{-ma}{\sqrt{2}}\right)\left[\exp\left(\frac{ima}{\sqrt{2}}\right) + \exp\left(\frac{-ima}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Hence by Cauchy's residue theorem,

$$\int_C f(z)\, dz = \int_T f(z)\, dz + \int_{-R}^{R} f(x)\, dx = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Taking limit as $R \to \infty$ and using (1), we get

$$\int_{-\infty}^{\infty} \frac{x^3 e^{imx}}{x^4 + a^4}\, dx = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Equating imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\, dx = \pi \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

or $$\int_{0}^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\, dx = \frac{\pi}{2} \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

12.4 Multivalued Function and its Branches

The familiar fact that sin q and cos q are periodic functions with period 2p, is responsible for the non-uniqueness of θ in the representation $z = |z|e^{i\theta}$ i.e. $z = re^{i\theta}$. Here, we shall discuss non-uniqueness problems with reference to the function arg z, log z and z^a. We know that a function w = f(z) is multivalued when for given z, we may find more than one value of w. Thus, a function f(z) is said to be single-valued if it satisfies

$$f(z) = f(z(r, \theta)) = f(z(r, \theta + 2\pi))$$

otherwise it is classified as multivalued function.

For analytic properties of a multivalued function, we consider domains in which these functions are single valued. This leads to the concept of branches of such functions. Before discussing branches of a many valued function, we give a brief account of the three functions arg z, log z and z^a.

Argument Function

For each $z \in \mathbb{C}$, $z \neq 0$, we define the argument of z to be

$$\arg z = [\arg z] = \{\theta \in R : z = |z|e^{i\theta}\}$$

Notes

the square bracket notation emphasizes that arg z is a set of numbers and not a single number. i.e. [arg z] is multivalued. In fact, it is an infinite set of the form $\{\theta + 2n\pi : n \in I\}$, where θ is any fixed number such that $e^{i\theta} = \frac{z}{|z|}$.

For example, arg i = $\{(4n+1)\pi/2 : n \in I\}$

Also, $\arg\left(\frac{1}{z}\right) = \{-\theta : \theta \in \arg z\}$

Thus, for $z_1, z_2 \neq 0$, we have

$$\arg (z_1 z_2) = \{\theta_1 + \theta_2 : \theta_1 \in \arg z_1, \theta_2 \in \arg z_2\}$$

$$= \arg z_1 + \arg z_2$$

and $\quad \arg = \arg z_1 - \arg z_2$

For principal value determination, we can use Arg z = θ, where $z = |z|\, e^{i\theta}$, $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$). When z performs a complete anticlockwise circuit round the unit circle, θ increases by 2π and a jump discontinuity in Arg z is inevitable. Thus, we cannot impose a restriction which determines θ uniquely and therefore for general purpose, we use more complicated notation arg z or [arg z] which allows z to move freely about the origin with θ varying continuously. We observe that

$$\arg z = [\arg z] = \text{Arg } z + 2n\pi, n \in I.$$

Logarithmic Function

We observe that the exponential function e^z is a periodic function with a purely imaginary period of $2\pi i$, since

$$e^{z+2\pi i} = e^z.\, e^{2\pi i} = e^z, e^{2\pi i} = 1.$$

i.e. $\exp(z + 2\pi i) = \exp z$ for all z.

If w is any given non-zero point in the w-plane then there is an infinite number of points in the z-plane such that the equation

$$w = e^z \tag{1}$$

is satisfied. For this, we note that when z and w are written as $z = x + iy$ and $w = \rho\, e^{i\phi}$ $(-\pi < \phi \leq \pi)$, equation (1) can be put as

$$e^z = e^{x+iy} = e^x\, e^{iy} = r\, e^{i\phi} \tag{2}$$

From here, $e^x = \rho$ and $y = \phi + 2n\pi$, $n \in I$.

Since the equation $e^x = \rho$ is the same as $x = \log_e \rho = \log \rho$ (base e understood), it follows that when $w = \rho\, e^{i\phi}(-\pi < \phi \leq \pi)$, equation (1) is satisfied if and only if z has one of the values

$$z = \log \rho + i(\phi + 2n\pi), n \in I \tag{3}$$

Thus, if we write

$$\log w = \log \rho + i(\phi + 2n\pi), n \in I \tag{4}$$

we see that exp (log w) = w, this motivates the following definition of the (multivalued) logarithmic function of a complex variable.

Notes

The logarithmic function is defined at non-zero points $z = re^{i\theta}$ $(-\pi < \theta \leq \pi)$ in the z-plane as

$$\log z = \log r + i\,(\theta + 2n\pi),\ n \in I \tag{5}$$

The principal value of log z is the value obtained from (5) when n = 0 and is denoted by Log z. Thus,

$$\text{Log } z = \log r + i\theta \text{ i.e. Log } z = \log |z| + i \text{ Arg } z \tag{6}$$

Also, from (5) & (6), we note that

$$\log z = \text{Log } z + 2n\pi i,\ n \in I \tag{7}$$

The function Log z is evidently well defined and single-valued when $z \neq 0$.

Equation (5) can also be put as

$$\log z = \{\log |z| + i\theta : \theta \in \arg z\}$$

or $$[\log z] = \{\log |z| + i\theta : \theta \in [\arg z]\} \tag{8}$$

or $$\log z = \log |z| + i\theta = \log |z| + i \arg z \tag{9}$$

where $\theta = \theta + 2np$, $\theta = \text{Arg } z$.

From (8), we find that

$$\log 1 = \{2n\pi i,\ n \in I\},\ \log(-1) = \{(2n+1)\ pi,\ n \in I\}$$

In particular, Log 1 = 0, Log (–1) = πi. Similarly log, log i = {(un+1) πi/2, n$\in$I}, log (–i) = {un-1) πi/2, n$\in$I} In particular, Log i = πi/2, Log (–i) = –πi/2.

Thus, we conclude that complex logarithm is not a bona fide function, but a multifunction. We have assigned to each $z \neq 0$ infinitely many values of the logarithm.

Complex Exponents

When $z \neq 0$ and the exponent a is any complex number, the function z^a is defined by the equation.

$$w = z^a = e^{a \log z} = \exp(a \log z) \tag{1}$$

where log z denotes the multivalued logarithmic function. Equation (1) can also be expressed as

$$w = z^a = \{e^{a(\log |z| + i\theta)} : \theta \in \arg z\}$$

or $$[z^a] = \{e^{a(\log |z| + i\theta)} : \theta \in [\arg z]\}$$

Thus, the multivalued nature of the function log z will generally result in the many-valuedness of z^a. Only when a is an integer, z^a does not produce multiple values. In this case, z^a contains a single point z^n. When $a = \dfrac{1}{n}$ (n = 2, 3,…), then

$$w = z^{1/n} = (r\, e^{i\theta})^{1/n} = r^{1/n} e^{i(\theta + 2m\pi)/n},\ m \in I$$

We note that in particular, the complex nth roots of ±1 are obtained as

$$w^n = 1 \quad \Rightarrow w = e^{2m\pi i/n},\ w^n = -1 \quad \Rightarrow w = e^{(2m+1)\pi i/n},\ m = 0,1,\ldots, n\text{-}1.$$

For example, $i^{-2i} = \exp(-2\, i \log i) = \exp[-2i\,(4n+1)\,\pi i/2]$

$$= \exp[(4n+1)\,\pi],\ n \in I$$

It should be observed that the formula

$$x^a\, x^b = x^{a+b},\ x, a, b, \in R$$

Notes

can be shown to have a complex analogue (in which values of the multi-functions involved have to be appropriately selected) but the formula

$$x_1^2 x_2^a = (x_1 x_2)^a, x_1, x_2, a \in R$$

has no universally complex generalization.

Branches, Branch Points and Branch Cuts

We recall that a multifunction w defined on a set $S \subseteq \forall$ is an assignment to each $z \in S$ of a set $[w(z)]$ of complex numbers. Our main aim is that given a multifunction w defined on S, can we select, for each $z \in S$, a point $f(z)$ in $[w(z)]$ so that $f(z)$ is analytic in an open subset G of S, where G is to be chosen as large as possible? If we are to do this, then $f(z)$ must vary continuously with z in G, since an analytic function is necessarily continuous.

Suppose w is defined in some punctured disc D having centre a and radius R i.e. $0 < |z - a| < R$ and that $f(z) \in [w(z)]$ is chosen so that f(z) is at least continuous on the circle g with centre a and radius r ($0 < r < R$). As z traces out the circle g starting from, say z_0, $f(z)$ varies continuously, but must be restored to its original value $f(z_0)$ when z completes its circuit, since f(z) is, by hypothesis, single valued. Notice also that if $z - a = r\, e^{i\theta(z)}$, where $\theta(z)$ is chosen to vary continuously with z, then $\theta(z)$ increases by $2p$ as z performs its circuit, so that $\theta(z)$ is not restored to its original value. The same phenomenon does not occur if z moves round a circle in the punctured disc D not containing a, in this case $\theta(z)$ does return to its original value. More generally, our discussion suggests that if we are to extract an analytic function from a multi-function w, we shall meet to restrict to a set in which it is impossible to encircle, one at a time, points a such that the definition of $[w(z)]$ involves the argument of $(z\text{-}a)$. In some cases, encircling several of these 'bad' points simultaneously may be allowable.

A branch of a multiple-valued function f(z) defined on $S \subseteq \not\subset$ is any single-valued function F(z) which is analytic in some domain $D \subset S$ at each point of which the value F(z) is one of the values of $f(z)$. The requirement of analyticity, of course, prevents $F(z)$ from taking on a random selection of the values of $f(z)$.

A branch cut is a portion of a line or curve that is introduced in order to define a branch F(z) of a multiple-valued function $f(z)$.

A multivalued function $f(z)$ defined on $S \subseteq \forall$ is said to have a branch point at z_0 when z describes an arbitrary small circle about z_0, then for every branch F(z) of f(z), F(z) does not return to its original value. Points on the branch cut for F(z) are singular points of F(z) and any point that is common to all branch cuts of f(z) is called a branch point. For example, let us consider the logarithmic function

$$\log z = \log r + iq = \log |z| + i \arg z \tag{1}$$

If we let α denote any real number and restrict the values of q in (1) to the interval $\alpha < \theta < \alpha + 2\pi$, then the function

$$\log z = \log r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi) \tag{2}$$

with component functions

$$u(r, \theta) = \log r \text{ and } v(r, \theta) = \theta \tag{3}$$

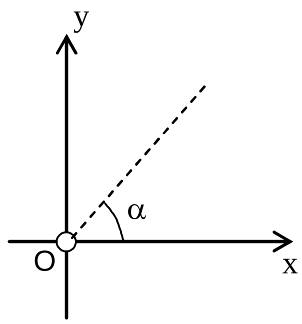
is single-valued, continuous and analytic function. Thus for each fixed a, the function (2) is a branch of the function (1). We note that if the function (2) were to be defined on the ray $\theta = a$, it would not be continuous there. For, if z is any point on that ray, there are points arbitrarily close to z at which the values of v are near to a and also points such that the values of v are near to

Notes

$\alpha + 2\pi$. The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The function

$$\text{Log } z = \log r + i\theta \ (r > 0, -\pi < \theta < \pi) \tag{4}$$

is called the principal branch of the logarithmic function in which the branch cut consists of the origin and the ray $\theta = \pi$. The origin is evidently a branch point of the logarithmic function.



For analyticity of (2), we observe that the first order partial derivatives of u and v are continuous and satisfy the polar form

$$u_r = \frac{1}{r}v_\theta, \qquad v_r = -\frac{1}{r}u_\theta$$

of the C-R equations. Further,

$$\frac{d}{dz}(\log z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}}$$

Thus, $\frac{d}{dz}(\log z) = \frac{1}{z}$ $(|z| = r > 0, < \arg z < a + 2p)$

In particular,

$$\frac{d}{dz}(\log z) = \frac{1}{z} \ (|z| > 0, -\pi < \text{Arg } z < \pi).$$

Further, since $\log \frac{1}{z} = -\log z$, ∞ is also a branch point of log z. Thus, a cut along any half-line from 0 to ∞ will serve as a branch cut.

Now, let us consider the function $w = z^a$ in which a is an arbitrary complex number. We can write

$$w = z^a = = e^{a \log z} \tag{5}$$

where many-valued nature of log z results is many-valuedness of z^a. If Log z denotes a definite branch, say the principal value of log z, then the various values of z^a will be of the form

$$z^a = e^{a(\text{Log } z + 2n\pi i)} = e^{a \text{ Log } z} e^{2\pi i a n} \tag{6}$$

where $\log z = \text{Log } z + 2n\pi i$, $n \in I$.

Notes

The function in (6) has infinitely many different values. But the number of different values of z^a will be finite in the cases in which only a finite number of the values $e^{2\pi ian}$, $n \in I$, are different from one another. In such a case, there must exist two integers m and m′ (m′ = m) such that $e^{2\pi iam} = e^{2\pi iam'}$ or $e^{2\pi ia(m-m')} = 1$. Since $e^z = 1$ only if $z = 2\pi in$, thus we get a (m-m′) = n and therefore it follows that a is a rational number. Thus, z^a has a finite set of values iff a is a rational number. If a is not rational, z^a has infinity of values.

We have observed that if $z = re^{i\theta}$ and a is any real number, then the branch

$$\log z = \log r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi) \tag{7}$$

of the logarithmic function is single-valued and analytic in the indicated domain. When this branch is used, it follows that the function (5) is single valued and analytic in the said domain. The derivative of such a branch is obtained as

$$\frac{d}{dz}(z^a) = \frac{d}{dz}[\exp\,(a \log z)] = \exp\,(a \log z)\frac{a}{z}$$

$$= a\frac{\exp(a \log z)}{\exp(\log z)} = a \exp\,[(a-1) \log z]$$

$$= az^{a-1}.$$

As a particular case, we consider the multivalued function $f(z) = z^{1/2}$ and we define

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2},\ r > 0,\ \alpha < \theta < \alpha + 2\pi \tag{8}$$

where the component functions

$$u(r, \theta) = \sqrt{r} \cos \theta/2,\ v(r, \theta) = \sqrt{r}\ \sin \theta/2 \tag{9}$$

are single valued and continuous in the indicated domain. The function is not continuous on the line $\theta = \alpha$ as there are points arbitrarily close to z at which the values of v (r, θ) are nearer to $\sqrt{r}\ \sin \alpha/2$ and also points such that the values of v(r, θ) are nearer to $-\sqrt{r}\ \sin \alpha/2$. The function (8) is differentiable as C-R equations in polar form are satisfied by the functions in (9) and

$$\frac{d}{dz}(z^{1/2}) = e^{-i\theta}(u_r + iv_r) = e^{i\theta}\left(\frac{1}{2\sqrt{r}}\cos\theta/2 + i\frac{1}{2\sqrt{r}}\sin\theta/2\right)$$

$$= \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2z^{1/2}}$$

Thus, (8) is a branch of the function $f(z) = z^{1/2}$ where the origin and the line $\theta = \alpha$ form branch cut. When moving from any point $z = re^{i\theta}$ about the origin, one complete circuit to reach again, at z, we have changed arg z by 2π. For original position $z = re^{i\theta}$, we have $w = \sqrt{r}\, e^{i\theta/2}$, and after one complete circuit, $w = \sqrt{r}\, e^{i(\theta+2\pi)/2} = -\sqrt{r}\, e^{i\theta/2}$. Thus, w has not returned to its original value and hence, change in branch has occurred. Since a complete circuit about z = 0 changed the branch of the function, z = 0 is a branch point for the function $z^{1/2}$.

Notes

12.5 Summary

- If f(z) = w has a solution z = F(w), then we may write

  f{F(w) } = w, F{ f(z)} = z. The function F defined in this way, is called inverse function of f.

- Inverse Function Theorem

  Let a function w = f(z) be analytic at a point $z = z_0$ where $f'(z_0) \neq 0$ and $w_0 = f(z_0)$.

  Then there exists a neighbourhood of w_0 in the w-plane in which the function w = f(z) has a unique inverse z = F(w) in the sense that the function F is single-valued and analytic in that neighbourhood such that $F(w_0) = z_0$ and

  $$F'(w) = \frac{1}{f'(z)}.$$

- If f(z) is analytic or has an isolated singularity at infinity and if C is a circle enclosing all its singularities in the finite parts of the z-plane, the residue of f(z) at infinity is defined by

  $$\text{Res } (z = \infty) = \frac{1}{2\pi i}\int_C f(z)\, dz, \qquad \Big|\ \text{or Res } (z = \infty) - \frac{1}{2\pi i}\int_C f(z)\, dz,$$

  Integration taken in positive sense

  the integration being taken round C in the negative sense w.r.t. the origin, provided that this integral has a definite value. By means of the substitution $z = w^{-1}$, the integral defining the residue at infinity takes the form

  $$\frac{1}{2\pi i}\int [-f(w^{-1})]\frac{dw}{w^2},$$

  taken in positive sense round a sufficiently small circle with centre at the origin.

- (i) If the function f(z) has a simple pole at z = a, then, Res (z = a) = $\lim_{z\to a}$ (z–a) f(z).

  (ii) If f(z) has a simple pole at z = a and f(z) is of the form $f(z) = \frac{\phi(z)}{\psi(z)}$ i.e. a rational function, then

  $$\text{Res } (z = a) = \lim_{z\to a} (z-a)\, f(z) = \lim_{z\to a} (z-a)\frac{\phi(z)}{\psi(z)}$$

  $$= \lim_{z\to a} \frac{\phi(z)}{\dfrac{\psi(z)-\psi(a)}{z-a}}$$

  $$= \frac{\phi(a)}{\psi'(a)},$$

  where $\psi(a) = 0$, $\psi'(a) \neq 0$, since $\psi(z)$ has a simple zero at z = a

Notes

(iii) If *f*(z) has a pole of order m at z = a then we can write

$$f(z) = \frac{\phi(z)}{(z-a)^m}$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

$$\text{Now, Res } (z = a) = b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^m}dz$$

$$= \frac{1}{\lfloor m-1}\,\frac{\lfloor m-1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^{m-1+1}}\,dz$$

$$= \frac{1}{\lfloor m-1}\,\phi^{m-1}(a) \qquad \text{[By Cauchy's integral formula for derivatives]}$$

- A function f(z) is said to be single-valued if it satisfies

  $$f(z) = f(z(r, \theta)) = f(z(r, \theta + 2\pi))$$

  otherwise it is classified as multivalued function.

12.6 Keywords

***n* zeros:** Every polynomial of degree n has exactly n zeros.

Inverse function: If f(z) = w has a solution z = F(w), then we may write f{F(w)} = w, F{f(z)} = z. The function F defined in this way, is called inverse function of f.

Residue at infinity: If f(z) is analytic or has an isolated singularity at infinity and if C is a circle enclosing all its singularities in the finite parts of the z-plane, the residue of f(z) at infinity is defined by

$$\text{Res } (z = \infty) = \frac{1}{2\pi i}\int_C f(z)\,dz,$$

Cauchy Residue Theorem: Let f(z) be one-valued and analytic inside and on a simple closed contour C, except for a finite number of poles within C. Then

$$\int_C f(z)\,dz = 2\pi i \text{ [Sum of residues of f(z) at its poles within C]}$$

Multivalued function: a function f(z) is said to be single-valued if it satisfies f(z) = f(z(r, θ)) = f(z(r, θ + 2π)) otherwise it is classified as multivalued function.

12.7 Self Assessment

1. Every polynomial of degree n has exactly ....
2. If f(z) = w has a solution z = F(w), then we may write f{F(w) } = w, F{ f(z)} = z. The function F defined in this way, is called ....

Notes

3. Integration taken in positive sense the integration being taken round C in the negative sense w.r.t. the origin, provided that this integral has a definite value. By means of the substitution $z = w^{-1}$, the integral defining the residue at infinity takes the form .... taken in positive sense round a sufficiently small circle with centre at the origin.

4. If the function f(z) has a simple pole at z = a, then, Res (z = a) = ...

5. If f(z) is analytic except at a finite number of singularities and if $f(z) \to 0$ uniformly as $z \to \infty$, then ...., where T denotes the semi-circle $|z| = R$, $I_m . z \geq 0$, R being taken so large that all the singularities of f(z) lie within T.

6. a function f(z) is said to be single-valued if it satisfies $f(z) = f(z(r, \theta)) = f(z(r, \theta + 2\pi))$ otherwise it is classified as ....

7. A .... is a portion of a line or curve that is introduced in order to define a branch F(z) of a multiple-valued function f(z).

8. A .... f(z) defined on $S \subseteq \forall$ is said to have a branch point at z_0 when z describes an arbitrary small circle about z_0, then for every branch F(z) of f(z), F(z) does not return to its original value.

12.8 Review Questions

1. Discuss the concept of fundamental theorem on algebra.
2. Describe the calculus of residues.
3. Discuss the multivalued functions and its branches.

Answers: Self Assessment

1. n zeros.
2. inverse function of f.
3. $\frac{1}{2\pi i}\int [-f(w^{-1})]\frac{dw}{w^2}$,
4. $\lim_{z \to a} (z-a) f(z)$
5. $\lim_{R \to \infty} \int_T e^{imz} f(z)\, dz = 0$, $m > 0$
6. multivalued function.
7. branch cut
8. multivalued function

12.9 Further Readings

Books

Ahelfors, D.V. : Complex Analysis

Conway, J.B. : Function of one complex variable

Pati, T. : Functions of complex variable

Shanti Narain : Theory of function of a complex Variable

Tichmarsh, E.C. : The theory of functions

H.S. Kasana : Complex Variables theory and applications

P.K. Banerji : Complex Analysis

Serge Lang : Complex Analysis

H. Lass : Vector & Tensor Analysis

Shanti Narayan : Tensor Analysis

C.E. Weatherburn : Differential Geometry

T.J. Wilemore : Introduction to Differential Geometry

Bansi Lal : Differential Geometry.

Notes