ADVANCE STATISTICAL METHODS IN ECONOMICS

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SYLLABUS

ADVANCE STATISTICAL METHODS IN ECONOMICS

Objectives:

The course aims to equip the students with statistical tools and concepts that help in decision making. The emphasis is on their application in business.

Sr. No.	Content
1	Index number: Introduction and Use of index numbers and their types, Methods: Simple (unweighted) Aggregate Method, Weighted aggregate method, Methods: Simple (unweighted) Aggregate Method, Methods: Simple Average of Price Relatives, Methods: Weighted Average of Price Relatives, Test of consistency: Unit test, Time Reversal Test, Factor Reversal Test and Circular, Cost of Living index and its uses. Limitation of Index Numbers
2	Time Series Analysis: Introduction and components of time series, Time Series Methods: Graphic, method of semi-averages, Time Series Methods: Principle of Least Square and its application, Methods of Moving Averages
3	Theory of Probability: Introduction and uses, Additive and Multiplicative law of probability
4	Theory of Estimation: Point estimation, Unbiasedness, Consistency, Efficiency and Sufficiency, Method of point estimation and interval estimation
5	Types of Hypothesis: Null and Alternative, types of errors in testing hypothesis, Level of significance

Content

Unit 1: Correlation Analysis Vs. Regression Analysis

CONTENTS

Objectives

After reading this unit students will be able to:

- Describe Correlation and Regression Analysis.
- Explain Correlation Analysis Vs. Regression Analysis.

Introduction

Correlation and regression analysis are related in the sense that both deal with relationships among variables. The correlation coefficient is a measure of linear association between two variables. Values of the correlation coefficient are always between – 1 and + 1. A correlation coefficient of + 1 indicates that two variables are perfectly related in a positive linear sense, a correlation coefficient of – 1 indicates that two variables are perfectly related in a negative linear sense, and a correlation coefficient of 0 indicates that there is no linear relationship between the two variables. For simple linear regression, the sample correlation coefficient is the square root of the coefficient of determination, with the sign of the correlation coefficient being the same as the sign of b 1, the coefficient of x l in the estimated regression equation.

Neither regression nor correlation analyses can be interpreted as establishing cause-and-effect relationships. They can indicate only how or to what extent variables are associated with each other. The correlation coefficient measures only the degree of linear association between two variables. Any conclusions about a cause-and-effect relationship must be based on the judgment of the analyst.

When once a relationship between two variables is ascertained, it is quite likely that estimating the value of one for some given value of other is expected. This can be done with the help of regression. It measures the average relationship between two or more variables in terms of original units of the data. The dictionary meaning of the term 'Regression' is to revert or return back. The term was used for the first time by Sir Francis Galton in 1877. In statistics, the technique of Regression is used in all those fields where two or more variables have the tendency to go back to the mean. While correlation measures the direction and strength of the relationship between two or more variables, regression involves methods by which estimates are made of the values of a variable from the knowledge of the values of one or more other variables. Along with this, measurement of the error involved in the estimation process are also included. This means, the regression technique can be used for the prediction on the basis of the average relationship.

1.1 Correlation Analysis

So far we have studied problems relating to one variable only. In practice, we come across a large number of problems involving the use of two or more than two variables. If two quantities vary in such a way that movements in one are accompanied by movements in the other, these quantities are correlated. For example, there exists some relationship between age of husband and age of wife, price of a commodity and amount demanded, increase in rainfall up to a point and production of rice, an increase in the number of television licences and number of cinema admissions, etc. The statistical tool with the help of which these relationships between two or more than two variables are studied is called **correlation.** The measure of correlation, called the correlation coefficient, summarizes in one figure the direction and degree of correlation. Thus correlation analysis refers to the techniques used in measuring the closeness of the relationship between the variables. A very simple definition of correlation is that given by A.M. Tuttle. He defines correlation as: "An analysis of the covariation of two or more variables is usually called *correlation*".

The problem of analysing the relation between different series can be broken down into three steps:

(1) Determining whether a relation exists and, if does, measuring it.

(2) Testing whether it is significant.

(3) Establishing the cause and effect relation, if any.

In this unit, only the first aspect will be discussed. For second aspect a reference may be made in the unit on Tests of Significance. The third aspect in the analysis, that of establishing the cause-effect relation, is beyond the scope of statistical analysis. An extremely high and significant correlation between the increase in smoking and increase in lung cancer would not prove that smoking causes lung cancer. The proof of a cause and effect relation can be developed only by means of an exhaustive study of the operative elements themselves.

It should be noted that the detection and analysis of correlation *(i.e.,* covariation) between two statistical variables requires relationships of some sort which associate the observations in pairs, one of which pair being a value of each of the two variables. In general, the pairing relationship may be of almost any nature, such as observations at the same time or place over a period of time or different places.

Notes The computation concerning the degree of closeness is based on the regression equation. However, it is possible to perform correlation analysis without actually having a regression equation.

Utility of the Studt of Correlation

The study of correlation is of immense use in practical life because of the following reasons:

1. Most of the variables show some kind of relationship. For example, there is relationship between price and supply, income and expenditure, etc. With the help of correlation analysis, we can measure in one figure the degree of relationship existing between the variables.
2. Once we know that two variables are closely related, we can estimate the value of one variable given the value of another. This is done with the help of regression equations discussed in Unit 8.
3. Correlation analysis contributes to the economic behaviour, aids in locating the critically important variables on which others depend, may reveal to the economist the connection by which disturbances spread and suggest to him the paths through which stabilizing forces become effective.

   In business, correlation analysis enables the executive to estimate costs, sales, prices and other variables on the basis of some other series with which these costs, sales, or prices may be functionally related. Some guesswork can be removed from decisions when the relationship between a variable to be estimated and the one or more other variables on which it depends are close and reasonably invariant.

However, it should be noted that coefficient of correlation is one of the most widely used and also one of the most widely *abused* of statistical measures. It is abused in the sense that one sometimes overlooks the fact that *r* measures nothing but the strength of *linear* relationships and that it does not necessarily imply a cause-effect relationship.

4. Progressive development in the methods of science and philosophy has been characterised by increase in the knowledge of relationships or correlations. Nature has been found to be a multiplicity of interrelated forces.

Correlation and Causation

Correlation analysis helps in determining the degree of relationship between two or more variables— it does not tell us anything about cause and effect relationship. Even a high degree of correlation does not necessarily mean that a relationship of cause and effect exists between the variables or, simply stated, correlation does not necessarily imply causation of functional relationship though the existence of causation always implies correlation. By itself it establishes only *covariation.* The explanation of a significant degree of correlation may be due to any one or a combination of the following reasons:

1. **The correction may be due to pure chance, especially in a small sample:** We may get a high degree of correlation between two variables in a sample but in the universe there may not be any relationship between the variables at all. This is especially so in case of small samples. Such a correlation may arise either because of pure random sampling variation or because of the bias of the investigator in selecting the sample. The following example shall illustrate the point:

Income (Rs.)	**Weight (*lb.*)**
10,000	120
20,000	140
30,000	160
40,000	180
50,000	200

The above data show a perfect positive relationship between income and weight, *i.e.,* as the income is increasing the weight is increasing and the ratio of change between two variables is same.

2. **Both the correlated variables may be influenced by one or more other variables:** It is just possible that a high degree of correlation between variables may be due to the same causes affecting each variable or different causes affecting each with the same effect. For example, a high degree of correlation between the yield per acre of rice and tea may be due to the fact that both are related to the amount of rainfall. But none of the two variables is the cause of the other.

3. **Both the variables may be mutually influencing each other so that neither can be designated as cause and the other the effect:** There may be a high degree of correlation between the variables but it may be difficult to pin point as to which is the cause and which is the effect. This is especially likely to be so in case of economic variables. For example, such variables as demand and supply, price and production, etc., mutually interact. To take a specific case, it is a well known principle of economics that as the price of a commodity increases its demand goes down and so price is the cause, and demand the effect. But it is also possible that increased demand of a commodity due to growth of population or other reasons may force its price up. Now the cause is the increased demand, the effect the price. Thus at times it may become difficult to explain from the two correlated variables which is the cause and which is the effect because both may be reacting on each other.

The above points clearly bring out the fact that correlation does not manifest causation of functional relationship. By itself, it establishes only covariation. Correlation observed between variables that could not conceivably be causally related are called *spurious* or *non-sense correlation.* More appropriately we should remember that it is the *interpretation* of the degree of correlation that is spurious, not the degree of correlation itself. The high degree of correlation indicates only the mathematical result. We should reach a conclusion based on logical reasoning and intelligent investigation on significantly related matters. It should also be noted that errors in correlation analysis include not only reading causation into spurious correlation but also interpreting spuriously a perfectly valid association.

1.2 Regression Analysis

Meaning and Definition

The term regression was for the first time used by Sir Francis Galton in 1877 while studying the relationship between the height of fathers and sons. He carried out a study on height of one thousand fathers and sons and revealed that tall fathers tend to have tall sons and short fathers short sons, but the average height of the sons of a group of tall fathers is less than that of the fathers and the average height of the sons of a group of short fathers is greater than that of the fathers. This line describing the tendency to regress or going back was called as a regression line by Galton. The term is still used to describe that line drawn for a group of points to represent the trend present, however, today it does not necessarily have the original implication of stepping back (for which Galton had used this term). In modern times term 'estimating line' is coming to be used instead of 'regression line'. On examining a few definitions, the term regression as used in statistics can be clearly described.

(1) As described by *Morris Hamburg,* "The term regression analysis refers to the methods by which estimates are made of the values of a variable from a knowledge of values of one or more other variables and to the measurement of the errors involved in this estimation process."

(2) According to *Taro Yamane,* "One of the most frequently used techniques in economics and business research, to find, relation between two or more variables that are related casually, is regression analysis."

On the basis of the above definitions, it has become very clear that regression analysis is done for estimating or predicting the unknown value of one variable from the known value of the other variable. The variable which is used to predict the variable of interest is called the independent variable or explanatory variable and the variable we are trying to predict is called the dependent or explained variable.

Did u know? In the words of *Ya Lum Chou,* "Regression analysis attempts to establish the nature of the relationship between variables, that is, to study the functional relationship between the variable and thereby provide a mechanism for prediction or forecasting."

1.3 Correlation Analysis Vs. Regression Analysis

Most of the times, the correlation and regression analysis are confused with one another, probably because of the fact that both of them study about the relationship between two variables. By studying the points of differentiation, this would become clear:

(1) Correlation coefficient measures the degree of covariability between X and Y. The regression analysis, on the other hand, studies the nature of relationship between X and Y so that one may be predicted on the basis of the other.

(2) Correlation only ascertains the degree of relationship between two variables and it not be made clear that one variable is the cause and the other is the effect. But in regression analysis, one variable is taken as dependent while other as independent so that the cause and effect relationship can be studied.

(3) In correlation $r_{xy} = r_{yx}$ but regression coefficients b_{xy} is never equal to b_{yx}.

$$b_{xy} \neq b_{yx}.$$

(4) Correlation may be found to exist between two variables by chance with no practical relevance. But in regression the results are never by chance.

(5) Correlation coefficient is independent of origin and change of scale. Regression coefficient is independent of change of scale but not of origin.

Some Similarities: (1) Coefficient of correlation for two variables shall take the same sign as regression coefficients. (2) If, at a given level of significance, the value of regression coefficients is significant, the value of correlation coefficient shall also be significant at that level.

On examining the various definitions, it reveals that regression is a tool which helps in estimating or predicting the unknown value of one variable from the known value of the other variable. It differs from correlation as the later only tell the direction and extent of relationship between two variables whereas regression is a step further.

Self-Assessment

1. Indicate whether the following statements are True or False [T/F]:

(*i*) Correlation always signifies a cause and effect relationship between the variables.

(*ii*) If r is negative both the variables are decreasing.

(*iii*) Regression analysis reveals average relationship between two variables.

(*iv*) The terms 'dependent' and 'independents' do not imply that there is necessarily any cause and effect relationship between the variables.

(*v*) In regression analysis b_{xy} stands for regression coefficient of X on Y.

1.4 Summary

- Correlation analysis refers to the techniques used in measuring the closeness of the relationship between the variables. A very simple definition of correlation is that given by A.M. Tuttle. He defines correlation as: "An analysis of the covariation of two or more variables is usually called *correlation*".
- The computation concerning the degree of closeness is based on the regression equation. However, it is possible to perform correlation analysis without actually having a regression equation.
- Correlation analysis contributes to the economic behaviour, aids in locating the critically important variables on which others depend, may reveal to the economist the connection by which disturbances spread and suggest to him the paths through which stabilizing forces become effective.
- Progressive development in the methods of science and philosophy has been characterised by increase in the knowledge of relationships or correlations. Nature has been found to be a multiplicity of interrelated forces.
- Correlation observed between variables that could not conceivably be causally related are called *spurious* or *non-sense correlation*. More appropriately we should remember that it is the *interpretation* of the degree of correlation that is spurious, not the degree of correlation itself. The high degree of correlation indicates only the mathematical result. We should reach a conclusion based on logical reasoning and intelligent investigation on significantly related matters. It should also be noted that errors in correlation analysis include not only reading causation into spurious correlation but also interpreting spuriously a perfectly valid association.
- In modern times term 'estimating line' is coming to be used instead of 'regression line'. On examining a few definitions, the term regression as used in statistics can be clearly described.
- The variable which is used to predict the variable of interest is called the independent variable or explanatory variable and the variable we are trying to predict is called the dependent or explained variable.

1.5 Key-Words

1. Correlation : In the world of finance, a statistical measure of how two securities move in relation to each other. Correlations are used in advanced portfolio management.

1.6 Review Questions

1. Define Correlation. Discuss its uses.
2. What is regression ? Explain clearly the significance of this concept with the help of an example.
3. Distinguish clearly between regression and correlation analysis giving suitable examples.
4. What are the similarities between correlation and regression analysis ?
5. Describe Correlation and Causation.

Answers: Self-Assessment

1. (*i*) F (*ii*) F (*iii*) T (*iv*) T (*v*) T

1.7 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 2 : Index Number–Introduction and Use of Index Numbers and their Types

CONTENTS

Objectives

After reading this unit students will be able to :

- Discuss the Introduction of Index Number.
- Know the Use of Index Number.
- Explain the Types of Index Number.

Introduction

Any change in the level of a phenomenon with respect to time, geographical location etc. is measured with the help of a statistical device called Index numbers. It was for the first time used to compare the changes in prices for the year 1750 with the price level of year 1500 in Italy. It was constructed by Carli. However, today it is used to measure the change in level of any phenomena may it be changes national income, expenditure, cost of living, incidences of crimes, number of accidents and so on. Index numbers are said to be barometers which measure the charge in the level of a phenomenon.

2.1 Introduction to Index Numbers

Index numbers have become today one of the most widely used statistical devices. Though originally developed for measuring the effect of change in prices, there is hardly any field today where index numbers are not used. Newspapers headline the fact that prices are going up or down, that industrial production is rising or falling, that imports are increasing or decreasing, that crimes are rising in a particular period compared to the previous period as disclosed by index numbers. They are used to feel the pulse of the economy and they have to be used as indicator of inflationary or deflationary tendencies. In fact, they are described as *barometers of economic activity, i.e.,* if one wants to get an idea as to what is happening to an economy he should look to important indices like the index number of industrial production, agricultural production, business activity, etc.

An index number may be described as a specialized average designed to measure the change in the level of a phenomenon with respect to time, geographic location or other characteristics such as income, etc. Thus, when we say that the index number of wholesale prices is 125 for the period Dec. 2005 compared to Dec. 2004, it means there is a net increase in the prices of wholesale commodities to the extent of 25 per cent.

For a proper understanding of the term index number, the following points are worth considering:

(1) **Index numbers are specialized averages :** As explained in unit 4 central value, an average is a single figure representing a group of figures. However, to obtain an average, items must be comparable; for example, the average weight of men, women and children of a certain locality has no meaning at all. Furthermore, the unit of measurement must be the same for all the items. Thus an average of the weight expressed in *kg*., *lb*., etc., has no meaning. However, this is not so with index numbers. Index numbers are used for purposes of comparison in situations where two or more series are expressed in different units or the series are composed of different types of items. For example, while constructing a consumer price index the various items are divided into broad heads, namely (i) Food, (ii) Clothing, (iii) Fuel and Lighting, (iv) House Rent, and (v) Miscellaneous. These items are expressed in different units : thus under the head 'food' wheat and rice may be quoted per quintal, ghee per *kg*., etc.

Similarly, cloth may be measured in terms of metres or yards. An average of all these items expressed in different units is obtained by using the technique of index numbers.

(2) **Index numbers measure the change in the level of a phenomenon :** Since index numbers are essentially averages they describe in one simple figure the increase or decrease in the level of a phenomenon under study. Thus if the consumer price index of working class for Delhi has gone up to 125 in 2005 compared to 100 in 2004 it means that there is a net increase of 25% in the prices of commodities included in the index. Similarly, if the index of industrial production is 108 in 2005 compared to 100 in 2004 it means that there is a net increase in industrial production to the extent of 8%. It should be carefully noted that even where an index is showing a net increase, it may include some items which have actually decreased in value and others which have remained constant.

(3) **Index numbers measure the effect of change over a period of time :** Index numbers are most widely used for measuring changes over a period of time. Thus we can find out the net change in agricultural prices from the beginning of first plan period to the Ninth plan period, *i.e.,* 1997-2002. Similarly, we can compare the agricultural production, industrial production, imports, exports, wages, etc., at two different times. However, it should be noted the index numbers not only measure changes over a period of time but also compare economic conditions of different locations, different industries, different cities or different countries. But since the basic problems are essentially the same and since most of the important index numbers published by the Government and private research organisations refer to data collected at different times, we shall consider in this chapter index numbers measuring changes relative to time only. However, methods described can be applied to other cases also.

Did u know? "Index numbers are devices for measuring differences in the magnitude of a group of related variables."

Meaning and Definition

Index numbers are the specialised averages designed to measure the changes in a group of related variables over a period of time. Some important definitions of index number are given below. They would help in understanding about what index numbers are :

(1) According to *Spiegel,* "An index number is a statistical measure designed to show changes in a variable or a group of related variables with respect to time, geographic location or other characteristics such as income, profession etc."

(2) As per *Morris Homburg,* "In its simplest form, an index number is nothing more than a relative number or a 'relative' which expresses the relationship between two figures, where one of the figures is used as base."

(3) *A. M. Tuttle* suggests, "Index number is a single ratio (usually in percentages) which measures the combined (*i.e.,* averaged) change of several variables between two different times, places or situations."

(4) As per *Patterson,* "In its simplest form, an index number is the ratio of two index numbers expressed as a per cent. An index number is a statistical measure — a measure designed to show changes in one variable or in a group of related variables over time or with respect to geographic location or other characteristic."

From the above definitions it is very clear that the index numbers are specialised averages designed to measure change in a group of related variables over a period of time.

Features of Index Number

To understand what an index number is the following features are worth considering :

(1) Index numbers are specialised averages,

(2) Index numbers measure the effect of changes over a period of time,

(3) Index numbers measure the net change in a group of related variables.

2.2 Purpose or Use of Index Number

The definitions and features of index number stated above makes it very clear that index numbers measure changes. In this way they are indispensible fools in the hands of economists and business analysts who constantly work for change, of course, towards betterment. The various uses of index numbers are highlighted below :

(1) Index numbers reveal trends and tendencies : The trend of the phenomenon under study can be obtained by measuring the changes over a period of time. On the basis of this analysis can be done. For example, by examining the index numbers of industrial production, business activity etc. their trend can be understood and analysed.

(2) Index numbers help in measuring suitable policies : The above use of index number which reveals the trends and tendencies help in framing suitable policies so as to achieve the said goal. For example, by knowing about the rising trend of imports, suitable policy can be formulated to prevent it.

(3) Index numbers are used in deflating : Index numbers are very useful in deflating *i.e.,* they are used to adjust the original data for price changes, or to adjust wages for cost of living changes and thus transform nominal wages into real wages. Moreover nominal income can be transformed into real income and nominal sales into real sales through appropriate index numbers.

(4) Index number are used in forecasting future economic activity : Along with studying the past and the present variables of the economy, index numbers are also useful in estimating the future economic activity. The long-term variations, trends etc. help in estimating the coming problems.

On the basis of the above, it may be concluded that index numbers are strong tools into the hands of the economists and business analysts with the help of which they can measure changes in certain phenomenon over a period of time or through a geographical location. This enables them to form suitable policies to obtain the desired results. Past, present and future trends and tendencies are revealed with the help of index numbers and they are also useful in deflating. In the words of *Kafka* and *Simpson,* "Index numbers are today one of the most widely used statistical devices. They are used to feel the pulse of the economy and they have come to be used as indicators of inflationary or deflationary tendencies."

Problems in the Construction of Index Numbers

Index numbers are constructed in the form of specialised averages with definite purpose and with some base period. Before constructing index numbers, it is necessary to know about the various problems which arise in its construction so that they can be minimised. Some of the important problems are :

(1) Selection of base period : Base period is the one against which comparisons are made. It may be year a month or a day. The index for base period is always taken to be 100. It is essential to choose an appropriate base before constructing index numbers. The base is selected as per the object of the index, but following considerations should also be made — (a) The base period should not be a very old period. It must be a fairly recent period, so that comparisons are between similar set of circumstances. (b) The base period selected should be a normal period *i.e.,* that period should be free from abnormalities like war, earthquake, boom or depression etc. (c) It has to be predecided whether fixed base or chain base index has to be prepared. In fixed base, the base year remains fixed, while in chain-base, the base year for each time period is the index of the preceeding time period *i.e.,* the base is not fixed, it changes with each time period (a year, a month a week or a day).

(2) **Purpose of Index numbers :** While constructing index number the purpose for which it is constructed must be clearly defined. This is because there are no all-purpose index and every index is of limited and particular use. Lack of clearity of purpose would lead to confusion and wastage of time with no fruitful results.

(3) **Selection of number of items :** To study the change in a certain phenomenon, it is not possible to include each and every item leading to change. For example, while constructing price index, change in price of each and every item cannot be included. The selection of commodities should be such that they are representative of the tastes, preferences and habits etc. of the group of people regarding whom the index is constructed. In this way, it becomes a big problem as to which items to be included and which to be excluded ? Again, by having purpose of index properly defined, the selection of items can be eased. Index numbers would give false results if at one time one set of quantities are used and at other time other set of quantities are used.

(4) **Choice of average :** Another problem faced while constructing an index is the decision as to which average, mean, median, mode, geometric mean or harmonic mean, should be used for constructing the index. Each one has its own advantages and drawbacks. Theoretically, geometric mean is considered to be most suitable because of the following reasons — (a) While constructing index numbers ratios of relative changes are taken into consideration and geometric mean gives equal weights to equal ratio of change, (b) index numbers that use geometric mean can be reversed which makes base shifting easily possible, (c) geometric mean is not influenced much by violent fluctuations in the values of individual items. However, use of arithmetic mean is also popular because it is simpler to compute than geometric mean.

(5) **Selection of appropriate weights :** The term 'weight' refers to the relative importance of the different items in the construction of the index. All items are not of equal importance and hence it is necessary to device some suitable method which is done by allocating weights. In case of weighted indices, specific weights are assigned. Implicit or explicit methods of assigning weights can be used. Quantity or value weights can be assigned. If aggregative method is used, quantities are used as weights and in averaging of price relatives method of constructing index, value weights are used. Moreover, it has to be decided whether the weights would be fixed or fluctuating. In this way selection of appropriate weights forms a very crucial problem in constructing index.

(6) **Selecting appropriate formula :** There is a large range of formulae available to construct index numbers. Choosing among these to prepare index is another problem while constructing index numbers. The choice of formula should depend not only on purpose but also on the availability of the data. Theoretically, Fisher's method is the 'ideal' method for constructing index numbers. However, depending upon the purpose and availability of data, other methods may also be used.

(7) **Obtaining quotations :** The change in the level of certain phenomenon can be measured only when proper data regarding quotations is available. For example, while preparing price index, it is essential to obtain proper price quotations of the selected commodities. Or while preparing expenditure index, information about expenditure should be available. In the absence of real information, the results may be misleading. This poses to be another problem while constructing index numbers.

On the basis of the above it can be said that most of the problems arise as a result of availability of alternatives. That is, in the presence of alternatives it becomes difficult to choose the best possible alternative, for example, which average to use, which method to weight must be used, which base year to be selected, what items should be selected, which formula should be chosen and so on. If clearity of purpose is assured most of these problems can be solved. Regarding obtaining proper quotations, efforts should be made to choose reliable sources, and items chosen should be standardised and graded, so that quotations are easily and accurately obtained.

2.3 Types of Index Numbers

Price Relatives

One of the simplest types of index numbers is a *price relative.* It is the ratio of the price of a *single* commodity in a given period or point of time to its price in another period or point of time, called the *reference period* or *base period.* If prices for a period, instead of a point of time, are considered, then suitable price average for the period is taken and these prices are expressed in the same units.

If p_0 and p_n denote the price of a commodity during the base period or reference period (0) and the given period (n) then the *price* relative of the period n with respect to (w.r.t) the base period 0 is

defined by price relative in percentage (of period n w.r.t. period 0) = $\frac{p_n}{p_0} \times 100$ ... (1)

and is denoted by $p_{0/n}$.

For example, if the retail price of fine quality of rice in the year 1980 was Rs. 3.75 and that for the year 1983 was Rs. 4.50 then

$$p_{1980\,|1983} = \frac{\text{Rs. } 4.50}{\text{Rs. } 3.75} \times 100 = 120\%$$

For example, the exchange rate of a U.S. dollar was Rs. 10.00 in July 1984 (period J) and was Rs. 12.50 in December 1984 (period D) then the price relative of a dollar in December w.r.t. that in July is given by

$$p_{\text{J}|\text{D}} = \frac{\text{Rs.}12.50}{\text{Rs.}10.00} \times 100 = 125\%$$

Quantity Relatives

Another simple type of index numbers is a quantity relative, when we are interested in changes in quantum or volumes of a commodity such as quantities of production or sale or consumption. Here the commodity is used in a more general sense. It may mean the volume of goods (in tonnes) carried by roadways, the number of passenger miles travelled, or the volume of export to or import from a country. In such cases we consider of quantity or volume relatives. If quantities or volumes are for a period instead of a point of time, a suitable average is to be taken and the quantities or volumes are to be expressed in the same units. If q_0 and q_n denote the quantity or volume produced, consumed or transacted during the base period (0) and the given period (n) then *quantity relative* of the period n w.r.t. the base period 0 is defined by quantity relative in percentage (of period n w.r.t. period 0) =

$$\frac{q_n}{q_0} \times 100 \qquad \text{... (2)}$$

and is denoted by $q_{0|n}$.

Value Relatives

A value relative is another type of simple index number, usable when we wish to compare changes in the money value of the transaction, consumption or sale in two different periods or points of time. Multiplication of the quantity q by the price p of the commodity produced, transacted or sold gives the total money value pq of the production, transaction or sale. If instead of point of time, period of time is considered, a suitable average is to be taken and is to be expressed in the same units.

If p_0 and q_0 denote the price and the quantity of the commodity during the base period (0) and if p_n and q_n denote the corresponding price and quantity during a given period (n), then the total value $v_0 = p_0 q_0$ and $v_n = p_n q_n$, and the value relative of the period n w.r.t. the base period 0 is defined by

$$\text{value relative in percentage (of period } n \text{ w.r.t. period 0)} = \frac{v_n}{v_0} \times 100$$

$$= \frac{p_n q_n}{p_0 q_0} \times 100 \quad \ldots (3)$$

and is denoted by $v_{0|n}$.

Properties of Relatives

Let $p_a, p_b, p_c, \ldots$ denote the prices, in the periods $a, b, c, \ldots$ respectively, $q_a, q_b, q_c, \ldots$ denote the quantities and $v_a, v_b, v_c, \ldots$ the volumes for the corresponding periods.

The relatives satisfy some properties which are directly obtained from the definitions. We shall state the results for price relatives and write similar results for the quantity and value relatives :

1. Identity property :

   The price relative of a given period w.r.t. the same period is 1, that is

   $$p_{a/a} = 1.$$

2. Time reversal property :

   If the base period and the reference period are interchanged, then the product of the corresponding relatives is unity (one is the reciprocal of the other). That is :

   $$p_{a|b} \times p_{b|a} = 1$$

   or $$p_{b|a} = \frac{1}{p_{a|b}}$$

   Here

   $$p_{a|b} = \frac{p_b}{p_a} \text{ and } p_{b|a} = \frac{p_a}{p_b} \text{ and so the result follows.}$$

3. Circular or Cyclic property :

   We have

   $$p_{a|b} = \frac{p_b}{p_a},\ p_{b|c} = \frac{p_c}{p_b},\ p_{c|a} = \frac{p_a}{p_c}$$

   and so $$p_{a|b} \times p_{b|c} \times p_{c|a} = 1$$

   That is, if the periods *a, b, c,* are in cyclic order then the product of the three relatives w.r.t. the preceding period as base period is unity.

   This holds for *any* number of periods in cyclic order.

4. Modified Circular or Cyclic property :

   We have

   $$p_{a|b} \times p_{b|c} = \frac{p_b}{p_a} \times \frac{p_c}{p_b} = \frac{p_c}{p_a} = p_{a|c}$$

   More generally

   $$p_{a|b} \times p_{b|c} \times p_{c|d} = p_{a|d}\ .$$

Self-Assessment

1. Fill in the blanks :

(*i*) Index numbers are .... averages.

(*ii*) Historically the first index was constructed in ....

(*iii*) Theoretically the best average in the construction of index number is ....

(*iv*) The index numbers are descriptive measures or ....

(*v*) Index numbers are .... of economic activity.

2.4 Summary

- Index numbers have become today one of the most widely used statistical devices. Though originally developed for measuring the effect of change in prices, there is hardly any field today where index numbers are not used. Newspapers headline the fact that prices are going up or down, that industrial production is rising or falling, that imports are increasing or decreasing, that crimes are rising in a particular period compared to the previous period as disclosed by index numbers. They are used to feel the pulse of the economy and they have to be used as indicator of inflationary or deflationary tendencies. In fact, they are described as *barometers of economic activity, i.e.,* if one wants to get an idea as to what is happening to an economy he should look to important indices like the index number of industrial production, agricultural production, business activity, etc.
- to obtain an average, items must be comparable; for example, the average weight of men, women and children of a certain locality has no meaning at all. Furthermore, the unit of measurement must be the same for all the items. Thus an average of the weight expressed in *kg*., *lb*., etc., has no meaning. However, this is not so with index numbers. Index numbers are used for purposes of comparison in situations where two or more series are expressed in different units or the series are composed of different types of items.
- Index numbers are most widely used for measuring changes over a period of time. Thus we can find out the net change in agricultural prices from the beginning of first plan period to the Ninth plan period, *i.e.,* 1997-2002. Similarly, we can compare the agricultural production, industrial production, imports, exports, wages, etc., at two different times. However, it should be noted the index numbers not only measure changes over a period of time but also compare economic conditions of different locations, different industries, different cities or different countries. But since the basic problems are essentially the same and since most of the important index numbers published by the Government and private research organisations refer to data collected at different times, we shall consider in this chapter index numbers measuring changes relative to time only. However, methods described can be applied to other cases also.
- The trend of the phenomenon under study can be obtained by measuring the changes over a period of time. On the basis of this analysis can be done. For example, by examining the index numbers of industrial production, business activity etc. their trend can be understood and analysed.
- Index numbers are very useful in deflating *i.e.,* they are used to adjust the original data for price changes, or to adjust wages for cost of living changes and thus transform nominal wages into real wages. Moreover nominal income can be transformed into real income and nominal sales into real sales through appropriate index numbers.
- It may be concluded that index numbers are strong tools into the hands of the economists and business analysts with the help of which they can measure changes in certain phenomenon over a period of time or through a geographical location. This enables them to form suitable policies to obtain the desired results. Past, present and future trends and tendencies are revealed

Notes

with the help of index numbers and they are also useful in deflating. In the words of *Kafka* and *Simpson*, "Index numbers are today one of the most widely used statistical devices. They are used to feel the pulse of the economy and they have come to be used as indicators of inflationary or deflationary tendencies."

- Index numbers are constructed in the form of specialised averages with definite purpose and with some base period. Before constructing index numbers, it is necessary to know about the various problems which arise in its construction so that they can be minimised.
- Base period is the one against which comparisons are made. It may be year a month or a day. The index for base period is always taken to be 100. It is essential to choose an appropriate base before constructing index numbers.
- To study the change in a certain phenomenon, it is not possible to include each and every item leading to change. For example, while constructing price index, change in price of each and every item cannot be included. The selection of commodities should be such that they are representative of the tastes, preferences and habits etc. of the group of people regarding whom the index is constructed. In this way, it becomes a big problem as to which items to be included and which to be excluded ? Again, by having purpose of index properly defined, the selection of items can be eased. Index numbers would give false results if at one time one set of quantities are used and at other time other set of quantities are used.
- The term 'weight' refers to the relative importance of the different items in the construction of the index. All items are not of equal importance and hence it is necessary to device some suitable method which is done by allocating weights. In case of weighted indices, specific weights are assigned. Implicit or explicit methods of assigning weights can be used. Quantity or value weights can be assigned. If aggregative method is used, quantities are used as weights and in averaging of price relatives method of constructing index, value weights are used. Moreover, it has to be decided whether the weights would be fixed or fluctuating. In this way selection of appropriate weights forms a very crucial problem in constructing index.
- The change in the level of certain phenomenon can be measured only when proper data regarding quotations is available. For example, while preparing price index, it is essential to obtain proper price quotations of the selected commodities. Or while preparing expenditure index, information about expenditure should be available.
- One of the simplest types of index numbers is a *price relative.* It is the ratio of the price of a *single* commodity in a given period or point of time to its price in another period or point of time, called the *reference period* or *base period.* If prices for a period, instead of a point of time, are considered, then suitable price average for the period is taken and these prices are expressed in the same units.
- Simple type of index numbers is a quantity relative, when we are interested in changes in quantum or volumes of a commodity such as quantities of production or sale or consumption. Here the commodity is used in a more general sense. It may mean the volume of goods (in tonnes) carried by roadways, the number of passenger miles travelled, or the volume of export to or import from a country. In such cases we consider of quantity or volume relatives. If quantities or volumes are for a period instead of a point of time, a suitable average is to be taken and the quantities or volumes are to be expressed in the same units.
- A value relative is another type of simple index number, usable when we wish to compare changes in the money value of the transaction, consumption or sale in two different periods or points of time. Multiplication of the quantity q by the price p of the commodity produced, transacted or sold gives the total money value pq of the production, transaction or sale. If instead of point of time, period of time is considered, a suitable average is to be taken and is to be expressed in the same units.

2.5 Key-Words

1. Index Number : In economics and finance, an index is a statistical measure of changes in a representative group of individual data points. These data may be derived from any number of sources, including company performance, prices, productivity, and employment. Economic indices (index, plural) track economic health from different perspectives. Influential global financial indices such as the Global Dow, and the NASDAQ Composite track the performance of selected large and powerful companies in order to evaluate and predict economic trends.

2.6 Review Questions

1. What are index numbers ? How are they constructed ?
2. Define index numbers. Analyse the use of index numbers.
3. What are the various problems faced in construction of index numbers ?
4. "Index numbers are devices for measuring differences in the magnitude of a group of related variables". Discuss this statement and point out the use of index numbers.
5. What are the various types of index numbers.

Answers: Self-Assessment

1. (*i*) Specialised (*ii*) 1764
   (*iii*) Geometric mean (*iv*) Changes (*v*) Barometers

2.7 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 3: Methods—Simple (Unweighted) Aggregate Method and Weighted Aggregate Method

CONTENTS

Objectives

After reading this unit students will be able to:

- Discuss Simple (Unweighted) Aggregate Method.
- Explain Weighted Aggregate Method.

Introduction

A large number of formulae have been devised for constructing index numbers. Broadly speaking, they can be grouped under two heads:

(a) Unweighted indices, and (b) Weighted indices.

In the unweighted indices weights are not expressly assigned where again the weighted indices weights are assigned to the various items. Each of these types may further be divided under two heads:

(1) Simple Index Numbers

Simple Index Number is that Index number in which all the items are assigned equal importance. In other words, weights are not assigned to the different commodities and as such it is also called unweighted Index Number.

There are two methods of calculating Simple Index Number.

(i) Simple aggregate method

(ii) Simple average of price relative method.

(2) Weighted Index Numbers

In constructing simple index numbers, all commodities are given equal importance but in practice, all commodities don't have equal importance. For example, for a consumer, wheat is more important than vegetable or pulse. Similarly, clothes are more important than a video. To express the relative importance of different commodities, weights on some definite basis are used. When index numbers are constructed taking into consideration the importance of different commodities, then they are called weighted index numbers. There are two methods of constructing weighted index numbers:

(i) Weighted Aggregative Method

(ii) Weighted Average of Price Relative Method.

Notes

3.1 Simple (Unweighted) Aggregate Method

This is the simplest method of constructing Index Number. In this method *the total of current year prices for the various commodities is divided by the total of base year prices,* the resultant so obtained is multiplied by 100 to get the Index Numbers for the current year in terms of percentage.

Simbolically,

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

Where, P_{01} = Current year price Index Number based upon base year;

ΣP_1 = Sum total of current year prices; and ΣP_0 = Sum total of base year prices.

In Index Number 0 is used for base year and 1 is used for current year.

Example 1: Given the following data, and assuming 1991 as the base year, find out index value of the prices of different commodities for the year 1995.

Commodity	A	B	C	D	E
Prices in 1991 (Rs.)	50	40	10	5	2
Prices in 1995 (Rs.)	80	60	20	10	6

Solution: **Construction of a Simple Index Number-Simple Aggregate Method**

Commodities	**1991 (or Base Year) P_0 (Rs.)**	**1995 (or Current Year) P_1 (Rs.)**
A	50	80
B	40	60
C	10	20
D	5	10
E	2	6
Total	**ΣP_0 = 107**	**ΣP_1 = 176**

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100 = \frac{176}{107} \times 100 = 164.48$$

Thus, Price Index No. = 164.48

It means that prices, in general has increased by 64.48%.

Example 2: From the following data construct an index for 2005 taking 2004 as base.

Commodities	A	B	C	D	E
Prices in 2004 (Rs.)	50	40	80	110	20
Prices in 2005 (Rs.)	70	60	90	120	20

Solution: **Construction of Price index**

Commodities	**Prices in 2004 P_0 (Rs.)**	**Prices in 2005 P_1 (Rs.)**
A	50	70

B	40	60
C	80	90
D	110	120
E	20	20
Total	**$\Sigma P_0 = 300$**	**$\Sigma P_1 = 360$**

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

$$\Sigma P_1 = 360,\ \Sigma P_0 = 300$$

$$P_{01} = \frac{360}{300} \times 100 = 120$$

This means that as compared to 2004, in 2005 there is a net increase in the prices of commodities included in the index to the extent of 20%.

Merits and Demerits of Simple Aggregate method: Simple aggregative method of index number construction is very easy but it can be applied only when the prices of all commodities have been expressed in the same unit. If units are different, the results will be misleading:

Limitations of Simple Aggregate method: There are two main limitations of the simple aggregative index.

(i) In this type of index, the items with the large unit. Prices exert the greatest influence.

(ii) No consideration is given to the relative importance of the commodities.

3.2 Weighted Aggregative Index Numbers

These indices are of the simple aggregative type with the fundamental difference that weights are assigned to the various items included in the index. There are various methods of assigning weights and consequently a large number of formulae for constructing index numbers have been devised of which some of the more important ones are:

1. Laspayres method.
2. Paasche method.
3. Dorbish and Bowley method.
4. Fisher's ideal method, and
5. Marshall-Edgeworth method.

All these methods carry the name of persons who have suggested them.

1. **Laspeyres Method:** It is the most important of all types of index numbers. In this method the base year quantities are taken as weights. The formula for constructing the index is:

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

Steps:

(i) Multiply the current year prices of various commodities with base year weights and obtain $\Sigma p_1 q_0$.

(ii) Multiply the base year prices of various commodities with base year weights and obtain $\Sigma p_0 q_0$.

(iii) Divide $\Sigma p_1 q_0$ by $\Sigma p_0 q_0$ and multiply the quotient by 100. This gives us the price index.

Laspeyres Index attempts to answer the question: What is the change in aggregate value of the base period list of goods when valued at given period prices ? This index is very widely used in practical work.

2. **Paasche Method:** In this method the *current year* quantities are taken as weights. The formula for constructing the index is:

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

Steps:

(i) Multiply the current year prices of various commodities with base year weights and obtain $\Sigma p_1 q_1$.

(ii) Multiply the base year prices of various commodities with the base year weights and obtain $\Sigma p_0 q_1$.

(iii) Divide $\Sigma p_1 q_1$ by $\Sigma p_0 q_1$ and multiply the quotient by 100.

In general this formula answers the question: What would be the value of the given period list of goods when valued at base-period prices ?

Comparison of Laspeyres and Paasche Methods. From a practical point of view, Laspeyres index is often preferred to Paasche's for the simple reason that in Laspeyres index weights (q_0) are the base-year quantities and do not change from one year to the next. On the other hand, the use of Paasche index requires the continuous use of new quantity weights for each period considered and in most cases these weights are difficult and expensive to obtain.

An interesting property of Laspeyres and Paasche indices is that the former is generally expected to *overestimate* or to leave an upward bias, whereas the latter tends to *underestimate, i.e.,* shows a downward bias. When the prices increase there is usually a reduction in the consumption of those items for which the increase has been the most pronounced, and hence, by using base year quantities we will be giving too much weight to the prices that have increased the most and the numerator of the Laspeyres index will be too large. When the prices go down, consumers often shift their preference to those items which have declined the most and, hence, by using base-period weights in the numerator of the Laspeyres index we shall not be giving sufficient weights to the prices that have gone down the most and the numerator will again be too large. Similarly because people tend to spend less on goods when their prices are rising the use of the Paasche or current weighting produces an index which tends to underestimate the rise in prices, *i.e.,* it has a downward bias. But the above arguments do not imply that Laspeyres index must necessarily be larger than the Paasche's.

Unless drastic changes have taken place between the base year and the given year, the difference between the Laspeyre's and Paasche's will generally be small and either could serve as a satisfactory measure. In practice, however, the base year weighted Laspeyre's type index remains the most popular for reasons of its practicability. The Paasche type index can only be constructed when up-to-date data for the weights are available. Furthermore, the price index of a given year can be compared only with the base year. For example, let P_{82} =102, P_{83} = 130, and P_{84} = 145. Then P_{83} and P_{84} are using different weights and cannot be compared with each other.

If these indices had been obtained by the Laspeyre's formula they could be compared because in that case the weights are the same base-year weights (q_0). For these reasons, in practice the Paasche formula is usually not used and the Laspeyre type index remains most popular for reasons of its practicability.

3. **Dorbish and Bowley's Method:** Dorbish and Bowley have suggested simple arithmetic mean of the two indices (Laspeyres and Paasche) mentioned above so as to take into account the influence of both the periods, *i.e.,* current as well as base periods. The formula for constructing the index is:

$$P_{01} = \frac{L+P}{2}$$

where L = Laspeyre's Index

P = Paasche's Index

or

$$P_{01} = \frac{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$$

4. **Fisher's Ideal Index:** Prof. Irving Fisher has given a number of formulae for constructing index numbers and of these he calls one as the 'ideal' index. The Fisher's Ideal Index is given by the formula:

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

or

$$P_{01} = \sqrt{L \times P}$$

It shall be clear from the above formula that Fisher's Ideal Index is the geometric mean of the Laspeyre and Paasche indices. Thus in the Fisher method we average geometrically formulae that are in opposite directions.

Did u know? Dorbish and Bowley have suggested simple arithmetic mean of the two indices (Laspeyres and Paasche) mentioned above so as to take into account the influence of both the periods, *i.e.*, current as well as base periods.

The above formula is known as 'Ideal' because of the following reasons:

(i) It is based on the geometric mean which is theoretically considered to be the best average for constructing index numbers.

(ii) It takes into account both current year as well as base year prices and quantities.

(iii) It satisfies both the time reversal test as well as the factor reversal test as suggested by Fisher.

(iv) It is free from bias. The two formulae (Laspeyre's and Paasche's) that embody the opposing type and weight biases are, in the ideal formula, crossed geometrically, *i.e.*, by the averaging process that of itself has no bias. The result is the complete cancellation of biases of the kinds revealed by time reversal and factor reversal tests.

It is not, however, a practical index to compute because it is excessively laborious. The data, particularly for the Paasche segment of index, are not really available. In practice, statisticians will continue to rely upon simple, although perhaps less exact, index number formulae.

5. **Marshall-Edgeworth Method:** In this method also both the current year as well as base year prices and quantities are considered. The formula for constructing the Index is:

$$P_{01} = \frac{\Sigma(q_0 + q_1)p_1}{\Sigma(q_0 + q_1)p_0} \times 100$$

or, opening the brackets

$$P_{01} = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1} \times 100$$

It is a simple, readily constructed measure, giving a very close approximation to the results obtained by the ideal formula.

Example 3: Construct index numbers of price from the following data by applying (1) Laspeyre's method, (2) Passche method, (3) Bowley method, (4) Fisher's Ideal method, and (5) Marshall-Edgeworth method.

	2004		**2005**	
Commodities	**Price**	**Quantity**	**Price**	**Quantity**
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

Solution:

Calculation of Various Indices

Commodities	**2004**		**2005**					
	Price p_0	**Quantity** q_0	**Price** p_1	**Quantity** q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	2	8	4	6	32	16	24	12
B	5	10	6	5	60	50	30	25
C	4	14	5	10	70	56	50	40
D	2	19	2	13	38	38	26	26
					Σp_1q_0 = 200	Σp_0q_0 = 160	Σp_1q_1 = 130	Σp_0q_1 = 103

1. Laspeyre's Method: $P_{01} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$

$$\Sigma p_1q_0 = 200,\ \Sigma p_0q_0 = 160$$

$$P_{01} = \frac{200}{160} \times 100 = 125$$

2. Paasche Method: $P_{01} = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$

$$\Sigma p_1q_1 = 130,\ \Sigma p_0q_1 = 103$$

$$P_{01} = \frac{130}{103} \times 100 = 126.21$$

3. Bowley's Method: $P_{01} = \frac{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} + \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2} \times 100$

$$= \frac{\frac{200}{160} + \frac{130}{103}}{2} \times 100$$

$$= \frac{1.25 + 1.262}{2} \times 100 = \frac{2.512}{2} \times 100 = 125.6$$

or $$P_{01} = \frac{L + P}{2} = \frac{125 + 126.21}{2} = 125.61$$

4. Fisher's Ideal Method:

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100 = \sqrt{\frac{200}{160} \times \frac{130}{103}} \times 100$$

$$= \sqrt{1.578} \times 100 = 1.256 \times 100 = 125.6$$

5. Marshall-Edgeworth Method:

$$P_{01} = \frac{\Sigma(q_0 + q_1)p_1}{\Sigma(q_0 + q_1)p_0} \times 100$$

$$= \frac{200 + 130}{160 + 103} \times 100 = \frac{330}{263} \times 100 = 125.48.$$

Example 4: Using appropriate formula, construct Index Numbers for the year 1994 on the basis of year 1992 of the following data:

Year	Article 1		Article II		Article III	
	Price	**Quantity**	**Price**	**Quantity**	**Price**	**Quantity**
1992	5	10	8	6	6	3
1994	4	12	7	7	5	4

Solution: Since we are given price and quantity data for base as well as current year, the suitable index will be the fisher's Ideal Index.

	1992 Base Year		1994 Current Year					
Article	**Price** p_0	**Quantity** q_0	**Price** p_1	**Quantity** q_1	p_0q_0	p_0q_1	p_1q_0	p_1q_1
I	5	10	4	12	50	60	40	48
II	8	6	7	7	48	56	42	49
III	6	3	5	4	18	24	15	20
Total					Σp_0q_0 = 116	Σp_0q_1 = 140	Σp_1q_0 = 97	Σp_1q_1 = 117

According to Fisher's Ideal Formula,

Index Number for 1994

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100 = \sqrt{\frac{97}{116} \times \frac{117}{140}} \times 100$$

$$= \sqrt{0.6969} \times 100 = 83.6$$

Example 5: From the following data, calculate price index numbers for the current year by using.

(a) Paasche's Method

(b) Laspeyre's Method

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	8	50	20	60
B	2	15	6	10
C	1	20	2	8
D	2	10	5	8
E	1	40	3	30

Solution:

Commodities	Base year		Current year					
	p_0	q_0	p_1	q_1	p_0q_0	p_0q_1	p_1q_0	p_1q_1
A	8	50	20	60	400	480	1000	1200
B	2	15	6	10	30	20	90	60
C	1	20	2	8	20	8	40	16
D	2	10	5	8	20	16	50	40
E	1	40	3	30	40	30	120	90
					510	554	1300	1406

(a) Paasche's Method

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100 = \frac{1406}{554} \times 100 = 253.79$$

(b) Laspeyre's Method

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{1300}{510} \times 100 = 254.90$$

Example 6: Calculate the weighted price Index from the following data:

Materials required	Unit	Quantity Required	1990 (Rs.)	1995 (Rs.)
Cement	100 Qtl.	500 *lb.*	5.0	8.0
Timber	*c.ft.*	2,000 *c.ft.*	9.5	14.2
Steel sheets	*c.wt.*	50 *c.wt.*	34.0	42.0
Bricks	per' 000	20,000	12.0	24.0

Solution: Since the weight are fixed, we apply Kelly's method for computing Index.

Calculation of Weighted Price Index

Material	Unit	Quantity	Price During 1990	Price During 1995		
		q	P_0	P_1	p_0q	p_1q
Cement	100 Qtl.	5	5.0	8.0	25	40
Timber	*c.ft.*	2,000	9.5	14.2	19,000	28,400
Steel Sheet	*c.wt.*	50	34.0	42.0	1,700	2,100
Bricks	per' 000	20	12.0	24.0	240	480
					$\sum p_0q = 20,965$	$\sum p_1q = 31,020$

$$P_{01} = \frac{\sum p_1q}{\sum p_0q} \times 100; \quad \sum p_1q = 31,020; \quad \sum p_0q = 20,965$$

$$\therefore \quad P_{01} = \frac{31,020}{20,965} \times 100 = 147.96$$

Example 7: Construct index numbers of price from the following data by applying:

1. Laspeyres method,
2. Paasche method,
3. Bowley's method,
4. Fisher's Ideal method, and
5. Marshall-Edgeworth method.

Commodity	2006 Price	2006 Quantity	2007 Price	2007 Quantity
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

Solution:

Commodity	2006 Price p_0	2006 Qty. q_0	2007 Price p_1	2007 Qty. q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	2	8	4	6	32	16	24	12
B	5	10	6	5	60	50	30	25
C	4	14	5	10	70	56	50	40
D	2	19	2	13	38	38	26	26
					$\sum p_1q_0 = 200$	$\sum p_0q_0 = 160$	$\sum p_1q_1 = 130$	$\sum p_0q_1 = 103$

Notes

1. Laspeyres Method: $P_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$; where $\sum p_1 q_0 = 200$, $\sum p_0 q_0 = 160$

$$P_{01} = \frac{200}{160} \times 100 = 125$$

2. Paasche's Method: $P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$; where $\sum p_1 q_1 = 130$, $\sum p_0 q_1 = 103$

$$P_{01} = \frac{130}{103} \times 100 = 126.21$$

3. Bowley's Method: $P_{01} = \frac{\frac{\sum p_1 q_0}{\sum p_0 q_0} + \frac{\sum p_1 q_1}{\sum p_0 q_1}}{2} \times 100$

$$= \frac{\frac{200}{160} + \frac{130}{103}}{2} \times 100$$

$$= \frac{1.25 + 1.262}{2} \times 100 = \frac{2.512}{2} \times 100 = 125.6$$

$$P_{01} = \frac{L + P}{2} = \frac{125 + 126.2}{2} = 125.6$$

4. Fisher's Ideal Method:

$$P_{01} = \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} + \frac{\sum p_1 q_1}{\sum p_0 q_1}} \times 100 = \sqrt{\frac{200}{160} \times \frac{130}{103}} \times 100$$

$$= \sqrt{1.578} \times 100 = 1.2561 \times 100 = 125.61$$

5. Marshall-Edgeworth Method:

$$P_{01} = \frac{\sum (q_0 + q_1) p_1}{\sum (q_0 + q_1) p_0} \times 100 = \frac{\sum p_1 q_0 + \sum p_1 q_1}{\sum p_0 q_0 + \sum p_0 q_1}$$

$$= \frac{200 + 130}{160 + 103} \times 100 = \frac{330}{263} \times 100 = 125.48$$

Example 8: Compute Laspeyres, Paasche's, Fisher's and Marshall-Edgeworth's Index Numbers from the following data:

Item	Base Year		Current Year	
	Price (Rs.)	Quantity	Price (Rs.)	Quantity
A	5	25	6	30
B	3	8	4	10
C	2	10	3	8
D	10	4	3	5

Solution: **Computation of Laspeyres, Paasche's, Fisher's Index**

Item	p_0	q_0	p_1	q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	5	25	6	30	150	125	180	150
B	3	8	4	10	32	24	40	30
C	2	10	3	8	30	20	24	16
D	10	4	3	5	12	40	15	50
					Σp_1q_0 = 224	Σp_0q_0 = 209	Σp_1q_1 = 259	Σp_0q_1 = 246

Laspeyres Index: $P_{01} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0}\times 100 = \frac{224}{209}\times 100 = 107.17$

Paasche's Index: $P_{01} = \frac{\Sigma p_1q_1}{\Sigma p_0q_1}\times 100 = \frac{259}{246}\times 100 = 105.28$

Fisher's Index: $P_{01} = \sqrt{L\times P} = \sqrt{107.17\times 105.28} = 106.22$

Marshall-Edgeworth's Index: $P_{01} = \frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)}\times 100 = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1}\times 100$

$= \frac{224+259}{209+246}\times 100 = 106.15$

Example 9 : Prepare Index Number for 2010 on the basis of 2005, where the following information is given:

Year	Article I		Article II		Article III	
	Price	**Quantity**	**Price**	**Quantity**	**Price**	**Quantity**
2005	5	10	8	6	6	4
2010	4	12	7	7	5	3

Solution : Fisher's Ideal Index Number for 2010 — (Base 2005)

Article	2005 (Base Year)		2010		p_0q_0	p_1q_0	p_0q_1	p_1q_1
	Price p_0	**Qnty.** q_0	**Price** p_1	**Qnty.** q_1				
I	5	10	4	12	50	40	60	48
II	8	6	7	7	48	42	56	49
III	6	4	5	3	24	20	18	15
					122 (Σp_0q_0)	102 (Σp_1q_0)	134 (Σp_0q_1)	112 (Σp_1q_1)

Fisher's Ideal Index Number for 2010

or $$P_{01} = \sqrt{\frac{\sum p_1q_0 \times \sum p_1q_1}{\sum p_0q_0 \times \sum p_0q_1}} \times 100$$

$$= \sqrt{\frac{102}{122} \times \frac{112}{134}} \times 100$$

$$= \sqrt{.84 \times .84} \times 100 = .84 \times 100 = 84$$

Self-Assessment

1. Which of the following statements is True or False:
   (*i*) The base period should be a normal period.
   (*ii*) Unweighted indices are actually implicitly weighted.
   (*iii*) Bowley's index is the geometric mean of Laspeyre and Paasche Index.
   (*iv*) Marshall-Edgeworth's method satisfies the time reversal test.
   (*v*) In the Paasche's Price index, the weights are determined by quantities in the base period.

3.3 Summary

- *Simple Index Number is that Index number in which all the items are assigned equal importance.* In other words, weights are not assigned to the different commodities and as such it is also called unweighted Index Number.
- In constructing simple index numbers, all commodities are given equal importance but in practice, all commodities don't have equal importance. For example, for a consumer, wheat is more important than vegetable or pulse. Similarly, clothes are more important than a video. To express the relative importance of different commodities, weights on some definite basis are used. When index numbers are constructed taking into consideration the importance of different commodities, then they are called weighted index numbers.
- This is the simplest method of constructing Index Number. In this method *the total of current year prices for the various commodities is divided by the total of base year prices,* the resultant so obtained is multiplied by 100 to get the Index Numbers for the current year in terms of percentage.
- Simple aggregative method of index number construction is very easy but it can be applied only when the prices of all commodities have been expressed in the same unit.
- These indices are of the simple aggregative type with the fundamental difference that weights are assigned to the various items included in the index. There are various methods of assigning weights and consequently a large number of formulae for constructing index numbers have been devised of
- From a practical point of view, Laspeyres index is often preferred to Paasche's for the simple reason that in Laspeyres index weights (q_0) are the base-year quantities and do not change from one year to the next. On the other hand, the use of Paasche index requires the continuous use of new quantity weights for each period considered and in most cases these weights are difficult and expensive to obtain.
- When the prices increase there is usually a reduction in the consumption of those items for which the increase has been the most pronounced, and hence, by using base year quantities we will be giving too much weight to the prices that have increased the most and the numerator of the Laspeyres index will be too large.
- Unless drastic changes have taken place between the base year and the given year, the difference between the Laspeyre's and Paasche's will generally be small and either could serve as a satisfactory measure. In practice, however, the base year weighted Laspeyre's type index remains

the most popular for reasons of its practicability. The Paasche type index can only be constructed when up-to-date data for the weights are available. Furthermore, the price index of a given year can be compared only with the base year. For example, let P_{82} =102, P_{83} = 130, and P_{84} = 145. Then P_{83} and P_{84} are using different weights and cannot be compared with each other.

- Prof. Irving Fisher has given a number of formulae for constructing index numbers and of these he calls one as the 'ideal' index.
- It shall be clear from the above formula that Fisher's Ideal Index is the geometric mean of the Laspeyre and Paasche indices. Thus in the Fisher method we average geometrically formulae that are in opposite directions.
- A practical index to compute because it is excessively laborious. The data, particularly for the Paasche segment of index, are not really available. In practice, statisticians will continue to rely upon simple, although perhaps less exact, index number formulae.

3.4 Key-Words

1. Aggregate method : The term Aggregate Method refers the way price and volume data are handled when daily prices are gathered into weekly, monthly or even longer-term aggregate files. These settings are included in UA Preferences. Click "Aggregate Method" to see your current setting and adjust as desired.
2. Weighted index number : When all commodities are not of equal importance. We assign weight to each commodity relative to its importance and index number computed from these weights is called weighted index numbers.

3.5 Review Questions

1. Discuss the Simple aggregative method of constructing Price index number.
2. What are weighted index numbers ? Describe various types of weighted aggregative index numbers.
3. Why Fisher's formula for computing index numbers is said to be ideal ?
4. Distinguish between Laspeyre's and Paasche's index.
5. Explain the situations in which weighted and unweighted index numbers are useful.

Answers: Self-Assessment

1. (*i*) T (*ii*) T (*iii*) F (*iv*) T (*v*) F

3.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Notes

Unit 4: Methods: Simple (Unweighted) Aggregate Method

CONTENTS

Objectives

After reading this unit students will be able to:

- Explain Simple Aggregate Method.

Introduction

Simple Index Number is that Index number is which all the items are assigned equal importance. In other words, weights are not assigned to the different commodities and as such it is also called unweighted Index Number.

There are two methods of calculating Simple Index Number.

- Simple aggregate method.
- Simple average of price relative method.

4.1 Simple Aggregate Method

This is the simplest method of constructing Index Number. In this method the total of current year prices for the various commodities is divided by the total of base year prices, the resultant so obtained is multiplied by 100 to get the Index Numbers for the current year in terms of percentage.

Simbolically,

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100$$

Where, P_{01} = Current year pric Index Number based upon base year,

$\sum P_1$ = Sum total of current year prices; $\sum P_0$ = Sum total of base year prices.

In Index Number 0 is used for base year and 1 is used for current year.

Example 1: Given the following data, and assuming 1991 as the base year, find out index value of the prices of different commodities for the year 1995.

Commodity	A	B	C	D	E
Prices in 1991 (Rs.)	50	40	10	5	2
Prices in 1995 (Rs.)	80	60	20	10	6

Notes

Solution: Construction of a Simple Index Number-Simple Aggregate Method:

Commodities	1991 (or Base Year) P_0 (Rs.)	1995 (or Current Year) P_1 (Rs.)
A	50	80
B	40	60
C	10	20
D	5	10
E	2	6
Total	$\Sigma P_0 = 107$	$\Sigma P_1 = 176$

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100 = \frac{176}{107} \times 100 = 164.48$$

Thus, Price Index No. = 164.48

It means that prices, in general has inceased by 64.48%.

Example 2: Construct price index number for 1990 based on 1981 using Simple Agregative Method:

Commodities	Price in 1981 (in Rs.)	Price in 1990 (in Rs.)
A	50	80
B	40	60
C	10	20
D	5	10
E	2	8

Solution:

Construction of Price Index Number

Commodities	Price in 1981 (P_0)	Price in 1990 (P_1)
A	50	80
B	40	60
C	10	20
D	5	10
E	2	8
Total	$\Sigma P_0 = 107$	$\Sigma P_1 = 178$

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100 = \frac{176}{107} \times 100 = 166.48$$

Merits and Demerits of Simple Aggregate Method: Simple aggregative method of index number construction is very easy but it can be applied only when the prices of all commodities have been expressed in the same unit. If units are different, the retults will be misleading?

Example 3: Given the following data, and assuming 1991 as the base year, find out index value of the prices of different commodities for the year 1995.

Commodity	A	B	C	D	E
Prices in 1991 (Rs.)	50	40	10	5	2
Prices in 1995 (Rs.)	80	60	20	10	6

Solution: **Construction of a Simple Index Number-Simple Aggregate Method**

Commodities	**1991 (or Base Year) P_0 (Rs.)**	**1995 (or Current Year) P_1 (Rs.)**
A	50	80
B	40	60
C	10	20
D	5	10
E	2	6
Total	$\Sigma P_0 = 107$	$\Sigma P_1 = 176$

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100 = \frac{176}{107} \times 100 = 164.48$$

Thus, Price Index No. = 164.48

It means that prices, in general has increased by 64.48%.

Example 4: From the following data construct an index for 2005 taking 2004 as base.

Commodities	A	B	C	D	E
Prices in 2004 (Rs.)	50	40	80	110	20
Prices in 2005 (Rs.)	70	60	90	120	20

Solution: **Construction of Price index**

Commodities	**Prices in 2004 P_0 (Rs.)**	**Prices in 2005 P_1 (Rs.)**
A	50	70
B	40	60
C	80	90
D	110	120
E	20	20
Total	$\Sigma P_0 = 300$	$\Sigma P_1 = 360$

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

$$\Sigma P_1 = 360,\ \Sigma P_0 = 300$$

$$P_{01} = \frac{360}{300} \times 100 = 120$$

This means that as compared to 2004, in 2005 there is a net increase in the prices of commodities included in the index to the extent of 20%.

Merits and Demerits of Simple Aggregate method: Simple aggregative method of index number construction is very easy but it can be applied only when the prices of all commodities have been expressed in the same unit. If units are different, the results will be misleading:

Limitations of Simple Aggregate method: There are two main limitations of the simple aggregative index.

(i) In this type of index, the items with the large unit. Prices exert the greatest influence.

(ii) No consideration is given to the relative importance of the commodities.

Self-Assessment

1. Tick (√) the correct statements

(*i*) Simple index number is that number in which all the items are assigned equal.

(*ii*) In Lasreyers method, the base year quantities are taken as weights.

(*iii*) In pass the method the current year quantities are taken as weights.

(*iv*) In Marshall-Edgeworth Method both the current year as well as base year prices ae considered.

4.2 Summary

- Simple aggregative method of index number construction is very easy but it can be applied only when the prices of all commodities have been expressed in the same unit.
- Simple Index Number is that Index number is which all the items are assigned equal importance. In other words, weights are not assigned to the different commodities and as such it is also called unweighted Index Number.

  There are two methods of calculating Simple Index Number.

  (*i*) Simple aggregate method.

  (*ii*) Simple average of price relative method.
- This is the simplest method of constructing Index Number. In this method the total of current year prices for the various commodities is divided by the total of base year prices, the resultant so obtained is multiplied by 100 to get the Index Numbers for the current year in terms of percentage.

4.3 Key-Words

1. Index Number : Index Number is that Index number is which all the items are assigned equal importance. In other words, weights are not assigned to the different commodities and as such it is also called unweighted Index Number.
2. Price index : Index that tracks inflation by measuring price changes. Examples include the Consumer Price Index and the Producer Price Index.
3. Consumer price index (CPI) : A measure of changes in the purchasing-power of a currency and the rate of inflation. The consumer price index expresses the current prices of a basket of goods and services in terms of the

prices during the same period in a previous year, to show effect of inflation on purchasing power. It is one of the best known lagging indicators. See also producer price index.

4. Producer price index (PPI) : Relative measure of average change in price of a basket of representative goods and services sold by manufacturers and producers in the wholesale market. A family of three indices (finished goods, intermediate goods, and raw materials or crude commodities), it is used as an indicator of rate of inflation or deflation. In contrast to the consumer price index (CPI) which measures price changes from the consumer's perspective, PPI measures them from the seller's perspective. Older name wholesale price index.

4.4 Review Questions

1. What do you mean by Simple Index Number? Discuss its methods.
2. What is Simple aggregate method? Explain with examples.

Answers: Self-Assessment

1. (*i*) √ (*ii*) √ (*iii*) √ (*iv*) √

4.5 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Notes

Unit 5Methods—Simple Average of Price Relatives

CONTENTS

Objectives

After reading this unit students will be able to:

- Explain Simple Average of Price Relatives.
- Know the Merits and Limitations of Simple Average of Price Relatives Method.

Introduction

When this method is used to construct a price index first of all price relatives are obtained for the various items included in the index and then an average of these relatives is obtained using anyone of the measures of central value.

5 .1Simple Average of Price Relatives Method

In this method, we can use either Arithmetic Mean or Geometric Mean as the average of relatives.

(a) **Using Arithmetic Mean:** The arithmetic average has the advantage of simplicity but it is too much affected by the extreme values. It gives too much weight to increasing prices and little to decreasing ones. According to this method, we first find out price relative for each commodity and then take simple average of all price relatives. A price relative is the percentage ratio of the price of a variable in the current year to the price in the base year. Thus,

$$P_{01} = \frac{\Sigma\left(\frac{P_1}{P_0} \times 100\right)}{N}$$

$$\Rightarrow P_{01} = \frac{\Sigma P}{N}$$

[where, $\frac{P_1}{P_0} \times 100$ = Price Relative = P; N = Number of Commodities; P_1 = Current Year's Price; P_0 = Base Year's Price.]

Example 1: Given the following data and using the Price Relative method, construct an Index for the year 1993 in relation to 1983 price.

Notes

Commodities	Wheat (Per Qt.)	Ghee (Per *kg*.)	Milk (Per *kg*.)	Rice (Per Qt.)	Sugar (Per *kg*.)
1983 Prices (Rs.)	100	8	0.50	200	1
1993 Prices (Rs.)	200	40	4	800	6

Solution:

Construction of a Simple Index Number Average of Price Relative Method

Commodities	Base year 1983 Price P_0	1993 Price P_1	Price Relative of 1993 in relation to 1983 $P = \frac{P_1}{P_0} \times 100$
Wheat	100 (Per Qt.)	200 (Per Qt.)	$\frac{200}{100} \times 100 = 200$
Ghee	8 (Per *kg*.)	40 (Per *kg*.)	$\frac{40}{8} \times 100 = 500$
Milk	0.50 (Per *kg*.)	4 (Per *kg*.)	$\frac{4}{0.5} \times 100 = 800$
Rice	200 (Per Qt.)	800 (Per Qt.)	$\frac{800}{200} \times 100 = 400$
Sugar	1 (Per *kg*.)	6 (Per *kg*.)	$\frac{6}{1} \times 100 = 600$
N = 5			$\Sigma\left(\frac{P_1}{P_0} \times 100\right) = \Sigma P = 2500$

$$P_{01} = \frac{\Sigma\left(\frac{P_1}{P_0} \times 100\right)}{N} = \frac{\Sigma P}{N} = \frac{2500}{5} = 500$$

(b) **Using Geometric Mean:** *The Geometric Mean is used when items in a group are considered from the view point of their relative difference rather than that of their absolute difference*. For example, if the price of a commodity increases by 50% and that of another falls by 50%, the arithmetic average of relatives will neither rise nor fall implying that there has been no change in the price level. But in fact both the prices have changed. The Geometric Mean of relatives would in this case show that there has been a change in the price.

When Geometric mean is used, then the following formula is used:

$$\log P_{01} = \frac{\Sigma \log\left[\frac{P_1}{P_0} \times 100\right]}{N}, \text{ then}$$

$$P_{01} = \frac{\text{Antilog}\,\Sigma\log\left[\frac{P_1}{P_0}\times 100\right]}{N}$$

If $\left(\frac{P_1}{P_0}\times 100\right)$ is represented by P, then

$$P_{01} = \text{Antilog}\,\frac{\Sigma\log P}{N}$$

The following example will illustrate the application of above rules.

Example 2: In the above example 2, Calculate Index Number using Geometric Mean as Average of Relatives.

Solution:

Commodities	**1983 Base Year Price P_0**	**1993 Current Year Price P_1**	**$P = \frac{P_1}{P_0}\times 100$**	**log P**
Wheat	100	200	$\frac{200}{100}\times 100 = 200$	2.3010
Ghee	8	40	$\frac{40}{8}\times 100 = 500$	2.6990
Milk	0.50	4	$\frac{4}{0.5}\times 100 = 800$	2.9031
Rice	200	800	$\frac{800}{200}\times 100 = 400$	2.6021
Sugar	1	6	$\frac{6}{1}\times 100 = 600$	2.7782
				$\Sigma\log P = 13.2834$

$$\log P_{01} = \frac{\Sigma\log P}{N}$$

$$\log P_{01} = \frac{13.2834}{5}$$

$$\log P_{01} = \text{Antilog}\,[2.6567]$$

$$P_{01} = 453.63$$

Example 3: From the following data construct an Index for 2005 taking 2004 as base by the average of relatives methods using (a) arithmetic mean, and (b) geometric mean for averaging relatives.

Notes

Commodities	Price in 2004 (Rs.)	Price in 2005 (Rs.)
A	50	70
B	40	60
C	80	90
D	100	120
E	20	20

Solution:

(a) INDEX NUMBER USING ARITHMETIC MEAN OF PRICE RELATIVES

Commodities	Price in 2004 (Rs.) P_0	Price in 2005 (Rs.) P_1	Price $\frac{P_1}{P_0} \times 100$
A	50	70	140.0
B	40	60	150.0
C	80	90	112.5
D	110	120	109.1
E	20	20	100.0
			$\Sigma \frac{P_1}{P_0} \times 100 = 611.6$

$$P_{01} = \frac{\Sigma \frac{P_1}{P_0} \times 100}{N} = \frac{611.6}{5} = 122.32$$

(b) INDEX NUMBR USING GEOMETRIC MEAN OF PRICE RELATIVES

Commodities	Price in 2004 P_0	Price in 2005 P_1	Price Relatives P	Log P
A	50	70	140.0	2.1461
B	40	60	150.0	2.1761
C	80	90	112.5	2.0512
D	110	120	109.1	2.0378
E	20	20	100.0	2.0000
				log P = 10.4112

$$P_{01} = \text{Antilog}\left[\frac{\Sigma \log P}{N}\right]$$

$$= \text{Antilog}\left[\frac{10.4112}{5}\right] = \text{Antilog } 2.0822 = 120.9$$

Although arithmetic mean and geometric mean have both been used, the arithmetic mean is often preferred because it is easier to compute and much better known. Some economists, notably F.Y. Edgeworth, have preferred to use the median which is not affected by a single extreme value. Since the argument is important only when an index is based on a very small number of commodities, it generally does not carry much weight and the median is seldom used in actual practice.

5.2 Merits and Limitations of Simple Average of Price Relative Method

Merits

This method has the following two advantages over the previous method:

1. Extreme items do not influence the index. Equal importance is given to all the items.
2. The index is not influenced by the units in which prices are quoted or by the absolute level of individual prices. Relatives are pure numbers and are, therefore, divorced from the original units. Consequently, index numbers computed by the relatives method would be the same regardless of the way in which prices are quoted. This simple average of price relatives is said to meet what is called the *units test*.

Limitations

Despite these merits this method is not very satisfactory because of two reasons:

1. Difficulty is faced with regard to the selection of an appropriate average. The use of the arithmetic mean is considered as questionable sometimes because it has an upward bias. The use of geometric mean involves difficulties of computation. Other averages are almost never used while constructing index numbers.
2. The relatives are assumed to have equal importance. This is again a kind of concealed weighting system that is highly objectionable since economically same relatives are more important than others.

Self-Assessment

1. Fill in the Blanks:

(*i*) Theoretically the best average in the construction of index number is.

(*ii*) A price relative is the percentage ratio of the price of a variable in the .... year to the price in the .... year.

(*iii*) Weighted average of relatives can be combined to form a new ....

(*iv*) The index is not influenced by the unit in which .... are quoted.

(*v*) The average may be arithmetic mean, median, mode or ....

5.3 Summary

- The arithmetic average has the advantage of simplicity but it is too much affected by the extreme values. It gives too much weight to increasing prices and little to decreasing ones. According to this method, we first find out price relative for each commodity and then take simple average of all price relatives. A price relative is the percentage ratio of the price of a variable in the current year to the price in the base year.
- *The Geometric Mean is used when items in a group are considered from the view point of their relative difference rather than that of their absolute difference*. For example, if the price of a commodity

increases by 50% and that of another falls by 50%, the arithmetic average of relatives will neither rise nor fall implying that there has been no change in the price level. But in fact both the prices have changed. The Geometric Mean of relatives would in this case show that there has been a change in the price.

- Although arithmetic mean and geometric mean have both been used, the arithmetic mean is often preferred because it is easier to compute and much better known. Some economists, notably F.Y. Edgeworth, have preferred to use the median which is not affected by a single extreme value. Since the argument is important only when an index is based on a very small number of commodities, it generally does not carry much weight and the median is seldom used in actual practice.
- The index is not influenced by the units in which prices are quoted or by the absolute level of individual prices. Relatives are pure numbers and are, therefore, divorced from the original units. Consequently, index numbers computed by the relatives method would be the same regardless of the way in which prices are quoted. This simple average of price relatives is said to meet what is called the *units test*.
- Difficulty is faced with regard to the selection of an appropriate average. The use of the arithmetic mean is considered as questionable sometimes because it has an upward bias. The use of geometric mean involves difficulties of computation. Other averages are almost never used while constructing index numbers.
- The relatives are assumed to have equal importance. This is again a kind of concealed weighting system that is highly objectionable since economically same relatives are more important than others.

5.4 Key-Words

1. Arithmetic mean : In mathematics and statistics, the arithmetic mean, or simply the mean or average when the context is clear, is the central tendency of a collection of numbers taken as the sum of the numbers divided by the size of the collection. The collection is often the sample space of an experiment. The term "arithmetic mean" is preferred in mathematics and statistics because it helps distinguish it from other means such as the geometric and harmonic mean.
2. Geometric mean : In mathematics, the geometric mean is a type of mean or average, which indicates the central tendency or typical value of a set of numbers by using the product of their values (as opposed to the arithmetic mean which uses their sum). The geometric mean is defined as the nth root (where n is the count of numbers) of the product of the numbers.

5.5 Review Questions

1. Discuss steps of simple average of price relative method of constructing index numbers.
2. What are the merits and limitations of simple average of price relative method.
3. Explain the role of weights in the construction of general price index numbers.
4. What is simple average of price relative method of constructing index numbers ? Explain by using arithmetic mean.
5. What is simple average of price relative method of constructing index numbers? Explain by using geometric mean.

Answers: Self-Assessment

1. (*i*) Geometric mean (*ii*) Current, base (*iii*) Index
   (*iv*) Prices (*v*) Geometric mean

5.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 6: Methods—Weighted Average of Price Relatives

Notes

CONTENTS

Objectives

After reading this unit students will be able to:

- Describe Weighted Average of Price Relatives.
- Explain Quantity Index Number.

Introduction

In the weighted aggregative methods discussed earlier price relatives were not computed. However, like unweighted relative method it is also possible to compute weighted average of relatives. For the purpose of averaging we may use either the arithmetic mean or the geometric mean.

6.1 Weighted Average of Price Relative Method

In order to compute index number by Weighted Average of Relatives Method, following steps are necessarily be taken: (1) Express each item of the period for which the index number is being calculated as a percentage of the same item in the base period. (2) Multiply the percentage as obtained in step (1) for each item by the weight that has been assigned to that item. (3) Add the results obtained in step (2), (4) Divide the sum obtained in step (3) by the sum of weights used to obtain the index number.

When arithmetic mean is used,

$$P_{01} = \frac{\Sigma PV}{\Sigma V}$$

where P is price relative $\frac{p_1}{p_0} \times 100$ and V is value weights $p_0 q_0$.

When geometric mean is used,

$$P_{01} = \text{Antilog}\left[\frac{\Sigma V \log P}{\Sigma V}\right]$$

where $P = \frac{p_1}{p_0} \times 100$, $V = p_0 q_0$

Notes

Example 1: From the following data compute price index by applying weighted average of Price relative method using:

(a) arithmetic mean, and

(b) geometric mean.

Commodities	p_0 **Rs.**	q_0	p_1 **Rs.**
Sugar	6.0	10 *kg.*	8.0
Rice	3.0	20 *kg.*	3.2
Milk	2.0	5 *lt.*	3.0

Solution:

(a) Index number using weighted arithmetic mean of Price Relatives

Commodities	p_0	q_0	p_1	$p_0 q_0$ V	$\frac{p_1}{p_0} \times 100$ P	**PV**
Sugar	Rs. 6.0	10 *kg.*	Rs. 8.0	60	$\frac{8}{6} \times 100$	8,000
Rice	Rs. 3.0	20 *kg.*	Rs. 3.2	60	$\frac{3.2}{3} \times 100$	6,400
Milk	Rs. 2.0	5 *lt.*	Rs. 3.0	10	$\frac{3}{2} \times 100$	1,500
				$\Sigma V = 130$		$\Sigma PV = 15{,}900$

$$P_{01} = \frac{\Sigma PV}{\Sigma V} = \frac{15{,}900}{1300}$$

$$= 122.31.$$

This means that there has been a 22.3 percent increase in prices over the base level.

(b) Index Number using Geometric mean of Price Relatives

Commodities	p_0	q_0	p_1	**V**	**P**	**Log P**	**V. Log P**
Sugar	Rs. 6.0	10 *kg.*	Rs. 8.0	60	133.3	2.1249	127.494
Rice	Rs. 3.0	20 *kg.*	Rs. 3.2	60	106.7	2.0282	121.692
Milk	Rs. 2.0	5 *lt.*	Rs. 3.0	10	150.0	2.1761	21.761
				ΣV = 130			$\Sigma V. \log P$ = 270.947

$$P_{01} = \text{Antilog}\left[\frac{\Sigma V. \log P}{\Sigma V}\right]$$

$$= \text{Antilog}\left[\frac{270.947}{130}\right] = \text{Antilog } 2.084 = 121.3.$$

The result obtained by applying the Laspeyre's method would come out to be the same as obtained by weighted arithmetic mean of price relative method (as shown below):

PRICE INDEX BY LASPEYRE'S METHOD

Commodities	p_0	q_0	p_1	p_1q_0	p_0q_0
Sugar	Rs. 6.0	10 *kg.*	Rs. 8.0	80	60
Rice	Rs. 3.0	20 *kg.*	Rs. 3.2	64	60
Milk	Rs. 2.0	5 *lt.*	Rs. 3.0	15	10
				$\Sigma p_1q_0 = 159$	$\Sigma p_0q_0 = 130$

$$P_{01} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{159}{130} \times 100 = 122.31$$

The answer is the same as that obtained by weighted arithmetic mean of price relatives method.

Merits of Weighted Average of Relative Indices

The following are the special advantages of weighted average of relative indices over weighted aggregative indices:

(1) When different index numbers are constructed by the average of relatives method, all of which have the same base, they can be combined to form a new index.

(2) When an index is computed by selecting one item from each of the many sub-groups of items, the values of each sub-group may be used as weights. Then only the method of weighted average of relatives is appropriate.

(3) When a new commodity is introduced to replace the one formerly used, the relative for the new item may be spliced to the relative for the old one, using the former value weights.

(4) The price or quantity relatives for each single item in the aggregate are, in effect, themselves a simple index that often yields valuable information for analysis.

Did u know? Price index numbers measure and permit comparison of the price of certain goods; quantity index numbers, on the other hand, measure and permit comparison of the physical volume of goods produced or distributed or consumed.

6.2 Quantity Index Numbers

Price index numbers measure and permit comparison of the price of certain goods; quantity index numbers, on the other hand, measure and permit comparison of the physical volume of goods produced or distributed or consumed. Though price indices are more widely used, production indices are highly significant as indicators of the level of output in the economy or in parts of it.

In constructing quantity index numbers, the problems confronting the statistician are analogous to those involved in price indices. We measure changes in quantities, and when we weigh we use prices or values as weights. Quantity indices can be obtained easily by changing p to q and q to p in the various formulae discussed above.

Thus when Laspeyre's method is used

$$Q_{01} = \frac{\Sigma q_1p_0}{\Sigma q_0p_0} \times 100$$

When Paasche's formula is used

$$Q_{01} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1} \times 100$$

When Fisher's formula is used

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} \times 100$$

These formulae represent the *quantity index* in which the quantities of the different commodities are weighted by their prices. However, any other suitable weights can be used instead.

Example 2: Compute by suitable method the index number of quantity from the data given below:

Commodities	2004		2005	
	Price	**Total Value**	**Price**	**Total Value**
A	8	80	10	110
B	10	90	12	108
C	16	256	20	340

Solution : Since we are given the value and the price we can obtain quantity figure by dividing value by price for each commodity. We can then apply Fisher's method for finding out quantity index.

COMPUTATION OF QUANTITY INDEX BY FISHER'S METHOD

Commodities	2004		2005					
	p_0	q_0	p_1	q_1	q_1p_0	q_0p_0	q_1p_1	q_0p_1
A	8	10	10	11	88	80	110	100
B	10	9	12	9	90	90	108	108
C	16	16	20	17	272	256	340	320
					Σq_1p_0 = 450	Σq_0p_0 = 426	Σq_1p_1 = 558	Σq_0p_1 = 528

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} \times 100$$

$$Q_{01} = \sqrt{\frac{450}{426} \times \frac{558}{528}} \times 100$$

$$= \sqrt{1.116} \times 100 = 1.0564 \times 100$$

$$= 105.64.$$

Example 3: Compute Price index by applying weighted average of price relatives:

Commodities	p_0	q_0	p_1
Sugar	10	6 *kg.*	15
Rice	20	10 *kg.*	25
Milk	10	8 *kg.*	14

Solution: Computing price index by applying weighted average of Price Relatives.

Commodities	p_0	q_0	p_1	p_1q_0	p_0q_0
Sugar	10	6	15	90	60
Rice	20	10	25	250	200
Milk	10	8	14	112	80
				$\Sigma p_1q_0 = 452$	$\Sigma p_0q_0 = 340$

$$p_{01} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{452}{340} \times 100$$

$$= 132.94$$

Example 4: Using the following data construct in index for 2010 taking 2009 as base by the average of relatives method using arithmetic and geometric mean:

Commodity	Price in 2009	Price in 2010
A	500	700
B	400	600
C	800	900
D	110	120
E	20	20

Solution:

Commodity	p_0	p_1	$p_1/p_0 \times 100$	$\log p_1/p_0 \times 100$
A	500	700	140	2.1461
B	400	600	150	2.1761
C	800	900	112.5	2.0512
D	110	120	109.1	2.0378
E	20	20	100	2.0000
				$\frac{\Sigma p_1}{p_0} \times 100 = 611.6$

$$\Sigma \log \frac{p_1}{p_0} \times 100 = 10.4112$$

Using arithmetic mean, $P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right)}{N} = \frac{611.6}{5} = 122.32.$

Using geometric mean,

$$P_{01} = \text{antilog}\left[\frac{\Sigma \log \frac{p_1}{p_0} \times 100}{N}\right] = \text{antilog}\left[\frac{10.4112}{5}\right].$$

$$= \text{antilog } 2.0822$$

$$P_{01} = 120.9.$$

Example 5: From the following data construct a price index number of the group of four commodities using the appropriate formula:

Commodity	Base Year		Current Year	
	Price per unit	Expenditure (Rs.)	Price per unit	Expenditure (Rs.)
A	2	40	5	75
B	4	16	8	40
C	1	10	2	24
D	5	25	10	60

Solution: Since we are given base year and current year price and expenditure fishers ideal formula is appropriate for index.

Commodity	p_0	q_0	p_1	q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	2	20	5	15	100	40	75	30
B	4	4	8	5	32	16	40	20
C	1	10	2	12	20	10	24	12
D	5	5	10	6	50	25	60	30
					Σp_1q_0 = 202	Σp_0q_0 = 91	Σp_1q_1 = 199	Σp_0q_1 = 92

Quantity q is calculated by the following method:

$$q = \frac{\text{Expenditure}}{\text{Price per unit}}$$

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \cdot \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

$$= \sqrt{\frac{202}{91} \times \frac{199}{92}} \times 100$$

$$= 2.1912 \times 100 = 219.12.$$

Example 6: From the following data, compute price index by supplying weighted average of price relatives method using: (a) arithmetic mean, (b) geometric mean.

Commodity	p_0 (Rs.)	q_0	p_1 (Rs.)
A	3.0	20 *kg.*	4.0
B	1.5	40 *kg.*	1.6
C	1.0	10 *lt.*	1.5

Solution: **(a) Index number using weighted arithmetic mean of price relatives** **Notes**

Commodity	p_0	q_0	p_1	$V = p_0q_0$	$P = p_1/p_0 \times 100$	PV
A	3	20 *kg.*	4.0	60	4/3 × 100	8,000
B	1.5	40 *kg.*	1.6	60	1.6/1.5 × 100	6,400
C	1.0	10 *lt.*	1.5	10	1.5/1 × 100	1,500
				$\sum V = 130$		$\sum PV = 15,900$

$$P_{01} = \frac{\sum PV}{\sum V} = \frac{15,900}{130} = 122.31.$$

This means that there is a 122.31% increase in price over base year.

(b) Index number using geometric mean of price relatives.

Commodity	p_0	q_0	p_1	$V = p_0q_0$	$P = \frac{p_1}{p_0} \times 100$	log P	V log P
A	3	20 *kg.*	4.0	60	133.33	2.1249	127.404
B	1.5	40 *kg.*	1.6	60	106.7	2.0282	121.602
C	1.0	10 *lt.*	1.5	10	150.0	2.1761	21.761
				$\sum V = 130$			$\sum V \log P = 270.947$

$$P_{01} = \text{Antilog}\left[\frac{\sum V \cdot \log P}{\sum V}\right].$$

$$= \text{Antilog}\left[\frac{270.947}{130}\right].$$

$$= \text{Antilog } 2.084 = 120.9.$$

Self-Assessment

1. Fill in the blanks:

(*i*) Laepeyre's index is based on ....

(*ii*) Fisher's ideal index is ....

(*iii*) If with a rise of 10% in prices the wages are increased by 20%, the real wage increase is by ....

(*iv*) .... index is known as the 'Ideal' formula for constructing index numbers.

(*v*) The reference period is the period against which .... are made.

6.3 Summary

- In the weighted aggregative methods discussed earlier price relatives were not computed. However, like unweighted relative method it is also possible to compute weighted average of relatives. For the purpose of averaging we may use either the arithmetic mean or the geometric mean.

- When an index is computed by selecting one item from each of the many sub-groups of items, the values of each sub-group may be used as weights. Then only the method of weighted average of relatives is appropriate.
- The price or quantity relatives for each single item in the aggregate are, in effect, themselves a simple index that often yields valuable information for analysis.
- Though price indices are more widely used, production indices are highly significant as indicators of the level of output in the economy or in parts of it.
- In constructing quantity index numbers, the problems confronting the statistician are analogous to those involved in price indices. We measure changes in quantities, and when we weigh we use prices or values as weights. Quantity indices can be obtained easily by changing p to q and q to p in the various formulae discussed above.

6.4 Key-Words

1. Relative method : Index numbers measure changes or differences and are used in a variety of contexts. The Office for National Statistics (ONS) produces index numbers principally in the field of economics. Economists are interested in how changes in the monetary value of economic transactions can be attributed to changes in price (to measure inflation) and changes in quantity (to measure sales volume or economic output). Index numbers typically measure these changes over time. However, index numbers can also be used to make other comparisons, such as between regions of the UK.

6.5 Review Questions

1. Describe weighted Average of Price Relatives method to compute index numbers.
2. What are quantity or volume index numbers ?
3. What are the merits of weighted average of price relative method.
4. What is weighted average of price relative. Method of compute index number? Explain by using weighted arithmetic mean of Price Relatives.
5. Explain weighted average of Price Relative Method by using Geometric mean of Price Relatives.

Answers: Self-Assessment

1. (*i*) base year quantities (*ii*) geometric mean (*iii*) less than 10%
   (*iv*) Fisher's ideal index (*v*) comparisons

6.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 7: Test of Consistency: Unit Test, Time Reversal Test, Factor Reversal Test and Circular Test

Notes

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Objectives

After reading this unit students will be able to:

- Define Unit Test and Time Reversal Test.
- Know Factor Reversal Test and Circular Test.

Introduction

We have read in previous unit 19 that the relatives have certain important properties. What is true for an individual commodity should also be true for a group of commodities. The index number as an aggregative relative should also satisfy the same set of properties.

A number of mathematical criteria for judging the adequacy of an Index Number formula have been developed by statisticians. In fact, the problem is that of selecting the most appropriate one in a given situation. The following test are suggested for selecting an appropriate index.

(i) Unit Test

(ii) Time Reversal Test

(iii) Factor Reversal Test

(iv) Circular Test.

7.1 Unit Test

This test requires that the formula for constructing an index should be independent of the units in which the prices are quoted. All formulae of weighted aggregate method except simple aggregative method satisfy this test.

7.2 Time Reversal Test

Prof. Fisher has stated Time Reversal Test. 'The test is that the formula for calculating an Index Number should be such that will give the same ratio between one point of comparison and the other, no matter which of the two is taken as base. Time Reversal means that if we change the base year to the current year and *vice versa* then the product of the indices should be equal to unity. In other words, Simple Aggregative Method does not satisfy this test. The index number reckoned forward should be the reciprocal of that reckoned backward'. Thus, an ideal Index Number formula should work both ways, *i.e.*, forward as well as backward.

Mathematically, the following relation should be satisfied:

$$\frac{P_{01}}{100} \times \frac{P_{10}}{100} = 1 \text{ or } P_{01} \times P_{10} = 1$$

Laspeyre's Method

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \text{ and } P_{10} = \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1}; \; P_{01} \times P_{10} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \neq 1$$

$\because \quad P_{01} \times P_{10} \neq 1$ hence, test is not satisfied.

Paasche's Method

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \text{ and } P_{10} = \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}$$

Here, also $P_{01} \times P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \neq 1$, hence, test is not satisfied.

The test is not satisfied by Laspeyre's and the Paasche's method. However, Fisher's Method satisfies the test.

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \text{ and } P_{10} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}}$$

i.e., $\quad P_{01} \times P_{10} = \sqrt{1}$

or $\quad P_{01} \times P_{10} = 1$

Since, $P_{01} \times P_{10} = 1$, the Fisher's Ideal Index satisfies the test.

7.3 Factor Reversal Test

This is another test suggested by Fisher. According to Fisher Just as our formula should permit the interchange of two factors without giving inconsistent results, so without to permit interchange of prices and quantities without giving inconsistent results, *i.e.*, the two results multiplied together should give the true value ratio.

Mathematically,

$$\frac{P_{01}}{100} \times \frac{Q_{01}}{100} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \text{ or } P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

Laspeyre's & Paasche's method does not satisfy this test also like the above test.

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \text{ and } Q_{01} = \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0}$$

Here $$P_{01} \times Q_{01} \neq \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

Paasche's Method

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \text{ and } Q_{01} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}$$

Here also $$P_{01} \times Q_{01} \neq \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

This test is also satisfied by Fisher's Method only.

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}$$

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$$

$$P_{01} \times Q_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times \sqrt{\frac{\Sigma q_1 p_0}{\Sigma p_0 q_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$$

$$\sqrt{\left(\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}\right)^2} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \text{Value Index}$$

Hence, Fisher's Formula satisfies this test.

7.4 Circular Test

Another test of adequacy is circular test. This test is an extension of the time reversal test. It requires that if an index is constructed for the year '*b*' on year '*a*' and for the year '*c*' on base year '*b*' we should get the same result as if we calculate directly an Index for '*c*' on base year '*a*' without going through *b* as an intermediary.

Mathematically,

Let year *a*, *b* and *c* are denoted by 0, 1 and 2 respectively.

Condition of the test:

$$P_{01} \times P_{12} \times P_{20} = 1$$

For example, suppose prices have doubled in year 1 as compared to year 0, and again prices have doubled in year 2 as compared to year 1, in such a case if we correlated the prices of year 2 and 0 then we will find that prices of year 0 were $\frac{1}{4}$ of the prices of year 2.

Mathematically,

$$P_{01} = 2; \; P_{12} = 2; \; P_{20} = \frac{1}{4}$$

Here, $$P_{01} \times P_{12} \times P_{20} = 2 \times 2 \times \frac{1}{4} = 1$$

The test is not satisfied by any of the Laspeyre's, Paasche's or Fisher's Method. However, the simple aggregative method and the fixed weight aggregative method (Kelly's method) satisfy this test.

Example 1: Compute Fisher's ideal number and prove that it satisfies factor reversal and time reversal test.

Commodity	Price		Quantity	
	2002	2003	2002	2003
A	10	12	12	15
B	7	5	15	20
C	5	9	24	20
D	16	14	5	5

Solution:

Commodity Item	Price		Quantity		$p_n q_0$	$p_0 q_0$	$p_n q_n$	$p_0 q_n$
	2002	2003	2002	2003				
A	10	12	12	15	144	120	180	150
B	7	5	15	20	75	105	100	140
C	5	9	24	20	216	120	180	100
D	16	14	5	5	70	80	70	80
				Total	$\Sigma p_n q_0$ 505	$\Sigma p_0 q_0$ 425	$\Sigma p_n q_n$ 530	$\Sigma p_0 q_n$ 470

Fisher's price index

$$F_p = P_{on}^F = \sqrt{\frac{\Sigma p_n q_0}{\Sigma p_0 q_0} \cdot \frac{\Sigma p_n q_n}{\Sigma p_0 q_n}} \times 100$$

$$= \sqrt{\frac{505}{425} \times \frac{530}{470}} \times 100$$

$$= \sqrt{1.188 \times 1.128} \times 100$$

$$= 115.8$$

(i) Time reversal test

$$P_{no} = \sqrt{\frac{\Sigma p_0 q_n}{\Sigma p_n q_n} \cdot \frac{\Sigma p_0 q_0}{\Sigma p_n q_0}}$$

$$= \sqrt{\frac{470}{530} \times \frac{525}{505}}$$

$$\therefore \quad P_{no} \times P_{on} = \sqrt{\frac{505}{425} \cdot \frac{530}{470} \cdot \frac{470}{530} \cdot \frac{429}{505}}$$

$$= \sqrt{1} = 1$$

Notes

(ii) Factor reversal test

$$P_{on} = \sqrt{\frac{505}{425} \times \frac{530}{470}}$$

$$Q_{on} = \sqrt{\frac{470}{425} \times \frac{530}{505}}$$

$$P_{no} \times P_{on} = \frac{\Sigma p_n q_n}{\Sigma p_0 q_0}$$

Hence, the data satisfies both time reversal and factor reversal tests.

Example 2: The following figures relate to the prices and quantities of certain commodities. Construct an appropriate index number and find out whether it satisfies the time reversal test.

Commodities	2004		2005	
	Price	Quantity	Price	Quantity
A	30	50	32	50
B	25	40	30	35
C	18	50	16	55

Solution:

INDEX NUMBER BY FISHER'S IDEAL METHOD

Commodities	2004		2005					
	p_0	q_0	p_1	q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	30	50	32	50	1,600	1,500	1,600	1,500
B	25	40	30	35	1,200	1,000	1,050	875
C	18	50	16	55	800	900	880	990
					Σp_1q_0 = 3,600	Σp_0q_0 = 3,400	Σp_1q_1 = 3,530	Σp_0q_1 = 3,365

$$P_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times 100$$

$$= \sqrt{\frac{3{,}600}{3{,}400} \times \frac{3{,}530}{3{,}365}} \times 100 = \sqrt{1.111} \times 100 = 1.054 \times 100 = 105.4$$

Time reversal test is satisfied when

$$P_{01} \times P_{10} = 1$$

$$P_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} = \sqrt{\frac{3{,}600}{3{,}400} \times \frac{3{,}530}{3{,}365}}$$

$$P_{10} = \sqrt{\frac{\Sigma p_0q_1}{\Sigma p_1q_1} \times \frac{\Sigma p_0q_0}{\Sigma p_1q_0}} = \sqrt{\frac{3{,}365}{3{,}530} \times \frac{3{,}400}{3{,}600}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{3,600}{3,400} \times \frac{3,530}{3,365} \times \frac{3,365}{3,530} \times \frac{3,400}{3,600}} = \sqrt{1} = 1.$$

Hence time reversal test is satisfied by the above formula.

Example 3: From the following data calculate Fisher's ideal index and prove that it satisfies both the time reversal and factor reversal tests.

Commodity	2004		2005	
	Price	**Qty.**	**Price**	**Qty.**
A	4	8	5	8
B	5	10	6	12
C	3	6	4	7
D	8	5	10	4

Solution:

Calculation of Fishers's Ideal Index

Commodity	p_0	q_0	p_1	q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	4	8	5	8	40	32	40	32
B	5	10	6	12	60	50	72	60
C	3	6	4	7	24	18	28	21
D	8	5	10	4	50	40	40	32
					Σp_1q_0 = 174	Σp_0q_0 = 140	Σp_1q_1 = 180	Σp_0q_1 = 145

Fisher's Ideal Index, *i.e.*, $P_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times 100$

$$\Sigma p_1q_0 = 174,\ \Sigma p_0q_0 = 140,\ \Sigma p_1q_1 = 180,\ \Sigma p_0q_1 = 145$$

Substituting the values:

$$P_{01} = \sqrt{\frac{174}{140} \times \frac{180}{145}} \times 100 = 1.2429 \times 100 = 124.29$$

Time Reversal Test

Time Reversal Test is satisfied if $P_{01} \times P_{10} = 1$

$$P_{10} = \sqrt{\frac{\Sigma p_0q_1}{\Sigma p_1q_1} \times \frac{\Sigma p_0q_0}{\Sigma p_1q_0}} = \sqrt{\frac{145}{180} \times \frac{140}{174}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{174}{140} \times \frac{180}{145} \times \frac{145}{180} \times \frac{140}{174}} = \sqrt{1} = 1.$$

Hence Time Reversal Test is satisfied.

Factor Reversal Test

Factor Reversal Test is satisfied when:

$$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} = \sqrt{\frac{145}{140} \times \frac{180}{174}}$$

$$P_{01} \times Q_{01} = \sqrt{\frac{174}{140} \times \frac{180}{145} \times \frac{145}{140} \times \frac{180}{174}} = \frac{180}{140}$$

$$\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \frac{180}{140}.$$

Hence Factor Reversal Test is satisfied.

Example 4: From the following data construct Fisher's Ideal Index Number and show how it satisfies Time Reversal Test and Factor Reversal Test.

Items	Base Year		Current Year	
	Price Per unit Rs.	Total Expenditure Rs.	Price per unit Rs.	Total Expenditure Rs.
1	2	40	5	75
2	4	16	8	40
3	1	10	2	24
4	5	25	10	60

Solution: Divide expenditure by price to get quantity figures and then calculate Fisher's Ideal Index.

Item	p_0	q_0	p_1	q_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
1	2	20	5	15	100	40	75	30
2	4	4	8	5	32	16	40	20
3	1	10	2	12	20	10	24	12
4	5	5	10	6	50	25	60	30
					Σp_1q_0 = 202	Σp_0q_0 = 91	Σp_1q_1 = 199	Σp_0q_1 = 92

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

$$= \sqrt{\frac{202}{91} \times \frac{199}{92}} \times 100 = 2.1912 \times 100 = 219.12.$$

Time Reversal Test

Time reversal test is satisfied if $P_{01} \times P_{10} = 1$

$$P_{10} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}} = \sqrt{\frac{92}{199} \times \frac{91}{202}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{202}{91} \times \frac{199}{92} \times \frac{92}{199} \times \frac{91}{202}} = \sqrt{1} = 1.$$

Hence time reversal test is satisfied.

Factor Reversal Test

Factor reversal test is satisfied if:

$$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} = \sqrt{\frac{92}{91} \times \frac{199}{202}}$$

$$P_{01} \times Q_{01} = \sqrt{\frac{202}{91} \times \frac{199}{92} \times \frac{92}{91} \times \frac{199}{202}} = \frac{199}{91}$$

Hence factor reversal test is satisfied.

Self-Assessment

1. Tick the correct Answer

(*i*) A good index number is one that satisfies

(*a*) Unit test (*b*) time reversal test

(*c*) factor reversal test (*d*) circular test

(*e*) all the test form.

(*ii*) Time reversal test is satisfied when

(*a*) $P_{01} \times P_{10} = 0$ (*b*) $P_{01} \times P_{10} = 1$

(*c*) $P_{01} \times P_{10} > 1$ (*d*) $P_{01} \times P_{10} < 1$

(*iii*) The circular test is satisfied when

(*a*) $P_{12} \times P_{23} \times P_{31} = 0$ (*b*) $P_{12} \times P_{23} \times P_{13} = 1$

(*c*) $P_{12} \times P_{23} \times P_{31} = 1$ (*d*) $P_{12} \times P_{32} \times P_{31} = 0$

(*iv*) The circular test is an extension of the

(*a*) unit test (*b*) time reversal test

(*c*) Factor reversal test (*d*) None of these

(*v*) Factor reversal test is satisfied when

(*a*) $P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$ (*b*) $\Sigma p_1 q_0$

(*c*) $P_{01} \times Q_{10} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$ (*d*) None of these

7.5 Summary

- A number of mathematical criteria for judging the adequacy of an Index Number formula have been developed by statisticians. In fact, the problem is that of selecting the most appropriate one in a given situation.
- This test requires that the formula for constructing an index should be independent of the units in which the prices are quoted. All formulae of weighted aggregate method except simple aggregative method satisfy this test.
- Time Reversal means that if we change the base year to the current year and *vice versa* then the product of the indices should be equal to unity. In other words, Simple Aggregative Method does not satisfy this test. The index number reckoned forward should be the reciprocal of that reckoned backward'. Thus, an ideal Index Number formula should work both ways, *i.e.*, forward as well as backward.
- According to Fisher Just as our formula should permit the interchange of two factors without giving inconsistent results, so without to permit interchange of prices and quantities without giving inconsistent results, *i.e.*, the two results multiplied together should give the true value ratio.
- Another test of adequacy is circular test. This test is an extension of the time reversal test. It requires that if an index is constructed for the year '*b*' on year '*a*' and for the year '*c*' on base year '*b*' we should get the same result as if we calculate directly an Index for '*c*' on base year '*a*' without going through *b* as an intermediary.

7.6 Key-Words

1. Coefficient of variation (CV) : The standard deviation divided by the mean.
2. Collinearity : The condition in which the independent variables are (usually highly) correlated with each other.
3. Column totals : The total number of observations occurring in a column of a contingency table.

7.7 Review Questions

1. Explain time reversal test and factor reversal tests. Show that Fisher's Ideal index number satisfies both.
2. What do you mean by time reversal test for index numbers ? Show that laspeyre and paasche index numbers do not satisfy it and that Fisher's Ideal index does.
3. What are the various tests of adequacy of index number formulae ? Describe each briefly.
4. Distinguish between Laspeyre's and Paasche's index.
5. What are the differences between time reversal test and factor reversal test?

Answers: Self-Assessment

1. (*i*) (*e*) (*ii*) (*b*) (*iii*) (*c*) (*iv*) (*b*) (*v*) (*a*)

7.8 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.

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Unit 8: Cost of Living Index and Its Uses and Limitation of Index Numbers

CONTENTS

Objectives

After reading this unit students will be able to:

- Explain cost of Living Index and its Uses.
- Describe the Limitation of Index Numbers.

Introduction

One of the main types of index numbers in use is the cost of living index number (CLI). This is also known as consumer price index number (CPI). Gradually the expression CLI is being replaced by CPI; it is a special index number of retail prices in which only prices of selected commodities are considered which enter into the consumption pattern of a particular group of people. Thus different items enter into the "market basket of goods'', of different groups. Different groups of people have different CLI numbers. The market basket of goods includes goods and services needed for maintaining a certain standard of living for that group over a period of time. The CLI measures changes in the cost of maintaining the standard of living for that group.

In India CLI numbers are being constructed for three groups of people. These index numbers are

(1) The working class cost of living index numbers

(2) The middle-class cost of living index numbers

(3) The cost of living index numbers of the Central Government employees.

Did u know? The commodities of selected items for the group constitute what is known as "market basket of goods" for that group.

We shall describe them later. The basket of goods is divided into five major groups-food, housing, fuel and light, clothing and other goods and services.

In the U.S.A. a "Consumer Price Index for Urban Wage Earners and Clerical Workers" is constructed regularly. The commodities are divided into 8 major groups — food, housing, dress, transportation, medical care, personal care, reading and recreation and other goods and services, The special problems that arise in the construction of cost of living index numbers for a group lie in determining the market basket of goods and services needed for a person of the group for maintaining a certain standard of living. While transport may be an item for city dwellers, this may not be so in the case of villagers in a developing country.

The following points need be considered in the selection of items for a group of people: (1) items taken should be such as to represent the habits, tastes and traditions of the average person in the group; (2) the economic and social importance of the goods and services are also to be examined; (3) items should be such that they are not likely to vary in quality in appreciable degree over two different places or different periods of time; and (4) items should be fairly large in number so as to represent adequately the standard of living for the groups). A fairly reasonable number should be selected. Again after determining the items to be included in the basket, the question that arises is the determination of suitable weights for different items in the basket.

To determine the weights, a proper study of consumption habits of persons of that group is to be made. The usual procedure is to conduct "family budget surveys". Such surveys help in determining the items to be entered into the consumption pattern of the group and also help in determining the weights to be assigned to different items. One problem may occur regarding different *qualities* of the same type of commodity. Another problem may concern items of common use which do not occur in both the base period and the given period.

For a detailed account of this topic refer to Banerjee (1975).

8.1 Cost of Living Index and Its Uses

Meaning and Need

The consumer price index numbers, also known as cost of living index number, are generally intended to represent the average change over time in the prices paid by the ultimate consumer of a specified basket of goods and services. The need for constructing consumer price indices arises because the general index numbers fail to give an exact idea of the effect of the change in the general price level on the cost of living of different classes of people, since a given change in the level of prices affects different classes of people in different manners. Different classes of people consume different types of commodities and even the same type of commodities are not consumed in the same proportion by different classes of people. For example, the consumption pattern of rich, poor and middle class people varies widely. Not only this, the consumption habits of the people of the same class differ from place to place. For example, the mode of expenditure of a lower division clerk living in Delhi may differ widely from that of another clerk of the same category living in, say, Chennai. The consumer price index helps us in determining the effect of rise and fall in prices on different classes of consumers living in different areas. The construction of such an index is of great significance because very often the demand for a higher wage is based on the cost of living index and the wages and salaries in most countries are adjusted in accordance with the consumer price index.

The consumer price index numbers were earlier known as cost of living index numbers. But this name was not a happy one since the cost of living index does not measure the actual cost of living nor the fluctuations in the cost of living due to causes other than the change in the price level ; its object is to find out how much the consumers of a particular class have to pay more for a certain basketful of goods and services in a given period compared to the base period. At present, the three terms, namely, cost of living index, consumer price index and retail price index, are in use in different countries with practically no difference in their connotation. However, the term 'consumer price index' is the most popular of the three.

Notes To bring out clearly this fact, the Sixth International Conference of Labour Statisticians recommended that the term 'cost of living index' should be replaced in appropriate circumstances by the terms *'Price of living index'*, *'cost of living price index'*, or *'consumer price index'*.

It should be clearly understood at the very outset that two different indices representing two different geographical areas cannot be used to compare actual living costs of the two areas. A higher index for one area than for another with the same period is no indication that living costs are higher in the one

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than in the other. All it means is that as compared with the base periods, prices have risen in one area than in another. But actual costs depend not only on the rise in prices as compared with the base period, but also on the actual cost of living for the base period which will vary for different regions and for different classes of population.

Utility of the Cost of Living Index

The Consumer Price Indices are of great significance as can be seen from the following:

(1) The most common use of these indices is in wage negotiations and wage contracts. Automatic adjustments of wage or dearness allowance component of wages are governed in many countries by such indices.

(2) At Governmental level, the index numbers are used for wage policy, price policy, rent control, taxation and general economic policies.

(3) The index numbers are also used to measure changing purchasing power of the currency, real income, etc.

(4) Index numbers are also used for analysing markets for particular kinds of goods and services.

Construction of a Consumer Price Index or Cost of Living Index:

The following are the steps in constructing a consumer price index:

(1) Decision about the class of people for whom the index is meant: It is absolutely essential to decide clearly the class of people for whom the index is meant, *i.e.,* whether it relates to industrial workers, teachers, officers, etc. The scope of the index must be clearly defined. For example, when we talk of teachers, we are referring to primary teachers, middle class teachers, etc., or to all the teachers taken together. Along with the class of people it is also necessary to decide the geographical area covered by the index. Thus in the example taken above it is to be decided whether all the teachers living in Delhi are to be included or those living in a particular locality of Delhi, say, Chandni Chowk or Karol Bagh, etc.

Task What do you mean by cost of living index?

(2) Conducting family budget enquiry: Once the scope of the index is clearly defined the next step is to conduct a family budget enquiry covering the population group for whom the index is to be designed. The object of conducting a family budget enquiry is to determine the amount that an average family of the group included in the index spends on different items of consumption. While conducting such an enquiry, therefore, the quantities of commodities consumed and their prices are taken into account. The consumption pattern can thus be easily ascertained. It is necessary that the family budget enquiry amongst the class of people to whom the index series is applicable should be conducted during the base period. The Sixth International Conference of Labour Statisticians held in Geneva in 1946 suggested that the period of enquiry of the family budgets and the base periods should be identical as far as possible.

The enquiry is conducted on a random basis. By applying lottery method some families are selected from the total number and their family budgets are scrutinized in detail. The items on which the money is spent are classified into certain well accepted groups, namely,

(i) Food (ii) Clothing (iii) Fuel and Lighting

(iv) House Rent (v) Miscellaneous.

Each of these groups is further divided into sub-groups. For example, the broad group 'food' may be divided into wheat, rice, pulses, sugar, etc. The commodities included are those which are generally consumed by people for whom the index is meant. Through family budget enquiry an average budget is prepared which is the standard budget for that class of people. While constructing the index only such commodities should be included as are not subject to wide variations in quality or to wide seasonal alterations in supply and for which regular and comparable quotations of prices can be obtained.

(3) **Obtaining price quotations:** The collection of retail prices is a very important and, at the same time, very tedious and difficult task because such prices may vary from place to place, shop to shop and person to person. Price quotations should be obtained from the localities in which the class of people concerned reside or from where they usually make their purchases. Some of the principles recommended to be observed in the collection of retail price data required for purposes of construction of cost of living indices are described below

(a) The retail prices should relate to a fixed list of items and for each item, the quality should be fixed by means of suitable specification.

(b) Retail prices should be those actually charged to consumers for cash sales.

(c) Discount should be taken into account if it is automatically given to all customers.

(d) In a period of price control or rationing, where illegal prices are charged openly, such prices should be taken into account along with the controlled prices.

The most difficult problem in practice is to follow principle (a), *i.e.*, the problem of keeping the weights assigned and qualities of the basket of goods and services constant with a view to ensuring that only the effect of price change is measured. To conform to uniform qualities, the accepted method is to draw up detailed descriptions for specifications of the items priced for the use of persons furnishing or collecting the price quotations.

Since prices form the most important component of cost of living indices, considerable attention has to be paid to the methods of price collection and to the price collection personnel. Prices are collected usually by special agents or through mailed questionnaire or in some cases through published price lists. The greatest reliance can be placed on the price collection through special agents as they visit the retail outlets and collect the prices from them. However, these agents should be properly selected and trained and should be given a manual of instructions as well as manual of specifications of items to be priced.

Notes Appropriate methods of price verification should be followed such as '*check pricing*' in which price quotations are verified by means of duplicate prices obtained by different agents or '*purchase checking*' in which actual purchases of goods are made.

After quotations have been collected from all retail outlets an average price for each of the items included in the index has to be worked out. Such averages are first calculated for the base period of the index and later for every month if the index is maintained on a monthly basis. The method of averaging the quotations should be such as to yield unbiased estimates of average prices as being paid by the group as a whole. This, of course, will depend upon the method of selection of retail outlets and also the scope of the index.

In order to convert the prices into index numbers the prices or their relatives must be weighted. The need for weighting arises because relative importance of various items for different classes of people is not the same. For this reason, the cost of living index is always a weighted index. While conducting the family budget enquiry the amount spent on each commodity by an average family is ascertained and these constitute the weights. Percentage expenditures on the different items constitute the individual weights' allocated to the corresponding price relative and the percentage expenditure on the five groups constitute the 'group weight'.

Methods of Constructing the Index

After the above mentioned problems are carefully decided the index may be constructed by applying any of the following methods:

(1) Aggregate Expenditure Method or Aggregative Method; and

(2) Family Budget method or The Method of Weighted Relatives.

1. **Aggregate Expenditure Method:** When this method is applied the quantities of commodities consumed by the particular group in the base year are estimated which constitute the weights. The prices of commodities for various groups for the current year are multiplied by the quantities

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consumed in the base year and the aggregate expenditure incurred in buying those commodities is obtained. In a similar manner the prices of the base year are multiplied by the quantities of the base year and aggregate expenditure for the base period is obtained. The aggregate expenditure of the current year is divided by the aggregate expenditure of the base year and the quotient is multiplied by 100. Symbolically,

$$\text{Consumer Price Index} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

There is in fact the Laspeyre's method discussed earlier. This method is the most popular method for consumer price index.

2. **Family Budget Method:** When this method is applied the family budgets of a large number of people for whom the index is meant are carefully studied and the aggregate expenditure of an average family on various items is estimated. These constitute the weights. The weights are thus the value weights obtained by multiplying the prices by quantities consumed (*i.e.*, p_0q_0). The price relatives for each commodity are obtained and these price relatives are multiplied by the value weight for each item and the product is divided by the sum of the weight. Symbolically,

$$\text{Consumer Price Index} = \frac{\Sigma PV}{\Sigma V}$$

where, $P = \frac{p_1}{p_0} \times 100$ for each item

V = Value weights, *i.e.*, p_0q_0.

This method is the same as the weighted average of price relative method discussed earlier.

It should be noted that the answer obtained by applying the aggregate expenditure method and the family budget method shall be the same.

Example 1: Construct the consumer price index number of 2005 on the basis from the following data using (i) the average expenditure method, and (ii) the family budget method:

Commodity	Quantity consumed in 2004	Unit	Price in 2004		Price in 2005	
			Rs.	Paise	Rs.	Paise
A	6 Quintal	Quintal	5	75	6	0
B	6 "	"	5	0	8	0
C	1 "	"	6	0	9	0
D	6 "	"	8	0	10	0
E	4 Kg.	Kg.	2	0	1	50
F	1 Quintal	Quintal	20	0	15	0

Solution:

Computation of Consumer Price Index Number for 2005
(Base 2004 = 100) By the Aggregate Expenditure Method

Commodities	Quantities consumed q_0	Unit	Price in 2004 p_0	Price in 2005 p_1	p_1q_0	p_0q_0
A	6 Qtl.	Qtl.	5.75	6.00	36.00	34.50
B	6 "	"	5.00	8.00	48.00	30.00

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C	1 "	"	6.00	9.00	9.00	6.00
D	6 "	"	8.00	10.00	60.00	48.00
E	4 Kg.	Kg.	2.00	1.50	6.00	8.00
F	1 Qtl.	Qtl.	20.00	15.00	15.00	20.00
					$\Sigma p_1 q_0$ = 174	$\Sigma p_0 q_0$ = 146.50

$$\text{Consumer Price Index} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{174}{146.5} \times 100 = 118.77$$

Construction of consumer Price Index Number for 2005 (Base 2004 = 100)
By the Family Budget Method

Articles	**Quantities consumed q_0**	**Unit**	**Price in 2004 p_0**	**Price in 2005 p_1**	$\frac{p_1}{p_0} \times 100$ p	$p_0 q_0$ V	**PV**
A	6 Qtl.	Qtl.	5.75	6.0	104.35	34.5	3,600
B	6 "	"	5.00	8.0	160.00	30.0	4,800
C	1"	"	6.00	9.0	150.00	6.0	900
D	6 "	"	8.00	10.0	125.00	48.0	6,000
E	4 Kg.	Kg.	2.00	1.5	75.00	8.0	600
F	1 Qtl.	Qtl.	20.00	15.0	75.00	20.0	1,500
						ΣV = 146.5	ΣPV = 17,400

$$\text{Consumer Price Index} = \frac{\Sigma PV}{\Sigma V} = \frac{17,400}{146.5} = 118.77$$

Thus, the answer is the same by both methods. However, the reader should prefer the aggregate expenditure method because it is far more easier to apply compared to the family budget method.

Example 2: An enquiry into the budgets of the middle class families in a city in India gave the following information:

Expenses on:	**Food** 35%	**Rent** 15%	**Clothing** 20%	**Fuel** 10%	**Misc.** 20%
Price in 2004:	450	90	225	75	120
Price in 2005:	435	90	195	69	135

What change in the cost of living figures of 2005 has taken place as compared to 2004?

Solution:

Construction of Cost of Living Index for 2005 with 2004 the Base

Expenses on	Price in Rs. 2004	Price in Rs. 2005	Price Relative $\frac{p_1}{p_0} \times 100$ P	W	PW
Food	450	435	96.67	35	3383.45
Rent	90	90	100.00	15	1500.00
Clothing	225	195	86.67	20	1733.40
Fuel	75	69	92.00	10	920.00
Misc.	120	135	112.50	20	2250.00
				$\sum W = 100$	$\sum PW = 9786.85$

$$\text{Cost of Living Index} = \frac{\sum PW}{\sum W}$$

$$\sum PW = 9786.85, \sum W = 100$$

$$\text{Index} = \frac{9786.85}{100} = 97.87$$

Thus a fall of (100 – 97.87), *i.e.*, 2.13% has taken place in 2005 as compared to 2004.

Example 3 : Construct the cost of living index number from the data given below:

Group	Index	Expenditure
1. Food	550	46%
2. Clothing	215	10%
3. Fuel and Lighting	220	7%
4. House Rent	150	12%
5. Miscellaneous	275	25%

Solution:

Construction of Cost of Living Index Number

Group	Index Number I	Expenditure W	IW
Food	550	46	25,300
Clothing	215	10	2,150
Fuel and Lighting	220	7	1,540
House Rent	150	12	1,800
Miscellaneous	275	25	6,875
		$\sum W = 100$	$\sum IW = 37,665$

$$\text{Cost of Living Index} = \frac{\sum IW}{\sum W} = \frac{37,665}{100} = 376.65.$$

Notes

Precautions while using Consumer Price Index of Cost of Living Index

Quite often the consumer price indices are misinterpreted. Hence while using these indices the following points should be kept in mind

1. As pointed out earlier the consumer price index measures changes in the retail prices only in the given period compared to base period —it does not tell us anything about variations in living standard at two different places. Thus, if the cost of living index for working class for Mumbai is 175 and Delhi 150 for the same period and for the same class of people, it does not necessarily mean that living costs are higher in Mumbai as compared to Delhi.
2. While constructing the index it is assumed that the quantities of the base year are constant and hold good for current year also. But this assumption does not appear to be very logical because the pattern of consumption, goes on changing with the change in fashion, introduction of new commodities in the market, etc. It is desirable, therefore, that while constructing the index the current year quantities are taken into account. But this is a difficult task. The Sixth International Conference of Labour Statisticians recommended that the pattern of consumption should be examined and the weights adjusted, if necessary, at intervals of not more than ten years to correspond changes in the consumption pattern. The index also does not take into account changes in qualities. Unlike changes in consumption pattern changes in qualities of goods and services are more frequent and when a marked change in the quality of items occurs appropriate adjustment should be made to ensure that the index takes into account changes is qualities also. But in practice it is a difficult proposition to follow and, therefore, constant qualities are assumed at two different dates which again is a shaky assumption.
3. Like any other index the consumer price index is based on a sample. While constructing the index, sampling is used at every stage in the selection of commodities, in obtaining price quotations, selecting families for family budget enquiry, etc. The accuracy of the index thus hinges upon the use of sampling methods. The consumption pattern derived from the expenditure data of a sample of households covered in the course of family budget enquiry has to be representative of all the items in the average budget, the localities from which price data are collected have to be representative of all the localities from which the population group make purchases, the retail outlets from which prices are collected have to be representative of all the retail outlets patronised by the population group, etc. However, it is often difficult to ensure perfect representativeness and in the absence of this the index may fail to provide the real picture.

Example 4: The following are the group index numbers and the group weights of the budget of an average working class family. Construct the cost of living index number.

Group	**Index**	**Weight**
Food	352	48
Fuel and Lighting	220	10
Clothing	230	8
Rent	160	12
Miscellaneous	190	15

Solution:

Construction of Cost of Living Index

Group	**Index** **I**	**Weight** **W**	**IW**
Food	352	48	16,896
Fuel and Lighting	220	10	2,200

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Clothing	230	8	1,840
Rent	160	12	1,920
Miscellaneous	190	15	2,850
		$\sum W = 93$	$\sum IW = 25,706$

$$\text{Cost of Living Index} = \frac{\sum IW}{\sum W} = \frac{25,706}{93} = 276.41.$$

Example 5: The percentage expenses on different commodities consumed by the middle class families of a certain city and the group index numbers in 2005 as compared with the base year 2004 are as follows:

Commodities	% Expenses	Index
Food	45	410
Rent	15	150
Clothing	12	343
Fuel and Lighting	8	248
Miscellaneous	20	285

Calculate the consumer index for the year 2005.

Solution:

Calculation of Consumer Price Index

Commodities	% Expenses W	Index I	IW
Food	45	410	18,450
Rent	15	150	2,250
Clothing	12	343	4,116
Fuel and Lighting	8	248	1,984
Miscellaneous	20	285	5,700
	$\sum W = 100$	$\sum IW = 32,500$	

$$\text{Consumer Price Index} = \frac{\sum IW}{\sum W} = \frac{32,500}{100} = 325.$$

Example 6: Construct the cost of living index number from the following group data:

Group	Weights	Group Index No.
(1) Food	47	247
(2) Fuel and Lighting	7	293
(3) Clothing	8	289
(4) House Rent	13	100
(5) Miscellaneous	14	236

Notes

Solution: Construction of Cost of Living Index

Group	Weights w	Group Index I	IW
Food	47	247	11,609
Fuel and Lighting	7	293	2,051
Clothing	8	289	2,312
House Rent	13	100	1,300
Miscellaneous	14	236	3,304
	$\sum W = 89$		$\sum IW = 20,576$

$$\text{Cost of Living Index} = \frac{\sum IW}{\sum W} = \frac{20,576}{89} = 231.19.$$

Example 7: The data below show the percentage increases in price of a few selected food items and the weights attached to each of them. Calculate the index number for the food group.

Food Items	Rice	Wheat	Dal	Ghee	Oil	Spices	Milk	Fish	Vegetables	Refreshments
Weight	33	11	8	5	5	3	7	9	9	10
Percentage increase in price	180	202	115	212	175	517	260	426	332	279

Using the above food index and the information given below, calculate the cost of living index number.

Group	Food	Clothing	Fuel & Light	Rent & Rates	Miscellaneous
Index	–	310	220	150	300
Weight	60	5	8	9	18

Solution:

Calculations for Food Index

Food Items	Weight W	Percentage Increase	Current Index* I	IW
Rice	33	180	280	9240
Wheat	11	202	302	3322
Dal	8	115	215	1720
Ghee	5	212	312	1560
Oil	5	175	275	1375
Spaces	3	517	617	1851
Milk	7	260	360	2520
Fish	9	426	526	4734
Vegetables	9	332	432	3888
Refreshments	10	279	379	3790
	$\sum W = 100$			$\sum IW = 34000$

$$\text{Food Index} = \frac{\sum IW}{\sum W} = \frac{34000}{100} = 340$$

Construction of Cost of Living Index

Group	Index I	Weights W	IW
Food	340	60	20,400
Clothing	310	5	1,550
Fuel and Light	220	8	1,760
Rent and Rates	150	9	1,350
Miscellaneous	300	18	5,400
		$\sum IW = 100$	$\sum IW = 30,460$

$$\text{Cost of Living Index} = \frac{\sum IW}{\sum W} = \frac{30460}{100} = 304.6$$

* Current index has been obtained by adding 100 to the percentage increase in the various food items.

Example 8: Calculate the Cost of Living Index Number from the following data:

Items	Price		Weights
	Base Year	Current Year	
Food	30	47	4
Fuel	8	12	1
Clothing	14	18	3
House Rent	22	15	2
Miscellaneous	25	30	1

Solution:

Construction of Cost of Living Index

Items	p_0	p_1	$\frac{p_1}{p_0} \times 100$ p	W	PW
Food	30	47	156.67	4	626.68
Fuel	8	12	150.00	1	150.00
Clothing	14	18	128.57	3	385.71
House Rent	22	15	68.18	2	136.36
Miscellaneous	25	30	120.00	1	120.00
				$\sum W = 11$	$\sum PW = 1418.75$

$$\text{Cost of Living Index} = \frac{\sum PW}{\sum W} = \frac{1418.75}{11} = 128.98.$$

Notes

Example 9: (a) From the chain base index numbers given below, prepare fixed base index numbers:

Year:	2001	2002	2003	2004	2005
Index:	110	150	140	200	150

(b) From the chain base index number given below, construct fixed base index numbers:

Year:	2001	2002	2003	2004	2005
Chain Base Indices:	80	110	120	90	140

Solution:

CONSTRUCTION OF COST OF LIVING INDEX

Expenses	**2004** p_0	**2005** p_1	$\frac{p_1}{p_0} \times 100$	**W**	**PW**
Food	150	174	116	35	4060
Rent	50	60	120	15	1800
Clothing	100	125	125	20	2500
Fuel	20	25	125	10	1250
Misc	60	90	150	20	3000
				$\Sigma W = 100$	$\Sigma PW = 12610$

Cost of Living Index $\frac{\Sigma PW}{\Sigma W} = \frac{12610}{100}$ = 126.1. Thus as compared to 2004 the cost of living index has risen by 26.1 per cent in 2005.

8.2 Limitations of Index Numbers

Though the index numbers are of great significance, the reader must also be aware of their limitations so that he avoids errors of interpretation. The chief limitations of index numbers are:

1. Since index numbers are generally based on a sample, it is not possible to take into account each and every item in the construction of the index.
2. While taking the sample random sampling is seldom used. This is so because to sample from a population of literally millions of commodities and services, the random procedure could neither be practical nor representative. Typically, indices are constructed from samples deliberately selected. This is likely to introduce errors and every effort must be made to minimise these errors.
3. It is often difficult to take into account changes in the quality of products. With the passage of time tastes and habits of people also change with the result that very often old commodities go out of use and new commodities are introduced. In a really typical index, qualities of commodities should remain the same over a period of time because differences in quality would mean differences in prices also. But very often it is not practicable and it makes comparisons over long periods less reliable.
4. A large number of methods have been designed for constructing index numbers and different methods of computation give different results. Very often the selection of an appropriate formula creates problems and in the interest of comparability, it is necessary to ensure that the same

formula is adopted over a period of time for constructing a particular index. There is no index number method which is most satisfactory from all the various points of view which may logically or practically be taken. Index numbers are averages, and all averages are basically compromises between opposing extremes or forces.

5. Just like other statistical tools, index numbers can also be manipulated in such a manner as to draw the desired conclusions. Choosing a freak year is a favourite trick of those who use statistics to mislead. A dishonest capitalist could choose a record year of profits as base and so "prove" subsequent profits to be pitifully low. Similarly, in order to prove that the current prices are intolerably high a dishonest trade unionist may choose a year of exceptionally low prices as base.
6. Since in the construction of index numbers a large number of factual questions are involved, lack of adequate and accurate data in most cases becomes a serious limitation of the index itself. In most of the cases one cannot collect the data himself and, therefore, one has to rely on a published source. Ordinarily, we draw upon many sources of data which are geographically dispersed. Problems of comparability and reliability thus multiply and the chances of spurious results are increased. One mistake may "bias" the index such as including the price of one commodity for one time period, or the price of a slightly different commodity for another period, or taking the manufacturer's price at one time and wholesaler's or retailer's price another time.
7. Comparisons over long periods are not reliable.

Self-Assessment

1. Fill in the blanks:

(*i*) Theoretically the best average in the cost of living index numbers is ....

(*ii*) Cost of living index = $\dfrac{\text{...............}}{\Sigma p_0 q_0} \times 100$.

(*iii*) Kelly's method of constructing index involves the formula P_{01} = ... where q = ...

(*iv*) Cost of living index help in determaining .... wages.

(*v*) Cost of living index help in determing the purchasing power of ....

8.3 Summary

- One of the main types of index numbers in use is the cost of living index number (CLI). This is also known as consumer price index number (CPI). Gradually the expression CLI is being replaced by CPI; it is a special index number of retail prices in which only prices of selected commodities are considered which enter into the consumption pattern of a particular group of people. The commodities of selected items for the group constitute what is known as "market basket of goods" for that group. Thus different items enter into the "market basket of goods'', of different groups. Different groups of people have different CLI numbers. The market basket of goods includes goods and services needed for maintaining a certain standard of living for that group over a period of time. The CLI measures changes in the cost of maintaining the standard of living for that group.
- In the U.S.A. a "Consumer Price Index for Urban Wage Earners and Clerical Workers" is constructed regularly. The commodities are divided into 8 major groups — food, housing, dress, transportation, medical care, personal care, reading and recreation and other goods and services, The special problems that arise in the construction of cost of living index numbers for a group lie in determining the market basket of goods and services needed for a person of the group for maintaining a certain standard of living. While transport may be an item for city dwellers, this may not be so in the case of villagers in a developing country.
- To determine the weights, a proper study of consumption habits of persons of that group is to be made. The usual procedure is to conduct "family budget surveys". Such surveys help in

determining the items to be entered into the consumption pattern of the group and also help in determining the weights to be assigned to different items. One problem may occur regarding different *qualities* of the same type of commodity. Another problem may concern items of common use which do not occur in both the base period and the given period.

- The consumer price index numbers, also known as cost of living index number, are generally intended to represent the average change over time in the prices paid by the ultimate consumer of a specified basket of goods and services. The need for constructing consumer price indices arises because the general index numbers fail to give an exact idea of the effect of the change in the general price level on the cost of living of different classes of people, since a given change in the level of prices affects different classes of people in different manners. Different classes of people consume different types of commodities and even the same type of commodities are not consumed in the same proportion by different classes of people.
- The consumer price index numbers were earlier known as cost of living index numbers. But this name was not a happy one since the cost of living index does not measure the actual cost of living nor the fluctuations in the cost of living due to causes other than the change in the price level ; its object is to find out how much the consumers of a particular class have to pay more for a certain basketful of goods and services in a given period compared to the base period. To bring out clearly this fact, the Sixth International Conference of Labour Statisticians recommended that the term 'cost of living index' should be replaced in appropriate circumstances by the terms *'Price of living index', 'cost of living price index',* or *'consumer price index'*. At present, the three terms, namely, cost of living index, consumer price index and retail price index, are in use in different countries with practically no difference in their connotation. However, the term 'consumer price index' is the most popular of the three.
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- The scope of the index must be clearly defined. For example, when we talk of teachers, we are referring to primary teachers, middle class teachers, etc., or to all the teachers taken together. Along with the class of people it is also necessary to decide the geographical area covered by the index. Thus in the example taken above it is to be decided whether all the teachers living in Delhi are to be included or those living in a particular locality of Delhi, say, Chandni Chowk or Karol Bagh, etc.
- The consumption pattern can thus be easily ascertained. It is necessary that the family budget enquiry amongst the class of people to whom the index series is applicable should be conducted during the base period. The Sixth International Conference of Labour Statisticians held in Geneva in 1946 suggested that the period of enquiry of the family budgets and the base periods should be identical as far as possible.
- The commodities included are those which are generally consumed by people for whom the index is meant. Through family budget enquiry an average budget is prepared which is the standard budget for that class of people. While constructing the index only such commodities should be included as are not subject to wide variations in quality or to wide seasonal alterations in supply and for which regular and comparable quotations of prices can be obtained.

- Price quotations should be obtained from the localities in which the class of people concerned reside or from where they usually make their purchases. Some of the principles recommended to be observed in the collection of retail price data required.
- Since prices form the most important component of cost of living indices, considerable attention has to be paid to the methods of price collection and to the price collection personnel. Prices are collected usually by special agents or through mailed questionnaire or in some cases through published price lists. The greatest reliance can be placed on the price collection through special agents as they visit the retail outlets and collect the prices from them. However, these agents should be properly selected and trained and should be given a manual of instructions as well as manual of specifications of items to be priced. Appropriate methods of price verification should be followed such as '*check pricing*' in which price quotations are verified by means of duplicate prices obtained by different agents or '*purchase checking*' in which actual purchases of goods are made.
- In order to convert the prices into index numbers the prices or their relatives must be weighted. The need for weighting arises because relative importance of various items for different classes of people is not the same. For this reason, the cost of living index is always a weighted index. While conducting the family budget enquiry the amount spent on each commodity by an average family is ascertained and these constitute the weights. Percentage expenditures on the different items constitute the individual weights' allocated to the corresponding price relative and the percentage expenditure on the five groups constitute the 'group weight'.
- The Sixth International Conference of Labour Statisticians recommended that the pattern of consumption should be examined and the weights adjusted, if necessary, at intervals of not more than ten years to correspond changes in the consumption pattern. The index also does not take into account changes in qualities. Unlike changes in consumption pattern changes in qualities of goods and services are more frequent and when a marked change in the quality of items occurs appropriate adjustment should be made to ensure that the index takes into account changes is qualities also. But in practice it is a difficult proposition to follow and, therefore, constant qualities are assumed at two different dates which again is a shaky assumption.
- The consumption pattern derived from the expenditure data of a sample of households covered in the course of family budget enquiry has to be representative of all the items in the average budget, the localities from which price data are collected have to be representative of all the localities from which the population group make purchases, the retail outlets from which prices are collected have to be representative of all the retail outlets patronised by the population group, etc. However, it is often difficult to ensure perfect representativeness and in the absence of this the index may fail to provide the real picture.
- While taking the sample random sampling is seldom used. This is so because to sample from a population of literally millions of commodities and services, the random procedure could neither be practical nor representative. Typically, indices are constructed from samples deliberately selected. This is likely to introduce errors and every effort must be made to minimise these errors.
- A large number of methods have been designed for constructing index numbers and different methods of computation give different results. Very often the selection of an appropriate formula creates problems and in the interest of comparability, it is necessary to ensure that the same formula is adopted over a period of time for constructing a particular index. There is no index number method which is most satisfactory from all the various points of view which may logically or practically be taken. Index numbers are averages, and all averages are basically compromises between opposing extremes or forces.

8.4 Key-Words

1. Combinations : The number of ways objects can be selected without regard to order.
2. Combinatorics : The branch of mathematics dealing with the number of different ways objects can be selected or arranged.

8.5 Review Questions

1. What is meant by cost of living Index number ? What are its uses ?
2. How are cost of living Index number constructed ?
3. Explain briefly the various methods of construction of cost of living index number.
4. What are the limitations of index numbers.
5. What do you understand by cost of living index numbers ? Describe briefly the various steps involved in their construction.

Answers: Self-Assessment

1. (*i*) Median (*ii*) $\sum p_1 q_0$

   (*iii*) $\dfrac{\sum p_1 q}{\sum p_0 q} \times 100$, where $q = \dfrac{q_1 + q_2 \ldots . q_n}{n}$

   (*iv*) read (*v*) money

8.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 9: Time Series Analysis—Introduction and Components of Time Series

CONTENTS

Objectives

After reading this unit students will be able to:

- Know the Introduction to Time Series Analysis.
- Discuss Components of Time Series.
- Explain an Illustration Involving All Components.

Introduction

One of the major managerial responsibilities is the design and implementation of policies for the achievement of the short-term and long-term goals of the business firm. Previous performances must be studied so as to generate or forecast future business activity. Given a projection of the pattern and the level of future business activity, the desirability of alternative actions can then be investigated. For example, we may be interested to project sales activity levels with maintenance of adequate but not excessive inventory levels. Labour and material requirements must be projected. Need of working capital must be anticipated, and appropriate arrangements for financing investigated. The suitability and timing of capital intensive projects must be carefully evaluated. And lastly, once a strategy has been selected, control procedures must be incorporated to enable the firm to reassess the validity of the original projected values and the extent to which the actual results vary on a continuous basis. The quality of the forecasts or projections the management can make is strongly related to the information that can be extracted and used from past data. Time series analysis is one of the quantitative methods used to determine the patterns in data collected over a period of time. Thus, a time series consists of a set of chronological observations of a statistical series recorded either at successive points in time or over successive periods of time.

9.1 Introduction to Time Series Analysis

A series of observations recorded over time is known as a *time series*. The data on the population of a country over equidistant time points constitute a time series, e.g. the population of India recorded at the ten-yearly censuses. Some other examples of time series are: annual production of a crop, say, rice over a number of years, the wholesale price index over a number of months, the turn-over of a firm over a number of months, the sales of a business establishment over a number of weeks, the daily maximum temperature of a place over a number of days, and so on. In fact, economic data are, in general, recorded over time and are released at regular intervals. These constitute economic time series.

Notes

The objectives of analysis of a time series are (l) to give a general description of the past behaviour of the series, (2) to analyse the past behaviour, and (3) to attempt to forecast the future behaviour on the basis of the past behaviour; however, great caution is needed to do this.

A *forecast* does not tell what will happen but indicates what *would* happen if the past behaviour (as reflected in trends etc.) continues.

The techniques of time series analysis have largely been developed by economists. Empirical investigations dealing with economic theory are largely dependent on time series analysists. Social scientists, in general, do not have the privilege of conducting studies through laboratory experimentation. Studies are to be based on time series data collected over time in such cases. For example, trade cycles are important to economists and others in business and commerce. The exact behaviour of the cycles and their causes are of interest to them. Various theories explaining the phenomena are put forward. Analysis of time series provides an important tool for testing the theories and the explanations. Consumer behaviour is studied mainly with the help of time series data.
The analysis of time series plays an important role in empirical investigations, leading to quantitative revolution, in economics and in several other areas of social sciences and even in biological sciences. Thus political economy (usually known as economics) has been described as 'the oldest of the arts, the newest of the sciences-indeed the queen of the social sciences.'
We shall now discuss the techniques used in the analysis of time series. We begin with the main components or characteristic movements in a time series.

Did u know? The analysis of time series is of interest in several areas, such as economics, commerce, business, sociology, geography, meteorology, demography, public health, biology, and so on.

Objectives or Importance of Time Series Analysis

In the words of **Prof. Hirsch,** *"The main objective in analysing of time series is to understand, interpret and evaluate changes in economic phenomena in the hope of most correctly anticipating the course of future events."*

Following are the main objectives of time series analysis.

(1) **Study of the Past Behaviour of the Data:** The purpose of time series analysis is to study the past behaviour of the data to easily understand what changes have taken place in the past.

(2) **To Forecast Future Behaviour:** The second objective of time series analysis is to predict the future behaviour of a particular variable. Time series can play an important role not in making short range estimates for a year or two a head but also estimating the probable seasonal variations within a year.

(3) **Comparison with other Series:** Time series analysis is helpful in making a comparison between the behaviour of different time series. We can make this comparison by knowing the causes of variations in two time series.

(4) **Study of Present Fluctuations:** Time series analysis is helpful in studying the present fluctuations in the economic variables like, national income, cost, prices, production, etc. It enables us to know achievements and failures regarding a particular variable.

(5) **Estimation of Trade Cycles:** The basic objective of time series analysis is to estimate the trade cycles. The businessmen can avoid their losses and get profits with the help of the estimation of trade cycles.

9.2 Components of Time Series

Variations in Time Series

The term time senes are used to refer to any group of statistical information collected at regular intervals of time.

Notes

There are four kinds of changes, or variations, involved in time series analysis. They are:

(i) Secular trend

(ii) Cyclical fluctuation (variation)

(iii) Seasonal variation

(iv) Irregular variation

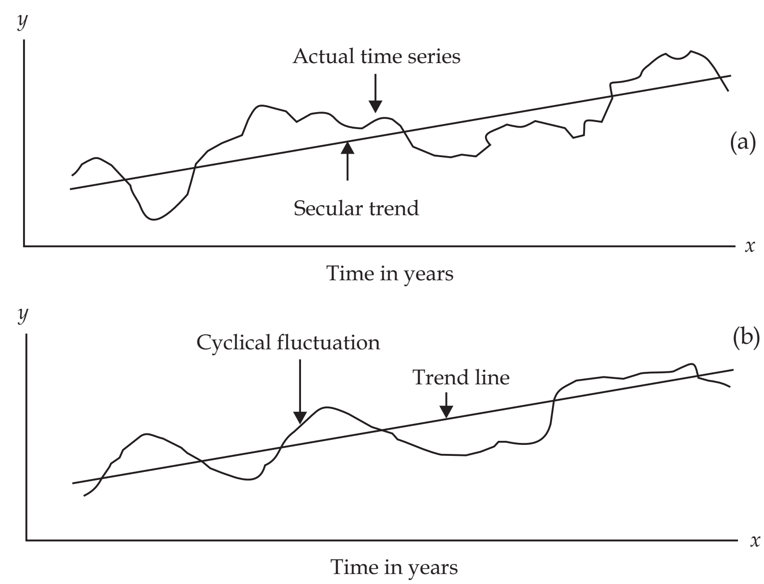
With the secular trend, the value of the variable tends to increase or decrease over a long period of time. The steady increase in the cost of living recorded by the consumer price index is an example of secular trend. From year to year, the cost of living varies a great deal; but, if we consider a long-term period, we see that the trend is towards steady increase. Other examples of secular trend are steady increase of population over a period of time, steady growth of agricultural food production in India over the last ten to fifteen years of time. Figure 1 *(a)* shows a secular trend in an increasing but fluctuating time series.

The second type of variation that can be observed in a time series is cyclical fluctuation. The most common example of cyclical fluctuation is the business cycle. Over a period of time, there are years when the business cycle has a peak above the trend line, and at other times, the business activity is likely to slump, touching a low point below the trend line. The time between touching peaks or failing to low points is generally 3 to 5 years, but it can be as many as 15 to 20 years. Figure 1 *(b)* illustrates a typical pattern of cyclical fluctuation. It should be noted that the cyclical movements do not follow any definite trend but move in a somewhat unpredictable manner.

The third kind of fluctuation that can occur in a time series data is the seasonal variation. Seasonal variation involves patterns of change within a year, that tend to be repeated from year to year. For example, sale of umbrellas is on the increase during the months of June and July every year because of the seasonal requirement. Since these are regular patterns, they are useful in forecasting the future production runs. Figure 1 *(c)* gives the seasonal variation in time series.

Irregular variation is the fourth type of change that can be observed in a time series data. These variations may be due to (i) random fluctuations: irregular random fluctuations refer to a large number of minute environmental influences (some uplifting, some depressing) operating on a series at any one time–no one of which is significantly important in and of itself to warrant singling out for individual treatment, and (ii) non-recurring irregular influences that exert a significant one time impact on the behaviour of a time series and as such must be explicitly recognized. The events included in this category are floods, strikes, wars, and so on, which influence the time series data.

The above four variations are generally considered as interacting in a multiplicative manner to produce observed values of the overall time series:



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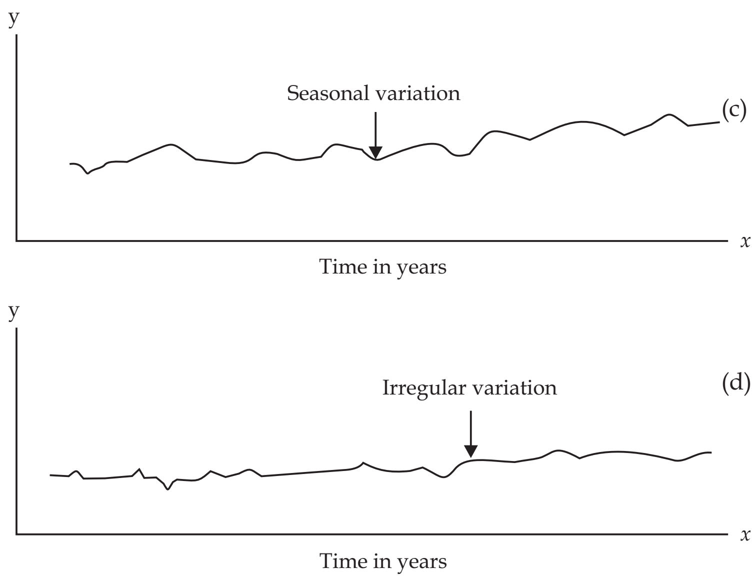


Figure 1: Time Series Variations

Multiplicative model: O = T × C × S × I.

where

O = observed value of time series

T = trend component

C = cyclical component

S = seasonal component

I = irregular component

Other types of models that are possible are:

Additive model : O = T + C + S + I

Combination model : O = T × C × S + I

O = (T + C) × S × I

However, we shall restrict our discussion to the multiplicative model.

Trend Analysis (Secular trend)

Secular trend represents the long-term variation of the time series. One way to describe the trend component in a time series data is to fit a line to a set of points on a graph. An approach to fit the trend line is by the method of least squares.

Following are the three major reasons for studying secular trend in a time series data:

1. Study of secular trend allows us to describe a historical pattern in the data. There are many situations when we can use past trend to evaluate the success of a previous management policy. For example, a multinational organisation may evaluate the effectiveness of the recruitment policy by examining its past enrollment trends.
2. Studying secular trend permits us to project past patterns, or trends into the future. Information of the past can tell us a great deal about the future. Examination of the growth of industrial production in the country, for example, help us to estimate the production for some future years.
3. In many situations, studying the secular trend of time series allows us to eliminate the trend component from the series. This makes it easier for us to study the other components of the time series. If we want to determine the seasonal variation in the sale of shoes, the elimination of the trend component gives us more accurate idea of the seasonal component.

Notes

Trends can be linear, or they can be curvilinear, *i.e.*, parabola. A straight line has a constant rate of change, while a parabola represents a changing rate of change. In fact, parabola shows a trend that is increasing at an increasing rate. The decision as to whether the trend should be a straight line or a parabola is an important one. It certainly is a subjective matter, and one has to be careful to choose the right trend to represent the data. Once the decision is made to the implied nature of the trend (linear or non-linear), one computes the rate of change and any errors in the choice of the trend curve will affect the results. Computation of the rate of change and measuring the trend is a process known as "fitting a curve to the data".

Before we discuss curve fitting, however, let us first consider the general types of trends that are available to represent the data. We shall identify three types of trends which are very popular in the analysis:

The Linear Trend (Straight Line, Constant Rate of Change)

This type of trend is represented in Figure 2, which shows two linear trend curves, (1) sloping upwards and (2) sloping downwards. The mathe-matical model for these linear trend is

$$Y = a + bX \quad \text{... (1)}$$

where *X* and Y are variables, *a* is the Y-intercept (*i.e*., the value of Y when X is equal to zero), and *b* is the rate of change of Y for unit change in X, or the constant rate of change.

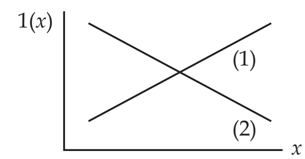


Figure 2: Pattern of Linear Trend.

The Parabolic Trend (Changing Rate)

Figure 3 represents the pattern of this trend. In this pattern (1) the trend is slopping upwards, indicating that the trend is increasing at an increasing rate of change, while in (2) the trend is slopping downwards, indicating that the trend is increasing at a decreasing rate. The mathematical representation of this curve is given by

$$Y = a + bX + cX^2 \quad \text{... (2)}$$

where *X* and *Y* are variables, *a* is the Y-*intercept* (*or the value of* Y when X is zero), and *b* and *c* are the rates of change of *Y* at given values of *X* which vary with different values of *X.*

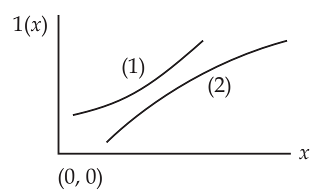


Figure 3:

The Exponential or Logarithmic Trend (Constant Percentage Rate of Change)

The mathematical model for this curve is*:*

$$Y = ab^X \quad \text{... (3)}$$

It is known as an exponential curve because the variable X appears as an exponent in the expression—unlike the other two cases stated earlier, where the variable X appeared as a factor. Plotting this curve on a semi-log graph sheet would produce a straight line (Figure 4), while plotting it on an ordinary graph sheet produces Figure 5. Taking logarithms of both sides of the expression (3), we get

Notes

$$\log Y = \log a + X \log b \quad \ldots (4)$$

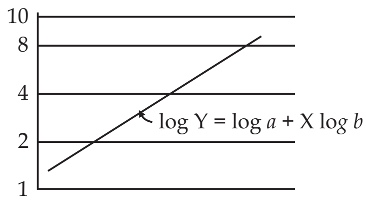


Figure 4: Exponential Trend on a Semi-log Graph.

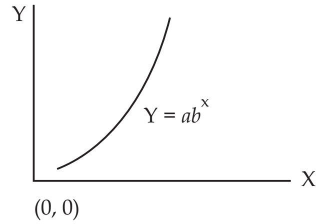


Figure 5: Exponential Trend on an Ordinary Graph.

If (4) is plotted on an ordinary graph sheet, it will produce a straight line (Figure 6), where log a is again the Y intercept and log b is the rate of change. In this case, since this expression is a straight line in a semi-logarithmic chart which shows percentage change, b (the rate of change) is a constant percentage rate of change.

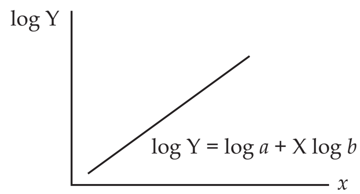


Figure 6: Logarithmic of Exponential Trend on an Ordinary Graph.

We shall study each of the curves individually and develop methods for computing their rates of change. Basically, there are four methods for fitting the treand in time series. These are:

(i) Free hand method

(ii) Method of semi-averages

(iii) Method of moving averages

(iv) Method of least squares.

These four types of secular trend are discussed in units 23, 24 and 25.

Uses of Secular Trend

The following are the main uses of secular trend:

(a) Basis of Fluctuations: The trend values are regarded as normal values. These normal values provide the basis for determining the nature of fluctuations. In other words, it can be found whether the fluctuations are regular or irregular. So general tendency of the data can be analysed with the help of secular trend.

(b) Help in Forecasting: The trend values help in business forecasting and planning of the future operations. This becomes possible since trend values describe the underlined behaviour pattern of the series in past.

(c) Effect of Short-term Variations: By eliminating the trend component in a time series, we can study the effect of short-term variations.

(d) It Facilitates Comparison: The trend analysis facilitates the comparison of two or more time series over different period of time. It helps us in drawing important conclusions about them.

Cyclical Variation

Cyclical variation is that component of a time series that tends to oscillate above and below the secular trend line for periods longer than one year and that they do not ordinarily exhibit regular periodicity. The periods and amplitudes may be quite irregular.

Notes

In a time series of annual data, only the secular trend, cyclical and irregular components are considered since the seasonal variation makes a complete, regular cycle within each year and thus do not affect the annual data. As secular trend can be described by a trend line, it is possible to isolate the remaining, cyclical and irregular components from the trend. For simplicity, we shall assume that the cyclical component explains most of the variations left unexplained by the trend component. However, in situations where this assumption does not hold good, methods such as Fourier analysis and spectral analysis can be used to analyse cyclical component in a time series. (These advanced techniques are beyond the scope of this text.)

If we use a time series composed of annual data, we can find the fraction of the trend by dividing the original value (Y) by the corresponding trend value (Y_{cal}) for each observation in the time series. We then take the percentage of this value by multiplying by 100. This gives us the measure of cyclical variation as a per cent of trend. Mathematically, this can be expressed as:

$$\text{Per cent of trend} = \frac{Y}{Y_{cal}} \times 100 \qquad \ldots (5)$$

where Y is the actual observation of the time series data, and Y_{cal} is the calculated trend value in the time series.

The above procedure used to identify cyclical variation is called the residual method.

Example 1: Let us consider the data given in Table 1 which refers to the yield per hectare of a certain foodgrain in an Indian state during 1970 to 1978. The third column in this table refers to the values of linear trend for each time period. The trend line has been developed using the method discussed in the previous section. It can be noted from the graph drawn (see Figure 7) with the actual (Y) and the trend (Y_{cal}) values for the nine years, the actual values move above and below the trend line.

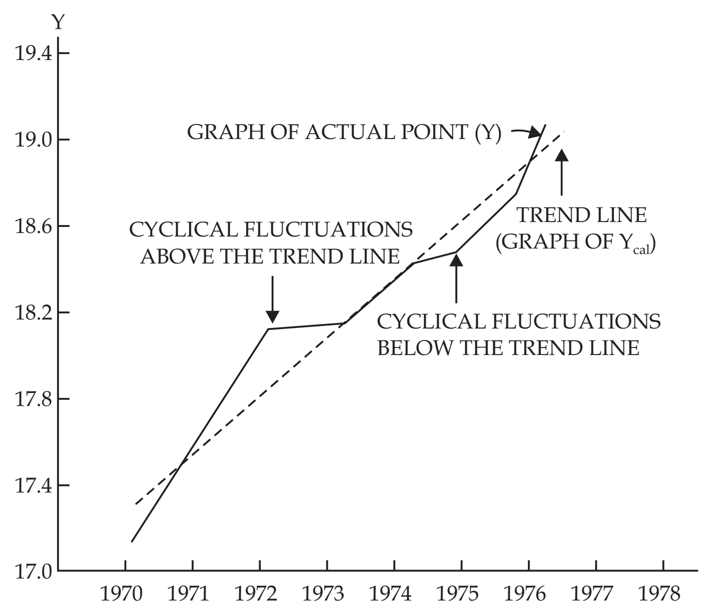


Figure 7: Cyclical Fluctuations Around the Trend Line.

From column 4 of Table 22.1 we can see the variation in actual yield around the estimated trend (98.8 to 101.7). We can attribute these cyclical variations to such factors as rainfall, humidity, etc. However, since these factors are relatively unpredictable, we cannot forecast any specific patten of variation using the method of residuals.

The above two methods of cyclical variation, per cent of trend and relative cyclical residual, are percentages of the trend. For example, in 1976 the per cent of trend indicated that the actual yield was 99.5 per cent of the expected yield for that year, while for the same year, the relative cyclical residual indicated that the actual yield was 0.5 per cent short of the expected yield during the year. It must be noted here that the methods described above are only used for describing the past cyclical variations and not for predicting future cyclical variations.

Use of Cyclical Variations

The following are the main uses of cyclical variations:

(a) **Aid to Policy Formation:** The study of cyclical variations is extremely useful in framing suitable policies for stabilizing the level of business activity. One can avoid the periods of booms and depressions since both are bad for an economy.

(b) **Helps in Studying Fluctuations of Business:** The cyclical variations are very helpful in studying the characteristics of fluctuations of a business. One can come to know how sensitive is the business to general cyclical influences ? The general pattern of a particular firm's production, profits, sales, raw material prices, etc. can also be known.

(c) **Helps in Forecasting:** The cyclical variations are helpful in forecasting and estimating about the future behaviour. Accurate forecasting is a prerequisite for successful business.

(d) **Knowledge of Irregular Fluctuations:** The study of cyclical variations is helpful in analysing and isolating the effects of irregular fluctuations. One can come to know either the variations are unpredictable or are caused by other isolated special occurrences like floods, earthquakes, strikes, wars, etc.

Seasonal Variation

Seasonal variations are those forces affecting time series that are the result of man made or physical phenomena. The major characteristic of seasonal variations is that they are repetitive and periodic, the period is less than one year, say a week, a month or a quarter. Seasonal variations can affect a time during November is normal, it can examine the seasonal pattern in the previous years and get the information it needs.

1. It is possible to establish the pattern of past changes. This helps us to compare two time intervals that would otherwise be too dissimilar. For example, if a business house wants to know whether the slump in sales during November is normal, it can examine the seasonal pattern in previous years and get the information it needs.
2. Seasonal variations help us to project past patterns into the future. In the case of long range decisions, secular trend analysis may be adequate. However, for short-run decisions, the ability to predict seasonal fluctuations is essential. For example, consider the case of a wholesale food dealer who wants to maintain a minimum adequate stock of all food items. The ability to predict short-run patterns, such as the demand of food items during Diwali, or at Christmas, or during the summer, is very useful to the management of the store.
3. Once the existence of the seasonal pattern has been established, it is possible to eliminate its effects from the time series. This elimination helps us to calculate the cyclical variation that takes place each year. When the effect of the seasonal variation has been eliminated from the time series, we have deseasonalized time series.

In order to measure the seasonal variation, we use the ratio-to-moving average method. This method provides an index that describes the degree of seasonal variation. The index is based on a mean of 100, with the degree of seasonality measured by variations away from the base.

The method of the ratio-to-moving average for computing the indices of seasonal variation is a procedure whereby the different components in the series are measured and are isolated or eliminated. Subsequently, the seasonal effect is identified and expressed in percentage form. We first take a series in which seasonal pattern is suspected and plot this series on a graph to identify the recurrence of the pattern. To identify the seasonal component, the data could be in quarters or months or any other time period less than a year.

Example 2 : Consider the data given in Table 3. Compute the seasonal index of quarterly sales of the departmental store by the method of ratio-to-moving average.

The first step in computing the seasonal index is to calculate the four-quarter moving totals for the series. This total is written in between quarters II and III in column 4 of Table 3. However, it could be "dropped down" one line to avoid the problem of having data between the lines.

Secondly, we compute the four-quarter moving average by dividing each of the four-quarter totals by four. We then find the centred four-quarter moving average so as to centre the moving averages against the periods. The seasonal and irregular components have thus been smoothened out. Figure 8 demonstrates how the moving average has smoothened the peaks and troughs of the original time series. The dotted line represents the cyclical and trend components.

Table 3: Computation of Ratios to Moving Averages to Quarterly Sales of a Departmental Store

(Rs. in lakhs)

Year	Quarter	Sales	4-quarter moving total	4-quarter moving average	Centred 4-quarter moving average	Ratio-to-moving average in percentage
1974	I	6.83				
	II	6.26	25.53	6.38	6.35	96.2
	III	6.11	25.24	6.31	6.25	101.3
	IV	6.33	24.77	6.19	6.13	106.7
1975	I	6.54	24.30	6.08	6.05	95.7
	II	5.79	24.12	6.08	6.06	93.1
	III	5.64	24.39	6.10	6.16	99.8
	IV	6.15	24.87	6.22	6.30	108.1
1976	I	6.81	25.51	6.38	6.45	97.2
	II	6.27	26.06	6.52	6.54	96.0
	III	6.28	26.26	6.57	6.52	102.8
	IV	6.70	25.90	6.48	6.39	109.7
1977	I	7.01	25.25	6.31	6.25	94.6
	II	5.91	24.71	6.18	6.04	93.2
	III	5.63	23.65	5.91	5.86	105.1
	IV	6.16	23.22	5.81	5.74	103.7
1978	I	5.95	23.70	5.74	6.00	91.3
	II	5.48	24.34	6.08		
	III	6.11				
	IV	6.80				

The next step is to calculate the percentage of the actual value to the moving average value for each quarter in the time series having a four-quarter moving average entry.

One should note at this stage that the first two and the last two quarters do not have the corresponding moving averages. This step allows us to recover the seasonal component for the quarters.

The fourth step is to collect all the percentages of actual to moving average values and to arrange them by quarter. We then calculate the mean for each quarter. These computations have been shown in Table 4.

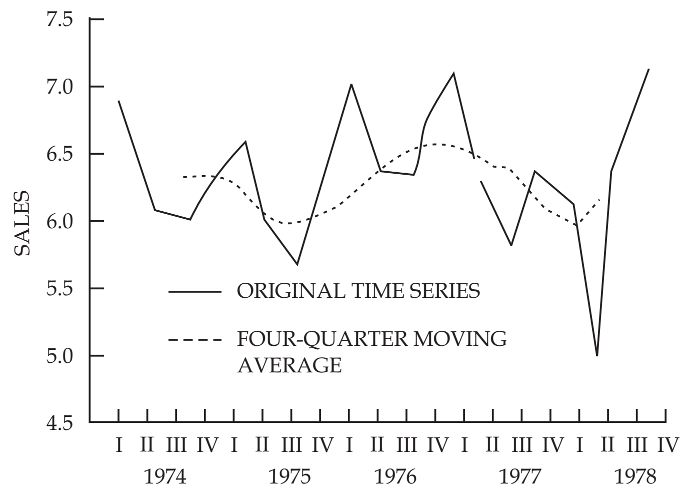


Figure 8: Moving Average: Smoothening the Original Data in Table 3

Table 4: Computation of Seasonal Index for Data Given in Table 3

Year	Quarter I	Quarter II	Quarter III	Quarter lV
1974	–	–	96.2	101.3
1975	106.7	95.7	93.1	99.8
1976	108.1	97.2	96.0	102.8
1977	109.7	94.6	93.2	105.1
1978	103.7	91.3	–	–
Mean	**107.1**	**94.7**	**94.6**	**102.2**
Seasonal Index	**107.5**	**95.1**	**95.0**	**102.5**

The final step is to adjust these quarterly means slightly. One can find out that the sum of the four quarterly means is 398.5 (from Table 4). However, the base for an index is 100 and therefore the four quarterly means should total 400 so that their mean is 100. To correct this error, we multiply each of the quarterly indices in Table 4 by an adjusting constant, the constant being computed by dividing the desired sum of indices (400) by the actual sum (398.5). In this case, this constant is 1.0038. Thus, we get the seasonal indices for each of the quarters. This has been given in the last row of Table 4.

There are a few other methods for computing the seasonal index. One of them is the link-relative method. However, the computation of seasonal indices by this method is slightly complicated. The steps involved in this method are as follows:

1. Calculate the seasonal link relatives for each seasonal value by the following formula:

$$\text{Link relative} = \frac{\text{Current seasons' value}}{\text{Previous season's value}} \times 100$$

2. Calculate the average of the link relatives for each season.

Notes

3. Convert these averages into chain relatives on the basis of the first season.
4. Calculate the chain relative of the first season on the basis of the last season.
5. A correction is applied to each of the relatives that have been computed in the earlier step. For this correction, the chain relative of the first season calculated by the first method is deducted from the chain relative of the first season calculated by the second method. The difference is divided by the number of seasons in a year. The resulting figure multiplied by 1, 2, 3 etc. is deducted respectively from the chain relatives of the 2nd, 3rd, 4th, etc.
6. The seasonal indices are obtained when the corrected chain relatives are expressed as percentage of their relative averages.

Example 3 : Calculate the seasonal index for the data given in Table 3 by the link-relative method.

Solution : The computation of seasonal indices has been explained below:

Table 5: Computation of Seasonal Indices by Link Relative Method

	Quarters			
Year	**I**	**II**	**III**	**IV**
1974	–	91.7	97.6	103.6
1975	103.3	88.5	97.4	109.0
1976	110.7	92.1	100.2	106.7
1977	104.6	84.3	95.3	109.3
1978	96.6	92.1	111.5	111.3
Mean chain	103.8	89.7	100.4	108.0
Relatives	100	$\frac{100 \times 89.7}{100}$ = 89.7	$\frac{89.7 \times 104.4}{100}$ = 90.1	$\frac{90.1 \times 108.0}{100}$ = 97.3
Corrected chain relatives	100.0	89.7 – 0.25 = 87.45	90.1 – 0.5 = 89.6	97.3 – 0.75 = 96.55
Corrected Seasonal index	100.0	$\frac{87.45}{93.4} \times 100$ = 93.6	$\frac{89.6}{93.4} \times 100$ = 95.9	$\frac{96.55}{93.4} \times 100$ = 103.4

The correction factor has been calculated as follows:

Chain relative of the first quarter on the basis of first quarter = 100.0

Chain-relative on the basis of the last quarter $= \frac{103.8 \times 97.3}{100}$

= 101.0

The difference between these chain relatives = 101.0 –100.0

= 1.0

Difference per quarter $= \frac{1.0}{4}$

= 0.25

Seasonal indices have been corrected as follows:

$$\text{Average of chain relatives} = \frac{100.0 + 87.45 + 89.6 + 96.55}{4}$$

$$= 93.4$$

$$\text{Corrected seasonal index} = \frac{\text{Corrected chain relative}}{93.4} \times 100$$

The other alternative method for determining the seasonal indices is the ratios-to-trend method. This method assumes that seasonal variation for a given season is a constant fraction of the trend. With the basic multiplicative model, O = TSCI, it is argued that the trend can be eliminated by dividing each observation by its corresponding trend value. The ratios resulting from this computation compose SCI. Each of these ratios to trend is a one-based relative, that is pure number with a unity base. Next, an average is computed for each season. This averaging process eliminates cyclical and random (irregular) fluctuations from the ratios to trend. Thus, these averages of ratios to trend contain only the seasonal component. These averages, therefore, constitute the seasonal indices. However, slight corrections can be incorporated in order to adjust these ratios to average to unity.

A simplest but a crude method of computing a seasonal index is to calculate the average value for each season, and express the averages as percentages so that all the seasonal percentages can add up to 100 multiplied by the number of seasons.

The seasonal indices are used to remove the seasonal effects from a time series. Before identifying either the trend or cyclical components of a time series, one must eliminate the seasonal variation. To do this, we divide each of the actual values in the series by the appropriate seasonal index. Once the seasonal effect has been eliminated, the deseasonalized values that remain in the series reflect only the trend, cyclical, and irregular components of the time series. With the help of the deseasonalized values we can project the future.

Uses of Seasonal Variations

The following are the main uses of seasonal variations:

(a) **Knowledge of the Pattern of Change:** A study of the seasonal variations helps in determining the pattern of the change. It can be known whether the change is stable or gradual or abrupt.

(b) **Helps in the Study of Cyclical Fluctuations:** The cyclical and irregular variations can be accurately studied only after eliminating seasonal components from a time series.

(c) **Aid of Policy Decisions:** The seasonal variations aid in formulating policy decisions. These are also useful in planning future variations. For example, the manufacturer may decide to cut the prices during slack season and providing incentives in the off-season. They may also incur huge expenditure in advertising off-seasonal use of the product.

(d) **Knowledge of the Nature of Change:** The study of seasonal variations provides a better understanding of the nature of variations. For example, in the absence of the knowledge of seasonal variations a seasonal upswing may be mistaken as an indication of better business conditions. Similarly, a seasonal slump may be mistaken as an indication of deterioration in business conditions. While in fact both these changes are seasonal and not of permanent nature.

Distinction between Cyclical and Seasonal Variations

Notes

The following are the main points of distinction between seasonal and cyclical variations:

(i) **Duration of Variations:** Cyclical variations have a duration of two to fifteen years, whereas seasonal variations have a duration of one year only.

(ii) **Degree of Accuracy:** Cyclical variations cannot be accurately estimated because of lack of their regularity whereas seasonal variations can be estimated with a high degree of accuracy.

(iii) **Regularity:** There is no regularity in the periodicity of cyclical variations whereas there is regular periodicity in seasonal variations.

(iv) **Causes of Variations:** The main causes of cyclical variations are economic whereas seasonal variations take place because of weather conditions and customs and traditions.

(v) **Activities of Preceding Variations:** Cyclical variations depend upon the activities of the preceding period whereas seasonal variations do not depend on the activities of preceding period.

Irregular Variation

The last component of a time series is the irregular variation. After eliminating the trend, cyclical, and seasonal variations from a time series, we have an unpredictable element left in the series. Irregular variation, generally, occurs over a short interval of time period and follows a random pattern. For example, a strike in an industrial unit may push down its production and consequently, the sales. Some other causes for these variations are flood draught, fire, war or other unforeseeable events.

Because of the unpredictability of irregular variation, attempt has not been made to study it mathematically. However, we can often isolate its causes, although in some situations it is difficult to identify such causes. But it should be noted that over a period of time, these random fluctuations tend to counteract each other and thus we may have a time series free of irregular variation.

9.3 An Illustration Involving all Components

As an illustration for studying all the components of a time series, we shall work out a problem involving all the components. An engineering firm producing farm equipments wants to predict future sales based on the analysis of its past sales pattern. The sales effected by the firm during the past five years is given in Table 6.

Table 6: Quarterly Sales of an Engineering Firm during 1975 to 1979

(Rs. in lakhs)

	Quarters			
Year	**I**	**II**	**III**	**IV**
1975	5.5	5.4	7.2	6.0
1976	4.8	5.6	6.3	5.6
1977	4.0	6.3	7.0	6.5
1978	5.2	6.5	7.5	7.2
1979	6.0	7.0	8.4	7.7

The procedure involved in this study consists of:

1. deseasonalizing the time series,
2. fitting the trend line, and
3. identifying the cyclical variation around the trend line.

The steps involved in deseasonalizing the time series are given in Table 7 and Table 8. These steps have been already discussed in Seasonal variation.

Table 7: Computation of Ratio-to-Moving Averages to Quarterly Sales Data of Table 6

Year	Quarter	Actual sales	Centred 4-quarter moving total	Centred 4-quarter moving average	Ratio-to moving average in percentage
1975	I	5.5			
	II	5.4			
	III	7.2	23.8	6.0	120.0
	IV	6.0	23.5	5.9	101.7
1976	I	4.8	23.2	5.8	82.8
	II	5.6	22.5	5.6	100.0
	III	6.3	21.9	5.5	114.5
	IV	5.6	21.9	5.5	101.8
1977	I	4.0	22.6	5.7	70.2
	II	6.3	23.4	5.9	06.8
	III	7.0	24.4	6.1	114.8
	IV	6.5	25.1	6.3	103.2
1978	I	5.2	25.5	6.4	81.3
	II	6.5	26.1	6.5	100.0
	III	7.5	26.8	6.7	111.9
	IV	7.2	27.5	6.9	104.3
1979	I	6.0	28.2	7.1	84.5
	II	7.0	28.9	7.2	97.2
	III	8.4			
	IV	7.7			

Table 8: Computation of Seasonal Indices for Quarterly Sales Data of Table 6.

	Quarters			
Year	**I**	**II**	**III**	**IV**
1975	–	–	120.0	101.7
1976	82.8	100.0	114.5	101.8
1977	70.2	106.8	114.8	103.2
1978	81.3	100.0	111.9	104.3
1979	84.5	97.2	–	–
Mean	**79.7**	**101.0**	**115.3**	**102.8**
Seasonal index	**79.9**	**101.3**	**115.6**	**103.2**

Table 9: Deseasonalized Sales of the Engineering Firm

Notes

Year	Quarter	Actual sales	Seasonal index 100	Deseasonalized sales
1975	I	5.5	0.799	6.9
	II	5.4	1.013	5.3
	III	7.2	1.156	6.2
	IV	6.0	1.032	5.8
1976	I	4.8	0.799	6.0
	II	5.6	1.013	5.5
	III	6.3	1.156	5.4
	IV	5.6	1.032	5.4
1977	I	4.0	0.799	5.0
	II	6.3	1.013	6.2
	III	7.0	1.156	6.0
	IV	6.5	1.032	6.3
1978	I	5.2	0.799	6.5
	II	6.5	1.013	6.4
	III	7.5	1.156	6.5
	IV	7.2	1.032	7.0
1979	I	6.0	0.799	7.5
	II	7.0	1.013	6.9
	III	8.4	1.156	7.3
	IV	7.7	1.032	7.5

The second step in identifying the components of the time series is to develop the trend line. For this purpose, we use the least squares technique to the deseasonalized time series. Table 9 gives the deseasonalized time series.

With the values from Table 9. we now find the equation for the linear trend. These computations have been shown in Table 10.

Table 10: Computations of the Trend from the Data of Table 9.

Year	Quarter	Deseasonalized sales (Y)	X*	X^2	XY
1975	I	6.9	– 19	361	– 131.1
	II	5.3	– 17	289	– 90.1
	III	6.2	– 15	225	– 93.0
	IV	5.8	– 13	169	– 75.4
1976	I	6.0	– 11	121	– 66.0
	II	5.5	– 9	81	– 49.5
	III	5.4	– 7	49	– 37.8
	IV	5.4	– 5	25	– 27.0
1977	I	5.0	– 3	9	– 15.0
	II	6.2	– 1	1	– 6.2
	III	6.0	1	1	6.0
	IV	6.3	3	9	18.9

1978	I	6.5	5	25	32.5
	II	6.4	7	49	44.8
	III	6.5	9	81	58.5
	IV	7.0	11	121	77.0
1979	I	7.5	13	169	97.5
	II	6.9	15	225	103.5
	III	7.3	17	289	124.1
	IV	7.5	19	361	142.5
Total		**125.6**		**2660**	**114.2**

* Since there are even number of periods in the series, we have assigned modified values to X.

$$b = \frac{\sum XY}{\sum X^2} = \frac{114.2}{2660.0} = 0.04$$

$$a = \frac{\sum Y}{n} = \frac{125.6}{20} = 6.3$$

Thus, the straight line trend equation is: $Y = a + bX$

that is, $Y_{cal} = 6.3 + 0.04\ X$

We have now been able to identify the seasonal and trend components in the time series. It is now required to find the cyclical variation around the trend line. This component is identified by measuring deseasonalized variation around the trend line. The cyclical variation is computed with the help of the residual method and the same is given in Table 11.

Table 11: Computations of Cyclical Variation

Year	Quarter	Deseasonalized sales (Y)	$Y_{cal} = a + bX$*	Per cent of trend $\frac{Y}{Y_{cal}} \times 100$
1975	I	6.9	5.54	124.5
	II	5.3	5.62	94.3
	III	6.2	5.70	108.8
	IV	5.8	5.78	100.3
1976	I	6.0	5.86	102.4
	II	5.5	5.94	92.6
	III	5.4	6.02	89.7
	IV	5.4	6.10	88.5
1977	I	5.0	6.18	80.9
	II	6.2	6.26	99.0
	III	6.0	6.34	94.6
	IV	6.3	6.42	98.1
1978	I	6.5	6.50	100.0
	II	6.4	6.58	97.3
	III	6.5	6.66	97.6
	IV	7.0	6.74	103.9

1979	I	7.5	6.82	111.0
	II	6.9	6.90	100.0
	III	7.3	6.98	104.6
	IV	7.5	7.06	106.2

The irregular variation is assumed to be short-term and relatively insignificant. We have, thus, described the time series in this problem using the trend, cyclical, and seasonal components. Figure 22.9 represents the original time series, its four-quarter moving average (containing the trend and cyclical components), and the trend line.

* Appropriate value of X is taken as given in Table 22.10.

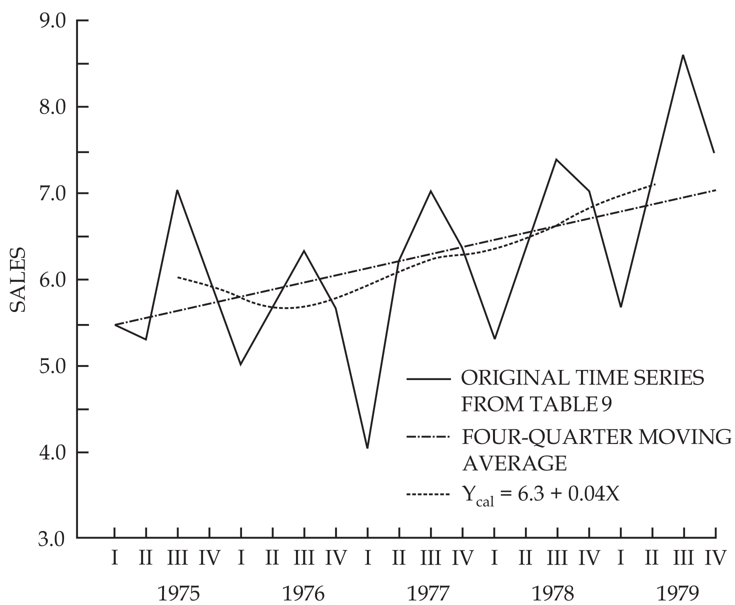


Figure 9: Original Time Series. Trend Line, and Four Quarter Moving Average for Sales Data of Table 6.

Suppose, now, the management of the engineering firm is interested in estimating the sales for the second and third quarters of 1980. Following procedure, then, will have to be adopted to estimate these figures:

$$Y_{cal} = a + bX$$

i.e. $Y = 6.3 + 0.04\,(23)$

(2nd quarter, 1980)

$= 7.22$

and $Y = 6.3 + 0.04\,(25)$

(3rd quarter, 1980)

$= 7.30$

Thus, the deseasonalized sales estimates for second and third quarters of 1980 are Rs. 7.22 lakhs and Rs. 7.30 lakhs, respectively. These estimates will now have to be seasonalized for the second and third quarters respectively. This is done in the following way:

1980-second quarter:

Seasonalized sales estimate $= 7.22 \times 1.013$

$= 7.31$

1980-third quarter:

Seasonalized sales estimate = 7.30 × 1.156

= 8.44

On the basis of the above analysis, the sales estimates of the engineering firm for the second and third quarters of 1980 are Rs. 7.31 lakhs and Rs. 8.44 lakhs, respectively. It should be noted here that these estimates have been obtained by taking the trend and seasonal variations into account. Further the cyclical and irregular components have not been taken into account in these estimates.

The procedure described earlier for the cyclical variation will only help us to study the past behaviour and does not help us in predicting the future behaviour. As stated in the earlier section, the irregular variations cannot be studied mathematically.

Notes Time series analysis is helpful in studying the present fluctuations in the economic variables like, national income, cost, prices, production, etc. It enables us to know achievements and failures regarding a particular variable.

Self-Assessment

1. Fill in the blanks:

(*i*) A time series consists of data arranged ....

(*ii*) The four components of time series are .... , ... , ... , and ....

(*iii*) The additive model of components is ....

(*iv*) Secular trend is referred for .... trend.

(*v*) Forces of rythmic nature cause ....

9.4 Summary

- A series of observations recorded over time is known as a *time series.* The data on the population of a country over equidistant time points constitute a time series, e.g. the population of India recorded at the ten-yearly censuses. Some other examples of time series are: annual production of a crop, say, rice over a number of years, the wholesale price index over a number of months, the turn-over of a firm over a number of months, the sales of a business establishment over a number of weeks, the daily maximum temperature of a place over a number of days, and so on.
- The analysis of time series is of interest in several areas, such as economics, commerce, business, sociology, geography, meteorology, demography, public health, biology, and so on. The techniques of time series analysis have largely been developed by economists. Empirical investigations dealing with economic theory are largely dependent on time series analysists. Social scientists, in general, do not have the privilege of conducting studies through laboratory experimentation. Studies are to be based on time series data collected over time in such cases. For example, trade cycles are important to economists and others in business and commerce. The exact behaviour of the cycles and their causes are of interest to them. Various theories explaining the phenomena are put forward. Analysis of time series provides an important tool for testing the theories and the explanations. Consumer behaviour is studied mainly with the help of time series data.
- The second objective of time series analysis is to predict the future behaviour of a particular variable. Time series can play an important role not in making short range estimates for a year or two a head but also estimating the probable seasonal variations within a year.
- With the secular trend, the value of the variable tends to increase or decrease over a long period of time. The steady increase in the cost of living recorded by the consumer price index is an

example of secular trend. From year to year, the cost of living varies a great deal; but, if we consider a long-term period, we see that the trend is towards steady increase. Other examples of secular trend are steady increase of population over a period of time, steady growth of agricultural food production in India over the last ten to fifteen years of time.

- Seasonal variation involves patterns of change within a year, that tend to be repeated from year to year. For example, sale of umbrellas is on the increase during the months of June and July every year because of the seasonal requirement. Since these are regular patterns, they are useful in forecasting the future production runs.
- Secular trend represents the long-term variation of the time series. One way to describe the trend component in a time series data is to fit a line to a set of points on a graph. An approach to fit the trend line is by the method of least squares.
- In many situations, studying the secular trend of time series allows us to eliminate the trend component from the series. This makes it easier for us to study the other components of the time series. If we want to determine the seasonal variation in the sale of shoes, the elimination of the trend component gives us more accurate idea of the seasonal component.
- The trend values are regarded as normal values. These normal values provide the basis for determining the nature of fluctuations. In other words, it can be found whether the fluctuations are regular or irregular. So general tendency of the data can be analysed with the help of secular trend.
- The trend analysis facilitates the comparison of two or more time
- Cyclical variation is that component of a time series that tends to oscillate above and below the secular trend line for periods longer than one year and that they do not ordinarily exhibit regular periodicity. The periods and amplitudes may be quite irregular.
- Another method used to measure the cyclical variation is the relative cyclical residual method. In this method, the percentage deviation from the trend is found for each value.
- The cyclical variations are very helpful in studying the characteristics of fluctuations of a business. One can come to know how sensitive is the business to general cyclical influences ? The general pattern of a particular firm's production, profits, sales, raw material prices, etc. can also be known.
- The study of cyclical variations is helpful in analysing and isolating the effects of irregular fluctuations. One can come to know either the variations are unpredictable or are caused by other isolated special occurrences like floods, earthquakes, strikes, wars, etc.
- Seasonal variations are those forces affecting time series that are the result of man made or physical phenomena. The major characteristic of seasonal variations is that they are repetitive and periodic, the period is less than one year, say a week, a month or a quarter. Seasonal variations can affect a time during November is normal, it can examine the seasonal pattern in the previous years and get the information it needs.
- Seasonal variations help us to project past patterns into the future. In the case of long range decisions, secular trend analysis may be adequate. However, for short-run decisions, the ability to predict seasonal fluctuations is essential. For example, consider the case of a wholesale food dealer who wants to maintain a minimum adequate stock of all food items. The ability to predict short-run patterns, such as the demand of food items during Diwali, or at Christmas, or during the summer, is very useful to the management of the store.
- The method of the ratio-to-moving average for computing the indices of seasonal variation is a procedure whereby the different components in the series are measured and are isolated or eliminated. Subsequently, the seasonal effect is identified and expressed in percentage form. We first take a series in which seasonal pattern is suspected and plot this series on a graph to identify the recurrence of the pattern. To identify the seasonal component, the data could be in quarters or months or any other time period less than a year.
- The seasonal indices are used to remove the seasonal effects from a time series. Before identifying either the trend or cyclical components of a time series, one must eliminate the seasonal variation.

To do this, we divide each of the actual values in the series by the appropriate seasonal index. Once the seasonal effect has been eliminated, the deseasonalized values that remain in the series reflect only the trend, cyclical, and irregular components of the time series. With the help of the deseasonalized values we can project the future.

- A study of the seasonal variations helps in determining the pattern of the change. It can be known whether the change is stable or gradual or abrupt.
- The seasonal variations aid in formulating policy decisions. These are also useful in planning future variations. For example, the manufacturer may decide to cut the prices during slack season and providing incentives in the off-season. They may also incur huge expenditure in advertising off-seasonal use of the product.
- The last component of a time series is the irregular variation. After eliminating the trend, cyclical, and seasonal variations from a time series, we have an unpredictable element left in the series. Irregular variation, generally, occurs over a short interval of time period and follows a random pattern. For example, a strike in an industrial unit may push down its production and consequently, the sales. Some other causes for these variations are flood draught, fire, war or other unforeseeable events.

9.5 Key-Words

1. Conditional odds : The odds of success given some level of another variable.
2. Conditional probability : The probability of one event given the occurrence of some other event.
3. Confidence interval : An interval, with limits at either end, with a specified probability of including the parameter being estimated.
4. Confidence limits : An interval, with limits at either end, with a specified probability of including the parameter being estimated.

9.6 Review Questions

1. What is time series ? What is the need to analyse the time series ?
2. Define time series. What are the preliminary adjustments that should be made before analysing time series ?
3. What are the various components of time series ? Explain.
4. What is secular trends ? Point out the significance of its study.
5. What do you mean by cyclical variation ? What are the methods of measuring such variations.

Answers: Self-Assessment

1. (*i*) Chronologically

(*ii*) Secular trend (T), Seasonal variations (S), Cyclical variations (C) and Irregular variations (I)

(*iii*) T + S + C + I

(*iv*) Cyclical variations (*v*)Seasonal variation

9.7 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 10 : Time Series Methods—Graphic, Method of Semi-averages

CONTENTS

Objectives

After reading this unit students will be able to:

- Describe Graphic Method.
- Explain Semi-average Method.

Introduction

Fitting a trend curve involves assuming that a given time series exhibits a certain trend movement which, were it not for cyclical, irregular and seasonal fluctuations would have been a linear or non-linear form. Therefore, we first assume that the data to be plotted on a graph exhibit a certain trend form (linear, parabolic or exponential) and then an attempt is made to measure this trend. Measuring a trend actually means computing the constants of the equation that we have chosen to be representative of the trend in the data. However, it should be remembered that if we choose a wrong curve for the data, then the constants of the equation computed would be wrong, and any forecasting made on the basis of this equation would be wrong.

10.1 Graphic

Freehand method is also called the graphic method in the sense that the trend line is determined by inspecting the graph of the series. According to this method, the trend values are determined by drawing freehand straight line through the time series data that is judged by the analyst to represent adequately the long-term movement in the series. Once the freehand trend line is drawn, a trend equation for the line can be approximated. This is done by first reading off the trend values of the first and the last period from the chart with reference to the freehand line. For this purpose, the first period is usually considered the origin. Thus, the trend value for the first period is the value of *a* for the equation. Then the difference between the trend values of the first and the last period is obtained. This difference represents the total change in variable Y throughout the whole duration of the series. Therefore, when this difference is divided by the number of periods in the series, the result represents the average change in Y per unit time period. This is the value of *b* in the equation. The trend line for many series may be satisfactorily drawn provided the fluctuations around the general drift are so small that the path of the trend is clearly defined. The main trouble with this method is that it is too subjective. Even an expert in this subject may draw different lines at different times for the same series. There is no formal statistical criterion whereby the adequacy of such a line can be judged.

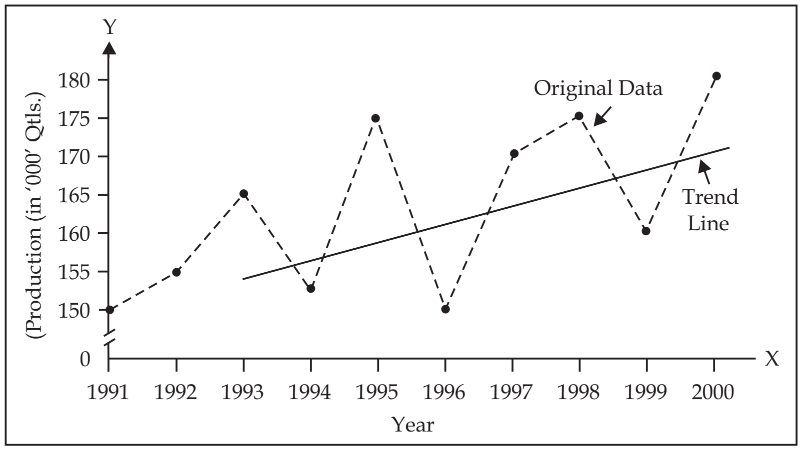
Notes

Furthermore, although this method appears simple and direct, in actuality. it is very time consuming to construct a freehand trend curve if a careful and conscientious job is to be done. For these reasons, the freehand method is not recommended for fitting a trend line.

Example 1: Draw a time series graph relating to the following data and fit the trend by freehand method.

Year :	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Production ('000 Qtls) :	150	155	165	152	174	150	170	175	160	178

Solution:



Merits of Graphic Method

(1) **Simplest :** This method of estimating trend is the simplest of all the methods of measuring trend. It involves no calculation at all since it is purely non-mathematical.

(2) **Easy to Fit Trend :** This method makes possible rapid approximations of trend that are relatively reliable. It gives a better expression of the secular movements.

(3) **Flexible :** This method is more flexible than rigid mathematical function, hence fits the curve more closely to the data *i.e* this method can be used even in cases where the size of the series is lengthy.

Demerits of Graphic Method

(1) **Trend Values are not Definite :** This method involves no calculation or mathematical formulae, hence the trend value cannot be definite. Different persons can draw different trend lines from the same original data.

(2) **Subjective Method :** This method is a subjective method. Being subjective, it has little value as a basis of future analysis of time series.

(3) **Lack of Accuracy :** This method lacks accuracy. Therefore, it is not suitable where a high degree of accuracy is desired. This method gives us an approximate picture of the tendency in the long run. This method should therefore be used only by experienced persons.

Limitations of Freehand Method

1. This method is highly subjective because the trend line depends on the personal judgment of the investigator and, therefore, different persons may draw different trend lines from the same set of data. Moreover, the work cannot be left to clerks and it must be handled by skilled and experienced people who are well conversant with the history of the particular concern.

2. Since freehand curve fitting is subjective it cannot have much value if it is used as basis for predictions.

3. Although this method seems to be quite simple, in actual practice it is very time-consuming to construct a freehand trend if a careful and conscientious job is to be done. It is only after long experience in trend fitting that a person should attempt to fit a trend line by inspection.

> *Notes* To determine the trend values by the semi-average method, the series in question is first divided into two equal segments; then the arithmetic mean for each part is computed.

10.2 Semi-averages Method

To determine the trend values by the semi-average method, the series in question is first divided into two equal segments; then the arithmetic mean for each part is computed. Lastly, a straight line passing through these two averages is drawn to provide the trend for the series. Each average provides the trend value for the middle time period of the corresponding segment. When the time series includes an odd number of periods, there are three methods for separating the series :

(a) Add half of the value of the middle period to the total value of each part.

(b) Add the total value of the middle period to the total value of each part.

(c) Drop the value of the middle period from the computations of the averages.

With the semi-average method, the middle time unit is considered as the origin, and the values of the Y-intercept and the slope of the straight line are derived by applying the following equations :

$$a = \frac{S_1 + S_2}{t_1 + t_2} \quad \text{... (1)}$$

$$b = \frac{S_2 - S_1}{t_1(n - t_2)} \quad \text{... (2)}$$

where t_1 and t_2 refer to the number of time units for the first and second segments in the series; S_1 and S_2 refer to the corresponding partial sums respectively; and n is the total number of periods in the series.

Example 2: For the purpose of example the above method, we shall use the data given in Table 1. This series contains 15 years and is divided into two parts with 7 years in each, the middle year being dropped.

Table 1 : Computation of Trend by Semi-average Method for the Data Relating to Number of Persons Registered in an Employment Exchange in an Indian State During 1965-1979 (Figures Given in Thousands)

Year	X	No. of persons	Semi-average	Trend value
1965	– 7	10.5		11.6
1966	– 6	15.3		12.3
1967	– 5	13.5		12.9
1968	– 4	12.9	13.6	13.6
1969	– 3	11.1		14.3
1970	– 2	15.9		14.9
1971	– 1	16.0		15.6
1972	0	16.5		16.3
1973	1	16.0		17.0
1974	2	16.4		17.6
1975	3	19.9		18.3
1976	4	21.7	19.0	19.0

1977	5	18.7		19.6
1978	6	18.6		20.3
1979	7	21.5		21.0

The arithmetic mean for the first half is 13.6 and that for the second is 19.0. The straight line trend drawn through these two points is given in Figure.

Furthermore, with these two average values, we get :

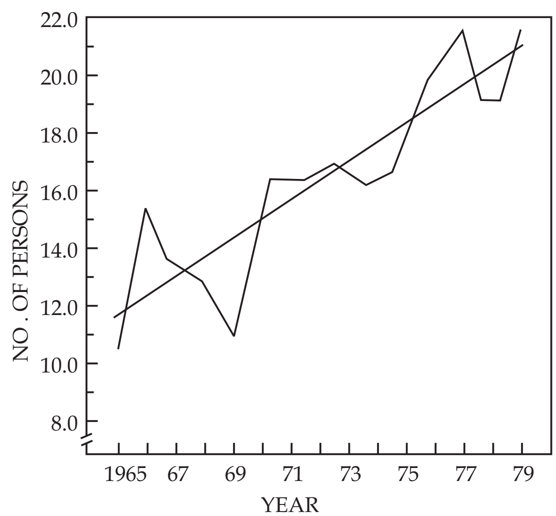
$$a = \frac{95.2 + 132.8}{7 + 7} = 16.3$$

$$b = \frac{132.8 - 95.2}{7(15 - 7)} = 0.67$$

Thus, the trend becomes

$$Y_{cal} = 16.3 + 0.67\ X$$

Trend by the Method of Semi-Averages



Trend value for each year can now be determined by substituting the X value for that period in the above trend equation. The trend values have been calculated for all the years and the same has been given in the last column of the Table 23.1.

This method of determining the trend is not a subjective one. The slope of the trend line now depends upon the difference between the averages that are computed from the original values, with each average as typical of the level of that segment of the data. However, this method is not entirely free from drawbacks. The major drawback here is due to the arithmetic mean which can be unduly affected by the extreme values in the series. If one part of the series contains more depressions or fewer prosperities than the other, then the trend line is not a true representation of the secular movements of the series. Therefore, the trend values obtained by this method are not accurate enough for the purpose either of forecasting the future trend or of eliminating the trend from the original data.

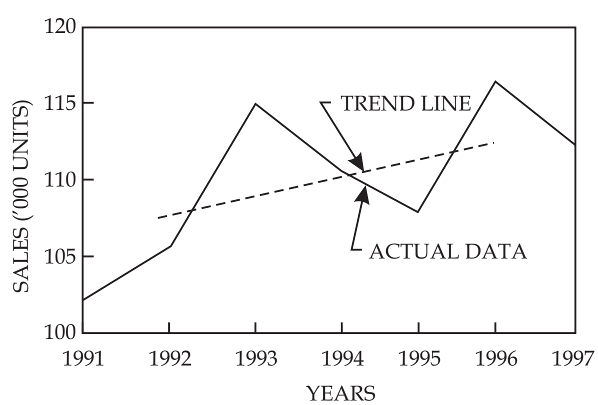
Example 3: Fit a trend line to the following data by the method of semi-averages : **Notes**

Year	Sales of Firm A (Thousand Units)
1991	102
1992	105
1993	114
1994	110
1995	108
1996	116
1997	112

Solution: Since seven years are given the middle year shall be left out and an average of first three years (1991-93) and the last three years (1995-97) shall be obtained. The average of the first three years is $\frac{102+105+114}{3} = \frac{321}{3}$ = 107 and the average of the last three years is $\frac{108+116+112}{3} = \frac{336}{3}$ = 112. Thus we get two points 107 and 112 which shall be plotted corresponding to their respective middle years, *i.e.*, 1992 and 1996. By joining these two points we shall obtain the required trend line. The line can be extended and can be used either for prediction or for determining intermediate values.

The actual data used and the trend line are also shown on the following graph :

TREND BY THE METHOD OF SEMI-AVERAGES



When there are even number of years like 6, 8, 10, etc., two equal parts can easily be formed and an average of each part obtained. However, when the average is to be centred there would be some problem in case the number of years is 8, 12, etc. For example, if the data relates to 1994, 1995, 1996 and 1997 which would be the middle year ? In such a case the average will be centered corresponding to 1^{st} July 1995, *i.e.*, middle of 1995, and 1996. The following example shall illustrate the point.

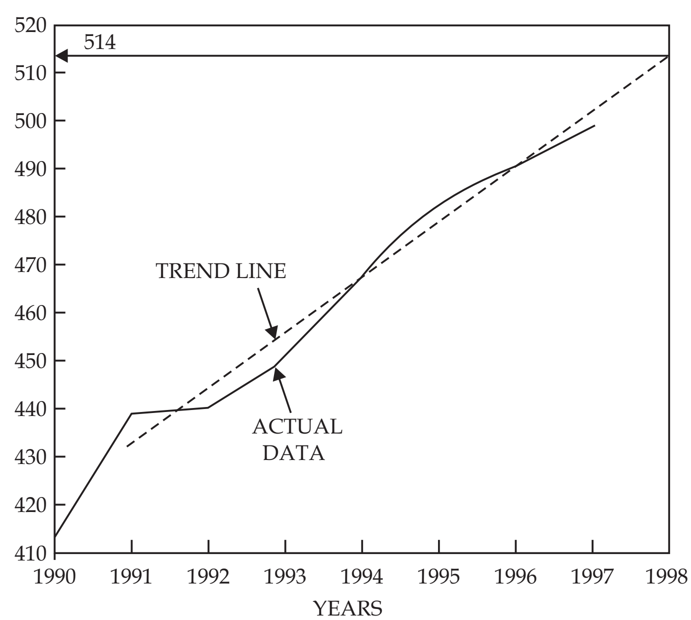
Example 4: Fit a trend line by method of semi-averages for the data given below. Estimate the population for 1998. If the actual figure for that year is 520 million, account for the difference between the two figures.

Year	Population (in million)	
1990	412	
1991	438	
1992	444	$\frac{1748}{4} = 437$
1993	454	
1994	470	
1995	482	
1996	490	$\frac{1942}{4} = 485.5$
1997	500	

Solution: The average of the first four years is 437 and that of the last four years 485.5. These two points shall be taken corresponding to the middle periods, *i.e.*, 1st July, 1991 and 1st July, 1995.

The estimate of population for 1998 by projecting the semi-average trend line is 514 million. The actual figure given to us is 520 million. The difference is due to the fact that time series analysis helps us to get the best possible estimates on certain assumptions which may come out to be true or not depending upon how far those assumptions have been realised in practice.

Trend by the Method of Semi-Averages



Example 5: The sale of a commodity in tonnes varied from January 1997 to December 1997 in the following manner :

280	300	280	280	270	240
230	230	220	200	210	200

Fit a trend line by the method of semi-averages.

Solution:

Notes

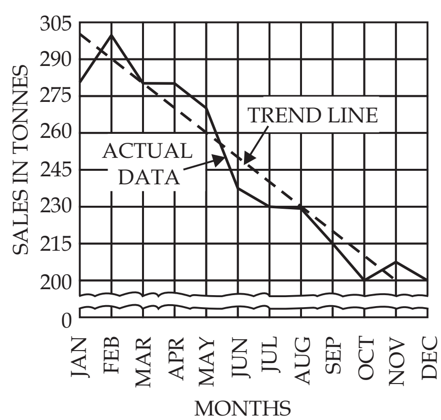
Calculation of Trend Values by the Method of Semi-Averages

Months			Sales in tonnes
January	280		
February	300		1,650, (Total of
March	280		first six months)
April	280		
May	270		
June	240		
July	230		
August	230		1,290, (Total of
September	220		last six months)
October	200		
November	210		
December	200		

Average of the first half = $\frac{1650}{6}$ = 275 tonnes.

Average of the second half = $\frac{1290}{6}$ = 215 tonnes.

These two figures, namely, 275 and 215, shall be plotted at the middle of their respective periods, *i.e.*, at the middle of March-April and that of September-October, 1997. By joining these two points we get a trend line which describes the given data.



Merits of Semi-average Method

Merits

1. This method is simple to understand compared to the moving average method and the method of least squares.
2. This is an objective method of measuring trend as everyone who applies the method is bound to get the same result (of course, leaving aside the arithmetic mistakes).

Demerits of Semi-average Method

1. **Extreme Values :** This method is based on arithmetic mean. Therefore, it is greatly affected by extreme values of the items.
2. **Straight-line Relationship :** This method assumes a straight-line relationship between the two points plotted on the graph, regardless of the fact whether such relationship exists or not.
3. **Influence of Cycle :** In this method, there is no assurance that the influence of cycle is eliminated. The danger is greater when the time period represented by average is small.

   In short, despite of above demerits, this method is definitely better than freehand curve method.

Limitations of Semi-average Method

1. This method assumes straight line relationship between the plotted points regardless of the fact whether that relationship exists or not.
2. The limitations of arithmetic average shall automatically apply. If there are extremes in either half or both halves of the series, then the trend line is not a true picture of the growth factor. This danger is greatest when the time period represented by the average is small. Consequently, trend values obtained are not precise enough for the purpose either of forecasting the future trend or of eliminating trend from original data.

   For the above reasons if the arithmetic averages of the data are to be used in estimating the secular movement, it is sometimes better to use moving averages than semi-averages.

Self-Assessment

1. Fill in the blanks:

(*i*) Graphic method is flexible and can be used for linear as well as .... trends.

(*ii*) The values of the time series are plotted on a graph paper in the form of a ....

(*iii*) In graphic method, the seasonal and cyclical and irregular variations are ....

(*iv*) Semi-average means average of the two .... of the series.

(*v*) Semi-average method is based on arithmetic ....

10.3 Summary

- Measuring a trend actually means computing the constants of the equation that we have chosen to be representative of the trend in the data. However, it should be remembered that if we choose a wrong curve for the data, then the constants of the equation computed would be wrong, and any forecasting made on the basis of this equation would be wrong.
- Freehand method is also called the graphic method in the sense that the trend line is determined by inspecting the graph of the series. According to this method, the trend values are determined by drawing freehand straight line through the time series data that is judged by the analyst to represent adequately the long-term movement in the series. Once the freehand trend line is drawn, a trend equation for the line can be approximated. This is done by first reading off the trend values of the first and the last period from the chart with reference to the freehand line. For this purpose, the first period is usually considered the origin. Thus, the trend value for the first period is the value of a for the equation. Then the difference between the trend values of the first and the last period is obtained.
- The trend line for many series may be satisfactorily drawn provided the fluctuations around the general drift are so small that the path of the trend is clearly defined. The main trouble with this method is that it is too subjective. Even an expert in this subject may draw different lines at different times for the same series. There is no formal statistical criterion whereby the adequacy of such a line can be judged. Furthermore, although this method appears simple and direct, in

actuality. it is very time consuming to construct a freehand trend curve if a careful and conscientious job is to be done. For these reasons, the freehand method is not recommended for fitting a trend line.

- This method of estimating trend is the simplest of all the methods of measuring trend. It involves no calculation at all since it is purely non-mathematical.
- This method is more flexible than rigid mathematical function, hence fits the curve more closely to the data *i.e* this method can be used even in cases where the size of the series is lengthy.
- This method lacks accuracy. Therefore, it is not suitable where a high degree of accuracy is desired. This method gives us an approximate picture of the tendency in the long run. This method should therefore be used only by experienced persons.
- This method is highly subjective because the trend line depends on the personal judgment of the investigator and, therefore, different persons may draw different trend lines from the same set of data. Moreover, the work cannot be left to clerks and it must be handled by skilled and experienced people who are well conversant with the history of the particular concern.
- Although this method seems to be quite simple, in actual practice it is very time-consuming to construct a freehand trend if a careful and conscientious job is to be done. It is only after long experience in trend fitting that a person should attempt to fit a trend line by inspection.
- Lastly, a straight line passing through these two averages is drawn to provide the trend for the series. Each average provides the trend value for the middle time period of the corresponding segment. When the time series includes an odd number of periods.
- This method of determining the trend is not a subjective one. The slope of the trend line now depends upon the difference between the averages that are computed from the original values, with each average as typical of the level of that segment of the data. However, this method is not entirely free from drawbacks. The major drawback here is due to the arithmetic mean which can be unduly affected by the extreme values in the series. If one part of the series contains more depressions or fewer prosperities than the other, then the trend line is not a true representation of the secular movements of the series. Therefore, the trend values obtained by this method are not accurate enough for the purpose either of forecasting the future trend or of eliminating the trend from the original data.
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- This method is simple to understand compared to the moving average method and the method of least squares.
- This is an objective method of measuring trend as everyone who applies the method is bound to get the same result (of course, leaving aside the arithmetic mistakes).
- This method assumes a straight-line relationship between the two points plotted on the graph, regardless of the fact whether such relationship exists or not.
- In this method, there is no assurance that the influence of cycle is eliminated. The danger is greater when the time period represented by average is small.

10.4 Key-Words

1. Measures of location : Another term for measures of central tendency.
2. Median (Med) : The score corresponding to the point having 50% of the observations below it when observations are arranged in numerical order.
3. Median location : The location of the median in an ordered series.

10.5 Review Questions

1. Explain briefly the graphic method for determining the trend.
2. What are the merits and demerits of graphic method ?
3. Describe the Semi-average method. How the method of Semi-averages help analysing a Time Series.
4. What are the limitations of graphic method ?
5. Explain the Semi-average method of determing trend.

Answers: Self-Assessment

1. (*i*) non-linear (*ii*) histogram (*iii*) eliminated
   (*iv*) halves (*v*) mean

10.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 11: Time Series Methods—Principle of Least Square and Its Application

Notes

CONTENTS

Objectives

After reading this unit students will be able to:

- Explain the Principle of Least Square and its Applications.
- Know Merits, Demerits and Limitations of the Method of Least Square.

Introduction

The earlier discussed methods for trend analysis have certain defects, particularly in providing a satisfactory projection for the future. To overcome this defect, a convenient method is to follow a mathematical approach. The device for getting an objective fit of a straight line to a series of data is the least squares method. It is perhaps the most commonly employed and a very satisfactory method to describe the trend. The Mark off theorem states that for a given condition, the line fitted by the method of least squares is the line of "best" fit in a well-defined sense. The term "best" is used to mean that the estimates of the constants *a* and *b* are the best linear unbiased estimates of those constants.

11.1 Principle of Least Square and its Application

In the least squares method, the sum of the vertical deviations of the observed values from the fitted straight line is zero. Secondly, the sum of the squares of all these deviations is less than the sum of the squared vertical deviations from any other straight line. The method of least squares can be used for fitting linear and non-linear trends as well.

To determine the values of *a* and *b* in a linear equation by the least squares method, we are required to solve the following two normal equations simultaneously:

$$\sum Y = an + b\sum X$$

$$\sum XY = a\sum X + b\sum X^2$$

In the case of time series analysis, the solution of *a* and *b* from these two equations is simplified by using the middle of the series as the origin. Since the time units in a series are usually of uniform duration and are consecutive numbers, when the middle point is taken as the origin, the sum of time units, *i.e.*, $\sum X$, will be zero. As a result, the above two normal equations reduce to:

$$\sum Y = an$$

$$\sum XY = b\sum X^2$$

Notes

Therefore, we can get

$$a = \frac{\sum Y}{n} \quad \text{... (1)}$$

$$b = \frac{\sum XY}{\sum X^2} \quad \text{... (2)}$$

From the above expressions, we can state that *a*, the value of Y at the origin, is the arithmetic mean of the Y variable. The value of *b*, of course, is the average amount of change in the trend values per unit of time.

It should be noted that in computing the trend it is convenient to use the middle of the series as the origin. If the series contains an odd number of periods, the origin is the middle of the given period. If an even number of periods is involved, the origin is set between the two middle periods.

It is important to recognize that the least squares technique requires that the type of line desired be specified. Once this has been done, the technique generates the line of best fit of that type, that is, the least squares line. Thus, for any set of data, one could generate: (a) a least squares straight line, (b) a geometric least squares straight line, and (c) a least squares second-degree parabola. If one were to compute each of the above and determine which of the three has the least sum of squared deviations, the least squares criterion could also be used to indicate which type of line provides the best fit.

In case of a straight line trend, for any set of data, it is sufficient to compute $\sum Y$, $\sum X$, $\sum XY$, and $\sum X^2$. We shall generate these values for the data in Example 1.

Example 1: Fit a linear trend to the data given in Table 24.1. In this Example, the middle point, *i.e.*, the year 1973, is taken as the origin, thus simplifying our computations. Therefore, the coefficients for the least squares straight line are:

$$a = \frac{\sum Y}{n} = \frac{206.53}{11} = 18.78$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{-23.21}{110} = -0.21$$

Table 24.1: Annual Sales of Electronic Calculators by an Indian Manufacturer

(Rs. in lakhs)

Year	Y	X	X^2	XY
1968	20.15	– 5	25	– 100.75
1969	19.49	– 4	16	– 77.96
1970	19.41	– 3	9	– 58.23
1971	19.54	– 2	4	– 39.08
1972	18.74	– 1	1	– 18.74
1973	18.00	0	0	0
1974	18.44	1	1	18.44
1975	18.81	2	4	37.62
1976	18.29	3	9	54.87
1977	17.68	4	16	70.72
1978	17.98	5	25	89.90
Total	**206.53**	**0**	**110**	**– 23.21**

To obtain the value of the trend line for any given period, we simply substitute the value of X for that year in the equation $Y_{cal} = 18.78 - 0.21\,X$, where Y_{cal} is the calculated value of Y. As an example, for the year 1973, X = 0.

Thus

$$Y_{cal}\,(1973) = 18.78 - 0.21 \times 0$$
$$= 18.78$$

As has been stated earlier, our interest in trend analysis is not only confined to determining the growth pattern of a series in the past but also concerned with forecasting future trend values. To forecast the trend values, we use the same technique that was used to compute the trend values for the periods covered by the series. For example, if we are interested in forecasting the trend value of annual sale of electronic calculators for 1980, we observe that 1980 is 7 years ahead of the origin of the trend equation we have established above, and thus

$$Y_{est}\,(1980) = 18.78 - 0.21 \times 7$$
$$= 17.31$$

We may, thus, state that, under the assumption that the same trend factors that produced the trend equation for 1968 and 1978 will remain operative, the average annual sales of electronic calculators by an Indian manufacturer will have a trend value of Rs. 17.31 lakhs during 1980.

> *Notes* The straight line trend studied earlier is appropriate when there is reason to believe that the time series is changing, on the average, by equal absolute amounts in each time period.

However, in many situations the growth is such that the absolute amounts of the Y variable increases more rapidly in later time periods than in earlier ones. When this is the case, if a trend is to be fitted to the original data on the natural graph, a curvilinear instead of the simple linear description would be necessary. However, the same data when plotted on a semilogarithmic scale, may very often reveal linear average relationship. In other words, a straight line would seem to describe the trend of the series plotted on a semilogarithmic chart, such a growth pattern in the series can be expressed by a geometric straight line.

To fit a geometric trend line, trend values are computed from the logarithms of the data (Y) instead of the original data. Thus, the trend values obtained by this method will be logarithms of trend values and could be related to logarithms of the data. However, in practice, it is more useful to take the antilogs which will give the trend values in natural numbers and that can be compared well with the original values of the series.

In fitting a straight line to the logarithms of the data by the least squares method, the procedure used is identical with that for an arithmetic straight line studied earlier. The only difference is that logarithms of the original data instead of the natural numbers are used throughout. The normal equations now are:

$$\Sigma(\log Y) = n \log a + \log b\,(\Sigma X)$$

$$\Sigma(X \log Y) = \log a(\Sigma X) + \log b\left(\Sigma X^2\right)$$

By setting the origin at the middle of the series, the formulae for the Y intercept and the slope become

$$\log a = \frac{\Sigma \log Y}{n} \qquad \text{... (3)}$$

$$\log b = \frac{\sum X \log Y}{\sum X^2} \qquad \text{... (4)}$$

With the values of *a* and *b*, the logarithmic straight line equation can now be written as

$$\log Y = \log a + X \log b \qquad \text{... (5)}$$

The property of *b* for a geometric straight line is of considerable significance. When it has been converted into a pure percentage it defines the annual rate of growth or decline of the series. Being an abstract measure, it allows comparison of trends all described by straight lines on ratio paper, of series with different original units. Series, such as production of all kinds, population or national income, become immediately comparable, and conclusions about the direction and magnitude of economic activities can be easily drawn. This measure provides an effective device for the study of the socio-economic changes in a country.

Did u know? Mathematical curves are useful to describe the general movement of a time series, but it is doubtful whether any analytical significance should be attached to them, except in special cases.

So far we have studied the method of fitting an arithmetic or geometric straight line to a time series. But many time series are best described by curves, and not by straight lines. In such situations, the linear model does not adequately describe the change in the variable as time changes. To study such cases, we often use a parabolic curve, which is described mathematically by a second degree equation. Such a curve is given in Figure 1. The general mathematical form of the second-degree parabolic trend is

$$Y = a + bX + cX^2 \qquad \text{... (6)}$$

where *a* is still the Y intercept, *b* is the slope of the curve at the origin, and *c* is the rate of change in the slope. It should be noted that, just as *b* is a constant in the first-degree curve *c* is a constant in the second-degree curve.

The *c* value determines whether the curve is concave or convex and the extent to which the curve departs from linearity.

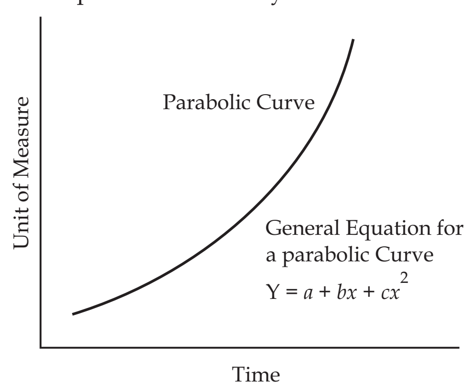


Figure 1: Form and Equation for a Parabolic Curve.

We use the least squares method to determine the second-degree equation to describe the best fit. The three normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^2$$

$$\sum X^2 Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

Notes

If the middle of the time series is taken as origin, as done earlier, the above normal equations reduce to

$$\sum Y = na + c\sum X^2$$

$$\sum XY = b\sum X^2$$

$$\sum X^2Y = a\sum X^2 + c\sum X^4$$

Simplifying, we get:

$$c = \frac{n\sum X^2Y - \sum X^2 \sum Y}{n\sum X^4 - \left(\sum X^2\right)^2} \quad \text{... (7)}$$

$$a = \frac{\sum Y - c\sum X^2}{n} \quad \text{... (8)}$$

$$b = \frac{\sum XY}{\sum X^2} \quad \text{... (9)}$$

Example 2: Consider India's exports of engineering goods during the years 1980 to 1986 given in Table 24.2. We shall fit a parabolic trend to describe the exports of engineering goods.

Table 24.2: India's Exports of Engineering Goods

(in crores of rupees)

Year	Exports Y	X	X^2	X^4	XY	X^2Y	Y_{cal}
1980	116.6	– 3	9	81	– 349.8	1049.4	120.28
1981	126.0	– 2	4	16	– 252.0	504.0	112.71
1982	130.0	– 1	1	1	– 130.0	130.0	137.10
1983	176.0	0	0	0	0	0.0	193.45
1984	299.0	1	1	1	299.0	299.0	281.76
1985	404.0	2	4	16	808.0	1616.0	402.03
1986	550.0	3	9	81	1650.0	4950.0	554.26
Total	**1801.6**	**0**	**28**	**196**	**2025.2**	**8548.4**	

Substituting the values from Table 24.2 into expressions for *a*, *b* and *c*, we get

$$c = \frac{7 \times 8548.4 - 28 \times 1801.6}{7 \times 196 - 28 \times 28} = 15.98$$

$$a = \frac{1801.6 - 15.98 \times 28}{7} = 193.45$$

$$b = \frac{2025.2}{28} = 72.33$$

With these values, the trend equation becomes

$$Y_{cal} = 193.45 + 72.33X + 15.98X^2$$

The trend values are obtained by substituting the X's and X^2's into the trend equation. These values have been given in the last column of Table 24.2.

Suppose we want to forecast India's exports of engineering goods for the year 1989, we observe that 1989 is 6 years ahead of the origin for the equation established above. Thus, when this value of X (= 6), is substituted into the second degree equation, we get

$$Y_{cal}\ (1989) = 193.45 + 72.33\ X + 15.98\ X^2$$
$$= 193.45 + 72.33\ (6) + 15.98\ (6)^2$$
$$= 1202.70$$

Based upon the past trend, we can conclude that India's exports of engineering goods during 1989 would be Rs. 1202.70 crores. This extra-ordinarily large forecast suggests, however, that we must be more careful in forecasting with a parabolic curve than when using a linear trend. The slope of the second degree equation is continually increasing. Therefore, the parabolic curve may become a poor estimator as we attempt to predict further into the future.

Example 3: Below are given the figures of production in (thousand quintals) of a sugar factory:

Year:	1992	1993	1994	1995	1996	1997	1998
Production (in '000 quintals):	80	90	92	83	94	99	92

(i) Fit a straight line trend to these figures.

(ii) Plot these figures on a graph and show the trend line.

(iii) Estimate the production in 2001.

Solution:

(i) FITTING THE STRAIGHT LINE TREND

Year X	**Production ('000 *qtls.*) Y**	**X**	**XY**	**X^2**	**Trend values Y_c**
1992	80	– 3	– 240	9	84
1993	90	– 2	– 180	4	86
1994	92	– 1	– 92	1	88
1995	83	0	0	0	90
1996	94	+ 1	+ 94	1	92
1997	99	+ 2	+ 198	4	94
1998	92	+ 3	+ 276	9	96
N = 7	$\Sigma Y = 630$	$\Sigma X = 0$	$\Sigma XY = 56$	$\Sigma X^2 = 28$	$\Sigma Y_c = 630$

The equation of the straight line trend is

$$Y_c = a + bX$$

Since $\Sigma X = 0$

$$a = \frac{\Sigma Y}{N},\ b = \frac{\Sigma XY}{\Sigma X^2}$$

Here $\Sigma Y = 630$, N = 7, $\Sigma XY = 56$, $\Sigma X_2 = 28$,

$\therefore \qquad a = \frac{630}{7} = 90$

and $\qquad b = \frac{56}{28} = 2$

Hence the equation of the straight line trend is

$Y_c = 90 + 2X$

For $\quad X = -3, Y_c = 90 + 2(-3) = 84$

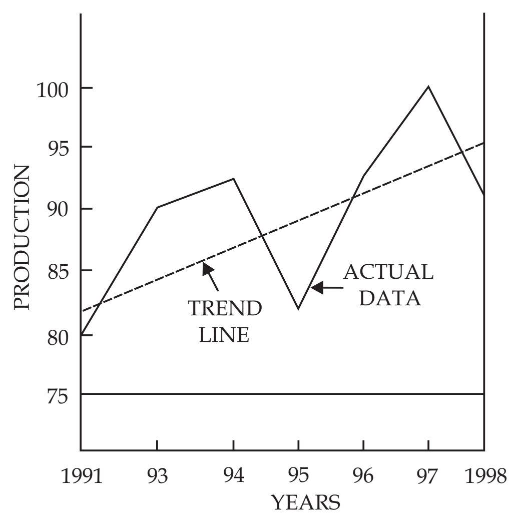
For $\quad X = -2, Y_c = 90 + 2(-2) = 86$

For $\quad X = -1, Y_c = 90 + 2(-1) = 88$

Similarly, by putting X = 0, 1, 2, 3, we can obtain other trend values. However, since the value of *b* is constant, only first trend value need be obtained and then if the value of *b* is positive we may continue adding the value of *b* to every preceding value. For example, in the above case for 1992 the calculated value of Y is 84. For 1993 it will be 84 + 2 = 86; for 1994 it will be 86 + 2 = 88, and so on. If *b* is negative then instead of adding we will deduct.

(ii) The graph of the above data is given below.

Linear Trend by the Method of Least Squares



For 2001 X would be + 6

$Y_{2001} = 90 + 2(6) = 102$ thousand quintals.

Example 4: Apply the method of least squares to obtain the trend value from the following data and show that $\sum(Y - Y_c) = 0$:

Year	Sales (in lakh tonnes)
1993	100
1994	120
1995	110
1996	140
1997	80

Also predict the sales for the year 1999.

Notes

Solution:

Calculation of Trend Values by the Method of Least Squares

Year	Sales	Deviations from middle year X	XY	X^2	Y_c	$(Y - Y_c)$
1993	100	– 2	– 200	4	114	– 14
1994	120	– 1	– 120	1	112	+ 8
1995	110	0	0	0	110	0
1996	140	+ 1	+ 140	1	108	+ 32
1997	80	+ 2	+ 160	4	106	– 26
N = 5	$\sum Y = 550$	$\sum X = 0$	$\sum XY = -20$	$\sum X^2 = 10$		$\sum(Y - Y_c) = 0$

1992 93 94 96 97 1998

The equation of the straight line trend is

$$Y_c = a + bX$$

Since $\sum X = 0$,

$$a = \frac{\sum Y}{N}, \; b = \frac{\sum XY}{\sum X^2}$$

$$\sum Y = 550, N = 5, \sum XY = -20, \sum X^2 = 10.$$

Substituting the values

$$a = \frac{550}{5} = 110$$

$$b = -\frac{20}{10} = -2$$

The required equation is

For 1993, *i.e.*, X = – 2, Y = 110 – 2 (– 2) = 114.

Since *b* is negative the other trend values will be obtained by *deducting* the value of *b* from the preceding value. Thus for 1994 the trend value will be 114 – 2 = 112 (since the value of *b* is negative). For 1999 likely sales = 100 lakh tonnes (since X would be + 5 for 1999).

Example 5: Fit a straight line trend by the method of least squares to the following data. Assuming that the same rate of change continues. What would be the predicted earnings for the year 1998 ?

Year	1987	1988	1989	1990	1991	1992	1993	1994
Earnings (Rs. lakhs.)	38	40	65	72	69	60	87	95

Do not plot the trend values on the graph.

Solution: **Notes**

Fitting of Straight Line Trend by the Method of Least Squares

Year	Earnings (Rs. Lakhs) Y	Deviations from 1990-5	Deviations multiplied by 2 X	XY	X^2
1987	38	– 3.5	– 7	– 266	49
1988	40	– 2.5	– 5	– 200	25
1989	65	– 1.5	– 3	– 195	9
1990	72	– 0.5	1	– 72	1
1991	69	+ 0.5	+ 1	+ 69	1
1992	60	+ 1.5	+ 3	+ 180	9
1993	87	+ 2.5	+ 5	+ 435	25
1994	95	+ 3.5	+ 7	+ 665	49
N = 8	$\Sigma Y = 526$		$\Sigma X = 0$	$\Sigma XY = 616$	$\Sigma X^2 = 168$

$$Y_c = a + bX$$

$$a = \frac{\Sigma Y}{N} = \frac{526}{8} = 65.75$$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{616}{168} = 3.67$$

$$Y = 65.75 + 3.67\,X.$$

For 1998 X will be + 15

When X is + 15, Y will be

$$Y = 65.75 + 3.67\,(15)$$
$$= 65.75 + 55.05 = 120.8.$$

Thus the estimated earnings for the year 1998 are Rs. 120.8 lakhs.

The same result will be obtained if we do not multiply the deviations by 2. But in that case our computations would be more difficult as would be seen below

Year	Sales in thousands of rupees Y	Deviations from 1991-5 X	XY	X^2
1987	38	– 3.5	– 133.00	12.25
1988	40	– 2.5	– 100.00	6.25
1989	65	– 1.5	– 97.50	2.25
1990	72	– 0.5	– 36.00	0.25
1991	69	+ 0.5	+ 34.50	0.25
1992	60	+ 1.5	+ 90.00	2.25
1993	87	+ .5	+ 217.50	6.25
1994	95	+ 3.5	+ 332.50	12.25
N = 8	$\Sigma Y = 526$	$\Sigma X = 0$	$\Sigma XY = 308$	$\Sigma X^2 = 42.00$

$$a = \frac{\sum Y}{N} = \frac{526}{8} = 65.75$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{308}{42} = 7.33$$

The advantage of this method is that the value of *b* gives annual increment of charge rather than 6 monthly increment, as in the first method discussed above. Hence, we will not have to double the value of *b* to obtain yearly increment. It is clear from the above illustration that in the first case the value of *b* is half of what we obtain from the second method. (*b*) was 3.67 in the first case and 7.33 in the second case.

11.2 Merits, Demerits and Limitations of the Method of Least Squares

Merits

1. This is a mathematical method of measuring trend and hence there is no possibility of subjectiveness.
2. The line obtained by this method is called *the line of best fit* because it is this line from where the sum of the positive and negative deviations is zero and sum of the squares of the deviations least, *i.e.*, $(Y - Y_c) = 0$ and $(Y - Y_c)^2$ least.

Demerits

Mathematical curves are useful to describe the general movement of a time series, but it is doubtful whether any analytical significance should be attached to them, except in special cases. It is seldom possible to justify on theoretical grounds any real dependence of a variable on the passage of time. Variables do change in a more or less systematic manner over time, but this can usually be attributed to the operation of other explanatory variables. Thus many economic time series show persistent upward trends over time due to a growth of population or to a general rise in prices, *i.e.*, national income and the trend element can to a considerable extent be eliminated by expressing these series per capita or in terms of constant purchasing power. For these reasons mathematical trends are generally best regarded as tools for describing movements in time series rather than as theories of the causes of such movements that follow, that it is extremely dangerous to use trends forecast future movements of a time series. Such forecasting, involving as it does extrapolation, can be valid only if there is theoretical justification for the particular trend as an expression of a functional relationship between the variable under consideration and the time. But if the trend is purely descriptive of past behaviour, it can give few clues about future behaviour. Sometimes the projection of a trend leads to absurd results which is *prima facie* evidence that the trend could not be maintained.

Hence, mathematical methods of fitting trend are not foolproof. In fact, they can be the source of some of the most serious errors that are made in statistical work. They should never be used unless rigidly controlled by a separate logical analysis. Trend fitting depends upon the judgement of the statistician, and a skilfully made free-hand sketch is often more practical than a refined mathematical formula.

Self-Assessment

1. Gompertz curve is a .... curve which denoted as ....
2. Equation for non-linear curve is ....
3. The two normal equations to calculate the values of '*a*' and '*b*' in Y = *a* + *b*X are .... and ....
4. A polynomial of the form $Y = a + bY + CX^2$ is called a ....
5. The line obtained by method of least squares is known as the line of ....

11.3 Summary

Notes

- The device for getting an objective fit of a straight line to a series of data is the least squares method. It is perhaps the most commonly employed and a very satisfactory method to describe the trend. The Mark off theorem states that for a given condition, the line fitted by the method of least squares is the line of "best" fit in a well-defined sense. The term "best" is used to mean that the estimates of the constants *a* and *b* are the best linear unbiased estimates of those constants.
- In the least squares method, the sum of the vertical deviations of the observed values from the fitted straight line is zero. Secondly, the sum of the squares of all these deviations is less than the sum of the squared vertical deviations from any other straight line. The method of least squares can be used for fitting linear and non-linear trends as well.
- It should be noted that in computing the trend it is convenient to use the middle of the series as the origin. If the series contains an odd number of periods, the origin is the middle of the given period. If an even number of periods is involved, the origin is set between the two middle periods.
- It is important to recognize that the least squares technique requires that the type of line desired be specified. Once this has been done, the technique generates the line of best fit of that type, that is, the least squares line.
- The line obtained by this method is called *the line of best fit* because it is this line from where the sum of the positive and negative deviations is zero and sum of the squares of the deviations least, *i.e.*, $(Y - Y_c) = 0$ and $(Y - Y_c)^2$ least.
- It is seldom possible to justify on theoretical grounds any real dependence of a variable on the passage of time. Variables do change in a more or less systematic manner over time, but this can usually be attributed to the operation of other explanatory variables. Thus many economic time series show persistent upward trends over time due to a growth of population or to a general rise in prices, *i.e.*, national income and the trend element can to a considerable extent be eliminated by expressing these series per capita or in terms of constant purchasing power.
- Hence, mathematical methods of fitting trend are not foolproof. In fact, they can be the source of some of the most serious errors that are made in statistical work. They should never be used unless rigidly controlled by a separate logical analysis. Trend fitting depends upon the judgement of the statistician, and a skilfully made free-hand sketch is often more practical than a refined mathematical formula.

11.4 Key-Words

1. Homogeneity of regression : The assumption that the regression line expressing the dependent variable as a function of a covariate is constant across several groups or conditions.
2. Homogeneity of variance : The situation in which two or more populations have equal variances.

11.5 Review Questions

1. Discuss the method of least squares for the measurement of trend.
2. Write the normal equations to determine the values of *a* and *b* in the trend equation $y = a + bx$, given the *n* observations.
3. Explain the principle of least square method and its application.
4. What are the limitations of least square method ?
5. Discuss the merits and demerits of least square method.

Answers: Self-Assessment

1. (*i*) $Y_c = a + bx + cx^2$ (*ii*) $Y = a + bx$

   (*iii*) $\sum Y = Na + b\sum X$ and $\sum XY = a\sum X + b\sum X^2$

   (*iv*) halves (*v*) Best fit

11.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 12 : Methods of Moving Averages

CONTENTS

Objectives

After reading this unit students will be able to :

- Discuss the Methods of Moving Averages.
- Know Merits, Demerits and Limitations of Moving Average Method.

Introduction

When a trend is to be determined by the method of moving averages, the average value for a number of years (or month or weeks) is secured, and this average is taken as the normal or trend value for the unit of time folling at the middle of the period covered in the calculation of the average.

This method may be considered as an artificially constructed time series in which each periodic figure is replaced by the mean of the value of that period and those of a number of preceding and succeeding periods. The computation of moving averages is simple and straight-forward. The properties and the utility of discussed below.

12.1 Method of Moving Averages

Moving average method is quite simple and is used for smoothing the fluctuations in curves. The trend values obtained by this method are very much accurate. Like semi-average method, this method also employs arithmetic means of items. But here we find out the moving averages from the time series. A moving average of a time series is a new series obtained by finding out successively the average of a number of the original successive items choosen on the basis of periodicity of fluctuations, dropping off one item and adding on the next at each stage.

The moving average may be for three, four, five, six, seven, ... years and so on according to the size and the periodicity of fluctuations of the data. Suppose moving average is to be calculated for three years. We will take the average of first three years and will place it against the middle year of the three. Now leave the first item and add the next item of the series and take the average of these items and place it against the middle year of the three. We will go on in this way, taking the average after leaving one preceeding year, till the end. The formula for calculating moving averages is 3-yearly moving average.

$$\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{c+d+e}{3}, \frac{d+e+f}{3}, \ldots$$

Example 1 : The data given in Table 25.1 refers to a hypothetical data assumed to have a uniform cyclical duration of 5 years and equal amplitude of 2 units. Three-year and five-year moving averages are fitted to the data. The procedure for calculating three-year moving averages is explained below :

(a) Compute the three-year moving totals. This is done by adding up the values of the first three years and centering it at the second year. This is the first three-year moving total. Then the first year value is deleted and fourth year value is included to form the second three-year moving total, which is centred at the third year. In a similar way, the computation moves through the end of the series. The three-year moving totals are entered in column 3 of Table 1.

Table 1 : Computation of Three-year and Five-year Moving Averages for the Hypothetical Data

Year (1)	Original value (2)	Three-year moving total (3)	Three-year moving average (4)	Five-year moving total (5)	Five-year moving average (6)
1	3				
2	4	12	4.0		
3	5	13	4.3	19	3.8
4	4	12	4.0	20	4.0
5	3	11	3.7	21	4.2
6	4	12	4.0	22	4.4
7	5	15	5.0	23	4.6
8	6	16	5.3	24	4.8
9	5	15	5.0	25	5.0
10	4	14	4.7	26	5.2
11	5	15	5.0	27	5.4
12	6	18	6.0	28	5.6
13	7	19	6.3	29	5.8
14	6	18	6.0	30	6.0
15	5	17	5.7	31	6.2
16	6	18	6.0	32	6.4
17	7	21	7.0	33	6.6
18	8	22	7.3	34	6.8
19	7	21	7.0		
20	6				

(b) The three-year moving averages are obtained by dividing each of the three-year moving totals by 3. These values for our hypothetical example is given in column 4 of Table 1.

A similar procedure is used to compute the five-year moving averages. In a five-year moving average, the value of each year is replaced by the mean of the value of the five successive years of which two precede and two succeed the given year. Both five-year moving totals and moving averages are centred in the middle of the respective five-year periods, with the first five-year moving total and the moving average entered in the third year. It is to be noted that in computing the moving averages for an even number of periods, the procedure is slightly more complicated. For example, the calculation of a 12-period (year or month) moving, average starts with adding up the first 12-period values in the series to form a 12-period moving total. The second moving

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total is obtained by dropping the value of the first period from, and adding the value of the thirteenth period to the first moving total, and so on until all the moving total have been obtained from the series. Then each moving total is divided by 12 to get the 12-period moving averages. The moving totals and moving averages so obtained fall in between two periods. However, data that are typical of a period should be centred at the middle of the period. Thus, to compute moving averages when an even number of time units is used, an additional step is required to centre the averages at the middle of each time unit. For a 12-period moving average, centred averages are obtained by adding the two averages at a time and dividing each sum by 2. Thus, the first centred 12-period moving average would fall on the seventh period of the series, the second on the eighth period, and so on.

The results of the computations made earlier are plotted with the original data in Figure 1. Both the sets of moving averages may be considered as the statistical expression of secular trend of our hypothetical data.

With the help of Table 25.1 and Figure 1, we can make the following conclusions about the characteristics of moving average :

1. A moving average of equal length period will completely eliminate the periodic fluctuations.
2. A moving average of equal length will be linear if the series changes on the average by a constant per time unit and its fluctuations are periodic.
3. Even when the data show periodic fluctuations, a moving average of unequal length period, no matter how small the difference is between the duration of periodicity of original series, and the length of the moving average, the moving average cannot completely remove the periodic variations in the original series. The averaging process then only tends to smooth out somewhat the short-run highs and lows.

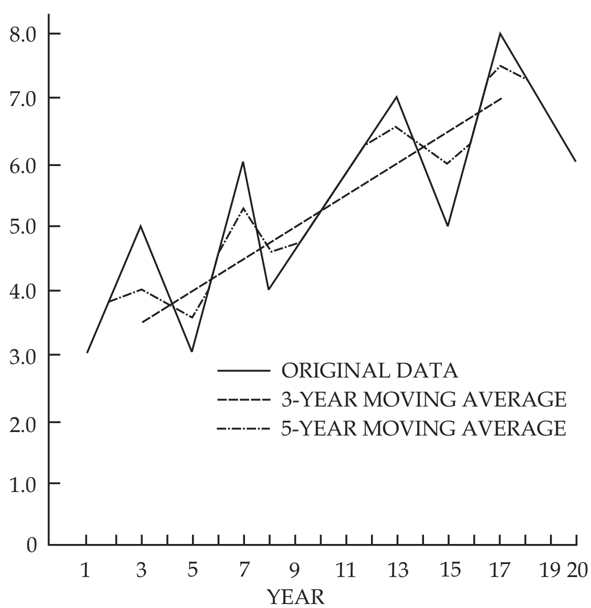

Figure 1 : Three-year and Five-year Moving Average Trends Fitted to Data in Table 1

Thus, we can see that the moving average may constitute a satisfactory trend for a series that is basically linear and that is regular in duration and amplitude. Further, there are two other disadvantages in the use of the moving average as a measure of trend. First, in computing moving averages, we loose some years at the beginning and end of the series. The second drawback is that the moving average is not represented by some mathematical formulae and, therefore, is not capable of objective future projections. Since one of the major objectives of trend analysis is that of forecasting, the moving average is no longer in wide use as a trend measure. However, the method of moving averages is a very useful technique in analyzing a time series data. First of all, in problems in which

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the trend of the time series is clearly not linear and in which we are concerned only with the general movements of the time series, whether it is a trend or a cycle or both, it is customary to study the smoothing behaviour of the series by the use of moving average. Secondly, the characteristic of a moving average is the basis of seasonal analysis, which shall be discussed in a subsequent section.

Selections of Period in Moving Average Method

The most important point in the average method is the selection of period. The selection of period depends upon the periodicity of data. The period of moving average method can be divided into two parts :

(A) **Odd Period of Moving Average :** Odd period is the period of three, five, seven, nine years and so on. In case of odd period moving average, no difficulty is faced while placing the computed average. The determination of trend in odd period moving average can be made clear with the help of following example :

Example 2 : Calculate trend values by 3-yearly moving average from the following data :

Year :	1980	1981	1982	1983	1984	1985	1986
Sales (‘000 Units) :	5	7	9	12	11	10	8

Solution :

Years	**Sales (000’ Units)**	**3-yearly totals**	**3-yearly moving average**
1980	5	–	–
1981	7	21	21/3 = 7.00
1982	9	28	28/3 = 9.33
1983	12	32	32/3 = 10.67
1984	11	33	33/3 = 11.00
1985	10	29	29/3 = 9.67
1986	8	–	–

The trend value is shown in Figure 2.

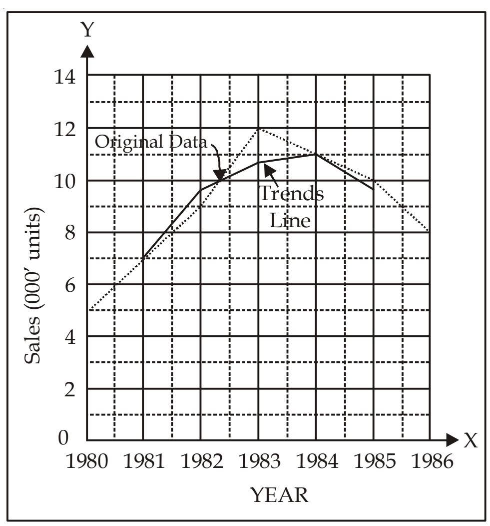

Figure 2 :

(B) **Even Period of Moving Average :** The procedure of calculation of moving average of even number of years, say, four, six, eight, and so on, is different from the procedure of odd number of years. Suppose moving average is to be calculated for four years. We will take the total of

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first four years and will be placed in between second and third years, *i.e.*, in the middle of four years. Leaving the first year, calculate the total of next four years and so on. It is important to note that these calculated totals are placed between two years. Then we adjust these moving totals. For this we compute two yearly moving totals of four yearly moving totals. We take the total of first and second four years moving totals and write against the third year and then the second and third four yearly moving totals are totalled and written against the fourth year and so on. The two yearly moving totals of four yearly moving totals are then divided by eight and this gives us the centered four yearly moving average. The series so obtained is known as an estimate of the trend.

Did u know? Moving Averages method is that it is difficult to determine the proper period of moving average. If a wrong period is selected, there is every likelihood that conclusions may be misleading.

Characteristics of Moving Averages

The main characteristics of moving averages are as under :

(i) If the original data, when plotted on a graph, give a straight line, the moving averages will simply reproduce the original line.

(ii) If the original series gives a curve which is concave, the moving average curve will be below it.

(iii) If the original series gives a convex curve, the moving average curve will be above it.

(iv) In a series having regular fluctuations, the moving average completely eliminates them, if the period selected for it coincides with the period when the fluctuations repeat themselves.

(v) No particular period of a moving average will eliminate the fluctuations completely. But greater the period, the greater will be the reduction in the irregular fluctuations. Because the duration of business cycles always remain changing.

12.2 Merits, Demerits and Limitations of Moving Average Method

Merits of Moving Average Method

(i) **Simple and Easy :** This method is quite simple and easy to calculate as compared with all the mathematical methods of fitting a trend.

(ii) **Flexible :** This method is highly flexible in the sense that if a few more figures are added to the series, entire calculations are not changed. Only thing, we have to do is to extend the process so as to calculate further trend values.

(iii) **High Degree of Accuracy :** This method is associated with a high degree of accuracy and it can be made the basis for further analysis of time series.

(iv) **Automatic Elimination of Fluctuations :** If and when the period of moving average is equivalent to the period of the fluctuations, the cyclical fluctuations are completely eliminated.

(v) **Irregular Trend :** This method is considered to be the most effective method if the trend of a series is very irregular.

(vi) **Practical Method :** This method is most commonly used and in a very long series, it is the only practical method.

Demerits of Moving Average Method

(i) **Period of Moving Average :** The main limitation or demerit of this method is that it is difficult to determine the proper period of moving average. If a wrong period is selected, there is every likelihood that conclusions may be misleading.

(ii) Ignores the Extreme Figures : This method ignores the extreme figures *i.e*. figures for first and last few years. Then the trend value for all the years cannot be computed. Hence, perfectly accurate trend line cannot be drawn.

(iii) Extreme Values : In this method, moving averages are calculated by using the arithmetic average. Thus, like arithmetic average it is also affected by extreme values of the series.

(iv) Non-linear Trend : If the basic trend in the data is not linear this method will produce a bias in the trend. According to **Wough**, "*If the trend line is concave downward (like the side of a bowl), the value of the moving average will always be too high; if the trend line is concave downward (like the side of a derby pot), the value of the moving average will always be too low.*"

(v) Forecasting : Since moving average is not represented by a mathematical function, this method is of little use in forecasting. Thus, it does not fulfil the basic objective of trend analysis.

(vi) Conditions : There are certain conditions for the use of this method. But these conditions are seldom met.

In short, despite of above cited demerits, it is becoming popular in the analysis of seasonal variations.

Limitations of Moving Average

1. Trend values cannot be computed for all the years. The longer the period of moving average, the greater the number of years for which trend values cannot be obtained. For example, in a three-yearly moving average, trend value cannot be obtained for the first year and last year, in a five-yearly moving average for the first two years and the last two years, and so on. It is often these extreme years in which we are most interested.
2. Great care has to be exercised in selecting the period of moving average. No hard and fast rules are available for the choice of the period and one has to use his own judgment.
3. Since the moving average is not represented by a mathematical function this method cannot be used in forecasting which is one of the main objectives of trend analysis.
4. Although theoretically we say that if the period of moving average happens to coincide with the period of cycle, the cyclical fluctuations are completely eliminated, but in practice. Since the cycles are by no means perfectly periodic, the lengths of the various cycles in any given series will usually vary considerably and, therefore, no moving average can completely remove the cycle. The best results would be obtained by a moving average whose period is equal to the average length of all the cycles in the given series. However, it is difficult to determine the average length of the cycle until the cycles are isolated from the series.
5. Finally, when the trend situation is not linear (a straight line) the moving average lies either above or below the true sweep of the data. Consequently, the moving average is appropriate for trend computations only when :
   (a) the purpose of investigation does not call for current analysis or forecasting.
   (b) the trend is linear, and
   (c) the cyclical variations are regular both in period and amplitudes.

Self-Assessment

Fill in the blanks–

1. The period of moving average is to be decided in the light of the .... of the cycle.
2. Moving average method is used for smoothing the .... in the curve.
3. The formula for calculating moving averages is .... moving average.
4. Moving average method is .... as compared to the method of least squares.
5. Moving averages are calculated by using the .... average.

12.3 Summary

Notes

This method may be considered as an artificially constructed time series in which each periodic figure is replaced by the mean of the value of that period and those of a number of preceding and succeeding periods. The computation of moving averages is simple and straight-forward.

- Moving average method is quite simple and is used for smoothing the fluctuations in curves. The trend values obtained by this method are very much accurate. Like semi-average method, this method also employs arithmetic means of items. But here we find out the moving averages from the time series. A moving average of a time series is a new series obtained by finding out successively the average of a number of the original successive items choosen on the basis of periodicity of fluctuations, dropping off one item and adding on the next at each stage.
- The most important point in the average method is the selection of period. The selection of period depends upon the periodicity of data.
- Odd period is the period of three, five, seven, nine years and so on. In case of odd period moving average, no difficulty is faced while placing the computed average.
- The procedure of calculation of moving average of even number of years, say, four, six, eight, and so on, is different from the procedure of odd number of years. Suppose moving average is to be calculated for four years. We will take the total of first four years and will be placed in between second and third years, *i.e.*, in the middle of four years. Leaving the first year, calculate the total of next four years and so on. It is important to note that these calculated totals are placed between two years. Then we adjust these moving totals. For this we compute two yearly moving totals of four yearly moving totals. We take the total of first and second four years moving totals and write against the third year and then the second and third four yearly moving totals are totalled and written against the fourth year and so on. The two yearly moving totals of four yearly moving totals are then divided by eight and this gives us the centered four yearly moving average. The series so obtained is known as an estimate of the trend.
- No particular period of a moving average will eliminate the fluctuations completely. But greater the period, the greater will be the reduction in the irregular fluctuations. Because the duration of business cycles always remain changing.
- This method is highly flexible in the sense that if a few more figures are added to the series, entire calculations are not changed. Only thing, we have to do is to extend the process so as to calculate further trend values.
- The main limitation or demerit of this method is that it is difficult to determine the proper period of moving average. If a wrong period is selected, there is every likelihood that conclusions may be misleading.
- The main limitation or demerit of this method is that it is difficult to determine the proper period of moving average. If a wrong period is selected, there is every likelihood that conclusions may be misleading.
- In this method, moving averages are calculated by using the arithmetic average. Thus, like arithmetic average it is also affected by extreme values of the series.
- Trend values cannot be computed for all the years. The longer the period of moving average, the greater the number of years for which trend values cannot be obtained. For example, in a three-yearly moving average, trend value cannot be obtained for the first year and last year, in a five-yearly moving average for the first two years and the last two years, and so on. It is often these extreme years in which we are most interested.
- Although theoretically we say that if the period of moving average happens to coincide with the period of cycle, the cyclical fluctuations are completely eliminated, but in practice. Since the cycles are by no means perfectly periodic, the lengths of the various cycles in any given series will usually vary considerably and, therefore, no moving average can completely remove the cycle. The best results would be obtained by a moving average whose period is equal to the

average length of all the cycles in the given series. However, it is difficult to determine the average length of the cycle until the cycles are isolated from the series.

12.4 Key-Words

1. Independent variables : Those variables controlled by the experimenter.
2. Independent events : Events are independent when the occurrence of one has no effect on the probability of the occurrence of the other.
3. Inferential statistics : That branch of statistics that involves drawing inferences about parameters of the population(s) from which you have sampled.

12.5 Review Questions

1. Describe the moving average method for smoothing time series data. Explain how this method is applied in the isolation of trend, if an appropriate period is choosen.
2. Explain the concept and advantages of moving average method of obtaining secular trend.
3. What do you understand by time series analysis ? How the method of moving averages help analysing a time series ?
4. What are the merits and Demerits of moving average method ?
5. What are the limitations of moving average method.

Answers: Self-Assessment

1. (*i*) length (*ii*) fluctuations (*iii*) 3-yearly
   (*iv*) simple (*v*) arithmetic

12.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002

Unit 13: Theory of Probability: Introduction and Uses

Notes

CONTENTS

Objectives

After reading this unit students will be able to:

- Introduce Theory of Probability.
- Discuss the Uses of Theory of Probability.

Introduction

In day-to-day life, we all make use of the word 'probability'. But generally people have no definite idea about the meaning of probability. For example, we often hear or talk phrases like, "Probability it may rain today"; "It is likely that the particular teacher may not come for taking his class today"; "there is a chance that the particular student may stand first in the university examination"; "it is possible that the particular company may get the contract which it bid last week"; "most probably I shall be returning within a week"; "it is possible that he may not be able to join his duty". In all the above statements, the terms—possible, probably, likely, chance, etc., convey the same meaning, *i.e.*, the events are not certain to take place. In other words, there is involved an element of uncertainty or chance in all these cases. A numerical measure of uncertainty is provided by the theory of probability. The aim of the probability theory is to provide a measure of uncertainty. The theory of probability owes its origin to the study of games of chance like games of cards, tossing coins, dice, etc. But in modern times, it has great importance is decision making problems.

26.1 Introduction to Theory of Probability

We have understood the difference between descriptive and inferential statistics. The study of probability provides a basis for inferential statistics. Inferential statistics involves sample selection, computing sample statistic on the basis of the concerned sample, and then inferring population parameter on the basis of the sample statistic. We do this exercise because population parameter is unknown. We try to estimate the unknown population parameter on the basis of the known sample statistic. This procedure works on uncertainty. By applying some defined statistical rules and procedures, an analyst can assign the probability of obtaining a result. To make rational decisions, a decision maker must have a deep understanding of probability theory. This understanding enhances his capacity to make optimum decisions in an uncertain environment. This unit focuses on the basic concept of probability which will serve as the foundation of probability distributions. A sound knowledge of probability and probabilistic distributions also helps in developing probabilistic decision models.

Concept of Probability

We live in a world dominated by uncertainty. Change is the only permanent phenomenon. We can never predict the nature and direction of change in our lives. Sometimes change is planned, but more often, change is unplanned. Even in cases of planned change, it is not possible to avoid uncertainty. There is a perceived need to be accurate (up to an extent) and prepared in this uncertain environment. Our need to cope with this unavoidable uncertainty of life has led to the study of **probability theory**. There might have been many occasions when we have said that the chances are 50-50 or there is a 70% chance of India winning the match, and so on. By making these statements, we try to attach some probability of the event happening or not happening. If we look at the wider picture, all these statements are related to the concept of probability. Therefore, there is a general understanding about the concept of probability, but there is a problem in terms of its proper application-oriented understanding.

In simple words, **probability** is the likelihood or chance that a particular event will or will not occur. The theory of probability provides a quantitative measure of uncertainty or likelihood of occurrence of different events resulting from a random experiment, in terms of quantitative measures ranging from 0 to 1. This means that the probability of a certain event is 1 and the probability of an impossible event is 0. In other words, a probability near 0 indicates that an event is unlikely to occur whereas a probability near 1 indicates that an event is almost certain to occur. For example, suppose an event is the success of a new product launched. A probability 0.90 indicates that the new product is likely to be successful whereas a probability of 0.15 indicates that the product is unlikely to be successful in the market. A probability of 0.50 indicates that the product is just as likely to be successful as not.

> *Notes* Probability is a concept that we all understand. In our daily life, we use words like chance, possibility, likelihood, and of course, probability.

Some Basic Concepts

Before we give definition of the word probability, it is necessary to define the following basic concepts and terms widely used in its study:

(1) An Experiment

When we conduct a trial to obtain some statistical information, it is called an experiment.

Examples: (i) Tossing of a fair coin is an experiment and it has two possible outcomes: Head (H) or Tail (T).

(ii) Rolling a fair die is an experiment and it has six possible outcomes: appearance of 1 or 2 or 3 or 4 or 5 or 6 on the upper most face of a die.

(iii) Drawing a card from a well shuffled pack of playing cards is an experiment and it has 52 possible outcomes.

(2) Events

The possible outcomes of a trial/experiment are called events. Events are generally denoted by capital letters A, B, C, etc.

Examples: (i) If a fair coin is tossed, the outcomes - head or tail are called events.

(ii) If a fair die is rolled, the outcomes 1 or 2 or 3 or 4 or 6 appearing up are called events.

(3) Exhaustive Events

The total number of possible outcomes of a trial/experiment are called exhaustive events. In other words, if all the possible outcomes of an experiment are taken into consideration, then such events are called exhaustive events.

Examples: (i) In case of tossing a die, the set of six possible outcomes, *i.e.*, 1, 2, 3, 4, 5 and 6 are exhaustive events.

(ii) In case of tossing a coin, the set of two outcomes, *i.e.*, H and T are exhaustive events.

(iii) In case of tossing of two dice, the set of possible outcomes are 6 × 6 = 36 which are given below:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

(4) Equally-Likely Events

The events are said to be equally-likely if the chance of happening of each event is equal or same. In other words, events are said to be equally likely when one does not occur more often than the others.

Examples: (i) If a fair coin is tossed, the events H and T are equally-likely events.

(ii) If a die is rolled, any face is as likely to come up as any other face. Hence, the six outcomes -1 or 2 or 3 or 4 or 5 or 6 appearing up are equally likely events.

(5) Mutually Exclusive Events

Two events are said to be mutually exclusive when they cannot happen simultaneously in a single trial. In other words, two events are said to be mutually exclusive when the happening of one excludes the happening of the other in a single trial.

Examples: (i) In tossing a coin, the events Head and Tail are mutually exclusive because both cannot happen simultaneously in a single trial. Either head occurs or tail occurs. Both cannot occur simultaneously. The happening of head excludes the possibility of happening of tail.

(ii) In tossing a die, the events 1, 2, 3, 4, 5 and 6 are mutually exclusive because all the six events cannot happen simultaneously in a single trial. If number 1 turns up, all the other five *(i.e.,* 2, 3, 4, 5, or 6) cannot turn up.

(6) Complementary Events

Let there be two events A and B. A is called the complementary event of B and B is called the complementary event of A if A and B are mutually exclusive and exhaustive.

Examples: (i) In tossing a coin, occurrence of head (H) and tail (T) are complementary events.

(ii) In tossing a die, occurrence of an even number (2, 4, 6) and odd number (1, 3, 5) are complementary events.

(7) Simple and Compound Events

In case of simple events, we consider the probability of happening or not happening of single events.

Example: If a die is rolled once and A be the event that face number 5 is turned up, then A is called a simple event

In case of compound events, we consider the joint occurrences of two or more events.

Example: If two coins are tossed simultaneously and we shall be finding the probability of getting two heads, then we are dealing with compound events.

(8) Independent Events

Two events are said to be independent if the occurence of one does not affect and is not affected by the occurence of the other.

Examples: (i) In tossing a die twice, the event of getting 4 in the 2nd throw is independent of getting 5 in the first throw.

(ii) In tossing a coin twice, the event of getting a head in the 2nd throw is independent of getting head in the 1st throw.

(9) Dependent Events

Two events are said to be dependent when the occurence of one does affect the probability of the occurence of the other events.

Examples: (i) If a card is drawn from a pack of 52 playing cards and is not replaced, this will affect the probability of the second card being drawn.

(ii) The probability of drawing a king from a pack of 52 cards is $\frac{4}{52}$ or $\frac{1}{13}$. But if the card drawn (king) is not replaced in the pack, the probability of drawing again a king is $\frac{3}{51}$.

Definition of Probability

The probability is defined in the following three different ways:

(1) Classical or Mathematical Definition

(2) Empirical or Relative Frequency Definition

(3) Subjective Approach.

(1) Classical or Mathematical Definition

This is the oldest and simplest definition of probability. This definition is based on the assumption that the outcomes or results of an experiment are equally likely and mutually exclusive.

According to Laplace, "Probability is the ratio of the favourable cases to the total number of equally likely cases". From this definition, it is clear that in order to calculate the probability of an event, we have to find the number of favourable cases and it is to be divided by the total number of cases. For example, if a bag contains 6 green and 4 red balls, then the probability of getting a green ball will be 6/4 + 6 = 6/10 because the total number of balls are 10 and the number of green balls is 6.

Symbolically,

$$P(A) = p = \frac{\text{Number of Favourable Cases}}{\text{Total Number of Equally Likely Cases}} = \frac{m}{n}$$

Where, $P(A)$ = Probability of occurrence of an event A

m = Number of favourable cases

n = Total number of equally likely cases

Similarly,

$$P(\bar{A}) = q = 1 - P(A) = 1 - \frac{m}{n}$$

Where, $P(\bar{A}) = q$ = Probability of non-occurrence of an event A.

From the above definition, it is clear that the sum of the probability of happening of an event called success *(p)* and the probability of non-happening of an event called failure *(q)* is always one (1), *i. e.*, $p + q = 1$. If *p* is known, we can find *q* and if *q* is known, then we can find *p*. In practice, the value of *p* lies between 0 and 1, *i.e.*, $0 \le p \le 1$. To quote **Prof. Morrison, "If an event can happen in *m* ways and fail to happen in *n* ways, then probability of happening is $\frac{m}{m+n}$ and that of its failure to happen is $\frac{n}{m+n}$".**

Limitations of Classical Definition

Following are the main limitations of classical definition of probability:

(1) If the various outcomes of the random experiment are not equally-likely, then we cannot find the probability of the event using classical definition.

(2) The classical definition also fails when the total number of cases are infinite.

(3) If the actual value of N is not known, then the classical definition fails.

(2) Empirical or Relative Frequency Definition

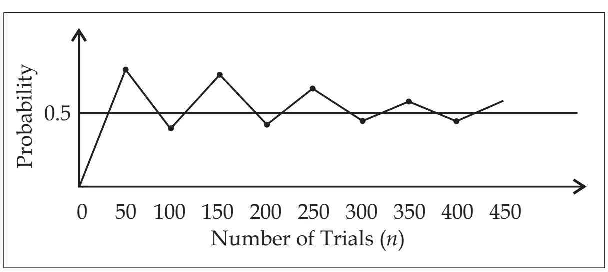
This definition of probability is not based on logic but past experience and experiments and present conditions. If vital statistics gives the data that out of 100 newly born babies, 55 of them are girls, then the probability of the girl birth will be 55/100 or 55%. **To quote Kenny and Keeping, "If event has occured *r* times in a series of *n* independent trials, all are made under the same identical conditions, the ratio *r/n* is the relative frequency of the event. The limit *of* *r/n* as *n* tends to infinity is the probability of the occurence of the event".**

Did u know? **Probability is the limit of the relative frquency of success in infinite sequences of trials.**

Symbolically,

$$P(A) = \underset{n \to \infty}{\text{Limit}} \frac{r}{n}$$

For example, if a coin is tossed 100 times and the heads turn up 55 times, then the relative frequency of head will be $\frac{55}{100}$ = 0.55. Similarly, if a coin is tossed 1000 times and if the head turns up 495 times, then the relative frequency will be $\frac{495}{1000}$ = 0.495. In 10,000 tosses, the head turns up 5085 then the relative frequency will be 0.5085. Thus as we go on increasing the number of trials, there is a tendency mat the relative frequency of head would approach to 0.50. The following figure illustrate the idea:



From the above figure, it is clear that as the number of trials increases, the probability of head tends to approach 0.5 and when the number of trials is infinite, *i.e.*, $n \to \infty$, the probability of getting head is equal to 0.5.

(3) Subjective Approach

According to this approach, probability to an event is assigned by an individual on the basis of evidence available to him. Hence probability is interpreted as a measure of degree of belief or confidence that a particular individual reposes in the occurrence of an event. But the main problem here is that different persons may differ in their degree of confidence even when same evidence is offered.

13.2 Uses of Theory of Probability

The theory of probability has its origin in the games of chance related to gambling such as tossing a die, tossing a coin, drawing a card from a deck of 52 cards and drawing a ball of a particular colour from a bag. But in modern times, it is widely used in the field of statistics, economics, commerce and social sciences that involve making predictions in the face of uncertainty. The importance of probability is clear from the following points:

(1) Probability is used in making economic decision in situations of risk and uncertainty by sales managers, production managers, etc.

(2) Probability is used in theory of games which is further used in managerial decisions.

(3) Various sampling tests like Z-test, t-test and F-test are based on the theory of probability.

(4) Probability is the backbone of insurance companies because life tables are based on the theory of probability.

Thus, probability is of immense utility in various fields.

Probability Scale

The probability of an event always lies between 0 and 1, *i.e.*, $0 \le p \le 1$. If the event cannot take place, *i.e.*, impossible event, then its probability will be zero, *i.e.*, P(E) = 0 and if the event is sure to occur, then its probability will be one, *i.e.*, P(E) = 1.

Calculation of Probability of an Event

The following steps are to be followed while calculating the probability of an event:

(1) Find the total number of equally likely cases, *i.e.*, *n*

(2) Obtain the number of favourable cases to the event,. *i.e.*, *m*

(3) Divide the number of favourable cases to the event (*m*) by the total number of equally likely cases (*n*). This will give the probability of an event.

Symbolically,

Probability of occurrence of an event E is:

$$P(E) = \frac{\text{Number of favourable cases to E}}{\text{Total number of equally likely cases}} = \frac{m}{n}$$

Similarly, Probability of non-occurrence of event E is:

$$P(\overline{E}) = 1 - P(E)$$

The following examples will illustrate the procedure:

Example 1: Find the probability of getting a head in a tossing of a coin.

Solution: When a coin is tossed, there are two possible outcomes - Head or Tail.

Total number of equally likely cases = n = 2

Number of cases favourable to H = m = 1

$\therefore \quad P(H) = \frac{m}{n} = \frac{1}{2}$

Example 2: What is the probability of getting an even number in a throw of an unbiased die ?

Solution: When a die is tossed, there are 6 equally likely cases, *i.e.*, 1, 2, 3, 4, 5, 6.

Total number of equally likely cases = $n = 6$

Number of cases favourable to even points (2, 4, 6) = $m = 3$

$\therefore$ Probability of getting an even number = $\frac{3}{6} = \frac{1}{2}$

Example 3: What is the probability of getting a king in a draw from a pack of cards ?

Solution: Number of exhaustive cases = $n = 52$

There are 4 king cards in an ordinary pack.

$\therefore$ Number of favourable cases = $m = 4$

$\therefore$ Probability of getting a king = $\frac{4}{52} = \frac{1}{13}$

Example 4: From a bag containing 5 red and 4 black balls. A ball is drawn at random. What is the probability that it is a red ball ?

Solution: Total No. of balls in the bag = 5 + 4 = 9

No. of red balls in the bag = 5

$\therefore$ Probability of getting a red ball = $\frac{5}{9}$

Example 5: A bag contains 5 black and 10 white balls. What is the probability of drawing (i) a black ball, (ii) a white ball ?

Solution: Total number of balls = 5 + 10 = 15

(i) P (black ball) = $\frac{\text{No. of black balls}}{\text{Total No. of balls}} = \frac{5}{15} = \frac{1}{3}$

(ii) P (white ball) = $\frac{\text{No. of white balls}}{\text{Total No. of balls}} = \frac{10}{15} = \frac{2}{3}$

Example 6: In a lottery, there are 10 prizes and 90 blanks. If a person holds one ticket, what are the chances of

(i) getting a prize

(ii) not getting a prize

Solution: Total No. of tickets = 10 + 90 = 100

(i) Probability of getting a prize:

No. of prizes = 10

$\therefore$ No. of favourable cases = 10

Total No. of cases = 100

Required Probability = $\frac{10}{100} = \frac{1}{10} = 0.1$

(ii) The probability of not getting a prize:

No. of Blanks = 90

$\therefore$ Number of favourable cases = 90

Notes

Total Number of cases = 100

Required Probability = $\frac{90}{100}$ = 0.9

Example 7: What is the probability of getting a number greater than 4 with an ordinary die ?

Solution: Number greater than 4 in a die are 5 and 6.

$\therefore$ Number of favourable cases = 2

Total number of cases = 6

Required Probability = $\frac{2}{6} = \frac{1}{3}$

Example 8: Find the probability of drawing a face card in a single random draw from a well shuffled pack of 52 cards.

Solution: There are 52 cards in a pack of cards.

Total number of cases = 52

Number of favourable cases (face cards include the Jack, Queen and King in each) = 12

Required Probability = $\frac{12}{52} = \frac{3}{13}$

Example 9: A card is drawn from an ordinary pack of playing cards and a person bets that it is a spade or an ace. What are odds against his winning this bet ?

Solution: Total number of cases = 52

Since there are 13 spades and 3 aces (one ace is also present in spades), Therefore the favourable cases = 13 + 3 = 16

The probability of winning the bet = $\frac{16}{52} = \frac{4}{13}$

The probability of losing the bet = $1 - \frac{4}{13} = \frac{9}{13}$

Hence, odds against winning the bet = $\frac{9}{13} : \frac{4}{13}$ = 9: 4

Example 10: A single letter is selected at random from the word 'PROBABILITY'. What is the probability that it is a vowel ?

Solution: There are 11 letters in the word 'PROBABILITY' out of which 1 is be selected.

$\therefore$ Total No. of words = 11

There are four vowels *viz.* O, A, I, I. Therefore favourable number of cases = 4

Hence, the required probability = $\frac{4}{11}$

Example 11: Find the probability of drawing an ace from a set of 52 cards.

Solution: Number of exhaustive cases (n) = 52

There are 4 ace cards in an ordinary pack.

$\therefore$ Favourable cases (n) = 4

$\therefore$ Probability of getting an ace = $\frac{4}{52} = \frac{1}{13}$

Example 12: What is the probability that a leap year selected at random will contain 53 Sundays ?

Solution: Total number of days in a leap year = 366

Number of weeks in a year = $\frac{366}{7} = 52\frac{2}{7}$

= 52 weeks and 2 days

Following may be the 7 possible combinations of these two extra days:

(i) Monday and Tuesday
(ii) Tuesday and Wednesday
(iii) Wednesday and Thursday
(iv) Thursday and Friday
(v) Friday and Saturday
(vi) Saturday and Sunday
(vii) Sunday and Monday

A selected leap year can have 53 Sundays if these two extra days happen to be a Sunday

Total possible outcomes of 2 days = n = 7

Number of cases having Sundays = m = 2

$\therefore$ The required probability = $\frac{2}{7}$

Use of Bernoulli's Theorem in Theory of Probability

Bernoulli's theorem is very useful in working out various probability problems. This theorem states that if the probability of happening of an event in one trial or experiment is known, then the probability of its happening exactly, 1, 2, 3, ... r times in n trials can be determined by using the formula:

$$P(r) = {}^{n}C_{r}\, p^{r} \cdot q^{n-r} \qquad r = 1, 2, 3, \ldots n$$

where,

P (r) = Probability of r successes in n trials.

p = Probability of success or happening of an event in one trial.

q = Probability of failure or not happening of the event in one trial.

n = Total number of trials.

The following examples illustrate the applications of this theorem:

Example 13: The chance that a ship safely reaches a port in 1/5. Find the probability that out of 5 ships expected at least one would arrive safely.

Solution: Given, $n = 5, p = \frac{1}{5}, q = 1 - \frac{1}{5} = \frac{4}{5}$

P (at least one ship arriving safely) = 1 – 1 (none arriving safely)

$$= 1 - \left[{}^{5}C_{0}(p)^{0} \cdot (q)^{5} \right)$$

$$= 1 - \left[{}^{5}C_{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{5} \right] = 1 - \left(\frac{4}{5}\right)^{5}$$

$$= 1 - \left(\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5}\right) = 1 - \frac{1024}{3125} = \frac{2101}{3125}$$

Example 14: Find the probability of throwing 6 at least once in six throws with a single die.

Solution: p = probability of throwing 6 with a single die = $\frac{1}{6}$

$$q = 1 - \frac{1}{6} = \frac{5}{6}$$

$$n = 6, p = \frac{1}{6}, q = \frac{5}{6}$$

p (at least one six)= 1 – P [none six in 6 throws]

$$= 1 - \left[{}^6C_0\left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^6\right] = 1 - \left(\frac{5}{6}\right)^6$$

Example 15: Three dice are thrown. What is the probability that at least one of the numbers turning up being greater than 4 ?

Solution: p = probability of a number greater than 4 (*i.e.,* 5 and 6) in a throw of one die

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \quad n = 3, p = \frac{1}{3}, q = \frac{2}{3}$$

P (at least one number greater than 4) = 1 – P (none of the number greater than 4)

$$= 1 - \left[{}^3C_0\left(\frac{1}{3}\right)^0 \cdot \left(\frac{2}{3}\right)^3\right]$$

$$= 1 - \left(\frac{2}{3}\right)^3 = 1 - \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{19}{27}$$

Example 16: A and B play for a prize of Rs. 1000. A is to throw a die first and is to win if he throws 6. If he fails B is to throw and is to win if throws 6 or 5. If he fails A is to throw again and to win if he throws 6, 5 or 4 and so on. Find their respective expectations.

Solution: Probability of A's winning in the 1st throw (*i.e.*, he throws 6) = $\frac{1}{6}$

Probability of B's winning in the 2nd throw (*i.e.*, he throws 6 or 5) = $\frac{5}{6} \times \frac{2}{6} = \frac{5}{18}$

Probability of A's winning in the 3rd throw (6 or 5 or 4) = $\frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} = \frac{5}{18}$

Probability of B's winning in the 4th throw (6 or 5 or 4 or 3) = $\frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{4}{6} = \frac{5}{27}$

Probability of A's winning in the 5th throw (6 or 5 or 4 or 3 or 2)

$$= \frac{5}{6}\times\frac{4}{6}\times\frac{3}{6}\times\frac{2}{6}\times\frac{5}{6} = \frac{25}{324}$$

Probability of B's winning in the 6th throw (6 or 5 or 4 or 3 or 2 or 1)

$$= \frac{5}{6}\times\frac{4}{6}\times\frac{3}{6}\times\frac{2}{6}\times\frac{1}{6}\times\frac{6}{6} = \frac{5}{324}$$

A's total chances of success = $\frac{1}{6}+\frac{5}{18}+\frac{25}{324} = \frac{169}{324}$

B's total chances of success = $\frac{5}{18}+\frac{5}{27}+\frac{5}{324} = \frac{155}{324}$

For a prize of Rs. 1,000

A's expectation = $p \times m = \frac{169}{324}\times 1{,}000$ = Rs. 521.6

B's expectation = $p \times m = \frac{155}{324}\times 1{,}000$ = Rs. 478.4

Example 17: A and B play for a prize of Rs. 99. The prize is to be won by a player who first throws 6 with one die. A first throws and if he fails B throws and if he fails A again throws and so on. Find their respective expectations.

Solution: The probability of throwing 6 with a single die = $\frac{1}{6}$

The probability of not throwing 6 with single die = $1-\frac{1}{6} = \frac{5}{6}$

If A is to win, he should throw 6 in the 1st, 3rd, or 5th...throws

If B is to win, he should throw 6 in the 2nd, 4th, 6th...throws

A's chance of success in given by

$$= \frac{1}{6}+\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)+\ldots\infty$$

$$= \frac{1}{6}\cdot\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\ldots\infty\right] \qquad \text{[Infinite GP series: S = } 1+a+a^2+\ldots\infty\text{]}$$

$$= \frac{1}{6}\cdot\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right] = \frac{1}{6}\times\frac{36}{11} = \frac{6}{11} \qquad \left(\therefore S_\infty = \frac{1}{1-a} = \frac{\text{First Term}}{1-\text{Common Ratio}}\right)$$

A's expectation = $p \times m = \frac{6}{11}\times 99$ = Rs. 54

B's chance of success is given by

$$= \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^5\cdot\left(\frac{1}{6}\right)+\ldots\infty$$

$$= \frac{5}{6} \times \frac{1}{6} \cdot \left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots \infty \right]$$

$$= \frac{5}{6} \times \frac{1}{6} \cdot \left[\frac{1}{1 - \left(\frac{5}{6}\right)^2} \right] = \frac{5}{6} \times \frac{1}{6} \times \frac{36}{11} = \frac{5}{11}$$

B's expectation = Rs. $99 \times \frac{5}{11}$ = Rs. 45.

Example 18: A bag contains 6 black and 9 white balls. A person draws out 2 balls. If on every black ball he gets Rs. 20 and on every white ball Rs. 10, find out his expectation.

Solution: There may be the following three options for drawing 2 balls:

(i) Both are white, (ii) Both are black, (iii) One is white and other is black.

(i) Both balls are white

$$P\ (2W) = p = \frac{{}^9C_2}{{}^{12}C_2} = \frac{12}{35}$$

$$\text{Expectation} = p \times m = \frac{12}{35} \times 10 \times 2 = \text{Rs. } 6.86$$

(ii) Both balls are black

$$P\ (2B) = p = \frac{{}^6C_2}{{}^{15}C_2} = \frac{1}{7}$$

$$\text{Expectation} = p \times m = \frac{1}{7} \times 20 \times 2 = \text{Rs. } 5.71$$

(iii) One ball is white and the other is black

$$P\ (1W\ 1B) = p = \frac{{}^6C_1 \times {}^9C_1}{{}^{15}C_2} = \frac{18}{35}$$

$$\text{Expectation} = p \times m = \frac{18}{35} \times (20 + 10) = \text{Rs. } 15.43$$

Total Expectation = 6.86 + 5.71 + 15.43 = Rs. 28

Example 19: If it rains, a taxi driver can earn Rs. 1000 per day. If it is fair, he can lose Rs. 100 per day. If the probability of rain in 0.4, what is his expectation ?

Solution: The distribution of earnings (X) is given as:

X	$X_1 = 1000$	$X_2 = -100$
P	$P_1 = 0.4$	$P_2 = 1 - 0.4 = 0.6$

$\therefore$ E (X) $= P_1 X_1 + P_2 X_2$

$= 0.4 \times 1000 + 0.6 \times (-100) =$ Rs. 340

Example 20: A petrol pump dealer sells an average petrol of Rs. 80,000 on a rainy day and an average of Rs. 95,000 at a clear day. The probability of clear weather is 76% on Tuesday. What will be the expected sale ?

Solution: The distribution of earnings (X) is given as:

X	X_1 = 80,000	X_2 = 95,000
P	1 – 0.76 = 0.24	0.76

$$E(X) = 80{,}000 \times 0.24 + 95{,}000 \times 0.76$$
$$= \text{Rs. } 91{,}400$$

Example 21: A player tosses 3 fair coins. He wins Rs. 12 if 3 heads appear, Rs. 8 if 2 heads appear and Rs. 3 if 1 head appears. On the otherhand, he loses Rs. 25 if 3 tails appear. Find the expected gain of the player.

Solution: If *p* denotes the probability of getting a head and X denotes the corresponding amount of winning, then the distribution of X is given by:

Heads:	0H	1H	2H	3H
Favourable Events	TTT	HTT, THT, TTH	HHT, HTH, THH	HHH
P	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$	$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$	$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$
X Winning amount	– 25	3	8	12

The expected gain of the player is given by:

$$E(X) = \frac{1}{8}(-25)+\frac{3}{8}(3)+\frac{3}{8}(8)+\frac{1}{8}(12)$$

$$= \frac{-25+9+24+12}{8} = \frac{20}{8} = \frac{5}{2} = \text{Rs. } 2.50.$$

Example 22: A player tosses two fair coins. He wins Rs. 5 if 2 heads appear, Rs. 2 if one head appear and Rs. 1 if no head appear. Find his expected gain of the player.

Solution: If *p* denotes the probability of getting a head and X denotes the corresponding amount of winning, then the probability distribution of X is given by:

Heads:	0H	1H	2H
Favourable Events	TT	HT, TH	HH
P	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$
X	1	2	5

The expected gain of the player is given by:

$$E(X) = P_1 X_1 + P_2 X_2 + P_3 X_3$$

$$= \frac{1}{4}\times 1+\frac{1}{2}\times 2+\frac{1}{4}\times 5 = \text{Rs. } 2.50$$

Notes

Example 23: A survey conducted over the last 25 years indicated that in 10 years, the winter was mild, in 8 years it was cold and in the remaining 7 it was very cold. A company sells 1000 woolen coats in a mild year, 1300 in a cold year and 2000 in a very cold year. If a woolen coat costs Rs. 173 and is sold for Rs. 248, find the yearly expected profit of the company.

Solution:

State of Nature	Prob. P (X)	Sale of woollen coat	Profit (X)
Mild winter	$\frac{10}{25} = 0.4$	1000	1000 × (248 – 173)
Cold winter	$\frac{8}{25} = 0.32$	1300	1300 × (248 – 173)
Very cold winter	$\frac{7}{25} = 0.28$	2000	2000 × (248 – 173)

∴ Expected profit is given by

$$E(X) = 1000 \times 75 \times 0.4 + 1300 \times 75 \times 0.32 + 2000 \times 75 \times 0.28$$

$$= 30,000 + 31,200 + 42,000 = \text{Rs. } 1,03,200$$

Self-Assessment

1. Which of the following statements are true or false:

(*i*) The classical approach to probability is the oldest and simplest.

(*ii*) The probability of throwing eight with a single dice is 1/6.

(*iii*) The modern probability theory has been developed automatically in which probability is an undefined concept.

(*iv*) In most field of research, a priori probability is employed.

(*v*) Dependent events are those in which the outcome of one does not affect and is not affected by the other.

13.3 Summary

- "It is likely that the particular teacher may not come for taking his class today"; "there is a chance that the particular student may stand first in the university examination"; "it is possible that the particular company may get the contract which it bid last week"; "most probably I shall be returning within a week"; "it is possible that he may not be able to join his duty". In all the above statements, the terms—possible, probably, likely, chance, etc., convey the same meaning, *i.e.,* the events are not certain to take place. In other words, there is involved an element of uncertainty or chance in all these cases. A numerical measure of uncertainty is provided by the theory of probability. The aim of the probability theory is to provide a measure of uncertainty. The theory of probability owes its origin to the study of games of chance like games of cards, tossing coins, dice, etc. But in modern times, it has great importance is decision making problems.
- In simple words, **probability** is the likelihood or chance that a particular event will or will not occur. The theory of probability provides a quantitative measure of uncertainty or likelihood of occurrence of different events resulting from a random experiment, in terms of quantitative measures ranging from 0 to 1. This means that the probability of a certain event is 1 and the probability of an impossible event is 0. In other words, a probability near 0 indicates that an event is unlikely to occur whereas a probability near 1 indicates that an event is almost certain

to occur. For example, suppose an event is the success of a new product launched. A probability 0.90 indicates that the new product is likely to be successful whereas a probability of 0.15 indicates that the product is unlikely to be successful in the market. A probability of 0.50 indicates that the product is just as likely to be successful as not.

- **According to Laplace, "Probability is the ratio of the favourable cases to the total number of equally likely cases".** From this definition, it is clear that in order to calculate the probability of an event, we have to find the number of favourable cases and it is to be divided by the total number of cases.

- The theory of probability has its origin in the games of chance related to gambling such as tossing a die, tossing a coin, drawing a card from a deck of 52 cards and drawing a ball of a particular colour from a bag.

13.4 Keywords

1. Dependent variables : The variable being measured. The data or score.
2. Depth : Cumulative frequency counting in from the nearer end.
3. Design matrix : A matrix of coded or dummy variables representing group membership.

13.5 Review Questions

1. Define Probability. Discuss the importance of probability in decision-making.
2. Give the classical definition of probability and state its limitations.
3. State and prove the theorem of total probability for mutually exclusive events.
4. Describe the uses of theory of probability.
5. What are the use of Bernoulli's theorem in theory of probability ?

Answers: Self-Assessment

1. (*i*) T (*ii*) F (*iii*) T
   (*iv*) T (*v*) F

13.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 14: Additive and Multiplicative Law of Probability

CONTENTS

Objectives

After reading this unit students will be able to:

- Discuss Additive Rule of Probability.
- Explain Multiplicative Rule of Probability: Conditional Probability.

Introduction

Often it is easier to compute the probability of an event from known probabilities of other events. This can be well observed if the given event can be represented as the **union of two other events** or as the complement of an event. Two such rules used for simplifying the computation of probabilities of events are:

1. Addition Rule of Probability
2. Multiplication Rule of Probability

14.1 Addition Rule of Probability

For any two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{... (1)}$$

or

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad \text{... (2)}$$

In case A and B are mutually exclusive events, then $P(A \cap B) = 0$ and the addition rule of probability in (1) becomes

$$P(A \cup B) = P(A) + P(B) \quad \text{... (3)}$$

Proof: For any two events A and B, we can write

$$A = (A \cap B)(A \cap \overline{B})$$

$$\therefore \quad P(A) = P(A \cap B) + P(A \cap \overline{B}) \quad \text{... (a)}$$

Using axiom (iii) of probability as the events $(A \cap B)$ and $(A \cap \overline{B})$ are mutually exclusive.

Similarly **Notes**

$$B = (A \cap B) \cup (\overline{A} \cap B)$$

$\therefore$ $$P(B) = P(A \cap B) + P(\overline{A} \cap B) \quad \text{... (b)}$$

The events $(A \cap B)$ and $(\overline{A} \cap B)$ being mutually exclusive. Thus, from (a) and (b), one gets

$$P(A) + P(B) = P(A \cap B) + P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B) \quad \text{... (c)}$$

Now the last three terms on R.H.S. of (c), *i.e.*, $P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B)$ represent the probability of occurrence of the events A or B or both A and B, *i.e.*, $P(A \cup B)$. Thus, replacing these three terms by $P(A \cup B)$, equation (c) can be written as

$$P(A) + P(B) = P(A \cap B) + P(A \cup B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{... (d)}$$

The rule in (d) is called the **addition of rule probability.**

If A and B are mutually exclusive, $P(A \cap B) = 0$ and the addition rule of probability becomes

$$P(A \cup B) = P(A) + P(B) \quad \text{... (e)}$$

Example 1: What is the probability of getting an odd number in tossing a die ?

Solution: There are three odd numbers on a die, *i.e.*, 1, 3 and 5. Let A, B and C be the respective events of getting 1, 3 and 5. Thus, P (A) = 1/6, P (B) = 1/6 and P (C) = 1/6. Since A, B and C are mutually exclusive therefore

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

Example 2: An urn contains 4 white and 2 red balls. Two balls are drawn randomly with replacement. Find the probability that

(i) both balls are white

(ii) both balls will be of the same colour.

Solution: Here total number of balls = 6

Two balls can be drawn out of 6 in $^6C_2$ ways *i.e.*, 15 ways. Let A be the event that both balls are white. The number (i) of ways of selecting 2 balls out of 4 is $^4C_2$ *i.e.*, 6 ways.

$\therefore$ $$P(A = \text{both balls white}) = \frac{6}{15}$$

(ii) The number of ways of selecting 2 balls out of 4 white balls is $^4C_2$ = 6 ways

The number of ways of selecting 2 both out of 2 red balls is $^2C_2$ = 1 way

$$\therefore \ P(\text{both balls will be of the same colour}) = \frac{^4C_2}{^6C_2} + \frac{^2C_2}{^6C_2}$$

$$= \frac{6}{15} + \frac{1}{15}$$

$$= \frac{7}{15}$$

Example 3: Find the probability of getting more than 4 in tossing a die.

Solution: The numbers more than 4 on a die are 5 and 6.

Let A and B be the respective events of getting 5 and 6.

Thus, $P(A) = \frac{1}{6}, P(B) = \frac{1}{6}$

Also A and B are mutually exclusive.

Thus, $P(A \text{ or } B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Example 4: A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability of getting an ace or a spade.

Solution: Let A be the event of getting an ace and B of getting a spade. Then A = set of all aces, B = set of all spades and $A \cap B$ = set of an ace of spade.

Clearly, $n(A) = 4$, $n(B) = 13$ and $n(A \cap B) = 1$.

Also, $n(S) = 52$. Therefore,

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}, P(B) = \frac{n(B)}{n(S)} = \frac{13}{52} \text{ and } P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{52}$$

Thus, the required probability

P (an ace or a spade) $= P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 5: A construction company is bidding for two contracts, A and B. The probability that the company will get contract A is 3/5, will get contract B is 1/4 and the probability that the company gets both the contracts is 1/8. What is the probability that the company will get contract A or B ?

Solution: Let A and B be the respective events of getting the contracts A and B. Then, we are given that

$$P(A) = 3/5, (B) = 1/4 \text{ and } P(A \cap B) = 1/8$$

Thus, the required probability that the company will get a contract A or B is

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{5} + \frac{1}{4} - \frac{1}{8} = \frac{29}{40}$$

Example 6: A bag contains 30 balls numbered from 1 to 30. One ball is drawn at random. Find the probability that number of the drawn ball is a multiple of (i) 4 or 9 (ii) 5 or 6.

Solution: (i) Let A be the event that the drawn number is a multiple of 4, then A = {4, 8, 12, 16, 20, 24, 28}. Further let B be the event that the drawn number is a multiple of 9, *i.e.*, B = {9, 18, 27}.

Also $A \cap B = \phi$, null set and n (S) = 30.

$$\therefore \; P(A) = \frac{n(A)}{n(S)} = \frac{7}{30}, \; P(B) = \frac{n(B)}{n(S)} = \frac{3}{30}, \; P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{0}{30} = 0$$

Thus, the probability of the desired event

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{30} + \frac{3}{30} - \frac{0}{30} = \frac{10}{30} = \frac{1}{3}$$

(ii) Let A be the event that the drawn number is a multiple of 5 and B that the number is a multiple of 6.

$\therefore$ A = {5, 10, 15, 20, 25, 30}, B = {6, 12, 18, 24, 30}.

and $A \cap B$ = {30}. Thus, n (A) = 6, n (B) = 5, $n(A \cap B)$ = 1, n (S) = 30

$$\therefore \; P(A) = \frac{n(A)}{n(S)} = \frac{6}{30}, \; P(B) = \frac{n(B)}{n(S)} = \frac{5}{30},$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{30}$$

$\therefore$ P (multiple of 5 or 6) = P (A or B) = $P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{30} + \frac{5}{30} - \frac{1}{3} = \frac{10}{30} = \frac{1}{3}.$$

Example 7: Two boxes contain respectively 6 brown, 8 blue, 1 black balls and 3 brown, 7 blue and 5 black balls. One ball is drawn from each box. What is the probability that both the balls drawn are of the same colour.

Solution: In all there are 15 balls in one box and 15 balls in another box.

One brown ball from each box may be drawn in $\frac{^6C_1}{^{15}C_1} \times \frac{^3C_1}{^{15}C_1}$ ways

One blue ball from each box may be drawn in $\frac{^8C_1}{^{15}C_1} \times \frac{^7C_1}{^{15}C_1}$ ways

One black ball from each box may be drawn in $\frac{^1C_1}{^{15}C_1} \times \frac{^5C_1}{^{15}C_1}$ ways

All the three cases are mutually exclusive and thus the required probability

$$= \frac{^6C_1}{^{15}C_1} \times \frac{^3C_1}{^{15}C_1} + \frac{^8C_1}{^{15}C_1} \times \frac{^7C_1}{^{15}C_1} + \frac{^1C_1}{^{15}C_1} \times \frac{^5C_1}{^{15}C_1}$$

$$= \frac{6 \times 3}{225} + \frac{8 \times 7}{225} + \frac{1 \times 5}{225}$$

$$= \frac{18 + 56 + 5}{225} = \frac{79}{225}$$

14.2 Multiplicative Rule of Probability: Conditional Probability

Suppose that two dice are thrown. Then there are 36 sample points and the event A that the first die shows a five consists of 6 sample points (5, 1), (5, 2), (5, 3), (5, 4), (5, 5) and (5, 6). Thus P (A) = $\frac{6}{36}$ = $\frac{1}{6}$. The event B that the sum (total) of numbers in the two dice is 9 consists of the sample points (3, 6), (4, 5), (5, 4), (6, 3) and P (B) = $\frac{1}{9}$. Now suppose that we are given the information that the first die shows a five, then the event that the sum of the numbers shown on the faces of the two dice is nine is a *conditional event,* and this conditional event is denoted by (B | A). The probability of the conditional event is called *conditional probability.* For a conditional event, instead of the whole sample space we have only the sample points comprising of the enent A, *i.e.* the 6 sample points (5, 1), (5, 2), (5, 3), (5, 4), (5, 5) and (5, 6), and the conditional probability of each of these is $\frac{1}{6}$. Conditioned by the event A, *i.e.* that the first die shows a five, the (conditional) event that the sum is nine comprises of only one sample point (5, 4) *i.e*. the sample point common to both A and B. Here the conditional probability of getting a sum of nine given that the first die shows a five is $\frac{1}{6}$, or in symbols P (B | A) = $\frac{1}{6}$.

General formula: Consider a conditional event (B | A) *i.e*. the event B given that A has actually happend. Then for the happening of the event (B | A), the sample space is restricted to the sample points comprising the event A. The conditional probability P (B | A) is given by

$$P(B \mid A) = \frac{i}{j}, \text{ where}$$

i = number of sample points common to both A and B,

j = number of sample points comprising A,

n = total number of points in the whole (unrestricted) sample space S.

Thus, $$P(AB) = \frac{i}{n},\ P(A) = \frac{j}{n} \text{ and } P(B \mid A) = \frac{i}{j}$$

Dividing both the numerator and the denominator of P (B | A) = $\frac{i}{j}$ by n, the total number of sample points in the sample space S, we get

$$P(B \mid A) = \frac{i}{j} = \frac{i}{n} / \frac{j}{n}$$

$$= \frac{P(AB)}{P(A)} \qquad \text{... (4)}$$

It follows that

$$P(AB) = P(B \mid A)\, P(A). \qquad \text{... (5)}$$

Note: (1) These two results hold only if P (A) > 0.

(2) The results are proved under the tacit assumption that n is finite and that each sample point has equal probability $\frac{1}{n}$. It can be shown that the results hold in the general case (without these restrictions).

Example 8: Two dice are thrown. Find the probability that the sum of the numbers in the two dice is 10, given that the first die shows *six*.

Let A be the event that the sum of numbers in two dice is 10,

B be the event that the first die shows 6.

Then AB is the event that the sum is 10 and the first die shows 6 or which is equivalent to the event that first die shows 6 and the second 4.

We have $P(B) = \frac{1}{6}$, $P(AB) = \frac{1}{36}$

Thus the required probability = $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{1}{6}$. Thus the conditional probability of getting a sum of 10, given that the first shows 6 is $\frac{1}{6}$. The unconditional probability of getting a sum of 10 is $\frac{1}{12}$.

Example 9: Two coins are tossed. What is the conditional probability of getting two heads (event B) given that at least one coin shows a head (event A) ?

Event A comprises of 3 sample points (HH), (HT), (TH) so that $P(A) = \frac{3}{4}$; the event AB comprises of only one point (HH), so that $P(AB) = \frac{1}{4}$. Thus the conditional probability is

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Example 10: A box contains 5 black and 4 white balls. Two balls are drawn one by one *without replacement*, *i.e.* the first ball drawn is not returned to the box. Given that the first ball drawn is black, what is the probability that both the balls drawn will be black ?

Before the first draw the sample space consists of 9 points each with probability $\frac{1}{9}$. After the first draw the number of sample points reduces to 8 (as one ball is already out of the box) and the probability of each sample point is $\frac{1}{8}$.

Let A be the event that the first ball drawn is black then $P(A) = \frac{5}{9}$, since there are 5 black balls.

Let B be the event that the second ball drawn is black. Then the conditional event (B | A) implies drawing a black ball from the box which contains 4 black and 4 white balls.

Thus $$P(B | A) = \frac{4}{8} = \frac{1}{2}.$$

The event implies that both the balls drawn are black.

We are required to find P (AB). We have

$$P(AB) = P(B | A) \cdot P(A)$$

$$= \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}.$$

In the above example, consider that the ball drawn in the first draw is returned to the box, so that the composition of the box remains the same before each drawing. Here it is drawing *with replacement*. Now the conditional event (B | A), that the second ball is black is not affected by the event that the first draw resulted in a black ball since the ball drawn was returned. Thus the event B and the event A are *independent* and the knowledge of one does not affect the other. Then P (B | A) = P (B), independent of P (A). We have then

$$P(B) = \frac{P(AB)}{P(A)}$$

Or, $$P(AB) = P(A)\,P(B) \qquad \text{... (6)}$$

which gives the multiplication rule for independent events, A and B.

Example 11: What is the probability that in 2 throws of a die, six appears in both the dice ?

Let A and B be the event that 6 appears in the first and the second throws respectively; these events are independent. Then the event AB implies that six appears in both the dice.

$$P(AB) = P(A)\,P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Example 12: The probability of getting HHT in tossing 3 coins is thus $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, since the three events Head in first throw, Head in second and Tail in third are independent.

Example 13: Find the probability that in a family of 2 children (i) both are boys, (ii) both are of the same sex, assuming that the probability of a child being a boy or a girl is equal $\left(\text{equal to } \frac{1}{2}\right)$

(i) Let A_1, A_2 be the events respectively that the first and the second child is a boy. Then since A_1, A_2 are independent,

$$P(A_1A_2) = P(A_1)\,P(A_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Similarly if B_1, B_2 are the events respectively that the first and the second child is a girl, then B_1, B_2 are independent and

$$P(B_1B_2) = P(B_1)\,P(B_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

(ii) Now the event that both children are of the same sex is equivalent to the event that both are either boys or girls and these are mutually exclusive events. Thus

the required probability = $\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$.

Mutually exclusive events and independent events

These ideas are not equivalent ideas. We discuss them to bring out the difference between the two. When the happening of one event precludes the happening of the other event, the two events are mutually exclusive (or (disjoint). For two mutually exclusive events A and B

$$P(AB) = 0.$$

When the happening of one event has no effect on the probability of occurrence of happening of the other event, the two events are independent. For two independent events A and B,

$$P(AB) = P(A)\,P(B).$$

Two events can be mutually exclusive and not independent. Again two events can be independent and not mutually exclusive.

Suppose two coins are tossed. The events {H, H} $\equiv$ A (head on both coins) and the event {T, T} $\equiv$ B are mutually exclusive (because if A happens B cannot happen) and

$$P(AB) = 0$$

But P (A) = $\frac{1}{4}$, P (B) = $\frac{1}{4}$ and so P (AB) = 0 $\neq$ P (A) P (B) and hence A and B are not independent.

Consider the events A, B that 6 appears in the first and the second die respectively in throwing 2 dice together (example 11). The events A and B are independent, as

$$P(AB) = \frac{1}{36} = P(A)\,P(B) = \frac{1}{6}\times\frac{1}{6}.$$

Here P (AB) $\neq$ 0 and hence the events are not mutually exclusive.

The only way that two mutually exclusive events A, B can be independent is when both

$$P(AB) = 0 \text{ and } P(AB) = P(A)\,P(B)$$

hold simultaneously. Both of the above can hold simultaneously if at least one of P (A) or P (B) is zero. In case at least one of P (A) or P (B) is zero then the two events A and B mutually exclusive events as well as independent events.

Number of sample points in a combination of events or sets

Let N (A) denote the number of points in the set A.

Then, using Venn diagrams, it can be easily seen that

$$N(A\cup B) = N(A) + N(B) - N(AB). \quad \text{... (7)}$$

Example 14: Suppose that students in an Institution can enrol for one, two or none of the language courses, French (A), German (B). If 30% are enrolled for French, 20% for German and 10% for both French and German, then the number (in percentage) enrolled for at least one of the courses is given by

$$\begin{aligned} N(A\cup B) &= N(A) + N(B) - N(AB) \\ &= 30 + 20 - 10 \\ &= 40 \end{aligned}$$

and the percentage not enrolled for any of the courses is

$$100 - 40 = 60$$

Thus the probability that a student selected at random is (i) enrolled for at least one of the courses is 0.4 and (ii) not enrolled for any of the courses is 1 – 0.4 = 0.6. Given that a student is enrolled for at least one of the courses, the (conditional) probability that he is enrolled for French is

$$\frac{30}{40} = 0.75$$

Discrete Sample Space

So far we considered cases where the sample space contains a finite number of points. We consider the following example where this is not the case.

Example 15: A coin is tossed until a head appears. Describe the sample space. Find the probability that the coin will be tossed (a) exactly 4 times (b) at the most, 4 times. (c) What is the probability that head will appear if the coin is tossed an infinite number of times ?

The head may appear at the

(i) very first throw (H)

(ii) second throw, the first toss resulting in a tail (TH)

(iii) third throw, the first two tosses resulting in tails (TTH)

(iv) fourth throw, the first three tosses resulting in tails (TTTH)

and so on: an infinite number of throws may be needed to get a head. The sample space consists of an *infinite* number of the *sample* points

H, TH, TTH, TTTH, TTTTH, ...

The trials are independent. Assume that the coin is fair (unbiased).

The probability of the event H = $\frac{1}{2}$

The probability of the event TH = $\frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^2$

The probability of the event TTH = $\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^3$

The probability of the event TTTH = $\left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^4$

and so on.

(a) The probability that to get a head, the coin will be tossed exactly 4 times is $\left(\frac{1}{2}\right)^4$.

(b) The event that the coin will be tossed at most 4 times is a compositive event comprising of the 4 sample events H, TH, TTH and TTTH. Thus the required probability

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{15}{16}.$$

(c) Suppose that the coin is tossed as many times as is necessary to get a head. The required probability is

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ ...$$

Notes

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Thus it is certain that a head will ultimately appear if the coin is tossed indefinitely. The result holds even if the coin is biased.

Incidentally, it is verified that P (S) = 1.

Note: In this example we find that though the number of points are infinite, these can be arranged according to the sequence of natural numbers (such that there is one-one correspondence between the natural numbers and the sample points); such an infinity of numbers is called *denumerable infinity* (or *countable infinity*) of numbers.

Discrete Sample space

A sample space that consists of a finite number of sample points or a denumerably infinite number of sample points is called a *discrete sample space.*

Self-Assessment

1. Tick the Correct Answer:

(*i*) Addition theorem states that if two events A and B are mutually exclusive, the probability of occurrence of either A or B is given by:

(*a*) P (A) + P (B) (*b*) P (A) × P (B)

(*c*) P (A) – P (B) (*d*) P (A) × (B) – P (AB)

(*ii*) If two events A and B are independent, the probability that they will both occur is given by:

(*a*) P (A) + P (B) (*b*) P (A) × P (B)

(*c*) P (A) – P (B) (*d*) P (A) × (B) + P (AB)

(*iii*) If two events A and B are dependent, the conditional probability of B given A, *i.e.*, P (B | A) is calculated as:

(*a*) P (AB) | P (B) (*b*) P (A) | P (B)

(*c*) P (AB) | P (A) (*d*) P (A) | P (AB)

(*iv*) If two events A and B are dependent, the conditional probability of A given B, *i.e.*, P (A | B) is calculated as:

(*a*) P (B | A) (AB) (*b*) P (B) | P (A)

(*c*) P (AB) | P (A) (*d*) P (AB) | P (B)

(*v*) $^5C_2$ is equal to

(*a*) 20 (*b*) 10

(*c*) 30 (*d*) 100

14.3 Summary

- It is easier to compute the probability of an event from known probabilities of other events. This can be well observed if the given event can be represented as the **union of two other events** or as the complement of an event.
- The probability of the conditional event is called *conditional probability*. For a conditional event, instead of the whole sample space we have only the sample points comprising of the enent A, *i.e.* the 6 sample points (5, 1), (5, 2), (5, 3), (5, 4), (5, 5) and (5, 6), and the conditional probability

Notes

of each of these is $\frac{1}{6}$. Conditioned by the event A, *i.e.* that the first die shows a five, the (conditional) event that the sum is nine comprises of only one sample point (5, 4) *i.e*. the sample point common to both A and B. Here the conditional probability of getting a sum of nine given that the first die shows a five is $\frac{1}{6}$, or in symbols P (B | A) = $\frac{1}{6}$.

- Consider a conditional event (B | A) *i.e*. the event B given that A has actually happend. Then for the happening of the event (B | A), the sample space is restricted to the sample points comprising the event A. The conditional probability P (B | A) is given by

$$P(B|A) = \frac{i}{j}.$$

- These ideas are not equivalent ideas. We discuss them to bring out the difference between the two. When the happening of one event precludes the happening of the other event, the two events are mutually exclusive (or (disjoint). For two mutually exclusive events A and B P (AB) = 0.
- When the happening of one event has no effect on the probability of occurrence of happening of the other event, the two events are independent. For two independent events A and B,

  P (AB) = P (A) P (B).
- Two events can be mutually exclusive and not independent. Again two events can be independent and not mutually exclusive.

  Suppose two coins are tossed. The events {H, H} $\equiv$ A (head on both coins) and the event {T, T} $\equiv$ B are mutually exclusive (because if A happens B cannot happen) and P (AB) = 0. But P (A) = $\frac{1}{4}$, P (B) = $\frac{1}{4}$ and so P (AB) = 0 $\neq$ P (A) P (B) and hence A and B are not independent.
-
- A sample space that consists of a finite number of sample points or a denumerably infinite number of sample points is called a *discrete sample space.*

14.4 Key-Words

1. Effective sample size : The sample size needed in equal-sized groups to achieve the power when we have groups of unequal sizes. It will generally be less than the total number of subjects in the unequal groups.
2. Efficiency : The degree to which repeated values for a statistic cluster around the parameter.

14.5 Review Questions

1. State and prove the multiplicative theorem of probability. How is the result modified when the events are independents ?
2. State and prove the addition rule of probability.
3. Differentiate between the circumstances when the probabilities of two events (i) added, and (ii) multiplied.
4. Discuss the general rule for probability. What is its form if the concerned events are mutually exclusive ?
5. Distinguish between addition and multiplicative rule of probability.

Answers: Self-Assessment

1. (*i*) (*a*) (*ii*) (*b*) (*iii*) (*c*) (*iv*) (*d*) (*v*) (*b*)

14.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 15: Theory of Estimation: Point Estimation, Unbiasedness, Consistency, Efficiency and Sufficiency

CONTENTS

Objectives

After reading this unit students will be able to:

- Explain Point Estimation and Unbiasedness.
- Discuss Consistency, Efficiency and Sufficiency.
- Know the Application of Point Estimation.

Introduction

The topic of estimation in Statistics deals with estimation of population parameters like mean of a statistical distribution. It is assumed, that the concerned variable of the population follows a certain distribution with some parameter(s). For instance, it may be assumed that the life of the electric bulbs follows a normal distribution which has two parameters *viz.* mean (*m*) and standard deviation (σ). While one of the parameters, say, standard deviation is known to be equal to 200 hours from past experience, the other parameter, *viz.* the mean life of the bulbs, is not known, and which we wish to estimate.

Given a sample of observations $x_1, x_2, x_3, ..., x_n$, one is required to determine with the aid of these observations, an estimate in the form of a specific number like 2500 *hrs.*, in the above case. This number can be taken to be the best value of the unknown mean. Such single value estimate is called **'Point'** estimate. The estimation could also be in the form of an interval, say 2,300 to 2,700 *hrs*. This can be taken to include the value of the unknown mean. This called **'Interval Estimation'**. An example of point and interval estimation could be provided from our day-to-day conversation when we talk about commuting time to office. We do make statements like "It takes about 45 minutes ranging from 40 to 50 minutes depending on the traffic conditions." The statistical details of these two types of estimation are described below.

15.1 Point Estimation

A point estimate is a single value, like 10, analogous to a point in a geometrical sense. It is used to estimate a population parameter, like mean, with the help of a sample of observations.

It may be noted that the observations $x_1, x_2, x_3, ... x_n$ are random variables, and therefore, any function of these observations will also be a random variable. Any function of the sample observations is

Notes

called a **Statistic.** For example, the arithmetic mean $\bar{x}$ of the sample $x_1, x_2, x_3, ..., x_n$, is also a random variable, as also a Statistic. This is illustrated by the numerical example given below:

Let the population comprise only 3 values, say 1, 2 and 3. If a sample of size 2 is taken, then there are 3 possible samples *viz.* 1 & 2, 1 & 3, 2 & 3.

It may be noted from the following Table 1 that the sample means are much closer to each other (in the range from 1.5 to 2.5) than the population values (in the range from 1 to 3). This is quantified by the variance calculated in both the cases. While the variance of the population values is 2/3, the variance of sample means is only 1/6.

Table 1: Variance of Sample Means

Population Values	Arithmetic Mean of Population	Variance	Samples of Two values	Arithmetic Mean of the Three Samples	Variance of the Three Sample Means
1	2	2/3	1, 2	1.5	1/6
2			1, 3	2.0	
3			2, 3	2.5	

In general, if the variance of the population with finite units is σ^2, the variance of the sample means from the population is {(N – n)/(N – 1)} $\left(\sigma^2/n\right)$, where n is the size of each sample and N is the population size. In the above case, N = 3, and n = 2. Therefore, variance of sample mean = {(3 – 2)/ (3 – 1)} {(2/3)/2} = 1/3 × 1/2 = 1/6.

However, if the population size is large as compared to the sample, then the variance of the sample mean is simply σ^2/n.

Incidentally, the standard deviation of the sample mean is known as the **standard error** of the mean. It is a measure of the extent to which sample means could be expected to vary from sample to sample. No statistic can be guaranteed to provide a close value of the parameter on each and every occasion, and for every sample. Therefore, one has to be content with formulating a rule/method which provides good results in the long run or which has a high probability of success.

Did u know? Incidentally, while the method or rule of estimation is called an estimator like sample mean, the value which the method or rule gives in a particular case is called an estimate.

Between two estimators, the estimator with lesser variance is preferred as a value obtained through any sample is more likely to be near the actual value of the parameter. For example, in Figure 1, the estimator 'A' is preferred as its variation is lesser than 'B'.

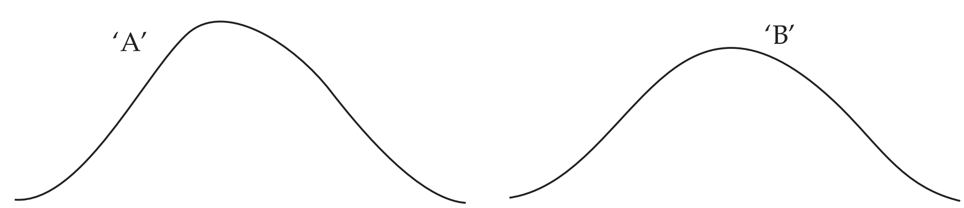


Figure 1: Distributions of Estimators 'A' and 'B'

The real exercise in estimation is to find an estimator. The merit of an estimator is judged by the distribution of estimates to which it gives rise *i.e.* by the properties of its sampling distribution as pointed above.

15.2 Unbiasedness, Consistency, Efficiency and Sufficiency

There can be more than one estimators of a population parameter. For example, the population mean (μ) may be estimated either by sample mean $(\overline{X})$ or by sample median (M) or by sample mode (Z), etc. Similarly, the population variance (σ^2) may be estimated either by the sample variance (s^2), sample S.D. (s), sample mean deviation, etc. Therefore, it becomes necessary to determine a good estimator out of a number of available estimators. A good estimator is one which is as close to the true value of the parameter as possible. A good estimator possess the following characteristics or properties:

(1) Unbiasedness

(2) Consistency

(3) Efficiency

(4) Sufficiency

Let us consider them in detail:

(1) Unbiased Estimator: An estimator $\hat{\theta}$ is said be unbiased estimator of the population parameter θ if the mean of the sampling distribution of the estimator $\hat{\theta}$ is equal to the corresponding population parameter θ. Symbolically,

$$\mu_{\hat{\theta}} = \theta$$

In terms of mathematical expectation, $\hat{\theta}$ is an unbiased estimator of θ if the expected value of the estimator is equal to the parameter being estimated. Symbolically,

$$E(\hat{\theta}) = \theta$$

Example 1: Sample mean $\overline{X}$ is an unbiased estimate of the population mean μ because the mean of the sampling distribution of the means $\mu_{\overline{X}}$ or $E(\overline{X})$ is equal to the population mean μ. Symbolically,

$$\mu_{\overline{X}} = \mu \text{ or } E(\overline{X}) = \mu$$

Example 2: Sample variance s^2 is a biased estimate of the population variance σ^2 because the mean of the sampling distribution of variance is not equal to the population variance. Symbolically,

$$\mu_{s^2} \neq \sigma^2 \text{ or } E(s^2) \neq \sigma^2$$

However, the modified sample variance $(\hat{s}^2)$ is unbiased estimate of the population variance σ^2 because

$$E(\hat{s}^2) = \sigma^2 \text{ where, } \hat{s}^2 = \frac{n}{n-1} \times s^2$$

Example 3: Sample proportion p is an unbiased estimate of the population proportion P because the mean of the sampling distribution of proportion is equal to the population proportion. Symbolically,

$$\mu_p = P \text{ or } E(P) = P$$

(2) Consistent Estimator: An estimator is said to be consistent if the estimator approaches the population parameter as the sample size increases. In other words, an estimator $\hat{\theta}$ is said to be consistent estimator of the population parameter θ, if the probability that $\hat{\theta}$ approaches θ is 1 an n becomes large and larger. Symbolically,

$$P\left(\hat{\theta} \to \theta\right) \to 1 \text{ as } n \to \infty$$

Note: A consistent estimator need not to be unbiased

A sufficient condition for the consistency of an estimator is that

(i) $E\left(\hat{\theta}\right) \to \theta$

(ii) $\text{Var}\left(\hat{\theta}\right) \to 0$ as $n \to \infty$

Example 4: Sample mean $\overline{X}$ is a consistent estimator of the population mean μ because the expected value of the sample mean approaches the population mean and the variance of the sample mean approaches zero as the size of the sample is sufficiently increased. Symbolically,

(i) $E\left(\overline{X}\right) \to \mu$

(ii) $\text{Var}\left(\overline{X}\right) = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$

Example 5: Sample median is also consistent estimator of the population mean because:

(i) $E(M) \to \mu$

(ii) Var (M) $\to$ 0 as $n \to \infty$

(3) Efficient Estimator: Efficiency is a relative term. Efficiency of an estimator is generally defined by comparing it with another estimator. Let us to take two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$. The estimator $\hat{\theta}_1$ is called an efficient estimator of θ if the variance of $\hat{\theta}_1$ is less than the variance of $\hat{\theta}_2$. Symbolically,

$$\text{Var}\left(\hat{\theta}_1\right) < \text{Var}\left(\hat{\theta}_2\right)$$

Then, $\hat{\theta}_1$ is called an efficient estimator.

Example 6: Sample mean $\overline{X}$ is an unbiased and efficient estimator of the population mean (or true mean) than the sample median M because the variance of the sampling distribution of the means is less than the variance of the sampling distribution of the medians.

The relative efficiency of the two unbiased estimators is given below:

We know that, $\text{Var}\left(\overline{X}\right) = \frac{\sigma^2}{n}$, $\text{Var}(M) = \frac{\pi}{2} \cdot \frac{\sigma^2}{n}$

$$\text{Efficiency} = \frac{\text{Var}(\overline{X})}{\text{Var}(M)} = \frac{\frac{\sigma^2}{n}}{\frac{\pi\sigma^2}{2n}} = \frac{2}{\pi} = \frac{14}{22} = \frac{7}{11} = 0.64 \left[\because \pi = \frac{22}{7}\right]$$

$$\therefore \quad \text{Var}(\overline{X}) = 0.64 \text{ Var (M)}$$

Therefore, sample mean $\overline{X}$ is 64% more efficiency than the sample median.

Hence, the sample mean is more efficient estimator of the population mean as compared to sample median.

(4) Sufficient Estimator: The last property that a good estimator should possess is sufficiency. An estimator $\hat{\theta}$ is said to be a 'sufficient estimator' of a parameter θ if it contains all the informations in the sample regarding the parameter. In other words, a sufficient estimator utilises all informations that the given sample can furnish about the population. Sample means $\overline{X}$ is said to be a sufficient estimator of the population mean.

15.3 Application of Point Estimation

The applications relating to point estimation are studied under two headings:

(1) Point Estimation in case of Single Sampling

(2) Point Estimation in case of Repeated Sampling.

(1) Point Estimation in case of Single Sampling: When a single independent random sample is drawn from the unknown population, the point estimate of the population parameter can be illustrated by the following examples:

Example 7: A sample of 10 measurements of the diameter of a sphere gave a mean $\overline{X}$ = 4.38 inches and a standard deviation = .06 inches. Determine the unbiased and efficient estimates of (a) the true mean (*i.e.*, population mean) and (b) the true variance (*i.e.*, population variance).

Solution: We are given: $n = 10$, $\overline{X} = 4.38$, $s = .06$

(a) The unbiased and efficient estimate of the true mean μ is given by:

$$\overline{X} = 4.38$$

(b) The unbiased and efficient estimate of the true variance σ^2 is:

$$\hat{s}^2 = \frac{n}{n-1} \cdot s^2$$

Putting the values, we get

$$\hat{s}^2 = \frac{10}{10-1} \times .06 = 1.11 \times 0.06 = .066$$

Thus, $\mu = 4.38$, $\sigma^2 = 0.666$

Example 8: The following five observations constitute a random sample from an unknown population:

Notes

6.33, 6.37, 6.36, 6.32 and 6.37 centimeters.

Find out unbiased and efficient estimates of (a) true mean, and (b) true variance.

Solution: (a) The unbiased and efficient estimate of the true mean (*i.e.* population mean) is given by the value of

$$\overline{X} = \frac{\Sigma X}{n} = \frac{6.33+6.37+6.36+6.32+6.37}{5} = \frac{31.75}{5} = 6.35$$

(b) The unbiased and efficient estimate of the true variance (*i.e.,* population variance) is:

$$\hat{s}^2 = \frac{\Sigma\left(X-\overline{X}\right)^2}{n-1}$$

where, $\hat{s}^2$ = modified sample variance.

$$= \frac{(6.33-6.35)^2+(6.37-6.35)^2+(6.36-6.35)^2}{5-1}$$

$$+(6.32-6.35)^2+(6.37-6.35)^2$$

$$= \frac{.0022}{4} = .00055\ cm^2$$

Example 9: The following data relate to a random sample of 100 students in Kurukshetra University classified by their weights (*kg*):

Weight (*kg*):	60 – 62	63 – 65	66 – 68	69 – 71	72 – 74
No. of Students:	5	18	42	27	8

Determine unbiased and efficient estimates of (a) population mean and (b) population variance.

Solution:

Calculation of Mean and Variance

Weight	No. of Students (*f*)	M.V. (*m*)	A = 67 *d* = *m* – A	*d′* = *d*/3	*fd′*	*fd′*^{2}
60 – 62	5	61	– 6	– 2	– 10	20
63 – 65	18	64	– 3	– 1	– 18	18
66 – 68	42	67	0	0	0	0
69 – 71	27	70	+ 3	+ 1	+ 27	27
72 – 74	8	73	+ 6	+ 2	+ 16	32
	n = 100				$\Sigma fd'$ = 15	$\Sigma fd'^2$ = 97

(a) The unbiased and efficient estimate of the population mean is given by the value:

$$\overline{X} = A + \frac{\Sigma fd'}{n} \times i$$

$$= 67 + \frac{15}{100} \times 3 = 64 + (0.45) = 67.45$$

(b) The unbiased and efficient estimate of the population variance is:

$$\hat{s}^2 = \frac{n}{n-1} \cdot s^2$$

where, $$s^2 = \frac{\Sigma fd'^2}{n} - \left(\frac{\Sigma fd'}{n}\right)^2 \times i^2$$

$$= \left[\frac{97}{100} - \left(\frac{15}{100}\right)^2\right] \times 3^2$$

$$= [0.97 - .0225] \times 9 = 8.5275$$

Now, $$\hat{s}^2 = \frac{n}{n-1} s^2 = \frac{100}{99} \times 8.5275 = 8.6136$$

Thus, $\mu = 67.45$, $\sigma^2 = 8.6136$

(2) Point Estimation in Case of Repeated Sampling: When large number of random samples of same size are drawn from the population with or without replacement, then the point estimates of the population parameter can be illustrated by the following examples:

Example 10: A population consists of five values: 3, 4, 5, 6 and 7. List all possible samples of size 3 without replacement from this population and calculate the mean $\overline{X}$ of each sample. Verify that sample mean $\overline{X}$ is an unbiased estimate of the population mean.

Solution: The population consists of the five values: 3, 4, 5, 6, 7. The total number of possible samples of size 3 without replacement are $5_{c_3} = 10$ which are shown in the following table:

Sample No. **(1)**	**Sample Values** **(2)**	**Sample Mean $(\overline{X})$** **(3)**
1	(3, 4, 5)	$\frac{1}{3}(3+4+5) = \frac{12}{3} = 4$
2	(3, 4, 6)	$\frac{1}{3}(3+4+6) = \frac{13}{3} = 4.33$
3	(3, 4, 7)	$\frac{1}{3}(3+4+7) = \frac{14}{3} = 4.67$
4	(3, 5, 6)	$\frac{1}{3}(3+5+6) = \frac{14}{3} = 4.67$
5	(3, 5, 7)	$\frac{1}{3}(3+5+7) = \frac{15}{3} = 5.0$
6	(3, 6, 7)	$\frac{1}{3}(3+6+7) = \frac{16}{3} = 5.33$

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7	(4, 5, 6)	$\frac{1}{3}(4+5+6) = \frac{15}{3} = 5.00$
8	(4, 5, 7)	$\frac{1}{3}(4+5+7) = \frac{16}{3} = 5.33$
9	(4, 6, 7)	$\frac{1}{3}(4+6+7) = \frac{17}{3} = 5.67$
10	(5, 6, 7)	$\frac{1}{3}(5+6+7) = \frac{18}{3} = 6.00$
Total	***k* = 10**	$\Sigma\overline{X}$ **= 50**

Mean of Sampling Distribution of Means = $\mu_{\overline{X}} = \frac{\Sigma\overline{X}}{k} = \frac{50}{10} = 5.$

Population Mean = $\mu = \frac{3+4+5+6+7}{5} = 5$

Since, $\mu_{\overline{X}} = \mu$, sample mean $\overline{X}$ is an unbiased estimate of the population mean μ.

Example 11: Consider a hypothetical population comprising three values: 1, 2, 3. Draw all possible samples of size 2 with replacement. Calculate the mean $\overline{X}$ and variance s^2 for each sample. Examine whether the two statistics ($\overline{X}$ and s^2) are unbiased and efficient for the corresponding parameters.

Solution: The population consists of three values: 1, 2 and 3. The total number of possible samples of size 2 with replacement are $N^n = 3^2 = 9$ which are given by

Sample No.	**Sample Values**	**Sample Mean** $(\overline{X})$	**Sample Variance** $s^2 = \frac{1}{2}\left[(x_1-\bar{x})^2+(x_2-\bar{x})^2\right]$	**Modified Sample Variance** $\left(\hat{s}^2 = \frac{n}{n-1}s^2\right)$
1.	(1, 1)	$\frac{1}{2}(1+1) = 1.0$	$\frac{1}{2}\cdot\left[(1-1)^2+(1-1)^2\right] = 0.00$	0.00
2.	(1, 2)	$\frac{1}{2}(1+2) = 1.5$	$\frac{1}{2}\cdot\left[(1-1.5)^2+(2-1.5)^2\right] = 0.25$	0.50
3.	(1, 3)	$\frac{1}{2}(1+3) = 2.0$	$\frac{1}{2}\cdot\left[(1-2)^2+(3-2)^2\right] = 1.00$	2.00
4.	(2, 1)	$\frac{1}{2}(2+1) = 1.5$	$\frac{1}{2}\cdot\left[(2-1.5)^2+(1-1.5)^2\right] = 0.25$	0.5
5.	(2, 2)	$\frac{1}{2}(2+2) = 2.0$	$\frac{1}{2}\cdot\left[(2-2)^2+(2-2)^2\right] = 0.00$	0.00
6.	(2, 3)	$\frac{1}{2}(2+2) = 2.5$	$\frac{1}{2}\cdot\left[(2-2.5)^2+(3-2.5)^2\right] = 0.25$	0.50

Notes

7.	(3, 1)	$\frac{1}{2}(3+1)$ = 2.0	$\frac{1}{2}\cdot\left[(3-2)^2+(1-2)^2\right]$ = 1.00	2.00
8.	(3, 2)	$\frac{1}{2}(3+2)$ = 2.5	$\frac{1}{2}\left[(3-2.5)^2+(2-2.5)^2\right]$ = 0.25	0.50
9.	(3, 3)	$\frac{1}{2}(3+3)$ = 3.0	$\frac{1}{2}\left[(3-3)^2+(3-3)^2\right]$ = 0.00	0.00
Total	***k* = 9**	$\Sigma\overline{X}$ **= 18**		$\Sigma\hat{s}^2$ **= 6**

(a) Mean of Sampling Distribution of Means = $\mu_{\overline{X}} = \frac{\Sigma\overline{X}}{k} = \frac{18}{9}$ = 2. Here, K = No. of samples.

Population Mean $\mu = \frac{1+2+3}{3}$ = 2.

Since, $\mu_{\overline{X}} = \mu$, sample mean $\overline{X}$ is an unbiased estimate of the population mean μ.

(b) Mean of the Sampling Distribution of Variance = $\mu_{s^2} = \frac{\Sigma s^2}{k} = \frac{3}{9} = \frac{1}{3}$

Population Variance $\sigma^2 = \frac{(1-2)^2+(2-2)^2(3-2)^2}{3} = \frac{2}{3}$

Since, $\mu_{s^2} \neq \sigma^2$, sample variance s^2 is not an unbiased estimate of the population variance $\left(\sigma^2\right)$.

But the modified sample variance defined as $\hat{s}^2 = \frac{n}{n-1}s^2$ will be unbiased estimate of the population variance σ^2 because:

$$\mu_{\hat{s}^2} = \frac{\Sigma\hat{s}^2}{k} = \frac{6}{9} = \frac{2}{3}$$

$$\sigma^2 = \frac{2}{3}$$

$$\therefore \quad \mu_{\hat{s}^2} = \sigma^2$$

Since $\mu_{\hat{s}^2} = \sigma^2$, the modified sample variation is an unbiased estimate of the population variance.

Example 12: Show that the sample mean $\left(\overline{X}\right)$ is an unbiased estimate of the population mean.

or

An independent random sample $x_1, x_2, x_3, ..., x_n$ is drawn from a population with mean μ. Prove that the expected value of the sample mean $\overline{X}$ equals the population mean μ.

Solution: A random sampling is one where each sample has an equal chance of being selected. We draw a random sample of size '*n*'.

Then,

$$E(\bar{x}) = E\left[\frac{x_1 + x_2 + ... + x_n}{n}\right] \text{ Where } x_1 \text{ is the sample observation.}$$

$$= \frac{1}{n} \cdot \left[E(x_1) + E(x_2) + ... E(x_n)\right]$$

Now the expected values of x_i (a member of the population) is population mean μ.

$$\therefore \; E(\bar{x}) = \frac{1}{n}[\mu + \mu + ... + \mu] \quad [\because E(x_1) = E(x_2) = ... E(x_n) = \mu]$$

$$= \frac{1}{n} \cdot [n\mu] = \mu \qquad [\because \Sigma C = C_1 + C_2 + ... C_n = nC]$$

Thus, sample mean $\overline{X}$ is an unbiased estimate of population mean.

Self-Assessment

1. Fill in the Blanks:

(*i*) The two types of estimates are .... and ....

(*ii*) The numerical value of a sample mean is said to be an estimate of the population .... figure.

(*iii*) A point estimate is a single number which is used as an estimate of the unknown .... parameter.

(*iv*) Point estimate provides one single value of the ....

(*v*) Parameter of a sample denoted by ....

15.4 Summary

- The topic of estimation in Statistics deals with estimation of population parameters like mean of a statistical distribution. It is assumed, that the concerned variable of the population follows a certain distribution with some parameter(s). For instance, it may be assumed that the life of the electric bulbs follows a normal distribution which has two parameters *viz.* mean (*m*) and standard deviation (σ). While one of the parameters, say, standard deviation is known to be equal to 200 hours from past experience, the other parameter, *viz.* the mean life of the bulbs, is not known, and which we wish to estimate.
- An example of point and interval estimation could be provided from our day-to-day conversation when we talk about commuting time to office. We do make statements like "It takes about 45 minutes ranging from 40 to 50 minutes depending on the traffic conditions." The statistical details of these two types of estimation are described below.
- A point estimate is a single value, like 10, analogous to a point in a geometrical sense. It is used to estimate a population parameter, like mean, with the help of a sample of observations.
- It may be noted that the observations $x_1, x_2, x_3, ... x_n$ are random variables, and therefore, any function of these observations will also be a random variable. Any function of the sample observations is called a **Statistic.**
- the standard deviation of the sample mean is known as the **standard error** of the mean. It is a measure of the extent to which sample means could be expected to vary from sample to sample.

- No statistic can be guaranteed to provide a close value of the parameter on each and every occasion, and for every sample. Therefore, one has to be content with formulating a rule/method which provides good results in the long run or which has a high probability of success.
- **Incidentally, while the method or rule of estimation is called an estimator like sample mean, the value which the method or rule gives in a particular case is called an estimate.**
- An estimator $\hat{\theta}$ is said be unbiased estimator of the population parameter θ if the mean of the sampling distribution of the estimator $\hat{\theta}$ is equal to the corresponding population parameter θ.
- An estimator is said to be consistent if the estimator approaches the population parameter as the sample size increases. In other words, an estimator $\hat{\theta}$ is said to be consistent estimator of the population parameter θ, if the probability that $\hat{\theta}$ approaches θ is 1 an n becomes large and larger.
- Efficiency is a relative term. Efficiency of an estimator is generally defined by comparing it with another estimator. Let us to take two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$. The estimator $\hat{\theta}_1$ is called an efficient estimator of θ if the variance of $\hat{\theta}_1$ is less than the variance of $\hat{\theta}_2$.
- The last property that a good estimator should possess is sufficiency. An estimator $\hat{\theta}$ is said to be a 'sufficient estimator' of a parameter θ if it contains all the informations in the sample regarding the parameter. In other words, a sufficient estimator utilises all informations that the given sample can furnish about the population. Sample means $\overline{X}$ is said to be a sufficient estimator of the population mean.

15.5 Key-Words

1. Deviation scores : Data in which the mean has been subtracted from each observation.
2. Descriptive statistics : Statistics which describe the sample data without drawing inferences about the larger population.

15.6 Review Questions

1. What is Estimation ? How many types of estimates are possible ?
2. Explain the properties of a good estimator ?
3. What do you understand by point estimator ?
4. Discuss the application of point estimation.
5. Distinguish between consistency and efficiency.

Answers: Self-Assessment

1. (*i*) Point estimate, interval estimate (*ii*) Mean (*iii*) Population
   (*iv*) Parameter (*v*) θ.

15.7 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

Unit 16: Methods of Point Estimation and Interval Estimation

Notes

CONTENTS

Objectives

After reading this unit students will be able to:

- Discuss the Methods of Point Estimation.
- Explain the Interval Estimation.

Introduction

The object of sampling is to study the features of the population on the basis of sample observations. A carefully selection sample is expected to reveal these features, and hence we shall infer about the population from a statistical analysis of the sample. This process is known as *Statistical Inference.*

There are two types of problems. Firstly, we may have no information at all about some characteristics of the population, especially the values of the parameters involved in the distribution, and it is required to obtain estimates of these parameters. This is the problem of *Estimation.* Secondly, some information or hypothetical values of the parameters may be available, and it is required to test how far the hypothesis is tenable in the light of the information provided by the sample. This is the problem of *Test of Hypothesis* or *Test of Significance.*

Suppose we have a random sample $x_1, x_2, \ldots x_n$ on a variable x, whose distribution in the population involves an unknown parameter θ. It is required to find an estimate of θ on the basis of sample values. The estimation is done in two different ways: (i) *Point Estimation,* and (ii) *Interval Estimation.* In point estimation, the estimated value is given by a single quantity, which is a function of sample observations (*i.e.* statistic). This function is called the '*estimator*' of the parameter, and the value of the estimator in a particular sample is called an '*estimate*'. In interval estimation, an interval within which the parameter is expected to lie is given by using two quantities based on sample values. This is known as *Confidence Interval,* and the two quantities which are used to specify the interval, are known as *Confidence Limits.*

16.1 Methods of Point Estimation

(1) Method of Maximum Likelihood

This is a convenient method for finding an estimator which satisfies most of the criteria discussed earlier. Let $x_1, x_2, \ldots x_n$ be a random sample from a population with p.m.f. (for discrete case) or p.d.f. (for continuous case) $f(x, \theta)$, where θ is the parameter. Then the joint distribution of the sample observations viz.

Notes

$$L = f(x_1,\theta), f(x_2,\theta), ... f(x_n,\theta) \quad ... (1)$$

is called the *Likelihood Function* of the sample.

The *Method of Maximum Likelihood* consists in choosing as an estimator of θ that statistic, which when substituted for θ, maximises the likelihood function L. Such a statistic is called a *maximum likelihood estimator* (m.l.e.). We shall denote the m.l.e. of θ by the symbol θ_0.

Since log L is maximum when L is maximum, in practice the m.l.e. of θ is obtained by maximising log L. This is achieved by differentiating log L partially with respect to θ, and using the two relations

$$\left[\frac{\partial}{\partial\theta}\log L\right]_{\theta=\theta_0} = 0, \left[\frac{\partial^2}{\partial\theta^2}\log L\right]_{\theta=\theta_0} < 0 \quad ... (2)$$

Properties of maximum likelihood estimator (m.l.e.)

(1) The m.l.e. is consistent, most efficient, and also sufficient, provided a sufficient estimator exists.

(2) The m.l.e. is not necessarily unbiased. But when the m.l.e. is biased, by a slight modification, it can be converted into an unbiased estimator.

(3) The m.l.e. tends to be distributed normally for large samples.

(4) The m.l.e. is invariant under functional transformations. This means that if T is an m.l.e. of θ, and $g(\theta)$ is a function of θ, then g (T) is the m.l.e. of $g(\theta)$.

Example 1: On the basis of a random sample find the maximum likelihood estimator of the parameter of a Poisson distribution.

Solution: The Poisson distribution with parameter m has p.m.f.

$$f(x, m) = \frac{e^{-m} \cdot m^x}{x!} \quad (x = 0, 1, 2, ... \infty)$$

The likelihood function of the sample observations is

$$L = f(x_1,m) \cdot f(x_2,m) . ... f(x_n,m)$$

$$\therefore \quad \log L = \log f(x_1,m) + \log f(x_2,m) + ... + \log f(x_n,m)$$

$$= \sum_{i=1}^{n} \log f(x_i,m)$$

$$= \sum\left[-m + x_i(\log m) - \log x_i!\right]$$

$$= -nm + (\log m)\Sigma x_i - \Sigma\log(x_i!)$$

Taking partial derivative of log L with respect to the parameter m,

$$\frac{\partial \log L}{\partial m} = -n + \frac{\Sigma x_i}{m} = -n + \frac{n\bar{x}}{m}$$

Now replacing m by m_0 and equating the result to zero,

$$\left[\frac{\partial \log L}{\partial m}\right]_{m = m_0} = -n + \frac{n\bar{x}}{m_0} = 0$$

Solving, we get $m_0 = \bar{x}$. Again,

$$\left[\frac{\partial^2 \log L}{\partial m^2}\right]_{m = m_0} = -\frac{n\bar{x}}{m_0^2} = -\frac{n\bar{x}}{\bar{x}^2} = -\frac{n}{\bar{x}} \text{ which is negative.}$$

This shows that log L is maximum at $m = m_0 = \bar{x}$. That is the m.l.e. of m is $m_0 = \bar{x}$, the sample mean.

Example 2: Find the maximum likelihood estimator of the variance σ^2 of a Normal population $N(\mu, \sigma^2)$, when the parameter μ is known. Show that this estimator is unbiased.

Solution: The p.d.f. of Normal distribution is

$$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \; ; \; (-\infty < x < +\infty)$$

The logarithm (to the base e) of the likelihood function L is

$$\log L = \sum_{i=1}^{n} \log f(x_i, \mu, \sigma^2)$$

$$= \sum\left[-\log\sigma - \frac{1}{2}\log(2x) - \frac{(x_i - \mu)^2}{2\sigma^2}\right]$$

$$= -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log(2x) - \frac{\Sigma(x_i - \mu)^2}{2\sigma^2}$$

Differentiating partially with respect to σ^2,

$$\frac{\partial}{\partial(\sigma^2)}\log L = -\frac{n}{2\sigma^2} + \frac{\sum(x_i - \mu)^2}{2(\sigma^2)^2}$$

The m.l.e. of σ^2 is obtained by solving

$$-\frac{n}{2\sigma^2} + \frac{\Sigma(x_i - \mu)^2}{2\sigma_0{}^4} = 0$$

$$\therefore \qquad \sigma_0{}^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{n}$$

It can be shown that $\left[\dfrac{\partial^2 \log L}{\partial(\sigma^2)^2}\right]_{\sigma^2 = \sigma_0^2} = \dfrac{-n}{2\sigma_0^4}$

which is negative. Thus the maximum likelihood estimator of σ^2 is

$$\sigma_0{}^2 = \sum\frac{(x_i - \mu)^2}{n}, \text{ (} \mu \text{ known)}$$

Notes

Again, since $x_1, x_2, \ldots x_n$ is a random sample and μ is the population mean, we have $E(x_i - \mu)^2 = \sigma^2$. Therefore,

$$E\left(\sigma_0{}^2\right) = \sum_{i=1}^{n} \frac{E(x_i - \mu)^2}{n} = \frac{\sum \sigma^2}{n} = \sigma^2$$

Thus, $\sigma_0{}^2$ is an unbiased estimator of σ^2.

Example 3: Find the m.l.e. of the parameters μ and σ^2 in random samples from a $N\left(\mu, \sigma^2\right)$ population, when both the parameters are unknown.

Solution: As in the preceding example,

$$\log L = -\frac{n}{2}\log\sigma^2 - n\log\sqrt{2\pi} - \frac{\sum(x_i - \mu)^2}{2\sigma^2}$$

$$\therefore \qquad \left[\frac{\partial \log L}{\partial \mu}\right]_{\mu=\mu_0} = \frac{-1}{2\sigma^2}\Sigma 2(x_i - \mu_0)(-1) = 0$$

This gives $\Sigma(x_i - \mu_0) = 0$; *i.e.*, $\mu_0 = \bar{x}$, the sample mean. The m.l.e. of the parameter μ is the sample mean $\bar{x}$. (Note that this estimator is *unbiased*).

Proceeding as in Example 2 we have $\sigma_0^2 = \frac{\Sigma(x_i - \mu)^2}{n}$. But since the parameter μ is not known, it is replaced by the m.l.e. $\mu_2 = \bar{x}$. The m.l.e. of σ^2 is now

$$\sigma_0{}^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = S^2$$

which is the sample variance. (Note that this estimator is *biased*).

Example 4: A tossed a biased coin 50 times and got head 20 times, while B tossed it 90 times and got 40 heads. Find the maximum likelihood estimate of the probability of getting head when the coin is tossed.

Solution: Let P be the unknown probability of obtaining a head. Using binomial distribution,

Probability of 20 heads in 50 tosses = ${}^{50}C_{20}\ P^{20}(1-P)^{30}$

Probability of 40 heads in 90 tosses = ${}^{90}C_{40}\ P^{40}(1-P)^{50}$

The likelihood function is given by the product of these probabilities:

$$L = {}^{50}C_{20} \cdot {}^{90}C_{40}\ P^{30}(1-P)^{30}$$

$$\therefore \qquad \log L = \log\left({}^{50}C_{20}{}^{90}C_{40}\right) + 60\log P + 80\log(1-P)$$

$$\text{Hence,} \qquad \frac{\partial \log L}{\partial P} = \frac{60}{P} - \frac{80}{1-P}$$

The maximum likelihood estimate P_0 is therefore obtained by solving

Notes

$$\frac{60}{P_0} - \frac{80}{1-P_0} = 0.$$

This gives $$P_0 = \frac{60}{140} = \frac{3}{7}.$$ **Ans.** 3/7

(2) Method of Moments

The *Method of Moments* consists in equating the first few moments of the population with the corresponding moments of the sample *i.e.* setting

$$\mu_r' = m_r' \qquad \text{... (3)}$$

where $\mu_r' = E(x^r)$ and $m_r' = \frac{\sum x_i^r}{n}$. Since the parameters enter into the population moments, these relations when solved for the parameters give the estimates by the method of moments. Of course, this method is applicable only when the population moments exist. The method is generally applied for fitting theoretical distributions to observed data.

Example 5: Estimate the parameter p of the binomial distribution by the method of moments (when n is known).

Solution: For the binomial distribution $\mu_1' = E(x) = np$. Also $m_1' = \bar{x}$. Setting $\mu_1' = m_1'$, we have $np = \bar{x}$. Thus

$$p = \frac{\bar{x}}{n}$$

i.e. the estimated value of p is given by the sample mean divided by the parameter n (known).

Example 6: Find the estimates of μ and σ in the Normal population $N(\mu, \sigma^2)$ by the method of moments.

Solution: Equate the first two moments of the population and the sample, $\mu_1' = m_1'$ and $\mu_2' = m_2'$,

i.e. $\mu_2 = m_2$. Thus

$$\mu = \bar{x} \text{ and } \sigma^2 = S^2, \text{ the sample variance.}$$

The parameters μ and σ are estimated by the sample mean $\bar{x}$ and the sample standard deviation S respectively.

16.2 Interval Estimation

In the theory of point estimation, developed earlier, any unknown parameter is estimated by a single quantity. Thus the sample mean $(\bar{x})$ is used to estimate the population mean (μ), and the sample proportion (p) is taken as an estimator of the population proportion (P). A single estimator of this kind, however good it may be, cannot be expected to coincide with the true value of the parameter, and may in some cases differ widely from it. In the theory of interval estimation, it is desired to find an interval which is expected to include the unknown parameter with a specified probability.

Let $x_1, x_2, \ldots x_n$ be a random sample from a population of a known mathematical form which involves an unknown parameter θ. We would try to find two functions t_1 and t_2 based on sample observations such that the probability of θ being included in the interval (t_1, t_2) has a given value, say c.

$$P(t_1 \le \theta \le t_2) = c \qquad \text{... (4)}$$

Notes

Such an interval, when it exists, is called a *Confidence Interval* for θ. The two quantities t_1 and t_2 which serve as the lower and upper limits of the interval are known as *Confidence Limits.* The probability (c) with which the confidence interval will include the true value of the parameter is known as *Confidence Coefficient* of the interval.

The significance of confidence limits is that if many independent random samples are drawn from the same population and the confidence interval is calculated from each sample, then the parameter will actually be included in the intervals in c proportion of cases in the long run. Thus the estimate of the parameter is stated as an interval with a specified degree of confidence.

The calculation of confidence limits is based on the knowledge of sampling distribution of an appropriate statistic. Suppose, we have a random sample of size n from a Normal population $N\left(\mu,\sigma^2\right)$, where the variance σ^2 is known. It is required to find 95% confidence limits for the unknown parameter μ. We know that the sample mean $(\bar{x})$ follows normal distribution with mean μ and variance σ^2/n, and so

$$z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

has a standard normal distribution. Since 95% of the area under the standard normal curve lies between the ordinates at $z = \pm 1.96$, we have

$$P\left[-1.96 \le \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \le 1.96\right] = 0.95$$

i.e. in 95% of cases the following inequalities hold

$$-1.96 \le \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \le 1.96$$

Separating out μ we get

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

The interval $\left[\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right]$ is known as the 95% confidence interval for μ, and the 95% confidence limits are

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$$

Again, 99% of area under the standard normal curve lies between the ordinates at $z = \pm 2.58$, and 99.73% (*i.e.* almost whole) of the area lies between $z = \pm 3$. Hence proceeding exactly in the same manner, the 99% confidence limits for μ are

$$\bar{x} \pm 2.58\frac{\sigma}{\sqrt{n}}$$

and almost sure limits for μ are:

$$\bar{x} \pm 3\frac{\sigma}{\sqrt{n}}$$

In fact, using values from the normal probability integral table (showing areas under standard normal curve), confidence limits corresponding to any specified percentage can be obtained. These are *exact* confidence limits.

In some cases, the population may not be truely a normal distribution, but the sampling distributions of statistics based on large samples are approximately normal. For example, the sample mean $(\bar{x})$ based on a large random sample drawn (with or without replacement) from any population is approximately normally distributed. Similarly, the sample proportion (p) calculated from a large random sample has approximately a normal distribution. It is therefore possible to utilise the properties relating to the percentage of area under the standard normal curve to find approximate confidence limits for the population mean μ and the population proportion P, provided the sample size n is large.

Approximate Confidence Limits (large samples) any Distribution

(1) for Mean μ:

95% confidence limits = $\bar{x} \pm 1.96(\text{S.E. of } \bar{x})$

99% confidence limits = $\bar{x} \pm 2.58(\text{S.E. of } \bar{x})$

Almost sure limits = $\bar{x} \pm 3(\text{S.E. of } \bar{x})$ ... (5)

(2) for Proportion P:

95% confidence limits = $p \pm 1.96(\text{S.E. of } p)$

99% confidence limits = $p \pm 2.58(\text{S.E. of } p)$

Almost sure limits = $p \pm 3(\text{S.E. of } p)$ ... (6)

(3) for Difference of Means $(\mu_1 - \mu_2)$:

95% confidence limits = $(\bar{x}_1 - \bar{x}_2) \pm 1.96(\text{S.E. of } \bar{x}_1 - \bar{x}_2)$

99% confidence limits = $(\bar{x}_1 - \bar{x}_2) \pm 2.58(\text{S.E. of } \bar{x}_1 - \bar{x}_2)$

Almost sure limits = $(\bar{x}_1 - \bar{x}_2 \pm 3)(\text{S.E. of } \bar{x}_1 - \bar{x}_2)$ ... (7)

(4) for Difference of Proportions $(P_1 - P_2)$:

95% confidence limits = $(p_1 - p_2) \pm 1.96(\text{S.E. of } p_1 - p_2)$

99% confidence limits = $(p_1 - p_2) \pm 2.58(\text{S.E. of } p_1 - p_2)$

Almost sure limits = $(p_1 - p_2) \pm 3(\text{S.E. of } p_1 - p_2)$ ... (8)

Notes (i) The '*probable limits*' (without any reference to the degree of confidence) may be taken to be '*almost sure limits*' in all the above cases.

(ii) The formulae for S.E. involve population parameters. If these parameters are not known, an approximate value of S.E. may be obtained by substituting the statistic for the corresponding parameter.]

Example 7: A sample of 6500 screws is taken form a large consignment and 75 are found to be defective. Estimate the percentage of defectives in the consignment and assign limits within which the percentage lies.

Solution: There are 75 defectives in a sample of size n = 600. Therefore, the sample proportion of defectives is

$$p = \frac{75}{600} = \frac{1}{8} = 12.5\%$$

This may be taken as an estimate of the percentage of defectives (P) in the whole consignment ('Point estimation').

The 'limits' to the percentage of defectives refer to the confidence limits, which may be given as $p \pm 3$ (S.E. of p).

$$\text{S.E. of } p = \sqrt{\frac{PQ}{n}}$$

$$= \sqrt{\frac{pq}{n}} \text{ approximately;}$$

(since the population proportion P is not known).

$$= \sqrt{\frac{\frac{1}{8}\left(1-\frac{1}{8}\right)}{600}} = \frac{1}{80}\sqrt{\frac{7}{6}} = .0135$$

Thus, the limits for P are

$$\frac{1}{8} \pm 3 \times .0135 = .125 \pm .0405$$

$$= .1655 \text{ and } .0845 = 16.55\% \text{ and } 8.45\%$$

The limits to the percentage of defectives in the consignment are 8.45% to 16.55% ('Interval estimation').

Ans. 12.5%; 8.45% to 16.55%.

Example 8: A random sample of 100 ball bearings selected from a shipment of 2000 ball bearings has an average diameter of 0.354 inch with a S.D. = .048 inch. Find 95% confidence interval for the average diameter of these 2000 ball bearings.

Solution: Theory: If a random sample of large size n is drawn *without replacement* from a *finite population* of size N, hen the 95% confidence limits for the population mean μ are $\bar{x} \pm 1.96$ (S.E. of x), where $\bar{x}$ denotes the sample mean and

$$\text{S.E. of } \bar{x} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}.$$

σ denoting the Standard Deviation (S.D.) of the population. Here,

Sample size (n) = 100, Population size (N) = 2000

Sample mean $(\bar{x})$ = 0.354, Sample S.D. (S) = .048

Since σ is not known, an approximate value of S.E. is obtained on replacing the population S.D. (σ) by the sample S.D. (S).

$$\text{S.E. of } \bar{x} = \frac{S}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} \text{ approximately}$$

$$= \frac{.048}{\sqrt{100}}\sqrt{\frac{2000-100}{2000-1}} = .0047$$

The 95% confidence limits for the population mean μ are

$$\bar{x} \pm 1.96(\text{S.E.of } \bar{x}) = 0.354 \pm 1.96 \times .0047$$

$$= 0.354 \pm .0092 = 0.3632 \text{ and } 0.3448$$

Thus, the 95% confidence interval is (0.3448 to 0.3632) inch.

Example 9: A random sample of 100 articles taken from an large batch of articles contains 5 defective articles. (a) Set up 96 per cent confidence limits for the proportion of defective articles in the batch. (b) If the batch contains 2696 articles set up 95% confidence interval for the proportion of defective articles.

Solution: (a) The 96% confidence limits for the population proportion (P) are given by $p \pm 2.05$ (S.E. of p), where p is the sample proportion.

$$\text{S.E. of } p = \sqrt{\frac{PQ}{n}}$$

Since the formula involves the unknown population proportion P, an approximate value of S.E. is obtained on replacing the population proportion (P) by the sample proportion (p). Putting n = 100 and p = 5/100 = .05, (q = 1 – p = .95)

$$\text{S.E. of } p = \sqrt{\frac{pq}{n}} = \sqrt{\frac{.05 \times .95}{100}} = .022$$

Hence, the 96% confidence limits for P are

$$p \pm 2.05 \text{ (S.E. of } p) = .05 \pm 2.05 \times .022 = .05 \pm .045$$

$$= .05 + .045 \text{ and } .05 - .045$$

$$= .095 \text{ and } .005$$

(b) The 95% confidence limits for proportion (P) are given by $p \pm 1.96$ (S.E. of p). But, when the population is of a finite size N,

$$\text{S.E. of } p = \sqrt{\frac{pq}{n}}\sqrt{\frac{N-n}{N-1}} \text{ (approximately)}$$

Here, n = 100, N = 2696, p = .05. Putting these values

$$\text{S.E. of } p = \sqrt{\frac{05 \times .95}{100}}\sqrt{\frac{2696-100}{2696-1}} = .022\sqrt{\frac{2596}{2696}}$$

$$= .022 \times .963 = .022 \times .98 = .0216 \text{ (approx.)}$$

Hence, the required 95% confidence limits for P are

$$p \pm 1.96 \text{ (S.E. of } p) = .05 \pm 1.96 \times .0216 = .092 \text{ and } .008$$

The 95% *confidence interval* for the proportion of defective articles is 008 to .092.

Ans. .005 and .095; .008 to .092

Example 10: 10 Life Insurance Polices in a sample of 200 taken out of 50,000 were found to be insured for less than Rs. 5,000. How many policies can be reasonably expected to be insured for less than Rs. 5,000 in the whole lot at 95%. Confidence level ?

Solution: Let us first find the confidence limits for the 'proportion' of life insurance policies insured for less than Rs. 5,000 in the whole lot. Here,

Population size (N) = 50,000,

Sample size (n) = 200

$$\text{Sample proportion } (p) = \frac{10}{200} = .05$$

Using,

$$\text{S.E. of } p = \sqrt{\frac{.05 \times .95}{200}} \sqrt{\frac{50,000 - 200}{50,000 - 1}}$$

$$= \sqrt{\frac{.05 \times .95 \times 49800}{200 \times 50,000}} \text{ (approx.)}$$

$$= .0154$$

95% confidence limits for the population proportion (P) are

$$p \pm 1.96 \text{ (S.E. of } p) = .05 \pm 1.96 \times .0154$$

$$= .05 \pm .030 = .080 \text{ and } .020$$

The means that out of the lot of 50,000, the 'proportion' of policies insured for less than Rs. 5000 lies between .020 and .080, with probability 95%. Thus the 'number' of such policies lies between 50,000 × .020 = 1000 and 50,000 × .080 = 4,000.

Ans. Between 1,000 and 4,000.

Exact Confidence Limits (any sample size) Normal Distribution

In the following cases, it is assumed that samples are drawn at random from Normal populations.

(5) for Mean μ : (s.d. known)

$$\text{95\% confidence limits less} = \bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right)$$

$$\text{99\% confidence limits} = \bar{x} \pm 2.58\left(\frac{\sigma}{\sqrt{n}}\right) \quad \text{... (9)}$$

(6) for Difference of Means $(\mu_1 - \mu_2)$: (s.d.s known)

$$\text{95\% confidence limits} = (\bar{x}_1 - \bar{x}_2) \pm 1.96\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\text{99\% confidence limits} = (\bar{x}_1 - \bar{x}_2) \pm 2.58\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad \text{... (10)}$$

Notes

Example 11: A random sample of size 10 was drawn from a normal population with an unknown mean and a variance of 44.1 (inch)^2. If the observations are (in inches): 65, 71, 80, 76, 78, 82, 68, 72, 65 and 81, obtain the 95% confidence interval for the population mean.

Solution: We are given $n = 10$, $\sigma^2 = 44.1$ and $\sum x_i = 738$.

$$\therefore \qquad \bar{x} = \frac{738}{10} = 73.8.$$

Since the population s.d. σ is known, using formula (9), 95% confidence limits for μ are given by

$$73.8 \pm 1.96\frac{\sqrt{44.1}}{\sqrt{10}} = 73.8 \pm 1.96 \times \sqrt{4.41}$$

$$= 73.8 \pm 1.96 \times 2.1$$

$$= 73.8 \pm 4.1 = 77.9 \text{ and } 69.7.$$

The 95% confidence interval for μ is therefore 69.7 to 77.9 inches.

(7) for Mean *m*: (s.d. unknown)

In random samples from a Normal population $N(\mu, \sigma^2)$

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n-1}}$$

follows t distribution with $(n - 1)$ degree of freedom, where S is the sample s.d. If $t_{.025}$ denotes the upper 2.5% point of t distribution with $(n - 1)$ d.f., then the 95% confidence interval for μ is obtained from

$$-t_{.025} \le \frac{\bar{x} - \mu}{S/\sqrt{n-1}} \le t_{.025}$$

Hence, for the population mean μ

$$\text{95\% confidence limits} = \bar{x} \pm t_{.025}\left(\frac{S}{\sqrt{n-1}}\right) \qquad \text{(11 a)}$$

$$\text{99\% confidence limits} = \bar{x} \pm t_{.005}\left(\frac{S}{\sqrt{n-1}}\right) \qquad \text{(11 b)}$$

(8) for Difference of Means $(\mu_1 - \mu_2)$: (common s.d. unknown)

Assuming that two independent samples are drawn from two Normal populations with means μ_1, μ_2 but a common *unknown* s.d. σ.

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_1}}}$$

follows t distribution with $(n_1 + n_2 - 2)$ d.f., where

$$s^2 = \frac{(n_1 S_1^2 + n_2 S_2^2)}{(n_1 + n_2 - 2)}$$

Notes

Hence, with 95% probability the following inequalities hold

$$-t_{.025} \le \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \le t_{.025}$$

from which the 95% confidence limits for $(\mu_1 - \mu_2)$ are

$$(\bar{x}_1 - \bar{x}_2) \pm t_{.025} \cdot s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \qquad \text{... (12 a)}$$

Similarly the 99% confidence limits for $(\mu_1 - \mu_2)$ are

$$(\bar{x}_1 - \bar{x}_2) \pm t_{.005} \cdot s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \qquad \text{... (12 b)}$$

Example 12: A random sample of 10 students of class II was selected from schools in a certain region, and their weights recorded are shown below (in. lb.): 38, 46, 45, 40, 35, 39, 44, 45, 33, 37. Find 95% confidence limits within which the mean weight of all such students in the region is expected to lie. (Given $t_{.025}$ = 2.262 for 9 d.f. and 2.228 for 10 d.f.).

Solution: From the given data, $\bar{x} = \frac{402}{10}$ = 40.2. To calculate the s.d. (S), we take deviations from 40, *i.e.* $d = x - 40$.

– 2, 6, 5, 0, – 5, – 1, 4, 5, – 7, – 3

$$\sum d = 2, \ \sum d^2 = 190$$

$$\therefore \quad S^2 = \frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2 = \frac{190}{10} - \left(\frac{2}{10}\right)^2 = 18.96$$

$$S = \sqrt{18.96} = 4.35.$$

Since the population s.d. σ is *unknown* the 95% confidence limits for μ are (see formula 12 a)

$40.2 \pm 2.262 \times 4.35/\sqrt{9}$ (degrees of freedom = 9)

= 40.2 ± 3.28 = 36.92 and 43.48

The 95% confidence limits for the mean weight are (in lb.) 36.9 and 43.5.

(9) for Variance σ^2

Case I:

(*mean known*) – In random samples from $N(\mu, \sigma^2)$ population, $\sum (x_i - \mu)^2/\sigma^2$ follows chi-square distribution with *n* degrees of freedom. If $\chi^2_{.975}$ and $\chi^2_{.025}$ denote the lower and the upper 2.5% points of chi-square distribution with *n* d.f., then with probability 95% we have

$$\chi^2_{.975} \le \sum \frac{(x_i - \mu)^2}{\sigma^2} \le \chi^2_{.025}$$

From which the 95% confidence interval for σ^2 is

$$\frac{\Sigma(x_i - \mu)^2}{\chi^2_{.025}} \le \sigma^2 \le \frac{\Sigma(x_i - \mu)^2}{\chi^2_{.975}} \qquad \text{... (13)}$$

Case II:

(*mean unknown*) — In this case $\frac{nS^2}{\sigma^2} = \frac{\Sigma(x_i - \bar{x})^2}{\sigma^2}$ follows chi-square distribution with $(n - 1)$ degrees of freedom. Using the lower and the upper 2.5% points of chi-square distribution with $(n - 1)$ d.f., we have with probability 95% the following inequalities

$$\chi^2_{.975} \le \frac{nS^2}{\sigma^2} \le \chi^2_{.025}$$

from which the 95% confidence interval for σ^2 can be given as

$$\frac{nS^2}{\chi^2_{.025}} \le \sigma^2 \le \frac{nS^2}{\chi^2_{.975}} \qquad \text{... (14)}$$

Example 13: The standard deviation of a random sample of size 12 drawn from a normal population is 5.5 Calculate the 95% confidence limits for the standard deviation (σ) in the population (Given $\chi^2_{.975}$ = 3.82 and $\chi^2_{.025}$ = 21.92 for 11 degrees of freedom).

Solution: Here n = 12 and the sample s.d. (S) = 5.5. Substituting the values in formula (14), the 95% confidence interval for σ^2 is

$$\frac{12 \times (5.5)^2}{21.92} \le \sigma^2 \le \frac{12 \times (5.5)^2}{3.82}$$

or, $$16.56 \le \sigma^2 \le 95.03$$

i.e., $$4.1 \le \sigma \le 9.7$$

The 95% confidence limits for the population s.d. (σ) are 4.1 and 9.7.

Example 14: A sample of size 8 from a normal population yields as the ubiased estimate of population variance the value 4.4. Obtain the 99% confidence limits for the population variance σ^2 (Given $\chi^2_{.975}$ = 0.99 and $\chi^2_{.005}$ = 20.3 for 7 d.f.).

Solution: Here n = 8, and the unbiased estimate s^2 = 4.4. So, $nS^2 = (n-1)s^2$ = 7 × 4.4 = 30.8.

Hence the 99% confidence limits for σ^2 are obtained from

$$\frac{nS^2}{\chi^2_{.005}} \le \sigma^2 \le \frac{nS^2}{\chi^2_{.975}}$$

or, $$\frac{30.8}{20.3} \le \sigma^2 \le \frac{30.8}{0.99}; \ \textit{i.e.,} \ 1.52 \le \sigma^2 \le 31.1$$

(10) for Variance-Ratio $\frac{\sigma_1^2}{\sigma_2^2}$: (means unknown)

If two independent random samples of sizes n_1 and n_2 are drawn from two Normal populations with unknown means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively, then $\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$ follows F distribution with degrees of freedom $(n_1 - 1, n_2 - 1)$. If $F_{.975}$ and $F_{.025}$ denote the lower and the upper 2.5% points of F distribution, we have with probability 95% the following inequalities

$$F_{.975} \le \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \le F_{.025} \quad \text{... (15)}$$

The 95% confidence interval for σ_1^2/σ_2^2 can be obtained from this as

$$\frac{1}{F_{.025}} \cdot \frac{s_1^2}{s_2^2} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{1}{F_{.975}} \cdot \frac{s_1^2}{s_2^2} \quad \text{... (16)}$$

where s_1^2 and s_2^2 denote unbiased estimators of σ_1^2, σ_2^2 respectively from the two samples.

Self-Assessment

1. Tick the Correct Answer

(*i*) What is the best description of a point estimate ?
- (*a*) any value from the sample used to estimate a parameter.
- (*b*) a sample statistic used to estimate a parameter.
- (*c*) the margin of error used to estimate a parameter.

(*ii*) Which best describes the lower endpoint of a confidence interval ?
- (*a*) point estimate
- (*b*) point estimate minus margin of error
- (*c*) point estimate plus margin of error

(*iii*) Which best describes the upper endpoint of a confidence interval ?
- (*a*) point estimate
- (*b*) point estimate minus margin of error
- (*c*) point estimate plus margin of error

(*iv*) Which value will be at the centre of a confidence interval
- (*a*) population parameter
- (*b*) point estimate
- (*c*) margin of error

(*v*) What is the relationship between a 95% confidence interval and a 99% confidence interval from the same sample ?
- (*a*) the 95% interval will be wider
- (*b*) the 99% interval will be wider
- (*c*) both intervals have the some width

16.3 Summary

- The object of sampling is to study the features of the population on the basis of sample observations. A carefully selection sample is expected to reveal these features, and hence we shall infer about the population from a statistical analysis of the sample. This process is known as *Statistical Inference.*
- In interval estimation, an interval within which the parameter is expected to lie is given by using two quantities based on sample values. This is known as *Confidence Interval,* and the two quantities which are used to specify the interval, are known as *Confidence Limits.*
- The *Method of Maximum Likelihood* consists in choosing as an estimator of θ that statistic, which when substituted for θ, maximises the likelihood function L. Such a statistic is called a *maximum likelihood estimator* (m.l.e.). We shall denote the m.l.e. of θ by the symbol θ_0.
- The parameters enter into the population moments, these relations when solved for the parameters give the estimates by the method of moments. Of course, this method is applicable only when the population moments exist. The method is generally applied for fitting theoretical distributions to observed data.
- In the theory of point estimation, developed earlier, any unknown parameter is estimated by a single quantity. Thus the sample mean $(\bar{x})$ is used to estimate the population mean (μ), and the sample proportion (*p*) is taken as an estimator of the population proportion (P). A single estimator of this kind, however good it may be, cannot be expected to coincide with the true value of the parameter, and may in some cases differ widely from it. In the theory of interval estimation, it is desired to find an interval which is expected to include the unknown parameter with a specified probability.
- The significance of confidence limits is that if many independent random samples are drawn from the same population and the confidence interval is calculated from each sample, then the parameter will actually be included in the intervals in *c* proportion of cases in the long run. Thus the estimate of the parameter is stated as an interval with a specified degree of confidence.

16.4 Key-Words

1. First order interaction	:	The interaction of two variables. Also known as a "simple interaction."
2. Fixed marginal totals	:	The situation in which the marginal totals in a contingency table are known before the data are collected and are not subject to sampling error.
3. Fixed model	:	Anova An analysis of variance model in which the levels of the independent variable are treated as fixed.

16.5 Review Questions

1. Discuss the methods of point estimation.
2. What is the difference between point estimation and interval estimation ? Is interval estimation better than point estimation ?
3. Explain the procedure of constructing a confidence interval for estimating population mean μ.
4. Explain interval estimation.
5. The central limit theorem for sample proportion can be used for estimating the population proportion. Elaborate.

Answers: Self-Assessment

1. (*i*) (*b*) (*ii*) (*b*) (*iii*) (*c*) (*iv*) (*b*) (*v*) (*b*)

16.6 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.

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Unit 17: Types of Hypothesis: Null and Alternative, Types of Errors in Testing Hypothesis and Level of Significance

CONTENTS

Objectives

After reading this unit students will be able to:

- Explain Null and Alternative Hypothesis.
- Know the Types of Errors in Testing Hypothesis.
- Discuss the Level of Significance.

Introduction

In unit 29 we showed how a sample could be used to develop point and interval estimates of population parameters. In this unit we continue the discussion of statistical inference by showing how hypothesis testing can be used to determine whether a statement about the value of a population parameter should or should not be rejected.

In hypothesis testing we begin by making a tentative assumption about a population parameter. This tentative assumption is called the null hypothesis and is denoted by H_0. We then define another hypothesis, called the alternative hypothesis, which is the opposite of what is stated in the null hypothesis. We denote the alternative hypothesis by H_1. The hypothesis testing procedure uses data from a sample to assess the two competing statements indicated by H_0 and H_1.

This unit shows how hypothesis tests can be conducted about a population mean and a population proportion. We begin by providing examples of approaches to formulating null and alternative hypotheses.

17.1 Null and Alternative Hypothesis

(a) **Null Hypothesis:** The null hypothesis asserts that there is no real difference in the sample and the population in the particular matter under consideration and that the difference found is accidental and unimportant arising out of fluctuations of sampling. The null hypothesis constitutes a challenge and the function of the experiment is to give the facts a chance to refute or fail to refute this challenge.

For example, if we want to find out whether the new vaccine has benefited the people or not, the null hypothesis, shall be set up saying that "the new vaccine has not benefited the people". The rejection of the null hypothesis indicates that the differences have statistical significance and the acceptance of the null hypothesis indicate that the differences are due to chance.

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(b) **Alternative Hypothesis:** The alternative hypothesis specifies those values that the researcher believes to hold true and hopes that the sample data would lead to acceptance of this hypothesis to be true. The alternative hypothesis may embrace the whole range of values rather than single point.

As per this definition, it is very difficult to find out which is null hypothesis and which one is alternative hypothesis.

However, for statistical convenience, the hypothesis these definitions are used.

The null hypotheses are represented by the symbol H_0 and the alternative hypothesis is represented by H_1.

Developing null and alternative hypotheses

In some applications it may not be obvious how the null and alternative hypotheses should be formulated. Care must be taken to structure the hypotheses appropriately so that the conclusion from the hypothesis test provides the information the researcher or decision-maker wants. Learning to formulate hypotheses correctly will take practice. The examples in this section show a variety of forms for H_0 and H_1 depending upon the application. Guidelines for establishing the null and alternative hypotheses will he given for three types of situations in which hypothesis testing procedures are commonly used.

Statistics in Practice

Monitoring the quality of latex condoms

Many consumer products are required by law to meet specifications set out in documents known as standards. This is particularly the case when there are issues of consumer safety, such as with electrical goods, children's toys or furniture (fire resistance). In less safety-critical cases, the standards may be permissive rather than obligatory, but manufacturers will often conform to the standards, and tell consumers so, as an assurance of quality. Many standards are established internationally and are embodied in documents published by the International Standards Organization (ISO). Companies based in the UK and other EU countries usually operate according to ISO standards.

The humble latex condom is the subject of ISO standard 4074: 2002. This lays down a range of specifications, relating to materials, dimensions, packaging and performance criteria including, for obvious reasons, freedom from holes. For an outline of quality testing procedures, read the relevant pages at www.durex.com, for example. ISO 4047: 2002 makes reference to other standards documents including frequent references to ISO 4859-1, which lays down specifications for the sampling schemes that must be used to ensure quality, such as in respect of freedom from holes, For the latter characteristic, ISO 4047: 2002 specifies an acceptable quality level (AQL) of no more than 0.25 per cent defective condoms in any manufacturing batch; in other words, a probability of no more than 1 in 400 that any particular condom will be defective.

As an example of the sampling specifications, suppose ISO 4859-1 requires that, from a batch of 10000 condoms, a random sample of 200 should be taken and examined individually for freedom from holes. This is likely to be a destructive test. Suppose ISO 4859-1 then stipulates that the whole batch from which the sample was drawn can be declared satisfactory, in respect of freedom from holes, only if the sample contains no more than one defective condom. If the sample does contain more than one defective condom, the whole batch must be scrapped, or further tests of quality must be done to gather further information about the overall quality of the batch.

A statistician would refer to this decision procedure as a hypothesis test. The working hypothesis is that the batch conforms to the AQL specified in ISO 4074: 2002.

If the sampled batch contains more than one defective condom, this hypothesis is rejected. Otherwise the hypothesis is accepted. In making the decision on the basis of sample evidence, the quality controller is taking two risks. One risk is that a batch meeting the AQL requirement will be incorrectly rejected. The second risk is that a batch not meeting the AQL requirement will be incorrectly accepted. The sampling schemes laid down in ISO 4859-1 are intended to clarify and restrict the level of risk involved, and to strike a sensible balance between the two types of risk.

In this chapter you will learn about the logic of statistical hypothesis testing.

Testing research hypotheses

Consider a particular model of ear that currently attains an average fuel consumption of 7 litres of fuel per 100 kilometres of driving. A product research group develops a new fuel injection system specifically designed to decrease the fuel consumption. To evaluate the new system, several will be manufactured, installed in cars, and subjected to research-controlled driving tests. Here the product research group is looking for evidence to conclude that the new system *decreases* the mean fuel consumption. In this case, the research hypothesis is that the new fuel injection system will provide a mean litres-per-100 *km* rating below 7; that is, $\mu < 7$. As a general guideline, a research hypothesis should be stated as the *alternative hypothesis*. Hence, the appropriate null and alternative hypotheses for the study are:

$$H_0: \mu \geq 7$$

$$H_1: \mu < 7$$

If the sample results indicate that H_0 cannot be rejected, researchers cannot conclude that the new fuel injection system is better. Perhaps more research and subsequent testing should be conducted. However, if the sample results indicate that H_0 can be rejected, researchers can make the inference that H_1: $\mu < 7$ is true. With this conclusion, the researchers gain the statistical support necessary to state that the new system decreases the mean fuel consumption. Production with the new system should be considered.

In research studies such as these, the null and alternative hypotheses should he formulated so that the rejection of H_0 supports the research conclusion. The research hypothesis therefore should be expressed as the alternative hypothesis.

Testing the validity of a claim

As an illustration of testing the validity of a claim, consider the situation of a manufacturer of soft drinks who states that bottles of its products contain an average of at least 1.5 litres. A sample of bottles will be selected, and the contents will be measured to test the manufacturer's claim. In this type of hypothesis testing situation, we generally assume that the manufacturer's claim is true unless the sample evidence is contradictory Using this approach for the soft-drink example, we would state the null and alternative hypotheses as follows.

$$H_0: \mu \geq 1.5$$

$$H_1: \mu < 1.5$$

In the sample results indicate H_0 cannot be rejected, the manufacturer's claim will not be challenged. However, if the sample results indicate H_0 can he rejected, the inference will be made that H_1: $\mu < 1.5$ is true. With this conclusion, statistical evidence indicates that the manufacturer's claim is incorrect and that the soft-drink bottles are being filled with a mean less than the claimed 1.5 litres. Appropriate action against the manufacturer may be considered.

In any situation that involves testing the validity of a claim, the null hypothesis is generally based on the assumption that the claim is true. The alternative hypothesis is then formulated so that rejection of H_0 will provide statistical evidence that the stated assumption is incorrect. Action to correct the claim should be considered whenever H_0 is rejected.

Testing in decision-making situations

In testing research hypotheses or testing the validity of a claim, action is taken if H_0 is rejected. In many instances, however, action must be taken both when H_0 cannot be rejected and when H_0 can be rejected. In general, this type of situation occurs when a decision-maker must choose between two

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courses of action, one associated with the null hypothesis and another associated with the alternative hypothesis. The quality-testing scenario outlined in the Statistics in Practice at the beginning of the unit is an example of this.

Suppose that, on the basis of a sample of parts from a shipment just received, a quality control inspector must decide whether to accept the shipment or to return the shipment to the supplier because it does not meet specifications. The specifications for a particular part require a mean length of two centimetres per part. If the mean length is greater or less than the two-centimeter standard, the parts will cause quality problems in the assembly operation. In this case, the null and alternative hypothesis would be formulated as follows.

$$H_0: \mu = 2$$

$$H_1: \mu \neq 2$$

If the sample results indicate H_0 cannot be rejected, the quality control inspector will have no reason to doubt that the shipment meets specifications, and the shipment will be accepted. However, if the sample results indicate H_0 should be rejected, the conclusion will be that the parts do not meet specifications. In this case, the quality control inspector will have sufficient evidence to return the shipment to the supplier. We see that for these types of situations, action is taken both when H_0 cannot be rejected and when H_0 can be rejected.

Summary of forms for null and alternative hypothesis

The hypothesis tests in this unit involve one of two population parameters: the population mean and the population proportion. Depending on the situation, hypothesis tests about a population parameter may take one of three forms; two include inequalities in the null hypothesis, the third uses only an equality in the null hypothesis. For hypothesis tests involving a population mean, we let μ_0 denote the hypothesized value and choose one of the following three forms for the hypothesis test.

$$H_0: \mu \geq \mu_0 \qquad H_0: \mu \leq \mu_0 \qquad H_0: \mu = \mu_0$$

$$H_1: \mu < \mu_0 \qquad H_1: \mu > \mu_0 \qquad H_1: \mu \neq \mu_0$$

For reasons that will be clear later, the first two forms are called one-tailed tests. The third form is called a two-tailed test.

In many situations, the choice of H_0 and H_1 is not obvious and judgment is necessary to select the proper form. However, as the preceding forms show, the equality part of the expression (either $\geq$, $\leq$ or =) *always* appears in the null hypothesis. In selecting the proper form of H_0 and H_1, keep in mind that the alternative hypothesis is often what the test is attempting to establish. Hence, asking whether the user is looking for evidence to support $\mu < \mu_0$, $\mu_1 > \mu_0$ or $\mu \neq \mu_0$ will help determine H_1. The following exercises are designed to provide practice in choosing the proper form for a hypothesis test involving a population mean.

17.2 Types of Errors in Testing Hypothesis

Ideally the hypothesis testing procedure should lead to the acceptance of the null hypothesis H_0 when it is true, and the rejection of H_0 when it is not. However, the correct decision is not always possible. Since the decision to reject or accept a hypothesis is based on sample data, there is a possibility of an incorrect decision or error. A decision-maker may commit two types of errors while testing a null hypothesis. The two types of errors that can be made in any hypothesis testing are shown in Table 1.

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Table 1: Errors in Hypothesis Testing

Decision	**State of Nature**	
	H_0 is True	**H_0 is False**
Accept H_0	Correct decision with confidence $(1-\alpha)$	Type II error (β)
Reject H_0	Type I error (α)	Correct decision $(1-\beta)$

Type I Error: This is the *probability of rejecting the null hypothesis when it is true* and some alternative hypothesis is wrong. The probability of making a Type I error is denoted by the symbol α. It is represented by the area under the sampling distribution curve over the region of rejection.

Hypothesis testing: The process of testing a statement or belief about a population parameter by the use of information collected from a sample(s).

Type I error: The probability of rejecting a true null hypothesis.

The probability of making a Type I error, is referred to as the level of significance. The probability level of this error is decided by the decision-maker before the hypothesis test is performed and is based on his tolerance in terms of risk of rejecting the true null hypothesis. The risk of making Type I error depends on the cost and/or goodwill loss. The complement $(1-\alpha)$ of the probability of Type I error measures the probability level of not rejecting a true null hypothesis. It is also referred to as *confidence level*.

Type II Error: This is the *probability of accepting the null hypothesis when it is false* and some alternative hypothesis is true. The probability of making a Type II is denoted by the symbol β.

The probability of Type II error varies with the actual values of the population parameter being tested when null hypothesis H_0 is false. The probability of committing a Type II error depends on five factors: (i) the actual value of the population parameter, being tested, (ii) the level of significance selected, (iii) type of test (one or two tailed test) used to evaluate the null hypothesis, (iv) the sample standard deviation (also called standard error) and (v) the size of sample.

A summary of certain critical values at various significance levels for test statistic *z* is given in Table 30.2.

Level of significance: The probability of rejecting a true null hypothesis due to sampling error.

Type II error: The probability of accepting a false null hypothesis.

Table 2: Summary of Certain Critical Values for Sample Statistic *z*

Rejection Region	**Level of Significance, α per cent**			
	α = 0.10	**α = 0.05**	**0.01**	**α = 0.005**
One-tailed region	± 1.285	± 1.645	± 2.33	± 2.58
Two-tailed region	± 1.645	± 1.96	± 2.58	± 2.81

Power of a Statistical Test

Another way of evaluating the goodness of a statistical test is to look at the complement of Type II error, which is stated as:

$$1-\beta = \text{P (reject } H_0 \text{ when } H_1 \text{ is true)}$$

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The complement $1-\beta$ of β, *i.e.* the probability of Type-II error, is called the *power of a statistical test* because it measures the probability of rejecting H_0 when it is true.

For example, suppose null and alternative hypotheses are stated as.

$$H_{0:}\ \mu = 80 \text{ and } H_{1:}\ \mu = 80$$

Power of a test: The ability (probability) of a test to reject the null hypothesis when it is false.

Often, when the null hypothesis is false, another alternative value of the population mean, μ is unknown. So for each of the possible values of the population mean μ, the probability of committing Type II error for several possible values of μ is required to be calculated.

Suppose a sample of size n = 50 is drawn from the given population to compute the probability of committing a Type II error for a specific alternative value of the population mean, μ. Let sample mean so obtained be $\bar{x}$ = 71 with a standard deviation, s = 21. For significance level, α = 0.05 and a two-tailed test, the table value of $z_{0.05} = \pm 1.96$. But the deserved value from sample data is

$$z = \frac{\bar{x}-\mu}{\sigma_{\bar{x}}} = \frac{71-80}{21/\sqrt{50}} = -3.03$$

Since $z_{cal} = -3.03$ value falls in the rejection region, the null hypothesis H_0 is rejected. The rejection of null hypothesis, leads to either make a correct decision or commit a Type II error. If the population mean is actually 75 instead of 80, then the probability of commiting a Type II error is determined by computing a critical region for the mean $\bar{x}_c$. This value is used as the cutoff point between the area of acceptance and the area of rejection. If for any sample mean so obtained is less than (or greater than for right-tail rejection region), $\bar{x}_c$, then the null hypothesis is rejected. Solving for the critical value of mean gives

$$z_c = \frac{\bar{x}_c-\mu}{\sigma_{\bar{x}}} \text{ or } \pm 1.96 = \frac{\bar{x}_c-80}{21/\sqrt{50}}$$

$$\bar{x}_c = 80 \pm 5.82 \text{ or } 74.18 \text{ to } 85.82$$

If μ = 75, then probability of accepting the false null hypothesis H_0: μ = 80 when critical value is falling in the range $\bar{x}_c$ = 74.18 to 85.82 is calculated as follows:

$$z_1 = \frac{74.18-75}{21/\sqrt{50}} = -0.276$$

The area under normal curve for $z_1 = -0.276$ is 0.1064.

$$z_2 = \frac{85.82-75}{21\sqrt{50}} = 3.643$$

The area under normal carve for z_2 = 3.643 is 0.4995

Thus the probability of committing a Type II error (β) falls in the region:

$$\beta = P(74.18 < \bar{x}_c < 85.82) = 0.1064 + 0.4995 = 0.6059$$

The total probability 0.6059 of committing a Type II error (β) is the area to the right of $\bar{x}_c$ = 74.18 in the distribution. Hence the power of the test is $1-\beta$ = 1 – 0.6059 = 0.3941 as shown in Figure 1 (b).

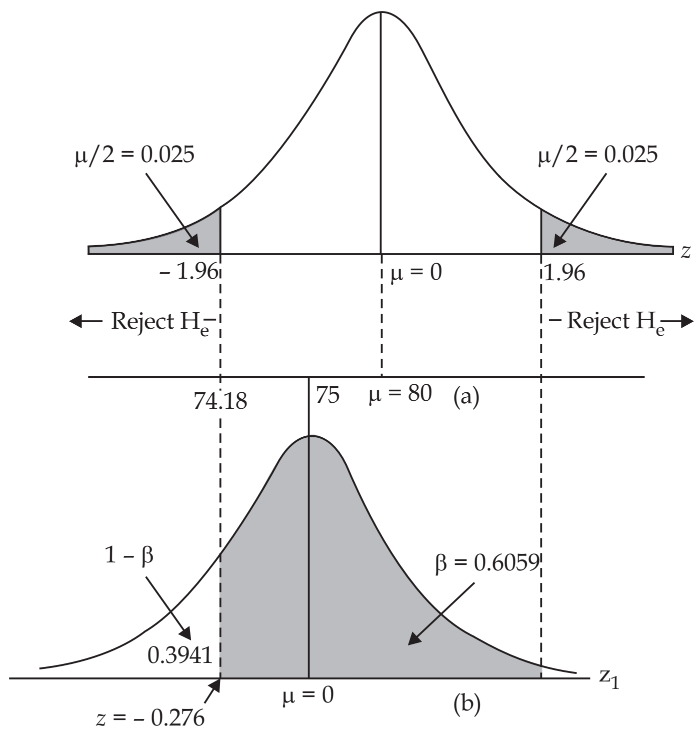

Figure 1 (a): Sampling distribution with H_0: μ = 80

Figure 1 (b): Sampling distribution with H_0: μ = 75

To keep α or β low depends on which type of error is more costly. However, if both types of errors are costly, then to keep both α and β low, then inferences can be made more reliable by reducing the variability of observations. It is preferred to have large sample size and a low α value.

Few relations between two errors α and β, the power of a test $1-\beta$, and the sample size n are stated below:

(i) If α (the sum of the two tail areas in the curve) is increased, the shaded area corresponding to β gets smaller, and vice versa.

(ii) The β value call be increased for a fixed α, by increasing the sample size n.

Special Case: Suppose hypotheses are defined as:

$$H_0: \mu = 80 \text{ and } H_1: \mu < 80$$

Given n = 50, s = 21 and $\bar{x}$ = 71. For α = 0.05 and left-tailed test, the table value $z_{0.05}$ = – 1.645. The observed z value from sample data is

$$z = \frac{\bar{x}-\mu}{\sigma_{\bar{x}}} = \frac{71-80}{21/\sqrt{50}} = -3.03$$

The critical value of the sample mean $\bar{x}_c$ for a given population mean μ = 80 is given by:

$$z_c = \frac{\bar{x}_c-\mu}{\sigma_{\bar{x}}} \text{ or } -1.645 = \frac{\bar{x}_c-80}{21/\sqrt{50}}$$

$\bar{x}_c = 75.115$

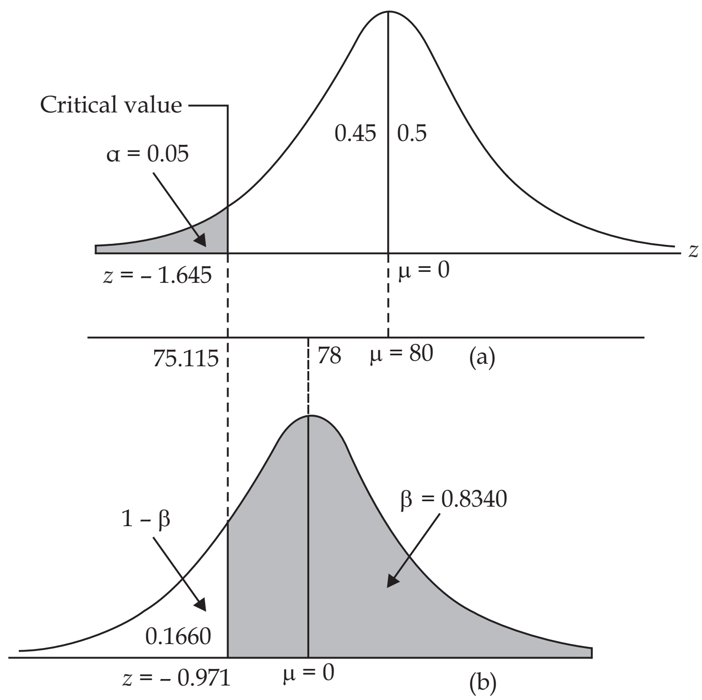

Figure 2 (a): Sampling distribution with H_0: $\mu = 80$

Figure 2 (b): Sampling distribution with H_0: $\mu = 78$

Figure 2 (a) shows that the distribution of values that contains critical value of mean $\bar{x}_c = 75.115$ and below which H_0 will be rejected. Figure 2 (b) shows the distribution of values when the alternative population mean value $\mu = 78$ is true. If H_0 is false, it is not possible to reject null hypothesis H_0 whenever sample mean is in the acceptance region, $\bar{x} \geq 75.151$. Thus critical value is computed by extending it and solved for the area to the right of $\bar{x}_c$ as follows:

$$z_1 = \frac{\bar{x}_c - \mu}{\sigma_{\bar{x}}} = \frac{75.115 - 78}{21/\sqrt{50}} = -\,0.971$$

This value of z yields an area of 0.3340 under the normal curve. Thus the probability = 0.3340 + 0.5000 = 0.8340 of committing a Type II error is all the area to right of $\bar{x}_c = 75.115$.

Remark: In general, if alternative value of population mean μ is relatively more than its hypothesized value, then probability of committing a Type II error is smaller compared to the case when the alternative value is close to the hypothesized value. The probability of committing a Type II error decreases as alternative values are greater than the hypothesized mean of the population.

17.3 The Level of Significance

In Section 30.2, we introduced hypothesis testing along rather traditional lines: we defined the parts of a statistical test along with the two types of errors and their associated probabilities α and $\beta(\mu_a)$. The problem with this approach is that if other researchers want to apply the results of your study using a different value for α then they must compute a new rejection region before reaching a decision concerning H_0 and H_a. An alternative approach to hypothesis testing follows the following steps: specify the null and alternative hypotheses, specify a value for α, collect the sample data, and determine the weight of evidence for rejecting the null hypothesis. This weight, given in terms of a probability, is called the level of significance (or ***p*-value**) of the statistical test. More formally, the level of significance is defined as follows: *the probability of obtaining a value of the test statistic that is as likely or more likely to reject H_0 as the actual observed value of the test statistic, assuming that the null hypothesis is true.* Thus, if the level of significance is a small value, then the sample data fail to support H_0 and our decision is to reject H_0. On the other hand, if the level of significance is a large value, then we fail to reject H_0. We must next decide what is a large or small value for the level of significance.

Decision Rule for Hypothesis Testing **Using the *p*-Value**

1. If the *p*-value $\leq \alpha$, then reject H_0.
2. If the *p*-value $> \alpha$, then fail to reject H_0.

We illustrate the calculation of a level of significance with several examples.

Example 1 : (a) Determine the level of significance (*p*-value) for the statistical test and reach a decision concerning the research hypothesis using α = .01.

(b) If the preset value of α is .05 instead of .01, does your decision concerning H_a change ?

Solution :

(a) The null and alternative hypotheses are

$$H_0: \mu \leq 380$$

$$H_a: \mu > 380$$

From the sample data, with *s* replacing σ, the computed value of the test statistic is

$$z = \frac{\bar{y} - 380}{\sigma/\sqrt{n}} = \frac{390 - 380}{35.2/\sqrt{50}} = 2.01$$

The level of significance for this test (*i.e.,* the weight of evidence for rejecting H_0) is the probability of observing a value of $\bar{y}$ greater than or equal to 390 assuming that the null hypothesis is true; that is, μ = 380. This value can be computed by using the *z*-value of the test statistic, 2.01, because *p*-value = $P(\bar{y} \geq 390$, assuming μ = 380) = $P(z \geq 2.01)$

Referring to Table 30.1 in the Appendix, $P(z \geq 2.01) = 1 - P(z < 2.01) = 1 - .9778 =$ .0222. This value is shown by the shaded area in Figure 3. Because the *p*-value is greater than α (.0222 > .01). we fail to reject H_0 and conclude that the data do not support the research hypothesis.

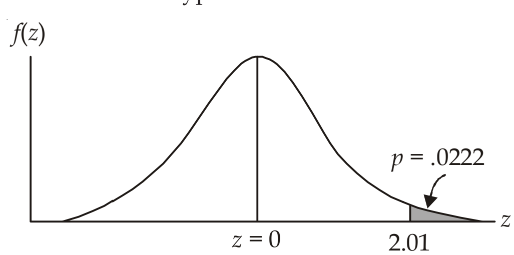

Figure 3: Level of significance for Example 1

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(b) Another person examines the same data but with a preset value for α = .05. This person is willing to support a higher risk of a Type I error, and hence the decision is to reject H_0 because the *p*-value is less than α $(.0222 \le .05)$. It is important to emphasize that the value of α used in the decision rule is *preset* and not selected after calculating the *p*-value.

As we can see from Example 1, the level of significance represents the probability of observing a sample outcome more contradictory to H_0 than the observed sample result. *The smaller the value of this probability, the heavier the weight of the sample evidence against H_0*. For example, a statistical test with a level of significance of p = .01 shows more evidence for the rejection of H_0 than does another statistical test with p = .20.

Example 2 : Using a preset value of α = .05, is there sufficient evidence in the data to support the research hypothesis ?

Solution : The null and alternative hypotheses are

$$H_0 : \mu \ge 33$$

$$H_\alpha : \mu < 33$$

From the sample data, with s replacing σ, the computed value of the test statistic is

$$z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} = \frac{31.2 - 33}{8.4/\sqrt{35}} = -1.27$$

The level of significance for this test statistic is computed by determining which values of $\bar{y}$ are more extreme to H_0 than the observed $\bar{y}$. Because H_α specifies μ less than 33, the values of $\bar{y}$ that would be more extreme to H_0 are those values less than 31.2, the observed value. Thus,

p-value = $P(\bar{y} \le 31.2$, assuming μ = 33) = $P(z \le -1.27)$ = .1020

There is considerable evidence to support H_0. More precisely, p-value = .1020 > .05 = α, and hence we fail to reject H_0. Thus, we conclude that there is insufficient evidence (p-value = .1020) to support the research hypothesis. Note that this is exactly the same conclusion reached using the traditional approach.

For two-tailed tests, $H_\alpha : \mu \ne \mu_0$, we still determine the level of significance by computing the probability of obtaining a sample having a value of the test statistic that is more contradictory to H_0 than the observed value at the test statistic. However, for two-tailed research hypotheses, we compute this probability in terms of the magnitude of the distance from $\bar{y}$ to the null value of μ because both values of $\bar{y}$ much less than μ_0 and values of $\bar{y}$ much larger than μ_0 contradict $\mu = \mu_0$. Thus, the level of significance is written as

$$p\text{-value} = P\left(|\bar{y} - \mu_0| \ge \text{ observed } |\bar{y} - \mu_0|\right) = P\left(|z| \ge |\text{computed } z|\right)$$

$$= 2P\left(z \ge |\text{computed } z|\right)$$

To summarize, the level of significance (p-value) can be computed as

Notes

Case 1	Case 2	Case 3
$H_0 : \mu \leq \mu_0$	$H_0 : \mu \geq \mu_0$	$H_0 : \mu = \mu_0$
$H_\alpha : \mu > \mu_0$	$H_\alpha : \mu < \mu_0$	$H_\alpha : \mu \neq \mu_0$
p-value: $P(z \geq \text{computed } z)$	$P(z \leq \text{computed } z)$	$2P(z \geq \lvert\text{computed } z\rvert)$

Example 3 : Using a preset value of α = .01, is there sufficient evidence in the data to support the research hypothesis ?

Solution : The null and alternative hypotheses are

$$H_0 : \mu = 190$$

$$H_\alpha : \mu \neq 190$$

From the sample data, with s replacing σ, the computed value of the test statistic is

$$z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} = \frac{178.2 - 190}{45.3/\sqrt{100}} = -2.60$$

The level of significance for this test statistic is computed using the formula on page 248.

$$p\text{-value} = 2P(z \geq \lvert\text{computed } z\rvert) = 2P(z \geq \lvert -2.60\rvert) = 2P(z \geq 2.60)$$

$$= 2(1 - .9953) = .0047$$

Because the p-value is very small, there is very little evidence to support H_0. More precisely, p-value = $.0047 \leq .05 = \alpha$, and hence we reject H_0. Thus, there is sufficient evidence (p-value = .0047) to support the research hypothesis and conclude that the mean cholesterol level differs from 190. Note that this is exactly the same conclusion reached using the traditional approach.

There is much to he said in favor of this approach to hypothesis testing. Rather than reaching a decision directly, the statistician (or person performing the statistical test) presents the experimenter with the weight of evidence for rejecting the null hypothesis. The experimenter can then draw his or her own conclusion. Some experimenters reject a null hypothesis if $p \leq .10$, whereas others require $p \leq .05$ or $p \leq .01$ for rejecting the null hypothesis. The experimenter is left to make the decision based on what he or she believes is enough evidence to indicate rejection of the null hypothesis.

Many professional journals have followed this approach by reporting the results of a statistical test in terms of its level of significance. Thus, we might read that a particular test was significant at the $p = .05$ level or perhaps the $p < .01$ level. By reporting results this way, the reader is left to draw his or her own conclusion.

One word of warning is needed here. The p-value of .05 has become a magic level, and many seem to feel that a particular null hypothesis should not be rejected unless the test achieves the .05 level or lower. This has resulted in part from the decision-based approach with α preset at .05. Try not to fall into this trap when reading journal articles or reporting the results of your statistical tests. After all, statistical significance at a particular level does not dictate importance or practical significance. Rather, it means that a null hypothesis can be rejected with a specified low risk of error. For example, suppose that a company is interested in determining whether the average number of miles driven per car per month for the sales force has risen above 2,600. Sample data from 400 cars show that $\bar{y}$ = 2,640 and s = 35. For these data, the z statistic for H_0: μ = 2,600 is z = 22.86 based on σ = 35; the level of significance is $p < .0000000001$. Thus, even though there has only been a 1.5% increase in the average

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monthly miles driven for each car, the result is (highly) statistically significant. Is this increase of any practical significance ? Probably not. What we have done is proved *conclusively* that the mean μ has increased slightly.

The company should not just examine the size of the *p*-value. It is very important to also determine the size of the difference between the null value of the population mean μ_0 and the estimated value of the population mean $\bar{y}$. This difference is called the estimated *effect size*. In this example the estimated effect size would be $\bar{y} - \mu_0$ = 2,640 – 2,600 = 40 miles driven per month. This is the quantity that the company should consider when attempting to determine if the change in the population mean has practical significance.

Throughout the text we will conduct statistical tests from both the decision-based approach and from the level-of-significance approach to familiarize you with both avenues of thought. For either approach, remember to consider the practical significance of your finding after drawing conclusions based on the statistical test.

Self Assessment

1. Fill in the Blanks:

(*i*) The first important step in the decision making procedure is to state the .... hypothesis.

(*ii*) When the hypothesis is true but the test rejects it, it is called .... error.

(*iii*) When the hypothesis is false and the test accepts it, it is called .... error.

(*iv*) The confidence with which an experimenter rejects. Or retains a null hypothesis depends upon the .... level adopted.

(*v*) The alternative hypothesis may embrace the whole range of value rather than .... point.

17.4 Summary

- In hypothesis testing we begin by making a tentative assumption about a population parameter. This tentative assumption is called the null hypothesis and is denoted by H_0. We then define another hypothesis, called the alternative hypothesis, which is the opposite of what is stated in the null hypothesis. We denote the alternative hypothesis by H_1. The hypothesis testing procedure uses data from a sample to assess the two competing statements indicated by H_0 and H_1.
- The null hypothesis asserts that there is no real difference in the sample and the population in the particular matter under consideration and that the difference found is accidental and unimportant arising out of fluctuations of sampling. The null hypothesis constitutes a challenge and the function of the experiment is to give the facts a chance to refute or fail to refute this challenge.
- The rejection of the null hypothesis indicates that the differences have statistical significance and the acceptance of the null hypothesis indicate that the differences are due to chance.
- The alternative hypothesis specifies those values that the researcher believes to hold true and hopes that the sample data would lead to acceptance of this hypothesis to be true. The alternative hypothesis may embrace the whole range of values rather than single point.
- The null hypotheses are represented by the symbol H_0 and the alternative hypothesis is represented by H_1.
- Care must be taken to structure the hypotheses appropriately so that the conclusion from the hypothesis test provides the information the researcher or decision-maker wants. Learning to formulate hypotheses correctly will take practice. The examples in this section show a variety of forms for H_0 and H_1 depending upon the application. Guidelines for establishing the null and alternative hypotheses will he given for three types of situations in which hypothesis testing procedures are commonly used.

Notes

- Many consumer products are required by law to meet specifications set out in documents known as standards. This is particularly the case when there are issues of consumer safety, such as with electrical goods, children's toys or furniture (fire resistance). In less safety-critical cases, the standards may be permissive rather than obligatory, but manufacturers will often conform to the standards, and tell consumers so, as an assurance of quality. Many standards are established internationally and are embodied in documents published by the International Standards Organization (ISO). Companies based in the UK and other EU countries usually operate according to ISO standards.
- In making the decision on the basis of sample evidence, the quality controller is taking two risks. One risk is that a batch meeting the AQL requirement will be incorrectly rejected. The second risk is that a batch not meeting the AQL requirement will be incorrectly accepted. The sampling schemes laid down in ISO 4859-1 are intended to clarify and restrict the level of risk involved, and to strike a sensible balance between the two types of risk.
- In research studies such as these, the null and alternative hypotheses should he formulated so that the rejection of H_0 supports the research conclusion. The research hypothesis therefore should be expressed as the alternative hypothesis.
- In any situation that involves testing the validity of a claim, the null hypothesis is generally based on the assumption that the claim is true. The alternative hypothesis is then formulated so that rejection of H_0 will provide statistical evidence that the stated assumption is incorrect. Action to correct the claim should be considered whenever H_0 is rejected.
- In testing research hypotheses or testing the validity of a claim, action is taken if H_0 is rejected. In many instances, however, action must be taken both when H_0 cannot be rejected and when H_0 can be rejected. In general, this type of situation occurs when a decision-maker must choose between two courses of action, one associated with the null hypothesis and another associated with the alternative hypothesis.
- The hypothesis tests in this unit involve one of two population parameters: the population mean and the population proportion. Depending on the situation, hypothesis tests about a population parameter may take one of three forms; two include inequalities in the null hypothesis, the third uses only an equality in the null hypothesis.
- The hypothesis testing procedure should lead to the acceptance of the null hypothesis H_0 when it is true, and the rejection of H_0 when it is not. However, the correct decision is not always possible. Since the decision to reject or accept a hypothesis is based on sample data, there is a possibility of an incorrect decision or error.
- The probability of making a Type I error, is referred to as the level of significance. The probability level of this error is decided by the decision-maker before the hypothesis test is performed and is based on his tolerance in terms of risk of rejecting the true null hypothesis. The risk of making Type I error depends on the cost and/or goodwill loss. The complement $(1-\alpha)$ of the probability of Type I error measures the probability level of not rejecting a true null hypothesis. It is also referred to as *confidence level*.
- The ability (probability) of a test to reject the null hypothesis when it is false. Often, when the null hypothesis is false, another alternative value of the population mean, μ is unknown. So for each of the possible values of the population mean μ, the probability of committing Type II error for several possible values of μ is required to be calculated.E
- However, if both types of errors are costly, then to keep both α and β low, then inferences can be made more reliable by reducing the variability of observations. It is preferred to have large sample size and a low α value.
- However, if both types of errors are costly, then to keep both α and β low, then inferences can be made more reliable by reducing the variability of observations. It is preferred to have large sample size and a low α value.

- An alternative approach to hypothesis testing follows the following steps: specify the null and alternative hypotheses, specify a value for α, collect the sample data, and determine the weight of evidence for rejecting the null hypothesis. This weight, given in terms of a probability, is called the level of significance (or ***p*-value**) of the statistical test. More formally, the level of significance is defined as follows: *the probability of obtaining a value of the test statistic that is as likely or more likely to reject H_0 as the actual observed value of the test statistic, assuming that the null hypothesis is true.* Thus, if the level of significance is a small value, then the sample data fail to support H_0 and our decision is to reject H_0. On the other hand, if the level of significance is a large value, then we fail to reject H_0. We must next decide what is a large or small value for the level of significance.

17.5 Key-Words

1. Harmonic mean : The number of elements to be averaged divided by the sum of the reciprocals of the elements.
2. Heavy tailed distribution : A distribution with a higher percentage of scores in the tails than we would expect in a normal distribution.
3. Heterogeneity of variance : A situation in which samples are drawn from populations having different variances.

17.6 Review Questions

1. What is meant by the terms hypothesis and a test of a hypothesis ?
2. What do you understand by null hypothesis and alternative hypothesis ? Explain developing null and alternative hypothesis.
3. Discuss the types of errors in testing hypothesis.
4. Define term 'level of significance'. How is it related to the probability of committing a Type I error.
5. How is power related to the probability of making a Type II error ?

Answers: Self-Assessment

1. (*i*) Null (*ii*) type I (*iii*) type II
   (*iv*) significance (*v*) single

17.7 Further Readings

Books

1. Elementary Statistical Methods; SP. Gupta, Sultan Chand & Sons, New Delhi - 110002.
2. Statistical Methods — An Introductory Text; Jyoti Prasad Medhi, New Age International Publishers, New Delhi - 110002.
3. Statistics; E. Narayanan Nadar, PHI Learning Private Limied, New Delhi - 110012.
4. Quantitative Methods—Theory and Applications; J.K. Sharma, Macmillan Publishers India Ltd., New Delhi - 110002.