LOVELY PROFESSIONAL UNIVERSITY

QUANTITATIVE METHODS FOR ECONOMISTS

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Printed by
LAXMI PUBLICATIONS (P) LTD.
113, Golden House, Daryaganj,
New Delhi-110002
for
Lovely Professional University
Phagwara

SYLLABUS

Quantitative Methods for Economists

Objectives

- To aware of students the mathematical aspects of Economics.
- To introduce the concept of interrelation and inter dependency of mathematical Economics.
- To increase understanding of the application of the mathematical properties of Economics.

S.No.	Topics
1.	Economic Applications of Integration
2.	**Differential Equations:** Introduction
	Solution – Variable Separable Case, Homogenous Case
	Matrices: Meaning and Types
	Transpose, Trace of a Matrix, Adjoint and Inverse of Matrix
	Cramer's Rule
	Determinants: Types and Properties
3.	Rank of a Matrix
	Application of Matrices in Economics
	Input - Output Analysis
4.	Hawkins – Simon Conditions
	Closed Economic Input - Output Analysis
	Introduction to Linear Programming
5.	Formulation of Linear Programming Problems

CONTENT

Tanima Dutta, Lovely Professional University

Unit 1: Introduction to Differential Equations and Solutions: Variable Separable Case and Homogeneous Equation

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know about Differential Equation of First Order and First Degree.
- Know about Exact Differential Equations.

Introduction

Differential Equations are the equations in which dependent, independent variable exists, and there are different derivatives of dependent variables with respect to one or more independent variables. The order of a differential equation is the highest order of different derivatives included in that equation. An equation will be called as linear if the derivatives of dependent variables are of first degree, otherwise it will be called as non-linear.

Function $f(x)$ will be called as the solution of different equations if it is replaced in any equation then it reduces the equation to the identity and the method of finding all the solutions will be called as the solution of differential equation.

Normal Solution: The solution of a differential equation in which the independent imaginary constant is equal to the order of differential equation, then it is called as Normal Solution.

Particular Solution: In normal solution, if any particular value is given to constants then it is called the Particular Solution of that equation.

Example: Find the differential equation of curve $y = Ae^x + B/e^x$ for different values of A and B.

Given value $$y = Ae^x + B/e^x \quad \ldots(1)$$

To get the differential equation for the values of A and B of above equation, we'll differentiate equation (1) twice. On differentiating equation (1)

$$\frac{dy}{dx} = Ae^x - Be^{-x} \quad \ldots(2)$$

On differentiating equation (2) again

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} \quad \ldots(3)$$

Note

On eliminating A and B from equation (2) and (3), we get –

$$\frac{d^2y}{dx^2} = y$$

This is our differential equation.

1.1 Differential Equation of First Order and First Degree

We represent the differential equation of first order and first degree in the following form –

$$M + N\,(dy/dx) = 0 \quad \text{or} \quad M\,dx + N\,dy = 0$$

Where M and N are constant and x and y are few functions. Though all the differential equations of first order can't be solved always, then also if they are available in the following form then their solution can be found out by few methods –

1.1.1 Variable Separable Case

$$f_1(x)\,dx = f_2(y)\,dy$$

If the differential equation is shown in the following form –

Where $f_1(x)$ and $f_2(y)$ are the functions of x and y respectively.

In such situation, we differentiate the both sides of the equation and then add an imaginary constant on any one side of the equation. Therefore

$$\int f_1(x)\,dx = \int f_2(y)\,dy + c$$

Notes Here 'c' is an imaginary constant.

Example 1: Find the solution of $\left[x/\sqrt{1+x^2}\right]dx = -\left[y/\sqrt{1+y^2}\right]dy$ **.**

Solution: On differentiating both sides

$$\int (1+x^2)^{-1/2}.\,x\,dx = -\int (1+y^2)^{-1/2}.\,y\,dy + c$$

Or $$\frac{1}{2}\int (1+x^2)^{-1/2}.(2x)\,dx = -\frac{1}{2}\int (1+y^2)^{-1/2}.(2y)\,dy + c$$

Or $\sqrt{1+x^2} + \sqrt{1+y^2} = c$, This is the solution of the given equation.

Example 2: Find the solution of $(1 + e^x)\,y\,dy = (1 + y)\,e^x\,dx$.

Solution: We can also write this equation as the following form –

$$\left(\frac{y}{1+y}\right)dy = \frac{e^x}{1+e^x}\,dx$$

On differentiating both sides

$$\int \frac{y}{1+y}\,dy = \int \frac{e^x}{1+e^x}\,dx + \text{constant}$$

Note

Or $\qquad y - \log(1 + y) = \log(1 + e^x) + \log c$

Or $\qquad y = \log[c(1 + y)(1 + e^x)]$

Or $\qquad c(1 + y)(1 + e^x) = e^y$ This will be the solution of above equation.

1.1.2 Homogeneous Equation

If an equation is represented in the form of $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, then it is called as Homogeneous equation. Here $f_1(x, y)$ and $f_2(x, y)$; are the functions of x similar degree of x and y. To solve such equations, we

Put $y = vx$, where $\frac{dy}{dx} = v + x\frac{dv}{dx}$

In such situation, the given equation is represented as following written form –

$$v + x\frac{dv}{dx} = f(v)$$

i.e.,

$$x\frac{dv}{dx} = f(v) - v$$

The variables can be separated here and then integrate them on the both sides –

$$\int \frac{dv}{f(v) - v} = \log x + c$$

Here, c is an imaginary value.

> *Did u know?* We put y/x on the place of v after differentiating the equation and that will be the solution of that equation.

Example 3: Find the solution of $(x^3 - 3xy^2)\,dx = (y^3 - 3x^3 y)\,dy$.

Solution: The given equation can also be written as the followng form –

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y} \qquad \ldots(i)$$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in equation (i)

$$v + x\frac{dv}{dx} = \frac{x^3 - 3xv^2x^2}{v^3x^3 - 3x^2vx} = \frac{x^3(1 - 3v^2)}{x^3(v^3 - 3v)} = \frac{1 - 3v^2}{v^3 - 3v}$$

Or

$$x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - 3v^2 - 3v^4 + 3v^2}{v^3 - 3v} = \frac{1 - v^4}{v^3 - 3v}$$

Or

$$\frac{dx}{x} = \frac{v^3 - 3v}{1 - v^4}\,dv = \left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{v^2 + 1}\right]$$

Note (By Partial Function)

On differentiating both side

$$\log x + \log c = \frac{1}{2}\log(v+1) + \frac{1}{2}\log(v-1) - \log(v^2+1)$$

Or $$\log(cx) = \frac{1}{2}\log\left[(v+1)^{1/2}(v-1)^{1/2}/v^2+1\right]$$

Or $$cx = \frac{(v^2-1)^{1/2}}{v^2+1} \text{ or } c^2x^2 = \frac{v^2-1}{(v^2+1)} \text{ or } c^2x^2(v^2+1)^2 = (v^2-1)$$

Or $$c^2x^2[y^2/x^2+1]^2 = [y^2/x^2-1] \qquad [v = y/x \text{ On putting}]$$

$$c^2(y^2+x^2) = (y^2-x^2)$$ This is the solution of above equation.

1.1.3 Equations Changeable to Homogeneous Form

If the differential is in the following form –

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}, \text{ where } \frac{a}{a_1} \neq \frac{b}{b_1}$$

Then we change it in the homogeneous equation on putting $x = x + h$ and $y = y + k$. (Here h and K are constant). Except it, also put $dy = DY$ and $dx = DX$ –

$$\frac{DY}{DX} = \frac{a(x+h)+b(y+k)+c}{a_1(x+h)+b_1(y+k)+c_1}$$

$$= \frac{ax+by+(ah+bk+c)}{a_1x+b_1y+(a_1h+b_1k+c_1)}$$

Above equation will be homogeneous if $ah + bk + c = 0$ and $a_1h + b_1k + c_1 = 0$

$$\frac{DY}{DX} = \frac{ax+by}{a_1x+b_1y}$$

Now we'll solve the equation on putting $Y = vX$ according the last example. At the last, on putting $X = x - h$ and $Y = y - k$, the solution of given equation would be fould.

If there is $\frac{a}{a_1} = \frac{b}{b_1} = m$ then the above given equation will be as following –

$$\frac{dy}{dx} = \frac{m(a_1x+b_1y)+c}{a_1x+b_1y+c_1}$$

To solve such differential equations, $v = a_1x + b_1y$ is considered.

Example 4: Find the solution of $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$.

Here, the situation is of , therefore, on putting $x = X + h$ and $y = Y + k$

$$\frac{DY}{DX} = \frac{(Y+k)-(X+h)+1}{(Y+k)-(X+h)+5} = \frac{Y-X+(k-h+1)}{Y-X(k-h+5)} \qquad \ldots(i)$$

Note

On putting $k - h + 1 = 0$ and $k + h + 5 = 0$, we get that $k = -3$, $h = -2k$ and the equation (i) will be as following on putting the value of h –

$$\frac{dY}{dX} = \frac{Y - X}{Y + X}$$

On putting $Y = vX$, we get

Or $$v + X\frac{dv}{dX} = \frac{v-1}{v+1} \text{ or } X\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{-1-v^2}{v+1}$$

Or $$\frac{v+1}{v^2+1}dv = -\frac{dx}{X} \text{ or } \frac{v}{v^2+1}dv + \frac{dv}{v^2+1} = -\frac{X}{dx}$$

On multiplying with 2 on both sides,

Or $$\frac{2v}{v^2+1}dv + \frac{2v}{v^2+1}dv = \frac{-2X}{dx}$$

On integrating both sides

$$\log(v^2+1) = 2\tan^{-1} v = -2\log X + c$$

Or $$\log(v^2+1) + 2\log X = -2\tan^{-1} v + c$$

Or $$\log[(v^2+1)X^2] = -2\tan^{-1} v + c$$

On putting thevalue of v

$$\log(Y^2 + X^2) = -2\tan^{-1}(Y/X) = c$$

On putting the values of Y and X

$\log[(y+3)^2 + (x+2)^2] + 2\tan^{-1}\{(y+3)/(x+2)\} = c$

This is the solution of given differential equation.

1.1.4 Linear Differential Equation

The Linear Differential Equation is represented in the following form –

$$\frac{dy}{dx} + PY = Q$$

Here P and Q are the functions of x only and y is a dependent variable. To solve this equation, on multiplying both sides with $e^{\int p\,dx}$

$$e^{\int p\,dx}\left(\frac{dy}{dx}\right) + e^{\int p\,dx} Py = e^{\int p\,dx}.Q$$

Or $$\frac{d}{dx}\left\{ye^{\int p\,dx}\right\} = Qe^{\int p\,dx}$$

On differentiating both sides with respect to 'x'

$$ye^{\int p\,dx} = \int Q\,e^{\int p\,dx}\,dx + c$$

This will be the solution of given equation.

Note

Example 5: Find the solution of $\frac{dy}{dx} + 2xy - e^{-x^2}$.

On comparing the above equation with $\frac{dy}{dx}$ + $PY = Q$, we get

$$P = 2x,\ Q = e^{-x^2}$$

On integrating both separately with respect to 'x'

$$\int P\,dx = 2\int x\,dx, \int Q\,dx = \int e^{-x^2}$$

$$\int P\,dx = 2\int \frac{x^2}{2} = x^2$$

Therefore, $$\int_e p\,dx = e^{x^2}$$

The solution of the given differential equation will be –

$$y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$$

[Here I.F. = Integrating Factor]

$$y\,e^{x^2} = \int e^{-x^2}\,e^{x^2}\,dx + c$$

Or $$y\,e^{x^2} = x + c$$

This is the solution of above equation.

1.1.5 Change in linear Form

If any differential equation can be converted in linear form then that can be solved by two methods –

(I) Barnauli Equation: If any equation is given in the following form –

$$\frac{dy}{dx} + PY = Q\,y^n$$

In this situation, we multiply with y^{-n} on the both sides.

$$y^{-n}\frac{dy}{dx} + PY^{-n+1} = Q \qquad \text{...(i)}$$

We put $y^{-n+1} = v$ Then there will be $\frac{dv}{dx}(1-n)y^{-n}\frac{dy}{dx}$

Now equation (i) will be in the following form –

$$\frac{1}{1-n}\frac{dv}{dx} + pv = Q$$

Or $$\frac{dv}{dx} + (1-n)\,pv = (1-n)\,Q \qquad \text{...(ii)}$$

This equation is a linear equation, where v is a dependent variable. The solution of this differential equation will be according to 1.1.4.

(II) If an equation is shown in the following form –

$$\frac{dv}{dx} + P\phi(y) = Q\,f(y)$$

Note

Where, P and Q are the functions of x only.

Then, for convering the above equation into linear equation, on dividing with $f(y)$ on the both sides of equation

$$\frac{1}{f(y)}\frac{dv}{dx} + P\frac{\phi(y)}{f(y)} = Q$$

In equation $\frac{\phi(y)}{f(y)} = v$ take it $\frac{dv}{dx} = \frac{d}{dx}\left\{\frac{\phi(y)}{f(y)}\right\}$

$$= k\frac{1}{f(y)}\frac{dy}{dx},\ \text{where } k \text{ is a constant.}$$

Therefore, we can write equation (ii) in the following form –

$$\frac{1}{k}\frac{dv}{dx} + Pv = Q$$

Or $\frac{dv}{dx} = k\,Pv = kQ$ which is a linear equation.

Example 6: Find the solution of $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$.

On dividing given equation by y^6

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{y^5} = x^2 \qquad ...(i)$$

Assume $\frac{1}{y^5} = v$ then $\frac{dv}{dx} = -\frac{5}{y^6}\frac{dy}{dx}$

Or $\frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dy}{dx}$

Putting in equation (i)

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \text{ or } \frac{dv}{dx} - \frac{1}{5}v = -5x^2$$

This is a linear equation in which v is a dependent variable.

Here $P = -5/x$ and $Q = -5x^2$

Therefore $\int Pdx = \int\left(-\frac{5}{x}\right)dx = -5\log x = \log x^5 = \log\left(\frac{1}{x^5}\right)$

∴ $\int_e P\,dx = e\log^{(1/5)} = \frac{1}{x^5}$

$$v\text{ (I.F)} = \int Q\text{ (I.F.) } dx + c$$

Or $v(1/x^5) = \int -5x^2\,(1/x^5)\,dx + c = -5\int x^{-3}dx + c$

Note

$$\left(\frac{1}{y^5}\right)\left(\frac{1}{x^5}\right) = -\frac{5}{2}\left(\frac{1}{x^2}\right) + c \qquad [\because v = 1/y^5]$$

Or $$\frac{1}{x^5y^5} + \frac{5}{2}\left(\frac{1}{x^2}\right) + c$$

This is the solution of differential equation.

1.2 Exact Differential Equations

The differential Equation $M\,dx + N\,dy = 0$ will be exact if there is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If the given equation is exact then we'll use following steps after it –

i. Integrate M with respect to x when y is constant.

ii. Integrate N with respect to y only and we'll integrate only the terms which don't have x.

iii. Add above both the integrations.

Therefore if the differential equation $M\,dx + N\,dy = 0$ is exact the its solution will be as following –

$\int M\,dx$ (Considering y as a constant) + $\int M\,dy$ (Only the terms which dont have x) $= c$

For example: Find the solution of $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$.

Here, $M = x^2 - ay$ and $N = (ax - y^2)$.

$$\frac{\partial M}{\partial y} = -a \text{ and } \frac{\partial N}{\partial x} = -a$$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Therefore the given equation is exact.

Now $\int M\,dx$ (Considering y as a constant) $= \int (x^2 - ay)\,dx = \frac{1}{3}x^3 - ay\,x$ …(i)

And $\int N\,dy$ (Only the terms which don't have x) $= \int y^2\,dy = \frac{1}{3}y^3$ …(ii)

Therefore, we'll get –

(i) + (ii) = c(imaginary constant)

Or $\frac{1}{3}x^3 - ayx + \frac{1}{3}y^3 = c$

Or $x^3 - 3\,ayx + y^3 = 3c$..

Task Find the solution of $\frac{dy}{dx} + \frac{1}{x}y = x^2\,y^6$. (**Ans.**: $\frac{1}{x^5y^5} + \frac{5}{2}\left(\frac{1}{x^2}\right) + c$)

Self Assessment

Note

1. Fill in the blanks:

1. Differential Equations are the equations in which ……………, independent variable exists.
2. An equation will be called as linear if the …………….. of dependent variables are of first degree.
3. The solution of a differential equation in which the independent imaginary constant is equal to the order of differential equation, then it is called as ………… Solution.
4. In normal solution, if any particular value is given to constants then it is called the …………………. solution of that equation.
5. If an equation is represented in the form of $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, then it is called as Homogeneous equation.

17.3 Summary

- Differential Equations are the equations in which dependent, independent variable exists, and there are different derivatives of dependent variables with respect to one or more independent variables. The order of a differential equation is the highest order of different derivatives included in that equation. An equation will be called as linear if the derivatives of dependent variables are of first degree, otherwise it will be called as non-linear.
- Normal Solution: The solution of a differential equation in which the independent imaginary constant is equal to the order of differential equation, then it is called as Normal Solution.
- Particular Solution: In normal solution, if any particular value is given to constants then it is called the Particular Solution of that equation.
- If the differential equation is shown in the following form –

  $$f_1(x)\, dx = f_2(y)\, dy$$

  Where $f_1(x)$ and $f_2(x)$ are the functions of x and y respectively.
- In such situation, we differentiate the both sides of the equation and then add an imaginary constant on any one side of the equation.

1.4 Keywords

- ***Homogeneous:*** Similar
- ***Separable:*** Able to be separate

1.5 Review Questions

1. Find the differential equation for the different values of A and B of $y = A\, e^x + B/e^x$.

   (**Ans.:** $\frac{d^2y}{dx^2} = y$)

2. Find the solution of $\left[\frac{x}{\sqrt{1+x^2}}\right] dx = -1 \left[\frac{y}{\sqrt{1+y^2}}\right] dy$. (**Ans.:** $\sqrt{1+x^2} + \sqrt{1+y^2} = c$)

3. Find the solution of $\frac{dy}{dx}$ + $2xy - e^{-x2}$. (**Ans.:** $ye^{x2} = x + c$)

Note

Answers: Self Assessment

1. Dependent
2. Differentiation
3. General
4. Special
5. Homogeneous

1.6 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics for Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman

Mathematics for Economist – Council for Economic Education

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 2: Matrices: Meaning and Types

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Learn the Order and Notation of a Matrix.
- Get the Knowledge about Kinds of Matrix.
- Understand the Important Properties of Matrices.
- Know the Addition and Subtraction of Matrices.
- Solve the Questions related with Matrix Multiplication.

Introduction

A Rectangular Arromy, which is arranged in rows and columns, is called Matrix.

The Permutation of mn numbers in m rows and n columns is called the matrix of $m \times n$ order.

$$\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{array}$$

Notes The numbers a_{11}, a_{12}, a_{13},…………. are called the Elements or Constituents of Matrix.

The rows and columns are made from the horizontal constituents and vertical constituents respectively. As -

Note

$A = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 3 \end{bmatrix}$ is a Rectangular Permutation in which 2 rows and 3 columns and the numbers of constituents are 2 × 3 = 6.

Similarly $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is also a Rectangular Permutation in which there are 3 rows and 1 column

And the number of entries are 3 × 1 = 3.

2.1 Order of Matrix

As there are two dimensions length and breadth in a rectangle similarly the dimension or order of matrix is 'number of rows × number of columns'. We indicate the matrices given in definition as *A*, *B*, *C*, etc. As Am *x n* , $A_{m \times n}$. $B_{2 \times 3}$. $C_{1 \times 3}$, etc.

2.2 Notation of a Matrix

If the number of rows and columns in any matrix are m and n respectively, then there will be $m \times n$ number of constituents in the rectangular permutation. If the place of constituents is a_{ij} in the i^{th} row and j^{th} column, then matrix is written as following -

$$A_{m \times n} = A = [a_{ij}]_{m \times n}$$

Where i = 1, 2, 3, ………., m and j = 1, 2, 3, ………….., n.

Did u know? Here, it is remarkable that Matrix is only the rectangular permutation (Written in well organized and brief form) of numbers. It have no numerical value.

2.3 Kinds of Matrix

1. **Square Matrix**

   If the number of rows and columns are equal in a matrix then it is called as Square Matrix. i.e., if $m = n$ then the arromy is called as the matrix of order n. if there is other matrices then they are called as rectangular matrices. For example,

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 7 & 2 & 5 \end{bmatrix}$$

   Both of above matrices are Square Matrices of order (2 × 2) and (3 × 3) respectively.

2. **Row Matrix**

   If $m = 1$, then there will be only one row in the matrix, it will be called as Row Matrix. For example -

$$A = [2\ 3]_{1 \times 2}$$

$$B = [-5\ 2\ 0]_{1 \times 3}$$

$$C = [1\ 3\ 0\ -7]_{1 \times 4}$$

Note

3. **Column Matrix**

   If $n = 1$, there will be only one column and many numbers of rows, it will be called as Column Matrix. For example -

$$A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{2 \times 1}, B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}_{3 \times 1}, C = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 7 \end{bmatrix}_{4 \times 1}$$

4. **Null or Zero Matrix**

   If all the elements in a matrix are zero, then it is called Zero or Null Matrix. For example -

$$A = [0\ 0], B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. **Identity or Unit Matrix**

   It is a Square Matrix in which all the diagonal elements are unit and others are zero. It is called as Identity or Unit Matrix. For example -

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. **Diagonal Matrix**

   If all the elements except the elements situated on the diagonal of a square matrix, are zero then it is called as Diagonal Matrix. For example -

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

   In short it is indicated as $[d_1, d_2, d_3, d_4]$.

7. **Scalar Matrix**

   The diagonal Matrix in which all the diagonal elements are same, is called Scalar Matrix. For example -

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$A = [a_{ij}]$ will be Scalar if $a_{ij} = 0$,when $i \neq j$; and $a_{ij} = 2$ when $i = j$

$$B = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$B = [a_{ij}]_{3 \times 3}$, where $a_{ij} = 0$ when $i \neq j$; and $a_{ij} = k$, when $i = j$, Then B is a Scalar Matrix.

Note

8. **Lower Triangular Matrix**

The Square Matrix in which all the elements above the diagonal are zero, is called as Lower Triangular Matrix. For example -

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ -2 & -1 & -3 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}, \text{ here } a_{ij} = 0; \text{ when } i < j$$

Task Write an example of identity matrix.

9. **Upper Triangular Matrix**

The Square Matrix in which all the elements of diagonal are zero, is called the Upper Triangular Matrix. For example -

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix}; B = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots. & a_{1n} \\ 0 & a_{21} & a_{23}\ldots. & a_{2n} \\ 0 & 0 & a_{33}\ldots. & a_{3n} \\ \ldots. & \ldots. & \ldots. & \ldots. \\ \ldots. & \ldots. & \ldots. & \ldots. \\ 0 & 0 & 0 & a_{mn} \end{bmatrix}$$

Square Matrix A (order $n \times n$); where element $a_{ij} = 0$, when $i < j$

10. **Trace of a Matrix**

The addition of all the elements on the principal diagonal of a Square Matrix, is called the Trace of a Matrix. For example -

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

The above Matrix, Trace of Matrix = $a + b + c$, in the square matrix A of order $n \times n$, is called as the Trace of Matrix = $\sum_{i=1}^{n} a_{ij}$, where $j = i$, and written as Tr.

11. **Symmetric Matrix**

The Square Matrix A (order $n \times n$) in which there is $a_{ij} = a_{ji}$ for each $(i - j)^{th}$ element, is called Symmetric Matrix. For example -

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

The above matrix is a Symmetric Matrix because $a_{ij} = a_{ji}$ *i.e.* $a_{21} = h = a_{12}$; $a_{32} = f = a_{23}$; $a_{22} = b = a_{22}$.

12. **Skew - Symmetric Matrix**

The Square Matrix A (order $n \times n$) in which there is $a_{ji} = a_{ij}$ for each $(i - j)^{th}$ element, is called Symmetric Matrix. For example -

$$A = \begin{bmatrix} 0 & g \\ -h & f \\ -g & 0 \end{bmatrix}$$

Note

The diagonal elements are $a_{11}, a_{22}, a_{33}, ..., a_{ij}$ and for all the values os the condition $a_{ij} = -a_{ji}$ and

$\therefore \quad 2a_{ij} = 0$ or $-a_{ij} = 0$

So all the diagonal elements of Skew - Symmetric Matrix are zero.

Self Assessment

1. Fill in the blanks:

1. A Rectangular Arromy, which is arranged in rows and columns, is called …………….
2. If the number of rows and columns in any matrix are m and n respectively, then there will be …………………… number of constituents in the rectangular permutation.
3. If the number of rows and columns are equal in a matrix then it is called as ………… Matrix.
4. If all the elements in a matrix are zero, then it is called …………………………. Matrix.
5. If all the elements except the elements situated on the diagonal of a square matrix, are zero then it is called as …………………. Matrix.

2.4 Important Properties of Matrices

1. **Equality of Matrices**

   Two matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{p \times q}$ will be equal if the order of both are the same i.e., $m = p$; $n = q$ and $a_{ij} = b_{ij}$.
2. **Equivalance Relations of Matrices**

   If Matrices A, B, C are confirmable i.e., having similar order; then

   (i) $A = A$ (Reflexive Relation)

   (ii) $A = B$?$\Leftrightarrow$ $B = A$ (Symmetric Relation)

   (iii) $A = B$ and $B = A \Leftrightarrow A = C$; Transitive Relation

Where, notation $\Rightarrow$ (if ………… then) and $\Leftrightarrow$ (iff = if and only if)

(a) $A = A$ means $a_{ij} = a_{ij}$ where $i = 1,2,3,\ldots\ldots m$ and $j = 1,2,3,\ldots\ldots n$. It is called Reflexive Relation.

(b) If there are $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then $A = B$ means $a_{ij} = b_{ij}$ or $b_{ij} = a_{ij}$., then $B = A$.

$\therefore \quad A = B \Rightarrow B = A$ it is called Symmetric Relation

(c) If $A = B$, it is called $B = C$ Symmetric Relation

$$a_{ij} = b_{ij} \text{ and } b_{ij} = c_{ij}$$

therefore $\quad a_{ij} = c_{ij} \quad \therefore \quad A = C$

The relationship in Matrices A, B, C which is Reflexive, Symmetric and Transitive, is called the Equivalance Relationship. For example -

From the definitions of equal matrices, the value of x, y and z are

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Therefore, the relative elements of two equal matrices are equal.

$\therefore \quad x + 3 = 0; 2y + x = -7; z - 1 = 3$ and $4a - 6 = 2a$

From these equations, $x = -3$; $y = -2$; $z = 4$ and $a = 3$.

Note

2.5 Addition and Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of order $m \times n$, then we'll show their addition $A + B$ from $C = [c_{ij}]$ of $m \times n$ order.

Where $c_{ij} = b_{ij}$,, $i = 1,2,3,\ldots m$; $j = 1,2,3,\ldots n$ for all values.

Here it is remarkable that if matrices are not in same order then their addition is not possible.

Example 1: Evaluate -

(*i*) [1 2 3] + [4 5 6];

(*ii*) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$; then $A + B = ?$

Solution: (i) Here $A = [1\ 2\ 3]$; $B = [4\ 5\ 6]$

$\therefore \quad A + B = [1 + 4\ 2 + 5\ 3 + 6] = [5\ 7\ 9]$

(ii) Here $A + B = C$; where $c_{ij} = a_{ij} + b_{ij}$

$$\therefore \quad C = \begin{bmatrix} 1+7 & 2+3 & 4+2 \\ 0+5 & 5+1 & 3+9 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$$

Field Properties of Matrix Addition

(i) If two matrices are of same order and conformable then the matrix from these addition is also of the similar order.

If $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$ then

$A + B = C \quad \Rightarrow \quad [a_{ij} b_{ij}]_{m \times n}$; (Closure Rule for Addition)

(ii) If two matrices are conformable (of similar order) then

$A + B = B + A$ (Commutative Rule For Addition)

If $A + B = [a_{ij} + b_{ij}]_{m \times n} = [b_{ij} + a_{ij}]_{m \times n} = B + A$ then the Addition of Conformable Matrices follow the Commutative Rule.

(iii) if three matrices A, B, C are conformable, then

$(A + B) + C = A + (B + C)$ (Associative Law for Addition)

Here $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$; $C = [c_{ij}]_{m \times n}$, then

$\therefore \quad (A + B) + C = [a_{ij} + b_{ij}]_{m \times n} + [c_{ij}]_{m \times n} = [a_{ij} + b_{ij} + c_{ij}]_{m \times n}$

And $\quad (A + B) + C = [a_{ij}]_{m \times n} + [b_{ij} + c_{ij}]_{m \times n} = [a_{ij} + b_{ij} + c_{ij}]_{m \times n}$

$\therefore \quad (A + B) + C = A + (B + C)$

(iv) If matrix A is of $m \times n$ order and Zero Matrix is also of same order then $A + 0 = A$; where the addition of zero matrix is identity element.

(v) If matrix A is of $m \times n$ order and $A + (-A) = 0$, then $-A$ Matrix is the additive inverse of matrix A.

Subtraction of Two Matrices

If $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$ then Subtraction of two matrices, is that matrix C in which $C = [c_{ij}]_{m \times n}$ and if $c_{ij} = a_{ij} = b_{ij}$

i.e., $c_{ij} = a_{ij} + (-b_{ij})$

The Subtraction of two matrices is found from the addition of the additive inverse of A and B i.e., $A - B$.

Note

Example 2: If $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ **and** $\begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$ **then evaluate $A - B$.**

Solution: $A - B = \begin{bmatrix} 1-7 & 2-3 & 4-2 \\ 0-5 & 5-1 & 3-9 \end{bmatrix} = \begin{bmatrix} -6 & -1 & 2 \\ -5 & 4 & -6 \end{bmatrix}$

2.6 Matrix Multiplication

If $A = [a_{ij}]$, $m \times n$ is the matrix of $m \times n$ order and $B = [b_{ik}]$, $n \times p$ is the matrix of $n \times p$ order, then the Multiplication of Matrices A and B is $AB = C$; where, $C = [c_{ik}]$, $m \times n$ is a matrix A of $m \times n$ order. i.e., if two matrices A and B are conformable in which the number of columns A is equal to the number of rows B, then we can multiply both the matrices and denote the product from AB.

Thus, the method to get the common element c_{ik} of Matrix Multiplication AB is -

To multiply and add the elements of ith row of A and kth column of B. it is called Row Multiplied Column Method.

Assume the $A = [a_{i1}, a_{i2}....a_{in}]$ and $B = \begin{bmatrix} b_{1k} \\ b_{2k} \\ : \\ b_{nk} \end{bmatrix}$ then the multiplication of AB will be -

$$[a_{i1}\, a_{i2}....a_{in}] \begin{bmatrix} b_{1k} \\ b_{2k} \\ : \\ b_{nk} \end{bmatrix} = \rightarrow \begin{bmatrix} : \\ : \\ c_{ik} \\ : \end{bmatrix}$$

Where $c_{ik} = a_{i1}\, b_{1k} + a_{i2}b_{2k}.... + a_{ij}b_{ik} + ... + a_{in}b_{nk}$.

It is attentable that The Multiplication AB is definite only then if the number of columns in matrix A is equal to the numbers of rows in Matrix B. Matrix A and B are comfortable for multiplication. Therefore we can multiply the one row matrix and column if the number of elements in both are equal.

If Matrix A is of $1 \times m$ order and B is of $n \times 1$ order then Matrix Multiplication $A.B$ will be of 1×1 order but the product of matrices B and A will be of $n \times n$ order.

$$A \,.\, B = [a_1, a_2....a_n] \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix} = [a_1b_1 + a_2b_2 + ... + a_nb_n]$$

$$B \,.\, A = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}_{1 \times n} [a_1a_2....a_n]_{1 \times n} = \begin{vmatrix} b_1a_1 & b_1a_2..... & b_1a_n \\ b_2a_1 & b_2a_2..... & b_2a_n \\ ..... & .... & ..... \\ ..... & .... & ..... \\ b_na_1 & b_na_2..... & b_na_n \end{vmatrix}.$$

Note

Example 3: If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}_{2 \times 3}$ **and** $B = \begin{vmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 2 & 3 & 1 \end{vmatrix}_{3 \times 3}$ **then evaluate** ***AB*** **-**

Solution: $AB = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2.4 + 1.3 + 3.2 & 2.1 + (-1) + 3.3 & 2.2 + 1.5 + 3.1 \\ (-1)4 + 3.3 + 1.2 & (-1) \,.\, 1 + 3(-1) + 1.3 & (-1)2 + 3.5 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 10 & 22 \\ 7 & 1 & 14 \end{bmatrix}_{2 \times 3}$$

Important Note - If Matrices A and B are the Square Matrices of same order then AB and BA both will be defined and both will be the square matrices of that order, then also in general $AB \neq BA$ i.e., Matrix Multiplication is not commutative.

Example 4: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$**, then evaluate** ***AB*****.**

Solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

It proves that Matrix Multiplication is a Zero Matrix but $A \neq 0$ and $B \neq 0$.

Associative Law of Matrix Multiplication

(i) **Associative Law of Matrix Multiplication**

If $A = [a_{ij}]_{, m \times n}$ is of $m \times n$ order, $B = [b_{ik}]$, $n \times p$ is of $n \times p$ order and $C = [c_{ik}]$, $p \times q$ is of $p \times q$ order, then $A \,.\, (B \,.\, C) = (A \,.\, B) \,.\, C$, both sides are the matrices of $m \times q$ order, i.e., Matrix Multiplication is Associative.

It can be proved very easily.

Assume that $B.C = [d_{ji}]$, where

$$d_{ji} = b_{ji}\, c_{i1} + b_{j2}c_{2i} + ... + b_{ip}C\pi = \sum_{k=1}^{p} b_{ik}c_{ki}$$

Therefore $A \,.\, (B \,.\, C) = [c_{ij}]$, where $c_{ji} = a_{i1}d_{1i} + ...\ a_{in}d_{ni}$

$$= \sum_{i=1}^{n} a_{ij}d_{ji} = \sum_{j=1}^{n} n_{ij} \sum_{k=1}^{P} B_{jk}c_{ki}$$

Because the number of elements is symmetric, so on changing the order of addition

$$= \sum_{k=1}^{p} \left(\sum a_{ji}b_{jk}\right) c_{ki} = (A.B) \,.\, C$$

(ii) **Distributive Law of Matrix Multiplication**

If $A = [a_{ij}]$, $m \times n$ is of $m \times n$ order, and $B = [b_{ik}]$, $C = [c_{jk}]$, $n \times p$, are of $n \times p$ order,

Note

Then $A(B + C) = AB + AC$

Because $B + C = [b_{ij}] + [c_{jk}] = [b_{jk}] + [c_{jk}]$

Therefore $A(B + C) = [d_{ik}]$,

Where $d_{ik} = a_{i1}(b_{1k} + c_{1k}) + ... + a_{in}(b_{nk} + c_{nk})$

$$= \sum_{j=1}^{p} a_{ij}(b_{jk} + c_{jk})$$

$$= \sum_{i=1}^{n} (a_{ij}b_{jk} + a_{jj}c_{jk}) = \sum_{i=1}^{n} a_{ij}b_{jk} + \sum_{i=1}^{n} a_{ij}c_{jk}$$

Therefore $[d_{ik}] = AB + AC$

Therefore $A(B + C) = AB + AC$

Similarly, if $B = [b_{ik}]]$ $C = [c_{jk}]$, are of $n \times p$ order and $A = [a_{ij}]$, is of $p \times n$ order, then $(B + C)A = BA + CA$.

Example 5: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ **then prove that** $AB \neq BA$**:**

Solution: $AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 0 + 2\times 2 + 3\times -1 & 0\times 0 + 1\times 2 + 1\times -1 & 1\times 0 + -1\times 2 + 0\times -1 \\ 1\times 1 + 2\times 1 + 3\times 0 & 0\times 1 + 1\times 1 + 1\times 0 & 1\times 1 + -1\times 1 + 0\times 0 \\ 1\times 2 + 2\times -1 + 3\times 1 & 0\times 2 + 1\times -1 + 1\times 1 & 1\times 2 + -1\times -1 + 0\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

Now, $BA = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0\times 1 + 1\times 0 + 2\times 1 & 2\times 1 + 1\times 0 + -1 - 1\times 1 & -1\times 1 + 0\times 0 + 1\times 1 \\ 0\times 2 + 1\times 1 + 2\times -1 & 2\times 2 + 1\times 1 + -1\times -1 & -1\times 2 + 0\times 1 + 1\times -1 \\ 0\times 3 + 1\times 1 + 2\times 0 & 2\times 3 + 1\times 1 + -1\times 0 & -1\times 3 + 0\times 1 + 1\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & -3 \\ 1 & 7 & -3 \end{bmatrix}$$

Therefore, $AB \neq BA$.

Note

Example 6: If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **then evaluate** AB**.**

Solution:

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0\times 0+1\times 2 & 1\times 0+0\times 2 & 0\times 0+1\times 2 \\ 0\times 1+1\times 1 & 1\times 1+0\times 1 & 0\times 1+1\times 1 \\ 0\times 2+1\times 0 & 1\times 2+0\times 0 & 0\times 2+1\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Self Assessment

2. State whether the following statements are True or False:

6. $A = A$; is Reflexive Relationship when matrix A is conformable.
7. $A = B \Leftrightarrow B = A$ is symmetric relationship, when A, B are conformable.
8. When Matrices A, B, C are conformable then $A = B$ and $B = C \Leftrightarrow A = C$ shows the Reflexive relationship.
9. If the three matrices A, B, C are conformable then $(A + B) + C = A + (B + C)$ shows the associative law of addition.
10. The Subtraction of two matrices is found from the addition of additive inverses of A and B i.e., $A + B$.

2.7 Summary

- A Rectangular Arromy, which is arranged in rows and columns, is called Matrix.
- The Permutation of mn numbers in m rows and n columns is called the matrix of $m \times n$ order.
- As there are two dimensions length and breadth in a rectangle similarly the dimension or order of matrix is 'number of rows × number of columns'. We indicate the matrices given in definition as A, B, C, etc. As $A_{m \times n}$. $B_{2 \times 3}$. $C_{1 \times 3}$, etc.
- If the number of rows and columns in any matrix are m and n respectively, then there will be $m \times n$ number of constituents in the rectangular permutation.
- If $n = 1$, there will be only one column and many numbers of rows, it will be called as Column Matrix.
- If the number of rows and columns are equal in a matrix then it is called as Square Matrix.
- If all the elements in a matrix are zero, then it is called Zero or Null Matrix.

Note

- It is a Square Matrix in which all the diagonal elements are unit and others are zero. It is called as Identity or Unit Matrix.
- If all the elements except the elements situated on the diagonal of a square matrix, are zero then it is called as Diagonal Matrix.
- The diagonal Matrix in which all the diagonal elements are same, is called Scalar Matrix.
- The Square Matrix in which all the elements above the diagonal are zero, is called as Lower Triangular Matrix.
- The Square Matrix in which all the elements of diagonal are zero, is called the Upper Triangular Matrix.
- The addition of all the elements on the principal diagonal of a Square Matrix, is called the Trace of a Matrix.
- The Square Matrix A (order $n \times n$) in which there is $a_{ij} = a_{ji}$ for each $(i - j)^{th}$ element, is called Symmetric Matrix.
- The Square Matrix A (order $n \times n$) in which there is $a_{ji} = a_{ij}$ for each $(i - j)^{th}$ element, is called Symmetric Matrix.

2.8 Keywords

- ***Matrix:*** Rectangular arromy
- ***Elements:*** Constituents

2.9 Review Questions

1. If $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$, then find A+B. (**Ans.:** $\begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$)

2. If $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then find A+B. (**Ans.:** $\begin{bmatrix} 3 & 7 & 11 \\ 5 & 9 & 13 \end{bmatrix}$)

3. If $A = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then find A–B. (**Ans.:** $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$)

4. If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}_{3 \times 3}$. Find AB. (**Ans.:** $\begin{bmatrix} 17 & 10 & 22 \\ 7 & 1 & 14 \end{bmatrix}_{2 \times 3}$)

Answers: Self Assessment

1. Matrix	2. $m \times n$	3. square	4. Zero
5. diagonal	6. True	7. True	8. False
9. True	10. False		

Note

2.10 Further Readings

Books

Mathematics for Economist – Yamane - Prentice Hall India

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics for Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani- Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman

Mathematics for Economist – Council for Economic Education

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 3: Transpose and Inverse of Matrix

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Transpose of Matrix.
- Get the Knowledge about Transpose of Product of Two Matrices.
- Understand the Regular Matrix.
- Solve the Questions Related to Inverse or Opposite of Matrix.
- Know the Things Related to Orthogonal Matrix.

Introduction

A Rectangular Arromy, which is arranged in rows and columns, is called Matrix.

There is the discussion on inverse or opposite of matrix. Under it, if there is a matrix of $m \times n$ order, then the $n \times m$ matrix that will be found on exchanging the rows and columns of it, will be called as Transpose of A.

If matrix A and B are adaptable for product then the transpose of AB will be equal to the product of transposes in inverse.

19.1 Transpose of Matrix

If A is a matrix of order m x n, then the matrix that will be found on exchanging the rows and columns of it, will be called as Transpose of A and will be denoted as A' or A^t or A^T.

Example: If

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 4 \end{bmatrix} \text{ then } A' = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix}$$

Note It is clear that element A_{ij}, which was in the i^{th} row and j^{th} column, will be on j^{th} row and i^{th} column of transpose matrix A'. Therefore if

$$A = [a_{ij}] \text{ and } A' = [a_{ij}] \quad \Rightarrow \quad a_{ij}' = a_{ij}$$

Note – If we'll transpose A' then we'll get A again.

$$\therefore \qquad (A')' = (A^T)^T = [A^t]^t = A$$

Similarly $[KA]' = KA'$; here K is a scaler.

Theorem 1 – If A and B both are the matrix of order $m \times n$, then

$$(A + B)' = A' + B'$$

Proof – We are known that A and B will be adaptable to addition if both are of the same order. So assume that

$A = [a_{ij}]$ and $B = [b_{ij}]$ then, $C = A + B = [c_{ij}]$, where $c_{ji} = a_{ji} + b_{ij}$

Now $A' = [a_{ji}]$, $B' = [b_{ji}]$, where $a_{ji} = a_{ij}$ and $b_{ji} = b_{ij}$

$$(A + B)' = C' = [c_{ji}] = [a_{ij} + b_{ij}] = [a_{ji}] + [b_{ji}] = A' + B'.$$

3.2 Transpose of Product of two Matrices

Theorem 2 – If matrix A and B are adaptable for product then the transpose of AB will be equal to the product of transposes in inverse i.e., if A and B are of order $m \times n$ and $n \times p$, then $(AB)' = A'B'$.

Proof – The order of AB is $m \times p$, then $(AB)'$ will be of order $p \times m$, the order of B' is $p \times n$ and that of A' is n x m, therefore $B'A'$ will be of order $p \times m$. It means both $(AB)'$ and $B'.A'$ are of same order.

Assume then $A = [a_{ij}]$, $B = [b_{jk}]$, then

The $(k – i)^{th}$ element of $(AB)'$ = The $(i – k)^{th}$ element of (AB)

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

$(B)' = (b_{kj})$, $A' = (a_{ji})$ where $b_{jk}' = b_{jk}'$, $a_{ji}' = a_{ij}'$

The $(k - i)^{th}$ element of $B'A'$ will be

$$= \sum_{j=1}^{n} a_{jk} b_{ij}$$

$$= \sum_{j=1}^{n} a_{jk} b_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

i.e., The $(k - i)^{th}$ element of $B'A$ = The $(k - i)^{th}$ element of $(AB)'$

$$(AB)' = B'.A'$$

This result can be used till the matrices of adaptable order of any number, i.e.,

$(ABC........L.M)' = M'\,L'........C'\,B'\,A'$

> *Notes* If $A = B$ then $(A^2)' = A' \,.\, A' = (A')^2$ similarly, on taking any positive integer power of A, $(A^k)' = (A')^k$.

Note

3.3 Regular Matrix

The square – matrix A will be called as Regular if there is $BA = I$, then there will also be $AB = I$. It will be proved as matrix – derivative if

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \delta_{up}; i, j = 1, 2, \ldots, n$$

Then
$$\sum_{k=1}^{n} a_{ik}b_{ij} = \delta_{ij}; i, j = 1, 2, \ldots, n$$

Which is not only true but is clear also.

Self Assessment

1. Fill in the blanks:

1. If matrix A and B are adaptable for product then the transpose of AB will be equal to the …………… of transposes in inverse.
2. The square – matrix A will be called as Regular if B is in such matrix existence when $BA = I$, where I is the ……………….. matrix of same order.
3. The inverse of a transpose matrix is ………………
4. If A is square matrix, then $AB = 0 \Rightarrow B = 0$
5. If $AA' = I$ then A is called ………………….. matrix.

3.4 Inverse or Opposite of Matrix

If the square – matrix is regular then there exist such square – matrix B that $BA = AB = I$.

Therefore we'll call the regular matrix A as Invertible and B as the inverse of A and denote it as A^{-1}. In this situation, the inverse of matrix is found out by following technique –

(i) Form a transpose of matrix on replacing the a_{ij} elements of matrix A from the co – factors c_{ji}. So made changed matrix is called Adjoint of A so that

$$\begin{bmatrix} c_{11} & c_{21} \ldots\ldots & c_{n1} \\ c_{12} & c_{22} \ldots\ldots & c_{n2} \\ \vdots & & \\ c_{1n} & c_{2n} \ldots\ldots & c_{nn} \end{bmatrix}$$

(ii) Divide the Adj. (A) from the matrix of A, $|A|$. (If $\neq 0$)

$$\frac{\text{Adj}(A)}{|A|} = \begin{bmatrix} \frac{c_{11}}{|A|} & \frac{c_{21}}{|A|} \ldots & \frac{c_{n1}}{|A|} \\ \frac{c_{12}}{|A|} & \frac{c_{22}}{|A|} \ldots & \frac{c_{n2}}{|A|} \\ \vdots & & \\ \frac{c_{1n}}{|A|} & \frac{c_{2n}}{|A|} \ldots & \frac{c_{nn}}{|A|} \end{bmatrix}$$

$\left|\frac{(A)}{|A|}\right|$ is the required A^{-1}.

Note

Example 1: If $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & \mathbf{12} \end{bmatrix}$ **then find** A^{-1}**.**

Solution: We know that $A^{-1} = \dfrac{\text{Adj.}A}{|A|}$

On making the Transpose of given Matrix

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix}$$

To find Adjoint, we'll find the co–factors of matrix A (a_{ij}) as following –

$$\text{Co - factors } 1 = +\begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} = 36 - 25 = 11$$

$$\text{Co - factors } 2 = -\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} = (24 - 15) = -9$$

$$\text{Co - factors } 3 = +\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = 10 - 9 = 1$$

$$\text{Co - factors } 2 = -\begin{bmatrix} 2 & 3 \\ 5 & 12 \end{bmatrix} = 24 - 15 = -9$$

$$\text{Co - factors } 3 = +\begin{bmatrix} 1 & 3 \\ 3 & 12 \end{bmatrix} = 12 - 9 = 3$$

$$\text{Co - factors } 5 = -\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -(5 - 6) = -(-1) = +1$$

$$\text{Co - factors } 3 = +\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = (10 - 9) = 1$$

$$\text{Co - factors } 5 = -\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -(5 - 6) = +1$$

$$\text{Co - factors } 12 = +\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = (3 - 4) = -1$$

$$Adj.\ A = \begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix} = 1\begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} - 2\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$= 1\,(36 - 25) - 2\,(24 - 15) + 3\,(10 - 9)$

$= (1 \times 11) - (2 \times 9) + (3 \times 1) = 11 - 18 + 3 = -4$

$$A^{-1} = \frac{Adj.\,A}{|A|} = -\frac{1}{4}\begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} & \frac{9}{4} & -\frac{1}{4} \\ \frac{9}{4} & -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Properties of Inverse Matrices

The inverse of an Invertible Matrix is unique. Unique means – if the inverse of A is B, then there is no other inverse of matrix A except B.

Proof – If it is possible, assume that B and C both are the inverse of matrix A. then

$$A\,B = BA = I$$

And $$AC = CA = I$$

$$AB = AC = \;\Rightarrow\; B(AB) = B(AC)$$

$\Rightarrow$ $$(BA)B = (BA)C$$

$\Rightarrow$ $$IB = IC$$

$\Rightarrow$ $$B = C$$

(ii) If A and B are the Invertible Matrices and their order is n, then their product is inverse of AB and $(AB)^{-1} = B^{-1}\,A^{-1}$.

Proof – Because A and B are invertible, therefore there exist A^{-1} and B^{-1}. Consequently

$(B^{-1}\,A^{-1})\,(AB) = B^{-1}\,(A^{-1}\,A)B$ Therefore the product is Homogeneous

$$= B^{-1}\,(IB)\,A^{-1}\,A = 1$$

$$= B^{-1}\,B = I$$

Again $$(AB)\,(B^{-1}\,A^{-1}) = A\,(BB^{-1})\,A^{-1}$$

$$= A(IA^{-1})$$

$$= AA^{-1} = I$$

Therefore AC is Invertible and $(AB)^{-1} = B^{-1}\,A^{-1}$.

Comment – (i) If A, B, C,…………AM are the invertible matrices of the same order, then $(A.B.C.....M) = M^{-1}\,...C^{-1}\,B^{-1}\,A^{-1}$

(ii) If A is a regular square matrix, then $AB = 0 \Rightarrow B = 0$;

If B is a regular square matrix, then $AB = 0 \Rightarrow A = 0$

(iii) The Transpose and inverse process in matrix is commutative.

$$(A')^{-1} = (A^{-1})^{1}.$$

3.5 Orthogonal Matrix

If $AA' = I$ then A is called Orthogonal Matrix.

Involuntary Matrix

If A is a square matrix and $A^2 = I$ (Corresponding or identity), then A is called Involuntary Matrix.

Note

Example 2: Show that the product for the following 2 × 2 matrices is commutative –

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Solution: Here

$$AB = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

$$= \begin{bmatrix} ax - by & ay + bx \\ -bx - ay & -by + ax \end{bmatrix}$$

And

$$BA = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} xa - yb & xb + ya \\ -ya - xb & -yb + xa \end{bmatrix}.$$

Self Assessment

2. State whether the following statements are True or False:

6. Form a transpose of matrix on replacing the a_{ij} elements of matrix A from the co – factors c_{ji}. So made changed matrix is called Adjoint of A.
7. The inverse of an Invertible Matrix is not unique.
8. If B is a regular square matrix, then $AB = 0 \Rightarrow A = 0$.
9. If $AA' = I$ then A is called Zero Matrix.
10. If A is a square matrix and $A^2 = I$ (Corresponding or identity), then A is called Involuntary Matrix.

3.6 Summary

- If A is a matrix of order $m \times n$, then the matrix that will be found on exchanging the rows and columns of it, will be called as Transpose of A and will be denoted as A' or A^t or A^T.
- If matrix A and B are adaptable for product then the transpose of AB will be equal to the product of transposes in inverse i.e., if A and B are of order $m \times n$ and $n \times p$, then $(AB)' = A'B'$.
- The square – matrix A will be called as Regular if there is $BA = I$ Where I is the identity matrix of same order.
- We'll call the regular matrix A as Invertible and B as the inverse of A and denote it as A^{-1}.
- The inverse of an Invertible Matrix is unique. Unique means – if the inverse of A is B, then there is no other inverse of matrix A except B.
- If A and B are the Invertible Matrices and their order is n, then their product is inverse of AB and $(AB)^{-1} = B^{-1} A^{-1}$.
- If A, B, C,…………AM are the invertible matrices of the same order, then $(A.B.C\ldots\ldots\ldots\ldots M) = M^{-1}\ldots\ldots C^{-1}B^{-1}A^{-1}$. If A is a regular square matrix, then $AB = 0 \Rightarrow B = 0$;
- If B is a regular square matrix, then $AB = 0 \Rightarrow A = 0$. The Transpose and inverse process in matrix is commutative.
- If $AA' = I$ then A is called Orthogonal Matrix.
- If A is a square matrix and $A^2 = I$ (Corresponding or identity), then A is called Involuntary Matrix.

Note

13.7 Keywords

- ***Scalar:*** Which has magnitude, but don't have direction
- ***Inverse:*** Opposite

3.8 Review Questions

1. Prove that $(A + B)' = A' + B'$
2. Prove that $(AB)' = B'.A'$
3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix}$ Then find A^{-1}. **(Ans.:** $\begin{bmatrix} \frac{-11}{4} & \frac{9}{4} & \frac{-10}{4} \\ \frac{9}{4} & \frac{-3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$**)**

Answers: Self Assessment

1. Product	2. Identity	3. Unique	4. Regular
5. Orthogonal	6. True	7. False	8. True
9. False	10. True		

3.9 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India

Mathematics for Economist Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Blum- Viva Publications

Mathematics Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani- Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Bloom.

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Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Note

Unit 4: Cramer's Rule

CONTENTS

Objectives

After reading this unit, students will be able to :

- Learn the Method to Solve the Simultaneous Equations.
- Known with Cramer's Rule.

Introduction

Cramer searches the easy method to solve the simultaneous equations that is known as Determinant Method. Equations can be solving easily by this methods.

4.1 Solution to Simultaneous Equations (Cramer's Rule)

Simultaneous equations can be solved easily by determinant method following:

Firstly we will solve the two variable simultaneous equations-

The determinant of the variable of x and y is-

$$a_1x + b_1y = c_1 \qquad \ldots(i)$$

$$a_2x + b_2y = c_2 \qquad \ldots(ii)$$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, for finding the value of x we put the invariable column in place of the column of coefficient. So determinant will be-

$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, same for find out the value of y we put the invariable column in place of the column of coefficient. We can write it like that,

$$\frac{x}{\text{the determinant in which the column of invariables in place of the column of the } x \text{ coefficient}} = \frac{y}{\text{the determinant in which the column of invariable in place of the column of the } y \text{ coefficient}} = \frac{1}{\text{the determinant of } x \text{ and } y}$$

Here shown that all invariable written at right hand side in the equations.

It can be solved by the Cramer rule like that-

$$x = \frac{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}},\ y = \frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$ It can also written like that-

$$\frac{x}{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}} = \frac{y}{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}} = \frac{1}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$

So we can also solve more than two variable equations

Example 1: Solve the following equation by determinant method.

$$\mathbf{3x + 4y = 5}$$
$$\mathbf{3x - 4y = 2}$$

Solution: The determinant of coefficient is-

$\begin{bmatrix} 3 & 4 \\ 3 & -4 \end{bmatrix}$ and column of invariable is- $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ so,

$$\frac{x}{\begin{bmatrix} 5 & 4 \\ 2 & -4 \end{bmatrix}} = \frac{y}{\begin{bmatrix} 3 & 5 \\ 3 & 2 \end{bmatrix}} = \frac{1}{\begin{bmatrix} 3 & 4 \\ 3 & -4 \end{bmatrix}}$$

or, $$\frac{x}{(-20-8)} = \frac{y}{(6-15)} = \frac{1}{(-12-12)}$$

and $$\frac{x}{-28} = \frac{y}{-9} = \frac{1}{-24}$$

so, $$x = \frac{-28}{-24} = \frac{7}{6}$$

$$y = \frac{-9}{-24} = \frac{3}{8}.$$

Task Write the two variable simultaneous equations.

Now, lets discuss about the set of following simultaneous equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

Above equation can be written as array according to Cramer's rule.

$$AX = z \qquad ...(i)$$

Note

Here A= set of a_{ij} coefficient,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

X = vector column of variable

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } C = \text{vector columnof variables}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Equation (i) again multiplied by A^{-1}.

$$A^{-1} AX = A^{-1} Z$$

or $$X = A^{-1} Z \qquad [\because AA^{-1} = I]$$

or $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Here C_{ij} is the intersection with respect to a_{ij}.

So, $$X = \frac{z_1c_{11} + z_2c_{21} + z_3c_{31}}{|A|}$$

So the value of x_2 and x_3 can be find out.

Example 2: Solve the following equation by determinant method

$$\mathbf{4x + 2y = 2} \qquad ...(i)$$

$$\mathbf{3x + 5y = 21} \qquad ...(ii)$$

Solution: On writing the above equations in matrix form

$$AX = Z$$

Here, $$A = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } Z = \begin{bmatrix} 2 \\ 21 \end{bmatrix}$$

On using the Cramer's Rule

$$X = A^{-1}Z$$

$$= \frac{\text{Adjoint } A}{|A|} . Z$$

To get A^{-1}, firstly we would have to get the Transpose of given matrix –

$$A' = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\text{Adjoint } A = \begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix}$$

Note

Co-factor of 4 = + 5

Co-factor of 3 = – 2

Co-factor of 2 = – 3

Co-factor of 5 = + 4

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{14}\begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} \frac{5}{14} & \frac{-2}{14} \\ \frac{-3}{14} & \frac{4}{14} \end{bmatrix}$$

Now $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{-2}{14} \\ \frac{-3}{14} & \frac{4}{14} \end{bmatrix}\begin{bmatrix} 2 \\ 21 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{5}{14} \times 2\right) + \left(\frac{-2}{14} \times 21\right) \\ \left(\frac{-3}{14} \times 2\right) + \left(\frac{4}{14} \times 21\right) \end{bmatrix} = \begin{bmatrix} \frac{5}{7} - 3 \\ \frac{-3}{7} + 6 \end{bmatrix} = \begin{bmatrix} \frac{-16}{7} \\ \frac{39}{7} \end{bmatrix}$$

Therefore $x = \frac{-16}{7}$ and $y = \frac{39}{7}$.

Example 3: Solve by Matrix Method

$x - 2y + 3z = 1$ ...(*i*)

$3x - y + 4z = 3$ ...(*ii*)

$2x + y - 2z = -1$ ...(*iii*)

Solution: On writing the above equation in matrix form

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

On using Cramer Rule

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{Co-factor } A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\text{Co-factor } 1 = +\begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} = +(-2-4) = -6$$

$$\text{Co-factor } 3 = -\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = -(4-3) = -1$$

Note

$$\text{Co-factor } 2 = + \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} = + (-8 + 3) = -5$$

$$\text{Co-factor } -2 = - \begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix} = - (-6 - 8) = +14$$

$$\text{Co-factor } -1 = + \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = + (-2 - 6) = -8$$

$$\text{Co-factor } 1 = - \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = - (4 - 9) = +5$$

$$\text{Co-factor } 3 = + \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = + (3 + 2) = +5$$

$$\text{Co-factor } 4 = - \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = - (1 + 4) = -5$$

$$\text{Co-factor } -2 = + \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} = + (-1 + 6) = +5$$

$$\text{Adjoint } A = \begin{bmatrix} -6 & -1 & -5 \\ +14 & -8 & +5 \\ +5 & -5 & +5 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 1\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} + 2\begin{bmatrix} 3 & 4 \\ 2 & -2 \end{bmatrix} + 3\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= 1 \times (2 - 4) + 2(-6 - 8) + 3(3 + 2)$$
$$= -26 - 28 + 15 = -15$$

Therefore

$$A^{-1} = \frac{\text{Adj.}A}{|A|} = \frac{1}{-15}\begin{bmatrix} -6 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{-15} & \frac{1}{15} & \frac{5}{15} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{5}{15} \\ -\frac{5}{15} & \frac{5}{15} & -\frac{5}{15} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Now,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Note

$$= \begin{bmatrix} -\frac{2}{5}\times 1+\frac{3}{15}-\frac{1}{3} \\ -\frac{14}{15}+\frac{24}{15}+\frac{1}{3} \\ -\frac{1}{3}+\frac{3}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{8}{15} \\ 1 \\ 1 \end{bmatrix}$$

Therefore, on solving further we'll get

$$x = -\frac{8}{15},\ y = 1,\ z = 1$$

Self Assessment

1. Fill in the blanks:

1. Simultaneous equations can be solved easily by ……………………. Method.
2. $\dfrac{x}{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}} = \dfrac{y}{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}$ = ---------
3. To get A^{-1}, firstly we would have to get the ………………………… of given matrix.

4.2 Summary

- Simultaneous equation can be solved easily by following determinant method

  Firstly we will solved the two variable simultaneous equations-

  $a_1x + b_1y = c_1$

  $a_2x + b_2y = c_2$

  The determinant of the variable of x and y is- $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- for finding the value of x we put the invariable column in place of the column of coefficient.

4.3 Keywords

- ***Simultaneous:*** Together

4.4 Review Questions

1. Solve the following equations:

(*i*) $4x + 2y = 2;\ 3x - 5y = 21$

(*ii*) $2x - 3y + 4z = 8$

$3x - 4y + 5z = -4$

$4x - 5y + 6z = 12$ (**Ans.:** (*i*) $x = 2, y = 3$ (*ii*) $x = 1, y = 2, z = 3$)

Note

2. Prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = a+b+c.$$

3. Solve the following equation by determinant method –

$$3x + 4y = 5$$

$$3x - 4y = 2$$

(**Ans.:** $x = \frac{7}{6}$, $y = \frac{3}{8}$)

Answers: Self Assessment

1. Determinant
2. $\dfrac{1}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$
3. Transpose

4.5 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics Economist – Makcal Harrison, Patrick Waldron.

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Mathematics for Economist – Council for Economic Education

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 5: Determinant: Types and Properties

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Understand Definition of Determinant.
- Know the Rows and Columns of a Determinant.
- Know Shape and Constituents of a Determinant.
- Expansion of a Determinant.
- Know the Questions Related to Cofactors.
- Understand Properties of Determinants.
- Calculate the Multiplications of Two Determinants.

Introduction

It is difficult to solve the parallel equations because as much as number of variables exist in these equations, the same amount of equations are available. To solve these equations in algebra, we do use a very special which is known to as Determinant.

5.1 Definition of Determinant

Think on the following exponential equations-

$$a_1x + b_1y = 0 \qquad \text{...(i)}$$

$$a_2x + b_2y = 0 \qquad \text{...(ii)}$$

From the above equations, for eliminations of x and y, equation (i) and equation (ii) are to be subtracted from a_2 and a_1 respectively.

Note

$$(b_1a_2 - b_2a_1)y = 0$$

$$b_1a_2 - b_2a_1 = 0$$

$$a_1b_2 - a_2b_1 = 0$$

Expression $(a_1b_2 - a_2b_1)$ has been expressed in the following manner:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Which is referred to as determinant

$$\therefore \qquad a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_2 \\ a_2 & b_2 \end{vmatrix}$$

The expression $(a_1b_2 - a_2b_1)$ is referred as expansion or value of this determinant.

In the above determinant there are two Rows and two Columns. Thus the above determinant is known as second order determinant. The expression $(a_1b_2 - a_2b_1)$ is known as the expansion of this determinant. $a_1.a_2b_1.b_2$ are the constituents of determinant and a_1, b_2 and a_2, b_1 are its elements.

Now consider on the following three exponential equation

$$a_1 + b_1y + c_1z = 0$$

$$a_2 + b_2y + c_2z = 0$$

$$a_3 + b_3y + c_3z = 0$$

Eliminating x, y, z from these three equations, we find following result

$$a_1(b_2c_3 - b_1(a_2c_3 - a_3c_2) + c_1(a_3b_2 - a_3b_2) = 0$$

or

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = 0$$

Expression of obtained value in the result can be shown as under:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This is called third order determinant. In third order determinant there are three rows and three columns.

> *Notes* Expression on the left side is called expansion or value of this determinant.

5.2 Rows and Columns of a Determinant

In a determinant, horizontal lines from top to bottom are called its first, second and third rows, which is expressed as R_1, R_2, R_3...... respectively and vertical lines from left side to right side are called its first, second and third columns, which are expressed as C_1, C_2, C_3......

5.3 Shape and Constituents of a Determinant

Note

Each determinant has a square shape. Therefore, the more the number of determinant, the more will be the rows and columns.

For example: In a third order determinant there are 3 rows and 3 columns and the number of constituents is 3^2 or 9. Therefore in an nth order determinant there would be n rows and n columns and the number of constituents is n^2.

> *Did u know?* Number of constituents in a determinant = (order of determinant)^2

5.4 Expansion of a Determinant

A second order determinant can be expressed as under

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 \diagdown & \\ & b_2 \end{vmatrix} - \begin{vmatrix} & b_2 \\ a_2 \diagup & \end{vmatrix} = a_1 b_2 - b_1 a_2$$

For example:

$$\begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = \begin{vmatrix} 4 \diagdown & \\ & 5 \end{vmatrix} - \begin{vmatrix} & 2 \\ -3 \diagup & \end{vmatrix} = (4 \times 5) - (-2 \times -3) = 20 - 6 = 14$$

Expansion of third order determinant

Assume a third order determinant

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expansion of determinant is generally relative to constituents of first row or constituents of first column

(i) Expansion in terms of First Row

Imagine $\Delta = a_1 A - b_1 B + c_1 C$ ...(i)

Where A, B, C express determinants respectively

While expansion, signs of steps are taken in +, - , +, - , …….. order,

A, B, C can be found in the following manner:

A = a_1 comes in which row and column, leaving that obtained determinant

B = b_1 comes in which row and column, leaving that obtained determinant

C = c_1 comes in which row and column, leaving that obtained determinant

Here, a_1 comes in first row and first column, thus leaving them obtained determinant is

$$|A| = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Note

Here, b_1 comes in first row and second column, thus leaving them obtained determinant is

$$|B| = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

And c_1 comes in first row and third column, thus leaving them obtained determinant is

$$|C| = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

From equation (i), expanding the given determinant we get

$$a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$a_1(b_2c_3 - c_2b_3) - b_1(a_1c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$a_1b_2c_3 - a_1c_2b_3 - b_1a_1c_3 - b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3$$

Task What would be the number of rows and columns of a second order determinants?

(ii) Expansion in Terms of First Column

Imagine $\Delta = a_1P - a_2Q + a_3R$ ...(ii)

Where *P, Q, R* displays determinants respectively

a_1, a_2 and a_3 falls in which rows and columns, leaving them and writing them as above

$$P = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, Q = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, R = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

From equation (ii)

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

$$= a_1b_2c_3 - a_1c_2b_3 - a_2b_1c_3 + a_2c_1b_3 + a_3b_1c_2 - a_3c_1b_2$$

Thus it is clear that with the application of both the methods expanding separately, we get the similar result. Here, it is noteworthy that there is a certain value of each determinant which is obtained after their expansion.

Method of finding co-determinant of some element

Suppose we have to find the co-determinant of constituent c_2 of above discussed third order determinant then in which row and column c_2 falls, leaving them together whatever determinant is left, that would be the co-determinant of constituent c_2.

Thus, co-determinant of constituent c_2 is as under

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

Note

In the same fashion co-determinant of all other constituents can be found. Note that while determining the co-determinant we don't need to consider the sign. Therefore each co-determinant has a (+) sign.

Self Assessment

1. Fill in the blanks:

1. In ……………. order determinant there are three rows and three columns.
2. In a determinant, …………….. lines from top to bottom are called its first, second and third rows.
3. ……………… lines from left side to right side are called its first, second and third columns.
4. Each determinant has a …………… shape.
5. There is a ………… value of each determinant which is obtained after their expansion.

5.5 Cofactors

As has been told earlier that while determining the co-determinant, we don't need to consider the sign. Now even if sign of co-determinants of constituents are taken into consideration, they become the co-factors of those constituents, which are expressed in C_1, C_2, C_3,.......

It is clear that there is only difference of sign between co-determinant and co-factor, whereas value of both are same.

Following is the definition of co-factors

Cofactors of constituent $a_{ij} = (-1)^{+} \times A_{ij}$

Where a_{ij} is the constituent of i^{th} row and j_{th} column and is the A_{ij} co-determinant of constituent a_{ij}

$$a_1 \text{ Cofactor } = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = C_1$$

$$a_2 \text{ Cofactor } = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = C_2$$

$$a_3 \text{ Cofactor } = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = C_3$$

Thus from equation (i), given determinant is = $a_1C_1 + a_2C_2 + a_3C_3$

5.6 Properties of Determinants

Properties of determinants are given below. With the help of these properties problems related to determinants can easily be solved. Following are the properties of determinants:

1. If all the rows and columns of a certain determinants are interchanged, there would be no change in the value of the determinants.
2. If the two adjacent rows or columns are mutually interchanged, the numerical value of the determinants remain same, but sign changes.
3. If all elements of any two rows or columns of any determinants are the multiple of a single number, then main determinants is multiplied by the same figure.

Note

4. Special attention is required on two rules related to this property:

   (i) If in a certain row or column of the determinants any common figure exists, then that is taken out of the determinants as a factor

   For example

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

   (ii) If row or column of the determinants is multiplied by any figure, then whole determinant should be divided by that figure.

   For example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{k}\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

5. If element of any row or column of determinants are total of two numbers, then the same determinants can be expressed as a total of two determinants of same order.

   For example

$$\begin{vmatrix} a_1+p & b_1 & c_1 \\ a_2+q & b_2 & c_2 \\ a_3+r & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}$$

6. If all elements of rows or columns of determinants are multiplied by a certain number and added to or subtracted from corresponding element of any other row and column, then the value of determinants remain unchanged.

   For example $\Delta' = \Delta$, whereas

$$\begin{vmatrix} a_1+mb_1-nc_1 & b_1 & c_1 \\ a_2+mb_2-nc_2 & b_2 & c_2 \\ a_3+mb_3-nc_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This property is of special importance while solving the problems related to determinants. According to this property, in determinants, in the elements of any row or rows, number of times corresponding value of any other row or column can be added or subtracted. The same process can be exercised for the columns.

5.7 Multiplication of Two Determinants

Multiplication of Two Determinants can be found only when both of them are of same order, else not.

Assume $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $B = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Two determinants of third order are there. Their multiplication can be expressed with AB, there will be a third order determinants which can be found in the following way

Note

First of all, keeping first row (a_1 b_1 c_1) of determinants A at constant, multiply its elements with corresponding elements of first, second and third row of determinants B and find out their total separately, as shown below:

	$(a_1b_1c_1)$	$(a_1b_1c_1)$	$(a_1b_1c_1)$
	×	×	×
	$(x_1y_1z_1)$	$(x_2y_2z_2)$	$(x_3y_3z_3)$
Total =	$a_1x_1+ b_1y_1 + c_1z_1$	$a_1x_2+ b_1y_2 + c_1z_2$	$a_1x_3+ b_1y_3 + c_1z_3$

All the above three totals will be the first, second and third element of first row of determinants AB.

This way, keeping second row ($a_2b_2c_2$) of determinants A at constant, multiply its elements with corresponding elements of first, second and third row of determinants B and find out their total separately, which will be the elements of second row of determinant AB

The same process can be adopted for third row ($a_3b_3c_3$) of A and all rows of B.

Thus, multiplication of A and B = AB

$$= \begin{vmatrix} a_1x_1+b_1y_1+c_1z_1 & a_1x_2+b_1y_2+c_1z_2 & a_1x_3+b_1y_3+c_1z_3 \\ a_2x_1+b_2y_1+c_2z_1 & a_2x_2+b_2y_2+c_2z_2 & a_2x_3+b_2y_3+c_2z_3 \\ a_3x_1+b_3y_1+c_3z_1 & a_3x_2+b_3y_2+c_3z_2 & a_3x_3+b_3y_3+c_3z_3 \end{vmatrix}$$

Working rule

With the help of above property, create Zero in any row or a column to the extent possible and expand the determinants relative to elements of the same row or column

In solving the problems first, second and third …… etc. rows are expressed as R_1, R_2, R_3,....... etc and first, second, third………….. etc columns are expressed as C_1, C_2, C_3,....... respectively and whatever working is done between the rows and columns of determinants, the same is written at the right side of the determinants

Example 1: Find out the value of the following determinants:

$$\begin{vmatrix} \mathbf{13} & \mathbf{16} & \mathbf{19} \\ \mathbf{14} & \mathbf{17} & \mathbf{20} \\ \mathbf{14} & \mathbf{18} & \mathbf{21} \end{vmatrix} = \mathbf{0}$$

Solution: Assume determinats = Δ

$$\therefore \quad \Delta \begin{vmatrix} 13 & 16 & 3 \\ 14 & 17 & 3 \\ 14 & 18 & 3 \end{vmatrix} (C_3 - C_2) = \begin{vmatrix} 13 & 3 & 3 \\ 14 & 3 & 3 \\ 14 & 3 & 3 \end{vmatrix} (C_2 - C_1) = 0 \quad (\therefore C_2 = C_3)$$

Example 2: Prove that:

$$\begin{vmatrix} \mathbf{23} & \mathbf{12} & \mathbf{11} \\ \mathbf{36} & \mathbf{10} & \mathbf{26} \\ \mathbf{63} & \mathbf{26} & \mathbf{37} \end{vmatrix} = \mathbf{0.}$$

Solution: Given is the determinant

$$\begin{vmatrix} 11 & 12 & 11 \\ 26 & 10 & 26 \\ 37 & 26 & 37 \end{vmatrix} \qquad \text{(Subtracting } C_2 \text{ from } C_1\text{)}$$

= 0 $\therefore$ C_1 and C_3 are equal

Note

Example 3: Find out the value of:

$$\begin{vmatrix} \mathbf{13} & \mathbf{18} & \mathbf{23} \\ \mathbf{14} & \mathbf{19} & \mathbf{24} \\ \mathbf{14} & \mathbf{20} & \mathbf{25} \end{vmatrix}$$

Solution: Given is the determinants

$$= \begin{vmatrix} 13 & 18 & 5 \\ 14 & 19 & 5 \\ 14 & 20 & 5 \end{vmatrix} \text{ (Subtracting } C_3 \text{ from } C_2)$$

$= 0 \therefore C_2$ and C_3 are equal

Example 4: Find out the value:

$$\begin{vmatrix} \mathbf{29} & \mathbf{26} & \mathbf{22} \\ \mathbf{25} & \mathbf{31} & \mathbf{27} \\ \mathbf{65} & \mathbf{54} & \mathbf{46} \end{vmatrix}.$$

Solution: Given is the determinants

$$= \begin{vmatrix} 3 & 26 & 22 \\ -6 & 31 & 27 \\ 9 & 8 & 46 \end{vmatrix}. \quad \text{(Subtracting } C_2 \text{ from } C_1)$$

$$= \begin{vmatrix} 3 & 4 & 22 \\ -6 & 4 & 27 \\ 9 & 8 & 46 \end{vmatrix} \quad \text{(Subtracting } C_2 \text{ from } C_1)$$

$$= 3 \times 4 \begin{vmatrix} 1 & 1 & 22 \\ -2 & 1 & 27 \\ 3 & 2 & 46 \end{vmatrix}. \quad \therefore 3 \text{ from } C_1 \text{ and } 4 \text{ from } C_2 \text{ is common}$$

$= 0 \therefore C_2$ and C_3 are equal

$$= 12 \begin{vmatrix} 1 & 1 & 22 \\ 0 & 3 & 71 \\ 0 & -1 & -20 \end{vmatrix}.$$

Adding $2R_1$ to R_2 and Subtracting $3R_1$ to R_3

$$= 12[3(-20) - 71(-1)] = 12[-60 + 71]$$

$$= 12 \times 11 = 132.$$

Example 5: Find out the value of:

$$\frac{\mathbf{1}}{(x+y)} \begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & x^2 & \mathbf{1} \\ \mathbf{3} & y^2 & \mathbf{1} \end{vmatrix}.$$

Solution: Given is the determinants

Note

$$\frac{1}{(x+y)}\begin{vmatrix} x^2 & 1 \\ y^2 & 1 \end{vmatrix}$$

Expanding with respect to first row

$$\frac{1}{(x+y)}.(x^2 - y^2) = \frac{(x-y)(x+y)}{(x+y)} = x - y$$

Example 6: Prove that:

$$\begin{vmatrix} \mathbf{1} & x & y+z \\ \mathbf{1} & y & z+x \\ \mathbf{1} & z & x+y \end{vmatrix} = \mathbf{0}.$$

Solution:

Determinants $= \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0.$ (Adding C_2 to C_3)

$$= (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$$= (x+y+z) \times 0 = 0.$$

$= 0$

$[\therefore\ C_1 = C_3 \therefore \Delta = 0]$

Example 7: Prove that:

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

Solution: Given determinants

$= -\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ (Subtracting C_1 from C_2) and (Subtracting C_1 from C_3)

$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$ (Expanding with respect to R_1)

$$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$$

$$= (b-a)(c-a)[(c+a)-(b+a)]$$

Note

$$= (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a) \qquad \text{(Expanding with respect to } R_1\text{)}$$

Example 8: Prove that:

$$\begin{vmatrix} \mathbf{1} & a & a^2 \\ \mathbf{1} & b & b^2 \\ \mathbf{1} & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$$

Solution: Apply processes $R_2 - R_1$ and $R_3 - R_1$ as in example 6

Example 9: Prove that:

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(b-c)(c-a)(a-b).$$

Solution : In C_1 a, C_2 b and C_3 c is common

$$\therefore \text{ Determinants } = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Further follow solution of example 7.

Example 10: Prove that:

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Solution: Given determinants

$$= \begin{vmatrix} 0 & 0 & 0 \\ a-b & b-c & c \\ a^3-a^3 & b^3-c^3 & c^3 \end{vmatrix}, (C_1 - C_2) \text{ and } (C_2 - C_3)$$

$$= \begin{vmatrix} a-b & b-c \\ a^2-b^2 & b^2-c^2 \end{vmatrix} \qquad \text{(Expanding it with respect to } R_1\text{)}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a^2ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

$$= (a-b)(b-c)[b^2+bc+c^2]-(a^2+ab+b^2)]$$

Note

$$= (a-b)(b-c)[bc + c^2 - a^2 - ab]$$

$$= (a-b)(b-c)[b(c-a) + (c-a)(c+a)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Example 11: Prove that:

$$\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2 - bc} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2 - ca} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2 - ab} \end{vmatrix} = \mathbf{0} \quad \text{or} \quad \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^2 - bc} & \boldsymbol{b^2 - ca} & \boldsymbol{c^2 - ab} \end{vmatrix} = \mathbf{0}.$$

Solution: L.H.S. $= \begin{vmatrix} 1 & a & a - bc \\ 0 & b-a & b^2 - a^2 - ca + bc \\ 0 & c-a & c^2 - a^2 - ab + bc \end{vmatrix}$, ($R_1 - R_2$ or $R_3 - R_1$)

$$= \begin{vmatrix} b-a & (b-a)(a+b+c) \\ c-a & (c-a)(a+b+c) \end{vmatrix} \qquad \text{(Expanding with respect to } C_1\text{)}$$

$$= (b-a)(a+b+c) - (b-a)(c-a)(a+b+c)$$

= 0 L.H.S

Example 12: Prove that:

$$\begin{vmatrix} \boldsymbol{a+b+2c} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix} = \mathbf{2}\,(\boldsymbol{a+b+c})^{\mathbf{3}}.$$

Solution: Given determinants

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix} \qquad (C_1 + C_2 + C_3 \text{ at add})$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \qquad C_1 + 2(a+b+c) \text{ of common}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \qquad (R_2 - R_1 \text{ or } R_3 - R_1)$$

$$= 2(a+b+c)\,[(b+c+a)\,(c+a+b) - 0] = 2(a+b+c)^3$$

Note

Example 13: Prove that

$$\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz.$$

Solution: From $R_1 - (R_2 + R_3)$, given determinants

$$= \begin{vmatrix} (y+z)-(y+z) & x-(x+2z) & x-(x+2y) \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2z & -2y \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 2z\begin{vmatrix} y & y \\ z & x+y \end{vmatrix} - 2y\begin{vmatrix} y & z+x \\ z & z \end{vmatrix} \quad \text{(Expanding with respect to } R_1\text{)}$$

$$= 2z[y(x+y) - yz] - 2y\,[yz - z(z+x)]$$

$$= 2yz[(x+y-z)] - 2yz\,(y-z-x)$$

$$= 2yz(x+y-z-y+z-x) = 4xyz$$

Self Assessment

2. State whether the following statements are True or False:

6. There is only difference of sign between co-determinant and co-factor, whereas values of both are same.
7. If all the rows and columns of a certain determinants are interchanged, then value of the determinants change.
8. If the two adjacent rows or columns are mutually interchanged, the numerical value of the determinants remain same, but sign changes.
9. If row or column of the determinants is multiplied by any figure, then whole determinant should be multiplied by that figure.
10. Multiplication of Two Determinants can be found only when both of them are of same order, else not.

5.8 Summary

- It is difficult to solve the parallel equations because as much as number of variables exist in these equations, the same amount of equations are available. To solve these equations in algebra, we do use a very special which is known to as Determinant.
- The expression is referred as expansion or value of this determinant.
- In a determinant, horizontal lines from top to bottom are called its first, second and third rows, which is expressed as respectively and vertical lines from left side to right side are called its first, second and third columns, which are expressed as C_1, C_2, C_3.....
- Expansion of determinant is generally relative to constituents of first row or constituents of first column.
- There is a certain value of each determinant which is obtained after their expansion.

Note

- There is only difference of sign between co-determinant and co-factor, whereas values of both are same.
- If all the rows and columns of a certain determinants are interchanged, there would be no change in the value of the determinants.
- If the two adjacent rows or columns are mutually interchanged, the numerical value of the determinants remain same, but sign changes.
- If all elements of any two rows or columns of any determinants are the multiple of a single number, then main determinants is multiplied by the same figure.
- Multiplication of Two Determinants can be found only when both of them are of same order, else not.

5.9 Keywords

- ***Expansion:*** Detailed
- ***Value:*** Price

5.10 Review Questions

1. Define determinants with example.
2. Write down the properties of determinants.
3. Find out the value of following determinants.

(a) $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 65 & 54 & 46 \end{vmatrix}$ **[Ans.: = 132]**

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ **[Ans.: =** $(a-b)(b-c)(c-a)$ **]**

4. Prove that: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

5. Prove that: $\begin{vmatrix} 23 & 12 & 11 \\ 36 & 10 & 26 \\ 63 & 26 & 37 \end{vmatrix} = 0$

Answers: Self Assessment

1. Third,	2. Horizontal,	3. Vertical,
4. Square,	5. Certain	6. True,
7. False,	8. True,	9. False
10. True		

Note

5.11 Further Readings

Books

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Unit 6: Rank of Matrix

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Know the Merits of the Rank.
- Understand the Merits of the Rank with Example.

Introduction

An important number related to matrices is referred as Rank. In a matrix A, m x n, n is column vector and m is a factor, then in that case linear independent set maximum column vector would be the Rank of the matrix, this can be shown as $r(A)$.

In other words, it can be said that maximum number of independent column is Rank of the matrix. An m x n matrix becomes non-singular when its rank reaches n.

6.1 Characteristics of Rank

- Only Zero matrix has a zero rank
- $\text{rank } (A) \leq \min(m, n) \ \{(A) \leq (m, n)\}$
- If A is a square matrix ($m = n$), A will be plural only when A's rank becomes n.
- Rank $(AB) \leq$ minimum (Rank A, Rank B)
- Rank (A) + Rank $(B) - n \leq$ Rank (AB)
- Rank (CA) = Rank (A) {C is constant figure}

Did u know? Maximum number of independent columns is known as Rank of the Matrix.

Example 1: Find out the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Note

Solution: Given that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Applying row operation

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_{21}(1-2),\ R_{22}(-1)]{} \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2(-1),\ R_3(-1)]{}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_{22}(-2)]{R_2(1/2)}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_{2-3}(-1),\ R_{12}(1)]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ here Matrix A keeps three non-zero rows

Therefore, Rank (A) is 3.

Example 2: If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ the find out the Rank of the matrix.

Solution: Given that A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to -R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

Here non-zero rows are two, hence Rank (A) will be 2.

Example 3: If $A = \begin{bmatrix} 5-x & 2 & 1 \\ 2 & 1-x & 0 \\ 1 & 0 & 1-x \end{bmatrix}$, then determine the rank for each value of x.

Solution: Given that

$$A = \begin{bmatrix} 5-x & 2 & 1 \\ 2 & 1-x & 0 \\ 1 & 0 & 1-x \end{bmatrix}$$

Based on three-column expanding the above table

$$= x(x-1)\,(x-6)\,|\,A\,| = 1\{0 - 1\,(1-x)\} - 0\,\{0-2\} + (1-x)\,\{(5-x)\,(1-x) - 4\}$$

Note

Or, if $x \neq 0, 1$ and 6 would be the Rank 3 of Matrix A

And if x = 0, 1, 6 then $\begin{bmatrix} 5-x & 2 \\ 1 & 0 \end{bmatrix} = -2 \neq 0$, Rank will be 2

Linear Dependence and Rank of Matrix:

Linear dependency is found in a rows (columns) of a matrix only when linear conjugation of those rows (columns) equal to zero vector, viz

$$\text{If } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$K_1 a_{11} + K_2 a_{12} + K_3 a_{13} = 0$$

$$K_1 a_{12} + K_2 a_{22} + K_3 a_{32} = 0$$

$$K_1 a_{13} + K_2 a_{23} + K_3 a_{33} = 0$$

Here among K_1, K_2 and K_3 at least value of one should be Zero.

Example 1: Find out the linear dependency and rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}.$$

Solution: While multiplying Row 1 (R_1) and deducting -1 from 2(R_2) and adding 3(R_3) then

$$-1R_1 - 1R_2 + R_3 = 0 \quad \text{...(i)}$$

$$-2R_1 + R_2 = 0 \quad \text{...(ii)}$$

$$-2R_1 + R_3 = 0 \quad \text{...(iii)}$$

R_1, R_2 and R_3 in equation (i) is not linear independence. Equation (ii) R_1 and R_2 and Equation (iii) R_1 and R_2 show the linear dependency. For Rank

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ viz Rank } (A) = 1 .$$

Example 2: If $A = \begin{bmatrix} 6 & 3 & 5 \\ -10 & 2 & 8 \\ 5 & 2 & 3 \end{bmatrix}$, then examine the linear dependency and find out the Rank of Matrix (A).

Solution: $C_1R_1 + C_2R_2 + C_3R_3 = 0$

$$6C_1 - 10C_2 + 5C_3 = 0 \quad \text{...(i)}$$

$$3C_1 + 2C_2 + 2C_3 = 0 \quad \text{...(ii)}$$

$$5C_1 + 8C_2 + 3C_3 = 0 \quad \text{...(iii)}$$

Note

Solving equation (i), (ii) and (iii)

$C_1 = -10$, C_2 1 and $C_3 = 14$, then linear dependency will be found

Viz and will be linear dependency because none of the linear are Zero viz Rank (A) = 2

6.2 Summary

- Only Zero matrix has a zero rank.
- Rank $(A) \leq \min(m, n)$ $\{(A) \leq (m, n)\}$.
- If A is a square matrix ($m = n$), A will be plural only when A's rank becomes n.
- Rank (AB) $\leq$ minimum (Rank A, Rank B).
- Rank (A) + Rank (B) – $n \leq$ Rank (AB).
- Rank (A) = Rank (A) {C is constant figure}.

6.3 Keywords

- ***Matrix Rank:*** Maximum number of independent columns.

6.4 Review Questions

1. Find out the rank of following matrix.

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 4 & 6 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

2. Verify the linear independence

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 3 & 8 & 4 \\ 5 & 6 & 6 \end{bmatrix}.$$

6.5 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Malcom, Nicolas, U C London.

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Mathematics for Economist – Yamane, Prentice Hall Publication.

Unit 7: Application of Matrices in Economics

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Understand the Application of Matrices in Economics.

Introduction

To solve the economical problems matrices are useful mathematical technique. With the help of it we can solve various economical problems. Whatever widest economical problem is linear, we can solve them with the help of matrices. Under this, we by using Cramer Rule (A^{-1}) we can solve the problems related to production, demand, supply, incoming and outgoing and national income is determined.

7.1 Application of Matrices in Economics

Example 1: Assume an equation for tri-regional economical model is given

$$Y = C + A_0$$
$$C = a + b\ (Y - T)$$
$$T = d + ty$$

Where $Y \rightarrow$ Income, C – Consumption, $T \rightarrow$ Tax Revenue, $t \rightarrow$ Rate of tax

A_0, a, b and c are constant, then calculate national income, consumption and tax revenue.

Solution: Given that $Y = C + A_0$

Or
$$y - C = A_0 \quad \text{...(i)}$$
$$-by + c + bT = a \quad \text{...(ii)}$$
$$-ty + T = d \quad \text{...(iii)}$$

Writing the equation (i), (ii) and (iii) in the following form

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} A_0 \\ a \\ d \end{bmatrix}$$

Note Here

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix}$$

Or $|A| = 1\ (1 + 0) + 1\ (-b + tb) = 1 - b + tb = 1 - b\ [1 - t]$

Applying Cramer Rule

$$Y = \frac{|A_1|}{|A|} \frac{\begin{vmatrix} A_0 & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{A_0(1) + 1(a - bd)}{1 - b(1-t)} = \frac{a - bd + A_0}{1 - b[1-t]}$$

$$C = \frac{|A_2|}{|A|} \frac{\begin{vmatrix} 1 & A_0 & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{a - bd - A_0(-b + bt)}{1 - b(1-t)}$$

$$T = \frac{|A_3|}{|A|} \frac{\begin{vmatrix} 1 & -1 & A_0 \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{d + (a - bd) + A_0(t)}{1 - b(1-t)}.$$ **Ans.**

Example 2: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 2 & 5 & 12 \end{bmatrix}$, then determine the value of A^{-1}

Transposing the given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 2 & 5 & 12 \end{bmatrix},$$

To find the adjoints, adjoints of matrix A (*aij*) will be determined in this way

$$\text{Adjoint 1} = \begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} = 36 - 25 = 11$$

$$\text{Adjoint 2} = -\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} = (12 - 15) = -9$$

$$\text{Adjoint 3} = +\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = 10 - 9 = 1$$

$$\text{Adjoint 2} = -\begin{bmatrix} 2 & 3 \\ 5 & 12 \end{bmatrix} = 24 - 15 = -9$$

$$\text{Adjoint 3} = +\begin{bmatrix} 1 & 3 \\ 3 & 12 \end{bmatrix} = 12 - 9 = 3$$

$$\text{Adjoint 5} = -\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -(5-6) = (-1) = +1$$

$$\text{Adjoint 3} = +\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = -(10-9) = 1$$

$$\text{Adjoint 5} = -\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -(5-6) = 1$$

$$\text{Adjoint 12} = +\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = (3-4) = -1$$

$$Adj.\ A = \begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix} = 1\begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} - 2\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= 1(36-25) - 2(24-15) + 3\,(10-9)$$

$$= (1\times 11) - (2\times 9) + (3\times 1) = 11 - 18 + 3 = -4$$

$$A^{-1} = \frac{Adj.\ A}{|A|} = -\frac{1}{4}\begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} & \frac{9}{4} & -\frac{1}{4} \\ \frac{9}{4} & -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$ **Ans.**

Example 3: Find out the optimal price of x, y and z by using Matrix Method

$$x - 2y + 3z = 1 \quad \text{...(i)}$$

$$3x - y + 4z = 3 \quad \text{...(ii)}$$

$$2x + y - 2z = -1 \quad \text{...(iii)}$$

Solution: Writing all the above equations in form of matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Note

Applying Cramer's Rule

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{Transpose of } A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\text{Adjoint } 1 = +\begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} = +(-2-4) = -6$$

$$\text{Adjoint } 3 = -\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = -(4-3) = -1$$

$$\text{Adjoint } 2 = +\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} = +(-8-3) = -5$$

$$\text{Adjoint } -2 = -\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix} = +(-6-8) = +14$$

$$\text{Adjoint } -1 = +\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = +(-2-6) = -8$$

$$\text{Adjoint } 1 = -\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = -(4-9) = +5$$

$$\text{Adjoint } 3 = +\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = +(3+2) = +5$$

$$\text{Adjoint } 4 = -\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = -(1+4) = -5$$

$$\text{Adjoint } -2 = +\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} = +(-1-6) = +5$$

$$\text{Adjoint A} = \begin{bmatrix} -6 & -1 & -5 \\ +14 & -8 & +5 \\ +5 & -5 & +5 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 1\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} - 2\begin{bmatrix} 3 & 4 \\ 2 & -2 \end{bmatrix} + 3\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$= 1 \times (2 - 4) + 2\,(-6 - 8) + 3\,(3 + 2)$

$= -2b - 28 = 15$

$= -15$

Therefore, **Note**

$$A^{-1} = \frac{Adj.A}{|A|} = \frac{1}{-15}\begin{bmatrix} -6 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{-15} & \frac{1}{15} & \frac{5}{15} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{5}{15} \\ -\frac{5}{15} & \frac{5}{15} & -\frac{5}{15} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Now

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2}{5}\times 1 & +\frac{3}{15} & -\frac{1}{3} \\ -\frac{14}{15} & +\frac{25}{15} & +\frac{1}{3} \\ -\frac{1}{3} & +\frac{3}{3} & +\frac{3}{3} \end{bmatrix} = \begin{bmatrix} \frac{-8}{15} \\ 1 \\ 1 \end{bmatrix}$$

Therefore, after solving we will get $x = -\frac{8}{15}, y = 1, z = 1$.

7.2 Summary

- If the economical problem is widest and linear, we can solve them with the help of matrices. Under this, we by using Cramer Rule (A^{-1}) we can solve the problems related to production, demand, supply, incoming and outgoing and national income is determined.

7.3 Keywords

- ***Matrix:*** Frame, Structure, Texture, Sequence.

7.4 Review Questions

1. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 6 \\ 3 & 6 & 13 \end{bmatrix}$, then find out A^{-1}.
2. Find out the optimal price of a, b and c by using Matrix Method.

   $a + 2b + 6c = 1$

   $3a - 4b - 2c = 1$

   $2a - b - 5c = -2$

Note

7.5 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Malcom, Nicolas, U C London.

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Unit 8: Input-Output Analysis

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand the Assumptions of Input-Output Analysis.
- Know the Leontief's Input-Output Closed Model.
- Get the Information Related to First set for Equilibrium Equation of Closed Model.
- Get the Information related to the Second Set of Equilibrium Equation of Closed Model.

Introduction

Before we get to know the explanation of Input-Output Analysis, it would be better for us to understand the meaning of Input and Output. Input means the demand from the producer for material to produce the product, whereas output means the result of efforts made towards production. According to **Prof J R Hicks** "Input means materials which are purchased by the producer for production, contrary to this output stands for what is sold by the producer". We can say that input is the cost for the firm and output is revenue.

With the help of Input-Output Analysis, we get inter-trade relation and inter-dependence of entire economy because output of one industry could be input for another industry. Similarly output of other industry could be the input for first industry. For example the output of coal industry could be the input of steel industry.

Notes Output of steel industry is input of coal industry.

8.1 Assumptions of Input-Output Analysis

Followings are the assumptions of Input-Output Analysis:

1. Economy is in balanced condition.
2. Economy is divided in two parts – inter-trade part and last demand part and each part can further be bifurcated.

Note

3. Each industry produces only one product and there is no joint production of two products.
4. Total output of an industry is used as input for any other industry.
5. Production is done under rule of constant return.
6. Technical development is constant, it means that input coefficients are constant
7. Here in production no external austerity or improvidences are created

Leontief's Static Input-Output Model – Open Model

Leontief's Static Input-Output Model is based on the above assumptions. This can be understood with an example. Suppose the economy is divided into three parts. Out of which agriculture and industry are inter-trades and domestic part is expressed as last demand part.

With the help of Input-Output Analysis we can understand this model. In the given table, output of all the three parts is shown in horizontal rows, whereas inputs are shown in vertical columns. Total of the first row is 300 units, which shows the total output of agriculture. Out of which 50 units are of agriculture, 200 units are related to industry and balance 50 units are utilized as input for domestic part. The second row of the table shows total production of industry. Production is done equal to 150 units in the industry, out of which 55 units for agriculture, 25 units are related to industry and 70 units are used in domestic part.

Similarly columns show the cost of these areas. First column shows that for total production of 300 units in the industry, cost of 125 units comes, out of which 5 units are related to agriculture, 55 units are for industry and 20 units related to domestic area. Second column shows for total production of 150 units in the industry, total cost comes equal to 255 units, out of which 200 units relate to agriculture, 25 for industry and 30 units for domestic part. Zero in third columns shows depicts that domestic part is a consuming area, where not sells are made

Table 8.1: Input-Output Table

		Purchase area			Total output or total receipt
	Areas	Agriculture(1)	Industry(2)	Last Demand (3)	
Sell area ↑	(1) Agriculture	50	200	50	300
	(2) Industry	55	25	70	150
	(3) Last Demand	20	30	0	50
	Total cost or Total input	125	255	120	500

With the help of above table general Transaction Matrix can be created

Table 8.2: Transaction Matrix

		Purchase area			Total output
	Areas	Agriculture(1)	Industry(2)	Last Demand (3)	
Sell area ↑	(1) Agriculture	x_{11}	x_{12}	D_1	X_1
	(2) Industry	x_{21}	x_{22}	D_2	X_2
	(3) Last Demand	x_{31}	x_{32}	D_0	X_3

If we take columns of above table, then we will get following production function

Note

$X_1 = f_1(x_{11}, x_{21}, x_{31})$ or $300 = f_1(50, 55, 20)$

$X_2 = f_2(x_{12}, x_{22}, x_{33})$ or $150 = f_2(200, 25, 30)$

This way total production can be divided in various parts in the following way (adding division of all parts in the rows)

$X_1 = x_{11} + x_{12} + D_1$

$X_2 = x_{21} + x_{22} + D_2$

$X_3 = x_{31} + x_{32}$

Here we assume that total production of i industry is utilized as input in *n* industries, in this codition

$X_1 = x_{11} + x_{12} + ... + x_{in} + D_1$

In Leontifs, the concept of constant coefficient has also a value. In this situation technical coefficient will be

$$a_{ij} = \frac{x_{ij}}{X_j}$$

Here, x_{11} = production of i^{th} industry which is utilized by j^{th} industry

X_1 = Total production of i^{th} industry

In the above Table -1 technical coefficient can be found in the following manner

Table 8.3: Technical Matrix

		Working area		Input-Output Coefficient	
		Agriculture(1)	Industry(2)	Last Demand (3)	Total production
Sell area ↑	Agriculture	0.16	1.33	50	600
	Industry	0.18	0.16	70	150
	Sell area	0.06	0.20	0	50

Method of finding technical coefficient is very simple. Here we divide input of desired are by total production of that area. For example, total production of agriculture area is 300 units and inputs are 50, 55 and 20 units, in this condition technical coefficient would be $\frac{50}{300} = 0.16$, $\frac{55}{300} = 0.18$ and $\frac{20}{300} = 0.06$. In the similar way it can be calculated for other areas.

Leontief's Input-Output Matrix can be shows in algebraic expression in the following manner:

Assume our general model is following:

$X_1 = x_{11} + x_{12} + ... x_{in} + D_1$ ...(8.1)

Here X_1 = total production of i^{th} area, where i = 1, 2, …… n

x_{ij} = production of i^{th} industry which is utilized by j^{th} industry

Model 24.1 can be divided for n^{th} areas in the following manner:

$X_1 = x_{11} + x_{12} + ... + x_{1n} + D_1$

Note

$$X_2 = x_{21} + x_{22} + ... + x_{2n} + D_2$$
$$X_n = x_{n1} + x_{n2} + ... + x_{nn} + D_n$$

We know the technical coefficient

$a_{ij} = \frac{x_{ij}}{X_j}$ here x_1 = total production of i^{th} area

Or $$x_{ij} = a_{ij} X_j$$

If $x_{11} = a_{11} X_1, X_{12} = a_{12} X_2$.

Now our binomial equation would be

$$X_1 = a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n + D_1$$
$$X_2 = a_{21}X_1 + a_{22}X_2 + ... + a_{2n} X_n + D_2$$
$$X_n = a_{n1}X_1 + a_{n2}X_2 + ... + a_{nn}X_n + D_n$$

General matrix can be written in the following way

$$X_1 = \sum_{j=1}^{n} a_{ij} X_j + D_n$$

Or $X = AX + D$ ...(8.2)

Here $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ $A = \begin{bmatrix} a_{12} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}$

From equation (24.2)

$$D = X - AX = (I - A) X$$

Here I means Idently Matrix, therefore

$$X = (I - A)^{-1}D$$ ...(8.3)

Did u know? Here $(I - A)$ is an Inverse Matrix.

8.1.2 Limitations of Input-Output Model

Although Leontief Input-Output Analysis plays an important role in economic analysis, but there are some limitation which are as under:

1. *Impracticability of the Assumptions* – Assumptions of Leontief model are impractical. It assumes that technological coefficients are constant, which means technology will be constant or change in production will result to change in means devoted for production. Similarly it is also assumed that capital requirement for all parts of the economy would be that, but capital requirement of each part varies, thus their requirement would also be different.
2. *Neglecting Certain Factors* – because of the rigid nature of this model, solution to increasing cost and different other problems are not possible.
3. *One Sided Analysis* – This model is considered to be one-sided analysis, because it takes into account only productive sector of economy. Thus it ignores unproductive sectors completely.

Note

4. *Neglecting Factors of Substitution* – assumption of constant technical coefficient neglects the possibilities of factors of substitution. But in reality it is seen that such type of possibility of factors of substitution exist for short term also. For long term its possibility grows much more.
5. *Absence of Linear Relations* – The model assumes that input of one part is output for others viz there is linear relation in parts, which is contrary to the fact, as because of indivisibility of factors, increase in outputs are always not equal to increase in inputs.
6. *Use of Physical Units* – In practical commodities and services are expressed in monetary form only. Therefore, forecasting of economy in composite form is difficult with the balanced equation by Input-Output Analysis. Besides, measurements of physical units for various commodities and services are different. Therefore to create an Input-Output Table and Technological Coefficient, one has got to face many difficulties.
7. *Complexity* – This analysis takes help of various constructed equations and mathematical techniques, for which knowledge of advanced maths and statistical methods is required. Thus, the technique becomes more complex.

Task Describe assumptions of Input-Output Analysis.

8.2 Leontief's Input-Output Closed Model

In Leontief's Input-Output Open Model we had considered domestic demand as a separate area. If it is also merged in the economy, then no other area will remain left which has relation to external area. In this condition, each good will have a nature of intermediate goods because under this arrangement (n + 1), produced outputs are used for production of these outputs. This condition is known as closed model.

In this situation, full competition is found in the whole economy and there is no government interference. Here each of the industries (n + 1) produces a different quantity x_i (i = 1, 2.... n + 1 of output. The last area is domestic area for which output is X_{n+1}. Suppose x_{ij} = output of i^{th} industry, which is sent to j^{th} industry.

Self Assessment

1. Fill in the blanks:

1. Input means the demand from the producer for material to …………………….. the product.
2. …………….. means the result of efforts made towards production.
3. Total output of an industry is used as …………… for any other industry.
4. Assumptions of ………………. model are impractical.
5. With the help of Input-Output Analysis, we get inter-trade relation and inter-…………….. of entire economy.

8.3 First Set for Equilibrium Equation of Closed Model

Mathematical form under the closed model

$$X_i = \sum_{j=1}^{n+1} X_{ij} \qquad \text{...(8.4)}$$

Note

$$(i = 1, 2, ... n + 1)$$

Under constant technological coefficient

$$x_{ij} = a_{ij} X_i \qquad ...(8.5)$$

Putting the value of equation 24.5 in equation 26.4

$$X_i = \sum_{j=1}^{n+1} a_{ij} X_j$$

Or

$$X_i = \sum_{j=1}^{n+1} a_{ij} X_j = 0 \qquad ...(8.6)$$

Equation (8.6) is the first set for equilibrium equation.

8.4 The Second Set of Equilibrium Equation of Closed Model

Here we will concentrate on P_i (i = 1, 2 .... n + 1). Here P_i is the labour rate of domestic area.

Here assuming that Receipts and Costs are equal in Industry, equilibrium equation can be derived in the following manner:

For i^{th} industry

$$\text{Receipts} = X_i P_i$$

$$\text{And costs} = \sum_{j=1}^{n+1} P_i X_{ij} = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i$$

Where $x_{ij} = a_{ij} X_i$

In equilibrium condition

$$X_i P_i = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i$$

or

$$X_i P_i = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i$$

or

$$X_i P_i = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i = 0 \qquad ...(8.7)$$

$$i = 1, 2, ...n$$

Equation (8.7) is the second set for equilibrium equation.

8.4.1 Short note on Leontief's Dynamic Model

In constant model, with the help of technical coefficient of various areas of economy, mutual dependency is studied. But these coefficients do not throw light on the actual requirement of stock in economy. They are even to unable to enlighten us about how much capital is required for the required factors which is to be consumed by the industry. These capital is required to maintain constant capital appropriation such as construction, machine etc or to maintain raw material stock for production. If under constant input-output model (open model) effect of capital is involved, then this arrangement becomes dynamic. Thus, capital investment is a unique characteristic of dynamic input-output model.

Thus, constructive equations which are used under Input-Output Constant Model (Open Model), involve capital requirement including time-interval for each areas. Therefore, this model is just expansion of Constant Input-Output Model (Leontief's Open Model).

Assumptions

Input-Output Dynamic Model is generalized form of constant model. Therefore its assumptions would be similar to constant model. As **(1)** each industry produces homogenous products and **(2)** for production of each product only one technology is available, where conjugation of factors of products are constant.

Assume $C_i(t)$ is the consumption for the current year which is produced by i^{th} industry. $X_i(t)$ is the total production of i^{th} industry during t time-period, which is used for three purposes **(i)** consumption $\{C_i\,(t + 1)\}$, for the next period, **(ii)** net stock $S_i(t + 1) - S_i\,(t)$ of capital products for n industries and **(iii)** current flow of industrial production in the economy. Under these situations, following will be the equilibrium equation:

$X_i(t) = C_i\,(t + 1) + S_i\,(t + 1) - S_i\,(t) + x_{11}\,(t) + x_{12}\,(t)$ here $i = 1, 2$

Now the question arises how do current productions $X_1\,(t)$ and $X_2\,(t)$ happen? In Leontief production function, total production is the product of two factors – **(i)** Raw material for the current year and **(ii)** stock of capital products

$$X_1\,(t) = f[x_{11}\,(t), x_{21}(t), S_{11}(t), S_{21}(t)]$$

$$X_2\,(t) = f[x_{12}\,(t), x_{22}(t), S_{12}(t), S_{22}(t)]$$

In an economy total capital stock would be the total of capital stock of all industries

$$S_i\,(t) = S_{i1}\,(t) + S_{i2}\,(t)$$

And we will obtain change rate in capital stock.

$$\Delta S_i(t) = S_i\,(t + 1) - S_i\,(t)$$

Capital stock coefficient can be inversed in following manner:

$$b_{ij} = \frac{S_{ij}}{X_j}$$

Here b_{ij} = capital stock coefficient, x_i = total production of j^{th} industry and refinement of production of j^{th} industry by i^{th} industry.

8.4.2 The Importance of Input-Output Models for a Planned Economy

The procedures of input-output analysis play a very important role in economic area. Discussions on economic principles, drawing of national papers, planning of economic plans, study of inter-relation and inter-dependence of industries, study of trade cycles etc are the areas, which has been possible because of application of this procedure. Today almost in every area of economy, this technique is being used, but at present this technique is especially being used in planned economy.

Today many socialist and other countries have adopted the basic concept of input-output technique to accomplish their programs related to economic development. Most of the socialist and communist economy has a structural thought that economy should be molded under convenient integrated principle. Therefore this is the liability of this central agency that it ascertains the requirements of the economy and according to the requirement make available different production factors so that the objective of maximum social welfare can be met. Therefore, this is essential that industrial activities be instructed in a planned manner.

It can be ascertained with the help of this technique that how much of the total production of one industry can be used as input by the other industry. With the help of input-output matrix, statisticians and planning officers get to understand the transactional relation of whole economy.

Note

Input-Output Analysis helps us understanding the relation among the various areas of the economy. Thus, this analysis is convenient to the planners to understand the internal structure of the economy. Unplanned economy is based on the system of Test and Error, but under the planned economy, mitigation of such type of errors are done through this analysis.

This is noteworthy fact that in national economic program concept of dynamic input-output is more important that constant input-output as because due to fast change in the development of economy the flow of economic structure can not remain constant. In this context, this technique will be more suitable to the economic programs for developing economies. The reason for this is that under the linear identical input-output model constant technical coefficient, it can be applied even in the absence of reliable even numbers.

Input-Output model is not only used for studying the mutual relation in different production areas, but also is used for accomplishment of different purposes. For study of the administrative part of national and international, this model is also used. Under this model we can get scheme for easy study of input-output flow relation between the different areas of any country with that of different country.

This analysis is also used for the study of railway freight. For the very years, in railway trade tables contribution of production areas has been taken by the various railway stations. Inter-area flow is replaced by the division of tonnes, which is sent from one station to another station.

From the above analysis it is clear that input-output matrix is used for the analysis of economic factors. Tables of this analysis can be elected in different ways in the following way, which different areas of economy such as mutual relation and dependence of trades, study of inter-flow of different countries can be done easily. Under the socialist conditions, input-output analysis is an essential tool (Oskar Lang: Introduction to Economics, 1959) for the investigation of internal adjustment of national plans.

Example 1: Find out the input-out coefficient from the given transfer table

Purchase area → / Production area ↓	Agriculture	Industry	Last Demand	Total output
Agriculture	300	600	100	1000
Industry	400	1200	400	2000

If the obtained demand changes to 200 and 800 respectively, then find out total production which will be equivalent to new demand

Solution: Applying the formula of technical coefficient

$$a_{ij} = \frac{x_{ij}}{X_j}$$

Here a_{ij} = Technical coefficient,

X_i = Total production of j^{th} area

x_{ij} = Total production of j^{th} area which is absorbed by the i^{th} area in its production

Table to determine Technical Coefficient

Sector	**Agriculture**	**Industry**	**Total Output**
Agriculture	300/ 1000 =0.3	600/2000 = 0.3	1000
Industry	400/1000 = 0.4	1200/2000 = 0.6	2000

If the last demand becomes 200 and 800 respectively for the agriculture and industry, then

Note

Sector	Agriculture	Industry	Last demand	Total Output
Agriculture	300	600	200	1100
Industry	400	1200	800	2400

Thus, Gross Ouput = 1100 + 2400 = 3500

Example 2: Table given below shows the inter-trade transactions of many crores of three sectors S_1, S_2 or S_3 of economy

	S_1	S_2	S_3	Last Demand	Total output
S_1	50	25	25	100	200
S_2	40	50	10	200	300
S_3	100	50	150	300	600

Calculate the coefficient matrix

Solution: We know that [technical coefficient = $a_{ij} = \frac{x_{ij}}{X_j}$]

Table to determine Technical Coefficient

	S_1	S_2	S_3	Total output
S_1	50/200 = 0.25	25/300 = 0.08	25/ 600 = 0.04	200
S_2	40/200 = 0.20	50/300 = 0.16	10/600 = 0.016	300
S_3	100/200 = 0.50	50/300 = 0.16	150/600 =0.25	600

Example 3: Technological matrix under Input-Output Model is given below:

Sector 1	Sector 2	Last demand	
Sector I	0.1	0.3	F_1
Sector II	0	0.2	F_2
Labor	0.9	0.8	-

If last demand = $F_1 = 0.5y + 100$

$F_2 = 0.3y + 204$

Then find out the equilibrium of revenue and output of different sectors. Compare the results if

Solution: Under equilibrium situation

$$F_1 = F_2$$

$\therefore$ $0.5y + 100 = 0.3y + 200$

or $5y + 1000 = 3y + 2000$

or $2y = 1000$

or $y = 500$

Again $F_1 = 0.5y + 100 = (0.5 \times 500) + 100 = 350$

$F_2 = 0.3y + 200 = (0.3 \times 500) + 200 = 350$

We know that

$$X = AX + D$$

or $X = (I - A)^{-1} D$

Note

Here

$X =$ area and I area II is total production

$I =$ Unit Matrix

$D =$ Last Demand

$A =$ Coefficient of Matrix

Now

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & -0.3 \\ 0 & 0.8 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{\text{Adjoint}}{\text{Determinant}}$$

$$\text{Det. of } (I - A) = \begin{bmatrix} 0.9 & -0.3 \\ 0 & 0.8 \end{bmatrix}$$

$$= 0.72$$

Transferred matrix $(I - A) = \begin{bmatrix} 0.9 & 0 \\ -0.3 & 0.8 \end{bmatrix}$

Co - factor of 0.9 = 0.8
Co - factor of 0.3 = 0
Co - factor 0 = 0.3
Co - factor 0.8 = 0.9

$\left[\therefore (-1)^{i+j} \text{ for cofactor sign} \right]$

$$\text{Adjoint of } (I - A) = \begin{bmatrix} 0.8 & 0.3 \\ 0 & 0.9 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{1}{0.72} \begin{bmatrix} 0.8 & 0.3 \\ 0 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0.8}{0.72} & \frac{0.3}{0.72} \\ 0 & \frac{0.9}{0.72} \end{bmatrix} = \begin{bmatrix} 1.11 & 0.41 \\ 0 & 1.25 \end{bmatrix}$$

Now $(I - A)^{-1} D = \begin{bmatrix} 1.11 & 0.41 \\ 0 & 1.15 \end{bmatrix} = \begin{bmatrix} 350 \\ 350 \end{bmatrix} = \begin{bmatrix} 532.0 \\ 437.5 \end{bmatrix}$

Thus, total production of first sector = 532 units and

Total production of second sector = 437.5 units

Note

if F_1 = 100 and F_0 = 200, then

$$X\,(I-A)^{-1}D = \begin{bmatrix} 1.11 & 0.41 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 192 \\ 250 \end{bmatrix}$$

Under this situation, total production of first sector = 192 units and

Total production of second sector = 250 units

Self Assessment

2. State whether the following statements are True or False:

6. $X_i = \sum_{j=1}^{n+1} X_{ij}$
7. $X_i P_i = \sum_{j=1}^{n+1} a_{ij}\, X_i\, P_i = 0$ equation is the second set of equilibrium equations.
8. Procedures of Input-Output Analysis does not play an important role in economic sector
9. Input-Output Analysis does not help us to understand the relation among the various sectors of the economy.
10. Technical coefficient = $a_{ij} = \frac{x_{ij}}{x_j}$.

8.5 Summary

- Input means the demand from the producer for material to produce the product, whereas output means the result of efforts made towards production.
- With the help of Input-Output Analysis, we get inter-trade relation and inter-dependence of entire economy because output of one industry could be input for another industry.
- Leontief's Static Input-Output Model is based on the above assumptions. This can be understood with an example.
- With the help of Input-Output Analysis we can understand this model. In the given table, output of all the three parts is shown in horizontal rows, whereas inputs are shown in vertical columns.
- Method of finding technical coefficient is very simple. Here we divide input of desired area by total production of that area.
- Assumptions of Leontief model are impractical. It assumes that technical coefficient are constant, which means technology will be constant or change in production will result to change in means devoted for production.
- In practical, commodities and services are expressed in monetary form only. Therefore, forecasting of economy in composite form is difficult with the balanced equation by Input-Output Analysis.
- In Leontief's Input-Output Open Model we had considered domestic demand as a separate area. If it is also merged in the economy, then no other area will remain left which has relation to external area.

Note

- In constant model, with the help of technological coefficient of various areas of economy, mutual dependency is studied.
- Input-Output Dynamic Model is generalized form of constant model. Therefore its assumptions would be similar to constant model.
- The procedures of input-output analysis play a very important role in economic area.
- Discussions on economic principles, drawing of national papers, planning of economic plans, study of inter-relation and inter-dependence of industries, study of trade cycles etc are the areas, which has been possible because of application of this procedure.
- Input-Output Analysis helps us understanding the relation among the various areas of the economy. Thus, this analysis is convenient to the planners to understand the internal structure of the economy.

8.6 Keywords

- ***Input:*** Receipts
- ***Output:*** Issuance Final product

8.7 Review Questions

1. Mathematically explain the first set of equilibrium equations of closed model.
2. What do you mean by the Leontief's Input-Output closed model?
3. Give a short note on Leontief's dynamic model.
4. Explain the importance of input-output model for a planned economy.
5. Mathematically explain the second set of equilibrium equations of closed model.

Answers: Self Assessment

1. Production	2. Output	3. Input	4. Leontief
5. Dependency	6. True	7. True	8. False
9. False	10. True		

8.8 Further Readings

Books

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Malcom, Nicolas, U C London.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

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Mathematics for Economist – Yamane, Prentice Hall Publication.

Unit 9: Conditions of Hawkins and Simon

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Conditions of Hawkins and Simon.
- Find the Technical Multiplications of Matrix.

Introduction

Sometime solution of in-out matrix comes to negative. If our out-result come to negative, it means that we are using more in than out, which is an unrealistic situation. Thus we can say that the system is not viable.

9.1 Conditions of Hawkins and Simon

Conditions of Hawkins and Simon can get us out from this situation. Our primary equation is $X = (I - A)^{-1}F$, then $(I - A)$can be written in following way

$$\begin{vmatrix} (1-a_{11}) & -a_{+2} & -a_{+3} & \dots & a_{1n} \\ -a_{21} & (1-a_{22}) & -a_{23} & \dots & a_{2n} \\ -a_{31} & -a_{32} & (1-a_{33}) & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & (1-a_{nn}) \end{vmatrix}$$

Then there would be two conditions of H.S.

(1) Table of matrix should always be positive

(2) Defacement element *i.e.* $(1 - a_{11})$, $(1 - a_{22})$, $(1 - a_{33})$ ...... $(1 - a_{nn})$ should be positive viz a_{11}, a_{22}, a_{33}, ... a_{nn} should always be less than 1

Note

Did u know? The above two conditions are called the conditions of Hawkins-Simon.

Example 1: If $(A) = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$

$$\therefore \qquad (I - A) = \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$$

And if value of table is $(I - A)$, then 0.06 – 0.18 = (–) 8.12 which is less than zero. Here H.S. condition does not fulfill.

Then there is no solution possible.

Task Write down the conditions of Hawkins-Simon.

Example 2: For the year 1990, following is the transaction of inter-trade

Given table of inter-trade transaction for the year 1990 has been created for an economy

Trade	**1**	**2**	**Last consumption**	**Total**
1	500	1600	400	2500
2	1750	1600	4650	8000
Labor	250	4800	-	5050
Total	**2500**	**8000**	**5050**	**15550**

9.2 To Find the Technical Multiplications of Matrix

Showing direct requirement, prepare the technical multiplication of matrix. Is there any solution available for this method?

Dividing all outputs of the sector from all inputs, showing direct requirement of each unit of output technical multiplication of matrix can be found.

Solution:

$$a_{11} = \frac{500}{2,500} = 0.20 \left(= \frac{X_{11}}{X_1} \right)$$

$$a_{12} = \frac{1,600}{8,000} = 0.20 \left(= \frac{X_{12}}{X_2} \right)$$

$$a_{21} = \frac{1,750}{2,500} = 0.70 \left(= \frac{X_{21}}{X_1} \right)$$

$$a_{22} = \frac{1,600}{8,000} = 0.20 \left(= \frac{X_{22}}{X_2} \right)$$

Note

Therefore, given technical matrix

Industry	1	2
1	0.20	0.20
2	0.70	0.20
labour	0.10	0.60

∴ $A =$ (the above matrix)

and $$(I - A) = \begin{pmatrix} 1 & -0.20 & -0.20 \\ & -0.70 & 1-0.20 \end{pmatrix} = \begin{pmatrix} 0.80 & -.20 \\ -0.70 & 0.80 \end{pmatrix}$$

∴ $$(I - A) = \begin{vmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{vmatrix}$$

$$= 0.80 \times 0.80 - 0.20 \times 0.70 = 0.50$$

Since $|I - A|$ is positive and each element of basic diagonal of $(I - A)$ will be positive, therefore conditions of Hawkins-Simon is fulfilled, thus there is a solution for practical oriented method.

Example: Given

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix}$$

And last demand are F_1, F_2, F_3 then find out the level of output?

And last demand is F_1, F_2 and F_3. With the regularity of the model, find out the level of output. What would be the level of output if $F_1 = 20$, $F_2 = 0$ and $F_3 = 100$?

Solution: $$= \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = (I - A)^{-1} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Now $$(I - A) = \begin{bmatrix} 0.9 & -0.3 & -0.1 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Thus co-divisive is

$$A_{11} = \begin{vmatrix} 0.8 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0.56$$

$$A_{12} = -\begin{vmatrix} 0 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0$$

$$A_{13} = +\begin{vmatrix} 0 & -0.8 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} -0.3 & -0.1 \\ 0 & 0.7 \end{vmatrix} = 0.21$$

$$A_{22} = \begin{vmatrix} 0.9 & -0.1 \\ 0 & 0.7 \end{vmatrix} = 0.63$$

Note

$$A_{23} = -\begin{vmatrix} -0.9 & -0.3 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} -0.3 & -0.1 \\ 0.8 & -0.2 \end{vmatrix} = 0.14$$

$$A_{32} = -\begin{vmatrix} -0.9 & -0.1 \\ 0 & -0.2 \end{vmatrix} = 0.18$$

$$A_{33} = \begin{vmatrix} 0.9 & -0.3 \\ 0 & 0.8 \end{vmatrix} = 0.72$$

Therefore the value of table is 0.9 × 0.56 = 0.504

Therefore, $$(I - A)^{-1} = \frac{1}{0.504}\begin{bmatrix} 0.56 & 0.21 & 0.14 \\ 0 & 0.63 & 0.18 \\ 0 & 0 & 0.72 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.11F_1 & + & 0.42F_2 & + & 0.28F_2 \\ 0 & + & 1.25F_2 & + & 0.36F_3 \\ 0 & + & 0 & + & 1.43F_3 \end{bmatrix}$$

From the given value of F_1, F_2 and F_3, we get

$$X_1 = 1.11F_1 + 0.42F_2 + 0.28F_3$$
$$= 1.11 \times 20 + 0 + 0.28 \times 100$$
$$= 50.2$$
$$X_2 = 1.25F_2 + 0.36F_3$$
$$0 + 0.36 \times 100 = 36 \text{ or}$$
$$X_3 = 1.43F_3 = 143.$$

Notes This is worth notable that if technical matrix is highly triangular, if all elements of main diagonal is zero or near to zero, then $(I - A)^{-1}$ matrix would be triangular. In this condition X_3 output will be totally dependent on the last demand of sector 3 and X_3 will be dependent on the last demand of sector 2 and sector 3.

Note

Example: In the above example, if final demands change by 10, 10, 10 then what will be the change in sector output?

We have $X = (I - A)^{-1} F$

$\therefore$ $\Delta X = (I - A)^{-1} \Delta F$

Where ΔX and ΔF are the conveyor of change in output and last demand, therefore

$$\Delta X = \begin{bmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

$$\Delta X_1 = 1.11 \times 10 + 0.42 \times 28 \times 10 = 18.1$$

$$\Delta X_2 = 0 + 1.25 \times 10 + 0.36 \times 10 = 16.1$$

$$\Delta X_3 = 14.3$$

Self Assessment

1. Fill in the blanks:

1. Sometime solution of in-out matrix comes to ………………..
2. If our out-result come to negative, it means that we are using …………. more than out.
3. Table of matrix should always be ………….

9.3 Summary

- Sometime solution of in-out matrix comes to negative. If our out-result come to negative, it means that we are using in more than out, which is an unrealistic situation. Thus we can say that the system is not viable.
- Our primary equation is $X = (I - A)^{-1}F$, then $(I - A)$ can be written in following way

$$\begin{bmatrix} (1-a_{11}) & -a_{+2} & -a_{+3} & a_{1n} \\ -a_{21} & (1-a_{22}) & -a_{23} & a_{2n} \\ -a_{31} & -a_{32} & (1-a_{33}) & a_{3n} \\ -a_{n1} & -a_{n2} & -a_{n3} & (1-a_{nn}) \end{bmatrix}$$

9.4 Keywords

- ***Condition:*** Constraints

9.5 Review Questions

1. Find out output, when following technical multiplication (A) and last demand is given

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.4 \\ 0.1 & 0.3 & 0.3 \end{bmatrix}_3 \quad F = \begin{bmatrix} 100 \\ 50 \\ 60 \end{bmatrix}$$

Note

2. With the help of following table calculate technical multiplication

↓ Purchase area → / Production area	Agriculture	Industries	Last Demand	Total production
(1) Agriculture	500	1000	200	1700
(2) Industries	700	1500	600	2800

Answers: Self Assessment

1. Negative 2. Input 3. Positive

9.6 Further Readings

Books

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Malcom, Nicolas, U C London.

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Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Unit 10: Closed Economy: Input-Output Model

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Prime Elements of Input-output Analysis.
- Understand closed Input-Output Model.

Introduction

Meaning of the word input is Producer's demand of material for production. Similarly, meaning of the word output is associated with the result of productivity. By the word cost is meant that material which the producer purchases for production whereas by the word output is meant that which the producer sells. Hence for any firm, input is cost and output is receivable.

Input-Output analysis was searched by W.W. Leontief in 1951. This analysis studies that on a given technique, how much production should be done in various areas which is completely taken for use in form of consumption by consumers and industries and here the level of satisfaction is the maximum.

10.1 Prime Elements of Input- Output Analysis

1. This analysis is applicable in balanced economy.
2. This analysis studies production activities and problems related to technique.
3. This analysis is based on universal searches.

Input Output Model:

For making its technique clear, two types of equations are used:

(A) **Comparative equation:** this equation clarifies that complete production of any industry is either consumed by itself or is consumed by other industries and outside areas. Assumed that number of other industries from other areas is n and industry is I, whose total production is x_i, then following comparative equation will be there:

$$X_1 = X_{11} + X_{12} + X_{13} + \dots + X_{1n} + F_1$$

$$X_2 = X_{21} + X_{22} + X_{23} + \dots + X_{2n} + F_2$$

$$\therefore \quad X_i = X_{i1} + X_{i2} + X_{i3} + \dots + X_{in} + F_i$$

here $F \Rightarrow$ Demand of outside area.

Note

(B) ***Structural Equation:*** for construction of this equation, support of technical co-efficient is taken. Assumed that i and j are two industries:

Technical co-efficient of industry i in industry j is

$$j = \frac{\text{Quantity of consumption of product of industry } i \text{ in industry } j}{\text{Total production of industry } i}$$

i.e.,
$$a_{ij} = \frac{X_{ij}}{X_j}$$

or
$$X_{iy} = a_{ij} \times X_j \qquad \ldots(A)$$

Here, a_{ij} is technical coefficient

The above give equation (A) is called structural equation. Now on converting comparative equation to structural equation:

$$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \ldots\ldots + a_{in}X_n + F_1$$
$$X_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \ldots\ldots + a_{2n}X_n + F_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$X_i = a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3 + \ldots\ldots + a_{in}X_n + F_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$X_n = a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \ldots\ldots + a_{nn}X_n + F_n$$

On writing in tabular form:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_i \\ X_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_i \\ \vdots \\ F_n \end{bmatrix}$$

or
$$X = AX + F \qquad \ldots(i)$$

It is the general equation of input-output model.

10.2 Closed Input- Output Model

If demand of outside area i.e. F, works like internal area, then that model is called closed input-output model. Just like final consumption area, supplies labour to other industries then it works like an industry. In this state, number of industries will be $n + 1$, if outside areas are represented by X_0 then,

$$X_0 = a_{00}X_0 + a_{01}X_1 + a_{02}X_2 + \ldots\ldots + a_{0n}X_n$$
$$X_1 = a_{10}X_0 + a_{11}X_1 + a_{12}X_2 + \ldots\ldots + a_{1n}X_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$X_n = a_{n0}X_0 + a_{n1}X_1 + a_{n2}X_2 + \ldots\ldots + a_{nn}X_n$$

or
$$X = AX$$

Note

or $X - AX = 0$

or $[I - A]X = 0$, here $I \Rightarrow$ identity Matrix

For example, consumption function analysis (closed model) assumed that two industries and a final demand is given:

–	1	2	Final Demand
1	0.1	0.3	F_1
2	0	0.2	F_2
Labour (L)	0.9	0.5	–

Now assumed that final demand is also operating like an industry

$$F_1 = 0.5\,y + \bar{C}_1$$

$$F_2 = 0.3\,y + \bar{C}_2$$

Here 0.5 and 0.3 MPC or $\bar{C}_1$ and $\bar{C}_2$ constant.

Then coefficient $k = \dfrac{1}{1-(0.5+0.3)} = \dfrac{1}{0.2} = 5$

Then new input-output model;

$$X_1 = 0.1X_1 + 0.3X_2 + 0.5\,y + \bar{C}_1$$

$$X_2 = 0.0X_1 + 0.2X_2 + 0.3\,y + \bar{C}_2$$

And labour rate is given to be 0.9 and 0.5 then $.9x_1 + 0.5x_2 = y$

Then, $X_1(1 - 0.1) - 0.3\,X_2 - 0.5y = \bar{C}_1$

$$-0.0X_1 + X_2(1 - 0.2) - 0.3\,y = \bar{C}_2$$

$$0.9X_1 + 0.5X_2 + y = 0$$

Then in tabular form:

$$\begin{bmatrix} X_1 \\ X_2 \\ y \end{bmatrix} = \frac{1}{0.144}\begin{bmatrix} 0.65 & 0.55 & 0.49 \\ 0.27 & 0.45 & 0.27 \\ 0.72 & 0.72 & 0.72 \end{bmatrix}\begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ 0 \end{bmatrix}$$

Or $$\begin{bmatrix} X_1 \\ X_2 \\ y \end{bmatrix} = \begin{bmatrix} 4.51 & 3.82 & 3.34 \\ 1.87 & 3.12 & 1.87 \\ 5.0 & 5.0 & 5.0 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ 0 \end{bmatrix}$$

Then balanced income $y = 5X\bar{C}_1 + 5X\bar{C}_2 = 5(\bar{C}_1 - \bar{C}_2)$. **Ans.**

10.3 Summary

- Meaning of the word input is Producer's demand of material for production. Similarly, meaning of the word output is associated with the result of productivity. By the word cost is meant that material which the producer purchases for production whereas by the word output is meant that which the producer sells. Hence for any firm, input is cost and output is receivable.

Note

10.4 Keywords

- ***Input:*** Material for production for any firm

10.5 Review Questions

1. Clarify the closed input-output model.
2. What will be the effect on production in increase of its cost?

10.6 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall, India.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics – Malcolm, Nicholas, U.C. London.

Mathematics for Economics – Karl P. Simone, Lawrence Bloom.

Mathematical Economics – Micheal Harrison, Patrick Waldaron.

Mathematics for Economist – Mehta and Madnani- Sultan Chand and Sons.

Mathematics for Economics and Finance – Martin Norman .

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Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Unit 11: Linear Programming

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Obtain Knowledge related to Meaning of Linear Programming, its Conditions and Generalisation.
- Obtain Knowledge of Application of Firm Theory.
- Know the Limitations of Linear Programming.

Introduction

Linear programming is a mathematical method which mathematician **George Dantzig** had developed in 1947 for making the plan of various activities of American air force related to the problem of providing supplies to the army. It was also developed for use in economic theory of firm, administrative economics, inter-state trade, general balance analysis, welfare economics and development planning. In this unit, linear programming relating to firm is being described.

11.1 Meaning of Linear Programming

Problems of maximisation and minimisation are also called problems of optimisation. The techniques that are adopted by the economists for solving these problems, they are known as linear programming. With some constraints remaining in form of linear inequalities, it is a mathematical technique for analysis of optimum decisions. In mathematical language, it is applicable on all those problems in which, despite of arrangements of linear inequalities expressed in form of some variable; there is need for solution of maximisation and minimisation. If two variables x and y are functions of z, then value of z will be maximum when value of z is less than any movement done from that point. Value of z is minimum when value of z is more than any movement. When there is change in per unit cost and price along with the size of output then problem is not linear and if there is no change in it with the output, then the problem is linear. In this way programming may be defined like this that it is that method which is optimum combination for sources of production of given output or is produced from the given plant and machinery decides the optimum combination of goods. It is also used for deciding technical diversity.

Note

11.2 Conditions and Generalisation

Use of linear programming technique depends upon some conditions and generalisation.

First, a definite objective is there. This objective may be to maximise profit or income or to minimise costs. It is called **objective function** or **tenterion function.** If one quantity is maximised, its negative quantity is minimised. Dual of each maximization problem is minimisation problem. Original problem is primal, which always has a dual. If primal problem is related to maximisation, then dual problem will be minimisation and vice-versa.

Second, alternative production process must be there for fulfilling the objective. Thought of process or activity is very important in linear programming. Process, "is a specific method of doing any economic work" it is "a physical activity of some kind like, consuming something, collecting something, purchasing something, throwing away something and producing something in a special way."

Notes Linear Programming technique helps the agency taking the decision in the thing that for fulfilling the objective, they may select the most efficient and economical process.

Third, some constraints or restraints of the problem are also important. It is the limitations or barriers related to the problem which tell that what cannot be done and doing what is important. These are also known as inequalities. In production they are often given quantities of land, labour and capital, which are used in most efficient process for fulfilling a definite objective.

Fourth, choice variables are also there. These are those institutions which are selected so that the objective function may be made maximum or minimum and all restraints may be satisfied.

Last, feasible and optimum solutions are there. on income of the consumer and price of good being given, all possible combination of goods, which he may purchase from feasibility, will be feasible solution. For consumer feasible solution of two goods is all those combinations which are located on the budget line or to its left while on isocost line they are either located on it or to its right.

In other words we may say that feasible solution is that, which satisfies all restraints. Best from all feasible solutions is the optimum solution. If a feasible solution makes the objective function maximum or minimum, then it is optimum solution. Best available method for searching optimum solution from all possible solutions is simplex method. This process famous by the name simplex method is extremely mathematical and technical. Main objective of linear programming is to find optimum solutions and studying its specialities.

Task What is Linear Programming?

11.3 Application to the Theory of the Firm

Till the neo classic theory of firm, did the analysis of the problem of decision maker, taking one or two variable at time. It was related to one production process at a time. In linear programming, production function goes beyond these limited areas of economic theory. It thinks over various capacities and restrictions that are created in the production process. It select between various complex production process for doing the maximisation of profits and minimisation of costs.

Assumptions: linear programming analysis of the firm is based on the following assumption:

i. Institutions taking decision have to face some constraints or restrictions. It may happen that there is borrowing, raw material or space constraint on its activities. Type of constraint actually depends on the nature of the problem. Mostly they are constant sources of production process.

ii. It moves assuming number of optional production process to be limited.

iii. One assumption of it is there are linear relations among different variables which mean that under a process there is stable proportionality between input-output.

iv. Prices of inputs and coefficients are given and stable. They are known definitely.

v. Concept of additivity is also stable in the core of linear programming which means that total resources used by all firms will be equal to the sum resources used by each personal firm.

vi. Linear programming techniques also consider continuity and divisibility in goods and resources.

vii. Institutional resources considered as stable.

viii. For programming a definite duration is assumed. For convenience and more accurate results duration is generally small though possibility of comparatively longer duration is not ended.

On these assumptions being given application of linear programming for theory of the firm is done for solution to three problems:

(1) ***Maximisation of Output:*** We assume that a firm is made to produce a good *Z* with the use of input *X* and *Y*. Its objective is to **maximise the production.** It has two optional production processes *C* (Capital intensive) and *L* (Labour intensive). Restraint, cost-expense is line *MP*. As has been shown in figure 11.1. Rest all assumptions related to linear programming technique (as told above) are applicable. The problem is being described in language of figure 11.1.

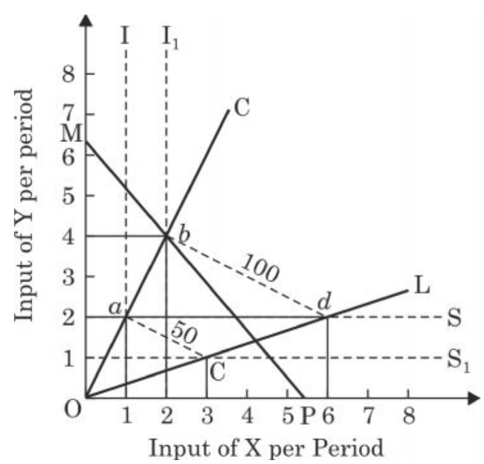

Figure 11.1

Units of cost (resources) per period have been measured on vertical *Y* axis and units of input *X* per period have been shown on the horizontal axis. If process *C* needs 2 units of *Y* with each unit of input *X*, then it will produce 50 units of good *Z*. If by doubling the inflow of *X and Y* units of *X* and units of *Y* are made 4, then outflow will also be doubled to 100 units of *Z*. these combinations of *X* and *Y* expressed by *a* and *b* establish production parameter scale **capital intensive process** line *OC*. At the other side original unit of *goods X* (50) may be produced through process *L* with the combination of one unit *of X* and *Y* and 100 units of *Z* may be produced by doubling *X and Y* with 2 units of *X* and 6 units of *Y*. these production parameters are established on line *OL* of labour intensive process which are expressed by combinations *c* and *d* of the prices. If at unit level of 50, points *a* and *c* are joined on linear rays *OC* and *OL*, they make isoquant (which is shown by dotted line). *t* in isoquant according to production level of 100 units is *bds.* Isocost *MP* represents cost-expenditure restraint and determines a limit of production capacity of the firm. Inside the area expressed by the triangle *obd*, firm may produce from any of the two available techniques *C* and *L.* firm will not be able to produce outside the "area of these feasible solutions." Optimum solution maximising the production of the firm will be at the point where isocost curve touches the isoquant curve of maximum production. In the figure above isocost curve *MP* touches the isoquant $I\text{-}_1bds$ at point b of process ray *OC.* It shows that firm, using 4 units of input *Y* and 2 units if input *X*, will use the capital intensive technique and produce 100 units of good *Z*.

Note

(2) ***Maximisation of Revenue:*** Take the other firm whose objective function is to maximise its revenue with restraints of limited capacity remaining. Consider that planning produces two goods *X* and *Y*. It has four departments in which capacity of each is fixed. Consider that these four departments are associated with making, compilation, polishing and packing which we name *A, B, C; D*. problem has been shown in figure 11.2

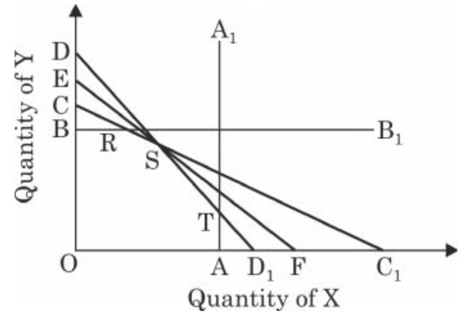

Figure 11.2

With restraints *A, B, C, D* remaining, *X* and *Y* are produced. Restraint A limits the production of good *X* to *OA*. Restraint *B* limits the production of good *Y* to *OB*. Restraint *C* limits production of both goods, *X* and *Y* to OC_1 and *OC* respectively whereas; restraint *D* limits their production to OD_1 and *OD* respectively. Area OATSRB expresses all those combinations of X and Y which may be produced without violating any constraint. It is the area of feasible production inside which X and Y may be produced but there is no possibility of production of any combination on any point outside this area.

Optimum solution may be looked for inside the feasible area by taking isoprofit line. Isoprofit line expresses all those combinations of X and Y which provide equal profit to the firm. Optimum solution is located on point S of highest isoprofit line EF inside the polygon OATSRB. Any point other than S will be situated outside the feasible production area.

(3) ***Minimisation of Cost:*** Food problem was first economic problem whose solution through linear programming was done through minimisation of cost. Consider that a consumer buys bread and butter at market price. Problem is that from various quantities of both foods cost of receiving total nutrients is made minimum.

Dotted linear solution of food problem is shown in figure 11.3. Bread (x_1) and butter (x_2) have been both measured respectively on both axis. Line AB expresses combination of less bread and more butter and line CD expresses combination of more bread and less butter. Feasible solution is located on deep line AZD or above it. Optimum solution is on point Z, where isocost line (dotted) RK is there which passes through the intersection point of AB and CD. If bread is expensive, possible solution may be on A and if butter is comparatively expensive, it can be on D. but in this problem, this solution will be on Z because here only cost is minimised.

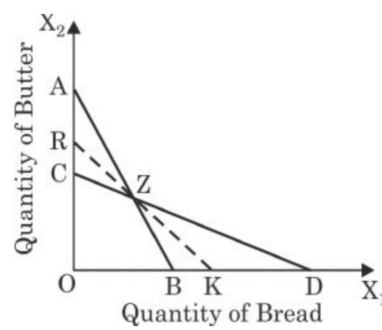

Figure 11.3

Self Assessment

1. Fill in the blanks:

1. Linear programming was developed in ....
2. .... was the father of linear programming.
3. Problems of maximisation and minimisation are also known as problems of ....
4. Optimum solution may be looked for inside the feasible area by taking .... line.
5. Food problem was first economic problem whose solution through .... programming was done through minimisation of cost.

Note

11.4 Limitations of Linear Programming

Linear programming proved to be a very profitable resource in economics. But it has its own limitations. In reality, because of many restraints, actual problems cannot be solved through linear programming. **First**, it is not easy to define a specific objective function. **Secondly**, even if a specific objective function is defined, knowing various social, institutional financial and other constraints popular in the path of the view of the objective is not an easy task. **Thirdly,** on a set of specific objectives and constraints being given, it is important that constraints may not be expressed directly in form of linear inequalities. **Fourthly**, even if described problems are worth crossing over also, main problem is of estimation in relation to various constant coefficients, which enter a linear programming problem like prices. **Fifth**, Main shortcoming of this technique is it is based on establishment of linear relation between inputs and outputs which means that relations of sum, multiplication and divisibility of found between various inputs and outputs. But these relations are not in each linear programming problem because in many problems non-linear relations are found. **Sixth**, technical goods and resources are based on assumption of free competition in the market. But in state of free competition is not found. **Seventh**, linear programming moves with the assumption of consideration in the economy, but in reality consideration are either decreasing or increasing. **Lastly**, it is a very complex mathematical and complex technique. Solution of a problem with linear programming expects maximisation and minimisation of a clear designated variable. Solution of this linear programming problem may also be found through complex methods like Simplex Method, in which many mathematical enumerations have to be done. For it special computational technique like electric computer of desk calculator is needed. Such calculators are not only expensive, but for operating them, specialists are also needed.

Did u know? Linear Programming model mostly presents trial and error solutions. Searching optimum solution of various economic problems in reality is difficult.

11.5 Summary

- Linear programming is a mathematical method which mathematician George Dantzig had developed in 1947 for making the plan of various activities of American air force related to the problem of providing supplies to the army. It was also developed for use in economic theory of firm, administrative economics, inter-state trade, general balance analysis, welfare economics and development planning.
- Problems of maximisation and minimisation are also called problems of optimisation. The techniques that are adopted by the economists for solving these problems, they are known as linear programming.

11.6 Keywords

- ***Optimise:*** As per requiement
- ***Primal:*** Main, chief

11.7 Review Questions

1. What is linear programming? Describe its conditions.
2. On which assumptions is linear programming analysis of firm based upon?
3. Describe maximisation of production.

Note

Answers: Self Assessment

1. 1947
2. George Dantzig
3. Optimisation
4. Isoprofit
5. Linear

11.8 Further Readings

Books

Mathematics for Economics – Malcolm, Nicholas, U.C. London.

Mathematics for Economics – Karl P. Simone, Lawrence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Simone and Bloom – Viva Publication.

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Mathematical Economics – Micheal Harrison, Patrick Waldaron.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Unit 12: Formulation of Linear Programming

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know Linear Programming Formulation

Introduction

Technique adopted by economists for solving the problems of maximisation and minimisation is called Linear Programming. It is a mathematical method. It was developed by a mathematician George Denzing in 1947.

Technique of knowing maximum or minimum value of any problem, on constraints being given, is known as Linear Programming Formulation. Here one is objective functional while the other is given with condition.

12.1 Linear Programming Formulation

Steps of Linear Programming Formulation may be understood through example. As we know that LPP (Linear Programming Problem) has three parts:

1. Objective Function: which we maximise or minimise
2. Structural constraint
3. Non- Negative Constraint

For e.g., Assumed that two things are X and Y whose price are ₹ 2 and ₹ 5, objective function

$$\text{Max} : f = 2x + 5y \qquad \text{(A)}$$

And structural constraint:

$$\begin{aligned} X + 4y &\leq 24 \\ 3X + y &\leq 21 \qquad \text{(B)} \\ X + y &\leq 9 \end{aligned}$$

And non-negetive constraint:

$$X \geq 0 \qquad \text{(C)}$$

And, $y \geq 0$

In the above problem, equation A represents objective function whose objective is to maximise and minimise the problem. While equation B shows that how much production of x and y is possible by combinations of different resources of production. It means it shows the constraint of resources of production and equation. C shows non-negetive constraint. Above process is known as Formulation of Linear Programming.

Note

Example: Convert the below given problem in form of Linear Programming

Vitamin (Type)	Vitamin in Per Kg food item I	II	Minimum daily vitamin requirement
A_1	10	4	20
A_2	5	5	20
A_3	52	6	12
Per Kg Price of Food Item	₹ 0.60	₹ 1.00	

Solution: $\text{Min } f = 0.6\,x_1 + x_2$

$x_2 = 0.6\,x_1 + x_2$

Constraint $10x_1 + 4x_2 \geq 20$

$5x_1 + 5x_2 \geq 20$

$2x_1 + 6x_2 \geq 12$

And, $x_1 \geq 0$

$x_2 \geq 0$

Non-negative constraint

12.2 Summary

- Technique adopted by economists for solving the problems of maximisation and minimisation is called Linear Programming. It is a mathematical method. It was developed by a mathematician George Denzing in 1947.

12.3 Keywords

- ***Linear Programming:*** Economical method for solving the problem associated with maximisation and minimisation.

12.4 Review Questions

Convert the below given table in linear programming:

Calcium (Type)	Calcium in Per Kg food item I	II	Minimum daily calcium requirement
Z_1	20	8	40
Z_2	10	5	40
Z_3	4	6	24
Per Kg Price of Food Item	₹ 1.20	₹ 2.00	

12.5 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics – Karl P. Simone, Lawrence Bloom.

Mathematical Economics – Michael Harrison, Patrick Waldran.

Mathematics for Economics – Malcolm, Nicholas, U.C. London.

Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Unit 13: Graphic Method

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Graphic Solution of Economic Problems.
- Understand the Minimisation of Cost.

Introduction

Take a firm which at given prices ₹ 12 and ₹ 15 respectively produce two goods *X* and *Y* per unit. For producing goods *X* firm need 12 units of input *A*, 6 units of input B and 14 units of input *C*. but for goods *Y*, 4 units of input *A*, 12 unit of input *B*, and 12 units of input *C*. Total available units of input *A* are 48, of *B* is 7 and of *C* is 84 units. Input-output data of this linear programming problem has been shown in table 29.1.

Table 13.1: Input-Output Data

Input	Number of units for producing goods		Units of total receivable costs
	Goods X	Goods Y	
A	12	4	48
B	66	12	72
C	14	12	84
Cost Per unit	₹ 12	₹ 15	-

Each Linear Programming problem has three parts. They are of the following type of the above given problem:

(i) ***Objective Function:*** Objective function tells this that if two goods X and Y bring ₹ 12 and ₹ 15 per unit cost then how much quantity of these goods be produced so the firm may acquire maximum cost or income. It may be written like this:

Maximise $R = 12X + 15Y$

(ii) ***The Constraints:*** The above table may now be converted in form of equations which express the constraints under which a firm works. These are called structural constraints.

Let us first take input A. Maximum available quantity of input *A* is 48 units. But quantity of both goods *X* and *Y* cannot be more than 48 units. Mathematically, because $12X + 4Y$ units

Note

cannot be more than 48, that is why constraint of input A will be $12X + 4Y \leq 48$. Through similar logic, inequalities of constraints of inputs B and C may be written. Hence there are three structural constraints of our problem:

$$12X + 4Y \leq 48 \quad \text{...(1)}$$

$$6X + 12Y \leq 72 \quad \text{...(2)}$$

$$14X + 12Y \leq 84 \quad \text{...(3)}$$

(iii) ***Non Negative Constraint:*** In Linear Programming problem, non-negative constraints are also there which are dependent on this assumption that in solution to the problem, many variables cannot have negative values. It means that production of goods X and Y may be zero or positive but it cannot be negative. Hence non negative constraint of our problem is $X \geq 0$ and $Y \geq 0$.

13.1 The Graphic Solution of the Problem

For graphic solution, let us write the above describes problem again:

Maximise $R = 12X + 15Y$

Subject to (*i*)

$$12X + 4Y \leq 48 \quad \text{...(1)}$$

$$6X + 12Y \leq 72 \quad \text{...(2)}$$

$$14X + 12Y \leq 84 \quad \text{...(3)}$$

$$X \geq 0,\ Y \geq 0$$

(ii) For expressing each inequality through a graph, we will leave inequality sign (≤) in each equation and take = sign. Hence equation (1) will be written as follows:

$$12X + 4Y = 48.$$

Or $X = 4$ (when $Y = 0$)

Similarly assuming that from all 48 units only goods Y is produced:

$$0 + 4Y = 48$$

Or $Y = 12$ (when $X = 0$)

Equation $12X + 4Y = 48$ has been shown by line AB in figure 13.1, where $OA = Y$ and $OB = 4X$. Any point of line AB, like T, satisfies equation $12X + 4Y = 48$ because area below and towards the left of this line AB satisfies the inequality equation $12X + 4Y \leq 48$.

Similarly on solving $6X + 12Y = 72$: $X = 12$ and $Y = 6$ are obtained. Which have been indicated by line CD in figure 13.1 where $OC = 6Y$ and $OD = 12X$ and on solving equation $14X + 12Y = 84$ we find, $X = 6$, $Y = 7$ which has been shown in figure 13.1 through line EF, where $OE = 7Y$ and $OF = 6X$.

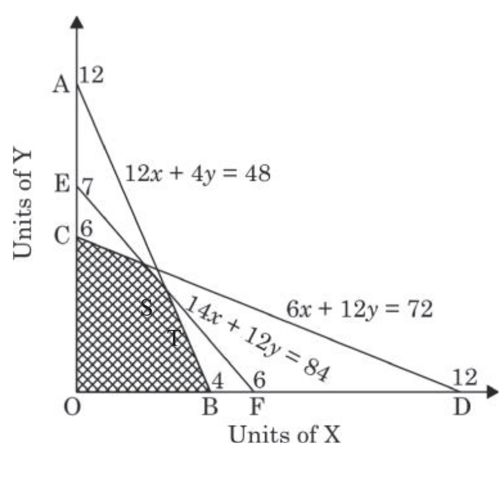

Figure 13.1

Feasible Region: Figure 13.1 shows that in shaded region all points that are surrounded by the three intersecting lines will satisfy each of the three inequalities. At point S line EF intersects line CD and at point T line CD intersects Line AB. In this way area OBTSC which is situated below the points intersecting these three line, S and T at the left side, satisfies the inequalities of all the three equations. This shaded region is called feasible area of production and each point which is inside it or on its boundary expresses feasible solution of the problem.

Optimum Solution: which point various points *B*, *T*, *S*, and *C* which express feasible solution is optimum point which will maximise the cost of the firm? How can this point be known through algebra?

Through equations (1) and (2) we know the co-ordinates of point *B* and *C* according to which $OB = 4X$ and $OC = 6Y$. For determining the coordinates of point *T*, let us take equations (1) and (2) in form of simultaneous equation (Because lines *AB* and *EF* intersect at point *T*) and solve them:

$$12X + 4Y = 48 \quad ...(1)$$

$$14X + 12Y = 84 \quad ...(3)$$

Multiplying equation (1) with 3 and subtracting equation (3) from it:

$$36X + 12Y = 144$$

$$14X + 12Y = 84$$

$$22X = 60$$

$$X = 2.73$$

Applying the value of $X = 2.73$ in equation 1,

$$12 \times 2.73 + 4Y = 48$$

$$32.76 + 4Y = 48$$

$$4Y = 48 - 32.76$$

Or $$4Y = 15.24$$

$$Y = 3.81$$

Hence, coordinates of point *T* are $X = 2.73$ and $Y = 3.81$. Similarly on solving equations (3) and (2), coordinates of Point S are, X=1.5 and Y= 5.25.

For searching the optimum combination of *X* and *Y*, let us substitute prices of *X* and *Y* (₹ 12 and ₹ 15 respectively) in values of points of these co-ordinates which have been calculated above. At point *B*, $X = 4$ and $Y = 0$. Substituting them in objective function $f = 12X + 15Y$:

(₹ 12) (4) + (₹ 15) (0) = ₹ 48 ...(4)

At point *T*, $X = 2.73$ and $Y = 3.81$, let us obtain in the same way:

(₹ 12) (2.73) + (₹ 15) (3.81) = ₹ 89-91 ...(5)

At point S, X = 1.5 and $Y = 5.25$, we obtain,

(₹ 12) (1.5) + (₹ 15) (5.25) = 96.75 ...(6)

At point *C*, $X = 0$, $Y = 6$:

(₹ 12) (0) + (₹ 15) (6) = ₹ 90 ...(7)

$$14X + 12Y = 84$$

$$6X + 12Y = 72$$

	(Primal Problem)		(Dual Problem)
Maximise Revenue	$R = 12X + 15Y$	Minimise Cost	$C = 48A + 72B + 84C$
Object to	$12X + 4Y \leq 48$	Subject to	$12A + 6B + 14C \geq 12$
	$6X + 12Y \leq 72$		$4A + 12B + 12C \geq 15$
	$14X + 12Y \leq 84$		$A \geq 0, B \geq 0, C \geq 0$
	$X \geq 0, Y \geq 0.$		

Students solve this dual problem themselves like the solution for the food problem.

13.2 Minimisation of Cost- Solution of the Food Problem

Food problem was the first economic problem whose solution through linear programming was done through cost equation. Consider that consumer buys bread and butter at market price. Problem is that from various quantities of both goods cost of receiving their net material is made minimum.

Note

Table 13.2: Data of Food problem

Nutrition element	Nutrient Material per unit Bread X_1	Butter X_2	Minimum ideal
Calorie (1000)	1	2	3
Protein (25 g)	2	8	8
Cost (₹ Per unit)	2	6	(?)

Consider that X_1 and X_2 express Bread and butter respectively of which amount of Calories and grams of proteins is given in table 13.2. Nutrition material of bread per half kilogram is 1000 calorie and quantity of protein is 50 g and of butter is 2000 calories and 200 g protein is per half kilogram. In ideal diet, per day 3000 calories and 200 g protein is needed. Market price of 500 g bread is ₹ 2 and price of per 500 g butter is ₹ 6.

Problem is that according to minimum diet ideal given in the last column of the above table what will be the best diet and what will be the minimum cost expressed by (?).

Total cost of the food

Minimise

Subject to

$$\left.\begin{aligned} C &= 2x_1 = 6x_2 \\ x_1 + 2x_2 &\geq 3 \\ 2x_1 + 8x_2 &\geq 8 \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned}\right\} \quad ...(1)$$

And the cost being minimised is C, which is the linear function of both variables x_1 and x_2. Side relations 3 and 8 are inequalities which express the minimum ideal food diet obtained by the given food. Problem is linear because despite of linear inequalities, non- variable are to be minimised. Out of the three, solution to any two situations may be obtained. For e.g. with one side relation remaining, cost C can be minimised: x_1+ $2x_2$= 3 on solving it, x_1=3 and x_2=3/2=1.5. In figure 13.2, it has been expressed through line *AB* where *OA*= 1.5x_2 and *OB* =3x_1

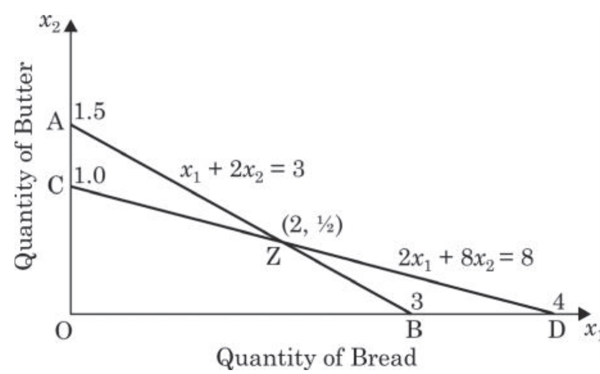

Figure 13.2

Second side relation is $2x_1 + 8x_2 = 8$ and on solving it, $x_1 = 4$ and $x_2 = 1$are obtained. It has been drawn through line *CD* in figure 13.2, which satisfies this equation where $OC = 1x_2\, OD = 4x_1$.

Hence in the figure, x_1, (Bread) has been measured on horizontal axis and x_2, (butter) has been measured on vertical axis. Line AB expresses equation $x_1 + 2x_2 = 3$

And line CD expresses equation $2x_1 + 8x_2 = 8$. Feasible solution will be on thick line AZD or above it. In our problem, it happen on point Z where both lines AB and CD intersect.

$$x_1 + 2x_2 = 3 \quad ...(1)$$

$$2x_1 + 8x_2 = 8 \quad ...(2)$$

To find out that feasible solution is on Z itself or on points A or B, we will solve both equations of the problem in form of simultaneous equation:

Example: Assumed that a producer wants to maximise its revenue under given constraints. Considered that a firm wants to produce two products X_1 andX_2 and for it three resources a, b, and c of the following type are given:

$$a = 40, b = 50, c = 42$$

We also assumed that for producing one unit of X_1 it will need each resource of following type-

$$a = 4, b = 10, c = 6$$

Similarly we also assumed that for producing unit of X_2 it will need each resource in following way:

$$a = 10, b = 5, c = 7$$

We also assumed that per unit cost of X_1 and X_2 is ₹ 5 and ₹ 7 respectively. In this way that producer will like to maximise his total revenue from the given resources. Mathematically,

$$\text{Max } TR = 5X_1 + 7X_2 = Z$$

Constraints on the firm have been given and firm will not use more than those resources.

$$4X_1 + 10X_2 \leq 40 \qquad ...(i)$$

$$10X_1 + 5X_1 \leq 50 \qquad ...(ii)$$

$$6X_1 + 7X_2 \leq 42 \qquad ...(iii)$$

Here, $X_1, X_2 \leq 0$

These constraints of the firm may be shown in the following manner:

First constraint tells this that resource 'a' which has been brought in production of X_1 and X_2; it cannot be more than 'total supply' of resource 'a'. This resource can be less than or equal to supply, not more. Similar will be in relation to other constraints. Here we will show a linear object with the help of graph:

First we will take out the coordinates of X_1 and X_2, for which we will remove inequalities of constraint equations and in this way we will obtain coordinates of all the three techniques. These techniques will express feasible solution of object problem. In the figure, on axis *X*, product X_1 and on *Y* axis Product X_2 has been measured. Here with the help of coordinates of constraints adopted by the firm has been marked in form of an easy line through graph. Z_1 is objective function.

Considered that producer brings a and b constraints in use then area *OMAG* will be obtained as feasible area for the firm. If firm use 'a' and 'c' constraints then the possible area will be OATK. Similarly if firm brings constraints b and c in use then area *ONLG* will be obtained.

If he takes all the three constraints a, b and c together in use, then he will obtain feasible area equal to OAMG. Feasible solution of firm will be under this area only.

If producer will want to search the feasible solution of product X_1 and X_2 in area *OAMG* (which has been shown by shaded area in the figure), it will not be correct because he must make maximum use of each technique. Hence feasible solution area will be on the boundary of area *OAMG*.

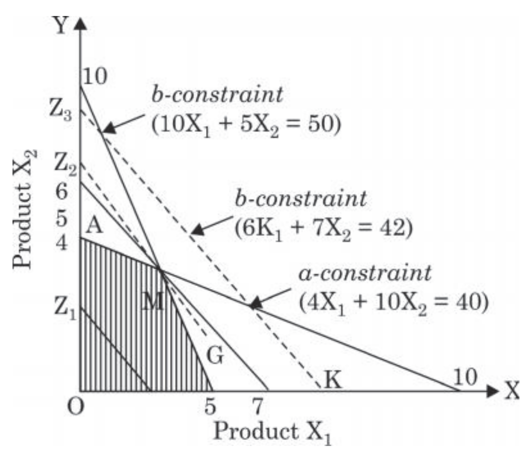

Figure 13.3

Now the question arises that, if he takes the objective function at the boundary of feasible area then for obtaining **optimum solution** of X_1 and X_2 firm should get only one point which will give only one solution of X_1 and X_2. For this, it will have to search corner solution. Hence in such situation, firm will change the objective function Z_1 equally. That point of the corner of feasible solution which the feasible solution touches, it will be an optimum solution of X_1 and X_2. Here in figure 29.3, at point *M*, optimum solution of X_1 and X_2 will be obtained.

Note

Self Assessment

1. Multiple Choice Questions:

1. In which field linear programming has proved very useful?
   - (a) In Economics
   - (b) In Science
   - (c) In Mathematics
   - (d) In Politics
2. Real problems cannot be solved through which technique due to many constraints?
   - (a) Differentiation
   - (b) Linear Programming
   - (c) Integration
   - (d) None of these
3. Which problem was the first economic problem whose solution through linear programming was done through cost equation?
   - (a) Money
   - (b) Residence
   - (c) Food
   - (d) Water

13.3 Summary

- Till Neo-classic theory of the firm in time, taking one or two variable did the analysis of the problem of decision making.
- Food problem was the first economic problem whose solution through linear programming was done through cost equation.
- Linear programming proved to a very profitable resource in economics, but it had its own limitations. In reality, because of many constraints real problems cannot be solved through linear programming technique.

13.4 Keywords

- ***Optimise:*** Required
- ***Primal:*** Main

13.5 Review Questions

1. Interpret the minimisation of cost.
2. Write down the limitations of linear Programming.
3. Present the minimisation of cost: Solution of the food problem.

Answers: Self Assessment

1. (a) 2. (b) 3. (c)

13.6 Further Readings

Books

Mathematics for Economist – Yamane- Prentice Hall, India.

Mathematical Economics – Micheal Harrison, Patrick Waldaron.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Simone and Bloom- Viva Publication.

Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Unit 14: Simplex Method

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand clearly Simplex Method.
- Understand the Steps of Method.

Introduction

Because of more numbers of inequations in linear programming method, Graphical method becomes more complicated. This method can be used for two, three or more inequations. Therefore because of large number of equations another mathematical method is used for a paired equation which is called Simplex Method.

14.1 Simplex Method

Suppose following is an Objective function which is to be maximized (or minimized)-

$$Z = c_1x_1 + c_2x_2 + \dots\dots\dots + c_nx_n \qquad \text{...(i)}$$

Here the values of all constants (C_i) are known.

Here we also supposed that m is a linear inequation in which there are n variables in each equation.

Thus Constraints are the following -

$$a_{i1} + a_{i2} + \dots a_{in}x_n \{\le = \ge\}\ b_i,\ i = 1, 2 \dots\dots\dots, m \qquad (14.1)$$

Notes Here one and only one sign (≤ = ≥) will be for each constraint. Value of variable will always be positive, i.e. $x_i \ge 0$, $j = 1, 2 \dots n$.

Each group of x_j which satisfies constraints will be called a solution. Any solution which satisfies Non-negativity restrictions will be called a feasible solution. In the same way any feasible solution which minimizes or Maximizes Objective Function Z will be called Optimal Feasible Solution.

Here we will try to find out the Optimal Feasible Solution of an Objective function within the given constraints. To find the Optimal Feasible Solution firstly we will find all the feasible Solutions, after

Note

that we will find one feasible solution which satisfies maximum (or minimum) of Objective function, that will be our Optimal Feasible Solution.

Like the number of variables will increase the same way number of Feasible solutions will also increase. If there are 6 variables in our problem then number of feasible solutions will be $|\underline{6}||\underline{3}||\underline{3} = 20$, where there were 3 yugmpad equations and three unknown values.

> *Task* What is Simplex Method?

Equation (i) can be rewritten as –

Objective Function $\sum_{j=1}^{n} c_j x_j$ (Maximum or Minimum)

Within the following constraints

$$\sum_{j=1}^{n} a_i x_j = b_j \qquad \text{...(14.2)}$$

$i = 1, 2 \text{........} m$

$j = 1, 2 \text{......} n$ (Making equations same here)

$j = 1$

Knowledge of Basic Theorems is necessary related to the following problems –

1. If the possible solution of problems of Linear programming is a Vector $X = [x_1, x_2, \text{....} x_n]$, which satisfies equation 14.1. Vector a_j associated with x_j can be defined as -

   $$a_j = [a_{1j} a_{2j} \text{........} a_{mj}]$$

2. Quantities $b_1\ b_2 \text{.......} b_m$ are the elements of a column vector which is called requirement vector. Here

   $$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ and } b \geq 0, \text{ where 0 is a zero vector.}$$

3. Coefficients $C_1\ C_2 \text{.........} C_n$ under Objective function are called the Prices Associated of variables $x_1, x_2 \text{......} x_n$ and the vector formed from them is called Price Vector represented by C. Thus

   $$C = [c_1, c_2, \text{......} c_n]$$

4. Set of Values of $x_1, x_2 \text{......} x_n$ which satisfies **Equation** 14.1 and the Non-negativity condition, is called feasible Solution.

5. That Feasible Solution, which optimizes Objective function (14.1), is called Optimal Solution. That is if feasible solution minimizes Objective function then it is called Minimum Feasible Solution and if it maximizes Objective function then it is called Maximum Feasible solution.

Note

6. If feasible solution which does not have positive more then m, then it is called Basic feasible Solution. Therefore if feasible solution is to be converted into Basic feasible Solution then it is necessary to vanish at least ($n - m$) variables.
7. In the Basic feasible Solution if m is positive then it is called Non-degenerate Basic feasible Solution. Non-Zero variables are called Basic Variables. When at least one Basic Feasible Solution vanishes then Basic feasible Solution is called degenerate Basic feasible Solution.
8. Solution which does not satisfies constraints Equation (30.1) and the Non-negativity condition is called Non-feasible solution.
9. If Equation has the sign (≤) then the variables used to make changes in its balanced equations, are called Slack-Variable.

Did u know? Those variables which are used to change dissimilation into equation, are called a Surplus Variables. These both types of variables in the combined form is called a Dummy variable. Whose numbering is equal to the number of variables in the Objective Function.

14.2 Calculating Steps of Method

1. Slack Variables should be used according to the need for changing dissimilation into equations.
2. If needed Artificial variables should be included. After that write constraints as $AX = b$, where $b \geq 0$. If any b is negative then make it positive by multiplying its equation by (– 1).
3. Calculate X_j by solving Initial Basic Feasible Solution and the value of $Z_j - C_j$ should be calculated for every column of A.
4. For Authorized Equation every $Z_j - C_j \geq 0$, then this Iteration is the Optimum Solution. Again if $Z_j - C_j > 0$ (for every non-based Variables) then proper solution is Unique otherwise optional Solution can exists.
5. Entering Vector and departing Vector should be selected.
6. Artificial Vector should be separated.

Example 1: Solve the following by Simplex Method –

Of which $$Z = 2X_1 - 3X_2 + 7X_2 \quad ...(1)$$

Minimum $$3X_1 - 4X_2 - 6X_3 \leq 2 \quad ...(2)$$

$$2X_1 - X_2 - 2X_3 \geq 11 \quad ...(3)$$

$$X_1 - 3X_2 - 3X_3 \leq 5 \quad ...(4)$$

So that $X_1, X_2, X_3 \geq 0$

Solution: First of all Slack and Surplus Variables will be used according to the need for changing dissimilation into equations. Since there are 3 Equation equations, therefore 3 Slack variables and 3 Surplus Variables will be used. Therefore including X_4, X_5 and X_6 given problem can be represented as the following –

$$2X_1 - 3X_2 + 6X_3 + 0X_4 + 0X_5 + 0X_6 \quad ...(5)$$

When equations are the following –

$$3X_1 - 4X_2 - 6X_3 + X_4 + 0X_5 + 0X_6 = 2 \quad ...(6)$$

$$2X_1 - X_2 + 2X_3 + 0X_4 - X_5 + 0X_6 = 11 \quad ...(7)$$

$$X_1 + 3X_2 - 2X_4 + 0X_4 + 0X_5 + X_6 = 5 \quad ...(8)$$

Note

For finding the feasible solution data is represented in the following Matrix form –

$$AX = b$$

Here $A = [a_{ij}] = [P_{ij}]$

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_6 \end{bmatrix} \text{ and } b = P_0 = \begin{bmatrix} 2 \\ 11 \\ 5 \end{bmatrix}$$

Or

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ 3 & -4 & -6 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & -1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_6 \end{bmatrix}$$

Since the Basis Matrix must be a Unit Matrix but it is not becoming possible from the known facts. Therefore the inclusion of a New Vector P_7 is necessary. Therefore Primary basis is P_4, P_7 and P_6. Therefore writing the following table –

Table I

(Basis Vector)	**↓C_j**	**C_j→ P_0**	**2 P_1**	**–3 P_2**	**6 P_3**	**0 P_4**	**0 P_5**	**0 P_6**	**0 P_7**
P_4	0	2	3	–4	–6	1	0	0	0
P_7	0	11	2	–1	2	0	–1	0	1
P_6	0	5	1	3	–2	0	0	1	0
Z_j		0	0	0	0	0	0	0	0
$Z_j - C_j$			–2	+3	–6	0	0	0	0

Here the value of Objective Function

$$Z_j = Z_o = 0$$

But it is not minimized, since $Z_j - C_j > 0$ in context to vector P_2. Therefore we will use θ trick to minimize P_2.

$$P_o = 2P_4 + 11P_7 + 5p_6 \quad \text{...(i)}$$

$$P_2 = -4P_4 + P_7 - 3P_6 \quad \text{...(ii)}$$

Multiplying eq. (ii) by θ

$$\theta P_2 = -4\theta\, P_4 + 5P_7 = 3\theta\, P_6 \quad \text{...(iii)}$$

Subtracting eq. (iii) from (i)

$$P_o = \theta P_2 + (2 - 4\theta)\, P_4 + (11 - \theta)\, P_7 + (5 - 3\theta)\, P_6 \quad \text{...(iv)}$$

From the three values of θ, $\frac{1}{2}$, 11 and $\frac{5}{3}$, $\theta = \frac{1}{2}$ is the related value. Therefore putting $\theta = \frac{1}{2}$ in eq. (iv)

Note

$$P_0 = \frac{1}{2}P_2 + 0P_4 + \frac{21}{2}P_7 + \frac{7}{2}P_6$$

$$= \frac{1}{2}P_2 = \frac{21}{2}P_7 + \frac{7}{2}P_6 \qquad \text{...(v)}$$

Equation (v) becomes new basis 11 in which P_2, P_7 and P_6 are included. Therefore putting P_2 in place of P_4 can be written in the following table form.

Here the value of Objective function $Z_j = Z_o = -\frac{3}{2}$

This is the desired value since $Z_j - C_j < 0$

$j = 1, 2, ... 7$

Table II

(Basis Vector)	↓C_j	C_j→	2	–3	6	0	0	0	0
		P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_2	–3	$\frac{1}{2}$	$\frac{3}{8}$	1	$\frac{6}{3}$	0	0	0	0
P_7	0	$\frac{21}{3}$	5	0	8	0	–3	0	1
P_6	0	$\frac{7}{2}$	$\frac{13}{12}$	0	$-\frac{1}{6}$	$\frac{1}{4}$	0	1	0
Z_j		$-\frac{3}{2}$	$\frac{9}{8}$	–3	$-\frac{9}{4}$	0	0	0	0
$Z_j - C_j$			$-\frac{7}{8}$	0	$-\frac{33}{4}$	0	0	0	0

Example 2: Solve the following Linear Programming equation –

Maximum $Z = 5X_1 + 4X_2$

Which $X_1 + 2X_2 \leq 8000$

$3X_1 + 2X_2 \leq 9000$

From which $X_1, X_2 > 0$

Solution: First of all dissimilarities of equations will be removed by the inclusion of Slack Variables. Now we can show the constraints in the following way –

$$X_1 + 2X_2 + 0X_3 = 8000$$

$$3X_2 + 2X_2 + 0X_4 = 9000$$

Note

Now on the basis of above facts following table will be made –

Table III

			Cost	5	4	0	0	
C_β	Solution	X_β	β	Y_1	Y_2	Y_3	Y_4	$\theta = \frac{\text{Sol.}}{Y_k}$
0	8000	X_3	Y_3	1	2	1	0	8000
0	9000	X_4	Y_4	3	2	1	0	$\frac{9000}{3} = 3000$
$\sum Z_j - C_j = \sum C_\beta (Y_j - C_j)$				–5	–4	0	0	

Here firstly we will include row Y_k. For this will calculate the value of 0. Value of q can be calculated by division of the value of Y_k.

$$\theta = \frac{8000}{1} = 8000$$

And
$$0 = \frac{9000}{3} = 3000$$

After that the row will include the minimum value. Here l th row (second row) will be removed by k th (i.e. 3) vector solution.

In second table, first of all we have to divide kth row with the pivot of second row of first table. Multiply new row of second table with rest rows of table. Then subtract first row from ith row. You will get the result of ith row. Repeat the all steps till you get the value for $Z_j - Cj$

Table IV

			Cost	5	4	0	0	
C_β	Solution	X_β	β	Y_1	Y_2	Y_3	Y_4	θ
0	00	X_3	Y_3	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	$\frac{15000}{4}$
0	3000	X_1	Y_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{9000}{2}$
	$Z_j - C_j$			$5 - 5$ $= 0$	$\frac{10}{3} - 4$ $= -\frac{2}{3}$	0	$\frac{5}{3}$	

For first row

8000	1	2	1	0
–3000	1	$\frac{2}{3}$	0	$\frac{1}{2}$
5000	0	$\frac{3}{4}-\frac{1}{3}$ on Subtracting		

This method will be repeated until then value of $Z_j - C_j$ comes negative. Since β is minimum in the first row and $Z_j - C_j$; is negative for Y_2. Therefore first row will be deleted from the table. Here the value of Pivot is 4/3 and when we repeat the method –

Table V

			Cost	**5**	**4**	**0**	**0**	
C_β	Solution	X_β	β	Y_1	Y_2	Y_3	Y_4	θ
4	$\frac{1500}{4}$	X_2	Y_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	
5		X_1	1	0	1	$-\frac{1}{2}$	$\frac{1}{6}$	
		$Z_j - C_j$		0	0	$\frac{1}{2}$	$\frac{1}{6}$	

For second row

3000	1	$\frac{2}{3}$	0	$\frac{1}{3}$
2500	1	0	$+\frac{1}{2}$	$-\frac{1}{6}$
500	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$

on Subtracting

Since here the value of $Z_j - C_j$ is positive. Column $X_2 = \frac{15000}{4}$ of solution and $X_1 = 500$ provides maximum value.

Now the maximum value of 2 will be found –

$$2 = 5X_1 + 4X_2 = (5 \times 500) + \frac{4 \times 15000}{5}$$

$$= 2500 + 15000 = 17500$$

Example 3: Maximize the profit. $Z = 4X + 3Y$.

Of which $x + \frac{7}{2}y \le 9$

$$2x + y \le 8$$

$$x + y \le 6$$

Note

Like $x \overline{<} 0, y \leq 0$

Solution: First of all Slack Variables will be used to balance the equation. Slack Variables are s_1, s_2, s_3 etc. Multiply these variables by zero, these are added just to balance the equations. Like

$$Z = 4x + 3y + os_1 = os_2 + os_3 = R$$

Primary table by Simplex Method

Table VI

$C_j \rightarrow$ ↓			4	3	0	0	0	
	Profit	**Qty**	**x**	**y**	**s_1**	**s_2**	**s_3**	**Ratio**
0	s_1	9	1	$\frac{7}{2}$	1	0	0	9
0	s_2	8	2	1	0	1	0	←4
0	s_3	6	1	1	0	0	1	6
	Z_j		0	0	0	0	0	
	$C_j - Z_j$		4	3	0	0	0	

Key Column ↑ Here Key Factor = 4

Note: Key factor = 4 since the value of this ratio is least.

Subject to $x + \frac{7}{2} y + s_1 + os_2 + os_3 = 9$

$$2x + y + os_1 + s_2 + os_3 = 8$$

$$x + y + os_1 + os_2 + s_3 = 6$$

Table VII

$C_j \rightarrow$ ↓			**4**	**3**	**0**	**0**	**0**	
	Profit	**Qty**	x	y	s_1	s_2	s_3	**Ratio**
4	x	4	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	8
0	s_1	5	0	3	1	$-\frac{1}{2}$	0	$\frac{5}{3}$
0	s_3	2	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	4
	Z_j		16	4	2	0	2	
	$C_j - Z_j$			0	1	0	–2	

Now the value of $C_j - Z_j$ is either Positive, zero or less. We will continue to make new table until the value of $C_j - Z_j$ becomes negative or zero.

Table VIII

	$C_j \rightarrow$ $\downarrow$		*4*	*3*	*0*	*0*	*0*	
	Profit	**Qty**	x	y	s_1	s_2	s_3	**Ratio**
3	y	$\frac{5}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{6}$	0	
4	x	$\frac{19}{6}$	1	0	$-\frac{1}{6}$	$\frac{7}{12}$	0	
0	s_3	$\frac{7}{6}$	0	0	$-\frac{1}{6}$	$\frac{5}{12}$	0	
	Z_j	$\frac{53}{3}$	4	3	$\frac{1}{3}$	$\frac{11}{6}$	0	
	$C_j - Z_j$		0	0	$-\frac{1}{3}$	$\frac{11}{6}$	0	

In the above table all values of $C_j - Z_j$ are either Negative or equal to Zero.

$x = 19/6$

$y = 5/3$

Therefore $s_3 = 7/6$

Working Notes
Table IX

	Qty	x	y	s_1	s_2	s_3
O.V.s_1	9	1	$\frac{7}{2}$	1	0	0
N.T. × 1	–4	–1	$-\frac{1}{2}$	–0	$-\frac{1}{2}$	–0
	5	0	3	1	$-\frac{1}{2}$	0
D.V.s_3	6	1	1	0	0	1
N.T. × 1	–4	–1	$-\frac{1}{2}$	–0	$-\frac{1}{2}$	–0
	2	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1

Note

Table X

O.V.	4	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
N.T. × $\frac{1}{2}$	$+\frac{5}{6}$	+0	$+\frac{1}{2}$	$+\frac{1}{6}$	$-\frac{1}{12}$	0
	–1	–1	–1	–1	+1	+1
	19/6	1	0	–1/6	7/12	0
O.V.s_3	2	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1
N.T. × $\frac{1}{2}$	+5/6	+0	+1/2	+1/6	–1/12	–0
	7/6	–0	0	–1/6	5/12	1
	5	0	3	1	$-\frac{1}{2}$	0
	38/3	4	0	–2/3	7/3	0
Z_j	53/3	4	3	1/2	+11/6	0

Note: O.V. = Original value,

N.T. = New table value.

Self Assessment

1. Fill in the blanks:

1. Because of more numbers of inequation in linear programming .... method becomes more complicated.
2. Any Solution which satisfies Non-Negativity constraints is called .... solution.
3. Any feasible solution which minimizes or Maximizes Objective Function will be called a .... Feasible Solution.
4. .... and Surplus Variables should beused according to the need for changing dissimilation into equations.
5. Solution which does not satisfies Non–negativity condition is called .... solution.

14.3 Summary

Because of more numbers of inequations in linear programming method, Graphical method becomes more complicated. This method can be used for two, three or more inequations. Therefore, because of large number of equations another mathematical method is used for Yugmpad equations which is called Simplex Method.

Note

14.4 Keywords

- ***Linear :*** Lined, in a line
- ***Method :*** Process

14.5 Review Questions

1. What do you understand by Feasible and Optimal Feasible Solution?
2. Write the different calculating steps of method.
3. Solve the following Linear Programming problem:

   Maximum $z = 5X_1 + 4X_2$

   Subject $X_1 + 2X_2 \leq 8000$

   $3X_1 + 2X_2 \leq 9000$

   Thus $X_1, X_2 > 0$ (**Ans.:** Maximum value: 17500)

Answers: Self Assessment

1.	Graphical	2.	Feasible	3.	Optimal
4.	Slack	5.	Non-feasible		

14.6 Further Readings

Books

Mathematical Economics – Michael Harrison, Patric Walderen.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publications.