MATHEMATICAL ECONOMICS

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SYLLABUS

Mathematical Economics

Objectives
- To aware of students the mathematical aspects of Economics.
- To introduce the concept of interrelation and inter dependency of mathematical Economics.
- To increase understanding of the application of the mathematical properties of Economics.

S.No.	Topics
1.	Type of Functions: Constant Function, Polynomial Functions, Rational Functions, Non-algebraic Function, Exponential Function, Log Function
	Limits and Continuity
2.	**Differentiation:** Simple
	Logarithmic Differentiation
	Second and Higher Order Differentiation
3.	**Differentiation:** Partial
	Homogeneous Function and Euler's Theorem
	Economic Applications of Differentiation
4.	Maxima and Minima of One Variable
	Maxima and Minima of Two Variables
	Constrained Maxima and Minima
5.	Economic Applications of Maxima and Minima
	Integration: Basic Rules of Integration
	Methods of Integration
	Integration as a Summation
	Definite Integration

CONTENT

Unit 1: Functions

Objectives

After reading this unit, students will be able to :

- Know the Functions and Related Quantities.
- Determine the Value of Functions.
- Know the Definition of Functions by Mapping.
- Determine the Domain and Range of Functions.
- Understand the use of Linear Functions in Economics.

Introduction

In many of the questions, we have to determine the effect of increase-decrease of an independent number on a number dependent on it. For example, area of circle always depends on the radius of it, because if radius is increased or decreased, then area of the circle will also decrease or increase according to the radius. Here radius of the circle is a number and area of the circle is another number, which are related to each other. Thus volume of a cylinder depends on its radius, area of square and volume of a cuboid depend on the length of its arm. The distance covered by a running train in dynamic velocity depends on the time taken. Velocity of a falling particle depends on the distance covered by it. Atmospheric pressure of a certain place depends on the alleviation of its height from sea-coast etc.

Note   With the change of value of one number, the rate of change in the value of other number and questions related to rate of such change and analysis and study of functions is referred to as Differential Calculus.

1.1   Quantities

There are two types of quantities:

1. Variable   2. Constant

1. **Variables –** Changing quantities are referred to as variables. Quantities for which values keeps on changing viz which can be given indefinite numerical values are Variables. These are general expressed with the last characters such as x, y, z, u, v, w etc. of English Words.

2. **Constants –** Numbers for which value is unchangeable under any process of mathematics, are constants. Constant quantities are of two types:

(i) Absolute constants

(ii) Arbitrary constants

Value of which in any problem is unchanged, is referred to as absolute constants *e.g.*

$5, -3, 1, 1, \sqrt{5}, \pi, \dfrac{2}{5}, e$ etc.

Value of which remains constant in a problem, but get different value in different problems, they are called arbitrary constants. These are expressed with the beginning letters such as a, b, c, d, etc. of English word.

1.2   Related Quantities

We know that the area of the circle depends on its radius. In other words, circles with different radii have different areas. Similarly, square of any positive number increases or decreases with any change in it. Here radius of circle and its area or number and its square are related quantities.

> *Notes*   Any two numbers, in which change in any one number affects another number, are referred to as related quantities.

Although both these two numbers are constants, but value of any one number can be changed liberally and the value of other number will change not independently but applying any rule. For example if we assume liberally 1, 2, 3, 4 …….. for the radius of the circle (r), then its area (A) applying the rule $A = \pi r^2$ would be π, 4π, 9π, 16π,..... respectively.

There are two types of constants:

(i) Independent Variables,   (ii) Dependent variables

If two constants x and y are related in a manner where one variable can be given any value liberally and the value of y depends on it, then x will be referred to as independent variables and y would be its dependent variables.

For example: Assume $y = 2x + 5$

Now giving x variable different values like 0, 1, 2, ….. etc. we get different results for y variable such 5, 7, 9, …… etc. respectively, which is completely depended on x.

Therefore, here x is independent variable and y is a dependent one.

1.3   Functions

The principle of functions is mutually based on the related quantities.

Assume area of a circle of any radius (r) is (A), then $A = \pi r^2$

Here r is a an independent variable and A is a dependent variable, since the area (A) of the circle is based on radius (r)

In this situation we say that variable (A) is function of (r), which in the language of mathematics is expressed in the following manner: $A = f(r)$

Thus, [A is a function of variable r]

1.4   Definition of Functions

Here two variables x (independent) and y (dependent) are related with the function (f) in such a way that for each value of x, a certain and unique value of y can be obtained, then y would be the f-function of x, which can be expressed as $y = f(x)$.

The functions of x are expressed with the symbols $f(x)$, $g(x)$, $\phi(x)$.....etc.

1.5   Explanation of Functions

If y is a function of x variable then $y = f(x)$.

Then x would be an independent variable and y is a dependent varialble.

If $y = 5x + 7$, then for each value of x, there is a certain value of y. therefore, y is said to be the function of x.

1.6   Functional Notations

Many symbols are used to express this function where y variable is the function of x variable. The symbol mostly used is

$y = f(x)$.

Which is read as "y is equal to function of x"

Other symbols are: $y = F(x), y = \phi(x), y = \psi(x), ...$, etc.

Did u know?   If for any value of x variable, there are more than one value, then y is not called the function of x, but the relation. For e.g. $y^2 = x$, here for each value of x, two values are obtained, such type of relations are sometimes also called multi-valued function.

1.7   Value of Functions

Assume any function of variable x is $y = f(x)$

Putting $x = a$ in $f(x)$ we get $f(a)$ which would be the value of function for $x = a$.

> **Value of $f(x)$ on $x = a$ would be $f(a)$**
>
> **Which is obtained replacing x with a in $f(x)$**

For *e.g.* assume $f(x) = 2x^2 - 3x + 5$

Note

Then $\qquad\qquad f(2) = 2(2)^2 - 3.(2) + 5,$ [putting 2 in place of x]

$$= 8 - 6 + 5 = 7$$

Self Assessment

1. Fill in the blanks:

(i) Changing numbers are called……………

(ii) Numbers, which remain same under any mathematical formula, are called……………

(iii) Any two numbers in which changing of one number affect second number are called ……………. numbers

(iv) The area of circle is depended on its ……………

(v) Value of the numbers which remains constant, is called………………

1.8 Definition of Functions by Mapping

If correspondence of each element (x) of Non-empty set X is obtained by any rule (f) for a certain and unique element (y) of Non-empty set, then f is called Mapping of non-empty set (x) of non-empty set (Y). Mapping shows the geometrical aspect of association of the elements of non-empty set X and Y, and the function shows its analytical aspect.

In other words,

> If X and Y are two non-empty sets and by any rule (f) element of X is associated with the a unique element of Y, then binary operation R is called the function of X and Y.

Thus, $R = \{(x, y): x \in X, \ y \in Y\}$ is the functions of X and Y.

For example: if $X = \{1, 2, 3\}$, $Y = \{3, 4, 5\}$, then with the function $y = x + 2$, element 1, 2, 3 of X will be associated with element 3, 4, 5 of Y. thus, binary operation would be the functions of X and Y and it can be written as:

$$f: X \rightarrow Y \text{ or } X \xrightarrow{\ f\ } Y$$

This way f is a rule by which element of X is associated with a unique element of Y.

1.9 Domain and Range of Functions

If function $f: X \rightarrow Y$ is defined, then non-empty set X is called domain of this mapping or function (f) and non-element set (Y) would be its range or co-domain.

Domain of function (f) = Element of actual values of X for which function is defined.

Range of function (f) = values of elements of Y corresponding to domain of all numbers x.

For Example: in $y = 2x + 1$, y is the function of x. If x holds positive bigger value than 1 and smaller than 5, domain element of the function would be $\{2, 3, 4\}$ and the range value would be $\{5, 7, 9\}$. Similarly domain of binary $R = \{(5, 8), (6, 9), (7, 10)\}$ operation will be $\{5, 6, 7\}$ and range value would be $\{8, 9, 10\}$

(Domain) (Range)
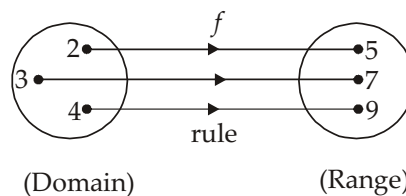
If domain of element of function x is associated with the element of y, then element (y) of this domain which is associated with x, is expressed with $f(x)$, where $f(x)$ expresses the value of function (f), which also can be written as under:

$$f : x \to y \ f : x \longrightarrow f(x)$$

1.10   Kinds of Functions

1. **Functions of single variable –** if variable y is only depended on variable x, then $y = f(x)$ is called functions of single variable. Similarly $y = f(\theta), s = f(t), v = f(t)$ etc. are functions of single variable.

2. **Functions of Many variable –** if variable u is depended on variable x and y, then $u = f(x, y)$ is called functions of many variable.

3. **Explicit Function –** function $y = f(x)$ is called explicit function, if it is possible to express y as independent element of x. for *e.g.* etc.

    $$y = x^2 + 2x - 5, \quad y = \cos x, \ y = \frac{x}{1+x^2}$$

4. **Implicit Function –** function $y = f(x)$ is called implicit function, if it is not possible to express y as independent element of x.

    For example: $y = x \sin(x + y)$, and $f(x, y) = 0, x^2 + y^2 - xy = 0$

5. **Even Function –** assume $y = f(x)$ is function of y.
    If $f(-x) = f(x)$

    Viz. if in place of x, putting $–x$ in $f(x)$, there is no change in sign of function, it is called even function.

    For example: $f(x) = \cos x = \cos(-x) = f(-x)$

6. **Odd Function –** assume $y = f(x)$ is function of x.

    For example: $f(x) = x^3 = -(x)^3 = -f(-x)$

    If $f(-x) = -f(x)$

    Viz. if in place of x, putting $–x$ in $f(x)$, the sign of function changed then it is called odd function.

7. **Algebraic Function –** Function with different exponential value of x variable, in which factors are certain is called algebraic function.

    For example: $f(x) = x^3 + 5x^2 + 2x$

    $f(x) = x^{3/2} + x^{-1/3}$

    $f(x) = (x + a)^{2/3}$

8. **Rational Function -** Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called rational function.

    $$f(x) = \frac{2x^3 + x + 7}{x^3 + 5x^2 + x + 5}$$

9. **Transcendental Function –** these are of following types:

    (i)   Trigonometrical Functions such as $\sin x$, $\cos x$, $\sec x$, $\sin 2x$, $\sec^2 x$ etc.

    (ii)   Inverse Circular Functions such as $\sin^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$ etc.

    (iii) Logarithmic Functions such as $\log_e x$, $\log_a x$, $\log_e (x^2 + 4x + 3)$ etc.

    (iv)   Exponential Functions such as $e^x, a^x, x^{\sin x}$, $(\sin x)^{\cos x}$ etc.

Note

10. **Defined Function On** – $x = a$, $f(x)$ is a defined function if putting $x = a$ in $f(x)$, the value of function $f(a)$ is equal to a certain and finite value, which is completely meaningful and real.

11. **Undefined function On** – $x = a\, f(x)$ is called undefined function if the value $f(a)$ is uncertain and meaningless for example

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty \times \infty, \infty - \infty, 0 \times \infty, 0^0, \infty^0, 1^\infty$$

For example

(i) $\quad y = \frac{\sin x}{x}$; then $x = 0, \left(\frac{0}{0}\right)$

(ii) $\quad y = \frac{x^2 + 3x + 4}{x^2 - 4}$; then $x = \infty \left(\frac{\infty}{\infty}\right)$

(iii) $\quad y = x^{\frac{1}{1-x}}$; then $x = 1, (1^\infty)$

(iv) $\quad y = \frac{1}{x} - \frac{1}{\sin x}$; then $x = 0, (\infty - \infty)$ etc.

12. **Constant Function** – $y = f(x) = c$ where c is a constant function shows a constant function. The value of y or $f(x)$ always remains constant for the value of x under constant function.

13. **Periodic Function** – Function $y = f(x)$ is called periodic function. If for each value of x $f(x + k) = f(x)$,

    Where k is an actual number which is not equal to 0 viz k is the period of the function.

14. **Function of the Function** – Assume $f(x)$ and $g(x)$ are the functions of variable x, then $f[g(x)]$ is the function of function which is obtained putting $g(x)$ in place of x in the function $f(x)$.

    Thus, $g[f(x)]$ is also the function of function which is obtained putting $f(x)$ in place of x in $g(x)$ Viz. if in place of x, putting $-x$ in $f(x)$, the sign of function changed then it is called odd function.

1.11 Operations in the Set of Functions

Assume S is the actual function of the sets with D domain. Then if f and g are the function of non-empty set S, then $f + g, f - g, fg, \dfrac{f}{g}$ will have the following meaning:

(i) $\quad (f + g)(x) = f(x) = g(x), \forall x \in D$

(ii) $\quad (f - g)(x) = f(x) = -g(x), \forall x \in D$

(iii) $\quad (fg)(x) = f(x) \cdot g(x), \forall x \in D$

    (it is clear that $f + g, f - g$ and fg have the same domain.

(iv) $\quad \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} \forall x \in D \sim S$

Where S is the solution of equation $g(x) = 0$. In other words S is a set where $x \in S \Rightarrow g(x) = 0$ or $S = \{x : g(x) = 0\}$

It is clear that $D \sim S$ is the domain of f/g viz those elements (x) are removed from D for which $g(x) = 0$.

For example: assume $f(x) = \dfrac{1}{x}$ and $\phi(x) = x^2$, are two function, where the algebraic total of $f(x)$ and

$\phi(x)$ is expressed as $f(x) + \phi(x)$ or $(f + \phi)(x)$ and would be $(f + \phi)(x) = \dfrac{1}{x} + x^2$

Further the multiplications of functions will be expressed as $f(x) \cdot \phi(x)$ and $f(x) \cdot \phi(x) = \dfrac{1}{2} . x^2 = x$

$f\{(\phi(x))\}$, describes the function of functions of x, where f, $\phi(x)$ is the function and again $\phi(x)$ is the function of x.

Examples with Solution

Example 1: If followings are functions?

(i) $y = \sqrt{x}$,   (ii) $y = x^3 + 5x$,   (iii) $R = \{(2,3), (5,6), (2, 11)\}$

(i) If any positive or negative value of $\sqrt{x}$ is taken, then $y = \sqrt{x}$ would be the function. If both the values are not taken together, then there won't be any function.

(ii) For each value of x, we will obtain a unique and certain value for $x^3 + 5x$. Therefore it is a function.

(iii) R is not function, because element 2 of non-empty set X is associated with the element 3 and 11 of Y, whereas each element of X should be associated with a certain and unique element of Y.

Example 2: Find out the domains of defined functions $x^2 - 1$, $\dfrac{1}{x^3 - 1}$, where x is real number.

Solution: (i) in $f(x) = x^2 - 1$, there is certain real value of function $f(x)$ for each real value of x, therefore its non-empty set is R.

(ii) In $f(x) = \dfrac{1}{x^3 - 1}$ the value of function $f(x)$ for value of x is uncertain, but if the value of x is all other balance real numbers, then value of $f(x)$ is also real. Therefore its domain is $R \sim \{1\}$.

Example 3: If x is a real number, then ascertain the domain of $f(x) = \dfrac{x^2 - 4}{x - 2}$.

Solution: where $x = 2$, then the value of x is $\dfrac{0}{0}$, which is an undefined value. Besides for each real value of x, there is a defined real value of function $f(x)$. Therefore, domain of the function is set of all real numbers except 2 which can be expressed as under:

$D(f) = R \sim \{2\}$

Example 4: If $x \in R$, then find out the domain of $f(x) = \dfrac{1}{x^2 - 9}$.

Solution: $\dfrac{1}{x^2 - 9} = \dfrac{1}{(x - 3)(x + 3)}$ therefore, $x = 3$ and $x = -3$, denominator of the fraction is obtained in zero (0), therefore, the values of $f(3)$ and $f(-3)$ become uncertain. Besides these two value, we

Note

obtain a certain value of function $f(x)$ for each real value of x, therefore domain of f is the complete set of real numbers except 3 and -3, which can be read as $D(f) = R \sim \{3, -3\}$

Example 5: If $f(x) = 5x^2 + 8x + 7$, then find out the value of $f(-9)$.

Solution:

$\because$ $$f(x) = 5x^2 + 8x + 7$$

$\therefore$ $$x = -9$$

$$f(-9) = 5(-9)^2 + 8(-9) + 7$$

$$= 405 - 72 + 7 = 340.$$ **Ans.**

Example 6: If $f(x) = \log, 'x'$, then prove that $f(uv) = f(u) + f(v)$.

Solution:

$\because$ $$f(x) = \log_e x$$

then $$f(u) = \log_e u \qquad \qquad ...(1)$$

then $$f(v) = \log_e v \qquad \qquad ...(2)$$

$$f(uv) = \log_e(uv)$$

$$= \log_e u + \log_e v = f(u) + f(v)$$

From equation (1) and (2) $f(uv) = f(u) + f(v)$

Example 7: If $f(x) = x^2 - x$, then find out the value of $f(y + 1) - y^2$.

Solution: here

here $$f(x) = x^2 - x$$

$\therefore$ $$f(y + 1) = (y + 1)^2 - (y + 1)$$

$$= y^2 + 2y + 1 - y - 1 = y^2 + y$$

$\therefore$ $$f(y + 1) - y^2 = y^2 + y - y^2 = y$$

Now $$f(y + 1) - y^2 = y.$$ **Ans.**

Example 8: If $f(x) = \dfrac{1}{(1 + \tan^2 x)}$ then find out the value of $f\left(\dfrac{\pi}{4}\right)$.

Solution:

$\because$ $$f(x) = \frac{1}{(1 + \tan^2 x)}$$

$\therefore$ $$f\left(\frac{\pi}{4}\right) = \frac{1}{1 + \tan^2\left(\frac{\pi}{4}\right)} = \frac{1}{1+1} = \frac{1}{2}.$$ **Ans.**

Task   If $f(x) = 5x^2 + 8x + 7$, then find out the value of $f(-10)$ **Ans.:** 427

Example 9: If $f(\theta) = \dfrac{1 - 2\tan\theta}{1 + 2\tan\theta}$, then find out the value of $f\left(\dfrac{\pi}{4}\right)$.

Solution:

$\because$ $$f(\theta) = \frac{1 - 2\tan\theta}{1 + 2\tan\theta}$$

$$\therefore \qquad f\left(\frac{\pi}{4}\right) = \frac{1 - 2\tan\dfrac{\pi}{4}}{1 + 2\tan\dfrac{\pi}{4}} = \frac{1 - 2(1)}{1 + 2(1)}$$

$$= \frac{1 - 2}{1 + 2} = -\frac{1}{3}. \qquad \textbf{Ans.}$$

Example 10: If $f(x) = \log\dfrac{1+x}{1-x}$, then prove that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

Solution: $\qquad\qquad\qquad \therefore \;\; f(x) = \log\dfrac{1+x}{1-x}$

Here putting $\dfrac{2x}{1+x^2}$ in place of x

$$f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1 + \dfrac{2x}{1+x^2}}{1 - \dfrac{2x}{1+x^2}}\right] = \log\left[\frac{1 + x^2 + 2x}{1 + x^2 - 2x}\right]$$

$$= \log\left[\left(\frac{1+x}{1-x}\right)^2\right] = 2\log\left(\frac{1+x}{1-x}\right)$$

$$= 2f(x). \qquad \textbf{Ans.}$$

Example 11: If $\phi(x) = \cot x$, then prove that $\phi(-x) = -\phi(x)$.

$\because \qquad\qquad\qquad\qquad \phi(x) = \cot x$

Solution: $\qquad\qquad \therefore \;\; \phi(-x) = \cot(-x) = -\cot x = -\phi(x). \qquad \textbf{Ans.}$

Example 12: If $f(x) = \dfrac{x}{x-1}$, then find out the value of $\dfrac{f(a/b)}{f(b/a)}$.

Solution: $\qquad\qquad f(a/b) = \dfrac{(a/b)}{(a/b)-1} = \dfrac{(a/b)}{(a-b)/b} = \dfrac{a}{a-b}$

And $\qquad\qquad\qquad = \dfrac{(b/a)}{(b/a)-1} = \dfrac{(b/a)}{(b-a)/a} = \dfrac{b}{b-a}$

$\therefore \qquad\qquad\qquad = \dfrac{a}{a-b} \cdot \dfrac{b-a}{b} = -\dfrac{a}{b}. \qquad \textbf{Ans.}$

Example 13: Find out the value of $f\left(\dfrac{1}{x}\right)$ if $f(x) = x^3 - \dfrac{1}{x^3}$.

Solution: $\qquad\qquad \because \;\; f(x) = x^3 - \dfrac{1}{x^3}$

Note

Putting $\dfrac{1}{x}$ in place of x

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \frac{1}{\left(\dfrac{1}{x}\right)^3} = \frac{1}{x^3} - x^3 .$$ **Ans.**

Example 14: If $f(x) = \dfrac{x}{x^2+1}$, then find out the value of $\left\{f\left(\dfrac{a}{b}\right) - f\left(\dfrac{b}{a}\right)\right\}$.

Solution: $\because$ $f(x) = \dfrac{x}{x^2+1}$

$\therefore$ $\left\{f\left(\dfrac{a}{b}\right) - f\left(\dfrac{b}{a}\right)\right\} = \dfrac{\dfrac{a}{b}}{\dfrac{a^2}{b^2}+1} - \dfrac{\dfrac{b}{a}}{\dfrac{b^2}{a^2}+1}$

$$= \frac{ab}{a^2+b^2} - \frac{ab}{a^2+b^2} = 0 .$$ **Ans.**

Example 15: Find out the value of $f(x) + f(-x)$, if $f(x) = x^3 - \dfrac{1}{x^3}$.

Solution: $f(x) = x^3 - \dfrac{1}{x^3}$

$\therefore$ $f(-x) = \dfrac{1}{(-x)^3} = -x^3 + \dfrac{1}{x^3}$

$\therefore$ $f(x) + f(-x) = x^3 - \dfrac{1}{x^3} - x^3 + \dfrac{1}{x^3} = 0 .$ **Ans.**

Example 16: Prove that $f(x) + f(b) = f\left(\dfrac{a+b}{1+ab}\right)$, if $f(x) = \log_e \dfrac{1-x}{x^2+1}$.

Solution: $f(x) = \log_e \dfrac{1-x}{1+x}$

$\therefore$ putting $x = a$ $f(x) = \log_e \dfrac{1-a}{1+a}$

Now putting $x = b$ $f(x) = \log_e \dfrac{1-b}{1+b}$

$\therefore$ $f(a) + f(b) = \log_e \dfrac{1-a}{1+a} + \log_e \dfrac{1-b}{1-b} = \log_e \left\{\dfrac{1-a}{1+a} \times \dfrac{1-b}{1-b}\right\}$

$$= \log_e \left\{\frac{1-a-b+ab}{1+a+b+ab}\right\} = \log_e \left\{\frac{(1+ab)-(a+b)}{(1+ab)+(a+b)}\right\}$$

$$= \log_e \left\{\frac{1-\dfrac{a+b}{1+ab}}{1+\dfrac{a+b}{1+ab}}\right\} = f\left(\frac{a+b}{1+ab}\right).$$ **Ans.**

Example 17: Prove that $\dfrac{f(a)-f(b)}{1-f(a)\cdot f(b)}=\dfrac{a-b}{a+b}$ **if** $f(x)=\dfrac{x-1}{x+1}$ **.**

Solution: $\because$

$$f(x) \;=\; \frac{x-1}{x+1}$$

$\therefore$

$$\frac{f(a)-f(b)}{1-f(a)\,f(b)} \;=\; \frac{\left(\dfrac{a-1}{a+1}\right)-\left(\dfrac{b-1}{b+1}\right)}{1-\left(\dfrac{a-1}{a+1}\right)\times\left(\dfrac{b-1}{b+1}\right)}$$

$$= \frac{\dfrac{(a-1)(b+1)-(b-1)(a+1)}{(a+1)\,(b+1)}}{1-\dfrac{(a-1)\,(b-1)}{(a+1)(b+1)}}$$

$$= \frac{(a-1)\,(b+1)-(b-1)\,(a+1)}{(a+1)\,(b+1)-(a-1)\,(b-1)}$$

$$= \frac{ab+a-b-1-ba-b+a+1}{(ab+a+b+1)-(ab-a-b+1)}$$

$$= \frac{2(a-b)}{2(a+b)}=\frac{a-b}{a+b}\,. \qquad\qquad \textbf{Ans.}$$

Example 18: If domain $=\{-2,-1,0,1,2\}$ **, and** $f(-2)=4,\ f(-1)=1,\ f(0)=0\ \ f(1)=1,\ f(2)=4,$ **then** f **expresses a function, prove and determine the range also.**

Solution: Given that domain is $=\{-2,-1,0,1,2\}$
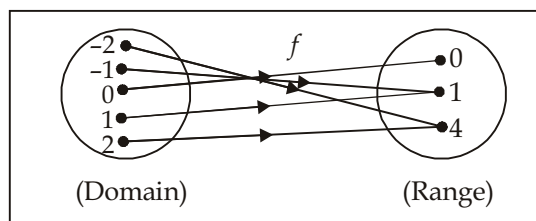
According to the question

$$f(-2) \;=\; 4 \;\Rightarrow f\text{-image of }-2=4$$
$$f(-1) \;=\; 1 \;\Rightarrow f\text{-image of }-1=1$$
$$f(0) \;=\; 0 \;\Rightarrow f\text{-image of }0=0$$
$$f(1) \;=\; 1 \;\Rightarrow f\text{-image of }1=1$$
$$f(2) \;=\; 4 \;\Rightarrow f\text{-image of }2=4$$

(Domain)   (Range)

Therefore *f* is expressed in the following explanation.

Note

Since by f, each element of domain, is connected with the unique element of set $\{0, 1, 4\}$ and any two or more than two elements of set $\{0, 1, 4\}$ is not connected with the domain.

Therefore, f expresses a function, which is defined by $f(x) = x^2$ formula whose range is $\{0, 1, 4\}$

Example 19: Find out the value of $f\{g(x)\} - g\{f(x)\}$, if $f(x) = 1 + 2x$ and $g(x) = \dfrac{1}{2}x$.

Solution: Putting $g(x)$ in place of x in the function $f(x) = 1 + 2x$

$$f\{g(x)\} = 1 + 2g(x) = 1 + 2\cdot\left(\frac{1}{2}x\right) = 1 + x$$

Thus, putting $f(x)$ in place of x in the function $g(x) = \dfrac{1}{2}x$

$$g\{f(x)\} = \frac{1}{2}\cdot f(x) = \frac{1}{2}(1 + 2x) = \frac{1}{2} + x$$

$\therefore$ $$f\{g(x)\} - g\{f(x)\} = (1 + x) - \left(\frac{1}{2} + x\right) = \frac{1}{2}.$$ **Ans.**

Example 20: If function $f : R \to R$ is defined in the following way

$$f(x) = \begin{cases} 3x - 1, & \text{if } x > 3 \\ x^2 - 2, & \text{if } 2 \leq x \leq 3 \\ 2x + 3, & \text{if } x < -2 \end{cases} \quad \text{then find out the value of } f(2), f(4), f(-1) \text{ and } f(-3)$$

Solution: Since set $(-2, 3)$ has 2 intervals, therefore here $f(x) = x^2 - 2$

$\therefore$ $\qquad\qquad\qquad\qquad\qquad\qquad f(2) = (2)^2 - 2 = 2$

Since set $(1, \infty)$ has 4 intervals, therefore $f(x) = 3x - 1$

$\therefore$ $\qquad\qquad\qquad\qquad\qquad\qquad f(4) = 3.4 - 1 = 11$

Since set $(-2, 3)$ has -1 intervals, therefore $f(x) = x^2 - 2$

Since $(-\infty, -2)$ has -3 intervals, therefore $f(x) = 2x + 3$

$\therefore$ $\qquad\qquad\qquad\qquad\qquad\qquad f(-3) = 2(-3) + 3 = -3.$ **Ans.**

Example 21: Find out the range of function $\sqrt{(x-2)(4-x)}$.

Solution: Assume $f(x) = \sqrt{(x-2)(4-x)}$

Where $x > 4$, then $f(x) = \sqrt{(\text{negative value})}$ = assumed value

Where $x < 2$, then $f(x) = \sqrt{(\text{negative value})}$ = assumed value

Therefore, $f(x)$ is real for $2 \leq x \leq 4$

Therefore, range of function $\sqrt{(x-2)(4-x)}$ is $2 \leq x \leq 4$. **Ans.**

Example 22: If $f : x \rightarrow x + 3$ and domain = $\{x : -2 \leq x \leq 2, x\}$, then find out its function (f) and range.

Solution: Range of f = $\{x : -2 \leq x \leq 2, x\} = \{-2, -1, 0, 1, 2\}$

Each element of domain f by f function is connected, therefore

Element –2 by function f is connected –2 + 3 = 1

Element –1 by function f is connected –1 + 3 = 2

Element 0 by function f is connected 0 + 3 = 3

Element 1 by function f is connected 1 + 3 = 4

Element 2 by function f is connected 2 + 3 = 5

Therefore the range of function = {1, 2, 3, 4, 5}

And function (f) = {(–2, 1), (–1, 2), (0, 3), (1, 4), (2, 5)}.

Questionnaire 1.1

1. Prove that $f(x) + f\left(\dfrac{1}{x}\right) = 0$ if $f(x) = x^2 - \dfrac{1}{x^2}$

2. Find out the value of $f(b) + f\left(\dfrac{1}{n}\right)$ if $f(b) = \dfrac{n}{1+n}$ **[Ans.: 1]**

3. If $f(x) = \log_e x$, then find out the value of $f(1)$ **[Ans.: 0]**

4. Find out the domain and range of function $f(x) = \dfrac{x}{1+x^2}$. If the function is single?   **[Ans.: R]**

5. Find out the value of $\dfrac{f\left(\dfrac{1}{b}\right)}{f\left(\dfrac{b}{a}\right)}$ if $f(x) = \dfrac{x}{x-1}$ **[Ans.: (–a/b)]**

6. Prove that $x = f(y)$, if $y = f(x) = \dfrac{x+2}{x-1}$

7. Prove that $f(x) \cdot f(-x) = 1$ if $f(x) = \dfrac{2x-3}{2x+3}$

8. Prove that $f(x + y + z) = f(x) \cdot f(y) \cdot f(z)$, if $f(x) = e^x$

9. Find out the values of $f(-1)$ and $f\left(\dfrac{1}{x}\right)$ if $f(x) = x^2 + \dfrac{1}{x^2}$ **[Ans.: 2, $\left(\dfrac{1}{x^2} + x^2\right)$]**

10. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, then find out the value of $f(6)$ **[Ans.: 30]**

11. Which of the following functions are not defined for the value of x?

   (a) $\dfrac{1}{x-3}$ (b) $\dfrac{1}{x^3-1}$ (c) $\dfrac{x^3-4}{x-2}$

   (d) $\tan x$ (e) $\sqrt{x}$ (f) $\dfrac{a^x-1}{x}$

Note

12. If $f(x) = 3x^2 + 2$ and $g(x) = 2x + 5$, then find out the value following:

   (a) $(f + g)\ (0),\ (f + g)\ (-2)$ (b) $(f - g)\ (3),\ (f - g)\ (-1)$

   (c) $(f\,g)\left(\dfrac{1}{2}\right),\ (f g)\ (1)$ (d) $\left(\dfrac{f}{g}\right)(2),\ \left(\dfrac{f}{g}\right)(-1)$

13. If $A = \{a, b, c, -1\}$, then explain if following relations are function of A in A. Describe the reason also:

   (a) $R_1 = \{(-1, c), (c, b), (a, b), (-1, -1)\}$

   (b) $R_2 = \{(a, b), (b, c), (c, -1), (-1, a)\}$

14. If $f: R \to R$, where $f(x) = \begin{cases} 2x + 5,\ x > 9 \\ x^2 - 1,\ x \in (-9, 9) \\ x - 4,\ x < -9 \end{cases}$ then find out the value of $f(3),\ f(12),\ f(-15)$ and

   $f\{f(5)\}$.

15. If $g : R \to R$, where $g(x) = \begin{cases} x^2 - x,\ x \geq 2 \\ x - 2,\ x < 2 \end{cases}$, find out the value of $g\ (5),\ g\ (0)$ and $g\ (-2)$ in 3,

   where $-3 \leq x < -1$ [**Ans.:** 20, –2, –4]

16. If $f(x) = \begin{cases} -6x - 3, \text{where} -1 \leq x \leq 0, \text{ then} \\ 3x - 3, \text{ where } \quad 0 < x \leq 1 \end{cases}$

   (a) Find out the domain of function f.

   (b) Find out the value of $f(-2),\ f\left(\dfrac{1}{2}\right), f(0),\ f(-1),\ f(2)$

   (c) Solve the equation $2f(x) + 3 = 0$

   Ans.: (a) $D = \{x : -3 \leq x \leq 1\}$ (b) $3, -3/2, -3, 3$ (c) $x = -1/4\ \&\ 1/2$

1.12   Graph of Functions

If $y = f(x)$, is the function of x, then assigning $x_1, x_2, \ldots, x_n, \ldots$ etc. to x, value of the function will become $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ or $(y_1, y_2, \ldots, y_n, \ldots)$ respectively. If as per the symbolic geometry, putting x on x-axis and y or $f(x)$ on y-axis, marking the points as $\{x_1, f(x_1)\}, \{x_2, f(x_2)\}, \ldots, \{x_n, f(x_n)\}, \ldots$ etc., a smooth curve is drawn, then the same curve will be call Graph of Functions.

For sketching the drawing of function $y = f(x)$, for the selected value of x, find out the value of y or $f(x)$ and draw a table

x	–3	–2	–1	0	1	2	3
$y = f(x)$	$f(-3)$	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$	$f(3)$

Now marking points $\{-3, f(-3)\},\ \{-2, f(-2)\}, \{-1,\ f(-1)\}, \{0, f(0)\}, \{1, f(1)\}, \{2, f(2)\}, \{3,\ f(3)\}$, etc. on a square paper, draw a smooth curve. This will be called Graph of Functions.
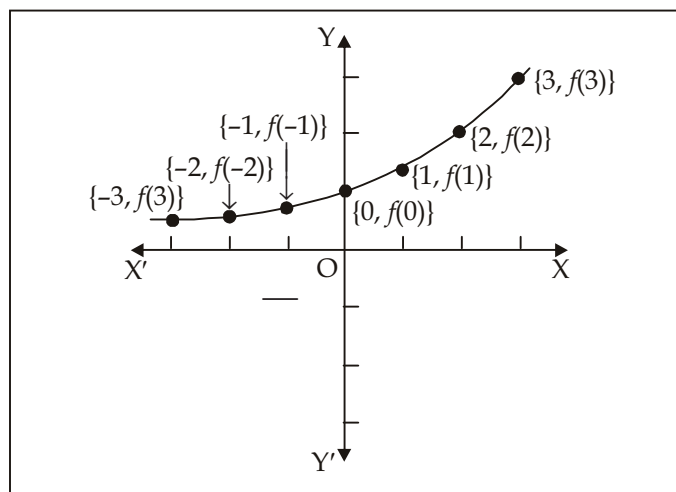
Example 1: Draw a sketch of function $y = 4x + 2$.

Solution : Assigning different value to x and determining values of y, following table will be drawn

x	−2	−1	0	1	2	3	$-\dfrac{1}{2}$
y	−6	−2	+2	6	10	14	0
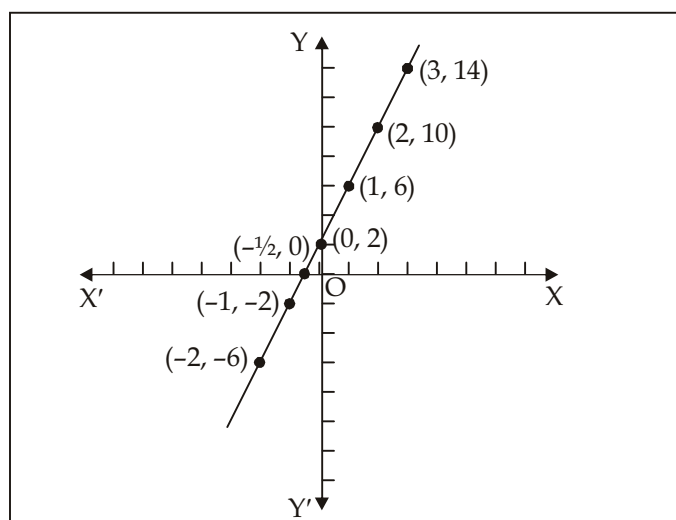
Now taking 1 box = 1 on x-axis and 2 boxes = 1 on y-axis, mark-up the points. Join them together. This is the final drawing. This is a simple line.

Example 2: Draw a sketch of function $y = x$.

Solution: Table for $y = x$

x	−4	−3	−2	−1	0	1	2	3	4	5
y	−4	−3	−2	−1	0	1	2	3	4	5

Note
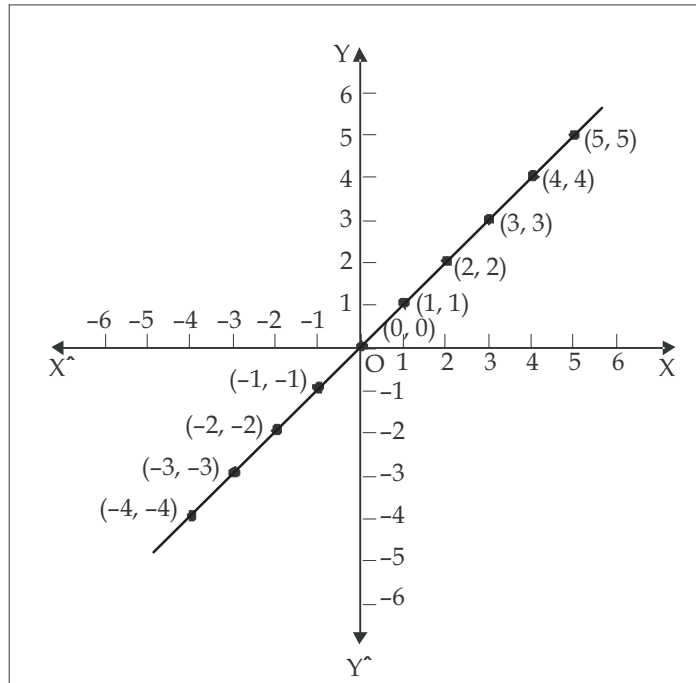
Example 3: Draw a sketch of $y = |x|$.

here $y = x$, if and $y = -x$, if $x < 0$

Table for $y = |x|$

x	4	3	–2	–1	0	1	2	3	4
y	4	3	2	1	0	1	2	3	4
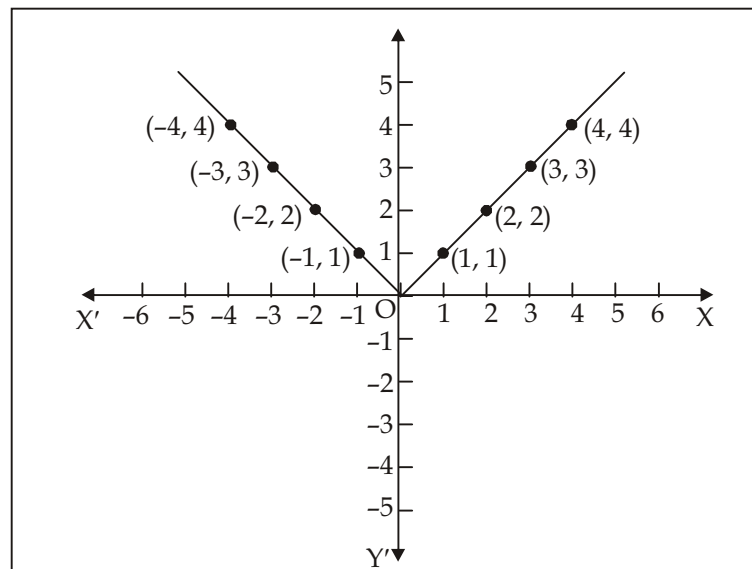
Example 4: Draw a sketch of function $y = \dfrac{|x|}{x}$ **if** $x \neq 0$.

Solution: This function is called Signum function, which can be expressed as under

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

Table for function $y = \dfrac{|x|}{x}$

x	−4	−3	−2	−1	1	2	3	4
y	−1	−1	−1	−1	1	1	1	1

Drawing of this table has been reverse figure.
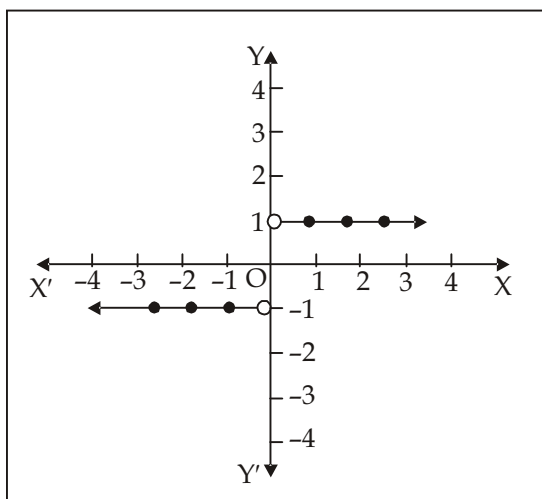
Example 5: Draw a sketch of $y^2 = x^3$.

Solution: Here if x is a negative, then value of y is imaginary. Therefore, we will take only positive value of x.

Table for $y^2 = x^3$

x	0	1	2	3	4
y	0	±1	±2.8	±5.2	±8

Note   Drawing of this table has been reverse figure.
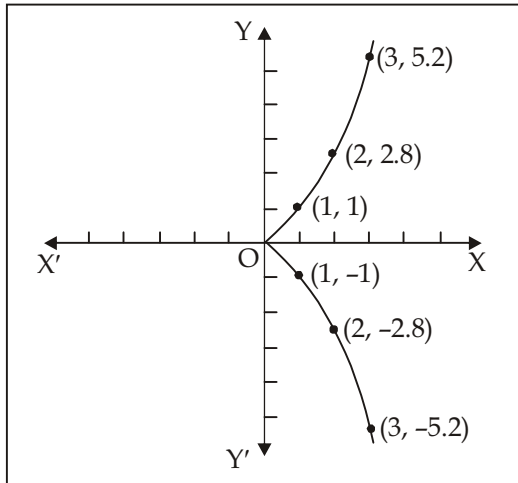
Example 6: Draw a sketch of $y = x^2 - 4x + 5$.

Solution: First of all create the following table

x	−4	−3	−2	−1	0	1	2	3	4	5	8
y	37	26	17	10	5	2	1	2	5	10	37

Here we see that as much as the value of x increases, the value of y also increases and marking $x \to \infty$, $y \to \infty$ when $x \to -\infty$, $y \to +\infty$, we will find the following smooth curve.
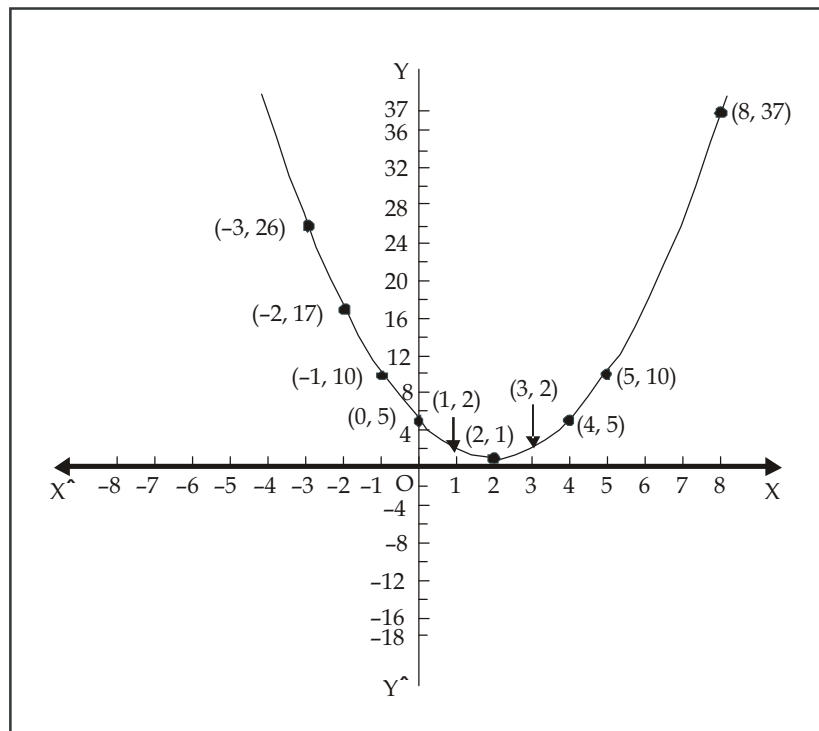
Example 7: Draw a sketch of $xy = 4$.

Solution: Here $y = \dfrac{4}{x}$, therefore if $x \to 0, y \to \infty$ and if $x \to \infty, y \to 0$

Table for $xy = 4$

x	$-\infty$	-4	-3	-2	-1	0	1	2	3	4	∞
x	0	-1	$-\dfrac{4}{3}$	-2	-4	∞	4	2	$\dfrac{4}{3}$	1	0
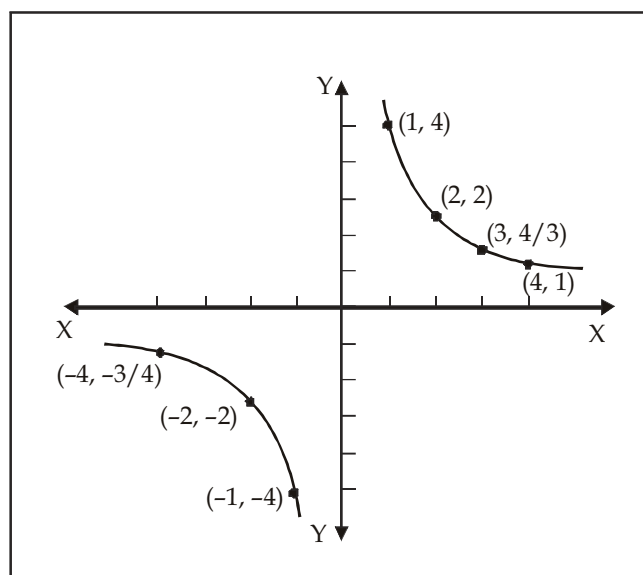
Example 8: Draw sketch of function $y = \dfrac{x^2 - 1}{x - 1}$.

Solution: Here function $x = 1$ is not defined. Besides for each value $y = \dfrac{(x - 1)(x + 1)}{x - 1} = (x + 1), (x \neq 1)$

is defined.

Table for function $y = \dfrac{x^2 - 1}{x - 1}$

x	0	1	2	3	-1	-2	-3	-4
y	1	2	3	4	0	-1	-2	-3

Note   Drawing has been shown in the following figure
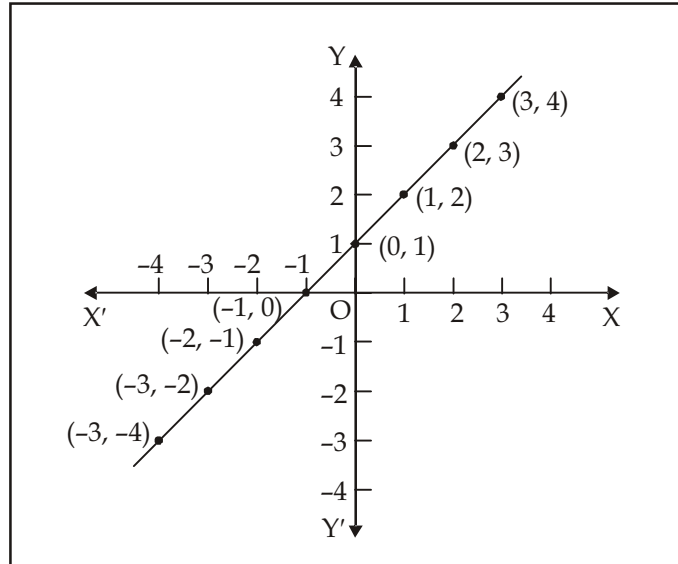
Example 9: Draw sketch of function $y = \dfrac{1}{1-x}$.

Solution: Assigning different value to x, and getting value of y, following table is prepared

Table for function $y = \dfrac{1}{1-x}$

x	-2	-1	$-\dfrac{1}{2}$	0	$\dfrac{1}{2}$	1	$\dfrac{3}{2}$	2
y	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1	2	∞	-2	-1

Drawing has been shown in the following figure
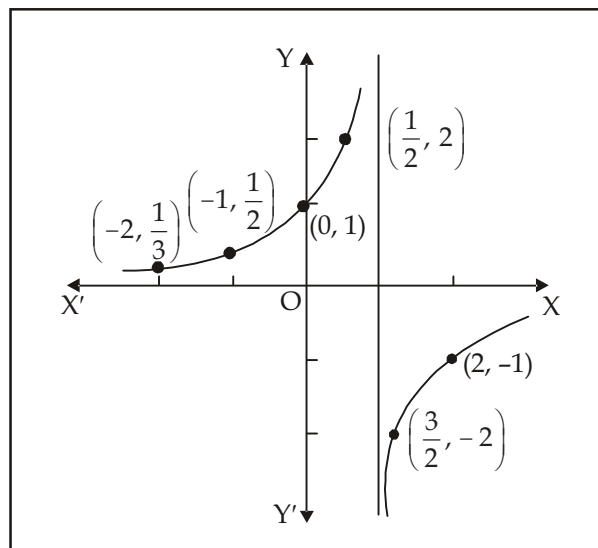
Example 10: Draw a sketch for curve $x^2 - y^2 = 9$.

Solution: Following is the given curve $x^2 - y^2 = 9$ or $(x^2/9) - (y^2/9)$ which is a hyperbola of $(x^2/a^2) - (y^2/b^2) = 1$
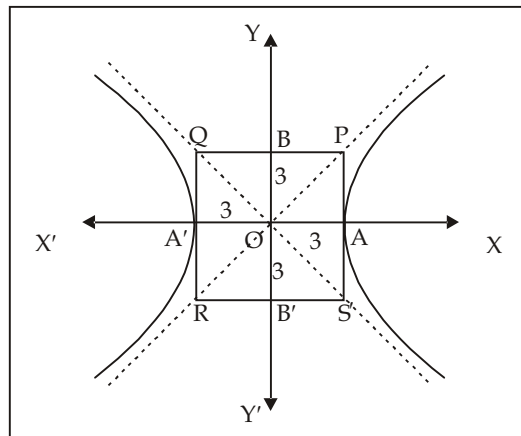
Comparing them

$$a^2 = 9 \text{ and } b^2 = 9$$

$\Rightarrow$ $\qquad\qquad\qquad\qquad a^2 = \pm 3 \text{ and } b = \pm 3$

$\therefore$ $\qquad\qquad\qquad\qquad\qquad = a = 3$

And $\qquad\qquad\qquad\qquad\qquad = b = 3$

Now *OA* on *x*-axis = *OA′* = 3 and

OB on *y*-axis = *OB′* = 3

Taking distances into consideration an square *PQRS* is drawn.

1.13 Use of Linear Functions in Economics

There are many uses of linear functions and simple lines in economics. The price of commodity in the market, is determined by demand and supply. If demand and supply function is given, then at the equilibrium level, we can determine such a level of price and quantity, where both the seller and customer would be satisfied.

Assume that demand function is $q = 16 - 4p$ $\hfill$ …(i)

And supply function is $q = -8 + 4p$ $\hfill$ … (ii)

Where *q* shows the quantity of commodity and *p* is meant for price.

Now we will try to derive such a value of *q* and *p*, which would satisfy both the equations (i) and (ii). For this both the equations need to be solved. In the state of equilibrium demand and supply

$$16 - 4p = -8 + 4p$$

Or $\qquad\qquad\qquad\qquad\qquad 8p = 24$

Or $\qquad\qquad\qquad\qquad\qquad p = \dfrac{24}{8} = 3$

Putting value of *p* in equation $q = 16 - (4 \times 3) = 16 - 12 = 4$

This way equilibrium price = 3 and quantity of demand and supply would be 4.

Note   We can also show these equations into the following figure

Graph
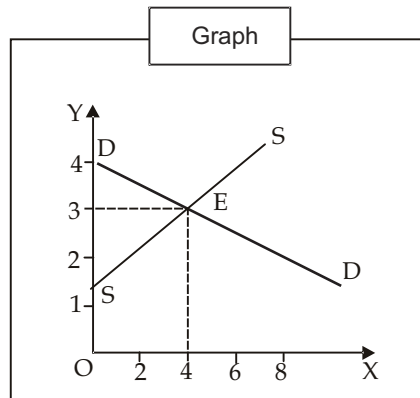
In the figure on *x*-axis quantity of commodity and on *y*-axis price of commodity is shown. *DD* and *SS* express demand function and *SS* supply function respectively. Both of them intersect each other on point E. therefore, equilibrium will also be obtained on point *E*. at this point *y*-axis will show equilibrium price and *x*-axis will show the quantity of demand and supply.

Therefore, equilibrium price = 3 and quantity of commodity = 4.

Functions used in the economics

(a)   Parabola

(b)   Hyperbola

(c)   Logarithmic Function

(d)   Exponential Function

(a)   **Parabola –** This is used heavily in economics. If *y* is dependent on *x* and its exponents, then *y* is called quardratic function. This can be expressed in the following manner:

$$y = a + bx + cx^2 \qquad \qquad ...(i)$$

Similarly If *x* is dependent on *y* and its exponents, then *x* is called quadratic function of *y*. This can be mathematically expressed in the following manner:

$$x = a + bx + cy^2 \qquad \qquad ...(ii)$$

Both (i) and (ii) shows the general result of parabola. Here it is pertinent that in the equation (i) of parabola *x* is not the factor of *y* and (ii) it is same in both x^2 or y^2.

Here it is noteworthy that the center of above equations will not be on main point as shown in the figure.

Graph
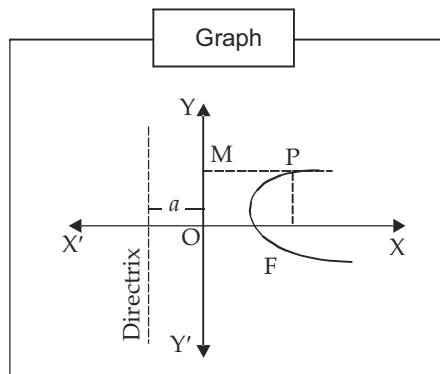
If equation of parabola is $y^2 = 4ax$, then the center of the equation will be on main point and parabola curve will be as per the figure. Similarly, if the equation of parabola is $x^2 = 4by$ then parabola curve will be as per the figure.

O is called the Vertix of parabola in the above figures. For Parabola a simple line parallel to OY axis, is placed on top opposite to the other side of the curve, which is called Directrix. Similarly, for curve $x^2 = 4by$ on the vertrix b is placed parallel to directrx OX.
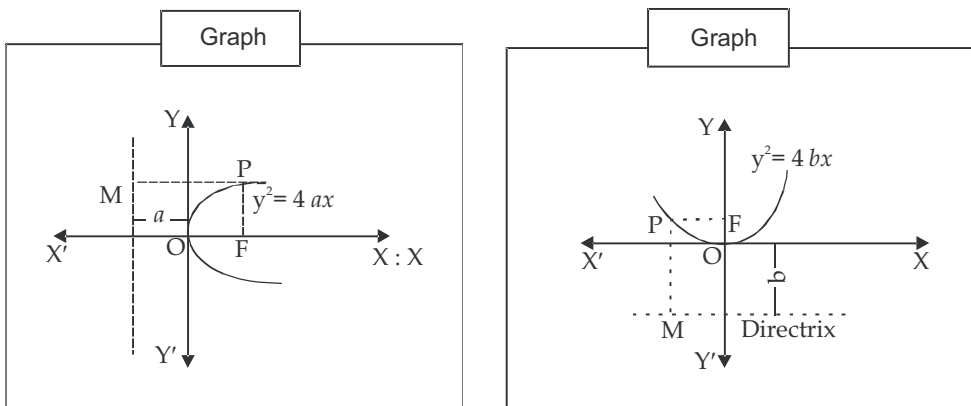
Note

The simple line which goes from Vertrix and is parallel to Directrix, is called the axis of the parabola. The axis divides the parabola in two equal parts. On the axis from top at the distance of a or b on the opposite side of the directrix placed point is called focus. The special property of the parabola is that focus of any point and directrix are placed equally on it.

(b) **Hyperbola –** Similar to parabola, hyperbola is also very much used in the economics. This function can be expressed mathematically in the following manner:

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where a, b are constants

Or

$$\dfrac{x^2}{a^2} - 1 = \dfrac{y^2}{b^2}$$

then

$$\dfrac{y^2}{b^2} = \dfrac{x^2 - a^2}{a^2}$$

$$y^2 = \dfrac{b^2}{a^2}(x^2 - a^2)$$

$$y = \pm \dfrac{b}{a}\sqrt{(x^2 - a^2)}$$

If we follow the following facts, then

(i)   Where $x = \pm a, y = 0$

(ii) $x > |a|$, assumed that $a = b = 3$ then $y = \pm\sqrt{(x^2 - 9)}$

(iii) If $x < 2$, then y's value would be imaginary, but for the other value of x, following values of y can be obtained

Value of x	value of y
+2 or –2	0 or 0
+3 or –3	+ $\sqrt{5}$ or $-\sqrt{5}$
+4 or –4	+ $\sqrt{12}$ or $-\sqrt{12}$

Note
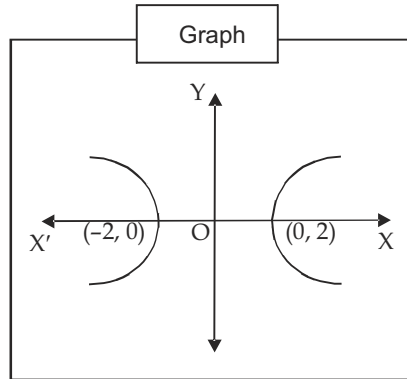
If main point (0, 0) is the center of hyperbola, then hyperbola curve would be of following way

If center-point is (h, k), then the equation of hyperbola would be of different type

$$\frac{(x-h^2)}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Generally, hyperbola function is expressed as $xy = a$, whereas $a > 0$.

(c) **Logarithmic Function –** When y is a function of log x, but not x, then that function is called logarithmic function. The general equation for a logarithmic function is under:

$y = \alpha + \beta \log x$. Here if $\alpha = 0$ and $\beta = 1$, then the equation will be $y = \log x$.

Assuming that logarithmic equation is $y = 5 + 5 \log_{10} x$ (here we have assumed $\alpha = 5$, $\beta = 5$) then

$$y = -5 + 5 \times 0 = -5 \qquad\qquad (x = 1)$$
$$y = -5 + 5 \times 1 = 0 \qquad\qquad (x = 10)$$
$$y = -5 + 5 \times 2 = 5 \qquad\qquad (x = 100)$$
$$y = -5 + 5 \times 3 = 10 \qquad\qquad (x = 1000)$$

The curve for this would be as per Figure

It is noteworthy that the value of x in this equation can not be 0 or negative because it has no logs. Most of the times e is taken as base for log.

The general equation for binary-logarithmic function is as under

$$\log y = \alpha + \beta \log x$$

Or $$\log y = \alpha \log x^\beta$$

$$\log y - x^\beta = \alpha$$

Or $$\log \frac{y}{x^\beta} = a$$

$\therefore$ $$\frac{y}{x^\beta} = e^a$$

$\therefore$ $$y = e^\alpha x^\beta = Ax^\beta$$

Where $A = e^x$

The special features of this function is as under

(i) If $\beta = 1$ or $y = \dfrac{A}{x}$ or $xy = A$, then it will become orthogonal hyperbola
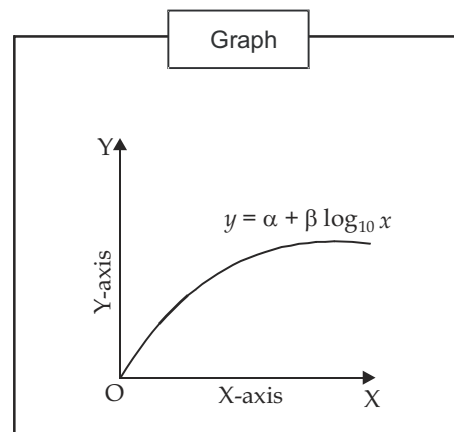
(ii) If β < 1 then y will increase at the rate of increase in x.

(iii) If 0 < β < 1, then y decreases at the rate of increase in x.

(iv) If β < 1, then y decreases when x increases.

Here it is pertinent to mention that positive exponent is production function and negative exponents are used in demand analysis. The special feature of this function is that exponent describes constant elasticity.

(d) **Exponential function –** These are also very much important in economics. In this function y is an exponential function of x instead of just a function. This can be expressed as

$y = \log_e x, y = e^x, y = e^{\sin x}$

If our exponential function is $y = b^x$, whereas b >, 1 then its curve will be as shown in the figure
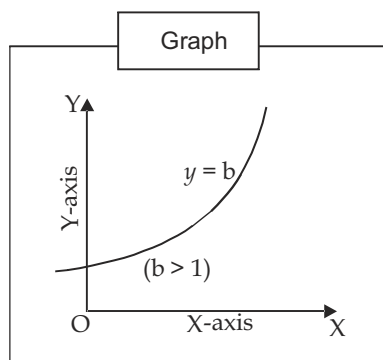
Assume exponential function $x = AB^y$, then this equation can be expressed in logarithmic for as under:

$$\log x = \log A + y \log B$$

∴ $$y \log B = \log x - \log A$$

Or $$y = \frac{\log x}{\log B} - \frac{\log A}{\log B}$$

$$\frac{1}{\log B} = β \text{ and } -\frac{\log A}{\log B} = α \text{ then}$$

Assume $$y = α + β \log x, \text{ which is our logarithmic function.}$$

Example 1: Assume that demand and supply are a simple line. If at the different prices, quantity of demand and price is shows as under, then determine the demand and supply equation.

Price (P)	Demand (D)	Supply (S)
2	40	50
5	20	60

Solution: Marking various points in the table above, we can give it a linear shape

	Demand (D)	Supply (S)
1	(40, 2)	(50, 2)
2	(20, 5)	(60, 5)

Note

Assume x = quantity and y = price, then equation of line joining (40, 2) and (20, 5) is

$$y \cdot y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Putting y_1, y_2, x_1, and x_2 in to the equation

$$y - 2 = \frac{5 - 2}{20 - 40}(x - 40)$$

Or

$$y - 2 = \frac{3}{-20}(x - 40)$$

Or

$$-20\ (y - 2) = 3\ (x - 40)$$

Or

$$-20\ y + 40 = 3x$$

Or

$$3x = 160 - 20\ y$$

Or

$$x = \frac{160}{3} - \frac{20}{3}y = 53.33 - 6.66y$$

If $x = D$ in place of demand and $y = p$ in place of price, the demand equation will be

$$D = 53.33 - 6.66\ p$$

Similarly supply line passed through (50, 2) and (60, 5) points, and equation for joining line for these points will be

$$y - 2 = \frac{5 - 2}{60 - 5}(x - 5)$$

Or

$$y - 2 = \frac{3}{10}(x - 50)$$

Or

$$10y - 20 = 3x - 150$$
$$3x = 130 - 10y$$
$$x = 13 + 3.33y$$

If $x = S$ in place of Supply quantity, and $y = p$ in place of price, then supply equation will be

$S = 13 + 3.33p$

Example 2: Find out the focus and directrix of following parabola

(i) $y^2 = 16x$ (ii) $x^2 = 20\ y$

Solution: (i) comparing $y^2 = 16x$

Thus, directrix is placed at the distance of – 4 at the opposite side from Top *viz* $x = -4$ and focus is situated opposite to the directrix at (4, 0)

(ii) comparing $x^2 = 4by$ to $x^2 = -20y$

$$b = 5$$

thus, directrix is placed at the distance of +5 at the opposite side of the curve from Top viz $y = 5$ and focus is situated at (0, 5).

Self Assessment

2. Multiple Choice Questions:

(i) If $f : x \to y$ has been defined, then what would you call non-empty set x of this function (f)?

    (a) Domain   (b) range

    (c) variable   (d) constant

(ii) If variable number y is depended on variable number x, then what would you call $y = f(x)$?

  (a) Multi-variable function (b) Single variable function

  (c) Explicit function (d) Implicit Function

(iii) Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called

  (a) algebraic function (b) Odd function

  (c) rational function (d) undefined Function

1.14 Summary

- The rate of change in the value of other number and questions related to rate of such change and analysis and study of functions is referred to as Differential Calculus.

- Changing quantities are referred to as variables. Quantities for which values keep on changing viz which can be given indefinite numerical values are Variables.

- Numbers for which value is unchangeable under any process of mathematics, are constants.

- The principle of functions is mutually based on the related quantities.

- If function $f : X \rightarrow Y$ is defined, then non-empty set X is called domain of this mapping or function (f) and non-element set (Y) would be its range or codomain.

- If variable y is only dependent on variable x, then $y = f(x)$ is called functions of single variable. Similarly $y = f(q)$, $s = f(t)$, $v = f(t)$ etc. are functions of single variable.

- If variable u is dependent on variable x and y, then $u = f(x, y)$ is called functions of many variable.

- Function with different exponential value of x variable, in which factors are certain is called algebraic function.

- Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called rational function.

- There are many uses of linear functions and simple lines in economics. The price of commodity in the market, is determined by demand and supply. If demand and supply function is given, then at the equilibrium level, we can determine such a level of price and quantity, where both the seller and customer would be satisfied.

- The simple line which goes from Vertrix and is parallel to Directrix, is called the axis of the parabola.

1.15 Keywords

- *Domain:* Affected area.
- *Range:* Series, limits of variation.

1.16 Review Questions

1. Express the following functions with graph:

  (a) $y = f(x) = 5 - 3x$; for range $x = -2$ to $x = 5$

  (b) $y = f(x) = 2x^2 - 5x + 1$; for range $x = -5$ to $x = 7$

  (c) $y = f(x) = \dfrac{24}{x}$ for range $x = 1$ to $x = 9$

Note

2. Determine the demand and supply function with the help of following table

Price (P)	Demand (D)	Supply (S)
1	100	50
2	50	75
3	0	100

3. Determine price and quantity with respect to equilibrium of demand and supply function of any market from the following:

   (i) $D = 15 - 3p$; $S = -10 + 2p$

   (ii) $D = 12 - 2p$; $S = -20 - 4p$

   (iii) $D = 50 - 4p$; $S = 2 + 10p - p^2$

4. The demand and supply function of any commodity is as under

   (i) $D = 100 - 10p$ $S = -12 + 9p$

5. Find out the vertex, focus and directrixs of parabola

   (i) $y = x^2 + 3x - 2$ **[Hints:** $-y = \left(x + \dfrac{3}{2}\right)^2 - \dfrac{17}{4}$ **]**

   (ii) $(x - 2)^2 = 4y - 16$

6. L is Labour and K is capital in the production function $y = 5_L{}^5 K^5$. If in short term capital is constant and $K = 100$, then draw the production function. Determine labour product (Y/L) and draw its curve.

Answers: Self Assessment

1. (i) variable,   (ii) constant   (iii) related,   (iv) radius,   (v) rational

2. (i) (a)   (ii) (b)   (iii) (c).

1.17   Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematical Economy – Michael Harrison, Patrik Walderon.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Unit 2: Limits and Continuity

Objectives

After reading this unit, students will be able to :

- Easily Solve the Problems Related to Limit of Function.
- Easily Solve the Inter-related Problems of Limit and Value at $x = a$ of Function $f(x)$.
- Understand the Theorem of Limits and Related Questions.
- Know the Method of Finding the Limit of any Function.
- Know the Geometrical Definition.
- Know the Method to Finding Continuity of a Function at any Point and Solve the Related Problems.
- Understand the Theorem on Continuous Functions and Related Problems.

Introduction

1. Explain the following geometrical series

   $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} +$ to infinite.

Note

Total of 1 term =   .5

Total of 2 terms =   .75

Total of 3 terms =   .875

Total of 4 terms =   .9375

Total of 5 terms =   .96875

Total of 6 terms =   .984375

Total of 7 terms =   .9921875

Total of 8 terms =   .99609375

Total of 9 terms =   .998046875

Total of 10 terms =   .9990234375

……………….. =   ………………

From the above it is apparent that as long as the term grows, total of the sequence moves towards 1, although the total could not be equal to 1.

Thus taking into account the sufficient number of terms, we can reduce the difference of total and 1 as much as we want.

2. Assume that in a circle with the given radius, a polygon is drawn. It is clear from the geometry that:

(a) The area of polygon can never exceed from the area of circle, even if the polygon has more number of arms.

(b) By increasing the number of arms indefinitely, we can reduce the difference of area between circle and polygon as much as we want.
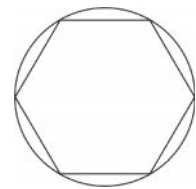
In calculus the fact can be expressed that the limit of area of polygon drawn under a circle, where the number of arms of polygon increases increasingly (viz moves towards indefinite), is the area of circle.

$\lim\limits_{n \to \infty}$ area of polygon =area of circle

3. Limit of a sequence.

Consider on the sequence 1, $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}$,...... for which $s_n = \dfrac{1}{n}$] where s_n expresses the sequence of nth term. We see that as the value of n grows the value of $\dfrac{1}{n}$ goes down. In fact, selecting n sufficiently greater we can scale down $\dfrac{1}{n}$ as much as we want. Here if $n > 10{,}000$ then $\dfrac{1}{n} < 0.0001$ and if $n > 10^8$ then $\dfrac{1}{n} < 10^{-8}$. Thus we see that as the $m$inclines towards infinite $\dfrac{1}{n}$ tends to zero (0). We see that limit of sequence $\{s_n\}$ is 0 and this is taken as $\lim\limits_{n \to \infty} s_n = 0$.

Now we will consider 0, $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}$,...... for which $s_n = \dfrac{n-1}{n}$. We see that as the n grows s_n, inclines closer to 1. In other words in fact, selecting n sufficiently greater we can scale down the statistical difference between s_n and 1 viz. $|s_n - 1|$ as much as we want. For this sequence.

$$\lim\limits_{n \to \infty} s_n = 1.$$

Generally for any sequences s_1, s_1, s_2, s_3,......, choosing n sufficiently greater, if we can reduce the difference $|s_n - A|$ of s_n and any number A as much as we want, A will be called the sequence of $\{s_n\}$ and this can be expressed as

$$\lim_{n \to \infty} s_n = A$$

Now the sequence is $1^2, 2^2, 3^2, 4^2, ... n^2, ...$

And considering on this we see that as the value of n grows, accordingly s_n grows. Taking n sufficiently greater we can maximize s_n as much as we want. Here $n > 100$, $s_n > 10^8$ if $n > 10^4$. In fact S_n here can be made greater that any greatest number. This can be expressed as $S_n \to \infty$ if $n \to \infty$ is expressed.

4. Before defining the limits of function, it will be better to consider the following example. Some functions are undefined for special value of x, such as .

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

It is clear that the value of $\frac{x^2 - 1}{x - 1}$ at $x = 1$ is $\frac{0}{0}$ viz indeterminate.

If $x = 0.9$ then the value of function $\frac{(0.9)^2 - 1}{0.9 - 1} = 1.9$,

If $x = 0.99$ then the value of function $\frac{(0.99)^2 - 1}{0.99 - 1} = 1.99$,

If $x = 0.999$ then the value of function $\frac{(0.999)^2 - 1}{0.999 - 1} = 1.999$,

If $x = 0.9999$ then the value of function etc. $\frac{(0.9999)^2 - 1}{0.9999 - 1} = 1.9999$ and so on.

x	0.9	0.99	0.999	0.9999	---
$f(x)$	1.9	1.99	1.999	1.9999	---

Thus we see that as the value of x goes closer to 1, the value of function $\frac{x^2 - 1}{x - 1}$ tends toward 2.

Now   if $x = 1.1$ then value of the function = $\frac{(1.1)^2 - 1}{1.1 - 1} = 2.1$

if $x = 1.01$ then value of the function = $\frac{(1.01)^2 - 1}{1.01 - 1} = 2.01$,

if $x = 1.001$ then value of the function = $\frac{(1.001)^2 - 1}{1.001 - 1} = 2.001$,

if $x = 1.0001$ then value of the function = $\frac{(1.0001)^2 - 1}{1.0001 - 1} = 2.0001$

Note

x	1.1	1.01	1.001	1.0001	---
$f(x)$	2.1	2.01	2.001	2.0001	---

Here we see that as the value of x decreases and goes down to 1, the value of function $\dfrac{x^2 - 1}{x - 1}$ reaches to 2.

Thus, whether the value of x decreases down to 1 from a little more than $1 + \varepsilon$ (where e is smallest independent positive number) or increases to 1 from a little less than $1 + \varepsilon$, the value of the function tends to a certain value 2. This certain value 2 is called limit of the function $\dfrac{x^2 - 1}{x - 1}$ at $x \to 1$. This can

be expressed as under $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$

2.1 Limit of a Function

Assume $y = f(x)$ is a function and $h_1, h_2, \ldots\ldots h_n, \ldots\ldots$ is a set of positive numbers, which value is continually decreasing viz

$$h_1 > h_2 > h_3 > \ldots\ldots > h_n > \ldots\ldots > 0 \qquad \ldots(1)$$

And which, choosing n sufficiently greater, can be made smaller as desired. In this state, as the h_n goes down, the value of function decreases
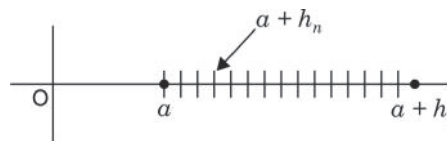
$$f(a + h_1), f(a + h_2), \ldots, f(a + h_n) \qquad \ldots(2)$$

If a number tends to A then this number is call right hand limit of function $f(x)$ at $x = a$ or this number A is called the right hand limit of function $f(x)$, when x tends to a. This can be expressed as:

$$\lim\limits_{x \to a + 0} f(x) = A = f(a + 0)$$

Here we have considered only those values of x, which is greater that a. (in the figure only a at the right side)
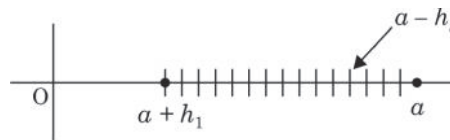
Now we will consider those values of x, which is smaller (viz in the figure only at the left side of a.) As h_n goes down, the value of function $f(a - h_1), f(a - h_2)\ldots\ldots, f(a - h_n),\ldots\ldots$ tends to B. This number B is called the left hand limit of function f(x), when x = a.

This is expressed as $\lim\limits_{x \to a - 0} f(x) = B = f(a - 0)$

If $A = B$ viz $\qquad \lim\limits_{x \to a + 0} f(x) = \lim\limits_{x \to a - 0} f(x)$

Then *A* is called limit of *f*(*x*) at *x = a*

Set $h_1, h_2, ..., h_n, ...$ is a sequence, for which limit is 0. Similarly second makes sequence (2). Here is it to be specially noted that for limit to exist, like sequence (1) *f*(*a* + *h_n*) every type of sequence should tend to *A*. viz the statistical difference of *f*(*a* – *h_n*) – *A*, choosing h_n sufficient smaller, can be reduced as desired. Assigning *a* + *h_n* {or *a* – *h_n*} = *x* or | *x* – *a* | = *h_n* we can define the limit as under

Definition – At *x = a*, limit of function *f*(*x*) is any number (assume A, which has the property that for each value of x for which | *x* – *a* | viz *x* – *a* is numerical value) sufficiently smaller (but not zero), |*f*(*x*) – A| viz the numerical value of *f*(*x*)-A is smaller as desired.

Limit can also be defined with the following

Second definition of limit

When *x* → *a* (when x tends to a), the limit of function f(x) is any number (Assume A), which has the property that for any independent positive smallest number ε, a second number δ greatest than 0 can be obtained, for which $|f(x) – A| < ε$ for every values of *x*,

$$0 < | x – a | < δ.$$

Notes   If at *x = a*, L is the limit (L) of *f*(*x*), then this can be expressed as

$$\lim_{x \to a} f(x) = L \text{ or } \lim_{x \to a} f(x) = L.$$

2.2   Right Hand and Left Hand Limits

2.2.1   Right Hand Limit

When the limit of function is obtained from the right hand of the independent variable, then it is called Right Hand Limit (R.H.L.) and applying positive (+) sign for the right side, this can be expressed as under

Right Hand Limit = *f*(*a* + 0) = $\lim_{x \to a^+} f(x) = l_1$.

2.2.2 Left Hand Limit

When the limit of function is obtained from the left hand of the independent variable, then it is called Left Hand Limit (L.H.L.) and applying negative (–) sign for the left side, this can be expressed as under

Left Hand Limit = *f*(*a* – 0) = $\lim_{x \to a^-} f(x) = l_2$.
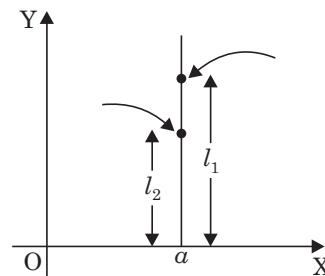
2.3   Working Rules for Finding Right Hand Limit and Left Hand Limit

   (i)   To obtain the limit of right and left hand, replace *x* variable with (*x* + *h*) ad (*x* – *h*) respectively in the function

   (ii)   Thus, obtained function *x*, should be replaced with point (assume a)

   (iii) Now at *h* → 0 determine the limit of function [viz function obtained by (ii) to be put in the above, put *h* = 0].

$\boxed{\textbf{Discriptive Examples}}$

Example 1: At $x = 2$, find out the right hand and left hand limit of function $f(x) = \dfrac{1}{2+x}$.

Solution:

Right Hand Limit	Left Hand Limit
I. $\quad f(x+h) = \dfrac{1}{2+(x+h)}$	I. $\quad f(x-h) = \dfrac{1}{2+(x-h)}$
II. $\quad f(2+h) = \dfrac{1}{2+(2+h)} = \dfrac{1}{4+h}$	II. $\quad f(2-h) = \dfrac{1}{2+(2-h)} = \dfrac{1}{4-h}$
III. $\quad \lim\limits_{h\to 0} f(2+h) = \lim\limits_{h\to 0} \dfrac{1}{4+h}$	III. $\quad \lim\limits_{h\to 0} f(2-h) = \lim\limits_{h\to 0} \dfrac{1}{4-h}$
Or $\quad f(2+0) = \dfrac{1}{4}$	Or $\quad f(2-0) = \dfrac{1}{4}$.

Example 2: At $x = 0$, find out the right hand and left hand limit of function $f(x) = x \cos\left(\dfrac{1}{x}\right)$.

Solution:

Right Hand Limit	Left Hand Limit
I. $\quad f(x+h) = (x+h) \cos\dfrac{1}{x+h}$	I. $\quad f(x-h) = (x-h)\cos\dfrac{1}{x-h}$
II. $\quad f(0+h) = h\cos\dfrac{1}{h}$	II. $\quad f(0-h) = -h\cos\left(-\dfrac{1}{h}\right)$
III. $\quad \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} h\cos\dfrac{1}{h}$	III. $\quad \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} (-h)\cos\left(-\dfrac{1}{h}\right)$
$\quad = \lim\limits_{h\to 0} h \times \lim\limits_{h\to 0} \cos\dfrac{1}{h}$	$\quad = \lim\limits_{h\to 0} (-h) \lim\limits_{h\to 0} \cos\left(-\dfrac{1}{h}\right)$
$\quad = 0$ [finite value in between -1 or 1]	$\quad = -0$ [finite value in between -1 or 1
Or $\quad f(0+0) = 0$	Or $\quad f(0-0) = 0$

2.4 Existence of Limit

If at $x = a$ both the limits of right and left hand of any function $f(x)$ exist and are equal, then at $x = a$, there is existence of limit of function $f(x)$ viz $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = l$ (Assumed)

Here l is called limit of the function and this is expressed as under $\lim\limits_{x\to a} f(x) = l$.

2.5 Distinction Between Limit and Value of a Function *f*(*x*) on *x* = *a*

$\lim\limits_{x \to a} f(x) = l$	$f(a)$
1. Limit of the function is a number, towards which function trends to respective when the value of independent variable x tends to a.	A number which is obtained putting $x = a$ in function $f(x)$
2. To know h limit, we get to study the values of to $f(x)$ in the small neighborhood of $x = a$ and conclude	To find out the value at $x = a$, we get find out the value of $f(x)$ at only $x=a$
3. At $x = a$, there can be existence of limit f function	At $x = a$ there can not be the value of function

2.6 Value and Limit of a Function

For the any value a of variable x, value and marginal value of function $f(x)$ should be different is not necessary. There are some kinds of function for which at $x = a$, value and marginal value of $f(x)$ are equal viz $f(a) = \lim\limits_{x \to a} f(x)$.

2.7 Theorems on Limits

Did u know? If $f(x)$ and $\phi(x)$ are two function and then $\lim\limits_{x \to a} f(x) = A$, $\lim\limits_{x \to a} f(x) = B$

then, (*i*) $\lim\limits_{x \to a} \{f(x) \pm f(x)\} = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} \phi(x) = A \pm B.$

(*ii*) $\lim\limits_{x \to a} \{k\, f(x)\} = k \lim\limits_{x \to a} f(x) = kA.$

(*iii*) $\lim\limits_{x \to a} \{f(x)\, \phi(x)\} = \lim\limits_{x \to a} f(x). \lim\limits_{x \to a} \phi(x) = AB$

Which has the property that limit of function f(x) is any number (A assumed)

(*iv*) $\lim\limits_{x \to a} \dfrac{f(x)}{\phi(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} \phi(x)} = \dfrac{A}{B}$] if $B \neq 0.$

(*v*) $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to a} f(h + a).$

2.8 Method of Finding the Limit of any Function

Although to find out the limit, we should find out the limits of right and left hand, but the intermediate level, we can often find out the limit directly.

If $y = f(x) = \dfrac{\phi(x)}{\Psi(x)}$, then to find out the limit, there would be following four types:

Type I: where $x \to a$: if $\phi(a) = 0$, $\Psi(a) = 0$, $f(a) = \dfrac{0}{0}$ then which is accessible. Limit of this be found out in the following way

Note

(i) Solving the function and assuming $x \neq a$, remove the common factors of numerator and denominator and

(ii) Then find out the value assigning a in place of x in the balance factors. This will be the limit of function $t \, x \rightarrow a$.

Type 2: Assigning $x = a + h$ to solve the limit of $f(x)$ at $x \rightarrow a$, here h is an indication of positive or negative increment. Find out the limit of $f(a+h)$. Where $h \rightarrow 0$ viz putting $h = 0$ in $f(a+h)$, get the value of function. This will be the limit of function at $x \rightarrow a$, because when

$$X = a + h \text{ and } x \rightarrow a \text{ then } h - 0$$

Self Assessment

1. Fill in the blanks:

(i) If any number tends to A, then at $x = a$ this number A of function $f(x)$ is called………..

(ii) If the limit of function is obtained from the value of left hand of independent variable, then it is called…………………

(iii) If at $x = a$ both the limits of right and left hand of any function $f(x)$ exist and are equal, then at $x = a$, there is ………………….. of limit of function $f(x)$.

(iv) For the any value a of variable x, value and …………… value of function $f(x)$ should be different is not necessary.

(v) Although to find out the limit, we should find out the limits of right and left hand, but the ………………….. level, we can often find out the limit directly.

EXAMPLES WITH SOLUTION

Sometimes it is not easy to divide with common factors. In this state applying the (ii) method, limit can be determined easily. Following examples make it clear. If it is not possible to divide the numerator and denominator with a common factor which is not zero, then after expansion in series or transformation, it can be possible.

Example 1: Find out the value of $\displaystyle\lim_{x \to a} \frac{x^3 - a^3}{x - a}$.

Solution:

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} \frac{(x - a)(x^2 + a^2 + ax)}{(x - a)}$$

$$= \lim_{x \to a} (x^2 + a^2 + ax), \qquad\qquad [\because x \neq a]$$

$$= a^2 + a^2 + a^2 = 3a^2. \qquad\qquad \textbf{Ans.}$$

Example 2: Find out the value of $\displaystyle\lim_{x \to 0} \frac{\log_e (1 + x) - x}{x^2}$.

Solution:

$$\lim_{x \to 0} \frac{\log_e (1 + x) - x}{x^2} = \lim_{x \to 0} \frac{\left[x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots \right] - x}{x^2}$$

$$= \lim_{x \to 0} \frac{-\dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots}{x^2} = \lim_{x \to 0} \frac{x^2 \left[-\dfrac{1}{2} + \dfrac{x}{3} - \ldots \right]}{x^2}$$

$$= \lim_{x \to 0} \left[-\frac{1}{2} + \frac{x}{3} - \cdots \right]$$

$$= -\frac{1}{2} + 0 - \cdots = -\frac{1}{2} .$$ **Ans.**

Task Find the value of $\displaystyle\lim_{x \to b} \frac{x^3 - b^3}{x - b}$ [**Ans.:** $3b^2$]

Example 3: Find out the value of $\displaystyle\lim_{x \to a} \left(\frac{x^2 - 2ax + a^2}{x - a} \right) .$

Solution: $\displaystyle\lim_{x \to a} \left(\frac{x^2 - 2ax + a^2}{x - a} \right) = \lim_{x \to a} \frac{(x - a)^2}{x - a}$

$$= \lim_{x \to a} (x - a) = a - a = 0.$$ **Ans.**

Example 4: Find out the value of $\displaystyle\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 + x - 6} .$

Solution: $\displaystyle\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 + x - 6} = \lim_{x \to 2} \frac{(x - 2)(x - 1)}{(x - 2)(x + 3)} = \lim_{x \to 2} \frac{(x - 1)}{(x + 3)}$ $[\because x \neq 2]$

$$= \frac{2 - 1}{2 + 3} = \frac{1}{5} .$$ **Ans.**

Type 3: Limit of Irrational Function

If $y = f(x)$ and $f(x)$ is an irrational function, then to find out the limit of $f(x)$ first of all, the function should be made rational and then applying 1 method, find out the limit

Example 5: Find out the value of $\displaystyle\lim_{x \to 0} \frac{\sqrt[3]{1 + x} - 1}{x} .$

Solution: $\displaystyle\lim_{x \to 0} \frac{\sqrt[3]{1 + x} - 1}{x} = \lim_{x \to 0} \frac{(1 + x)^{\frac{1}{3}} - 1}{x}$

$$= \lim_{x \to 0} \frac{1 + \dfrac{1}{3}x + \dfrac{\dfrac{1}{3}\left(\dfrac{1}{3} - 1\right)}{2!} x^2 + \cdots\cdots\infty - 1}{x}$$

$$= \lim_{x \to 0} \frac{1}{3} + \frac{\dfrac{1}{3}\left(-\dfrac{2}{3}\right) x}{2!} + \cdots\cdots \infty = \frac{1}{3} + 0 = \frac{1}{3} .$$ **Ans.**

Note

Example 6: Prove that $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/2} - (1-x)^{1/2}}{x} = 1.$

Solution: $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/2} - (1-x)^{1/2}}{x} = \lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$$= \lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \dfrac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \lim\limits_{x \to 0} \dfrac{(1+x) - (1-x)}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)} = \lim\limits_{x \to 0} \dfrac{2x}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)}$$

$$= \lim\limits_{x \to 0} \dfrac{2}{\sqrt{1+x} + \sqrt{1-x}} = \dfrac{2}{1+1} = 1. \qquad \textbf{Ans.}$$

Type 4: To know the limit of function when $x \to \infty$.

In this condition, first of all putting $x = \dfrac{1}{Z}$, change the form of limit from $x \to \infty$ to $Z \to 0$. Then applying method 3, find out the limit

Example 7: Find out the value of $\lim\limits_{x \to \infty} \dfrac{9x^2 + 3x + 7}{5x^2 + 2x + 1}.$

Solution: It is clear that if $x = \infty$ is put, then numerator and denominator would become $\dfrac{\infty}{\infty}$, therefore dividing both numerator and denominator with x^2

$$\lim\limits_{x \to \infty} \dfrac{9x^2 + 3x + 7}{5x^2 + 2x + 1} = \lim\limits_{x \to \infty} \dfrac{9 + \dfrac{3}{x} + \dfrac{7}{x^2}}{5 + \dfrac{2}{x} + \dfrac{1}{x^2}} = \dfrac{9 + 0 + 0}{5 + 0 + 0}$$

$$= \dfrac{9}{5}, \text{ whereas } x \to \infty$$

Since $x \to \infty$ therefore, $\dfrac{3}{x}, \dfrac{7}{x^2}, \dfrac{2}{5x}, \dfrac{1}{x^2}$ will tend to zero and function will tend to $\dfrac{9}{5}$

Therefore, $\lim\limits_{x \to \infty} \dfrac{9x^2 + 3x + 7}{5x^2 + 2x + 1} = \dfrac{9}{5}.$ \qquad **Ans.**

Example 8: Prove that $\lim\limits_{x \to a} \dfrac{x^m - a^m}{x - a} = ma^{m-1}$ **where m is a rational number.**

Solution:

First method: \qquad Left Hand $= \lim\limits_{x \to a} \dfrac{x^m - a^m}{x - a}$

Note

$$= \lim_{x \to a} \frac{(x - a)(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + ... + a^{m-1})}{(x - a)}$$

$$= \lim_{x \to a} (x^{m-1} + x^{m-2}a + x^{m-3}a^2 + ... + a^{m-1}) \text{ ,(when } x \neq a)$$

$$= a^{m-1} + a^{m-1} + ... + a^{m-1} = ma^{m-1} = \text{Left Hand.}$$

Second Method: $\lim_{x \to a} \dfrac{x^m - a^m}{x - a}$,   (if $x = a + h$ whereas $x \to a$, $h \to 0$)

$$= \lim_{h \to 0} \frac{(a + h)^m - a^m}{(a + h) - a} = \lim_{h \to 0} \frac{a^m \left(1 + \frac{h}{a}\right)^m - a^m}{h}$$

$$= \lim_{h \to 0} \frac{a^m \left\{1 + m\left(\frac{h}{a}\right) + \frac{m(m-1)}{2}\left(\frac{h}{a}\right)^2 + ... - 1\right\}}{h}$$

$$= \lim_{h \to 0} \frac{a^m h \left\{\frac{m}{a} + \frac{m(m-1)}{2}\frac{h}{a^2} + ...\right\}}{h}$$

$$= \lim_{h \to 0} a^m \left\{\frac{m}{a} + \frac{m(m-1)}{2!}\frac{h}{a^2} + ...\right\}, (h \neq 0)$$

$$= a^m \times \frac{m}{a} = ma^{m-1} = \text{Left Hand.}$$

Notes   To determine the limit of algebraic function, this is used in the form of major formula.

Example 9: Find out the value of $\lim_{x \to 0}\left(1 + \dfrac{p}{x}\right)^x$.

Solution: $\lim_{x \to \infty}\left(1 + \dfrac{p}{x}\right)^x$

Upper limit $\lim_{x \to \infty}\left(1 + \dfrac{1}{x}\right)^x = e$

$$\lim_{x \to \infty}\left(1 + \frac{p}{x}\right)^x = e^p, \text{ or } \lim_{x/p \to \infty}\left\{\left(1 + \frac{p}{x/p}\right)^{x/p}\right\}^p = e^p.$$ **Ans.**

Example 10: Find out the value of $\lim_{x \to 1}\left(\dfrac{\log x}{1 - x}\right)$.

Solution: $x = 1 + h$, where h is smallest

Note

If
$$\lim_{x \to 1} \frac{\log x}{1-x} = \lim_{h \to 0} \frac{\log(1+h)}{-h} = \lim_{h \to 0} \frac{h - \frac{h^2}{2} + \frac{h^3}{3} - \dots}{-h}$$

$$= \lim_{h \to 0} - \left(1 - \frac{h}{2} + \frac{h^2}{3} - \dots \right) \text{ (when } h \neq 0) = -1. \qquad \textbf{Ans.}$$

Example 11: Find out the value of $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x}$ **.**

Solution:
$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \frac{1}{x} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \right) \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[2 \left(x + \frac{x^3}{3!} + \dots \right) \right]$$

$$= \lim_{x \to 0} \left[2 \left(1 + \frac{x^2}{3!} + \dots \right) \right] \qquad [\because \quad x \neq 0]$$

$$= 2. \qquad \textbf{Ans.}$$

Example 12: Find out the value of $\lim_{x \to 0} \dfrac{e^x - 1}{x}$ **.**

Solution:
$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \infty \right) - 1}{x}$$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty \right)}{x} = \lim_{x \to 0} \frac{x \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots \infty \right)}{x}$$

$$= \lim_{x \to a} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots \infty \right)$$

$$= \text{(when } x \neq 0) = 1. \qquad \textbf{Ans.}$$

Example 13: Find out the value of $\lim_{x \to \infty} \dfrac{1 + 2 + 3 + \dots + x}{x^2}$ **.**

Solution:
$$\lim_{x \to \infty} \frac{1 + 2 + 3 + \dots + x}{x^2} = \lim_{x \to \infty} \frac{x(x+1)}{2x^2}$$

$$= \lim_{x \to \infty} \frac{x^2 \left(1 + \frac{1}{x} \right)}{2x^2} = \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{2}$$

$$= \frac{1}{2}. \qquad \left[\because \lim_{x \to \infty} \frac{1}{x} = 0 \right]$$

Therefore, the desired value is $= \dfrac{1}{2}$. $\qquad \textbf{Ans.}$

Example 14: Prove that $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} = \log_e \dfrac{a}{b}.$ **or find out the value of** $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x}.$

Solution: It is clear that when x = 0 then numerator and denominator become 0. Therefore expanding a^x and b^x with the help of exponential theory

$$\text{Left hand} = \lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{e^{x \log_e a} - e^{x \log_e b}}{x}$$

$$= \lim_{x \to 0} \frac{\left[1 + x \log_e a + \dfrac{(x \log_e a)^2}{2!} + ...\right] - \left[1 + x \log_e b + \dfrac{(x \log_e b)^2}{2!} + ...\right]}{x}$$

$$= \lim_{x \to 0} \left\{ \log_e a - \log_e b + \frac{x \left[(\log_e a)^2 - (\log_e b)^2\right]}{2!} + ... \right\}$$

$$= \log_e a - \log_e b = \log_e \left(\frac{a}{b}\right) = \text{Right hand} \qquad \textbf{[Proven]}$$

Example 15: Function f(x) is defined as under

$$f(x) = \begin{cases} 1, & \text{when} \quad x > 0 \\ -1, & \text{when} \quad x < 0 \\ 0, & \text{when} \quad x = 0 \end{cases}$$

Show that $\lim\limits_{x \to \infty} f(x)$ is not in existence

Solution: here $f(0 + h) = 1$.

So that   Right hand limit = $f(0 + 0) = \lim\limits_{x \to 0+0} f(x) = \lim\limits_{x \to 0+0} (1) = 1$

and $f(0 - h) = -1$

Therefore   Left hand limit = $f(0 - 0) = \lim\limits_{x \to 0-0} f(x) = \lim\limits_{x \to 0-0} (-1) = -1.$

$\because$ $\qquad\qquad\qquad\qquad \lim\limits_{x \to 0+0} f(x) \neq \lim\limits_{x \to 0-0} f(x)$

$\lim\limits_{x \to 0} f(x)$ is not in existence

Example 16: Show that $\lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$ **is not in existence.**

Solution: Where $x > 2, |x - 2| = (x - 2)$

$\therefore \qquad\qquad\qquad$ R.H.L. $= \lim\limits_{x \to 2+0} f(x) = \lim\limits_{x \to 2+0} \dfrac{|x - 2|}{x - 2} = \lim\limits_{x \to 2+0} \dfrac{x - 2}{x - 2} = 1$

$\qquad\qquad\qquad$ L.H.L. $= \lim\limits_{x \to 2-0} f(x) = \lim\limits_{x \to 2-0} \dfrac{|x - 2|}{x - 2} = \lim\limits_{x \to 2-0} \dfrac{-(x - 2)}{x - 2} = -1$

$\therefore \qquad\qquad\qquad$ R.H.L. $\neq$ L.H.L.

Therefore, $\lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$ is not in existence

Note

Example 17: Function $f(x)$ for which $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$, show that $\lim_{x \to 1} f(x) = 1$.

Solution: $\lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} f(1+h) = \lim_{h \to 0} (1+h)^2 = 1$

and $\lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} f(1-h) = \lim_{h \to 0} (1-h)^2 = 1$

$\lim_{x \to 1+0} f(x) = \lim_{x \to 1-0} f(x) = 1$

Therefore, $\lim_{x \to 1} f(x) = 1$. **Ans.**

Example 18: Function $f(x)$ is defined as under.

$$f(x) = \begin{cases} x, & \text{when } 0 \leq x < \dfrac{1}{2} \\ 0, & \text{when } x = \dfrac{1}{2} \\ 1-x, & \text{when } \dfrac{1}{2} < x \leq 1 \end{cases}$$

Find out the value of $\lim_{x \to 1/2} f(x)$

Solution: R.H.L. $= f\left(\dfrac{1}{2}+0\right) = \lim_{x \to 1/2+0} f(x)$

$$= \lim_{h \to 0} f\left(\dfrac{1}{2}+h\right) = \lim_{h \to 0}\left[1-\dfrac{1}{2}-h\right] = \lim_{h \to 0}\left[\dfrac{1}{2}-h\right] = \dfrac{1}{2}$$

L.H.L. $= f\left(\dfrac{1}{2}-0\right) = \lim_{x \to 1/2-0} f(x) = \lim_{h \to 0} f\left(\dfrac{1}{2}-h\right)$

$$= \lim_{h \to 0}\left(\dfrac{1}{2}-h\right) = \dfrac{1}{2}$$

∴ R.H.L. = L.H.L.

Therefore $\lim_{x \to 1/2} f(x) = \dfrac{1}{2}$. **Ans.**

Questionnaire 2.1

Find out the value of following:

1. $\lim_{x \to 0} (7x^2 - 5x + 1)$.

2. $\lim_{x \to 0} \dfrac{e^x - 1}{e^x}$.

3. $\lim_{x \to 0} \dfrac{e^{x/2} - 1}{x}$.

4. $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

5. $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

6. $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

7. $\lim_{x\to 0}\dfrac{y^2}{x}$ where $y^2 = ax + bx^2 + cx^3$.

8. $\lim_{x\to\infty}\dfrac{x^2}{1 + x^2}$.

9. $\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}$.

10. $\lim_{x\to a}\dfrac{x^4 - a^4}{x^2 - a^2}$.

11. $\lim_{x\to 1}\dfrac{x^3 - 1}{x^2 - 1}$.

12. $\lim_{x\to 0}\dfrac{x^3 - 8}{x - 2}$.

13. $\lim_{x\to a}\dfrac{x^5 - a^5}{x^2 - a^2}$.

14. $\lim_{x\to 1}\dfrac{x^n - 1}{x - 1}$.

15. $\lim_{x\to 2}\dfrac{x^2 - 5x + 6}{x^2 - 6x + 8}$.

16. $\lim_{x\to\infty}\dfrac{3x^2 + 2x - 5}{x^2 + 5x + 1}$.

17. $\lim_{x\to\infty}\left(1 - \dfrac{1}{x}\right)^x$.

18. $\lim_{x\to 0}\dfrac{x}{\sqrt{1 + x} - 1}$.

19. If $f(x) = |x|$, then show that $\lim_{x\to 0}\ f(x) = 0$.

20. If $f(x) = \dfrac{|x - 1|}{x - 1}$. then show that $\lim_{x\to 1}\ f(x)$ is not in existence

21. Function f(x) is defined as under:

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 2, & \text{if } x = 0 \\ 2 - x, & \text{if } x \ge 1 \end{cases}$$

Show that $\lim_{x\to 1} f(x) = 1$.

22. Prove that $\lim_{x\to\infty}\dfrac{e^{x^2} - 1}{e^{x^2} + 1} = 1$.

23. Find out the value of $\lim_{x\to 1}\dfrac{x^m - 1}{x^n - 1}$.

Answers

1. 1
2. 0
3. $\dfrac{1}{2}$
4. 3
5. 4

6. $\dfrac{1}{2}$
7. a
8. 1
9. 1
10. $2a^2$

11. $\dfrac{3}{2}$
12. 4
13. $\dfrac{5}{2}a^3$
14. n
15. $\dfrac{1}{2}$

16. 3
17. $\dfrac{1}{e}$
18. 2
23. $\dfrac{m}{n}$

2.9　Geometrical Definition

Continuity and Discontinuity: If drawing a graph of any function $f(x)$, the curve which we obtain. And it is in such a way that it does not break on any point $x = a$, then the function is called continuous on that point.
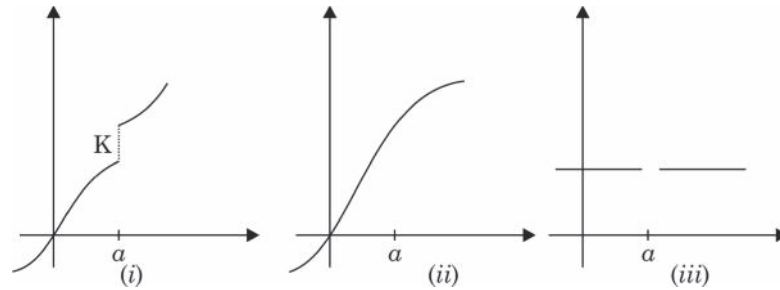
Contrary to this if at any point $x = a$, the graph of function $f(x)$ breaks, then at point $x = a$, function $f(x)$ will be called discontinuous. Looking at the graphs of the functions it is understood if in figure (i) of $f(x)$ at $x = a$, any height k is a jump that if we are moving from left hand to right hand of the graphs, then $x = a$ is a break and we have to lift the pencil from one side of $x = a$ to another side. The curve of function (iii) at $x = a$ is break. Function (ii) is not break at $x = a$ viz the curve is in such a way that it does not break. Therefore, at $x = a$, function $f(x)$ is continuous.

Example 1: If we draw the graph of function $y = x^2$, then this is a parabola as shown in the shape. On the graph of such functions if we from right to left or from left to right, then we will not observe any break and if we draw pencil on this graph from one side to another side, then in between we will not be required to lift the pencil. This is a continuous function.
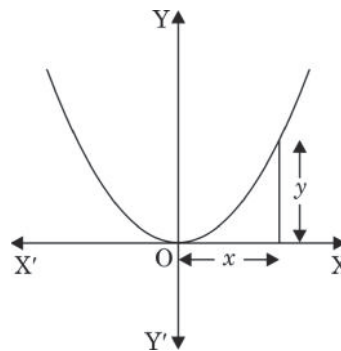
Example 2: If we draw the graph of , then we will find two parabola and one point (1,1) as shown in the figure. Here we move from right to left or left to right, then at $x = 1$ there is a break in sequence. And if we start moving from one side of $x = 1$ to another side, then at $x = 1$ we have to lift the pencil. This function is discontinuous.

$$y = \begin{cases} \dfrac{1}{2}x^2, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ 1 + \dfrac{1}{2}x^2, & \text{if } x > 1 \end{cases}$$
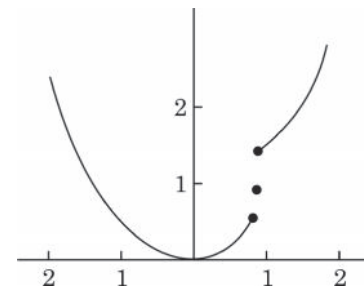
We see that if at any point the function is discontinuous then at the point there is a sudden jump in the value of function, whereas

at the change of value of x, the value of function changes gradually. Thus, if at any point $x = a$, **Note**
function is continuous then the value of $|f(x)\text{-}f(a)|$, can be scaled down at the small neighborhood of
a choosing the smallest value of x.

2.10 Continuity of a Function at any point

Cauchy's definition: A function $f(x)$ is called continuous at $x = a$. If for one chosen arbitrary positive
number ε, which is the smallest one, but not a zero, we can obtain a positive number δ based on ε that
$|f(x) - f(a)| < \varepsilon$, for every values of x, for which $0 < |x - a| < \delta$

Viz at any point x of interval $(a\text{-}\delta\ a\text{+}\delta)$, the positive difference of function $f(x)$ and $f(a)$ is smaller to
arbitrarily defined positive number e, then at x = a, function $f(x)$ is called continuous.

Alternate definition: Function $f(x)$ is called continuous at $x = a$ if there is existence of $\lim\limits_{x \to a}$ and that

is equal to the value of function at $x = a$.

Notes $f(x) = f(a)$ viz limit of $[f(x)$ where $x \to a] = $ [Value of function, where $x = a$]

Therefore, if $f(a+0) = f(a - 0)$, then $f(x)$ at $x = a$ is called continuous otherwise it is discontinuous.
Function $f(x)$ is called continuous on point $x = a$, if the function satisfies three conditions:

1. If at $x = a$, function $f(x)$ is defined viz there is a certain value of function at $x = a$

2. When the value of x tends to a, and $f(x)$ inclines towards any limit viz there is existence of $\lim\limits_{x \to a}$
    $f(x)$

3. $\lim\limits_{x \to a} f(x) = f(a)$

2.11 Geometrical Meaning of Continuity

Assume QR is the graph of function $f(x)$. In this graph
consider on point $P[a,f(a)]$. If ε is any arbitrarily taken
positive number, then lines $y = f(a) - \varepsilon$, $y = f(a) + \varepsilon$ will
be parallel to x-axis and down and up to P.

If each points of the graph of function $f(x)$ is between the
two lines $x = a - \delta$, $x = a + \delta$ is also available in between
$y = a - \varepsilon = a + \varepsilon$, then function $f(x)$ at $x = a$ is continuous.

For example: (i) Constant function $f(x) = c$, for each real
value of x is continuous.

(ii) function $f(x) = \sin x$ and $f(x) = \cos x$, for each real
value of x is continuous.
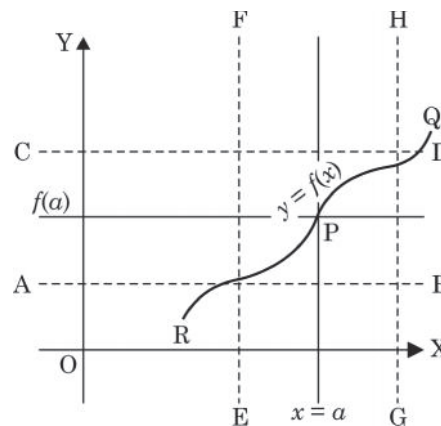
2.12 Method to Finding Continuity of a Function at any Point

From the definition of limit it is clear that existence of $\lim\limits_{x \to a} f(x)$ can happen only

when left hand limit of $f(x)$ = right hand limit of $f(x)$ viz

Note

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Viz $f(a - 0) = f(a + 0)$

Therefore, to show the continuity of function $f(x)$ at point $x = a$, we should show that at this point left hand limit of $f(x)$ = right hand limit of $f(x)$ = value of function viz

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Or $f(a - 0) = f(a + 0) = f(a)$

For Right Hand Limit in $f(x)$, put $x = a + h$, where $h \to 0$ where $x \to a$

For Left Hand Limit in $f(x)$, put $x = a - h$, where $h \to 0$ where $x \to a$

2.13   Continuity of a Function in an Interval

A function $f(x)$ is called continuous in an open interval (a,b) if it is continuous for every values of x in this interval (a,b)

Function $f(x)$ is called continuous in any closed interval, if

(i)   It is continuous for each value of x for which $a < x < b$

(ii) $\lim_{x \to a+0} f(x) = f(a)$

(iii) $\lim_{x \to b-0} f(x) = f(b)$.

Viz function is continuous in open interval (a,b) and at $x = a$ from right side and at $x = b$ from left hand side is continuous.

Discontinuity in an interval: Function $f(x)$ is called discontinuous in any interval if it is discontinuous at any or many points of interval.

2.14   Theorem on Continuous Functions

(i)   If both $f(x)$ and $g(x)$ at any point $x=a$ is continuous, then $f(x) \pm g(x)$ also will be continuous on $x = a$.

(ii)   If both $f(x)$ and $g(x)$ at any point $x=a$ is continuous, then $f(x)\, g(x)$ also will be continuous on $x = a$.

(iii)   If $f(x)$ at any point $x = a$ is continuous and k is a certain real number, then $kf(x)$ also will be continuous on $x = a$.

(iv)   If $f(x)$ and $g(x)$ at any point $x=a$ is continuous and $g(a) \neq 0$, then $\dfrac{f(x)}{g(x)}$ also will be continuous on $x = a$.

(v)   If $f(x)$ at $x=a$ is continuous and $f(a) \neq 0$ then $\dfrac{1}{f(x)}$ also will be continuous on $x = a$.

(vi)   If $f(x)$ at $x=a$ is continuous then at $x=a$ $f(x)$ is also continuous.

Example 1: Express that function $f(x) = x^2 + 1$ is continuous on $x = 2$.

Solution: $\lim_{x \to 2} f(x) = 5 = f(2)$ Therefore, function x = 2 is continuous

Example 2: At $x = 2$ function $f(x) = \dfrac{1}{x-2}$ is discontinuous. Prove.

Solution: (i) $f(2)$ is not defined (denominator is 0)

(ii) $\lim\limits_{x \to 2} f(x)$ has no existence (is equal $= \infty$)

leaving $x = 2$, at every point, function is continuous. Therefore at $x = 2$ function is discontinuous.
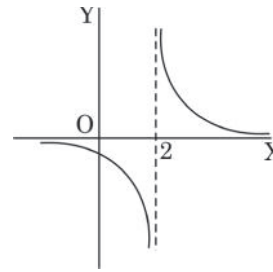
Example 3: Explain that at $x = 2$ function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous.

Solution: (i) $f(2)$ is not defined (numerator and denominator are zero)

(ii) $\lim\limits_{x \to 2} f(x) = 4$

Therefore function x= 2 is discontinuous

Discontinuity in example 3 can be escaped, because further $f(x) = \dfrac{x^2 - 4}{x - 2}$

defining the function at $f(2) = 4$

The graph of and $g(x) = x+2$ are same, whereas first one has hole.

Discontinuity in example 2 can not be ignored, because there is no existence of limit.
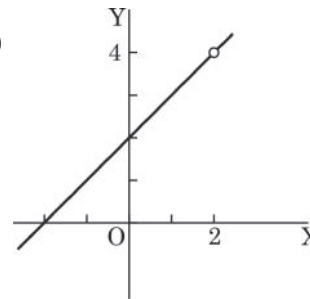
Example 4: If function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous on main point.

Solution: $f(x)$ is continuous on $x = a$ because $f(0) = 1$ (given)

And $\qquad \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 = f(0)$

Example 5: Explain that function $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$ is discontinuous on main point.

Solution: $f(x)$ is discontinuous on x = a because $f(0) = 1$

And $\qquad \lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

$\qquad\qquad = \lim\limits_{x \to 0} \dfrac{2 \sin x}{x} \cdot \cos x = 2.1 \cos 0^0 = 2$

Or $\qquad\qquad\qquad f(0) \neq \lim\limits_{x \to 0} f(x).$

Note

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Illustrations

</div>

Example 1: Draw the graph of $f(x) = \begin{cases} 2, & x < 0 \\ x, & x \geq 0 \end{cases}$ **. Prove that function is discontinuous on main point.**

Solution: It is easy to draw the graph of $f(x)$. It is shown in the figure. It is clear that at point $x = 0$ there is a jump in the graph. Now we will verify the limits of right and left hand side.

For $\quad x \geq 0$

$x \geq 0 \hspace{8em} f(x) = x$
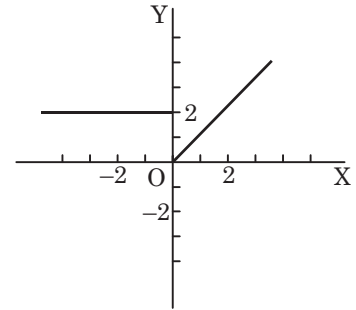
$$f(x + h) = (x + h)$$

$$f(0 + h) = (0 + h)$$

$$\left\{ \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} h \right.$$

$\therefore \hspace{10em} f(0 + 0) = 0 \hspace{8em} ...(i)$

Left Hand Limit at main point (for x<0)

$x < 0 \hspace{10em} f(x) = 2$

$$f(x - h) = 2$$

$$f(0 - h) = 2$$

$$\lim_{h \to 0} f(0 - h) = 2$$

$\therefore \hspace{10em} f(0 - 0) = 2 \hspace{8em} ...(ii)$

Thus from (i) and (ii) $\hspace{4em} f(0 + 0) \neq f(0 - 0)$

Therefore at x=0, function f(x) is discontinuous.

Example 2: If $f(x) = |x - 1| + |x + 2|$ **where x is any real number, then prove that function** $f(x)$ **is continuous on** $x = 1$ **and** $x = 2$ **.**

Solution:

Right Hand Limit at $x = 1$

$$f(x) = |x - 1| + |x + 2|$$

$$f(x + h) = |x + h - 1| + |x + h + 2|$$

$$f(1 + h) = |1 + h - 1| + |1 + h + 2|$$

$$\lim_{h \to 0} f(1 + h) = \lim_{h \to 0} [|h| + |3 + h|]$$

$$= |0| + |3 + 0| = 3$$

Left Hand Limit at $x = 1$

$$f(1 - h) = |1 - h - 1| + |1 - h + 2|$$

$$f(1 - h) = |1 - h| + |3 - h|$$

$$\lim_{h \to 0} f(1 - h) = \lim_{h \to 0} [|h| + |3 - h|]$$

$$= |0| + |3 - 0| = 3$$

$$\text{L.H.L.} = \text{R.H.L.}$$

Therefore $x = 1$ is continuous

Right Hand Limit at $x = -2$

$$f(x) = |x - 1| + |x + 2|$$
$$f(-2 + h) = |-2 + h - 1| + |-2 + h + 2|$$
$$= |h - 3| + |h|$$
$$\lim_{h \to 0} f(-2 + h) = \lim_{h \to 0} [|h - 3| + |h|]$$
$$= |-3| + |0| = 3$$

Left Hand Limit at x = -2

$$f(x) = |x - 1| + |x + 2|$$
$$f(-2 - h) = |-2 - h - 1| + |-2 - h + 2|$$
$$= |-3 - h| + |-h|$$
$$= |3 + h| + |h|$$
$$\lim_{h \to 0} f(-2 - h) = \lim_{h \to 0} [|3 + h| + |h|]$$
$$= 3$$
$$\text{L.H.S.} = \text{R.H.S.}$$

Therefore $x = 1$ is continuous

Example 3: Show that at $x = 1$, function $f(x) = \dfrac{1}{x - a}$, is discontinuous.

Solution: $\text{L.H.L.} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$

$$= \lim_{h \to 0} \frac{1}{(a - h) - a} = \lim_{h \to 0} \frac{1}{-h} = -\infty$$

$\text{R.H.L.} = \lim_{x \to a + 0} f(x) = \lim_{h \to 0} f(a + h)$

$$= \lim_{h \to 0} \frac{1}{(a + h) - a} = \lim_{h \to 0} \frac{1}{h} = \infty$$

∴ $\text{L.H.L.} \neq \text{R.H.L.}$

Therefore at $x = a$ function is discontinuous.   **Ans.**

Example 4: Express that at $x = 0$ $f(x) = |x|$ is continuous.

Solution: Here $f(0) = |0| = 0$

$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} |0 - h| = \lim_{h \to 0} (h) = 0$$

$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$$= \lim_{h \to 0} |0 + h| = \lim_{h \to 0} (h) = 0$$

∴ $\text{L.H.L.} = \text{R.H.L.} = f(0)$

Therefore at x = 0 function is continuous   **Ans.**

Example 5: Express that at $x = 1$ $f(x) = \begin{cases} x^2, \text{ if } x \neq 1, \\ 2, \text{ if } x = 1. \end{cases}$ is discontinuous.

Solution: Given that $f(1) = 2$

Note

$$\lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h)^2 = 1$$

Or

$$\lim_{x \to 1-0} f(x) = \lim_{h \to 0} (1-h) = \lim_{h \to 0} (1-h)^2 = 1$$

$\therefore$

$$\lim_{x \to 1} f(x) = 1 \neq f(1)$$

Thus function is discontinuous at x = 1.

Example 6: For which value of *k* function $f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4}, & x \neq 4 \\ k, & x = 4 \end{cases}$ is continuous at $x = 4$.

Solution: Since at $x = 4$, value of function $f(x)$ is k

$\therefore$ $\qquad\qquad\qquad\qquad f(4) = k$

$x = 4$ $\qquad\qquad$ R.H.L. $= \lim\limits_{x \to 4+0} f(x) = \lim\limits_{h \to 0} f(4+h)$

$$= \lim_{h \to 0} \frac{(4+h)^2 - 16}{4+h-4} = \lim_{h \to 0} \frac{16 + h^2 + 8h - 16}{h}$$

$$= \lim_{h \to 0} \frac{h(h+8)}{h} = \lim_{h \to 0} (h+8), \ \because h \neq 0$$

$$= 8$$

L.H.L. $= \lim\limits_{x \to 4-0} f(x) = \lim\limits_{h \to 0} f(4-h)$

$$= \lim_{h \to 0} \frac{(4-h)^2 - 16}{4-h-4} = \lim_{h \to 0} \frac{16 + h^2 - 8h - 16}{-h}$$

$$= \lim_{h \to 0} (-h + 8), h \neq 0$$

$$= 8$$

Given that at $x \neq 4$ function is continuous

$$\text{L.H.L.} = \text{R.H.L.} = f(4)$$
$$8 = 8 = k$$

Therefore, $\qquad\qquad\qquad k = 8.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **Ans.**

Task $\quad$ For which value of *k* function $f(x) \begin{cases} \dfrac{x^2 - 25}{x - 5}, & x \neq 5 \\ k, & x = 5 \end{cases}$ is continuous at $x = 5$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **Ans.: $k = 10$**

Example 7: Examine the continuity of function $f(x)$ at point $(0,0)$ when $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$

Solution: Here $\qquad\qquad f(0) = 1$

L.H.L. $= \lim\limits_{x \to 0-0} f(x) = \lim\limits_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} \frac{|0-h|}{0-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} f(x)$$

$$= \lim_{h \to 0} \frac{|0 - h|}{0 + h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

$\therefore$ R.H.L. $\neq$ L.H.L.

Thus at $x=0$ function is discontinuous

Therefore, LHL = RHL = $f(1)$

Therefore at $x = 1$ function is continuous.

Example 8: If at $x = 1$ $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$ is a continuous function, then find out the value of a

and b.

Solution: Since the given function $f(x)$ at $x = 1$ is continuous. Therefore

$$\lim_{x \to 1} f(x) = f(1)$$

$\Rightarrow$
$$\lim_{x \to 1+0} f(x) = f(1)$$

$\Rightarrow$
$$\lim_{h \to 0} f(1 + h) = 11$$

$$\lim_{h \to 0} 3a(1 + h) + b = 11$$

$$3a + b = 11 \qquad\qquad \text{...(i)}$$

Therefore
$$\lim_{x \to 1-0} f(x) = 11$$

$\Rightarrow$
$$\lim_{h \to 0} 5a(1 - h) - 2b = 11$$

$\Rightarrow$
$$5a - 2b = 11 \qquad\qquad \text{...(ii)}$$

Solving the equations (i) and (ii)

$$a = 3, b = 5. \qquad\qquad \textbf{Ans.}$$

Example 9: Function $f(x)$ at an interval [0,1] is defined as under:

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \dfrac{1}{2} - x, & \text{if } 0 < x < \dfrac{1}{2} \\ \dfrac{1}{2}, & \text{if } x = \dfrac{1}{2} \\ \dfrac{2}{3} - x, & \text{if } \dfrac{1}{2} < x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Find out the points where function is discontinuous.

Solution: (i) at $x = 0$ continuity $f(0) = 0$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \left\{ \frac{1}{2} - (0 + h) \right\} = \frac{1}{2} \neq f(0)$$

Therefore function is discontinuous at $x = 0$

Note

(ii) At $x = \dfrac{1}{2}$, continuity $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$

$$\text{L.H.L.} = \lim_{x \to 1/2 - 0} f(x) = \lim_{h \to 0} f\left(\dfrac{1}{2} - h\right)$$

$$= \lim_{h \to 0} \left\{ \dfrac{1}{2} - \left(\dfrac{1}{2} - h\right) \right\} = 0$$

$$\text{R.H.L.} = \lim_{x \to 1/2 + 0} f(x) = \lim_{h \to 0} f\left(\dfrac{1}{2} + h\right)$$

$$= \lim_{h \to 0} \left\{ \dfrac{2}{3} - \left(\dfrac{1}{2} + h\right) \right\} = \lim_{h \to 0} \left(\dfrac{1}{6} - h \right) = \dfrac{1}{6}.$$

∴ $\qquad$ L.H.L. ≠ R.H.L.

Thus, function is discontinuous at $x = \dfrac{1}{2}$

(iii) $\qquad$ At x = 1, continuity f(1) = 1

$$\text{L.H.L.} = \lim_{x \to 1 - 0} f(x) = \lim_{x \to 1 - 0} f(x)$$

$$= \lim_{h \to 0} \left\{ \dfrac{2}{3} - (1 - h) \right\} = -\dfrac{1}{3} \neq f(1) \qquad \text{**Ans.**}$$

Thus function is discontinuous at x = a.

Example 10: Examine the continuity of function $f(x)$ at $x = 0,1$ when

$$f(x) = \mid x \mid + \mid x - 1 \mid \text{ or}$$

$$f(x) = \begin{cases} 1 - 2x, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x < 1 \\ 2x - 1, & \text{if } x \geq 1 \end{cases}$$

Solution: $\qquad$ (i) $\qquad$ Continuity at x = 0

$\qquad\qquad$ f(0) = 1

$$\text{R.H.L.} = \lim_{x \to 0 + 0} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} (1) = 1$$

$$\text{L.H.L.} = \lim_{x \to 0 - 0} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} (1 + 2h) = 1$$

∴ $\qquad$ L.H.L. = $f(0)$ = R.H.L.

Thus $\qquad$ LHL = $f(0)$ = RHL

Therefore function is continuous at $x = 0$

(ii) Continuity at $x = 1$

$$f(1) = (2 \times 1 - 1) = 1,$$

$$\text{L.H.L.} = \lim_{x \to 1-0} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1) = 1.$$

$$\text{R.H.L} = \lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \{2\,(1+h) - 1\}$$

$$= \lim_{h \to 0} \{1 + 2h\} = 1$$

$\therefore$ L.H.L. = $f(1)$ = R.H.L.

Therefore function is continuous at $x = 1$.

Example 11: Show that the function $f(x)$, $\forall x \in R$ which is defined as under

$$(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x}}, & x \neq 0 \\[2mm] 0, & x = 0 \end{cases} \text{ is discontinuous at } x = 0.$$

Solution: At $x = 0$, given that

$$f(0) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{...(i)}$$

RHL at $x = 0$

$$f(x + h) = \frac{e^{1/(x+h)}}{1 + e^{1/(x+h)}}$$

$$f(0 + h) = \frac{e^{1/(0+h)}}{1 + e^{1(0+h)}}$$

$\therefore$ $\lim\limits_{h \to 0} f(0 + h) = \lim\limits_{h \to 0} \dfrac{e^{1/h}}{1 + e^{1/h}} = \lim\limits_{h \to 0} \dfrac{1}{e^{-1/h+1}}$

Therefore $f(0 + 0) = \dfrac{1}{0 + 1} = 1,$ $\left[\because \lim\limits_{h \to 0} e^{-1/h} = 0\right]$ ...(ii)

LHL at $x = 0$ $f(x - h) = \dfrac{e^{1/(x-h)}}{1 + e^{1/(x-h)}}$

$$f(0 - h) = \frac{e^{1/(0-h)}}{1 + e^{1/(0-h)}} = \frac{e^{-1/h}}{1 + e^{-1/h}}$$

$\therefore$ $\lim\limits_{h \to 0} f(0 - h) = \lim\limits_{h \to 0} \dfrac{e^{-1/h}}{1 + e^{-1/h}}$

Therefore $f(0 - 0) = \dfrac{0}{1 + 0} = 0$ ...(iii)

From (ii) and (iii) RHL $\neq$ LHL

Therefore the given function is discontinuous at $x = 0$.

Example 12: A function $f(x)$ is defined as under

$$f(x) = \begin{cases} x, x < 1 \\ 2 - x, 1 \le x \le 2 \\ x - \dfrac{1}{2} x^2 \; x > 2. \end{cases} \qquad \text{...(i)}$$

If $f(x)$ is continuous at $x = 1$ and $x = 2$?

Solution: first of all we will examine $x = 1$

Therefore at $x=1$, $f(x) = 2 - x$ to $f(1) = 2 - 1 = 1$

RHL at $x = 1$, thus for $x > 1$ $f(x) = 2 - x$

Now
$$f(1 + 0) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} 2 - (1 + h)$$

$$= \lim_{h \to 0} 1 - h = 1 \qquad \text{...(ii)}$$

LHL at $x = 1$, thus $f(x) = x$

$$f(1 - 0) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} (1 - h) = 1 \qquad \text{...(iii)}$$

Therefore, at $x = 1$ from (i), (ii) and (iii) $f(1) = f(1+0) = f(1-0)$

Therefore at $x = 1$ $f(x)$ is continuous

again $x = 2$ to $f(x) = 2 - x$, $f(2) = 2 - 2 = 0$ $\qquad \text{...(iv)}$

RHL at $x = 2$, thus for $x > 2$ $x - \dfrac{1}{2} x^2$

$$f(2 + 0) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} (2 + h) - \frac{1}{2} (2 + h)^2$$

$$= \lim_{h \to 0} - \frac{1}{2} h^2 - h = 0; \qquad \text{...(v)}$$

LHL at $x = 2$, thus for $x < 2$, $f(x) = 2 - x$

$$f(2 - 0) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} 2 - (2 - h) = 0. \qquad \text{...(vi)}$$

Therefore from (iv), (v), (vi) $f(2) = f(2 + 0) = f(2 - 0) = 0$.

Thus at $x = 2$ $f(x)$ is continuous.

<div style="text-align:center">

Questionnaire 2.2

</div>

Find out the point of discontinuity, if any from the following functions and draw the graph:

1. $f(x) = \begin{cases} -1, x \le 0 \\ +1, x > 0. \end{cases}$

2. $f(x) = |x - 1|$.

3. $f(x) = |x| + |x - 1|$.

Show that following functions are continuous at the points given against them: **Note**

4.   Function at x = 1, 3 $f(x) = x^2 - 7x + 3$.

5. $f(x) = \begin{cases} \dfrac{1}{x+2}, & x \neq -2 \\ 0, & x = -2 \end{cases}$

6.   If $f(x) = \begin{cases} x - 4, & \text{whereas } x \geq 5 \\ 5x - 24, & \text{whereas } x < 5 \end{cases}$, then show that at $x = 5, p\ f(x)$ is continuous function

7.   Prove that function $f(x)$ where $f(x) = \begin{cases} x - 1, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$ is continuous at every points. Draw the diagram of the function.

Examine the continuity of following functions

8. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \neq a \\ 2a, & x = a \end{cases}$

10. $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

11. $f(x) = \begin{cases} x, & x \geq 0 \\ x^2, & x < 0. \end{cases}$

12. $f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$

13. $f(x) = \begin{cases} \dfrac{|x - 3|}{x - 3}, & x \neq 3 \\ 0, & x = 3. \end{cases}$

14. $f(x) = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

15. $f(x) = \begin{cases} \dfrac{1}{1 - e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

16. $f(x) = \begin{cases} e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

17. $f(x) = \begin{cases} \dfrac{1}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

18. $f(x) = \begin{cases} \cos \dfrac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

19. $f(x) = \begin{cases} x \cos \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

20. $f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

21. $f(x) = \begin{cases} 5x - 4, & 0 < x \leq 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$

22. $f(x) = \begin{cases} x + 2, & -1 < x < 3 \\ 5, & x = 3 \\ 8 - x, & x > 3 \end{cases}$

23.   At $x = 1$ and $x = 2$, find out the continuity of function $f(x) = \begin{cases} 0, \text{ if } 0 < x < 1 \\ x, \text{ if } 1 \leq x < 2 \\ \dfrac{x^3}{4}, \text{ if } 2 \leq x < 3 \end{cases}$

24. Find out the continuity of function $f(x)$ at $x = 0, 1, 2$ when

$$f(x) = \begin{cases} -x^2, & \text{if } x \le 0 \\ 5x - 4, & \text{if } 0 < x \le 1 \\ 4x^2 - 3x, & \text{if } 1 < x < 2 \\ 3x + 4, & \text{if } x \ge 2. \end{cases}$$

25. If $f(x) = x^2 + 1$ where $x \ne 1$ and $f(x) = 3$ where $x = 1$, then find out that at point $x = 1$ whether the function is continuous or discontinuous.

Answers

1. 0,   2. None   3. None   8. Continuous

9. Continuous   10. Discontinuous,   11. Continuous   12. Discontinuous

13. x = 3, Discontinuous   14. X = 0, Continuous   15. Discontinuous   16. Discontinuous

17. X = 0, Continuous   18. Discontinuous   19. Discontinuous   20. Discontinuous

21. Continuous   22. Continuous   23. Yes

24. Continuous, $x = 1, 2$ and Discontinuous at $x = 0$25. Discontinuous

Self Assessment

2. Multiple Choice Questions:

(i)   The value of $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a}$

    (a) $3a^2$ (b) $3a$ (c) $3b^2$ (d) $3b$

(ii) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = $ ......

    (a) $\dfrac{1}{e}$ (b) e (c) $-e$ (d) ∞

(iii) $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} = \log e$....

    (a) $\dfrac{b}{a}$ (b) $\dfrac{1}{b}$ (c) $\dfrac{a}{b}$ (d) $-\dfrac{a}{b}$

(iv) If both $f(x)$ and $g(x)$ are continuous at any point $x = a$, then at $x = a$ what would be $f(x) \pm g(x)$

    (a) Continuous   (b) Discontinuous   (c) Relevant   (d) Irrelevant

(v) $\lim\limits_{x \to \infty} \dfrac{e^{x^2} - 1}{e^{x^2} + 1} = $ ......

    (a) 2   (b) $2e$ (c) 1   (d) $\dfrac{1}{e}$

2.15   Summary

- When the limit of function is obtained from the right hand of the independent variable, then it is called Right Hand Limit (R.H.L.) and applying positive (+) sign for the right side, this can be expressed as under

Right Hand Limit $= f(a + 0)$ **Note**

$$= \lim_{x \to a^+} f(x) = l_1.$$

- When the limit of function is obtained from the left hand of the independent variable, then it is called Left Hand Limit (LH.L.) and applying negative (–) sign for the left side, this can be expressed as under

  Left Hand Limit $= f(a - 0)$

  $$= \lim_{x \to a^-} f(x) = l_2.$$

- To obtain the limit of right and left hand, replace x variable with $(x+h)$ ad $(x–h)$ respectively in the function.

- Thus, obtained function x, should be replaced with point (assume a).

- Now at $h \to 0$ determine the limit of function [viz function obtained by (ii) to be put in the above, put $h = 0$].

- If at $x = a$ both the limits of right and left hand of any function $f(x)$ exist and are equal, then at

  $x = a$, there is existence of limit of function $f(x)$ viz $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$ (Assumed).

- Here l is called limit of the function and this is expressed as $\lim_{x \to a} f(x) = l$.

- We should find out the limits of right and left hand, but the intermediate level, we can often find out the limit directly.

- If it is not possible to divide the numerator and denominator with a common factor which is not zero, then after expansion in series or transformation, it can be possible.

- If drawing a graph of any function $f(x)$, the curve which we obtain. And it is in such a way that it does not break on any point $x = a$, then the function is called continuous on that point.

- A function $f(x)$ is called continuous in an open interval (a,b) if it is continuous for every values of x in this interval (a,b).

- Function $f(x)$ is called continuous in any closed interval, if

  (i)   It is continuous for each value of x for which $a < x < b$

  (ii) $\lim_{x \to a+0} f(x) = f(a)$

  (iii) $\lim_{x \to b-0} f(x) = f(b)$.

2.16 Keywords

- *Sequence:* Serial

- *Continually:* Continuously

2.17 Review Questions

1. Find out the value of $\lim_{x \to 0} \left(\dfrac{x^2 - 2ax + a^2}{x - a} \right)$ **Ans.: –a**

2. Prove that $\lim_{x \to 0} = \dfrac{a^x - b^x}{x} = \log_e \dfrac{a}{b} = \log_e$

Note

3. Find out the value of $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$ **Ans.:** 3

4. Examine the continuity of function $f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 3, & x = 0 \text{ but.} \end{cases}$ **Ans.:** Discontinuous

5. Show that at $x = 0$, $f(x) = |x|$ is continuous

Answers: Self Assessment

1. (i) Right Hand Limit,   (ii) Left Hand Limit,   (iii) Existence,

    (iv) Marginal,   (v) Intermediate

2. (i) (a),   (ii) (b),   (iii) (c)

    (iv) (a),   (v) (c)

2.18 Further Readings

Books

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

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Unit 3: Differentiation

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Objectives

Introduction

Objectives

After reading this unit, students will be able to :

●   Calculate Differential Coefficient.

●   Calculate Differential Coefficient of a Constant.

●   Calculate Differential Coefficient of x^n with Respect to x.

●   Calculate Differential Coefficient of Sum, Subtract, Multiplication and Division of two Functions.

Introduction

Assume $y = x^2$

Where $x = 2$, hence $y = 4$, where $x = 3$, $y = 9$

When x increases from 2 to 3, y also increases from 4 to 9. Any increase in the constant function is referred as Increment, which is indicated δx (delta x).

In the above example $\delta x = 3 – 2 = 1$ and $\delta y = 9 – 4 = 5$

If x, 1 changes to .8, $\delta x = 0.8 – 1 = – 0.2$

Note

3.1   Differential Coefficient

Assume $y = x^2$, is a function of x. Assume the prime value of $x = 3$.

The table below shows how the ratio of increment δx in x and δy in y changes. Here increment as 4, 3, 2, 1 etc. in the prime value of x has been considered.

δx	$x(x + \delta x)$	$y(y + \delta y)$	δy	$\dfrac{\delta y}{\delta x}$
4	7	49	40	10
3	6	36	27	9
2	5	25	16	8
1	4	16	7	7
0.1	3.1	9.61	.61	6.1
0.01	3.01	9.0601	.0601	6.01
0.001	3.001	9.006001	.006001	6.001
0.0001	3.0001	9.0006001	.00060001	6.0001
h	$3 + h$	$6 + 6h + h^2$	$6h + h^2$	$6 + h$

The table makes it clear that:

1. Wherever the decreasing value of δx approaches to zero, δy also decreases and reaches to ZERO.

2. But, the ratio of these values, instead of moving to ZERO approaches to particular value, which in this example is 6.

Therefore, we may conclude that whenever δx and as a result δy decreases, $\dfrac{\delta y}{\delta x}$ moves to value 6, in other words

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 6$$

$$y = x^2$$

$\therefore \qquad\qquad y + \delta y = (x + \delta x)^2$

$\therefore \qquad\qquad y + \delta y - y = (x + \delta x)^2 - x^2$

$\therefore \qquad\qquad \delta y = x^2 + 2x, \delta x + (\delta x)^2 - x^2$

$\therefore \qquad\qquad \delta y = 2x.\delta x + (\delta x)^2$

$\therefore \qquad\qquad \dfrac{\delta y}{\delta x} = 2x + \delta x$

$\therefore \qquad\qquad \lim_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0} (2x + \delta x)$

$\therefore \qquad\qquad \dfrac{dy}{dx} = 2x$

Thus, $\lim\limits_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right)$ is a differential coefficient or derivative of y with respect to x. To avoid any doubt with respect to independent variables, the differential coefficient or derivative with respect to x is written as $\lim\limits_{\delta x \to 0}\dfrac{\delta y}{\delta x} = \dfrac{d}{dx}(y)\dfrac{dy}{dx}$. This way the process of knowing the limit is known as differentiation or in other words method of calculating the differential coefficient of any product is referred as Differentiation.

Did u know? $\qquad \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0}\dfrac{\delta y}{\delta x}$

Students must understand the difference of $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0}\dfrac{\delta y}{\delta x}$ very carefully. $\dfrac{\delta y}{\delta x}$ is a division, for which

Numerator and denominator can be separated from each other, but $\dfrac{dy}{dx}$ is not a division, but only a

symbol to show the limiting value of $\dfrac{\delta y}{\delta x}$. δy can not be separated from δx. It would be incorrect to

read it as δy upon δx. $\dfrac{dy}{dx}$ is read as [δ-δx of y) and as have been told earlier this means $\dfrac{d}{dx}(y)$ or

differential coefficient of y with respect to x.

3.2 Definition

If $f(x)$ is the function of x and the same function of $x + \delta x$ is $f(x + \delta x)$, then limiting value is

$\lim\limits_{\delta x \to 0}\dfrac{f(x + \delta x) - f(x)}{\delta x}$, for the differential coefficient $f(x)$ with respect to x. Generally differential

coefficient is symbolized with $\dfrac{dy}{dx}, y', y_1, \dfrac{df(x)}{dx}, \dfrac{d}{dx}$ $f(x)$, $f(x)$, $Df(x)$, $f\,'$, etc. The method of calculating differential coefficient is referred as differentiating the function.

Thus, we can see that there are four steps for calculating differential coefficient

First step changing x into $x + \delta x$ and determining $f(x + \delta x)$

Second step determining difference of $f(x + \delta x) - f(x)$

Third step dividing difference by δx and determining $\dfrac{f(x + \delta x) - f(x)}{\delta x}$

Fourth step when δx approaches towards ZERO then determining the limiting value of ratio

here onwards incremental value δx of x would be replaced with h, so that students should not get

confused with δ and x. Thus, if $f(x)$ is the product of x, then $\lim\limits_{h \to 0}\dfrac{f(x + h) - f(x)}{h}$ is the differential

coefficient of $f(x)$. It needs to be remembered that above expression is considered as the product of h i.e. h is variable and x is constant.

3.3 Differential Coefficient of a Constant

Assume that c is a constant. Thus, here $f(x) = c$ now for every value of x, no change can happen in the constant value.

Therefore,

$$f(x + h) = c$$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0 .$$

$$\boxed{\frac{d}{dx} c = 0}$$

Thus, differential coefficient of a constant value is ZERO.

3.4 Differential Coefficient of the Production of a Constant and a Function

Assume that a is a constant value and $f(x)$ is the determined product, then

$$\frac{d}{dx} \{af(x)\} = \lim_{h \to 0} \frac{af(x+h) - af(x)}{h}$$

$$= \lim_{h \to 0} \left\{ a \frac{f(x+h) - f(x)}{h} \right\}$$

$$= a \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = a \frac{d}{dx} \{f(x)\}$$

$$\boxed{\frac{d}{dx} \{af(x)\} = a \frac{d}{dx} \{f(x)\}}$$

if
$$y = au$$

$$\boxed{\frac{dy}{dx} = a \frac{du}{dx}}$$

> **?**
>
> *Did u know?* Differential coefficient of Constant and multiplication of any product is equal to differential coefficient of product and multiplication of constant value.

3.5 Differential Coefficient of x^n with Respect to x

Assume $f(x) = x^2$, then $f(x + h) = (x + h)^n$

Thus, Differential Coefficient of x^n with respect to x $\frac{d}{dx}(x^n)$

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{(x + h/x)^n - 1}{h}, x^n \text{ ascertained}$$

Now, since $h \to 0$, we can assume $\dfrac{h}{x}$ is smaller to a unit. Therefore, expanding every value of n

$(1 + h/x)^n$ by using Binomial Theorem

$$\frac{d}{dx}x^n = \lim_{h \to 0}\frac{x^n}{h}\left\{1 + n.\frac{h}{x} + \frac{n(n-1)}{1.2}\frac{h^2}{x^2} + \ldots -1\right\}$$

$$= \lim_{h \to 0}\frac{x^n}{h}\left\{n.\frac{h}{x} + \frac{n(n-1)}{1.2}\cdot\frac{h^2}{x^2} + \ldots\right\}$$

$$= \lim_{h \to 0} x^n\left\{\frac{n}{x} + \frac{n(n-1)h}{1.2x^2} + \ldots\right\}$$

$$= \lim_{h \to 0} x^n\left\{\frac{n}{x} + h\times\right\}$$

$$= nx^{n-1}, \ h \to 0$$

$\therefore$

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

The special result of this is,

$$\boxed{\frac{d}{dx}x = 1}$$

Example 1: $\dfrac{d}{dx}(9x^7) = 9.\dfrac{d}{dx}(x^7) = 9.7\,x^{7-1} = 63x^6$

Example 2: $\dfrac{d}{dx}(-6x^2) = -6\dfrac{d}{dx}x^2 = -6.2\,x^{2-1} = -12x$

Example 3: $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{d}{dx}(x^{-1/2}) = -\dfrac{1}{2}\,x^{-1/2-1} = -\dfrac{1}{2}x^{-3/2}$

Task   Determine the value of $\dfrac{d}{dx}(5x^6)$ [**Ans.:** $30x^5$]

Self Assessment

1. Fill in the blanks:

(i)   The method for determining the differential coefficient of any function is referred as ………………..

(ii) Differential coefficient of Constant and multiplication of any product is equal to ………………………. of product and multiplication of constant value.

(iii) $\dfrac{dy}{dx} = \ldots\ldots\dfrac{du}{dx}$

(iv) $\dfrac{dy}{dx} = x^n = nx\ldots\ldots$

3.6   Differential Coefficient of Sum and Subtract of two Functions

Assume that $f(x)= f_1 \pm f_2(x)$

$\therefore$ $f(x+h) = f_1(x+h) \pm f_2(x+h)$

$$\frac{d}{dx}\{f(x)\} = \lim_{h \to 0} \frac{\{f_1(x+h) \pm f_2(x+h)\} - \{f_1(x) \pm f_2(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{\{f_1(x+h) - f_1(x)\} \pm \{f_2(x+h) - f_2(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{f_1(x+h) - f_1(x)}{h} \pm \lim_{h \to 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$\boxed{\frac{d}{dx}\{f(x)\} = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)}$$

Thus, Differential coefficient of addition or subtraction of any two product would be equal to addition or subtraction of differential coefficient of them

Assume that $f(x)= f_1(x) \pm f_2(x) \pm f_3(x) \pm ..... + f_n(x)$

Then $\boxed{\dfrac{d}{dx} f(x) = \dfrac{d}{dx} f_1(x) \pm \dfrac{d}{dx} f_2(x) \pm .... \pm \dfrac{d}{dx} f_n(x)}$

Notes   Differential Coefficient of Sum and Subtract of two or more than two functions would be equal to Sum and Subtract of differential coefficient of their separated value.

if $y = u + v$, and $y = u - v$

then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ in the same way $\dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

in expanded expression, if $y = u \pm v \pm w \pm....$ Then $a^2 + b^2$

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{du}{dx} \pm \frac{dw}{dx} \pm ....$$

EXAMPLES WITH SOLUTION

Example 1: Determine value of $\dfrac{d}{dx}(5x^7 + 2x)$.

Illustration $\dfrac{d}{dx}(5x^7 + 2x) = \dfrac{d}{dx}(5x^7) + \dfrac{d}{dx}2x = 5\dfrac{d}{dx}(x^7) + 2\dfrac{d}{dx}(x)$

$= 5.7x^6 + 2 = 35x^6 + 2$.   **Ans.**

Example 2: Determine value of $\dfrac{d}{dx}(x^5 - 4x^3 + 8x - 7)$.

Illustration $\dfrac{d}{dx}(x^5 - 4x^3 + 8x - 7) = \dfrac{d}{dx}x^5 + \dfrac{d}{dx}(-4x^3) + \dfrac{d}{dx}(8x) + \dfrac{d}{dx}(-7)$

$$= \dfrac{d}{dx}(x^5) + (-4)\dfrac{d}{dx}x^3 + 8\dfrac{d}{dx}(x) + \dfrac{d}{dx}(-7)$$

$$= 5x^4 - 4 \times 3x^2 + 8 \times 1 - 0$$

$$= 5x^4 - 12x^2 + 8 .$$ **Ans.**

Now we can determine differential coefficient of some function by using derivative limiting value. But we will notice afterwards that with the knowledge of standard form of differential coefficient, we can save time from evaluation

Example 3: Determine the differential coefficient of product $\left\{\dfrac{lx^2 + mx + n}{\sqrt{x}}\right\}$ **with respect to** *x.*

Illustration $\dfrac{d}{dx}\left\{\dfrac{lx^2 + mx + n}{\sqrt{x}}\right\} = \dfrac{d}{dx}\{lx^{3/2} + mx^{1/2} + mx^{-1/2}\}$

$$= l\dfrac{d}{dx}x^{3/2} + m\dfrac{d}{dx}x^{1/2} + n\dfrac{d}{dx}x^{-1/2}$$

$$= l\dfrac{3}{2}x^{1/2} + m \cdot \dfrac{1}{2}x^{1/2-1} + n \cdot \left(-\dfrac{1}{2}\right)x^{-1/2-1}$$

$$= \dfrac{3}{2}lx^{1/2} + \dfrac{1}{2}mx^{-1/2} - \dfrac{1}{2}nx^{-3/2}$$ **Ans.**

Task Determine the value of $\dfrac{d}{dx}(2x^7 + 5x)$ **[Ans.:** $14x^6 + 5$ **]**

Example 4: Determine the differential coefficient of product $1 + x + \left(\dfrac{x^2}{2!}\right) + \left(\dfrac{x^3}{3!}\right) + \left(\dfrac{x^4}{4!}\right) + ...$ **with**

respect to *x.*

$$\dfrac{d}{dx}\left\{1 + x + \left(\dfrac{x^2}{2!}\right) + \left(\dfrac{x^3}{3!}\right) + \left(\dfrac{x^3}{4!}\right) + ...\right\}$$

Illustration $= \dfrac{d}{dx}(1) + \dfrac{d}{dx}(x) + \dfrac{d}{dx}\left(\dfrac{x^2}{2!}\right) + \dfrac{d}{dx}\left(\dfrac{x^3}{3!}\right) + \dfrac{d}{dx}\left(\dfrac{x^4}{4!}\right) + ...$

$$= 0 + 1 + \dfrac{2x}{2!} + \dfrac{3x^2}{3!} + \dfrac{4x^3}{4!} + ...$$

Note

$$= 0 + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Ans.

Questionnaire 3.1

Determine the differential coefficient of following product with respect to x.

1. $3x^3, x^{-5}, x^9$

2. $3x^2, 6x^{-3}, \dfrac{x^5}{3}$

3. $x^{1/2}, x^{7/3}, x^{-5/2}$

4. $\sqrt{x^3}, \sqrt{x}, \sqrt{x^{-7}}$

5. $3x^{1/3}, 5x^{1/7}, 2x^{1/4}$

6. $\dfrac{7}{x^2}, \dfrac{5}{x^{3/2}}, \dfrac{1}{x}$

7. $x + \dfrac{2}{x}$

8. $x^m + a^n$

9. $ax^2 + bx + c$

10. $(ax)^m + (2b)^m$

11. $\dfrac{1}{2}\sqrt{x} + \sqrt{a}$

12. $y = (ax)^m + \left(\dfrac{b}{x}\right)^n$

13. $y = ax + (ax)^2 + (ax)^3 + \dots$

14. $y = a + \dfrac{a}{x} + \left(\dfrac{a}{x}\right)^2 + \left(\dfrac{a}{x}\right)^3 + \dots$

15. If $y = x^5 + 2x^4 + 7$ then determine the value of $\dfrac{dy}{dx}$ on $x = 0$

16. If $y = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots$, then determine the value of $\dfrac{dy}{dx}$.

Answers

1. $9x^2, -5x^{-6}, 9x^8$

2. $6x, -18x^{-4}, \dfrac{5}{3}x^4\, 3$

3. $\dfrac{1}{2\sqrt{x}}, \dfrac{4}{3}x^{\frac{4}{3}}, -\dfrac{5}{2}x^{\frac{-7}{2}}$

4. $\dfrac{3}{2}x^{\frac{1}{2}}, \dfrac{1}{2\sqrt{x}}, \dfrac{-7}{2}x^{\frac{-9}{2}}$

5. $x^{\frac{-2}{3}}, \dfrac{5}{6}x^{\frac{-6}{7}}, \dfrac{1}{2}x^{\frac{-3}{4}}$

6. $14x^{-3}, \dfrac{15}{2}x^{\frac{-5}{2}}, -\dfrac{1}{x^2}$

7. $1 - \dfrac{2}{x^2}$

8. mx^{m-1}

9. $2ax + b$

10. $ma^m x^{m-1}$

11. $\dfrac{1}{4\sqrt{x}}$

12. $a^m m x^{m-1} - n b^n x^{-n-1}$

13. $a + 2ax + 3ax^2 + \ldots$

14. $-\dfrac{a}{x^2} - \dfrac{2a^2}{x^3} - \ldots$

15. $\dfrac{dy}{dx} = 0$

16. $n_{c_2} + 2 n_{c_2} x + \ldots$

3.7   Differential Coefficient of Function e^x with Respect to x when e is Exponential

Here $f(x) = e^x$

Then $f(x + h) = e^{x+h}$,

Therefore, differential coefficient of e^x with respect to x is $\dfrac{d}{dx} e^x$

Further

$$\dfrac{d}{dx} e^x = \lim_{h \to 0} \dfrac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \dfrac{e^x e^h - e^x}{h} = \lim_{h \to 0} \dfrac{e^x (e^h - 1)}{h}$$

$$= \lim_{h \to 0} \dfrac{e^x \left(1 + \dfrac{h}{1!} + \dfrac{h^2}{2!} + \dfrac{h^3}{3!} + \ldots - 1\right)}{h}$$

$$= \lim_{h \to 0} e^x h \dfrac{\left(1 + \dfrac{1}{1!} + \dfrac{h}{2!} + \dfrac{h^2}{3!} + \ldots\right)}{h}$$

$$= \lim_{h \to 0} e^x (1 + h)$$

$$= e^x \left(\dfrac{1}{1!} + 0\right) = e^x$$

$\therefore$

$$\boxed{\dfrac{d}{dx}(e^x) = e^x}$$

3.8   Differential Coefficient of Function a^x with Respect to x when a is a Non-variable

Here $f(x) = a^x$

Then $f(x + h) = a^{x+h}$

Note

Therefore, differential coefficient of a^x with respect to x is $\dfrac{d}{dx}\,a^x$

again

$$\frac{d}{dx}\,a^x \;=\; \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x a^h - a^x}{h}$$

$$=\;\lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

$$=\;\lim_{h\to 0}\frac{a^x\left[1+\dfrac{h\log_e a}{1!}+\dfrac{h^2(\log_e a)^2}{2!}+.....-1\right]}{h}$$

$$=\;\lim_{h\to 0}\frac{a^x\left[\dfrac{h\log_e a}{1!}+\dfrac{h^2(\log_e a)^2}{2!}+...\right]}{h}$$

$$=\;\lim_{h\to 0}\frac{a^x\cdot h\left[\dfrac{\log_e a}{1!}+\dfrac{h(\log_e a)^2}{2!}+...\right]}{h}$$

$$=\;\lim_{x\to\infty}a^x\left[\frac{\log_e a}{1!}+h\ (\text{A convergent class})\right]$$

$$=\;a^x\left[\frac{\log_e a}{1!}+a\right]=a^x\log_e a$$

$\therefore$
$$\boxed{\;\frac{d}{dx}\,a^x = a^x\log_e a\;}$$

3.9   Differential Coefficient of Function $\log_e x$ with Respect to x, when base is Exponential

Here $\quad f(x)=\log_e x$

Then $\quad f(x+h)=\log_e(x+h)$

Further

$$\frac{d}{dx}\log_e x \;=\; \lim_{h\to 0}\frac{\log_e(x+h)-\log_e x}{h}$$

again

$$=\;\lim_{h\to 0}\frac{\log_e(1+h/x)-\log_e x}{h}$$

$$=\;\lim_{h\to 0}\frac{\log_e x+\log_e(1+h/x)-\log_e x}{h}$$

$$[\text{Formula is }\log_e(mn)=\log_e m+\log_e n]$$

$$= \lim_{h \to 0} \frac{\log_e (1 + h / x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{h}{x} - \dfrac{h^2}{2x^2} + \dfrac{h^3}{3x^3} - \dfrac{h^4}{4x^4} + \dots}{h}$$

$$\left[\text{Formula is } \log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right]$$

$$= \lim_{h \to 0} \frac{h \left[\dfrac{1}{x} - \dfrac{h}{2x^2} + \dfrac{h^2}{3x^3} - \dfrac{h^3}{4x^4} + \dots \right]}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^2} - \frac{h^3}{4x^4} + \dots \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{1}{x} - h \times \right\} = \frac{1}{x}$$

$\therefore$

$$\boxed{\frac{d}{dx} \log_e x = \frac{1}{x}}$$

Notes $\log_e x$ can also be indicated as *lnx*

3.10 Differential Coefficient of Function $\log_a x$ with Respect to x, when base of Logarithm is a Non-variable

Here $f(x) = \log_a x = (\log_e x) \log_a e$ by formula

$$= \log_a e \cdot \log_e x$$

$$\frac{d}{dx} \log_a x = \frac{d}{dx} \log_a e \log_e x = \log_a e \frac{d}{dx} \log_e x,$$

$$= \log_a e \cdot \frac{1}{x} \qquad\qquad \left[\because \frac{d}{dx} \log_e x = \frac{1}{x} \right]$$

$\therefore$

$$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e}$$

Note

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EXAMPLES WITH SOLUTION

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Example 1: Determine the differential coefficient of $6x^{1/3} + 2e^x$.

Solution:
$$\frac{d}{dx}(6x^{1/3} + 2e^x) = \frac{d}{dx}(6x^{1/3}) + \frac{d}{dx}(2e^x)$$

$$= 6.\frac{d}{dx}(x^{1/3}) + 2\frac{d}{dx}(e^x)$$

$$= 6.\frac{1}{3}x^{-2/3} + 2e^x = 2x^{-3/2} + 2e^x \qquad \textbf{Ans.}$$

Example 2: Determine the differential coefficient of $6 \log x - \sqrt{x} - 7$.

Solution: $\frac{d}{dx}(6\log x - \sqrt{x} - 7) = 6 \cdot \frac{d}{dx}\log x - \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(7)$

$$= 6.\frac{1}{x} - \frac{1}{2}x^{-1/2} - 0 = \frac{6}{x} - \frac{1}{2}x^{-1/2} \qquad \textbf{Ans.}$$

Example 3: Determine the differential coefficient of product $5\sqrt{x} + 7\log_e x - 11\log_a x$ with respect to x.

Solution: $\frac{d}{dx}(5\sqrt{x} + 7\log_e x - 11\log_a x) = \frac{d}{dx}(5x^{1/2}) + \frac{d}{dx}(7\log_e x) - \frac{d}{dx}(11\log_e x)$

$$= 5\frac{d}{dx}x^{1/2} + 7 \cdot \frac{d}{dx}\log_e x - 11 \cdot \frac{d}{dx}\log_e x$$

$$= \frac{5}{2}x^{-1/2} + \frac{7}{x} - 11\log_a e \cdot \frac{1}{x} \qquad \textbf{Ans.}$$

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Questionnaire 3.2

</div>

Determine the differential coefficient of following product with respect to x.

1. $e^x + \log_e x + l^x$

2. $\frac{1}{2x} + 7e^x$

3. $\frac{x^3 \log x + x - x^3 e^x}{x^3}$

4. $5\log_{10} x + 3$

5. $3\log_e x + x^{3/2} + 3$

6. $x(1 + x^2) + a^x$

7. $e^x + a^x + 1$

8. $\log_a x + \log_e x^2$

9. $\log_{10} x$

10. $7x^{-2/7} + \log_2 x$

11. $\frac{xe^x - 1}{x}$

12. $\sqrt{a + 2a^2 e^x + a^3 e^{2x}}$

13. $\sqrt[3]{1 + 3\log_a x + 3(\log_a x)^2 + (\log_a x)^3}$

14. $\log_e \sqrt{x} + a^x + 3$

15. $\frac{1}{a^x} + \log_a x$

$$\boxed{\textbf{Answers}}$$

1. $e^x + \dfrac{1}{x} + l^x \log_e l$

2. $-\dfrac{1}{2x^2} + 7e^x$

3. $\dfrac{1}{x} - 2x^{-2} - e^x$

4. $\dfrac{5}{x} \log_{10} e$

5. $\dfrac{3}{x} + \dfrac{3}{2} x^{1/2}$

6. $1 + 3x^2 + a^x \log_e a$

7. $e^x + a^x \log_e a$

8. $\dfrac{1}{x} \log_e a + \dfrac{2}{x}$

9. $\dfrac{1}{x} \log_e e$

10. $-2x^{-9/7} + \dfrac{1}{x} \log_2 e$

11. $e^x + x^{-2}$

12. $a^{3/2} e^x$

13. $\dfrac{1}{x} \log_a e$

14. $\dfrac{1}{2x} + a^x \log_e a$

15. $-a^{-x} \log_e a + \dfrac{1}{x} \log_a e$

3.11 Differential Coefficient of the Quotient of Two Functions

Assume

$$F(x) = \frac{f_1(x)}{f_2(x)} \quad f(x+h) = \frac{f_1(x+h)}{f_2(x+h)}$$

$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{\dfrac{f_1(x+h)}{f_2(x+h)} - \dfrac{f_1(x)}{f_2(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f_1(x+h) f_2(x) - f_1(x) f_2(x+h)}{h f_2(x+h) f_2(x)}$$

Adding and subtracting the product $f_1(x) f_2(x)$ in fraction

$$= \lim_{h \to 0} \frac{f_2(x)\{f_1(x+h) - f_1(x)\} - f_1(x)\{f_2(x+h) - f_2(x)\}}{h f_2(x+h) f_2(x)}$$

$$= \lim_{h \to 0} \frac{f_2(x)\left\{\dfrac{f_1(x+h) - f_1(x)}{h}\right\} - f_1(x)\left\{\dfrac{f_2(x+h) - f_2(x)}{h}\right\}}{f_2(x+h) f_2(x)}$$

$$= \frac{f_2(x) \cdot \dfrac{d}{dx} f_1(x) - f_1(x) \dfrac{d}{dx} f_2(x)}{[f_2(x)]^2}$$

Note   Viz differential coefficient of division of two functions

$$= \frac{\text{Denr. (Diff. coett. of Numr.)} - \text{Numr. (Diff. coeff. of Denr.)}}{\text{Square of Denominator}}$$

Did u know?

$$\left[\begin{array}{l} \text{If } y = \dfrac{u}{v} \\[2ex] \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \end{array}\right]$$

EXAMPLES WITH SOLUTION

Example 1: Calculate the value of $\dfrac{dy}{dx}$ **if** $y = \dfrac{e^x}{x}$.

Solution: $\dfrac{dy}{dx} = \dfrac{x - \dfrac{d}{dx}e^x - e^x\dfrac{d}{dx}x}{x^2} = \dfrac{xe^x - e^x \cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$.   **Ans.**

Example 2: Calculate the value of $\dfrac{d}{dx}\left(\dfrac{\sin x}{\log_e x}\right)$.

Solution:

$$\dfrac{d}{dx}\left(\dfrac{\sin x}{\log_e x}\right) = \dfrac{\log_e x \cdot \dfrac{d}{dx}(\sin x) - \sin x \cdot \dfrac{d}{dx}(\log_e x)}{(\log_e x)^2}$$

$$= \dfrac{(\log_e x)\cdot\cos x - (\sin x)\cdot\dfrac{1}{x}}{(\log_e x)^2}$$

$$= \dfrac{x\log_e x\cos x - \sin x}{x(\log_e x)^2} .$$ **Ans.**

Example 3: Determine the differential coefficient of product $\dfrac{x^n}{\log_e x}$ **with respect to** x.

Solution:

$$\dfrac{d}{dx}\left\{\dfrac{x^n}{\log_e x}\right\} = \dfrac{\log_e x\dfrac{d}{dx}x^n - x^n\dfrac{d}{dx}\log_e x}{(\log_e x)^2}$$

$$= \dfrac{\log_e x \cdot nx^{n-1}\cdot\dfrac{1}{x}}{(\log_e x)^2} = \dfrac{nx^{-1}\log_e x - x^n - 1}{(\log_e x)^2}$$

$$= \dfrac{x^{n-1}(n\log_e x - 1)}{(\log_e x)^2}$$ **Ans.**

Example 4: Prove that $x\dfrac{dy}{dx} = y(1-y)$ **if** $y = \dfrac{x}{x+5}$.

Solution: given that $y = \dfrac{x}{x+5}$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+5)\cdot\frac{d}{dx}(x)-x\cdot\frac{d}{dx}(x+5)}{(x+5)^2} = \frac{(x+5)\cdot 1 - x\cdot 1}{(x+5)^2}$$

$$= \frac{x+5-x}{(x+5)^2} = \frac{5}{(x+5)^2} \Rightarrow x\frac{dy}{dx} = \frac{5x}{(x+5)^2} \qquad \ldots(1)$$

And

$$= y(1-x) = \frac{x}{x+5}\left(1-\frac{x}{x+5}\right) = \frac{x}{x+5}\left(\frac{5}{x+5}\right) = \frac{5x}{(x+5)^2}$$

$$\ldots(2)$$

Therefore from (1) and (2)

$$x\frac{dy}{dx} = y(1-y)$$

Questionnaire 3.3

Determine the differential coefficient of the following with respect to x.

1. $\dfrac{1}{x^{1/4}}$

2. $\dfrac{x^n}{\log_e x}$

3. $\dfrac{x}{a^2+x^2}$

4. $\dfrac{x^2}{e^2+x^2}$

5. $\dfrac{e^x}{1+x^2}$

6. $\dfrac{e^x}{1+e^x}$

7. If $f(x)=\dfrac{x^3}{a^2-x^2}$, then evaluate $f'\left(\dfrac{a}{2}\right)$

8. If $y=\dfrac{x-4}{2\sqrt{x}}$, then evaluate the value of $\dfrac{dy}{dx}$ on $x = 4$. Can we find the value of $\dfrac{dy}{dx}$ on $x = 0$?

9. If $y=\dfrac{5x^2+6x+7}{2x^2+3x+4}$, then find the value of $\dfrac{dy}{dx}$

10. If $y=\dfrac{a^x}{x^n}$, then find the value of $\dfrac{dy}{dx}$

Determine the differential coefficient of following product with respect to x

11. (i) $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

    (ii) $\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

12. (i) $\dfrac{xe^x-1}{x}$

    (ii) $\dfrac{x}{\sqrt{1-x^2}}$

13. $\dfrac{1}{(x+a)\,(x+b)\,(x+c)}$

Note

$$\boxed{\textbf{Answers}}$$

1. $-\dfrac{1}{4}x^{-5/4}$

2. $\dfrac{nx^{n-1}\log_e x - x^{n-1}}{(\log_e 2)^2}$

3. $\dfrac{a^2 - x^2}{(a^2 + x^2)^2}$

4. $\dfrac{2xe^2}{(e^2 + x^2)^2}$

5. $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$

6. $\dfrac{e^x}{(1+e^x)^2}$

7. $\dfrac{11}{19}$

9. $\dfrac{3(x^2 + 4x + 1)}{(2x^2 + 2x + 4)^2}$

10. $\dfrac{a^2}{x^n}[\log_e a - \dfrac{n}{x}]$

11. (i) $\dfrac{\sqrt{a}}{\sqrt{x}(\sqrt{x} - \sqrt{x})^2}$ (ii) $\dfrac{-4}{[e^x - e^{-x}]^2}$

12. (i) $e^x + \dfrac{1}{x^2}$ (ii) $\dfrac{1}{(1-x^2)^{3/2}}$

13. $\dfrac{3x^2 + 2(a+b+c)x + (ab+bc+ca)}{(x+a)^2(x+b)^2(x+c)^2}$

Self Assessment

2. Multiple Choice Questions:

(i) $\dfrac{d}{dx}\{f(x)\} = \dfrac{d}{dx} f_1(x) \pm .....$

    (a) $\dfrac{d}{dx} f_2$ (b) $\dfrac{d}{dx} f_1$ (c) $\dfrac{d}{dx} f(x)$ (d) $\dfrac{d}{dx}\{f(x)\}$

(ii) What will be the value of $\dfrac{d}{d(x)}\left(\dfrac{1}{\sqrt{x}}\right)$?

    (a) $\dfrac{1}{2}x^{-\frac{3}{2}}$ (b) $-\dfrac{1}{2}x^{-\frac{3}{2}}$ (c) $-\dfrac{1}{2}x^{\frac{3}{2}}$ (d) $\dfrac{1}{2}x^{\frac{3}{2}}$

(iii) What will be differential coefficient of $3x^3$ with respect to x?

    (a) $6x^2$ (b) $3x^2$ (c) $9x^2$ (d) $9x$

(iv) $\dfrac{d}{dx}(e^x) = ...$

    (a) e (b) 1 (c) $e^{\frac{1}{x}}$ (d) e^x

(v) $\dfrac{d}{dx}\log_a x = ...\log_a e$

    (a) $\dfrac{1}{x}$ (b) x (c) $\log_a$ (d) $\log x$

3.12 Summary

- $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ is known as Derivative for the differential coefficient of y function with respect to x.

  To avoid any doubt with respect to independent variables, the differential coefficient or derivative $\lim\limits_{x \to 0} \dfrac{y}{x}$ with respect to x is written as $\dfrac{d}{dx}(y)$ or $\dfrac{dy}{dx}$. This way the process of knowing the limit is known as differentiation or in other words method of calculating the differential coefficient of any product is referred as Differentiation.

- If $f(x)$ is the function of x and the same function of $x + \delta x$ is $f(x + \delta x)$, then limiting value is

  $\lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x}$, for the differential coefficient $f(x)$ with respect to x.

- $\dfrac{d}{dx}\{f(x)\} = \dfrac{d}{dx} f_1(x) \pm \dfrac{d}{dx} f_2(x)$

- Differential coefficient of Constant and multiplication of any product is equal to differential coefficient of product and multiplication of constant value.

- $\dfrac{d}{dx}(e^x) = e^x$

- $\dfrac{d}{dx} a^x = a^x \log_e a$ *i.e.* differential coefficient of division of two function

3.13 Keywords

- *Differential coefficient:* Differentiation
- *Growth:* Increment

3.14 Review Questions

1. Find the value of $\dfrac{d}{dx}(-6x^2)$ **[Ans.: = –12x]**

2. Find the value of $\dfrac{d}{dx}(5x^6 + 2x)$ **[Ans.: = 30x⁵ + 2]**

3. Prove that $\dfrac{d}{dx} a^x = a^x \log_e a$

4. Find the differential coefficient of $6\log x - \sqrt{x} - 7$ **[Ans.: = $\dfrac{6}{x} - \dfrac{1}{2} x^{-\frac{1}{2}}$]**

5. If $y = \dfrac{x}{x+5}$, then prove that $x\dfrac{dy}{dx} = y(1 - y)$

Answers: Self Assessment

1. (i) Differentiation   (ii) Multiplication   (iii) *a*   (iv) *n* – 1   (v) $63x^6$

2. (i) *a*   (ii) (b)   (iii) *(c)*   (iv) *(d)*   (v) *(a)*

Note

3.15   Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Unit 4: Logarithmic Differentiation

Objectives

After reading this unit, students will be able to :

- Find Logarithmic Differentiation.
- Find Sum of Infinite Tables.
- Calculate Implicit Function.

Introduction

If you need to differentiate such functions in which exponent is also a function of that variable and a function that needs to be differentiated is the product or division of many functions, then we first need to find logarithm of those functions and then differentiate it. This process is known as logarithmic differentiation.

4.1   Logarithmic Differentiation

Let a function of x be equal to y. Then either take log of both the sides such that the exponent is the product or sum form, etc.

Now differentiate both sides wrt x and find $\dfrac{dy}{dx}$

Remember that to to differentiate the function of y wrt x, function is difrentiated wrt y and multiplied with $\dfrac{dy}{dx}$.

Caution log $(a + b) \neq$ log $a +$ log b

Therefore if then $y = x^x + (\sin x)^{\cos x}$ then $\log y \neq \log x^x + \log(\sin x)^{\cos x}$

Note   In such questions, differentiation of every element needs to be find separately and then only combined differentiation can be found out.

Notes   Some useful formulae

  (i) $\log(m.n) = \log m + \log n$

  (ii) $\log\left(\dfrac{m}{n}\right) = \log m - \log n$

  (iii) $\log(m)^n = n\log m.$

EXAMPLES WITH SOLUTION

Example 1: Find the differentiation of $e^x.\log_e x.\tan x$ with respect to x.

Solution: Let $y = e^x.\log_e \tan x$

By taking log of both sides

$\log y = \log e^x + \log(\log_e x) + \log \tan x$

By differentiating both sides

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{e^x}(e^x) + \frac{1}{\log_e x}\left(\frac{1}{x}\right) + \frac{1}{\tan x}(\sec^2 x)$$

$$= 1 + \frac{1}{x\log_e x} + \frac{\cos x}{\sin x}.\frac{1}{\cos^2 x}$$

$$= 1 + \frac{1}{x\log_e x} + \frac{1}{\sin x \cos x}$$

$$= 1 + \frac{1}{x\log_e x} + 2\operatorname{cosec} 2x$$

$\therefore$

$$\frac{dy}{dx} = y\left[1 + \frac{1}{x\log_e x} + 2\operatorname{cosec} 2x\right]$$

Therefore

$$\frac{dy}{dx} = e^x\log_e x\tan x\left[1 + \frac{1}{x\log_e x} + 2\operatorname{cosec} 2x\right].$$ **Ans.**

Example 2: Find the value of $y = x^{\sin^{-1}x}\dfrac{dy}{dx}$.

Solution: $y = x^{\sin^{-1}x}$

By taking log on both sides,

$\log y = \log x^{\sin^{-1}x} = \sin^{-1}x.\log x$

By differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = \sin^{-1}x.\frac{1}{x} + \log x.\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x}.\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}.\log x\right]$$

therefore

$$\frac{dy}{dx} = x^{\sin^{-1}x}\left[\frac{1}{x}\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}.\log x\right].$$ **Ans.**

Example 3: Find the differentiation of 10^x with respect to x, where x is a constant.

Solution: If $y = 10^x$

By taking log of both sides

$$\log y = \log 10^x$$

$$\log y = x\log 10$$

$$\log y = x$$

By differentiating both sides

$$\frac{1}{y}.\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y$$

By putting

$$y = 10^x$$

$$\frac{dy}{dx} = 10^x$$ **Ans.**

Example 4: Find differentiation of $\dfrac{1}{(x+a)(x+b)(x+c)}$ wrt x.

Solution: Let us assume $y = \dfrac{1}{(x+a)(x+b)(x+c)}$

By taking log of both sides

$$\log y = \log\frac{1}{(x+a)(x+b)(x+c)}$$

Or $$\log y = \log 1 - \log(x+a) - \log(x+b) - \log(x-c)$$

By differentiating both sides

$$\frac{1}{y}\frac{dy}{dx} = 0 - \frac{1}{x+a} - \frac{1}{x+b} - \frac{1}{x+c}$$

Or $$\frac{dy}{dx} = -y\left[\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c}\right]$$

Or $$\frac{dy}{dx} = -\frac{1}{(x+a)(x+b)(x+c)}\left[\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c}\right].$$ **Ans.**

Note

Example 5: If $(\sin y)^x = a.$ then find the value of $\dfrac{dy}{dx}$.

Solution: $(\sin y)^x = a.$

By taking log of both sides

$$\log(\sin y)^x = \log a$$

Or
$$x \log \sin y = \log a$$

By differentiating both sides wrt x

$$x \cdot \frac{1}{\sin y} \cdot \cos y \frac{dy}{dx} + \log \sin y = 0$$

$$x \cot y \frac{dy}{dx} = -\log \sin y$$

Or
$$\frac{dy}{dx} = -\frac{\log \sin y}{x \cot y}.$$
Ans.

Example 6: If $(\cos x)^y = (\sin y)^x$ then find the value of $\dfrac{dy}{dx}$.

Solution: $(\cos x)^y = (\sin y)^x$

By taking log of both sides

$$y \log \cos x = x \log \sin y$$

By differentiating both sides wrt x

$$y \frac{1}{\cos x} x^{-\sin x} + \log \cos x \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin y} \cdot \cos y \cdot \frac{dy}{dx} + \log \sin y.1$$

$$-y \tan x + \log \cos x \frac{dy}{dx} = x \cot y \frac{dy}{dx} + \log \sin y$$

$$\frac{dy}{dx}(\log \cos x - x \cot y) = (\log \sin y + y \tan x)$$

$$\frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}.$$
Ans.

Example 7: Find $\dfrac{dy}{dx}$ of $\log (xy) = x^2 + y^2$ wrt x.

$$\log(xy) = x^2 + y^2$$

$$\log x + \log y = x^2 + y^2$$

Solution: By differentiating both sides wrt x **Note**

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx} = = 2x+2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y}-2y\right) = 2x-\frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1-2y^2}{y}\right) = \frac{2x^2-1}{x}$$

$$\frac{dy}{dx} = \frac{y(2x^2-1)}{x(1-2y^2)}. \qquad \textbf{Ans.}$$

Questionnaire 4.1

Short Answer questions:

Find the differentiation of following functions wrt x:

1. x^x

2. $x^{\sin x}$

3. $(\log x)^x$

4. $(1+x)^x$

5. $(x-1)(x-2)(x-3)$

6. $\dfrac{x}{\sqrt{1+x^2}}$

7. $\sqrt{\dfrac{1-x}{1+x}}$

8. $x^x + a^x + x^a$

9. $\dfrac{x\sqrt{1+x}}{(1+x^2)^{3/2}}$

10. e^{x^x}

11. $\sqrt{\dfrac{x^2+x+1}{x^2-x+1}}$

12. $\dfrac{(x-a)(x-b)}{\sqrt{x-c}}$

13. $(x\log x)^{\log\log x}$

14. If $y=\sqrt{\dfrac{1-x}{1+x}}$ then prove that $(1-x^2)\dfrac{dy}{dx}+y=0$

15. If $y=10^{10^x}$ then find $\dfrac{dy}{dx}$

16. If $y=\dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}$ then find $\dfrac{dy}{dx}$

Answers

1. $[x^x(1+\log x)]$

2. $[\sin x(\cos x \log x + \frac{\sin x}{x})]$

3. $(\log x)^x(\log\log x + \frac{1}{\log x})$

4. $(1+x)^x\left[\frac{x}{1+x}+\log(1+x)\right]$

5. $(x-2)(x-3)+(x-11)(x-3)+x-1)(x-2)$

6. $\frac{1}{(1+x^2)^{3/2}}$

7. $\frac{-1}{(1+x)^{3/2}(1-x)^{1/2}}$

8. $x^x\log_e x + a^x\log_e a + ax^{a-1}$

9. $\frac{2+3x-4x^2-3x^3}{2\sqrt{(1+x)}(1+x^2)^{5/2}}$

10. $e^{2x}x^x(1+\log_e x)$

11. $\sqrt{\frac{x^2+x+1}{x^2-x+1}}\left(\frac{1-x^2}{x^4+x^2+1}\right)$

12. $\frac{(x-a)(x-b)}{\sqrt{x-c}}\left[\frac{1}{x-a}+\frac{1}{x-b}-\frac{1}{2(x-c)}\right]$

13. $(x\log x)(\log\log x - 1)$

15. $10^x.10^x(\log e\,10)^2$

16. $\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}\left[\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1\right]$

4.2 Sum of Infinite Tables

Example 1: If $y = x^{x^{x...\infty}}$ then prove that $x\dfrac{dy}{dx}=\dfrac{y^2}{1-y\log x}$.

Solution: $\qquad y = x^{x^{x...\infty}} = x^y$ because $x^{x...\infty} = y$.

By taking log $\log y = y\log x$

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{d}{dx}\log x + \log x\frac{dy}{dx}$$

$$= y\cdot\frac{1}{y}+\log x\frac{dy}{dx}$$

$\therefore\qquad\left(\dfrac{1}{y}-\log x\right)\dfrac{dy}{dx} = \dfrac{y}{x}.$

So that $\qquad x\dfrac{dy}{dx} = \dfrac{y^2}{1-y\log x}.$ **Hence Proved.**

Example 2: If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \, \infty}}},$ then prove that $\dfrac{dy}{dx} = \dfrac{\cos x}{2y-1}.$

Solution: Given that $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \infty}}}$

$\Rightarrow$
$$y = \sqrt{\sin x + y}$$

$$y^2 = \sin x + y$$

By differentiating both sides wrt x

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1} \qquad\qquad \textbf{Hence Proved}$$

Example 3: If $y = e^{x+e^{x+e^{x+\ldots \infty}}}$ then prove that $\dfrac{dy}{dx} = \dfrac{y}{1-y}.$

Solution: Given that $y = e^{x+e^{x+e^{x+\ldots \infty}}}$

$\Rightarrow \qquad y = e^{x+y}$

By taking log of both sides,

$$\log y = \log\{e^{x+y}\}$$

$$= (x+y)\log e$$

$$= x+y, \qquad\qquad [\because \log e = 1]$$

By differentiating both sides wrt x

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$\therefore \qquad\qquad \left(\dfrac{1}{y}-1\right)\dfrac{dy}{dx} = 1$

$\therefore \qquad\qquad \left(\dfrac{1-y}{y}\right)\dfrac{dy}{dx} = 1$

$\therefore \qquad\qquad \dfrac{dy}{dx} = \dfrac{y}{1-y}. \qquad\qquad \textbf{Hence Proved}$

Example 4: If $y = a^{x^{a^{x\ldots \infty}}}$, then prove that $\dfrac{dy}{dx} = \dfrac{y^2 \log y}{x(1 - y \log x \, . \, \log y)}.$

Solution: Here $y = a^{x^{a^{x\ldots \infty}}} = a^{x^y}$

Note

By taking log of both sides,

$\log y = x^y \log a$

By again taking log of both sides

$\log(\log y) = y \log x + \log(\log a)$

By differentiating wrt x

$$\frac{1}{\log y} \cdot \frac{1}{y} \frac{dy}{dx} = y \cdot \frac{1}{x} + \frac{dy}{dx}(\log x) + 0$$

$\therefore$
$$\left(\frac{1}{y \log y} \log x\right)\frac{dy}{dx} = \Rightarrow \left(\frac{1 - y \log x . \log y}{y \log y}\right)\frac{dy}{dx} = \frac{y}{x}$$

$\therefore$
$$\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y \log x . \log y)}.$$ **Hence Proved**

Self Assessment

1. Fill in the blanks:

(i) Such functions which are differentiated by taking logarithm are known as ……………………….
Differentiation.

(ii) $\log\left(\dfrac{m}{n}\right) = \log m - ....$

(iii) $\log(m)^n ......\log m$

(iv) .... $= \log m + \log n$

(v) $\log\left(\dfrac{25}{12}\right) = \log 25 - \log ....$

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Questionnaire 4.2

</div>

Short Answer Questions:

1. If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$ then prove that $(2y - 1)\dfrac{dy}{dx} = 1$.

2. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + ...\infty}}}$ then prove that $(2y - 1)\dfrac{dy}{dx} = \sec^2 x$

3. If $y = -\sqrt{x}^{\sqrt{x}^{\sqrt{x}...\infty}}$ then prove that $x\dfrac{dy}{dx} = 2\dfrac{y^2}{2 - y \log x}$

4. If $y = (\sin x)^{(\sin x)^{(\sin x)...\infty}}$ then prove that $\dfrac{dy}{dx} = \dfrac{y^2 \cot x}{1 - y \log(\sin x)}$

5. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + ... + \infty}}}$ then prove that $\dfrac{dy}{dx} = \dfrac{1}{x(2y - 1)}$

6. If $y = x^2 + \dfrac{1}{x^2 + \dfrac{1}{x^2 + ...}}$ then prove that $\dfrac{dy}{dx} = \dfrac{2xy^2}{1 + y^2}$

4.3 Implicit Function

If any such equation exists between x and y such that cannot be solved for y instantaneously then y is said to be the implicit function of x. In contrast if value of y can be found out in terms of x then y is said to be explicit function of x.

Did u know?   Differentiation of implicit function : To dy/dx find of implicit function, differentiate each element of the equation wrt x then by bringing the value of dy/dx to find one side, find its value.

EXAMPLES WITH SOLUTION

Example 1: If $ax^2 + 2hxy + by^2 = 0$ **, find the value of** dy/dx**.**

Solution: Given that : $ax^2 + 2hxy + by^2 = 0$

By differentiating wrt x

$$a.\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b \cdot \frac{d}{dx}(y^2) = 0$$

Or

$$2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + 2by \cdot \frac{dy}{dx} = 0$$

$$2(hx + by)\frac{dy}{dx} = -2(ax + hy)$$

Or

$$\frac{dy}{dx} = -\left(\frac{ax + hy}{hx + by}\right)$$ **Ans.**

Example 2: If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ **, then find the value of** dy/dx**.**

Solution: Given that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

By differentiating wrt x

$$a\frac{d}{dx}(x^2) + 2h \cdot \frac{d}{dx}(xy) + b \cdot \frac{d}{dx}(y^2) + 2g \cdot \frac{d}{dx}(x) + 2f \cdot \frac{d}{dx}(y) + \frac{d}{dx}(c) = 0$$

Or

$$a(2x) + 2h\left(x\frac{dy}{dx} + y\right) + b\left(2y\frac{dy}{dx}\right) + 2g \cdot 1 + 2f \cdot \frac{dy}{dx} + 0 = 0$$

$$2(hx + by + f)\frac{dy}{dx} = -2(ax + hy + g)$$

Or

$$\frac{dy}{dx} = -\frac{(ax + hy + g)}{(hx + by + f)}$$ **Ans.**

Note

Example 3: If $y = x^y$ prove that $x\dfrac{dy}{dx} = \dfrac{y^2}{1 - y\log x}$.

Solution: Given that $y = x^y$, by taking $\log y = y\log x$

By differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = y\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

Or

$$\left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

Therefore

$$x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}$$

Ans.

Example 4: If $\sin y = x\sin(a + y)$, then prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

Solution: Given that $\sin y = x\sin(a + y)$, or $x = \dfrac{\sin y}{\sin(a + y)}$

By differentiating wrt x

$$1 = \frac{\sin(a+y)\cdot\dfrac{d}{dx}\sin y - \sin y\cdot\dfrac{d}{dx}\sin(a+y)}{\{\sin(a+y)\}^2}$$

Or $\quad \sin^2(a+y) = \sin(a+y)\cdot\cos y\dfrac{dy}{dx} - \sin y\cdot\cos(a+y)\dfrac{dy}{dx}$

Or $\quad \sin^2(a+y) = \{\sin(a+y)\cos y - \sin y\cos(a+y)\}\dfrac{dy}{dx}$

Or $\quad \sin^2(a+y) = \sin(a+y-y)\dfrac{dy}{dx}$

Or $\quad \sin a\dfrac{dy}{dx} = \sin^2(a+y)$

Therefore $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$. Hence Proved.

Example 5: If $x^y = e^{x-y}$ then prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$.

Solution: Given that $x^y = e^{x-y}$, by taking log of both sides

$y\log_e x = (x - y)\log e$ or $y\log_e x = x - y$

Or $\quad y(1 + \log x) = x$ or $y = \dfrac{x}{1 + \log x}$

$\therefore$

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x(1/x)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Task   Find differentiation of $x^{\sin x}$ wrt x.

Ans.: $[\sin x(\cos x \log x + \dfrac{\sin x}{x})]$

Questionnaire 4.3

Find the value of $\dfrac{dy}{dx}$

1. $xy = c$

2. $x^2 + y^2 = a^2$

3. $3x^2 + y^2 = 5$

4. $5x^2 + 5y^2 - 11x - 9y - 12 = 0$

5. $x^n + y^n = a^n$

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

7. $y^2 = 4ax$

8. $x^{2/3} + y^{2/3} = a^{2/3}$

9. If $x^p y^q = (x+y)^{p+q}$ then prove $\dfrac{dy}{dx} = \dfrac{y}{x}$

10. $x^y + y^x = a^b$

11. $y^x = x^y$

12. $x^x + y^y = 1$

13. If $y\sqrt{(1-x^2)} + x\sqrt{(1-y^2)} = 1$ then prove $\dfrac{dy}{dx} = \sqrt{\dfrac{(1-y^2)}{(1-x^2)}} = 0$

14. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then prove that $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

15. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\dfrac{dy}{dx} = -(1+x)^{-2}$

Answers

1. $-\dfrac{y}{x}$

2. $-\dfrac{x}{y}$

3. $\dfrac{-3x}{y}$

4. $-\dfrac{(10x-11)}{10y-9}$

5. $-\left(\dfrac{x}{y}\right)^{n-1}$

6. $\dfrac{-b^2 x}{a^2 y}$

7. $\dfrac{2a}{y}$

8. $\dfrac{-y^{1/3}}{x^{1/3}}$

10. $\dfrac{-xy^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}$

11. $\dfrac{y(y - x \log x)}{x(x - y \log x)}$

12. $-\dfrac{x^x[1 + \log x]}{y^y(1 + \log y)}$

4.4 Parametric Function

Sometimes x and y are given in the form of function of a third variable. This third element is known as Parameter and this type of equation is known as parametric function. In this way we can find the $\dfrac{dy}{dx}$ without elimination of parameter.

Example 1: If $x = f_1(t)$ and $y = f_2(t)$ where t is independent variable and a and y are dependent variables.

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

Therefore differentiation of y wrt x is the multiple of differentiation of x and y wrt to the parameter.

EXAMPLES WITH SOLUTION

Example 1: If $x = at^2$ and $y = $ 2at, find the value of $\dfrac{dy}{dx}$.

Solution: Here $x = at^2$

Differentiating wrt parameter

$$\frac{dx}{dt} = 2at$$

And $y = 2at$

Differentiating both wrt t

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dt} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$$

$$= \frac{2a}{2at} = \frac{1}{t}$$ **Ans.**

Example 2: If $x = a \cos \theta$ and $y = b \sin\theta$ find value of $\dfrac{dy}{dx}$.

Solution: Here,

$$\frac{dx}{d\theta} = -a \sin \theta \text{ or } \frac{dy}{d\theta} = b \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{b \cos \theta}{-a \sin \theta} = -\frac{b}{a} \cot \theta$$ **Ans.**

$$\boxed{\textbf{Questionnaire 4.4}}$$

Find $\dfrac{dy}{dx}$ **of the following:**

1. $x = a \sec\theta, \ y = b \tan\theta$

2. $x = \cos t, \ y = \sin t$

3. $x = \log t, \ y = e^t + \cos t$

4. If $x = \dfrac{3at}{1+t^3}, \ y = \dfrac{3at^2}{1+t^3}$ then prove that

   (i) $\dfrac{dy}{dx} = \dfrac{t(2-t^3)}{1-2t^3}$

   (ii) find $\dfrac{dy}{dx}$ when $t = \dfrac{1}{2}$

$$\boxed{\textbf{Answers}}$$

1. $\dfrac{b}{a} \operatorname{cosec}\theta$ 2. $-\cot t$ 3. $t(e^t - \sin t)$ 4. (ii) $-3/2$

Self Assessment

2. Multiple Choice Questions:

(i) If there exists an equation between x and y such that cannot be solved for y instantaneously then y will be said to be what function of x?

   (a) Implicit   (b) Explicit

   (c) Equal   (d) group

(ii) If y can be calculated in terms of x, then y will be said to be what funtion of x?

   (a) Implicit   (b) Explicit

   (c) Equal   (d) group

(iii) What will be the value of $xy = c$

   (a) $-\dfrac{x}{y}$ (b) $\dfrac{x}{y}$

   (c) $-\dfrac{y}{x}$ (d) $\dfrac{y}{x}$.

4.5   Summary

- If you need to differentiate such functions in which exponent is also a function of that variable and a function that needs to be differentiated is the product or division of many functions, then we first need to find logarithm of those functions and then differentiate it.

- If any such equation exists between x and y such that cannot be solved for y instantaneously then y is said to be the implicit function of x. In contrast if value of y can be found out in terms of x then y is said to be explicit function of x.

   $\log (m \cdot n) = \log m + \log n$

Note

$$\log\left(\frac{m}{n}\right) = \log m - \log n$$

$$\log (m)^n = n \log m$$

● Sometimes x and y are given in the form of function of a third variable. This third element is known as Parameter and this type of equation is known as parametric function. In this way we can find the $\frac{dy}{dx}$ without elimination of parameter.

4.6 Keywords

● *Function:* Work

● *Infinite:* Which does not have an end

4.7 Review Questions

1. If $y = x^{\sin^{-1} x}$ then find $\frac{dy}{dx}$.

   (**Ans.:** $x^{\sin^{-1} x}\left[\frac{1}{x}\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\log x\right]$)

2. If $(\cos x)^y = (\sin y)^x$ find $\frac{dy}{dx}$

   (**Ans.:** $\frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$)

3. If $y = x^y$, Prove that $x\,\frac{dy}{dx} = \frac{y^2}{1 - y \log x}$

4. If $x = a \cos \theta$ and $y = b \sin \theta$ find $\frac{dy}{dx}$

   (**Ans.:** $-\frac{b}{a}\cot \theta$)

Answers: Self Assessment

1. (i) Logarithmic (ii) log n (iii) n (iv) log(m, n) (v) 12

2. (i) (a) (ii) (b) (iii) (c)

4.8 Further Readings

Books Mathematics for Economics- Council for Economic Education.

Mathematical Economics-Michael Harrison, Patrick Waldren.

Mathematics for Economics- Malcom, Nicholas, U.P. London

Mathematics for Economics-Carl P. Simon, Lawrence Bloom.

Mathematics for Economist- Mehta and Madnani-Sultan Chand and Sons.

Essential Mathematics for Economics-Nut Sedester, Peter Hamand, Prentice Hall Publication.

Mathematics for Economist-Simon and Bloom-Viva Publications.

Mathematics for Economist-Yamane-Prentice Hall India.

Mathematics for Economics and Finance-Martin Norman.

Unit 5: Second and Higher Order Differentiation

CONTENTS

Objectives

Introduction

5.1   Successive Differentiation

5.2   Summary

5.3   Keywords

5.4   Review Questions

5.5   Further Readings

Objectives

After reading this unit, students will be able to :

● Find out Successive Differentiation.

Introduction

If y is the product of x then $\dfrac{dy}{dx}$ would also be the product of x. The differential coefficient of this would be the second differential coefficient and then this will be referred as third differential coefficient.

5.1   Successive Differentiation

If y is the product of x then $\dfrac{dy}{dx}$ would also be the differential coefficient of y with respect to x, which further can be differentiated. The $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ differential coefficient of $\dfrac{dy}{dx}$ would be referred to as second differential coefficient. Thus, second differential coefficient of y, would be referred to as third differential coefficient. Thus, the different coefficient can be shown as $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \dots$.

Prime product $y = f(x)$	Simple Indication	Other Indication
First coefficient	$\dfrac{dy}{dx}$	$f'(x), Dy, y_1$
Second coefficient	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$	$f''(x), D^2y, y_2$
Third coefficient	$\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^2y}{dx^2}$	$f'''(x), D^3y, y_3$
………….	………….	………….
n^{th} coefficient	$\dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right) = \dfrac{d^n y}{dx^n}$	$f^n(x), D^n y, y_n$

Note

Thus, if $y = x^7$, then $\dfrac{dy}{dx} = 7x^6$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(7x^6) = 42x^5$$

And $\qquad \dfrac{d^3y}{dx^3} = \dfrac{d}{dx}(42x^5) = 210x^4$

EXAMPLES WITH SOLUTION

Example 1: If $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, then find out the value of $f'''(x)$.

Solution: given then

$$
\begin{aligned}
f(x) &= ax^5 + bx^4 + cx^3 + dx^2 + ex + f \\
f'(x) &= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e + 0 \\
f''(x) &= 20ax^3 + 12bx^2 + 6cx + 2d + 0 \\
f'''(x) &= 60ax^2 + 24bx + 6c = 0 \\
f''''(x) &= 120ax + 24b
\end{aligned}
$$

Ans.

Example 2: If $y = A \sin mx + B \cos mx$, then prove that $\dfrac{d^2y}{dx^2} + m^2 y = 0$.

Solution: given then $y = A \sin mx + B \cos mx$

$$
\begin{aligned}
\frac{dy}{dx} &= A \cos mx . \frac{d}{dx}(mx) + B(-\sin mx) \cdot \frac{d}{dx}(mx) \\
&= Am \cos mx - Bm \sin mx
\end{aligned}
$$

Further evaluating with respect to x

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= Am(-\sin mx) . \frac{d}{dx}(mx) - Bm \cos mx . \frac{d}{dx}(mx) \\
&= -Am^2 \sin mx - Bm^2 \cos mx \\
&= -m^2(A \sin mx + B \cos mx) = -m^2 y
\end{aligned}
$$

Therefore, $\dfrac{d^2y}{dx^2} + m^2 y = 0$.

Example 3: If $y = \sin(\sin x)$, then prove that $y_2 + y_1 \tan x + y \cos^2 x = 0$.

Solution: given that

$$
\begin{aligned}
y &= \sin(\sin x) \\
y_1 &= [\cos(\sin x)] \cdot \cos x \qquad\qquad\qquad\qquad \text{...(1)}
\end{aligned}
$$

On Differentiation $\qquad\qquad y_2 = [\cos(\sin x)] \cdot (-\sin x) + \cos x \cdot [-\sin(\sin x)] \cos x$

$$\qquad\qquad\qquad = -\sin x \cos(\sin x) - \cos^2 x \sin(\sin x)$$

$$= -\sin x \cos (\sin x) - y \cos^2 x, \qquad [\because y = \sin(\sin x)]$$

$$= -\sin x \cdot \frac{y_1}{\cos x} - y \cos^2 x, \qquad [\text{From (1)}]$$

Therefore, $y_2 + y_1 \tan x + y \cos^2 x = 0$.

Example 4: If $y = e^{ax} \sin bx,$ **, then prove that** $\dfrac{d^2 y}{dx^2} - 2a \dfrac{dy}{dx} + (a^2 + b^2) y = 0$.

Solution: given that $y = e^{ax} \sin bx$ ...(1)

Evaluating with respect to x

$$\frac{dy}{dx} = e^{ax} \frac{d}{dx}(\sin bx) + \sin bx . \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cos bx . \frac{d}{dx}(bx) + \sin bx . e^{ax} . \frac{d}{dx}(ax)$$

$$= b e^{ax} \cos bx + a e^{ax} \sin bx$$

$$= b e^{ax} \cos bx + ay, \quad [\text{From equation (1)}] \qquad ...(2)$$

Further evaluating with respect to x

$$\frac{d^2 y}{dx^2} = b \left[e^{ax} . \frac{d}{dx}(\cos bx) + \cos bx . \frac{d}{dx}(e^{ax}) \right] + a \frac{dy}{dx}$$

$$= b \left[e^{ax} .(-\sin bx) . \frac{d}{dx}(bx) + \cos bx . e^{ax} . \frac{d}{dx}(ax) \right] + a \frac{dy}{dx}$$

$$= b[-b e^{ax} \sin bx + a e^{ax} \cos bx] + a \frac{dy}{dx}$$

$$= -b^2 e^{ax} \sin bx + a(b e^{ax} \cos bx) + a \frac{d}{dy}$$

$$= -b^2 y + a \left(\frac{dy}{dx} - ay \right) + a \frac{dy}{dx} = 2a \frac{dy}{dx} - (a^2 + b^2) y,$$

$$= -b^2 y + a \left(\frac{dy}{dx} - ay \right) + a \frac{dy}{dx} = 2a \frac{dy}{dx} - (a^2 + b^2) y,$$

from equation (1) and (2)

Therefore, $\dfrac{d^2 y}{dx^2} - 2a \dfrac{dy}{dx} + (a^2 + b^2) y = 0$

Self Assessment

1. Fill in the blanks:

(i). $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is the .... differential coefficient of y.

(ii). $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ ....

(iii). $\dfrac{d}{dx}(...........)=\dfrac{d^2y}{dx^2}$

Questionnaire 5.1

1. If $y = 8x^3 + 4x^2 + 3x + 11$ then find out the value of $\dfrac{d^3y}{dx^3}$

2. If $y = ax^3 + bx^2 + cx + d$ then find out the value of $\dfrac{d^3y}{dx^3}$

3. If $y = x^2 \log x$, then prove that $\dfrac{d^3y}{dx^3} = \dfrac{2}{x}$

Determine the second differential coefficient of following product

4. (i) $x^3 \log x$ (ii) $x \log x$

5. $\sin(\cos x)$

6. $x^3 e^{4x}$

7. $\tan e^x$

8. Find out the nth differential coefficient of e^{nx}

9. If $y = A \sin px + B \cos px$, then prove that $\dfrac{d^2y}{dx^2} + p^2 y = 0$

10. If $x^3 + y^3 - 3axy = 0$, then prove that $\dfrac{d^2y}{dx^2} = \dfrac{2a^2xy}{(ax - y^2)^3}$

11. If y is the product of z and $z = ax$ then prove that $\dfrac{d^2y}{dx^2} = a^2 \dfrac{d^2y}{dz^2}$

12. If $y = (\sin^{-1} x)^2$, then prove that $(1 - x^2)y_2 - xy_1 = 2$

13. If $y = e^{\tan^{-1} x}$, then prove that $(1 + x^2)y_2 + (2x - 1)y_1 = 0$

14. If $\sqrt{x+y} + \sqrt{y-x} = c$, then prove that $\dfrac{d^2y}{dx^2} = \dfrac{2}{c^2}$

15. If $x^2 + xy + y^2 = a^2$, then prove that $\dfrac{d^2y}{dx^2} + \dfrac{6a^2}{(x+2y)} = 0$

16. If $y = \tan^{-1}\dfrac{1-2\log x}{1+2\log x} + \tan^{-1}\dfrac{3+2\log x}{1-6\log x}$, then prove that $\dfrac{d^2y}{dx^2} = 0$

17. If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, then prove that $\dfrac{d^2p}{d\theta^2} + p = \dfrac{a^2b^2}{p^3}$

18. If $y = e^{a\sin^{-1}x}$, then prove that $(1-x^2)y_2 - xy_1 - a^2y = 0$

19. If $y = \sin(m\sin^{-1}x)$, then prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$.

5.2 Summary

- If y is the product of x then $\dfrac{dy}{dx}$ would also be the differential coefficient of y with respect to

  x, which further can be differentiated. The $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ differential coefficient of $\dfrac{dy}{dx}$ would be referred to as **second differential coefficient.**

5.3 Keywords

- *Successive:* In a series
- *Miscellaneous:* Mixed

5.4 Review Questions

1. Find the value of $\dfrac{d^2y}{dx^2}$ if $y = x^3 - \dfrac{1}{x^3}$ 　　　　　　　　　[**Ans.:** $= 6x - 12x^{-5}$]

2. If $y = x^3\log x$, then prove that $\dfrac{d^4y}{dx^4} = \dfrac{6}{x}$

3. If $y = e^{ax}\sin bx$ then prove that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2+b^2)y = 0$

Answers: Self Assessment

1. (i) 2nd 　　　　　(ii) $\dfrac{d^3y}{dx^3}$ 　　　　　(iii) $\dfrac{dy}{dx}$

Note

5.5 Further Readings

Books

Mathematical Economy – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for economics – Council for Economic Education.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economics and Finance – Martin Norman.

Unit 6: Differentiation : Partial

CONTENTS

Objectives

Introduction

6.1   Differentiation of a Function in Respect to Other Function

6.2   Summary

6.3   Keywords

6.4   Review Questions

6.5   Further Readings

Objectives

After reading this unit students will be able to:

● Understand the Method of Differentiation of a Function in Respect to other Function.

Introduction

Differential coefficient of the first function relative to some other function is the ratio of the differential coefficient of the first function with respect to x to the differential coefficient of the second function with respect to x.

6.1   Differentiation of a Function in Respect to Other Function

Suppose $y_1 = f_1(x)$ and $y_2 = f_2(x)$

That is y_1 and y_2 are the functions of x, on differentiating both with respect to x

$$\frac{dy_1}{dx} = f_1'(x) \text{ and } \frac{dy_2}{dx} = f_1'(x)$$

Now the differential coffiecient of y_1 with respect to y_2 is $\dfrac{dy_1}{dy_2}$

Notes

$$\frac{dy_1}{dy_2} = \frac{\dfrac{dy_1}{dx}}{\dfrac{dy_2}{dx}} = \frac{f_1'(x)}{f_2'(x)}$$

Therefore $\dfrac{dy_1}{dy_2} = \dfrac{\text{Differential coefficient of } y_1 \text{ with respect to } x}{\text{Differential coefficient of } y_2 \text{ with respect to } x}$

Did u know?   Differential coefficient of the first function relative to some other function is the ratio of the Differential coefficient of the first function with respect to x to the Differential coefficient of the second function with respect to x.

Note

EXAMPLES WITH SOLUTION

Example 1: Find the differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x = \dfrac{1}{2}$.

Solution :
$$\frac{d(\tan^{-1}x)}{d(\sin^{-1}x)} = \frac{\dfrac{1}{1+x^2}}{\dfrac{1}{\sqrt{1-x^2}}} = \frac{\sqrt{1-x^2}}{1+x^2}$$

At
$$x = \frac{1}{2} = \frac{\sqrt{1-\dfrac{1}{4}}}{1+\dfrac{1}{4}} = \frac{\dfrac{\sqrt{3}}{2}}{\dfrac{5}{4}} = \frac{2\sqrt{3}}{5}$$
Ans.

Example 2: Find the differential coefficient of $e^{\tan x}$ with respect to $\sin x$.

Solution : Let us suppose $y_1 = e^{\tan x}$ and $y_2 = \sin x$

Here
$$dy_1 = \frac{d}{dx}e^{\tan x} = e^{\tan x} \cdot \sec^2 x$$

And
$$dy_2 = \frac{d}{dx}\sin x = \cos x$$

Therefore:
$$\frac{dy_1}{dy_2} = \frac{de^{\tan x}}{d\sin x} = \frac{e^{\tan x} \cdot \sec^2 x}{\cos x} = \frac{e^{\tan x}}{\cos^3 x}$$
Ans.

Example 3: Find the differential coefficient of $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$.

Solution : Suppose $y_1 = \tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ and $y_2 = \tan^{-1}x$

On putting, $\qquad x = \tan\theta$

$$y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\sec\theta-1}{\tan\theta} = \tan^{-1}\frac{1-\cos\theta}{\sin\theta}$$

$$= \tan^{-1}\frac{2\sin^2\dfrac{1}{2}\theta}{2\sin\dfrac{1}{2}\theta\cos\dfrac{1}{2}\theta}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$$

Note

$$\frac{dy_1}{dx} = \frac{d}{dx}\left[\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$\therefore \qquad = \frac{d}{dx}\left(\frac{\tan^{-1}x}{2}\right) = \frac{1}{2(1+x^2)}$$

And $$\frac{dy_2}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{dy_1}{dy_2} = \frac{d[\tan^{-1}\{\sqrt{1-x^2}-1\}/x]}{d(\tan^{-1}x)}$$

$$\therefore \qquad = \frac{\dfrac{1}{2(1+x^2)}}{\dfrac{1}{1+x^2}} = \frac{1}{2}.$$ **Ans.**

Task Find the differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x = \dfrac{1}{3}$

(**Ans. :** $\dfrac{3\sqrt{2}}{5}$)

Example 4: Find the differential coefficient of $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ **with respect to** $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right).$

Solution: Suppose $y_1 = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $y_2 = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

On putting $x = \tan\theta$

$$y_1 = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x$$

Then $$\frac{dy_1}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Also $$y_2 = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

Then $$\frac{dy_2}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

$$\frac{dy_1}{dy_2} = \frac{d[\tan^{-1}\{2x/(1-x^2)\}]}{d[\sin^{-1}\{2x/(1+x^2)\}]}$$

Note

$$= \dfrac{\dfrac{d}{dx}[\tan^{-1}\{2x/(1-x^2)\}]}{\dfrac{d}{dx}\sin^{-1}\{2x/(1+x^2)\}}$$

$$\therefore \qquad = \dfrac{2}{1+x^2} \times \dfrac{1+x^2}{2} = 1 .$$ **Ans.**

Example 5: Find the differential coefficient of function $f(x)=\sin^{-1}(2x\sqrt{1-x^2})$ **with respect to** $\sin^{-1}x$.

Solution : Suppose

$$y_1 = \sin^{-1}2x\sqrt{1-x^2}$$

$$x = \sin\theta$$

$$y_1 = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \sin^{-1}(2\sin\theta\cos\theta)$$

$$y_1 = \sin^{-1}(\sin 2\theta)$$

$$y_1 = 2\theta = 2\sin^{-1}x$$

$$\dfrac{dy_1}{dx} = \dfrac{2}{\sqrt{1-x^2}}$$

$$y_2 = \sin^{-1}x$$

Again

$$\dfrac{dy_2}{dx} = \dfrac{1}{\sqrt{1-x^2}}$$

Then

$$\dfrac{d\sin^{-1}(2x\sqrt{1-x^2})}{d\sin^{-1}x} = \dfrac{\dfrac{2}{\sqrt{1-x^2}}}{\dfrac{1}{\sqrt{1-x^2}}} = 2 .$$ **Ans.**

Example 6: If $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$ **then prove that** $\dfrac{dy}{dx}=\dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$.

Solution: Suppose $x^3=\sin\theta,\ y^3=\sin\phi,$ then $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$

Or

$$\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\phi} = a^3(\sin\theta-\sin\phi)$$

$$\cos\theta+\cos\phi = a^3(\sin\theta-\sin\phi)$$

$$2\cos\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2} = a^3.2\cos\dfrac{\theta-\phi}{2}\sin\dfrac{\theta-\phi}{2}$$

$$\Rightarrow \qquad \cot\dfrac{\theta-\phi}{2} = a^3$$

$$\Rightarrow \qquad \theta-\phi = 2\cot^{-1}a^3$$

$$\Rightarrow \qquad \sin^{-1}x^3-\sin^{-1}y^3 = 2\cot^{-1}a^3$$

On differentiating with respect to x

$\Rightarrow \qquad \dfrac{1}{\sqrt{1-x^6}}.3x^2 - \dfrac{1}{\sqrt{1-y^6}}.3y^2\dfrac{dy}{dx} = 0$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}.$ **Ans.**

Example 7: If $y = \dfrac{1}{1+x+x^2+x^3}$, then find the value of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $x = 0$.

Solution: $\qquad y = \dfrac{1}{1+x+x^2+x^3}$

$\qquad = (1+x+x^2+x^3)^{-1}$

$\qquad \dfrac{dy}{dx} = -(1+x+x^2+x^3)^{-2}(1+2x+3x^2)$

At $\quad x = 0$

$\qquad \dfrac{dy}{dx} = -1$

$\qquad \dfrac{d^2y}{dx^2} = 2(1+x+x^2+x^3)^{-3}(1+2x+3x^2)$

$$-(1+x+x^2+x^3)^{-2}(2+6x)$$

At $\quad x = 0$ $\qquad \dfrac{dy}{dx} = 0.$ **Ans.**

Example 8: If $y = x\cos(a+y)$, then prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\cos a}$ and at $x = 0$, $\dfrac{dy}{dx} = \cos a$.

Solution: $\qquad \sin y = x\cos(a+y)$

$\Rightarrow \qquad x = \dfrac{\sin y}{\cos(a+y)}$

On differentiating with respect to x

$\qquad 1 = \dfrac{\cos(a+y)\cos y\dfrac{dy}{dx} - \sin y \cdot \{-\sin(a+y)\}\dfrac{dy}{dx}}{\cos^2(a+y)}$

$\qquad 1 = \dfrac{\{\cos(a+y).\cos y + \sin(a+y).\sin y\}\dfrac{dy}{dx}}{\cos^2(a+y)}$

$\qquad 1 = \dfrac{\cos(a+y-y)\dfrac{dy}{dx}}{\cos^2(a+y)}$

$\qquad 1 = \dfrac{\cos a}{\cos^2(a+y)}\dfrac{dy}{dx}$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\cos a}$

Note   Again when $x = 0$

Then $\sin y = 0 \Rightarrow y = n\pi$

At, $x = 0$
$$\frac{dy}{dx} = \frac{\cos^2(a + n\pi)}{\cos a}$$

$\Rightarrow$
$$\frac{dy}{dx} = \frac{\cos^2 a}{\cos a} = \cos a$$

Example 9: If $y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$ **, then prove that**

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 4$$

Solution :
$$y = (\sin^{-1} x)^2 + (\cos^{-1} x)^2$$

$$\frac{dy}{dx} = \frac{2(\sin^{-1} x - \cos^{-1} x)}{\sqrt{1 - x^2}}$$

$\Rightarrow$
$$\sqrt{1 - x^2}\,\frac{dy}{dx} = 2(\sin^{-1} x - \cos^{-1} x)$$

On differentiating with respect to x,

$$\sqrt{1 - x^2}\,\frac{d^2y}{dx^2} + \frac{1}{2\sqrt{1 - x^2}}(-2x)\cdot\frac{dy}{dx} = 2\left[\frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}}\right]$$

$$\frac{(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx}}{\sqrt{1 - x^2}} = 2\cdot\frac{2}{\sqrt{1 - x^2}}$$

$\therefore$
$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 4$$

Self Assessment

1. Multiple Choice Questions:

(i)   What will be the differential coefficient of ax^7 with respect to x^7?

(a) a (b) x (c) x^7 (d) a^2

(ii)   What will be the differential coefficient of $\log x$ with respect to $\tan x$?

(a) $\dfrac{\sin^2 x}{x}$ (b) $\dfrac{\cos^2 x}{x}$ (c) $\dfrac{x}{\cos^2 x}$ (d) $\dfrac{x}{\sin^2 x}$

(iii)   Differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x = \dfrac{1}{2}$ will be

(a) $\dfrac{3\sqrt{2}}{5}$ (b) $\dfrac{5\sqrt{2}}{3}$ (c) $\dfrac{2\sqrt{3}}{5}$ (d) $\dfrac{5}{2\sqrt{3}}$

Questionnaire 6.1

Find the Differential coefficients of the following:

1. x^5 with respect to x^2

2. e^x with respect to $\sqrt{x}$

3. $x \sin^{-1} x$ with respect to $\sin^{-1} x$.

4. $\sin^{-1}\left(\dfrac{1-x}{1+x}\right)$ with respect to $\sqrt{x}$

5. $(\log \sin x)^{\sin x}$ with respect to $\sin x$

6. $\log (x^2 + 2x + 1)$ with respect to $(x^2 + 2x)$.

7. $\sec^{-1}\dfrac{1}{2x^2 - 1}$ with respect to $\sqrt{1 - x^2}$ at $x = \dfrac{1}{2}$

Answers

1. $\dfrac{5}{2}x^3$

2. $2\sqrt{x}\ e^x$

3. $x + \sqrt{1 - x^2} \cdot \sin^{-1} x$

4. $\dfrac{-2}{1 + x}$

5. $(\log \sin x)^{\sin x} \cdot [\log (\log \sin x + 1 \log \sin x)]$

6. $\dfrac{1}{x^2 + 2x + 1}$

7. 4

6.2 Summary

- Suppose $y_1 = f_1(x)$ and $y_2 = f_2(x)$ i.e. y_1 and y_2 are the two functions of x. On differentiating both with respect to x.

$$\frac{dy_1}{dx} = f_1'(x) \quad \frac{dy_2}{dx} = f_2'(x)$$

6.3 Keywords

- *Partial:* Unfair, divided

6.4　Review Questions

1.　Find the differential coefficient of $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$　　**[Ans. : $\dfrac{1}{2}$]**

2.　If $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$ then prove that $\dfrac{dy}{dx}=\dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$.

Answers: Self Assessment

(i)　a　　　　　　　　　(ii)　b　　　　　　　　　(iii)　c

6.5　Further Readings

Books　　　　　Mathematics for Economics – Malcom, Nicolas, U.C.London.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Simon and Bloom – Viva Publlications.

Unit 7: Homogeneous Function and Euler's Theorem

Objectives

After reading this unit students will be able to :

- Know the Definition of Homogeneous Function.
- Understand Euler's Theorem.
- Understand Cobb-Douglas Production Function.
- The C.E.S. (Constant Elasticity Substitution) Production Function.

Introduction

Maximum use of special production is referred to as Homogeneous.

Euler's Theorem describes when the every means of production are increased in a proportion, as a result production will increase in the same ratio.

Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means.

Cobb-Douglas Production Function also keeps an important place in economic area. In today's era economists are using Cobb-Douglas Production Function in various economical areas.

7.1   Homogeneous Function: Definition

Maximum use of special production is referred to as Homogeneous function.

For *e.g.* $f(x, y) = x^2 - y^2$

If $x \to tx$ and $y \to ty$ is put in the above production where t is positive constant, then

$$f(tx, ty) = (tx)^2 - (ty)^2 = t^2(x^2 - y^2)$$
$$= t^2 f(x, y)$$

If production is $f(x, y) = x^2 - y^2$

$$f(tx, ty) = t^3(x, y)$$
$$f(x, y) = x^n - y^n$$

Note

Then $f(tx, ty) = t^n f(x, y)$

Another example of homogeneous function

$$f(x, y) = x^2 + xy - 3y^2 \text{ then}$$

$$f(x, y) = x^3 + 3x^2y + y^3$$

And $\quad\quad\quad\quad\quad\quad f(tx, ty) = t^3 f(x, y)$

Production can be written as $\quad f(tx, ty) = t^n f(x, y)$

Notes   The above production is n category of $f(x, y)$. In economics majorly production of Zero Category is used.

Example 1: $\quad\quad f(x, y, z) = \dfrac{x}{z} + \dfrac{y}{z}$

Then, $\quad\quad\quad f(tx, ty, tz) = \dfrac{tx}{tz} + \dfrac{ty}{tz} = \dfrac{x}{z} + \dfrac{y}{z} = f(x,y,z) = t^0 f(x,y,z)$

Example 2: $\quad\quad f(x, y, z) = \dfrac{x^2}{yz} + \dfrac{y^2}{xz} + \dfrac{z^2}{xy}$

$$f(tx, ty, tz) = \dfrac{t^2x^2}{t^2yz} + \dfrac{t^2y^2}{t^2xz} + \dfrac{t^2z^2}{t^2xy}$$

$$= t^0 f(x, y, z)$$

Example 3: If q quantity, p price and y is income, then demand function is as under

$$q = f(p,y) = \dfrac{y}{kp} \text{ where } k \text{ is constant value, then}$$

$$f(tp, ty) = \dfrac{ty}{kp} = \dfrac{y}{kp} t^0 f(p,y)$$

$$q = f(p, y) = f(tp, ty)$$

Therefore, when price (p) and income (y) changes in same ratio, then there would be no change in demand (q).

7.2 Euler's Theorem

Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted. If every means of production is credited equal to its marginal productivity and total production is liquidated completely. In mathematical formula Euler's Theorem can be indicated. If production, $P = f(L, K)$ is Linear Homogeneous Function:

$$P = L\dfrac{\partial P}{\partial L} + K\dfrac{\partial P}{\partial K} \text{ in other words } P = LMP_L + KMP_K$$

Where P = Total Production, L = Unit of labor means, K = Unit of capital, or   **Note**

MP_L = Marginal productivity of Labor and $\dfrac{\partial P}{\partial K}$ = or MP_K = Marginal productivity of Capital

Assumptions

The theorem is based on following important assumptions:

1. The Law of Constant Returns of Scale is functional. This is possible only when production is linear homogeneous and is of a degree.

2. Market is fully competitive

3. When it considers the division of means of production

4. For the given period, techniques are constant

7.2.1 Diagrammatical Presentation of Euler's Theorem

Keeping all the above assumptions into consideration, Euler's Theorem can be shown in Figure. Assume that production is $P = f(L, K)$ then both the factors L and K affect total production P. First, We will see the effect of L factor over P where K is constant. Thereafter we will see the effect of K factor over P where L is constant
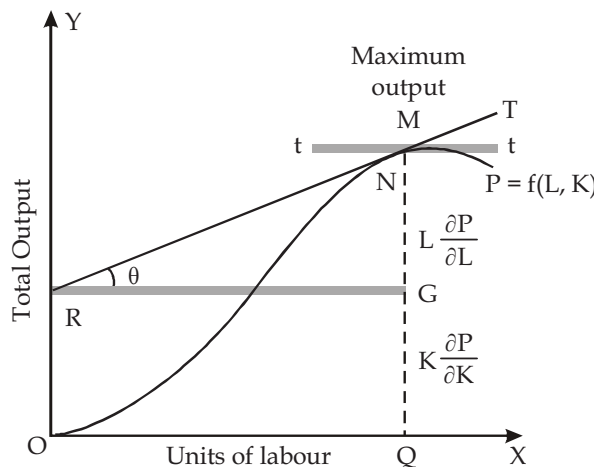
Figure 7.1

In Figure 7.1, X-axis contains the unit factor of L and Y-axis shows Total Production. *TP* is Total Production. At point M total production will be maximum. Assume at N point of total production curve *TP*, *TI* touching line is drawn. From N-point vertical line is drawn on X-axis, which meets at Q-point on X-axis.

Now, Slope on N-point $\dfrac{\partial P}{\partial K} = \tan \theta$

$$= \frac{NG}{RG} = \frac{NG}{OQ}$$

Now $L\dfrac{\partial P}{\partial L} = OQ\dfrac{NG}{OQ} = NG$ ...(i)

Note

Euler's Theorem

$$P = L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K}$$

Or

$$K\frac{\partial P}{\partial K} = P - L\frac{\partial P}{\partial L} = QN - NG = QG \qquad ...(ii)$$

Where *P* = Total production level

$$P = L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K}$$

Or

$$QN = NG + QG$$

Thus, total production *QN* is divisible in *NG* and *OG*. Where *NG* and *QG* are *L* factor is means and *K* factor is paid the value respectively. This way if marginal productivity of every factors of any firm is paid the value of its marginal product, then total production will be exhausted.

7.2.2 Mathematical Solution of Euler's Theorem

With the help of homogeneous production Euler's Theorem establishes a special relation in the principle of marginal productivity. If $u = f(x, y)$ *h* a homogeneous production of h degree, then theory will be shown as $= xf_{tx} + xf_{ty}$

$$\frac{\partial}{\partial t} \text{ (R.H.S.)} = \frac{\partial t^h}{\partial t} f(x,y) + t^h \frac{\partial f(x,y)}{\partial t} \text{ (L.H.S.)}$$

$$= ht^{h-1} f(x, y) - 0$$

$$= ht^{h-1} f(x, y)$$

Where L.H.S. means Left Hand side and R.H.S. means Right, then L.H.S. = R.H.S.

$$xf_{tx} + yf_{ty} = ht^{h-1} f(x, y)$$

Now if $t = 1$, then t may be any number

$$xf_x + yf_y = hf(x, y)$$

h is the degree of equation. If now we consider linear homogeneous function, then $h = 1$

then

$$xf_x + yf_y = f(x, y)$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

or   if we consider second equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} + ... = 2u$$

Generalizing for more than two constants

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} + ... = ku(x, y, z)$$

Where k is degree of equation $f(x, y, z) = 3x + 2y - 4z$

Is a linear homogeneous function

Then as per prime

$$xf_x + yf_y + zf_z = 1 \times (3x + 2y - 4z)$$
$$f_x = 3, f_y = 2, f_z = -4$$

Therefore, left side would be as under

$$\text{L.H.S.} = 3x + 2y - 4z = \text{R.H.S.}$$

7.2.3 Importance of Euler's Theorem

Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means. Now the question arises how all factors of production should be distributed in proportion to the total productivity so that total production is exhausted. In addition to that this helps in resolving short-term problems of production – like distribution of all factors of production and distribution of total production into all factors etc. with the help of this theory firm can ascertain how factors should be used. Factors of production should be utilized to the level where its price is equal to its marginal productivity. This way it also helps in determining the price factors of production.

Self Assessment

1. Fill in the blanks:

1. Maximum use of special production in economics is referred to as …………………….

2. ……….. Theorem establishes a special relation in the principle of marginal productivity.

3. Euler's Theorem has an important place in ……………….. area.

4. …………….. is made in conjugation with many means.

5. If production is made in …………….. with many means, it describes how all factors of production should be distributed in proportion to the total productivity so that total production is exhausted

7.3   Cobb-Douglas Production Function

Cobb-Douglas Production Function is used widely in economic area. This production function was developed by *C W* Cobb and D *H* Douglas. They studied various industries of the world to devise this function. This way, it is used as a Universal Law for production.

Did u know?   Cobb-Douglas Production Function is indicated as

$$P = AL^\alpha K^\beta u$$

Where P = Production, L = Labor, K = Capital u = Disturbance Terms and A is a constant value.

α and β are positive parameters. With these $\alpha > 0$, $\beta > 0$, $L > 0$ and $\alpha + \beta = 1$.

Some economists show this production function with u

$$P = AL^a K^b \ (\alpha > 0, \beta > 0)$$

If $\alpha + \beta = 1$

In Cobb-Douglas Production Function, there are two factors of production viz. $P = f(L, K)$

If both the sides are multiplied by λ (Lemda) then $\lambda P = f(\lambda L, \lambda K)$

Note

Then $\quad f(\lambda L, \lambda K) = A(\lambda L)^\alpha (\lambda K)^\beta u$

$$= \lambda^{\alpha+\beta} A L^\alpha K^\beta u$$

$$= \lambda \alpha + \beta P \qquad\qquad (\because\ P = AL^\alpha K^\beta u)$$

If $\alpha + \beta = 1$, then production would be under the constant result.

If $\alpha + \beta > 1$, then increase in production will happen

If $\alpha + \beta < 1$, then decrease in production will happen

7.3.1 Characteristics of Cobb Douglas Production Function

1. If factor of production is increased by some constant value *i.e.* λ, then productivity will also grow by λ. Cobb Douglas Production Function $P = L^\alpha K^\beta u$ is if Labor L and Capital *K* is increased by λ.

$$p^t = A(\lambda L)^\alpha (\lambda K)^\beta u$$

$$= \lambda^{\alpha+\beta} A L^\alpha K^\beta u$$

$$= \lambda A L^\alpha K^\beta u \qquad\qquad (\text{if } \alpha + \beta = 1)$$

$$= \lambda P$$
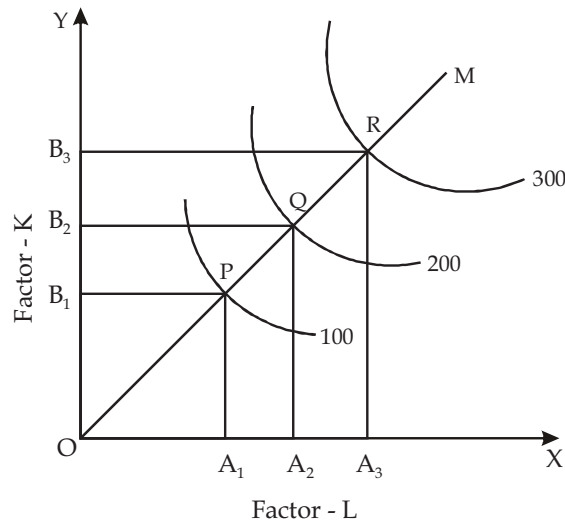
Figure 7.2

This way productivity will increase in the ratio o increase in factors. This can be seen in Figure 7.2. *X*-axis shows *L* (Labor) and *Y*-axis shows *K* (Capital). To produce 100 units OA_1 units of factor *L* and OB_1 units of factor *K* is taken. If the productivity is taken to 200 units, similarly *L* and *K* would also required to be doubled. Here OA_2 is just double of OA_1 units. Similarly OB_2 is also twice of OB_1. In a similar fashion for production of 300 units, triple unit of *A* and *B* would be required as can be seen in the figure 7.2.

This can be understood in other way. *OM* is the expansion path of production where *P, Q* and *R* meet the balance point. These balance point will be on equal distance viz $OP = PQ = QR$, which described that production will increase in the ratio of factors increased.

2. Production function is homogeneous and of a degree: **Note**

    If Production function is homogeneous and of a degree, it means that production will come under constant formula.

    Production function = $P = AL^{\alpha} K^{\beta} u$

    Taking log of two sides log $P = log\ A + \alpha \log L + \beta \log K + \log u$

Partially differentiating with respect to L and K separately

$$\frac{1}{P}\frac{\partial P}{\partial L} = \frac{\alpha}{L} \qquad \qquad ...(i)$$

And

$$\frac{1}{P}\frac{\partial P}{\partial K} = \frac{\beta}{L} \qquad \qquad (ii)$$

Writing (i) and (ii) further

$$L\frac{\partial P}{\partial L} = \alpha P \qquad \qquad ...(iii)$$

$$K\frac{\partial P}{\partial L} = \beta P \qquad \qquad ...(iv)$$

Adding equation (iii) and (iv)

$$\frac{L\partial P}{\partial P} + K\frac{\partial P}{\partial L} = P\ \alpha + P\ \beta = P(\alpha + \beta)$$

3. If Production function is homogeneous and of a degree, then elasticity of substitution will always be equal to unit. If production function is $P = AL^{\alpha}K^{\beta}\ u$, where $\alpha + \beta = 1$ we know that elasticity of substitution

    = σ = Change in ration of factor's quantity/ % change in price ratio of factor

$$\therefore \qquad \alpha = \frac{\partial(K\ /\ L)\ /\ (K\ /\ L)}{\partial(P_L\ /\ P_K)\ /\ P_L\ /\ P_K} = \frac{\partial(K\ /\ L)\ /\ K\ /\ L}{\partial R\ /\ R}$$

Where K/L = ratio of factor quantity

$R = P_L\ P_K$ = Price ratio of factor

We know that rate of marginal substitute technique = $\dfrac{\partial K}{\partial L}$

$$\therefore \qquad \frac{\partial K}{\partial L} = \frac{MP_L}{MP_K} = \frac{P_L}{P_K} = R$$

In other words

$$R = \frac{\partial P\ /\ \partial L}{\partial P\ /\ \partial K}$$

Our production function $\qquad P = AL^{\alpha}L^{\beta}u$

Differentiating with respect to L and K separately

$$\frac{\partial P}{\partial L} = \beta AL^{\alpha}K^{\beta}u$$

Note   Therefore

$$\frac{\partial P}{\partial L} = \alpha A L^{\alpha-1} K^{\beta-1} u$$

$$R = \frac{\partial P / \partial L}{\partial P / \partial K} = \frac{\alpha A L^{\alpha-1} K^{\beta} u}{\beta A l^{\alpha} K^{\beta} u} = \frac{\alpha}{L} \frac{K}{\beta}$$

$$= \frac{\alpha}{\beta}\left(-\frac{K}{L}\right)$$

$$\partial R = \alpha / \beta \partial (K / L)$$

Therefore,

$$\sigma = \frac{\partial(K / L) / K / L}{\partial(P_L / P_K) / P_L / P_K} = \frac{\partial(K / L) / K / L}{\partial R / R}$$

$$s = \frac{\partial(K / L) / K / L}{\dfrac{\partial(K / L) \cdot \alpha / b}{(K / L) \cdot \alpha / b}} = 1 \qquad\qquad \text{It's proved}$$

4.   For production capital and labour are important requirements – If capital becomes Zero, production will get into Zero

      Production function $= P = AL^{\beta}K^{\beta}u$

                                    $L = 0$

Assume $P = A.B.\ K^{\beta}\ u = 0$

                                      $K = 0$

Assume $P = AL^{\alpha} \cdot O \cdot u = 0$

Thus, for production both the factors are essential.

5.   If Production function $P = AL^{\alpha} K^{\beta} u$ is homogeneous and of a degree, then α and β reflects the position of labour and capital in the production.

      Production function is $P = AL^{\alpha} K^{\beta} u$

      Differentiating with respect to L and K separately

$$\frac{1}{P} = \frac{\partial P}{\partial L} \alpha \frac{1}{L}$$

$$\frac{1}{P} \frac{\partial P}{\partial K} = \beta \cdot \frac{1}{K}$$

$$\therefore \qquad \alpha = \frac{L}{P} \frac{\partial R}{\partial L} = \frac{\text{Labour}}{\text{product}} \times \text{Marginal product of labour}$$

$$= \frac{\text{Wage of labour}}{\text{Product}}$$

And

$$\beta = \frac{K}{P} \frac{\partial P}{\partial K} = \frac{\text{Capital}}{\text{Product}} \times \text{Marginal product of capital}$$

$$= \text{Labour share of total production}$$

$$= \text{Share of capital in production.}$$

6. Production function displays the elasticity of Labour and Capital. By the characteristics of   **Note**
    production function

$$\alpha = \frac{L}{P}\frac{\partial P}{\partial L}$$

$$= \frac{\partial P / P}{\partial L / L} = \text{ elasticity of labour}$$

$$\beta = \frac{K}{P}\frac{\partial P}{\partial K}$$

$$= \frac{\partial P / P}{\partial K / K} = \text{elasticity of capital}$$

The expansion path of Cobb Douglas Production Function is linear homogeneous and its passed through main point.

Cobb Douglas Production Function $P = AL^{\alpha}K^{\beta}u$

Taking log into both sides

$\log P = \log A + \alpha \log L + \beta \log K + \log u$

Differentiating with respect to L and K separately

$$\frac{1}{P}\frac{\partial P}{\partial L} = \frac{\alpha}{L} \qquad \qquad \text{...(A)}$$

$$\frac{1}{P}\frac{\partial P}{\partial K} = \frac{\beta}{K} \qquad \qquad \text{...(B)}$$

Writing both the equation (A) and (B) again

$$MP_L = \frac{\partial P}{\partial L} = P.\frac{\alpha}{L}$$

and

$$MP_K = \frac{\partial P}{\partial L} = P.\frac{\beta}{K}$$

$\therefore$

$$\frac{MP_L}{MP_K} = \frac{P_L}{P_K}$$

$$\frac{P\alpha / L}{\beta P / K} = \frac{P_L}{P_K}$$

Or

$$\frac{\alpha}{\beta}.\frac{K}{L} = \frac{P_L}{P_K}$$

Or

$$\alpha.KP_K = \beta.LP_L$$

Or

$$\alpha.KP_K - \beta LP_L = 0$$

Thus, Production Function is linear homogeneous and its passes through main point.

7.3.2 Economic significance of Cobb Douglas Production Function

Cobb Douglas Production Function has a very importance role in economic area. At present many economists are using Cobb Douglas Production Function in various economic areas. The use of this function is day-by-day is increasing especially in various industries and agriculture. This bring important information for these sectors. This also helps in framing various policies.

With the help of this function, we can also determine the Marginal Productivity and similarly it helps in determining principle of wages. Production function describes production technique. With the help of this function we can also determine whether any factor is paid the value with respect to its equality with the marginal productivity. In a same fashion it helps in agriculture to find the elasticity of economy. By this function we also display elasticity coefficients. These elasticity coefficients help us in comparing the international and internal areas.

As has already be described when function is linear and homogeneous and $\alpha + \beta = 1$, then production would be under the constant result, when $\alpha + \beta > 1$, then increase in production happens, and if $\alpha + \beta < 1$, then decrease in production happens. This way this function helps us in studying the rules of various results. Besides these it also fetches important information related to substitutability of various factors of production.

In short, this function plays an important role especially in agriculture and industries. This is used in determining the labour policies, inter-area comparison, substitutability of factors and degree of homogeneity.

7.3.3 Limitation of Cobb-Douglas Production Functions

Although Cobb Douglas Production Function is used widely in economic areas and its use is increasing in especially in various industries and agriculture, but some economists criticize this production function. Among them are Prof K.J. Arrow, H.B. Chenery, B.S. Minhas and R.M. Salow. Their main criticizes are:

1. The main demerit of this function is this that it considers only two factors of production i.e. Capital and Labour, whereas in reality other factors also have important role in production. In other words, this function does not apply to more than two factors. Besides it can be used only in construction industries. This way its use becomes narrow.

2. This function works under the constant result of formula. Rule of increase and decrease in result also apply to production function. But this function does not work under these rules.

3. Function is based on the assumptions that technical knowledge remains constant and no change in techniques happen in production. But the same can change in production. This way assumption of constant technique is irrelevant.

4. Cobb-Douglas Production Functions assume that all inputs are homogeneous. In reality all units of a factors are not homogeneous. For example some people are skilled and others are not in a labour population.

5. This does not determine any maximum level of production. Prof M. Chand says "Since, this does not ascertain the maximum level of P (Production), it would be practical and convenient not to use this function beyond a certain limit for statistical measurement of its values.

6. α and β of the function reflects the proportion of labour and capital in production. This becomes true only when market has a complete competition. But in case economy has a incomplete competition or monopoly, then above relation can not be obtained.

7. It takes into account only positive marginal productivity of factors and ignores the negative marginal productivity. Whereas marginal productivity of any factor can be zero or negative.

8. Last, the function is unable to produce information related to inter-relation of factors.

7.4 The Constant Elasticity Substitution (C.E.S.) Production Function

In the Cobb-Douglas Production Functions it has already been discussed that elasticity of substitution is always a unit in it. Here we will discuss a function where elasticity of substitution is not required. This is known as Constant Elasticity Substitution (C.E.S.) Production Function. This was devised by two groups of economists. First was K J Arrow, Chenery and B S Minhas and R M Salow, whereas second group consists M Brown, De Cani. Although they devised this function in other forms, but result were same. First group has shown the production function as:

$$P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v-\alpha}$$

$$(\gamma > 0, \ 0 < \delta < 1, \ \alpha > -1)$$

Where P = Production, C = Capital, N = Labour α = substitution parameter; γ = technical efficiency coefficient or efficiency parameter (this is considered as A of Cobb-Douglas Production Functions in C. E.S. function); δ = coefficiency of capital intensity (this is considered as α of Cobb-Douglas Production Functions in C. E.S. function)

$1-\delta$ = Labour Intensity Coefficient

v = Degree of Homogeneity

7.4.1 Properties of C.E.S. Production Function

1. If Production Function is linear homogeneous then substitution parameter α would be equal to constant $\left(\dfrac{1}{1+\alpha}\right)$ whereas production function is $P = \gamma(\delta C^{-\alpha} + (1-\delta)N^{-\alpha})^{-v/\alpha}$ provided $\gamma > 0, \ 0 < \delta$ and $\alpha > -1$

    Rational: According to definition elasticity of substitution

    $$\sigma = \frac{\partial \log(N/C)}{\partial \log R} = \frac{\partial(N/C)/N/C}{\partial R/R}$$

Here, $\dfrac{N}{C}$ = ratio of production factors and $R = \dfrac{P_C}{P_N}$ = Price Ratio

Now production function

$$P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha} \qquad \qquad ...(7.1)$$

Partially differentiating with respect to N

$$\frac{\partial P}{\partial N} = \gamma[-v/\alpha][\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha-1} \times [-\alpha(1-\delta)N^{-\alpha-1}]$$

$$= \frac{\gamma v}{\alpha}[\delta C^{-\alpha} + (1-\alpha)N^{-\alpha}]^{-v/\alpha-1}[\alpha(1-\delta)N^{-(\alpha+1)}] \ ...(7.2)$$

From equation 7.1

$$\left[\frac{P}{\gamma}\right] = [\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$$

Note

Or

$$\left[\frac{P}{\gamma}\right]^{-\alpha/v} = [\delta C^{-\alpha} + (1-\alpha)N^{-\alpha}]$$

Or

$$\left[\frac{P}{\gamma}\right]^{-(\alpha/v)(-v/\alpha-1)} = [\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha-1} \qquad ...(7.3)$$

Putting equation 7.3 into equation 7.2

Or

$$\frac{\partial P}{\partial N} = \gamma v \left[\frac{P}{\gamma}\right]^{1+\alpha/v} = (1-\delta)N^{-(1+\alpha)} \qquad ...(7.4)$$

Differentiating equation 7.1 further with respect to C

$$\frac{\partial P}{\partial C} = \gamma v[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha 1} \delta.C^{-1+\alpha}$$

$$= \gamma v \left[\frac{P}{\gamma}\right]^{1+\alpha/v} \delta C^{-(1+\alpha)} \quad [\text{from equation (7.3)}]$$

We know that

$$MTRS = \frac{\partial P / \partial C}{\delta P / \partial N} = \frac{\partial N}{\partial C} = R$$

$$R = \frac{\gamma v \left[\dfrac{P}{\gamma}\right]^{1+\alpha/v} \delta C^{-(1+\alpha)}}{\gamma v \left[\dfrac{P}{\gamma}\right]^{1+a/v} (1-\delta)N^{-(1+a)}}$$

Or

$$= \frac{d}{1-d}\left(\frac{C}{N}\right)^{(1+\alpha)}$$

Or

$$R = \frac{\delta}{1-\delta}\left(\frac{N}{C}\right)^{1+\alpha}$$

Taking log of both sides

$$\log R = \log\left(\frac{\delta}{1-\delta}\right) + (1+\alpha)\log\left[\frac{N}{C}\right]$$

$$= \log \delta' + (1+\alpha)\log G$$

Here,

$$\delta' = \frac{\delta}{1-\delta} \text{ and } G = \frac{N}{C}$$

Differentiating with respect to G

$$\frac{1}{R}\frac{\partial R}{\partial G} = \frac{1+\alpha}{G}$$

Or

$$\frac{G}{R}\cdot\frac{\partial R}{\partial G} = 1+\alpha$$

Or
$$\frac{\partial G}{G} / \partial R / R = \frac{1}{1+\alpha}$$

Or
$$\frac{\partial (NC) / N / C}{\partial R / R} = \frac{1}{1+\alpha}$$

∴   Elasticity of substitution $= \sigma = \dfrac{1}{1+\alpha}$ that's it

2.   Marginal Product of CES will always be positive, viz more than zero and never is negative.

Rationale: Production function $P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$

Differentiating with respect to N

$$\frac{\partial P}{\partial N} = \gamma^{\alpha/v} \cdot \gamma P^{1+\alpha/v}(1-\delta)N^{-\alpha-1} \text{ [from first characteristics]}$$

Or
$$MP_N = \gamma^{\alpha/v} \cdot v(1-\delta)P^{1+\alpha}N^{-(1+\alpha)}$$

$$= R_1 \frac{P^{1+\alpha/v}}{N^{1+\alpha/v}} \text{ here } R_1 = \gamma^{-\alpha/v}(1-\delta)$$

If production happens under constant result then $v = 1$

∴
$$MP_N = R_1 \frac{P^{1+\alpha}}{N^{1+\alpha}} = R_1 \left[\frac{P}{N}\right]^{1+\alpha}$$

Or
$$MP_N = R_1 \left(\frac{P}{N}\right)^{1/\sigma} \qquad \left[\text{here, } \sigma = \frac{1}{1+\alpha} \right]$$

∴
$$MP_N = \text{Positive or } MP_N > 0$$

$$P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$$

Differentiating with respect to C

$$MP_C = \frac{\partial P}{\partial C} = \gamma^{-\alpha/v} \cdot vp^1 + \alpha\delta C^{-\alpha-1}$$

$$= \gamma^{-\alpha/v} \cdot v\delta P^{-1+\alpha/v} C^{-(1+\alpha)} \text{ [from first characteristics]}$$

Or
$$MP_C = R_2 \frac{P^{1+\alpha/v}}{C^{1+\alpha/v}} \quad \text{ here, } R_2 = \gamma^{-\alpha/w} v\delta$$

If production function is working under constant result then $v = 1$

Therefore
$$MP_C = R_2 \frac{P^{1+\alpha}}{C^{1+\alpha}} = R_2 \left(\frac{P}{C}\right)^{1+\alpha}$$

$$= R_2 \left(\frac{P}{C}\right)^{1/\alpha} \qquad \left[\text{here, } \sigma = \frac{1}{1+\alpha} \right]$$

$$= \text{Positive}$$

Now
$$MP_C > O$$

Note

3. Marginal production of production function is always moves downward viz

$$\frac{\partial^2 P}{\partial M^2} < 0 \text{ and } \frac{\partial^2 P}{\partial C^2} < 0$$

Task What is the merit of C.E.S. Production function?

7.4.2 Advantages of CES Production Function over Cobb-Douglas Production Function

1. Compared to Cobb-Douglas Production Function, CES Production Function brings more general technique. Under CES Production Function, elasticity of substitution is constant and it is not necessary that this elasticity is equal to a unit.

2. Compared to Cobb-Douglas Production Function, CES Production Function keeps more important parameters. This way it has a wide area in substitutability and efficiency.

3. Cobb-Douglas Production Function and CES Production Function have a special form. If in CES function $a = 0$ then we will get Cobb-Douglas Production Function.

4. It is easy to find out the parameters under CES Production Function. Besides this function has removed all the problems and unrealistic assumptions of Cobb-Douglas Production Functions.

7.4.3 Limitation of CES Production Function

Although CES production function has removed all the problems and unrealistic assumptions of Cobb-Douglas Production Function and is widely used in economics, but yet it is criticized:

1. Like Cobb-Douglas Production Function, this function also considers only two factors of production (Labor and Capital). It does not apply to other factors of production. Prof H. Uzawa says it is difficult to apply this function on n^{th} factor of production.

   For example: if $A = \sqrt{ab}$ is production function, find out the demand of factor a and b, where their prices are constant at P_a and P_b. If demand curve $x = \beta - \alpha p$, what is the factor demand in term s of prices and constants.

   Solution: Given production function

   $$x = A\sqrt{ab}$$
   $$= Aa^{1/2}b^{1/2} \qquad \qquad \text{...(i)}$$

Differentiating a and b from equation (i) separately

$$\frac{\partial x}{\partial a} = \frac{1}{2}Aa^{-1/2}b^{1/2}$$

And

$$\frac{\partial x}{\partial b} = \frac{1}{2}Aa^{1/2}b^{-1/2} \qquad \qquad \text{...(ii)}$$

From equation (i) and (ii)

$$MP_a = \frac{1}{2}\frac{Aa^{1/2}b^{1/2}}{a} = \frac{x}{2a} \qquad \qquad \left(x = A\sqrt{ab}\right)$$

$$MP_b = \frac{1}{2}\frac{Aa^{1/2}b^{1/2}}{b} = \frac{x}{2b}$$

But we know that

$$\frac{MP_a}{MP_b} = \frac{P_a}{P_b}$$

$$\frac{x/2a}{x/2b} = \frac{P_a}{P_b}$$

Or

$$\frac{b}{a} = \frac{P_a}{P_b}$$

Or

$$b = \frac{P_a}{P_b} \text{ and } a = b\frac{P_b}{P_a}$$

Putting the value of a and b in equation (i)

$$\frac{x}{A} = a\sqrt{P_a/P_b}$$

Or

$$a = \frac{x}{A} = \sqrt{P_a/P_b} \qquad \qquad \text{...(iii)}$$

Thus

$$b = \frac{x}{A} = \sqrt{P_a/P_b} \qquad \qquad \text{...(iv)}$$

equation (iii) and (iv) displays the factor demand in form of factor price of a and b

Now, demand curve $\qquad \qquad x = \beta - \alpha p$

Since

$$TC = ap_a + a\frac{P_a}{P_b}.pb = 2ap_a$$

$$= 2p_a\frac{x}{A}\sqrt{P_b/P_a} \qquad \qquad \text{[from equation (iii)]}$$

Differentiating with respect to x

$$\frac{\partial(TC)}{\partial x} = \frac{2p_a}{A}\sqrt{p_b/p_a} = \frac{2}{A}\sqrt{(P_aP_b)}$$

$$MC = \frac{2}{A}\sqrt{(P_aP_b)}$$

Since

$$MC = p \qquad \qquad \text{(In case of full competition)}$$

$$x = \beta - \alpha p$$

Further

$$= \beta - \alpha\frac{2}{A}\sqrt{(p_a p_b)}$$

Putting the value of x in equation (i)

$$= \beta - \alpha\frac{2}{A}\sqrt{(p_a p_b)} = Aa^{1/2}b^{1/2} = Aa^{1/2}b^{1/2}\sqrt{P_a/P_b}$$

Or

$$= \frac{1}{A}\left(\beta - \frac{2\alpha}{A}\sqrt{(p_ap_b)}\right)\sqrt{p_a/p_b} = a$$

Note

Or
$$= \frac{1}{A}\left(\beta - \frac{2\alpha}{A}\sqrt{(p_a p_b)}\right)\sqrt{p_a / p_b} = a \qquad \text{...(v)}$$

Thus
$$b = \frac{1}{A}\left(\beta - \frac{2\alpha}{A}\sqrt{(p_a p_b)}\right)\sqrt{p_a / p_b} \qquad \text{...(vi)}$$

Equation (v) and (vi) displays the factor demand of a and b

Example 1: If production is in form of

$$Q = A\, K^\alpha L^\beta$$

then (A) Find out the marginal productivity of Capital (K) and Labour (L).

(B) Prove that there is elasticity of capital and labour in production function

Solution: Given production function

$$Q = AK^\alpha L^\beta \qquad \text{...(i)}$$

(A) Partially differentiating from equation with respect to K and L separately

$$\frac{\partial Q}{\partial K} = A\alpha K^{\alpha-1}L^\beta \qquad \text{...(ii)}$$

$$\frac{\partial Q}{\partial L} = A\beta K^\alpha L^{\beta-1} \qquad \text{...(iii)}$$

Function (ii) and (iii) displays marginal productivity of Capital (K) and Labour (L). writing this in simple form

$$MP_K = \frac{\partial Q}{\partial K} = \frac{\alpha}{K} \qquad (\therefore \quad Q = AK^\alpha L^\beta)$$

And
$$MP_L = \frac{\partial Q}{\partial K} = \frac{\beta}{K} \cdot Q$$

(B)   Production Elasticity of Capital

$$= \frac{K}{Q}\frac{\partial Q}{\partial K} = \frac{K}{Q} \cdot A\alpha K^{\alpha-1}L^\beta$$

$$= \frac{K}{AL^\alpha L^\beta} A\alpha K^{\alpha-1}L^\beta = \alpha\frac{A\alpha K^{\alpha-1}L^\beta}{A\alpha K^{\alpha-1}L^\beta} = \alpha$$

Production Elasticity of Labour

$$= \frac{L}{Q}\frac{\partial Q}{\partial L} = \frac{L}{Q} \cdot A\beta K^\alpha L^{\beta-1}$$

$$= \frac{L}{AK^\alpha L^\beta} A\beta K^\alpha L^{\beta-1}$$

$$= \frac{\beta}{AK^\alpha L^{\beta-1}} AK^\alpha L^{\beta-1} = \beta$$

This way in the given production function α and β shows the production elasticity of capital and labor.

Self Assessment

2. Multiple Choice Quesitons:

6. Who has been credited to devise the Cobb-Douglas Production Function?

    (a) C W Cobb and D H Douglas   (b) Cobb and Marshal

    (c) Douglas and Arastu   (d) Above all

7. Production function is

    (a) $P = A L^\beta K^\alpha u$ (b) $P = A L^\alpha K^\beta u$

    (c) $P = L^\alpha K^\beta$ (d) $P = A K^\beta u$

8. Marginal production of CES is always –

    (a) negative   (b) cubic

    (c) positive   (d) None of them

7.5 Summary

- Maximum use of special production is referred to as Homogeneous function.

- Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted.

- Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means.

- Cobb-Douglas Production Function is used widely in economic area. This production function was developed by C W Cobb and D H Douglas.

- If Production function is homogeneous and of a degree, it means that production will come under constant formula.

- Cobb Douglas Production Function has a very important role in economic area. At present many economists are using Cobb Douglas Production Function in various economic areas.

- Although Cobb Douglas Production Function is used widely in economic areas and its use is increasing in especially in various industries and agriculture, but some economists criticize this production function.

- In the Cobb-Douglas Production Functions it has already been discussed that elasticity of substitution is always a unit in it.

- Compared to Cobb-Douglas Production Function, CES Production Function brings more general technique.

7.6 Keywords

- *Homogeneous:* Undifferentiated, similar

- *Theorem:* practically which can be proved

7.7 Review Questions

1. Define homogeneous function with example.

2. Explain Euler's Theorem with realistic example.

3. Write down the mathematical solution of Euler's Theorem.

Note
4. Explain Cobb-Douglas Production Function.
5. Describe the economic importance of Cobb-Douglas Production Function.
6. Write down the limitation of Cobb-Douglas Production Function.
7. Explain the Constant Elasticity Substitution.

Answers: Self Assessment

1. Homogeneous
2. Euler's
3. Economic
4. Production
5. Homogeneous
6. (a)
7. (b)
8. (c)

7.8 Further Readings

Books
Mathematics for Economics – Council for Economic Education.
Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.
Mathematics for Economist – Carl P Simone, Lawrence Bloom.
Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.
Mathematics for Economist – Yamane, Prentice Hall Publication.
Mathematics for Economics and Finance – Martin Norman.
Mathematics for Economist – Simone and Bloom, Viva Publication.
Mathematics for Economist – Malcom, Nicolas, U C London.
Mathematical Economy – Michael Harrison, Patrick Walderan.

Unit 8: Use of Differentiation in Economics

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Objectives

After reading this unit students will be able to :

- Understand the usage of Differentiation in Economics.
- Explain Marginal Revenue and Elasticity of Demand.

Introduction

The use of differentiation in economics is growing day-by-day. The presented lesson discusses the use of differentiation in economics.

8.1   Use of Differentiation in Economics

The use of differentiation in economics is growing day-by-day. Based on the following points use of differentiation in economics can be cleared-

8.1.1 Elasticity

To calculate elasticity, differentiation is used in economics. If any product $y = f(x)$ is there, then with y, elasticity of x can be found in the following way:

$$E_x = \mathop{Lt}_{\Delta x \to 0} \frac{\Delta y \,/\, y}{\Delta x \,/\, x} = \frac{x}{y} \left(\frac{dy}{dx} \right)$$

Notes   If demand is $q = f(p)$, then elasticity of demand (E_d) can be calculated in the following manner:

$$E_d = \mathop{Lt}_{\Delta p \to 0} \frac{\Delta q / q}{\Delta p / p} = \frac{p}{q} \left(\frac{dy}{dx} \right)$$

Example 1: If demand is *q* = 300 -4.*p*², and price becomes *p* = 2, calculate the elasticity of demand.

Solution: Elasticity of demand $(E_d) = \dfrac{p\,dq}{q\,dp}$, here $q = 300 - 4p^2, \dfrac{dq}{dp} = -8p$

$$\therefore \qquad E_d = \left(\frac{p}{300 - 4p^2}\right) \times (\text{-}8p) = \frac{\text{-}8p^2}{300 - 4p^2}$$

If *p* = 2 then
$$E_d = \left(\frac{\text{-}8 \times 4}{300 - (4 \times 2)}\right) = \frac{\text{-}32}{292} = \frac{\text{-}8}{73}.$$

8.1.2 Marginal Revenue and Elasticity of demand

We know that Total Revenue (*TR*) = Price (*p*) x Quantity (*q*)

Differentiating with respect to *q*   Or *TR* = *p* × *q*

$$MR = \frac{d(TR)}{dq} = \frac{d}{dq}(d \times q)$$

$$MR = \left[p + q\frac{dp}{dq}\right] = p\left[1 + \frac{p}{q}\frac{dp}{dq}\right]$$

Or
$$MR = p\left[1 + \frac{1}{E_d}\right] \qquad\qquad \therefore \left[E_d = \frac{q}{p}\frac{dq}{dp}\right]$$

Or
$$MR = AR\left[1 + \frac{1}{E_d}\right]$$

?

Did u know?   Since elasticity of demand is always negative, therefore, the above relation can be expressed in the following manner

$$MR = AR\left[1 - \frac{1}{E_d}\right]$$

8.1.3 Finding Marginal Cost from Total Cost and Marginal output from Total output

Assume Total Cost (*C*) = *f*(*q*), *q* shows the quantity of total production

Then Marginal Cost (*MC*) = $\dfrac{d}{dq}(C)$

This way, Total Revenue (*R*) = $f(p,q)$

Then Marginal Revenue = $\dfrac{d}{dq}(R)$

Example 2: Assuming Total Cost is C = $15 + 10q - 9q^2 + q^3$, then find out the Marginal Cost (*MC*).

Solution: Total Cost = $15 + 10q - 9q^2 + q^3$

Differentiating with respect to q

$MC = \dfrac{dc}{dq} = 10 - 18q + 3q^2$.

Example 3: Assuming Total Revenue is R = $6q - 9q^2$ then find out the Marginal Revenue (*MR*) and taking total Cost, calculate the total time.

Solution: Total Cost = $R = 6q - 9q^2$,

Differentiating with respect to q

$$MR = \dfrac{dR}{dq} = 6 - 18q$$

In case of equilibrium *MR = MC*, therefore

$6 - 18q = 10 - 18q + 3q^2$

Or $3q^2 = 4$

$q^2 = \dfrac{4}{3}$ or $q = \pm\sqrt{4/3}$.

Self Assessment

1. Fill in the blanks:

  1.   Use of differentiation ………………… is growing day-by-day.

  2.   To calculate elasticity, …………………. is used in economics.

  3.   Revenue = Price x …………….

  4.   Price = ……………….. / Quantity

  5. $MR = \text{........}\left[1 - \dfrac{1}{E_d}\right]$,

8.1.4 Equilibrium in Monopoly: Finding Maximum Profit

With the help of differentiation, find out the equilibrium in Monopoly. We can easily show the maximum benefit in the following manner:

Equilibrium in Monopoly **MR = MC** viz

$\dfrac{d(R)}{dq} = \dfrac{d(C)}{dq}$

Total profit $\pi = R - C$

In case of maximum profit, following two conditions are essential-

$\dfrac{d\pi}{dq} = 0$ ...(i)

And $\dfrac{d^2\pi}{dq^2} < 0$ (negative)   …(ii)

Example 4: If demand is $p = 20 – 5q$ and average cost $AC = q$, then find out the equilibrium of monopoly and maximum profit.

Solution: (i) Given that demand $p = 20 – 5q$

And $\qquad\qquad\qquad\qquad\qquad AC = q$

Total Revenue $\qquad\qquad\qquad = p \times q$

$\qquad\qquad\qquad\qquad\qquad\quad = (20 – 5q)\, q = 20q – 5q^2$

Differentiating with respect to q

$$MR = \frac{d(R)}{dq} = 20 – 10q$$

Total cost $(C) = AC \times q = q \times q = q^2$

Differentiating with respect to q

$$MC = \frac{DC}{dq} = 2q$$

In equilibrium $\qquad\qquad\qquad MR = MC$

$\qquad\qquad\qquad\qquad\qquad 20 – 10q = 2q$

viz $\qquad\qquad\qquad\qquad\qquad\quad 8q = 20$

or $\qquad\qquad\qquad\qquad\qquad\quad q = \dfrac{20}{8} = 2.5$

(i) Total profit $\qquad\qquad\qquad (\pi) = R – C$

$\qquad\qquad\qquad\qquad\qquad\qquad = 20q – 5q^2 – q^2$

$\qquad\qquad\qquad\qquad\qquad\qquad = 20q – 6q^2$

(ii) for maximum profit $\qquad \dfrac{d\pi}{dq} = 0$ and $\dfrac{d^2\pi}{dq^2} < 0$

then $\qquad\qquad\qquad\qquad\quad \dfrac{d\pi}{dq} = 20 – 12q = 0$

or $\qquad\qquad\qquad\qquad\qquad 12q = 20$

or $\qquad\qquad\qquad\qquad\qquad q = \dfrac{20}{12} = \dfrac{5}{3}$

$\qquad\qquad\qquad\qquad\qquad\quad \dfrac{d^2\pi}{dq^2} = –12$

Therefore, at $q = \dfrac{5}{3}$ monopolized profit would be maximum.

Establish relation between $\dfrac{d(R)}{dq}$ and $\dfrac{d(C)}{dq}$.

Example 5: Given than Demand $Q = 23 – 4p + p^2$, where p and q are price and quantity respectively. What would be the elasticity of demand when the value of commodity is (i) ₹ 8 and (ii) ₹ 5.

Solution: Product of Demand given

$$Q = 25 - 4p + p^2 \left[\frac{p}{q} \, \alpha - \frac{dq}{p} \right]$$

Partially differentiating with respect to p

$$\frac{\partial Q}{\partial p} = -4 + 2p$$

We know that

$$\text{Elasticity of Demand} = \frac{\% \text{ change in demanded quality}}{\% \text{ change in price of the commodity}}$$

$$E_d = \frac{\partial Q / Q}{\partial p / p} = \frac{P}{Q} \frac{\partial Q}{\partial p}$$

$$= \frac{P}{Q}(-4 + 2p) = \frac{-4p + 2p^2}{25 - 4p + p^2}$$

$$Ed_{(p=8)} = \frac{-32 + 128}{25 - 32 + 64} = \frac{96}{57} = +1.6$$

$$Ed_{(p=5)} = \frac{-20 + 50}{25 - 20 + 25} = \frac{30}{360} = +1$$

Thus, when price is ₹ 8, elasticity of demand would be 1.6 and when price is ₹ 5, elasticity of demand would become a unit.

Example 6: When cross demand $Q = 150 – 15p$, then find out the elasticity of demand when $p = 4$.

Solution: Product of Demand given $Q = 150 – 15p$,

Differentiating with respect to p

$$\frac{\partial Q}{\partial p} = -15$$

We know that

Elasticity of Demand

$$(E_d) = \frac{P \partial Q}{Q \partial p} = \frac{-15p}{150 - 15p}$$

$$E_{d(p=4)} = \frac{-15 \times 4}{150 - (15 \times 4)} = \frac{-60}{150 - 60} = \frac{-60}{90} = -0.66$$

Question: Write short notes on (I) Cross elasticity of demand.

Answer

Cross elasticity of demand

Demand product shows that demand of any commodity is the product of price of that commodity. But demand of any commodity is also related to price of other related commodity. Cross demand tells that if the price of related commodity changes, in that case demand of that commodity also changes. Assume, here there are two commodities X and Y. Here because of the price of Y commodity, demand of X commodity changes, viz

$$E_c = \frac{\% \text{ Change in quality of X commodity}}{\% \text{ Change in price of Y commodity}}$$

Example 7: If the demand and supply of any fully competitive firm is following

Demand $P = 30 - x$

Supply $C = x^2 + 6x + 7$

Then at which level of production maximum profit can be earned and what would be the corresponding value of Price, Profit and Total Revenue?

Solution: In case of firm earning maximum profit, following condition should essentially be met with

$$\text{MR} = \text{MC viz } \frac{\partial \pi}{\partial x} = 0 \qquad \qquad \text{...(i)}$$

And
$$\frac{\partial^2 \pi}{\partial x^2} < 0 \qquad \qquad \text{...(ii)}$$

Here   Total Profit π = Total Revenue (R) = Total Cost (C)

Total Revenue (R) = Price x quantity of the commodity

$$= P.x = (30 - x) \cdot x = 30x - x^2$$

$$MR = \frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(30x - x^2) = 30 - 2x$$

Thus, $C = x^2 + 6x + 7$

$$MC = \frac{\partial C}{\partial x} = 2x + 6$$

In case of equilibrium $MR = MC$

$$30 - 2x = 2x + 6$$

Or $-4x = -30 + 6 = -24$

Or $x = \frac{24}{4} = 6$

The value of x can be found out in other way

$$\pi = R - c = (30x - x^2) - (x^2 + 6x + 7)$$

$$= 30x - x^2 - x^2 - 6x - 7$$

$$= 24x - 2x^2 - 7$$

Differentiating with respect to x

$$\frac{\partial \pi}{\partial x} = 24 - 4x$$

For maximum profit $\frac{\partial \pi}{\partial x} = 0$ or $24 - 4x = 0$ or $-4x = 24$ or $x = 6$

Second condition for maximum profit $\frac{\partial^2 \pi}{\partial x^2} < 0$

Thus, $\frac{\partial^2 \pi}{\partial x^2} = 4 < 0$, this way the condition is met with

Putting the value of $x = 4$ in Demand

$$p = 30 - x = 30 - 4 = 26$$

Total Revenue $(R) = 30x - x^2$

Putting $x = 6$

$$R = (30 \times 6) - (6 \times 6) = 180 - 36 = 144$$

Thus,

$$\pi = 24x - 2x^2 - 7 \, (x = 6)$$
$$= (24 \times 6) - (2 \times 6 \times 6) - 7$$
$$= 144 - 72 - 7 = 65 \, .$$

Example 8: In case of full competition, if the total cost of any firm is $C = 0.3x^3 - 3x^2 + 20x + 15$, then find out its supply.

We know that for supply

$P \geq AVC$ here P = Price, AVC = Average variable cost

We have that $AVC = \dfrac{TVC}{x} = \dfrac{TC - TFC}{x}$

Here TVC = Total Variable Cost, TC = Total Cost, TFC = Total Fixed Cost and x = quantity of the commodity. Thus

$$AVC = \dfrac{(0.3x^3 - 3x^2 + 20x + 15) - 15}{x}$$

$(\because TFC = 15)$

$$= 0.3x^2 - 3x + 20$$

$$\text{Minimizing } AVC \, \dfrac{\partial(AVC)}{\partial x} = 0 \text{ and } \dfrac{\partial^2(AVC)}{\partial x^2} > 0$$

Thus, $\dfrac{\partial(AVC)}{\partial x} = \dfrac{\partial(0.3x^2 - 3x + 20)}{\partial x} = 0.6x - 3 = 0$

Or $0.6x = 3$ thus, $x = 5$

thus, AVC will be minimum at $x = 5$

To calculate minimum AVC, put the value of $x = 5$

$$AVC = (3 \times 5 \times 5) - (3 \times 5) + 20$$
$$= 7.5 - 15 + 20 = 12.5$$

If, $P < AVC = 12.5,$ production level will be ZERO

If, $P > AVC = 12.5,$ then, supply level would be positive.

To calculate supply

$$MC = \dfrac{\partial C}{\partial x} = 3 \times 0.3x^2 - 6x + 20$$
$$= 0.9x^2 - 6x + 20$$

Note Since in case of equilibrium in full competition

$$P = MC$$

Thus $P = 0.9x^2 - 6x + 20$

Or $0.9x^2 - 6x + 20 - P = 0$

Using binomial equation to know the value of X,

$$x = \frac{6 \pm \sqrt{36 - 4 \times 0.9(20 - P)}}{2 \times .6}$$

$$= \frac{6 \pm \sqrt{36 - 3.6(20 - P)}}{1.2} = \frac{6 \pm \sqrt{36 - 72 + 3.6P}}{1.2}$$

$$= \frac{6 \pm \sqrt{-36 + 3.6P}}{1.2}$$

Thus, $P \geq 12.5$ supply $x = \dfrac{6 \pm \sqrt{3.6P - 36}}{1.2}$

And $P < 12.5$, then supply $x = 0$

Example 9: Demand and Cost of any monopoly are $P = 50 - 6q$, $x = 60 + 14q$ respectively.

What would be equilibrium level of issue price and profit? Prove the second condition of profit maximization.

Solution: Demand and cost is following

$$P = 50 - 6q \hspace{4cm} ...(i)$$
$$x = 60 + 14q \hspace{4cm} ...(ii)$$

Total Revenue

$$(R) = p \times q$$
$$= (50 - 6q)\,q = 50q - 6q^2 \hspace{2cm} ...(iii)$$

Profit

$$\pi = R - C$$
$$= 50q - 6q^2 - 60 - 14q$$
$$= 36q - 6q^2 - 60 \hspace{3cm} ...(iv)$$

For profit maximization

Differentiating equation (iv) with respect to q

$$\frac{d\pi}{dq} = 36 - 12q = 0$$

Or $12q = 36$

Or $q = \dfrac{36}{12} = 3$

Putting the value of q in equation (i)

Price = $p = 50 - (6 \times 3) = 50 - 18 = 32$

Putting the value of q in equation (iv)

Profit = $\pi = 36 \times 3 - 6 \times 3 \times 3 - 60$

$= 108 - 54 - 60$

$= 108 - 114 = -6$

Second condition of profit maximization

$$\frac{d^2\pi}{dq^2} < 0$$

Here, $\quad \frac{d^2\pi}{dq^2} = -12 < 0$

Example 10: Linear product of any monopoly firm is $p = 15 - 0.5q$ and Total cost is $C = 0.5q^2 + 5q + 10$, then find out the optimum value of issue (q), price (p), gross profit (π) and Total Revenue.

(a) **Under profit maximization**

(b) **Sales or revenue maximization**

Solution: (a) we know that

$$p = 15 - 0.5q \qquad \qquad \text{...(i)}$$

$$C = 0.5q^2 + 5q + 10 \qquad \qquad \text{...(ii)}$$

Thus Total Revenue

$$(R) = pq = (15 - 0.5q)\, q$$

$$= 15q - 0.5q^2 \qquad \qquad \text{...(iii)}$$

Total profit

$$(\pi) = R - C$$

$$= 15q - 0.5q^2 - 0.6q^2 - 5q - 10$$

$$= 10q - q^2 - 10 \qquad \qquad \text{...(iv)}$$

For profit maximization

$$\frac{d\pi}{dq} = 10 - 2q = 0$$

Or $\qquad \qquad \qquad 2q = 10$

Or $\qquad \qquad \qquad q = \dfrac{10}{2} = 5$

Putting the value of q in equation (i), (iii) and (iv)

$$p = 15 - (0.5 \times 5) = 15 - 2.5$$

$$= 12.5$$

$$R = 15 \times 5 - (0.5 \times 5 \times 5)$$

$$= 75 - 12.5 = 62.5$$

$$\pi = (10 \times 5) - (5 \times 5) - 10$$

$$= 50 - 35 = 15$$

(B)   Total Revenue $R = 15q - 0.5q^2$

R will become maximum, if

$$\frac{dR}{dq} = 15q - (0.5 \times 2)q = 0$$

Or $15 - q = 0$

Or $q = 15$

Putting the value of q in equation (i), (ii) and (iv)

$$\pi = 15 - (0.5 \times 15)$$
$$= 15 - 7.5 = 7.5$$
$$R = 15 \times 15 - 0.5 \times 15 \times 15$$
$$= 225 - 112.5$$
$$= 112.5$$
$$\pi = (10 \times 15) - (15 \times 15) - 10$$
$$= 150 - 225 - 10$$
$$= -85.$$

Example 11: Linear demand of any monopoly is $p = 12 - 0.4q$ and Demand $C = 0.6q^2 + 4q + 5$, then calculate issue (q), price (p) and gross profit (π) under profit maximization method.

Solution: We know that

$$p = 12 - 0.4q \qquad \qquad ...(i)$$
$$C = 0.6q^2 + 4q + 5 \qquad \qquad ...(ii)$$

Total Revenue

$$R = p \times q$$
$$= (12 - 0.4q) \times q$$
$$= 12q - 0.4q^2 \qquad \qquad ...(iii)$$

Total Profit

$$\pi = R - c$$
$$= (12q - 0.4q^2) - (0.6q^2 + 4q + 5)$$
$$= 12q - 0.4q^2 - 0.6q^2 - 4q - 5$$
$$= 8q - q^2 - 5 \qquad \qquad ...(iv)$$

For profit maximization

$$\frac{d\pi}{dq} = 8 - 2q = 0$$

Or $2q = 8$

$$q = \frac{8}{2} = 4$$

For second condition, $\frac{d^2\pi}{dq^2} = -2 < 0$

Putting the value of q in equation (i) and (iv)

$$p = 12 - (0.4 \times 4)$$
$$= 12 - 1.6 = 10.4$$

$$\pi = (8 \times 4) - (4 \times 4) - 5$$
$$= 32 - 16 - 5$$
$$= 16 - 5 = 11$$

Example 12: Demand of any monopoly is $p = 14 - 6q$ and Total cost is $C = 60 + 20q$ then calculate the optimum level of issue (q), price (p), Total Revenue (R) and profit (π). Tell us the second condition of profit maximization.

If monopoly becomes full competition, then what would be value of price (p), issue (q) and profit?

Solution: given values are

$$P = 140 - 6q \qquad\qquad\qquad ...(i)$$
$$C = 60 + 20q \qquad\qquad\qquad ...(ii)$$

Therefore, Total Revenue

$$R = P \times q$$
$$= (140q - 6q)q = 140q - 6q^2 \qquad\qquad ...(iii)$$

Total Profit

$$\pi = R - C$$
$$= 140q - 6q^2 - 60 - 20q$$
$$= 120q - 6q^2 - 60 \quad ...(iv) \qquad \text{For profit maximization}$$

$$\frac{d\pi}{dq} = 120 - 12q = 0$$

Or
$$12q = 120$$

Or
$$q = \frac{120}{12} = 10$$

Putting the value of q in equation (i), (iii) and (iv)

$$P = 140 - (6 \times 10) = 140 - 60 = 80$$
$$R = (140 \times 10) - (6 \times 10 \times 10)$$
$$= 1400 - 600 = 800$$
$$\pi = (120 \times 10) - (6 \times 10 \times 10) - 60$$
$$= 1200 - 600 - 60$$
$$= 1200 - 660 = 540$$

Second condition of profit maximization

$$\frac{d^2\pi}{dq^2} = -12 < 0$$

If monopoly becomes full competition, then in this condition of equilibrium would be $MC = P$ as in Full Competition $P = MR$

Now $\quad C = 60 + 20q$

Differentiating with respect to q $\quad MC = \dfrac{dC}{dq} = 20$

Note

Therefore, $20 = 140 - 6q$

Or $\quad 6q = 140 - 20 = 120$

$q = 20$ or $p = 20$

And $\quad \pi = -60$

Profit would be positive.

Example 13: Following is the demand and cost function of two separate markets

$$P_1 = 80 - 5q_1, P_2 = 180 - 29q_2 \text{ and } C = 50 + 20(q_1 + q_2)$$

In case of price difference, determine the price, production, marginal revenue and total profit of production of both the two markets.

Solution: We know that

$$P_1 = 80 - 5q_1 \quad\quad\quad …(i)$$
$$P_2 = 180 - 29q_2 \quad\quad\quad …(ii)$$
$$C = 50 + 20 (q_1 + q_2) \quad\quad\quad …(iii)$$

Total revenue of first market

$$R_1 = P_1 q_1 = (80 - 5q_1)\, q_1$$
$$= 80q_1 - 5q_1^2 \quad\quad\quad …(iv)$$

Total revenue of second market

$$R_2 = p_2 q_2 = (180 - 29q_2)\, q_2$$
$$= 180q_2 - 29q_2^2 \qu\quad\quad …(v)$$

By partially differentiating equation (iv) with respect to q_1, equaling it to Zero

$$MR_1 = \frac{\partial R_1}{\partial q_1} = 80 - 10q_2 = 0 \quad\quad\quad …(vi)$$

Or $\quad\quad\quad 10q_1 = 80$ or $q_1 = 8$

Similarly by partially differentiating equation (v), equaling it to Zero

$$MR_2 = \frac{\partial R_2}{\partial q_2} = 180 - 58q_1 = 0 \quad\quad\quad …(vii)$$

Or $\quad\quad\quad 58q_2 = 180$ or $q_2 = \dfrac{180}{58} = \dfrac{90}{29}$

Total Profit $\pi = R_1 + R_2 - C$

$$= 80q_1 - 5q_1^2 + 180q_2 - 29q_2^2 - 50 - 20q_1 - 20q_2$$

$$= 60q_1 - 5q_1^2 + 160q_2 - 29q_2^2 - 50 \quad\quad\quad …(viii)$$

Putting the value of q_1 and q_2 in equations (i), (ii), (vi), (vii) and (viii)

$$P_1 = 80 - 5 \times 8 = 80 - 40 = 40$$

$$P_2 = 180 - 29 \times \frac{90}{29} = 180 - 90 = 90$$

$$MR_1 = 80 - 10 \times 8 = 0$$

$$MR_2 = 180 - 58 \times \frac{90}{29} = 0$$

$$\pi = (60 \times 8) - (5 \times 8 \times 8) + 160 \times \frac{90}{29} - 29 \times \frac{90 \times 90}{29 \times 29} - 50$$

$$= 240 - 320 + \frac{14400}{29} - \frac{8100}{29} - 50$$

$$= -130 + \frac{14400 - 8100}{29}$$

$$= -130 + \frac{6300}{29} = -130 + 217\frac{7}{29}$$

$$= 87\frac{7}{29}$$

Example 14: Demand and Total Cost of a monopoly in two markets are as under

$$P_1 = 2 - q_1$$
$$P_2 = 9 - 6q_2$$
$$C = q_1 + q_2$$

In case of price difference in two markets determine the price, production (sale quantity), marginal revenue and profit of monopoly. Also find out the elasticity of demand for Market A and market B.

Solution: We know that

$$P_1 = 2 - q_1 \qquad \qquad ...(i)$$
$$P_2 = 9 - 6q_2 \qquad \qquad ...(ii)$$
$$C = q_1 + q_2 \qquad \qquad ...(iii)$$

Total Revenue for Market A $\qquad R_1 = P_1 q_1 = 2q_1 - q_1^2 \qquad \qquad ...(iv)$

Total Revenue for Market B $\qquad R_2 = P_2 q_2 = 9q_2 - 6q_2^2 \qquad \qquad ...(v)$

Separately differentiating equation (iv) and (v) separately with respect to q_1 and q_2:

$$MR_1 = \frac{\partial R_1}{\partial q_1} = 2 - 2q_1 \qquad \qquad ...(vi)$$

$$MR_2 = \frac{\partial R_2}{\partial q_2} = 9 - 12q_2 \qquad \qquad ...(vii)$$

Total profit

$$\pi = R_1 + R_2 - C$$

$$= q_1 + q_1^2 + 9q_2 - 6q_2^2 - q_1 - q_2$$

$$= q_1 - q_1^2 = 8q_2 - q_2^2 \qquad \qquad ...(viii)$$

Note   For profit maximization

By separately differentiating equation (viii) with respect to q_1 and q_2 equaling it to Zero

$$\frac{\partial \pi}{\partial q_1} = 1 - 2q_1 = 0 \text{ and } q_1 = \frac{1}{2}$$

$$\frac{\partial \pi}{\partial q_2} = 8 - 12q_2 = 0 \text{ and } q_2 = \frac{2}{3}$$

Putting the value of q_1 and q_2

$$P_1 = 2 - \frac{1}{2} = \frac{3}{2} = 1.5$$

$$P_2 = 9 - 6 \times \frac{2}{3} = 9 - 4 = 5$$

$$MR_2 = 2 - 2 \times \frac{1}{2} = 1$$

$$MR_2 = 9 - 12 \times \frac{2}{3} = 1$$

Here $MR_1 = MR_2$, but $P_1 \neq P_2$

Now
$$\left[\begin{array}{l} MR_1 = P_1\left(1 - \dfrac{1}{e_1}\right), e_1 = A \quad \text{here } e_1 = \text{elasticity of market } A \\[4mm] MR_2 = P_2\left(1 - \dfrac{1}{e_2}\right), e_2 = B \quad \text{here } e_2 = \text{elasticity of market } B \end{array}\right]$$

Putting value of MR_1, MR_2, P_1 and P_2

$$\left[\begin{array}{l} 1 = 1.5\left(1 - \dfrac{1}{e_1}\right) \text{ and } e_1 = 3 \\[4mm] 1 = 5\left(1 - \dfrac{1}{e_2}\right) \text{ and } e_2 = \dfrac{4}{5} \end{array}\right]$$

Thus elasticity of demand in Market A is more than the market, therefore the price of the commodity in Market A is less that market B.

Example 15: Demand and Total Cost of a monopolistic competitive firm are as under:

$$P = 36 - 5q$$
$$C = q^2 + 6q + 5$$

Determine price, production (q) and profit of the firm. Evaluate the monopolistic capacity of this firm? Determine its elasticity of demand.

Solution: We know that

$$P = 36 - 5q \qquad\qquad\qquad \text{...(i)}$$
$$C = q^2 + 6q + 5 \qquad\qquad\qquad \text{...(ii)}$$
$$\therefore \qquad R = Pq = 36q - 5q^2 \qquad\qquad\qquad \text{...(iii)}$$

$$\therefore \qquad MR = \frac{dR}{dq} = 36 - 10q \qquad\qquad\qquad \text{...(iv)}$$

And

$$\pi = R - C$$
$$= 36q - 5q^2 - q^2 - 6q - 5$$
$$= 30q - 6q^2 - 5 \qquad \text{...(v)}$$

For profit maximization

$$\frac{d\pi}{dq} = 30 - 12q = 0$$

$$12q = 30$$

Or

$$q = \frac{30}{12} = \frac{5}{2} = 2.5$$

Putting the value of q in equation (i), (iv) and (v)

$$P = 36 - 5 \times \frac{5}{2} = 36 - \frac{25}{2} = 36 - 12.5 = 23.5$$

$$MR = 36 - 10q$$

$$= 36 - 10 \times \frac{5}{2} = 36 - 25 = 11$$

$$\pi = 30q - 6q^2 - 5$$

$$= 30 \times \frac{5}{2} - 6 \times \frac{5}{2} \times \frac{5}{2} - 5$$

$$= 75 - \frac{75}{2} - 5$$

$$= 37.5 - 5 = 32.5$$

Since here $P > MR$, therefore monopolistic capacity of the firm is $= \dfrac{P - MR}{MR} \times 100$

$$= \frac{23.5 - 11}{11} \times 100$$

$$= \frac{12.5 \times 100}{11} = 113.6\%$$

In this condition, firm has the monopolistic capacity to increase the price by 113.6%

Thus,   elasticity of demand $(e) = \dfrac{P}{P - MR}$

$$= \frac{23.5}{23.5 - 11} = \frac{23.5}{12.5}$$

$$= 1.8.$$

Example 16: If following is the Demand and Cost:

$$P = 100 - 0.5 \ (q_1 + q_2)$$
$$C_1 = 5q_1$$
$$C_2 = 0.5q_2{}^2$$

Note

Then find out the value of q_1, q_2, P, π and π_2.

Find out the second level condition for both firms in case of profit maximization. Whether increase in production of a firm will affect the second firm to less productivity?

Solution: We know that

$$P = 100 - 0.5\,(q_1 + q_2)$$
$$C_1 = 5q_1$$
$$C_2 = 0.5q_2^2$$

Total revenue of first firm

$\therefore$ $$R_1 = Pq_1 = [100 - 0.5(q_1 + q_2)]q_1$$

$$\text{Second} = 100q_1 - 0.5q_1^2 - 0.5q_2$$

And total revenue of second firm

$$R_2 = Pq_2 = [100 - 0.5(q_1 + q_2)]q_2$$

$$= 100q_2 - 0.5q_1q_2 - 0.5q_2^2$$

Profit of first firm

$$\pi_1 = R_1 - C_1$$

$$= 100q_1 - 0.5q_1^2 - 0.5q_1q_2 - 5q_1$$

$$= 95q_1 - 0.5q_1^2 - 0.5q_1q_2$$

Profit of second firm

$$\pi_1 = R_2 - C_2$$

$$= 100q_2 - 0.5q_1q_2 - 0.5q_2^2 - 0.5q_2^2$$

$$= 100q_2 - 0.5q_1q_2 - q_2^2$$

Profit maximization

$$\frac{\partial \pi_1}{\partial q_1} = 95 - q_1 - 0.5q_2 = 0$$

Or $$q_1 + 0.5q_2 = 95 \qquad \qquad \text{...(i)}$$

$$\frac{\partial \pi_2}{\partial q_2} = 100 - 0.5q_2 - 2q_1 = 0$$

Or $$0.5q_1 + 2q_2 = 100 \qquad \qquad \text{...(ii)}$$

Multiplying equation (i) by 4

Or $$4q_1 + 2q_2 = 300 \qquad \qquad \text{...(iii)}$$

Subtracting equation (iii) from equation (iv)

$$3.5q_1 = 280$$

Or $$q_1 = \frac{280}{3.5} = 80$$

Putting the value of q_1 in equation (i)

$$80 + 0.5q_2 = 95$$

Or
$$0.5q_2 = 95 - 80 = 15 \text{ or } q_2 = \frac{15}{0.5} = 7.5$$

Putting the value of q_1 and q_2 in demand and profit

$$P = 100 - (q_1 + q_2)$$
$$= 100 - 0.5 (80 + 7.5)$$
$$= 100 - 0.5 \times 87.5 = 100 - 43.75 = 56.25$$
$$\pi_1 = 95 \times 80 - (0.5 \times 80 \times 80) - (0.5 \times 7.5 \times 80)$$
$$= 7600 - 3200 - 300 = 4100$$
$$\pi_2 = 100 \times 7.5 - 0.5 \times 7.5 \times 80 - 7.5 \times 7.5$$
$$= 75 - 300 - 56.25 = -281.25$$

Second condition

$$\frac{\partial \pi_1}{\partial q_1^2} = -1 < 0$$

$$\frac{\partial^2 \pi_2}{\partial q_2^2} = -2 < 0$$

By reciprocal method (i) and (ii)

$$q_1 = 95 - 0.5q_2$$

and
$$q_2 = \frac{100 - 0.5q_1}{2} = 50 - 0.25q_1$$

Since the slope of these curves is negative, therefore in case of increase in productivity of a firm, productivity of second firm will go down.

Self Assessment

2. State whether the following statements are True or False:

6. With the help of differentiation, level of equilibrium in monopoly is assessed.

7. $\dfrac{d(R)}{dq} \neq \dfrac{d(c)}{dq}$

8. $MR = MC$

9. $AVC = \dfrac{TC - TVC}{x}$

10. $AVC = \dfrac{TC - TFC}{x}$

8.2 Summary

- Differentiation is used in economics to determine elasticity.

- Demand product shows that demand of any commodity is the product of price of that commodity. But demand of any commodity is also related to price of other related commodity. Cross demand tells that if the price of related commodity changes, in that case demand of that commodity also changes.

8.3 Keywords

- *Use:* Application
- *Elasticity:* Resilience as like spring.

8.4 Review Questions

1. Explain the method of determining elasticity.

2. Establish the relation between Marginal Revenue and Elasticity of Demand.

3. How do we determine marginal cost from total cost and marginal revenue from total revenue?

4. If demand is $P = 20 - 5q$ and average cost is $AC = q$, then brief the equilibrium value of monopoly and maximum profit. (**Ans.:** Equilibrium value = 2.5, maximum profit = 5/3).

Answers: Self Assessment

1. Economics	2. Differentiation	3. Quantity	4. Revenue
5. AR	6. True	7. False	8. True
9. False	10. True		

8.5 Further Readings

Books

Mathematics for Economics – Carl P Simone, Lawrence Bloom.

Mathematics for Economist– Yamane, Prentice Hall India.

Mathematics for Economist–Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Council for Economic Education.

Essential Mathematics for Economics- Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist– Simone and Bloom, Viva Publication.

Mathematics for economics – Malcom, Nicolas, U C London.

Mathematical Economics – Michael Harrison, Patrick Walderan.

Unit 9: Maxima and Minima: One Variable

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Objectives

After reading this unit students will be able to :

- Understand the Concept of Maxima and Minima.

- Understand the Definition of Maxima and Minima.

- Know the Condition to Finding Maxima and Minima.

- Determine the Steps of Finding Maxima and Minima of the Function $y = f(x)$.

Introduction

If the height of your house is more than the houses situated in neighborhood (right or left), then the height of your house will be called maximum and contrary to this if the height is less, then it will be called Minimum.

9.1   Concept of Maxima and Minima

One of the main uses of mathematics is to determine the maxima and minima of any function. If any function grows to a certain value of its independent variable and decreases for the maximum value of its independent variable, then from its increasing state arriving at the state of decreasing function receives the maximum value. Similarly when function decreases to some certain point of independent variable and grows towards the next values, then arriving from the state of decreasing to an increasing state, the function obtains minimum value. This clearly describes that at its maximum point, the value of the function is maximum from the value of its immediate and small neighborhood and at its minimum point, the value of the function is minimum from the value of its immediate and small neighborhood.

For example: Assume that for each value of x, which is less than 1, function $y = x^3 - 6x^2 - 2, x$ viz $(x < 1)$, the value of y increases and for $y = 1 < x < 3$, the value of y decreases. Thus, among it, y obtains a maximum value.

Note

Assume $y = f(x)$ is a function. Drawing of the function $y = f(x)$ is given. At this curve four points A, B, C and D are there, at which touching line is parallel to x-axis.

Now the value of $f(x)$ is maximum at A and C and when A and C increases, the value of function stops increasing. Therefore at A and C, the value of function is called Maximum. Similarly at B and D, the value of function stops increasing viz increasing further, $f(x)$ starts increasing. Therefore at B and D, whatever value $f(x)$ obtains, they are called Minimum.
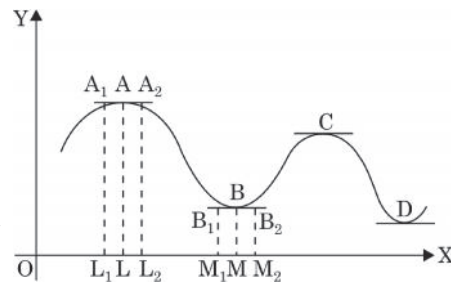
Maxima

Assume that for A point located at the curve $x = a = OL$

∴ $x = a$ $y = f(a) = AL$

Now at the left side of point A, from small neighborhood take any point A_1, for which $x = a - h = OL_1$, here h is small. Therefore,

∴ $x = a - h$ but $y = f(a - h) = A_1L_1 < AL$

Similarly at the left side of point A, from small neighborhood take any point A_2, for which

$$x = a + h = OL_2.$$

Therefore, at $x = a + h$ $y = f(a + h) = A_2L_2 < AL$

Now since A_1L_1 is smaller than AL, viz $A_1L_1 < AL$, therefore $f(a - h) < f(a)$ or $f(a) > f(a - h)$

And since A_2L_2 is smaller than AL, viz $A_2L_2 < AL$, therefore $f(a + h) < f(a)$ or $f(a) > f(a + h)$

Thus, at point A (for which $x = a$) the value of $f(x)$ viz. at each point on right or left of $f(a)$ A, corresponding value of f(x) is greater than $f(a - h)$ or $f(a + h)$.

9.2   Definition of Maxima and Minima

Maxima

Assume $y = f(x)$ is any given function, where $x = a$ is any given point.

Assume at the L.H.S. of point $x=A$, closest point is $x = a - h$, and at the R.H.S. of point $x=a$, closest point is $x = a + h$, where h is minimum number.

At point $x=a$, the value of $f(x) = f(a)$

At point x = $(a - h)$, the value of $f(x) = f(a - h)$

And at point $x = (a + h)$, the value of $f(x) = f(a + h)$

At point $x = a$, the value of function of $f(x)$ is called maximum, if

$$f(a - h) < f(a) > f(a+h)$$

viz at $x = a$, the value of $f(x)$, on the closest point at left and right, will be maximum fro $f(a-h)$ and f(a+h)

At $x = a$, any function $f(x)$ is called maximum where $f(a)$ is greater that all the given values which in the short neighborhood of each value of x, can accept $f(x)$

Minima

Assume that $x = a = OM$ for point B at the curve

Thus, at $x = a$, $y = f(a) = BM$

Now in small neighborhood at LHS of point B, take a point B_1 for which $x = a - h = OM_1$, where h is smallest.

Thus, at $x = a - h$, $y = f(a-h) = B_1M_1 > BM$

Now in small neighborhood at LHS of point B, take a point B_2 for which $x = a + h = OM_2$.

Thus, at $x = a + h$, $y = f(a+h) = B_2M_2 > BM$

Now since B_1M_1 is greater than BM, viz $B_1M_1 > BM$

Thus, $f(a-h) > f(a)$ or $f(a) < f(a-h)$

And since B_2M_2 is greater than BM, viz $B_2M_2 > BM$

Thus, $f(a+h) > f(a)$ or $f(a) < f(a+h)$

Thus at point B for which $x = a$, value of $f(x)$ viz the corresponding value of $f(x)$ at the left or right side of point B, $f(a)$ viz f(a-h) and $f(a+h)$ is smaller

At the point $x = a$, the function $f(x)$ is called minimum if

$$f(a - h) > f(a) < f(a + h)$$

viz At $x=a$, the value $f(a)$ of $f(x)$ is smaller than both the value $f(a-h)$ and $f(a+h)$ in its small neighborhood.

The maximum value of any function does not mean that it is the biggest value and similarly minimum value of does not mean that it's the smallest value. There can be many maximum and minimum value of any function and it is possible that a maximum value is smaller than minimum value. At *A*, maximum value of function or degree is there, it only means that in the small neighborhood of this point, its value is maximum and similarly in the small neighborhood of this point, its value is minimum.

9.3   Conditions for Finding Maxima and Minima

Following are the conditions to find maximum and minimum of function $y = f(x)$ at point $x = a$:

(i)   Necessary condition – the essential condition for both maximum and minimum is as under:

$$f'(x) = 0 \text{ or } \frac{dy}{dx} = 0$$

(ii) Sufficient condition - the sufficient condition for both maximum and minimum is as under:

For maximum

At x = a, the value of $\frac{d^2y}{dx^2}$ = negative value

For minimum

At $x = a$, the value of = $\frac{d^2y}{dx^2}$ positive value

Self Assessment

1. Fill in the blanks:

1.   One of the main uses of mathematics is to determine the maxima and minima of any ………….

2.   There can be many maximum and …………………… value of any function.

Note

3. For ………………….., at $x = a$, the value of $\dfrac{d^2y}{dx^2}$ = negative value.

4. For minimum at $x = a$, the value of $\dfrac{d^2y}{dx^2}$ = is …………….. value.

9.4 Conditions to Absence of Maxima or Minima

At point $x = a$, the value of function $y = f(x)$ will neither be maximum nor minimum if

The value of $\dfrac{d^2y}{dx^2} = 0$ and value of $\dfrac{d^3y}{dx^3} \neq 0$

Properties of Maximum and Minimum value

1. Maximum value comes after minimum value and minimum comes after maximum viz maximum and minimum comes in a sequence.

2. There will be a certain maximum or minimum value between the two equal values of function.

3. At point touching lines are parallel to x-axis where the maximum and minimum of function are there. Therefore, at such points value of $\dfrac{dy}{dx}$ will be 0, solving the equation after putting $\dfrac{dy}{dx} = 0$, value of x can be obtained, over which the value of the function is maximum or minimum.

4. At the maximum or minimum point of function the sign of $\dfrac{dy}{dx}$ changes. At maximum point it becomes negative from positive and contrary to this it becomes positive from negative.

Task Define the minimum.

9.5 Steps for Finding Maxima and Minima of the Function $y = f(x)$

(i) Calculating $\dfrac{dy}{dx}$ of $y = f(x)$

(ii) Determining various values of x from the equation obtain by assigning $\dfrac{dy}{dx} = 0$

(iii) Assume the different values of x are a_1, a_2, a_3 etc.

(iv) Obtaining, d^2y/dx^2 finding the value of $\dfrac{d^2y}{dx^2}$ on a_1, a_2, a_3 etc.

If for any value of x value of $\dfrac{d^2y}{dx^2}$ is positive, then function for that value of x is minimum and if it is negative, then value would be maximum.

(v) If for the value of x, $\dfrac{d^2y}{dx^2} = 0$, then for that value of x determining $\dfrac{d^3y}{dx^3}$. If for the value of x,

$\dfrac{d^3y}{dx^3} \neq 0$, then for that value of x, value of function would neither be maximum nor minimum.

(vi) If for the value of x, $\dfrac{d^3y}{dx^3} = 0$, then for that value of x determining $\dfrac{d^4y}{dx^4}$. If for the value of x,

is negative, $\dfrac{d^4y}{dx^4}$ then for that value of x, value of function would be maximum and if it positive, then value would be minimum. And if it is also zero, then the same process has to be repeated.

EXAMPLES WITH SOLUTION

Example 1: Prove that the maximum value is $\sqrt{2}$ for sin x + cos x .

Solution: Assume y = sin x + cos x

Differentiating both the sides with respect to x $\dfrac{dy}{dx} = \cos x - \sin x$

Further differentiating both the sides with respect to x $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$

For the maximum and minimum value of y, $\dfrac{dy}{dx} = 0$.

Thus,

$\therefore$ cos x – sin x = 0

Or   sin x = cos x

Or   tan x = 1

Or $x = \dfrac{\pi}{4}$

$x = \dfrac{\pi}{4}$ but the value $\dfrac{d^2y}{dx^2} = -\sin \dfrac{\pi}{4} - \cos \dfrac{\pi}{4}$

$= -\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}$

$= \dfrac{-2}{\sqrt{2}} = -\sqrt{2}$ = negative value

Therefore, at , $x = \dfrac{\pi}{4}$ the maximum value of y

the maximum value of y = sin $\dfrac{\pi}{4}$ + cos $\dfrac{\pi}{4}$ = $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}$ = $\sqrt{2}$.

Example 2: Find out the value of x for the maximum and minimum value of function $x^4 - 8x^3 + 22x^2 - 24x$.

Solution: Assume

$$y = x^4 - 8x^3 + 22x^2 - 24x$$

Differentiating both the sides with respect to x

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 \qquad \qquad ...(i)$$

For the maximum and minimum value of the function

$$\frac{dy}{dx} = 0$$

$\therefore$ $\qquad\qquad 4x^3 - 24x^2 + 44x - 24 = 0$

Or $\qquad\qquad 4(x^3 - 6x^2 + 11x - 6) = 0$

Or $\qquad\qquad x^3 - 6x^2 + 11x - 6 = 0$

Or $\qquad\qquad (x - 1)(x - 2)(x - 3) = 0$

$\therefore$ $\qquad\qquad x = 1, 2, 3$

Differentiating both the sides of equation 1 with respect to x

$$\frac{d^2y}{dx^2} = 12x^2 - 48x + 44$$

$x = 1$ but the value $\dfrac{d^2y}{dx^2}$ $= 12(1)^2 - 48(1) + 44$

$\qquad\qquad\qquad\qquad = 12 - 48 + 44$

$\qquad\qquad\qquad\qquad = 8 =$ positive value

Thus, the value of function is minimum at $x = 1$

$x = 2$ but the value $\dfrac{d^2y}{dx^2}$ $= 12(2)^2 - 48(2) + 44$

$\qquad\qquad\qquad\qquad = 48 - 96 + 44$

$\qquad\qquad\qquad\qquad = -4 =$ negative value

Therefore, the value of function is maximum at x = 2

$x = 3$ but the value $\dfrac{d^2y}{dx^2}$ $= 12(3)^2 - 48(3) + 44$

$\qquad\qquad\qquad\qquad = 108 - 144 + 44 = 8 =$ positive value

Therefore, at $x = 3$, the value of function is 0.

Example 3: Find out the maximum and minimum value of $y = x^3 - 2x^2 + x + 6$.

Solution: Assume $y = x^3 - 2x^2 + x + 6$

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

Or
$$\frac{d^2y}{dx^2} = 6x - 4$$

Now the value of $\frac{dy}{dx} = 0$

$3x^2 - 4x + 1 = 0$ Or $(3x - 1)(x - 1) = 0$, then $x = \dfrac{1}{3}$ or 1

At these points, function would be maximum or minimum

Now $x = \dfrac{1}{3}$ but the value $\dfrac{d^2y}{dx^2}$ of $6.\dfrac{1}{3} - 4 = -2 < 0$ (negative value $)$

$\therefore$ Function would be maximum $x = \dfrac{1}{3}$.

Or
$$x = 1 \text{ but } \frac{d^2y}{dx^2} = 6.1 - 4 = 2 > 0 \text{ (positive value })$$

Function would be minimum $x = 1$.

Maximum value $= f\left(\dfrac{1}{3}\right) = \left(\dfrac{1}{3}\right)^3 - 2.\left(\dfrac{1}{3}\right)^2 + \dfrac{1}{3} + 6 = \dfrac{166}{27}$ and

Minimum value $= f(1) = 1^3 - 2.1^2 + 1 + 6 = 6$.

Example 4: At which value of x, the value of function $2x^3 - 9x^2 + 12x - 3$, x is maximum or minimum?

Solution: Assume that $y = 2x^3 - 9x^2 + 12x - 3$

Differentiating the function with respect to x

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) \qquad\qquad ...(1)$$

For the maximum or minimum value of function

$$\frac{dy}{dx} = 0$$

$\therefore$ $6(x^2 - 3x + 2) = 0$ Or $x^2 - 3x + 2 = 0$

Or $(x - 1)(x - 2) = 0$ $\therefore$ $x = 1, 2$

Note   Now differentiating (1)

$$\frac{d^2y}{dx^2} = 6\,(2x - 3)$$

And for $x = 1$ $\frac{d^2y}{dx^2} = 6(2.1 - 3) = -6$ which is negative

Therefore, at $x = 1$ function is maximum. This maximum value

$$= 2.1^3 - 9.1^2 + 12.1 - 3 = 2 - 9 + 12 - 3 = 2$$

For for $x = 2$ $\frac{d^2y}{dx^2} = 6\,(2.2 - 3) = 6$ which is positive

Therefore at $x = 2$ function is maximum, and the minimum value of function is

$$= 2.2^3 - 9.2^2 + 12.2 - 3 = 1.$$

Example 5: For which value of x, the value of function $f(x) = x^5 - 5x^4 + 5x^3 - 1$, is maximum or minimum? Prove that at $x = 0$, the function is neither maximum nor minimum.

Solution: Assume that $y = x^5 - 5x^4 + 5x^3 - 1$

Differentiating both the sides with respect to x

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x^2 - 4x + 3) = 5x^2\,(x - 1)\,(x - 3)$$

For maximum and minimum value of function $\frac{dy}{dx} = 0$

∴ $5x^2\,(x - 1)\,(x - 3) = 0$ ∴ $x = 0, 1, 3$

Now $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$

Replacing $x = 1$

$$\frac{d^2y}{dx^2} = 10\,(2 - 6 + 3) = -10 \text{ negative}$$

Therefore At $x = 1$, function is maximum

Replacing $x = 3$

$$\frac{d^2y}{dx^2} = 10 \times 3\,(2 \times 3^2 - 6 \times 3 + 3) = 30(18 - 18 + 3) = 90 \text{ positive}$$

Therefore At $x = 3$, function is minimum

Replacing $x = 0$

$$\frac{d^2y}{dx^2} = 0, \text{ function is maximum and can not be said negative}$$

Now $\frac{d^3y}{dx^3} = 60x^2 - 120x + 30$

Replacing with $x = 0$

$$\frac{d^3y}{dx^3} = 0 - 0 + 30 = 30 \text{ which is not } 0$$

Therefore   At $x = 0$, function is neither maximum nor minimum

Example 6: Find out the maximum value of function $(x - 1)$ $(x - 2)$ $(x - 3)$.

Solution: Assume that $y = (x - 1)$ $(x - 2)$ $(x - 3) = x^3 - 6x^2 + 11x - 6$

Differentiating both the sides with respect to x

$$\frac{dy}{dx} = 3x^2 - 12x + 11$$

For the maximum or minimum value

$$\frac{dy}{dx} = 0$$

$\therefore$ $3x^2 - 12x + 11 = 0$

Or $x = \dfrac{12 \pm \sqrt{144 - 4 \times 3 \times 11}}{6} = \dfrac{12 \pm 2\sqrt{3}}{6}$

$\Rightarrow$ $x = 2 \pm \dfrac{1}{\sqrt{3}}$

$\therefore$ $x = 2 + \dfrac{1}{\sqrt{3}}$

Or $x = 2 - \dfrac{1}{\sqrt{3}}$

Now $\dfrac{d^2y}{dx^2} = 6x - 12$

$$x = 2 + \frac{1}{\sqrt{3}}$$

$$\frac{d^2y}{dx^2} = 6 \left(2 + \frac{1}{\sqrt{3}} \right) - 12 = 2\sqrt{3} \text{ positive}$$

At $x = 2 + \dfrac{1}{\sqrt{3}}$, function is minimum

And for $x = 2 - \dfrac{1}{\sqrt{3}}$

$$\frac{d^2y}{dx^2} = 6 \left(2 - \frac{1}{\sqrt{3}} \right) - 12 = -2\sqrt{3} \text{ negative}$$

Therefore at $x = 2 - \dfrac{2}{\sqrt{3}}$, function is maximum

Note

Now in the given function replacing $x = 2 - \dfrac{1}{\sqrt{3}}$

Maximum value

$$= \left(2 - \frac{1}{\sqrt{3}} - 1\right)\left(2 - \frac{1}{\sqrt{3}} - 2\right)\left(2 - \frac{1}{\sqrt{3}} - 3\right)$$

$$= \left(1 - \frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\left(-1 - \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}.$$

Example 7: Prove that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $(e)^{1/e}$.

Solution: Assume $\qquad y = \left(\dfrac{1}{x}\right)^x$

Taking logarithm of both sides based on 'e'

$\log y = \log\left(\dfrac{1}{x}\right)^x = x \log \dfrac{1}{x} = x \log x^{-1} = -x \log x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = -\left[x \cdot \frac{1}{x} + 1 . \log x\right]$$

$\therefore \qquad\qquad \dfrac{dy}{dx} = -(1 + \log x)\, y$

For the maximum and minimum value of y

$$\frac{dy}{dx} = 0$$

$\therefore \qquad\qquad -(1 + \log x)\,y = 0 \qquad\qquad$ Or $\qquad \log x = -1$

$\because \qquad\qquad\qquad y \neq 0$

Or $\qquad\qquad -\log x = 1$

Or $\qquad\qquad \log \dfrac{1}{x} = \log e \qquad\qquad \Rightarrow \qquad x = \dfrac{1}{e}$.

Now $\qquad\qquad \dfrac{d^2 y}{dx^2} = -\left[y \cdot \dfrac{1}{x} + (1 + \log x)\dfrac{dy}{dx}\right]$

Replacing with $x = \dfrac{1}{e}$

$$\frac{d^2 y}{dx^2} = -[ey + 0], \qquad \because \qquad \frac{dy}{dx} = 0$$

$\dfrac{d^2y}{dx^2} = -ey$ is negative, therefore for the function $x = \dfrac{1}{e}$ is maximum

Thus, in the referred function replacing with $x = \dfrac{1}{e}$

Maximum value $= \left(\dfrac{1}{\frac{1}{e}}\right)^{1/e} = (e)^{1/e}$ is proven.

Example 8: Find out the maximum value of $\dfrac{\log x}{x}$ **where** $0 < x < \infty$

Solution: Assume that $y = \dfrac{\log x}{x}$

$$\frac{dy}{dx} = \frac{x.(1/x) - (\log x).1}{x^2} = \frac{1 - \log x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2(-1/x) - (1 - \log x).2x}{x^4}$$

$$= \frac{-x - 2x + 2x \log x}{x^4} = \frac{2 \log x - 3}{x^3}$$

Putting , $\dfrac{dy}{dx} = 0$, $\dfrac{1 - \log x}{x^2} = 0$ or $1 - \log x = 0$ or $\log x = 1 = \log e, \therefore x = e$

At $x = e$ $\dfrac{d^2y}{dx^2} = \dfrac{2 \log e - 3}{e^3} = \dfrac{2 - 3}{e^3} = -\dfrac{1}{e^3}$ is negative

Therefore, at $x = +e$, function is maximum and its maximum value is $= \dfrac{\log e}{e} = \dfrac{1}{e}$.

Example 9: If at extremum values of $x = -1$ and $x = 2$ are $y = a \log x + bx^2 + x$ then find out the value of a and b.

Solution: $y = f(x) = a \log x + bx^2 + x$ $\Rightarrow$ $\dfrac{dy}{dx} = a.\dfrac{1}{x} + 2bx + 1$

For extremum $\dfrac{dy}{dx} = 0$,

$$\left(\frac{dy}{dx}\right)_{-1} = 0, \left(\frac{dy}{dx}\right)_{2} = 0$$

$\Rightarrow$ $\quad -a - 2b + 1 = 0$ ...(i)

Or $\quad \left(\dfrac{a}{2}\right) + 4b + 1 = 0$ ...(ii)

Note The value of example (i) and (ii)

$$a = -2, b = -\frac{1}{2}.$$ **Ans.**

Example 10: Find out the minimum or maximum values of function $x + \sin 2x$, $(0 < x < 2\pi)$.

Solution: Assume $\qquad y = x + \sin 2x$

$$\therefore \qquad\qquad \frac{dy}{dx} = 1 + 2 \cos 2x$$

Replacing with $\qquad\qquad \dfrac{dy}{dx} = 0$, $1 + 2\cos 2x = 0$ or $\cos 2x = -\dfrac{1}{2}$

Therefore $\qquad\qquad 2x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ $\qquad\qquad [\because 0 < x < 2\pi] \Rightarrow x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

Now $\qquad\qquad \dfrac{d^2 y}{dx^2} = -4 \sin 2x$

(1) when $x = \dfrac{\pi}{3}$

$$\frac{d^2 y}{dx^2} = -4 \sin \frac{2\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} \quad \text{negative}$$

Therefore at $x = \dfrac{\pi}{3}$ function is maximum

And in the given function replacing with $x = \dfrac{\pi}{3}$, maximum value of function

$$= \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \frac{2\pi + 3\sqrt{3}}{6}.$$

(2) where $x = \dfrac{2\pi}{3}$ then $\dfrac{d^2 y}{dx^2} = -4 \sin \dfrac{4\pi}{3} = -4\left(-\dfrac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ positive

Therefore, at $x = \dfrac{2\pi}{3}$, function is minimum

And in the given function replacing with $x = \dfrac{2\pi}{3}$, minimum value of function

$$= \frac{2\pi}{3} + \sin \frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} = \frac{4\pi - 3\sqrt{3}}{6}.$$ **Ans.**

Self Assessment

2. State whether the following statements are True or False:

5. At $x = a$ for maximum, the value of $\dfrac{d^2y}{dx^2}$ is positive.

6. At $x = a$ for minimum, the value of $\dfrac{d^2y}{dx^2}$ is negative.

7. There will be a certain maximum or minimum value between the two equal values of function.

8. At the maximum or minimum point of function the sign of $\dfrac{dy}{dx}$ changes.

9.6 Summary

● If the height of your house is more than the houses situated in neighbourhood (right or left), then the height of your house will be called maximum and contrary to this if the height is less, then it will be called Minimum.

● Function decreases to some certain point of independent variable and grows towards the next values, then arriving from the state of decreasing to an increasing state, the function obtains minimum value.

● Maximum value comes after minimum value and minimum comes after maximum viz maximum and minimum comes in a sequence.

● If for any value of x value of $\dfrac{d^2y}{dx^2}$ is positive, then function for that value of x is minimum and if it is negative, then value would be maximum.

9.7 Keywords

● *Maximum:* more value

● *Minimum:* less value

9.8 Review Questions

1. Find out the maximum and minimum value of $x^3 - 2x^2 + x + 6$

[**Ans.:** Maximum $\dfrac{166}{27}$, Minimum = 6]

2. Find out the maximum value of function $(x - 1)(x - 2)(x - 3)$.   [**Ans.:** Maximum $\dfrac{2}{3\sqrt{3}}$]

3. Prove that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $(e)^{1/e}$

4. At what values of x, function $2x^3 - 9x^2 + 12x - 3$, x is maximum or minimum

[**Ans.:** Maximum = 2, Minimum = 1]

Answers: Self Assessment

1. Function
2. Minima
3. Maxima
4. Positive
5. False
6. True
7. True
8. False

9.9   Further Readings

Books   Mathematics for Economics – Carl P Simone, Lawrence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Essential Mathematics for Economics- Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist– Yamane, Prentice Hall India.

Mathematics for Economist–Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist– Simone and Bloom, Viva Publication.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics – Malcom, Nicolas, U C London.

Mathematical Economics – Michael Harrison, Patrick Walderan.

Dilfraz Singh, Lovely Professional University

Unit 10: Maxima and Minima: Two Variables and Constrained Maxima and Minima with Lagrange's Multiplier

Note

CONTENTS

Objectives

Introduction

Objectives

After reading this unit, students will be able to :

- Understand the Conditions to finding Maxima and Minima.
- Understand Lagrange's Method.
- Know the Slutsky Equation.
- Understand the Slutsky Equation in Elasticity form.

Introduction

Suppose $U = f(xy)$ is any given function, and (x_0, y_0) are two given points on which Maxima and Minima has to be found and e < n are two positive constants then the function at the maxima (x_0, y_0) will be of the following form–

$f(x_0 - e, y_0 - n) < f(x_0, y_0)$ and $f(x_0, y_0) > f(x_0 + e, y_0 + n)$ and Minima $f(x_0 - e, y_0 - n) > f(x_0, y_0)$ and $f(x_0, y_0) < f(x_0 + e, y_0 + x)$ where $f(x_0, y_0)$ are the maxima and minima values of the function.

10.1   Conditions to finding Maxima and Minima

(A)   Necessary Condition: If $u = f(x,y)$

  Then $\partial u/\partial x = \partial u/\partial y = 0$

(B)   Sufficient Condition:

For maxima, $u = f(x, y)$ if and Necessary Condition $f x = 0$, and $f y = 0$

$$\partial^2 u/\partial x^2 < 0 \text{ and } \partial^2 u/\partial y^2 > 0$$

$\therefore$
$$\underset{\underset{A}{\downarrow}}{\frac{\partial^2 u}{\partial x^2}} \cdot \underset{(B)}{\frac{\partial^2 u}{\partial y^2}} > \underset{C}{\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2} \quad \text{or} \quad AB > C^2 \quad \text{or} \quad \boxed{f_{xx} \cdot f_{yy} > f_{xy}}$$

Note

For Minima,

$$u = f(x, y), fx = 0 \text{ and } f_y = 0$$

Then

$$\partial^2 u/\partial x^2 > 0 \text{ and } \partial^2 u/\partial yz > 0$$

∴ $$\partial^2 u/\partial x^2 . \frac{\partial^2 u}{\partial y^2} < \left(\frac{\partial^2 u}{\partial x \cdot \partial y}\right)^2 \quad \text{or} \quad \boxed{A.B < C^2}$$

or

$$f_{xx} \cdot f_{yy} < fxy$$

Example 1: Find the Maxima and Minima of $u = x^3 + x^2 + xy + y^2 + 4$.

Solution:

$$\frac{\partial u}{\partial x} = 3x^2 + 2x - y = 0 \qquad \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = -x + 2y = 0 \qquad \qquad ...(2)$$

On solving (1) and (2)

$$3x^2 + 2x - y = 0$$

$$-x + 2y = 0$$

If $x = 2y$ then putting it in the first equation

$$3(2y)^2 + 2(2y) - y = 0$$

$$y(12y + 3) = 0$$

When
$$y = 0 \text{ or } \quad y = -\frac{1}{4}$$

When
$$y = 0$$
$$x = 2y = 0$$

When
$$y = -\frac{1}{4}$$

Then
$$x = 2y = -\frac{1}{2}$$

In this way we have two points (0, 0) and (–1/2, –1/4) to find out Necessary condition. Now we will find that whether the above values satisfy the condition of maxima or minima or not.

$$\frac{\partial^2 u}{\partial x^2} = 6x + 2, \cdot \frac{\partial^2 u}{\partial y^2} = 2, \frac{\partial^2 u}{\partial x \partial y} = -1 \qquad \qquad ...(3)$$

For (0, 0) values;

$$\frac{\partial^2 u}{\partial x^2} = 6(0) + 2 = 2 > 0, \frac{\partial^2 u}{\partial y^2} = 2 > 0$$

$$\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} = 2 \cdot 2 = 4 > \left(\frac{\partial^2 u}{\partial x \, \partial y}\right)^2 = (-1)^2 = 1$$

In this way (0, 0) value satisfies the minima condition.

$$u = x^3 + x^2 - xy + y^2 + 4 = 4 \quad (0, 0)$$

will be minima value at (0 , 0) basic point.

For the points (–1/2, –1/4),   **Note**

$$\frac{\partial^2 u}{\partial x^2} = 6\left(-\frac{1}{2}\right) + 2 = -1, \frac{\partial^2 u}{\partial y^2} = 2$$

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \, \partial y}\right)^2 = (-1)2 - (-1) = -3 < 0$$

Point (–1/2, –1/4) gives the Saddle point.

Results can briefly be written as:

(Maximum)	(Minimum)
$f_x = 0, \quad f_y = 0$	$f_x = 0, \quad f_y = 0$
$f_{xx} < 0, \quad f_{yy} < 0$	$f_{xx} > 0, \quad f_{yy} > 0$
$f_{xx}f_{yy} - (f_{xy})^2 > 0$	$f_{xx}f_{yy} - (f_{xy})^2 > 0$
(Saddle Point)	(No Information)
$f_x = 0, \quad f_y = 0$	$f_x = 0, \quad f_y = 0$
$f_{xx}f_{yy} - (f_{xy})^2 < 0$	$f_{xx}f_{yy} - (f_{xy})^2 = 0$

Example 2: Find the maximum value of u with the help of the given function $y = x^3 + y^3 - 3x - 27y + 24$.

Solution: Conditions of first class

$$f_x = 3x^2 - 3 = 0, x^2 - 1 = 0$$
$$f_y = 3y^2 - 27 = 0, y^2 - 9 = 0$$

∴   (1, 3), (1, –3), (–1, 3), (–1, –3)

Conditions of second class

$$f_{xx} = 9x$$
$$f_{yy} = 9y$$
$$f_{xy} = 0$$

In context of (1 , 3)

$$f_{xx} = 9x = 9 > 0$$
$$f_{yy} = 9y = 27 > 0$$
$$f_{xx}f_{yy} - (f_{xy})^2 = 243 - 0 = 243 > 0$$

In this way, u is minima at point (1, 3)

In context of (1, – 3)

$$f_{xx} = 9x = 9 > 0$$
$$f_{yy} = 9y = -27 < 0$$
$$f_{xx}f_{yy} - (f_{xy})^2 = 9(-27) - 0 < 0$$

In this way, saddle point (neither minima nor maxima) will be at (1, –3)

In context of point (–1, 3)

$$f_{xx} = 9x = -9 < 0$$
$$f_{yy} = 9y = 27 > 0$$
$$f_{xx}f_{yy} - (f_{xy})^2 = (-9) (27) - 0 < 0$$

Saddle solution will be get at the point (1, –3)

Note

In context of (–1, –3)

$$f_{xx} = 9x = -9 < 0, \quad f_{yy} = 9y = -27 < 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = (-9)(-27) - 0 > 0$$

Value of *u* will be maxima at point (–1, –3)

Maxima and Minima with Constraint: Lagrange's Multiplier Method: Suppose utility function and income constraint is given as - $u = f(x,y)$ $P_x x + P_y y = M$

Here $u \rightarrow$ utility, $x, y \rightarrow$ things, M $\rightarrow$ Income, $p x$ and $p y \rightarrow$ costs of things

Here the consumer wants to maximize his utility, applying Lagrange's Multiplier on the given income constraint

$$v = f(x, y) + \lambda(M - P_x \cdot X - P_y \cdot y)$$

10.2 Constrained Maxima and Minima with Lagrange's Multiplier

Same results will also be found from Lagrange's method as those found from the following method. On taking gratification function and Budget Line.

$$V = f(q_1 q_2) + \lambda(y - p_1 q_1 - p_2 q_2)$$

Here *V* is the function of *V*, λq_1 and q_2 and λ is a Lagrange's Multiplier. Here our purpose is to maximize *V*. Therefore on partial differentiating *V* with respect to q_1, q_2 and λ and equating it to zero –

$$\frac{\partial V}{\partial q_1} = f_1 - \lambda p_1 = 0 \qquad \qquad ...(4)$$

$$\frac{\partial V}{\partial q_2} = f_2 - \lambda p_2 = 0 \qquad \qquad ...(5)$$

$$\frac{\partial V}{\partial \lambda} = y - p_1 q_1 - p_2 q_2 = 0 \qquad \qquad ...(6)$$

On taking equations (4) and (5)

$$f_1 = \lambda p_1 \text{ and } f_2 = \lambda p_2$$

On dividing both

$$\frac{f_1}{f_2} = \frac{p_1}{p_2} \text{ or } \frac{f_1}{f_2} = \frac{p_1}{p_2}$$

Here same result has been achieved as got from equation *A* in the first method.

On Total differentiating equation

$$f_{11}dq_1 + f_{12}dq_2 - p_1 d\lambda = \lambda dp_1$$

$$f_{21}dq_1 + f_{22}dq_2 - p_2 d\lambda' = dp_2$$

$$-p_1 dq_1 - p_2 dq_2 = -dy + q_1 dp_1 + q_2 dp_2$$

For Second Order Conditions, Bordered Hession determinant of Second Order should be there. Therefore

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} > 0 \qquad \qquad ...(7)$$

On expanding equation (7)

$f_{11} (0.f_{22} - p_2{}^2) - f_{12}(0.f_{21} - p_1 p_2) - p_1\{f_{21}(-p_2) - f_{22}(-p_1)\} > 0$ **Note**

Or $\qquad\qquad\qquad f_{11}p_2{}^2 - f_{12}p_1 p_2 + f_{21}p_1 p_2(-f_{22})(-p_1{}^2) > 0$

Or $\qquad\qquad\qquad f_{11}p_2{}^2 - 2f_{12}p_1 p_2 + f_{22} p_1{}^2 > 0$

Here same result has been achieved as got from equation in the first method.

Notes We can say that same results are received from both methods of Utility Maximization.

Example: Discuss mathematically the theory of consumer's behavior with the help of indifference curve technique.

Solution: Income of a consumer remains constant and a judgmatic consumer wants to attain maximum satisfaction from that income. That point at which consumer gets maximum satisfaction is called the Consumer Balance**.**

Consumer balance is in the case when, Budget line or Price line touches the indifference curve. In other words, Consumer is in the condition of balance when slope of Budget line and the slope of indifference curve are equal. Mathematically if Marginal rate of substitution (slope of indifference curve) and the value ratio of both things (slope of Budget line) are same then consumer remains in the balanced condition.

In the indifference curve technique slope of indifference curve is defined in terms of the ratio of marginal utilities and the slope of Budget line is defined in terms of ratio of prices. That is,

Slope of indifference curve = ratio of marginal utilities

$$= \frac{MU_1}{MU_2}$$

And slope of Budget line = ratio of prices

$$= \frac{p_1}{p_2}$$

Indifference Curve predication clarifies that similar property is received at every point of the curve. That is

$$u = f(q_1 \cdot q_2)$$

On total differentiating and equating to zero

$$du = f_1 dq_1 + f_2 dq_2 = 0$$

(du = 0 means that Marginal satisfaction is always zero) that is there is no difference in the utility.

Or $\qquad\qquad\qquad f_1 dq_1 = -f_2 dq_2$

Or $\qquad\qquad\qquad -\dfrac{dq_2}{dq_1} = -\dfrac{f_1}{f_2}$

$-\dfrac{dq_2}{dq_1}$ represents the slope of indifference curve that is called the Marginal rate of Substitution (MRS). Marginal rate of Substitution is that quantity of thing Q_2 which the consumer remains ready to sacrifice in order to achieve extra unit of Q_1. Negative sign of show that the slope of $-\dfrac{dq_2}{dq_1}$ indifference curve is downwards.

Note

Therefore f_1 and f_2 in the form of Cardinal is first partial differential or Marginal gratification. That is

$$f_1 = \frac{\partial U}{\partial q_1} = MU_1$$

$$f_2 = \frac{\partial U}{\partial q_2} = MU_2$$

And

$$\frac{f_1}{f_2} = \frac{MU_1}{MU_2}$$

Therefore

$$MRSq_1q_2 = \frac{MU_1}{MU_2} = \frac{\text{Marginal Utility of } Q_1}{\text{Marginal Utility of } Q_2}$$

Budget line can be defined by the following equation-

$$y = p_1q_1 + p_2q_2$$

Here y = income, $p_1 = q_1$ –value of thing and $p_2 = q_2$– value of thing.

On differentiating the above equation

$$dy = p_1 dq_1 + p_2 dq_2$$

$dy = 0$ since the income of consumer remains constant, there is no change in it. Therefore

$$p_1 dq_1 + p_2 dq_2 = 0$$

Or

$$p_1 dq_1 = -p_2 dq_2$$

Or

$$-\frac{dq_2}{dq_1} = \frac{p_1}{p_2}$$

Or

$$MRSq_1q_2 = \frac{p_1}{p_2}$$
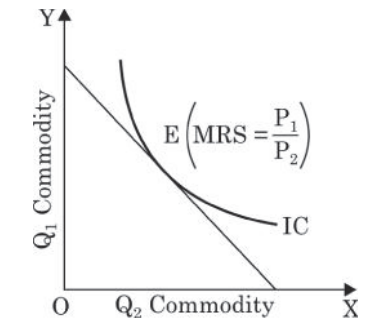
Diagram 10.1: Consumer Balance

That is slope of indifference curve = slope of budget line consumer balance predication can be represented by the following-

In the diagram 10.1 IC is a Indifference curve and Budget line is given that touches each other at point E. Tha is slope of both are equal at point E. $MRSq_1q_2 = \frac{p_1}{p_2}$ will be received at this point where consumer will get maximum satisfaction.

Self Assessment

1. Fill in the blanks:

1. Income of a consumer remains ....
2. That point at which consumer gets maximum satisfaction is called .... Balance.
3. Consumer balance is in the case when, Budget line or Price line touches the .... curve.
4. In the indifference curve technique slope of indifference curve is defined in terms of the ratio of .... utilities.
5. slope of indifference curve = ratio of marginal utilities ....

10.3   Slutsky Equation

If there is a change in price of anything then what will be the changes in the consumption of a consumer? Following effects are produced due to price effect – 1. Income Effect and 2. Substitution Effect.

If there is a decrease in the price of anything then the real income of a consumer increases. And contrast to this if there is an increase in the price of anything then the real income of a consumer decreases. In this way real income of the consumer changes with the price and so the consumption of a consumer will also change. This type of situation is called Income Effect.

If there is a price increase relative of a thing then the consumer uses more the substituted thing. And contrast to that if there is a price decrease relative of a thing then the consumer lessens the use of substituted thing. In this way there is a change in prices and income remains constant, and then more or less quantity of thing is used. This type of effect is called Substitution Effect.

Slutsky Equation is used to study these types of effects -

Utility Equation

$$U = f(q_1 \cdot q_2)$$

And Budget Constraint

$$y = p_1 q_1 + p_2 q_2$$

By using Lagrang's Multiplier undetermined equation –

$$Y = f(q_1, q_2) + \lambda(y - p_1 q_1 - p_2 q_2)$$

On differentiating equation with respect to q_1, q_2 and λ and equating to zero,

$$\frac{\partial V}{\partial q_1} = f_1 - \lambda p_1 = 0$$

$$\frac{\partial V}{\partial q_2} = f_2 - \lambda p_2 = 0 \qquad \qquad ...(8)$$

$$\frac{\partial V}{\partial \lambda} = y - p_1 q_1 - p_2 q_2 = 0$$

Now to calculate the price of consumer, income and substitution effect, differentiating complete equation –

$$f_{11} dq_1 + f_{12} dq_2 - p_1 d\lambda = \lambda dp_1$$
$$f_{21} dq_1 + f_{22} dq_2 - p_2 d\lambda = \lambda dp_2$$
$$-p_1 dp_1 - p_2 dq_2 = -dy + q_1 dp_1 + q_2 dp_2 \qquad \qquad ...(9)$$

We use Cramer's rule to solve this equation. According to this rule if $AB = C$ where A, B and C are matrix then-

$$B = A^{-1}C \text{ Where, } A^{-1} = \frac{adjA}{|A|}$$

Presenting equation (9) in the matrix form

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & p_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1 dp_1 + q_2 dp_2 \end{bmatrix} \qquad \qquad ...(10)$$

Note   Using the cramer's rule

$$\begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1 dp_1 + q_2 dp_2 \end{bmatrix}$$

Here,

$$\left[A^{-1} = \frac{[D_{ij}]}{|D|} \right]$$

Therefore

$$dq_1 = \frac{D_{11}\lambda dp_1 + D_{21}\lambda dp_2 + D_{31}(-dy + q_1 dp_1 + q_1 dp_2)}{|D|} \qquad ...(11)$$

$$dq_2 = \frac{D_{12}\lambda dp_1 + D_{22}\lambda dp_2 + D_{32}(-dy + q_1 dp_2 + q_2 dp_2)}{|D|} \qquad ...(12)$$

and

$$d\lambda = \frac{D_{13}\lambda dp_1 + D_{23}\lambda dp_2 + D_{33}(-dy + q_1 dp_1 + q_2 dp_2)}{|D|} \qquad ...(13)$$

dividing both sides of equation (11) by dp_1 and on considering p_2 and y constant $dy =$ and $dp_2 = 0$

$$\frac{\partial q_1}{\partial p_1} = \frac{D_{11}\lambda}{|D|} + q_1 \frac{D_{31}}{|D|} \qquad ...(14)$$

Here, $\frac{\partial q_1}{\partial p_1}$ = price effect since there is a change in q_1 because of the change in p_2 as p_2 and y remains constant.

In the same way

$$\frac{\partial q_1}{\partial p_2} = \frac{D_2\lambda}{|D|} + p_2 \frac{D_{32}}{|D|} \qquad ...(15)$$

dividing both sides of equation (11) by dy and considering p_1 and p_2 constant (that is)

Therefore

$$\frac{\partial q_1}{\partial y} = -\frac{D_{33}}{|D|} \qquad ...(16)$$

Here $\frac{\partial q_1}{\partial p_1}$, shows income effect since there is a change in q_1 because of the change in y as p_1 and p_2

remains constant. When there is any change in the price of anything then the satisfaction level of the consumer also changes. Therefore consumer balance also shifts on the other indifference curve. Suppose price change is compensated by income change and consumer remains on the same indifference curve, that is there is no change in his satisfaction, means $dU = 0$, therefore

$$U = f(q_1, q_2)$$

That means

$$dU = f_1 dq_1 + f_2 dq_2 = 0$$

But we know that $\frac{f_1}{f_2} = \frac{p_1}{p_2}$, therefore

$$p_1 dq_1 + p_2 dq_2 = 0$$

Putting this value in the third part of equation (9)

$$-dy + q_1 dp_1 + q_2 dp_2 = 0$$

Therefore from equation (11)   **Note**

$$\left(\frac{\partial q_1}{\partial p_1}\right) \text{ (Constant)} = \frac{D_{11}\lambda}{|D|}$$

Rewriting equation (14)

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial q_1}\right) U \text{ Constant } q_1 \left(\frac{\partial q_1}{\partial y}\right) \text{ price} = \text{constant} \qquad \dots (17)$$

Price effect   Substitution effect   income effect

Equation (17) is called Slutsky Equation.

> **Task**   Write the Utility Equation

10.4   Slutsky Equation in Elasticity Form

We know that demand of elasticity,

$$EP = \frac{p}{q} \cdot \frac{\Delta q}{\Delta p} = \frac{p}{q} \cdot \frac{dq}{dp}$$

Slutsky equation

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial p_1}\right) U = \text{Const.} - q_1 \left(\frac{\partial q_1}{\partial y}\right) p = \text{Constant}$$

Multiplying both sides by p_1/q_1

$$\frac{p_1}{q_2} \frac{\partial q_1}{\partial p_1} = \frac{p_1}{q_2}\left(\frac{\partial q_1}{\partial p_1}\right) U = \text{Const.} - \frac{q_1 p_1}{q_1}\left(\frac{\partial q_1}{\partial y}\right) p = \text{Constant}$$

Or $$\eta_{1p} = \eta_1 - \frac{p_1 q_1}{y} \eta_{1y}$$

Here η_{1p} = price Elasticity, η_{1s} is substitution elasticity and η_{1y} is income elasticity.

Generalization: Predication of the above two things can be expanded for *n* things. Suppose *n* consumer consumes things. In this case his utility function will be-

$$U = f(q_1, q_2 ..., q_n)$$

And budget line will be

$$y = p_1 q_1 + p_2 q_2 + ... + p_n q_n$$

Or $$= y - \sum_{i=1}^{n} p_i q_i = 0$$

Where $i = 1, 2 ..., n$

Using the Lagrange's Multiplier overall Utility function will be –

$$V = f(q_1, q_2, ..., q_n) + \left(y - \sum_{i=1}^{n} p_i q_i \right)$$

Note   For maximum satisfaction first partial differential should be zero means-

$$\frac{\partial V}{\partial q_i} = f_i - \lambda p_i = 0 \text{ same way } \frac{\partial V}{\partial p_j} = f_i - \lambda p_j = 0$$

Or $f_i = \lambda p_i$ and $f_j = \lambda p_j$

Thus $\dfrac{f_i}{f_j} = \dfrac{p_i}{p_j}$

?

Did u know?   Substitution rate of j thing for I goods is goods to their price ratio.

Bordered Hesison Determinant of Second order differentiation of maximum utilization should be negative.

Means

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} & -p_1 \\ f_{21} & f_{22} & \cdots & f_{2n} & -p_2 \\ \vdots & \vdots & & \vdots & \vdots \\ fn_1 & fn_2 & \cdots & fnn & -p_n \\ -p_1 & -p_2 & \cdots & -p_n & 0 \end{vmatrix} > 0$$

Slutsky equation in generalization will be

$$\frac{\partial q_i}{\partial p_i} = \left(\frac{\partial q_i}{\partial p_i}\right) u = \text{const.} - q\left(\frac{\partial q_i}{\partial y}\right) p = \text{const.}$$

In Elasticity form – Multiplying Slutsky equation both sides by

$$\frac{p_i}{q_i}\left(\frac{\partial q_i}{\partial p_i}\right) = \frac{p_i}{q_i}\left(\frac{\partial q_i}{\partial p_i}\right)_{u=\text{const.}} - q_i\frac{p_i}{q_i}\left(\frac{\partial q_i}{\partial y}\right)_{p=\text{const.}}$$

Note $\eta_{ip} = \eta_{is} - \dfrac{p_i q_i}{y}\eta_{1y}$

Here η_{ip} = Price elasticity

                               η_{is} = Substitution elasticity

                               η_{iy} = Common elasticity

Substitute Goods and Complementary Goods: If two things are substituted then substitution effect will be negative. In contrast, If two things are complementary then substitution effect will be positive. That is,

$$\left(\frac{\partial q_2}{\partial p_1}\right) U = \text{const.} > 1, \text{ Substituted goods}$$

$$\left(\frac{\partial q_2}{\partial p_1}\right) U = \text{const.} < 1, \text{ Complementary goods}$$

$$\left(\frac{\partial q_2}{\partial p_1}\right) U = \text{const.} = 0, \text{ Independent goods}$$

Normal Goods, Inferior Goods and Giffen Goods

We know that

Price effect = Substitution effect + Income effect

Income effect is negative for general things, that is

$$\left(\frac{\partial q_1}{\partial y}\right)_{\text{price = constant}} < 0$$

Because of the decrease in the prices negative income effect strengthens negative substitution effect. In contrast, Real income of consumer decreases with increase in price and substitution effect changes effectively to lessen the quantity q_1. That is,

$$\left(\frac{\partial q_1}{\partial p_1}\right) < 0, \text{ for general goods}$$

In contrast if income effect is negative income effect does not strengthens substitution effect then things will be cheap.

Those things whose income effect is more than substitution effect, that is, price of things drops with demand are called Giffen Goods.

Giffen goods and Inferior goods, in both situations income effect is negative, so if both things are equal, answer is no, since for Inferior goods substitution effect is more than income effect. That means inferior things follow the demand rule, in contrast, Giffen Goods do not follow demand rule.

Example 1: Is the $u = x^a\, y^b$ gratified function of two things where x and y are the quantities of these things. M is the quantity spend on things x and y then prove that demand of things –

$$x = \frac{aM}{(a+b)p_x} \text{ and } y = \frac{bM}{(a+b)p_y}$$

Where P_x and P_{yx} and y are the prices of things.

Solution: Given gratifed function is

$$U = x^a y^b$$

Separately partial differentiating with respect to 'x' and 'y'

$$MU_x = \frac{\partial U}{\partial x} = ax^{a-1}y^{b-1} \quad MU_y = \frac{\partial U}{\partial y} = bx^a y^{b-1}$$

In the condition of consumer's maximum satisfaction –

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

Or

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y} \qquad \qquad \text{...(i)}$$

And

$$x \cdot p_x + y \cdot p_y = M \qquad \qquad \text{...(ii)}$$

Putting the value of MU_x and MU_y in equation (i)

$$\frac{ax^{a-1}b}{bx^a y^{b-1}} = \frac{p_x}{p_y} \quad \text{or} \quad \frac{a}{b} \cdot \frac{y}{x} = \frac{p_x}{p_y}$$

Or

$$y = \frac{p_x}{p_y} \frac{b}{a} \cdot x$$

Note   Putting the value of y in equation (ii)

$$M = x \cdot px + \frac{p_x \, b}{p_y \, a} x \cdot Py \quad \text{or } aM = (a + b) \, p_x \cdot x$$

Or

$$x = \frac{aM}{(a + b)px} \text{ and } y = \frac{bM}{(a + b) \, py}$$

Example 2: If Utility function $u = xy$ and Budget constraint is $2x + y = 6$ then find the value of x, y and Utility.

Solution: Using the Langrange's Multiplier

$$z = xy + \lambda(2x + y - 6)$$

$$\frac{\partial z}{\partial x} = y + 2\lambda = 0$$

$$\frac{\partial z}{\partial y} = x + \lambda = 0$$

$$\frac{\partial z}{\partial \lambda} = 2x + y - 6 = 0$$

Solving all for x, y and λ, $x = 2/3$, $y = 4/3$ and $\lambda = -2/3$

Then, Maximum Utility $u = xy = \dfrac{2}{3}\left(\dfrac{4}{3}\right) = \dfrac{8}{9}$

It represents Equi Marginal Utility rule.

Self Assessment

2. Multiple Choice Question:

6. If there is a change in the price of anything then the change in the consumption of a consumer tells whose effect?

    (a) Price   (b) Consumption   (c) Consumer   (d) Thing

7. If there is a decrease in the price of anything then what will be the change in the real income of a consumer?

    (a) Decrease   (b) Increase   (c) equal   (d) None of these

8. What will be the change in the real income of a consumer if the price of goods increases?

    (a) Same   (b) Increase   (c) Decrease   (d) None of these

9. $E_p = \text{........} \dfrac{\Delta q}{\Delta p}$.

    (a) $\dfrac{q}{p}$ (b) $\dfrac{d}{q}$ (c) $\dfrac{q}{d}$ (d) $\dfrac{p}{q}$

10. Income Effect = Substitution Effect + …….

    (a) Income effect (b) Price effect   (c) Consumer Effect   (d) All

10.5 Summary

- $$\underset{A}{\frac{\partial^2 u}{\partial x^2}} \cdot \underset{(B)}{\frac{\partial^2 u}{\partial y^2}} < \left(\underset{C}{\frac{\partial^2 u}{\partial x \partial y}} \right)^2 \quad \text{or} \quad AB < C^2 \quad \text{or} \quad f_{xx} \cdot f_{yy} < f_{xy}$$

  For Minima,

  $$u = f(x, y), fx = 0 \quad \text{or} \quad f_y = 0$$

  Then,

  $$\partial^2 u / \partial x^2 > 0 \quad \text{and} \quad \partial^2 u / \partial yz > 0$$

  $$\therefore \qquad \partial^2 u / \partial x^2 \frac{\partial^2 u}{\partial y^2} < \left(\frac{\partial^2 u}{\partial x \cdot \partial y} \right)^2 \quad \text{or} \quad A.B < C^2$$

  or $\qquad\qquad f_{xx} \cdot f_{yy} < fxy$

- Consumer balance is in the case when, Budget line or Price line touches the indifference curve. In other words, Consumer is in the condition of balance when slope of Budget line and the slope of indifference curve are equal.

- In the indifference curve technique slope of indifference curve is defined in terms of the ratio of marginal utilities and the slope of Budget line is defined in terms of ratio of prices.

- Indifference Curve analysis clarifies that similar properties is received at every point of the curve.

- If there is a change in price of anything then what will be the changes in the consumption of a consumer? This is calculated by Income Effect.

- If there is a decrease in the price of anything then the real income of a consumer increases.

- If there is a price increase relative of a thing then the consumer uses more the substituted thing.

- When there is any change in the price of anything then the satisfaction level of the consumer also changes. Therefore consumer balance also shifts on the other indifference curve.

- If two things are substituted then substitution effect will be positive.

- Because of the decrease in the prices negative income effect strengthens negative substitution effect. In contrast, Real income of consumer decreases with increase in price and substitution effect changes.

- Those things whose income effect is more than substitution effect, that is, price of things drops with demand are called Giffen Goods.

- Giffen Goods and Inferior goods, in both situations income effect is negative.

10.6 Keywords

- *Complementary:* Supplementary
- *Income:* Earnings

10.7 Review Questions

1. Write Cramer's rule.
2. Explain Slutsky Equation.

3. What is Lagrange's Method?

4. Prove for minima

    $$f_{xx} \cdot f_{yy} < f_{xy}.$$

5. Find the maxima and minima values of $u = x^3 + x^2 - xy + y^2 + 4$.

Answers: Self Assessment

1. Constant

2. Consumer

3. Indifferent

4. Marginal

5. $\dfrac{MU_1}{MU_2}$

6. (a)

7. (b)

8. (c)

9. (d)

10. (a)

10.8 Further Readings

Books

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Simon and Bloom – Viva Publlications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Unit 11: Constrained Maxima and Minima

CONTENTS

Objectives

Introduction

Objectives

After reading this unit, students will be able to :

- Understand Profit Maximization.
- Understand Compensated Demand Function.

Introduction

Suppose a Firm purchase two products Q_1 and Q_2 of quantity q_1 and q_2 at the rates p_1 and p_2, then in this condition total proceeds will be

$$R = p_1 q_1 = p_2 q_3 \qquad \qquad ...(1)$$

Suppose a firm manufactures two outputs θ_1 and θ_2 from an input. In this condition the cost of two products (x) of firm in the context X will be the function of q_1 and q_2, i.e.,

$$x = h\,(q_1,\, q_2) \qquad \qquad ...(2)$$

In this condition to maximize the proceeds of a farm, on taking total proceeds and functions

$$V = p_1 q_1 = p_2 q_2 + \mu\,[x - h\,(q_1 q_2)\,\sigma] \qquad \qquad ...(3)$$

Here μ is a Lagrange Coefficient. By partially differentiating eq. (1) and putting it equal to zero

$$\frac{\partial V}{\partial q_1} = p_1 - \mu h_1 = 0$$

$$\frac{\partial V}{\partial q_2} = p_2 - \mu h_2 = 0 \qquad \qquad ...(4)$$

$$\frac{\partial V}{\partial \mu} = x - h\,(q_1, q_2) = 0$$

On taking equations (1) and (2)

$$\frac{p_1}{p_2} \;=\; \frac{h_1}{h_2} \;=\; RPT$$

(here, $RPT = -\,\dfrac{dq_2}{dq_1}$ The conversion rate of production)

Note

Therefore
$$\frac{p_1}{p_2} = \frac{\partial q_2 / \partial x}{\partial q_1 / \partial x} = RPT \qquad \qquad ...(5)$$

Equation (5) shows that the conversion rate of production is equal to the price ratio which tells that the proceeds line touches the product conversion curve.

First order conditions can be represented as –

$$\mu = \frac{p_1}{h_2} = \frac{p_2}{h_2}$$

Or
$$\mu = p\,\frac{\partial q_1}{\partial x} = p\,\frac{\partial q_2}{\partial x}$$

Since RPT in the Marginal Product can be represented in the following way

$$\frac{\partial q_1}{\partial x} = \frac{1}{h_1},\ \frac{\partial q_2}{\partial x} = \frac{1}{h_2}$$

> *Notes*   For RPT second order derivative, the condition is that Bordered HessionDeterminant for maximum profit should be positive.

Therefore

$$\begin{vmatrix} -\mu\,h_{11} & -\mu\,h_{12} & -h_1 \\ -\mu\,h_{21} & -\mu\,h_{22} & -h_2 \\ -h_1 & -h_2 & 0 \end{vmatrix} > 0 \qquad \qquad ...(6)$$

On expanding equation (6)

$$\mu\,(h_{11}\,h_2{}^2 - 2h_{12}\,h_1\,h_2 + h_{22}h_1{}^2) > 0 \qquad \qquad ...(7)$$

Or
$$h_{11}\,h_2{}^2 - 2h_{12}\,h_1 h_2 + h_{22}h_1{}^2 > 0 \qquad \qquad ... (8)\ (\text{Since } \mu > 0)$$

This way Firm satisfy equations (5) and (6) then firm will definitely be able to maximize its proceeds.

11.1   Profit Maximization

Profit θ_1 and θ_2 are the function of production quantity q_1 and q_2-

$$\pi = p_1 q_1 + p_2 q_2 - rh\,(q_1 q_2) \qquad \qquad ...(8)$$

By Partial derivative of equation (8) and placing them equal to zero

$$\frac{\partial \pi}{\partial q_1} = p_1 - rh_1 = 0 \qquad \qquad ...(i)$$

$$\frac{\partial \pi}{\partial q_2} = p_2 - rh_2 = 0 \qquad \qquad ...(ii)$$

On taking equations (1) and (2) we get –

$$r = \frac{p_1}{h_1} = \frac{p_2}{h_2}$$

Or
$$r = p_1\,\frac{\partial q_1}{\partial x} = p_2\,\frac{\partial q_2}{\partial x} \qquad \qquad ...(9)$$

According to Second-order conditions

$$\begin{vmatrix} -rh_{11} & -rh_{12} \\ -rh_{21} & -rh_{22} \end{vmatrix}$$

(here, $-rh_{11} < 0$)

On expanding Determinant

$$r^2 (h_{11}h_{12} - h_{12}{}^2) > 0$$

Since $r > 0$, second order conditions can be represented as –

$$h_{11} > 0 \; h_{11} \, h_{22} - h_{12}{}^2 > 0 \qquad\qquad ...(10)$$

Both shows that $h_{22} > 0$. In this condition the Marginal costs of every product increases. Conditions of equation (10) tells that Product Possibility curve roaches back from that point towards basic point where first order conditions of equation (9) gets satisfied. Firm will be in the condition of maximum profit.

Task If profits are θ_1 and θ_2 and the quantity of production are θ_1 and θ_2 then what is its function.

11.2 Compensated Demand Function

Compensated demand shows that quantity of thing that a customer purchase within the following fixed conditions (like Tax, Financial help).

If $\qquad\qquad U = q_1 \, q_2$ then expression

$$Z = p_1 \, q_1 + p_2 q_2 + \lambda \, (\overline{u}) - q_1 \, q_2$$

On partially differentiating and putting it equal to zero,

$$\frac{\partial Z}{\partial q_1} = p_1 - \lambda \, q_2 = 0$$

$$\frac{\partial Z}{\partial q_2} = p_2 - \lambda \, q_1 = 0$$

$$\frac{\partial Z}{\partial \lambda} = \overline{U} - q_1 \, q_2 = 0$$

On solving for q_1 and q_2 following will be the compensated Demand –

$$q_1 = \sqrt{\left(\frac{\overline{U}.p_2}{p_1}\right)}, q_2 = \sqrt{\left(\frac{\overline{U}.p_1}{p_2}\right)}$$

Example 1: Following are the Demand and Supply function

$$q = \frac{5}{2} - \frac{1}{2}p \text{ and } q = 2p - 3$$

Find the balanced price and quantity.

Solution: In the balanced condition the demand and supply of thing will be same, i.e.

Note

$$\frac{5}{2} - \frac{1}{2}p = 2p - 3 \text{ or } \frac{5}{2} + 3 = 2p + \frac{1}{2}p$$

Or

$$\frac{5}{2} + 3 = \frac{5}{2}p \text{ or } \frac{5}{2}p = \frac{11}{2}$$

Or

$$p = \frac{11}{5} = 2.2$$

Again $\quad q = 2p - 3$

Putting the value of P

$$q = \frac{22}{5} - 3 = \frac{22 - 15}{5} = \frac{7}{5} = 1.4.$$

Example 2: If the utility function and budget constraint for two things are $U = 3x^2 y^2$ and $2x + 3y = 18$ then find the quantities sought of things X and Y.

Solution: Given Utility function is

$$U = 3x^2y^2$$

And Budget Constraint is

$$2x + 3y = 18$$

Or $\quad 2x + 3y - 18 = 0$

Using the Lagrange coefficient, Demand function will be –

$$V = 3x^2 y^2 + \lambda (2x + 3y - 18) \qquad \text{...(i)}$$

Separate-Separate partial differentiation of utility function will be done in terms of $'x'$ $'y'$ and $'\lambda'$. After that to maximize utility on putting first derivative equal to zero –

$$\frac{\partial V}{\partial x} = 6x y^2 + 2\lambda = 0 \qquad \text{...(ii)}$$

$$\frac{\partial V}{\partial x} = 6x^2 y + 3\lambda = 0 \qquad \text{...(iii)}$$

$$\frac{\partial V}{\partial x} = 2x + 3y - 18 = 0 \qquad \text{...(iv)}$$

On taking equations (ii) and (iii)

$$\frac{-2\lambda}{-3\lambda} = \frac{6xy^2}{6x^2y} \text{ or } \frac{2}{3} = \frac{y}{x} \text{ or } 2x = 3y$$

Or $\quad x = (3/2) y$

On putting the value x of in equation (iv)

$$2 \times (3/2) y + 3y - 18 = 0$$

Or $\quad 6 y = 18 \text{ or } y = 18/6 = 3$

Again putting the value of y in equation (4)

$$2x + (3 \times 3) - 18 = 0$$

Or $\quad 2x = 9 \text{ or } x = 9/2 = 4.5$

Therefore the quantities sought of things x and y will be 4.5 and 3.

Example 3: Find the Firm's expansion paths expressed in terms of its total expenditure on its inputs in the given production function is $P = 12 \log L + 30 \log K$ and the input prices are $P_L = 2$ and $P_K = 5$.

Solution: Here main purpose of Manufacturer is to maximize his production (P) whereas his conditions are following –

$$TC = 2L + 5K \qquad \qquad ...(1)$$

By using the Lagrange coefficient

$$Z = 12 \log L + 30 \log K + \gamma (TC - 2L - 5K) \qquad \qquad ...(2)$$

By Separate-Separate partial differentiation of Z with respect to K and λ

$$\frac{\partial Z}{\partial L} = \frac{12}{L} - 2\lambda = 0 \qquad \qquad ...(i)$$

$$\frac{\partial Z}{\partial K} = \frac{30}{K} - 5\lambda = 0 \qquad \qquad ...(ii)$$

$$\frac{\partial Z}{\partial \lambda} = TC - 2L - 5K = 0 \qquad \qquad ...(iii)$$

On taking equations (i) and (ii)

$$\frac{12}{L} = 2\lambda \text{ or } \frac{30}{K} = 5\lambda$$

On splitting both

$$\frac{12/L}{30/K} = \frac{2}{5}$$

Or

$$\frac{2/L}{5/K} = \frac{2}{5}$$

Or

$$\frac{2}{L} = \frac{2}{K}$$

Or

$$L = K$$

On putting the value of L in equation (1)

$$TC = 2K + 5K = 7K$$

Or

$$K = \frac{TC}{7} \text{, in the same way } L = \frac{TC}{7}$$

$$\lambda = \frac{12}{2L} = \frac{6}{TC/7} = \frac{42}{TC}$$

(A) The parameter v of this function is influenced by the scale of operation and technical changes. Both of these influence the rate of return of the parameter but they cannot be differentiated themselves.

(B) Here we assume that replacement rate is influenced by technical changes and the ratio of instruments do not affect it but with behavioural studies it has been found that even ratio of instruments also influence it. In this way this function leaves an important fact.

(C) Lastly, the parameter δ of this function cannot be calculated. Apart from this, there is a great difficulty to verify the figures in this function.

Example 4: The production function is $P = AL^\alpha C^\beta$ where A, α and β are constant. Show that if the factors L, C are increased in the same proportion, the product increases in greater $\alpha + \beta$, equal or less proportion, according as is $\alpha + \beta$ greater than, equal to, or less than unity.

Solution: Production function is

$$P = AL^\alpha C^\beta$$

Consider if there is an increase in the instrument L and C then

$$P' = A (\psi L)^\alpha (\psi C)^\beta \qquad \text{(here } \psi > 0)$$

$$= \psi^{\alpha + \beta} AL^\alpha C^\beta$$

$$= \psi^{\alpha + \beta} P \qquad (\because P = AL^\alpha C^\beta)$$

If $\qquad\qquad\qquad \alpha + \beta = 1$ then

$$P' = \psi P$$

Therefore production will increase in the same ratio as sources will be increased. In this condition production will be according to the rule of constant by product of scale.

If $\alpha + \beta < 12\text{-}3$ i.e. $\alpha + \beta = \dfrac{1}{2}$ suppose then

$$P' = \psi^{1/2} P$$

In this condition production will increase lesser as sources will be increased. This condition will be according to the rule of decreasing by product of scale.

If, $\alpha + \beta > 1$ i.e. suppose then $\alpha + \beta = 2$ then

$$P' = \psi^2 P$$

Did u know? Production will be more than the increase in the sources and production will be operative within the rule of increasing by product of scale.

$$L = \frac{TC}{7}, K = \frac{TC}{7} \text{ and } \lambda = \frac{42}{TC}$$

These all will form the price desirable expansion path.

Example 5: If the profit rate (price of capital) remain unaltered , the ratio of amount of capital employed per unit of labor shifts from 10 : 10 to 12 : 11, given that rise in wages is 25% . Determine the elasticity of substitution.

Solution: We know that

$$\text{Elasticity of substitution} = \sigma = \frac{\partial \left(\frac{L}{K}\right)/L/K}{\dfrac{\partial (p_K / P_L)}{P_K / P_L}}$$

Here L and K represents labour and capital and P_L / P_K is the ratio of prices of source.

It is given that

Initially, $\dfrac{L}{K} = \dfrac{10}{10}$ and finally, $\dfrac{L}{K} = \dfrac{12}{11}$

$$\text{Change in } \frac{L}{K} = \frac{12}{11} - \frac{10}{10} = \frac{12 - 11}{11} = \frac{1}{11}$$

$$\therefore \qquad \frac{\partial\left(\frac{L}{K}\right)}{L/K} = \frac{\frac{1}{11}}{\frac{10}{10}} = \frac{1}{11}$$

Initially, $\dfrac{P_K}{P_L} = 1:1$ Now the wages increases to 25% then the ratio become $1.25:1 = 5/4$

$$\text{Change in } \frac{P_K}{P_L} = \partial\left(\frac{P_K}{P_L}\right) = \frac{5}{4} - \frac{1}{1} = \frac{5-4}{4} = \frac{1}{4}$$

$$\frac{\partial\left(\frac{P_K}{P_L}\right)}{\frac{P_K}{P_L}} = \frac{+\frac{1}{4}}{\frac{1}{1}} = +\frac{1}{4}$$

$$\sigma = \frac{1/11}{1/4} = \frac{4}{11} = 0.36$$

Self Assessment

1. Fill in the blanks:

1. $x = \dots\dots (q_1 q_2)$
2. $V = p_1 q_1 = p_2 q_2 + \mu \left[x - h \,(\dots\dots)\sigma\right]$
3. $\pi = \dots\dots + p_2 q_2 - rh \,(q_1 q_2)$
4. Elasticity of Substitution $= \sigma = \dfrac{\partial\left(\dfrac{L}{K}\right)/L/K}{\dots\dots}$

Solve the following Numericals:

1. It is known

   Utility Function, $U = q_1 q_2$

   Price of first thing, $P_1 = ₹\,4$

   Price of Second thing, $p_2 = ₹\,10$

   Income of customer, $Y = ₹100$

   Find the balanced level of consumption of things Q_1 and Q_2 and verify the condition of Maximization.

   Ans.: $\left[q_1 = \dfrac{25}{2}, q_2 = 5\right]$

2. If Utility function is $U = 4q_1 q_2 + 3q_1$ and income constraint is $60 = 2q_2 + 6q_1$ then for maximum utilization find the quantities of Q_1, Q_2

   Ans.: $[q_1 = 16.125,\ q_2 = -4.625]$

3. If desired property function is $U = (x_1, x_2) = 2 \log x_2$ then what will be the demand curve for first thing? If total income is W and the prices of two things are p_1 and p_2.

   Ans.: $\left[x_1 = \dfrac{2W}{p_1}, x_2 = \dfrac{W}{3p_2}\right]$

11.3   Summary

- Suppose a Firm purchase two products Q_1 and Q_2 of quantity q_1 and q_2 at the rates p_1 and p_2, then in this condition total proceeds will be

  $R = p_1 q_1 = p_2 q_2$

- Suppose a firm manufactures two outputs θ_1 and θ_2 from an input. In this condition the cost of two products (x) of firm in the context X will be the function of q_1 and q_2, i.e., $x = h(q_1, q_2)$

- Profit θ_1 and θ_2 are the function of production quantity q_1 and q_2-

  $\pi = p_1 q_1 + p_2 q_2 - rh(q_1 q_2)$

- Compensated demand shows that quantity of thing that a customer purchases within the following fixed conditions (like Tax, Financial help).

11.4   Keywords

- *Operation:* Process
- *Condition:* Bond

11.5   Review Questions

1. Describe Profit Maximization.
2. What is Compensated Demand Function? Describe.
3. If the Utility function of two things x and Y is $U = 3x^2 y^2$ and Budget constraint is $2x + 3y = 18$ then find the quantities sought of things X and Y?

   Ans.: $x = 4.5, y = 3$)

Answer: Self Assessment

1. h
2. $q_1 q_2$
3. $p_1 q_1$
4. $\dfrac{\partial(P_k / P_L)}{P_K / P_L}$

11.6   Further Readings

Books

Mathematical Economics – Michael Harrison, PatricWalderen.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publlications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics and Finance – Martin Norman.

Unit 12: Integration : Basic Rules of Integration

Objectives

After reading this unit, students will be able to :

- Solve Comprehensive Integrals.
- Understand Standard Integral.
- Solve Integration of x^n relative to x, where $n \neq 1$.
- Find out the Integration of Multiplication of a Constant and a Function.
- Do the Integration of the Sum and Subtract of the Function.

Introduction

Integration of the Function — The Inverse process of finding the differentiation of a function is called an Integration. In differential Maths we find the differential coefficient of a function. But in Integral Mathematics we have to find those functions whose differential coffecient is the given function.

For example, the differential coefficient of $\sin x$ with respect to x, is $\cos x$, then on integrating function $\cos x$ with respect to x integral will be $\sin x$.

Suppose $f(x)$ is some function of x, whose differential coefficient is $f'(x)$, i.e.

$$\frac{d}{dx}\{f(x)\} = f'(x)$$

Then we say that integral of f'(x) is, f (x).

It can be written in the symbol as :

$$\int f'(x)\,dx = f(x).$$

Sign '' $\int$ '' is the symbol of integration which is called the sign of integration. It represents the integration of a function. X in dx shows that integration is done with respect to variable x. If integration has to be done relative to some other variable, then that variable is kept in place of x.

Note

Signal $\int$ is the distorted form of English letter S. Basically integration is a special method of addition and the signal $\int$ is formed from the first letter of English word SUM.

Notes The process of integration of any function is called integration. The function to be integrated is called **integrand** and the function obtained after integration is called **integral**.

In $\int f'(x)\, dx = f(x)$ is $f'(x)$ integrand and $f(x)$ is integral

From the above interpretation it is clear that

$$\int \cos x\, dx = \sin x, \qquad\qquad \because \frac{d}{dx}\sin x = \cos x$$

$$\int \sin x\, dx = -\cos x, \qquad\qquad \because \frac{d}{dx}(-\cos x) = \sin x$$

$$\int e^x dx = e^x, \qquad\qquad \because \frac{d}{dx}e^x = e^x$$

And $$\int \frac{1}{t}dt = \log t, \qquad\qquad \because \frac{d}{dt}(\log t) = \frac{1}{t}$$

In the above functions $\cos x$, $\sin x$, e^x and $\frac{1}{t}$ are integrands and $\sin x$, $-\cos x$, e^x and $\log t$ are integrals.

If on any, function both integration and differentiation would be done then that function remains unaltered.

For example, $\dfrac{d}{dx}\left[\int \cos x\, dx\right] = \dfrac{d}{dx}\sin x = \cos x$

i.e. $\dfrac{d}{dx}\left[\int f'(x)\, dx\right] = \dfrac{d}{dx}f(x) = f'(x).$

It is clear from this example that integration and differentiation are opposite activities.

12.1 Comprehensive Integrals

From Differential Mathematics we know that, if the differential coefficient of $f(x)$ is $F(x)$, then the differential coefficient of f $(x) + c$ will also $F(x)$, where there is any arbitrary constant c.

i.e., if $$\frac{d}{dx}f(x) = F(x)$$

then $$\frac{d}{dx}\{f(x) + c\} = \frac{d}{dx}f(x) + \frac{d}{dx}(c) = \frac{d}{dx}f(x) + 0 = F(x)$$

$\therefore$ $$\int F(x)\, dx = f(x) + c,$$

Where c is an arbitrary constant which can have indefinite values.

Therefore the integral of any function is not unique, since many integrals of F (x) were obtained on giving different values to *c*, therefore if *f* (x) is any integral of F (x), then *f* (x) + *c* will be its Comprehensive Integral.

Did u know?   Arbitrary Constant c is called **Constant of Integration.**

12.2   Standard Integral

On the basis of Differential Coefficients of standard functions we will find the differentials of some functions, which are the following:

Integration	Differentiation
1- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c, (n \neq -1)$	$\because \quad \dfrac{1}{n+1} \cdot \dfrac{d}{dx}(x^{n+1}) = x^n$
2- $\int \dfrac{1}{x} dx = \log|x| + c$	$\because \quad \dfrac{d}{dx}(\log|x|) = \dfrac{1}{x}$
3- $\int e^x = e^x + c$	$\because \quad \dfrac{d}{dx}(e^x) = e^x$
4- $\int a^x dx = \dfrac{a^x}{\log_e a} + c$	$\because \quad \dfrac{d}{dx}\left(\dfrac{a^x}{\log_e a}\right) = a^x$
5- $\int \sin x \, dx = -\cos x + c$	$\because \quad \dfrac{d}{dx}(-\cos x) = \sin x$
6- $\int \cos x \, dx = \sin x + c$	$\because \quad \dfrac{d}{dx}(\sin x) = \cos x$
7- $\int \sec^2 x \, dx = \tan x + c$	$\because \quad \dfrac{d}{dx}(\tan x) = \sec^2 x$
8- $\int \text{cosec}^2 x \, dx = -\cot x + c$	$\because \quad \dfrac{d}{dx}(-\cot x) = \text{cosec}^2 x$
9- $\int \sec x \tan x \, dx = \sec x + c$	$\because \quad \dfrac{d}{dx}(\sec x) = \sec x \tan x$
10- $\int \text{cosec} x \cot x \, dx = -\text{cosec} x + c$	$\because \quad \dfrac{d}{dx}(-\text{cosec } x) = \text{cosec } x \cot x$
11- $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$	$\because \quad \dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
12- $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c$	$\because \quad \dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
13- $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$	$\because \quad \dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$ -

12.3 Integration of x^n relative to X, where $n \neq -1$

$\because$ $$\frac{d}{dx}x^{n+1} = (n+1)x^n$$

Or $$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n$$

$\therefore$ $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ , Where } n \neq -1.$$

Example 1: $\int dx = \int 1\, dx = \int x^0 dx = \dfrac{x^{0+1}}{0+1} + c = x + c$.

Example 2: $\int x^5 dx = \dfrac{x^{5+1}}{5+1} + c = \dfrac{x^6}{6} + c$.

Example 3: $\int x^{-10}\, dx = \dfrac{x^{-10+1}}{-10+1} + c = \dfrac{x^{-9}}{-9} + c = -\dfrac{1}{9}x^{-9} + c$.

Example 4: $\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \dfrac{x^{1/2+1}}{\frac{1}{2}+1} + c = \dfrac{2}{3}x^{3/2} + c$.

Example 5: $\int x^{2/3} dx = \dfrac{x^{2/3+1}}{\frac{2}{3}+1} + c = \dfrac{3}{5}x^{5/3} + c$.

Self Assessment

1. Fill in the blanks:

1. Opposite process of finding the differential of any function is called ....
2. The function to be integrated is called ....
3. $\int a^x \, dx = $ .... $+ c$
4. $\int$ .... dx .... $= -\cos x + c$
5. $\dfrac{d}{dx}x^{n+1} = ($.........$) x^n$

> **Questionnaire 12.1**

Very short answer questions.

Find the values of the following Integrals:

1. (a) $\int x\, dx$ (b) $\int x^4 \, dx$

(c) $\int x^a \, dx$

(d) $\int t^3 \, dt$

2. (a) $\int x^{-7} \, dx$

(b) $\int x^{-1/2} \, dx$

(c) $\int x^{-3/2} \, dx$

(d) $\int z^{-1/3} \, dz$

3. (a) $\int \dfrac{dx}{x^2}$

(b) $\int \dfrac{dt}{t}$

(c) $\int \dfrac{dx}{\sqrt{x}}$

(d) $\int \dfrac{dx}{x\sqrt{x}}$

4. (a) $\int 2^x \, dx$

(b) $\int 3^x \, dx$

(c) $\int b^x \, dx$

(d) $\int b^{x+a} \, dx$

5. (a) $\int \dfrac{dt}{\sqrt{1-t^2}}$

(b) $\int \dfrac{dz}{z\sqrt{z^2-1}}$

(c) $\int \dfrac{dy}{1+y^2}$

(d) $\int e^x \, dx$

6. (a) $\int \sec^2 t \, dt$

(b) $\int \operatorname{cosec}^2 z \, dz$

(c) $\int \sqrt{1-\cos^2 x} \, dx$

(d) $\int (1+\cot^2 x) \, dx$

7. (a) $\int \dfrac{dx}{\operatorname{cosec} x}$

(b) $\int \dfrac{dx}{\cos x \cot x}$

(c) $\int \dfrac{dx}{5^{-x}}$

(d) $\int \dfrac{dx}{\sec x}$.

8. Prove that $\int dx = x$, when $x = 0$.

Answers

1. (a) $\dfrac{x^2}{2}+c$ (b) $\dfrac{x^5}{5}+c$ (c) $\dfrac{x^{a+1}}{a+1}+c$ (d) $\dfrac{t^4}{4}+c$

2. (a) $-\dfrac{1}{6x^6}+c$ (b) $2x^{1/2}+c$ (c) $x^{-1/2}+c$ (d) $\dfrac{3}{2}z^{2/3}+c$

3. (a) $\dfrac{-1}{x}+c$ (b) $\log 1+1+c$ (c) $2x^{\frac{1}{2}}+c$ (d) $2x^{1/2}+c$

Note

4. (a) $\dfrac{2^x}{\log e^2} + c$ (b) $\dfrac{3^x}{\log e^3} + c$ (c) $\dfrac{b^x}{\log e^b} + c$ (d) $\dfrac{b^{x+a}}{\log e^b} + c$

5. (a) $\sin^{-1} t + c$ (b) $\sec^{-1} z + 1\, c$ (c) $\tan^{-1} x + c$ (d) $e^x + c$

6. (a) $\tan t + c$ (b) $-\cot z + c$ (c) $-\cos x + c$ (d) $-\cot x + c$

7. (a) $-\cos x + c$ (b) $\sec x + c$ (c) $\dfrac{5^x}{\log_e 5} + c$ (d) $\sin x + c$

12.4 Integration of the Multiplication of a Constant and a Function

If $\displaystyle\int f'(x)\,dx = f(x)$ if $\dfrac{d}{dx} f(x) = f'(x)$

$\because$ $\dfrac{d}{dx}\{a \cdot f(x)\} = a\dfrac{d}{dx} f(x)$

                    $= af'(x)$

$\because$ $\displaystyle\int af'(x)\,dx = a \cdot f(x) = a\int f'(x)\,dx$

                    $\displaystyle\int af'(x)\,dx = a\int f'(x)\,dx$

That is Integration of the Multiplication of a Constant and a Function is equal to the multiplication of the constant and integration of that function.

$\boxed{\textbf{EXAMPLES WITH SOLUTION}}$

Example 1: Integrate function $15x^4$ with respect to x.

Solution: $\displaystyle\int 15x^4\,dx = 15\int x^4\,dx$

                    $= \dfrac{15}{5}x^5 + c = 3x^5 + c.$ **Ans.**

Example 2: Find the value of $\displaystyle\int 7\dfrac{dx}{x}$.

Solution: $\displaystyle\int 7\dfrac{dx}{x} = 7\int \dfrac{dx}{x} = 7\log|x| + c$ **Ans.**

Example 3: Find the value of $\displaystyle\int 6\sin x\,dx$.

Solution: $\displaystyle\int 6\sin x\,dx = 6\int \sin x\,dx$

                    $= 6(-\cos x) + c = -6\cos x + c.$ **Ans.**

Example 4: Find the value of $\int 3e^x dx$.

Solution: $\int 3e^x dx \ = \ 3 \int e^x dx = 3e^x + c$ **Ans.**

Example 5: Integrate function e^{x+a} **with respect to** x.

Solution: $\int e^{x+a} dx \ = \ \int e^x . e^a dx$

$$= \ e^a \int e^x \, dx = e^a . e^x + c$$

$$= \ e^{x+a} + c.$$ **Ans.**

Task   Integrate the function $20x^4$ with respect to x.   (**Ans.:** $4x^5 + c$)

12.5   Integration of the Sum and Subtract of the Function

If $\int f_1'(x) \, dx \ = \ f_1(x)$ and $\dfrac{d}{dx} f_1(x) \ = f_1'(x)$

And $\int f_2'(x) \, dx \ = \ f_2(x)$ and $\dfrac{d}{dx} f_2(x) \ = f_2'(x)$

From the differential method of addition and subtraction of functions

$$\frac{d}{dx}\{f_1(x) \pm f_2(x)\} \ = \ \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)$$

$$= \ f_1'(x) \pm f_2'(x)$$

$\therefore$ $\int \{f_1'(x) \pm f_2'(x)\} \, dx \ = f_1(x) \pm f_2(x) = \int f_1'(x) \, dx \pm \int f_2'(x) \, dx$

Therefore $\boxed{\int \{f_1'(x) \pm f_2'(x)\} \, dx \ = \ \int f_1'(x) dx \pm \int f_2'(x) \, dx}$

This method in the same way is true for two or more functions.

Did u know?   Integration of addition or subtraction of any functions is equal to the addition or subtraction of the integration of those functions.

Therefore $\int [f_1'(x) \pm f_2'(x) \pm f_3'(x) \pm ....] \, dx$

$$= \ \int f_1'(x) \, dx \pm \int f_2'(x) \, dx \pm \int f_3'(x) \, dx \pm .....$$

EXAMPLES WITH SOLUTION

Example 1: Integrate the function $x^6 + \dfrac{1}{x} - e^x + 1$ **with respect to** x.

Solution: $\displaystyle\int \left(x^6 + \frac{1}{x} - e^x + 1 \right) dx \; = \; \int x^6 dx + \int \frac{1}{x} dx - \int e^x dx + \int dx$

$$= \frac{1}{7}x^7 + \log|x| - e^x + x + c$$ **Ans.**

Example 2: Find the value of $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\, dx$.

Solution: $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\, dx \; = \; \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x . \cos^2 x}\, dx$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \sec^2 x \; dx + \int \operatorname{cosec}^2 x \; dx$$

$$= \tan x - \cot x + c.$$ **Ans.**

Example 3: Find the value of $\displaystyle\int \left(3e^x - \frac{1}{5x} + \sec x \tan x \right) dx$.

Solution: $\displaystyle\int \left(3e^x - \frac{1}{5x} + \sec x \tan x \right) dx$

$$= 3\int e^x dx - \frac{1}{5}\int \frac{1}{x} dx + \int \sec x \tan x \; dx$$

$$= 3e^x - \frac{1}{5}\log|x| + \sec x + c.$$ **Ans.**

Example 4: Find the value of $\displaystyle\int (ax^2 + bx + c)\, dx$.

Solution: $\displaystyle\int (ax^2 + bx + c)\, dx \; = \; a\int x^2\, dx + b\int x\, dx + c\int 1\, dx$

$$= a\left(\frac{1}{3}x^3 \right) + b\left(\frac{1}{2}x^2 \right) + cx + d$$

$$= \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx + d,$$

where d is integration constant. **Ans.**

Example 5: Find the value of $\int \left(x - \dfrac{1}{x}\right)^2 dx$.

Solution: $\int \left(x - \dfrac{1}{x}\right)^2 dx = \int \left(x^2 - 2 + \dfrac{1}{x^2}\right) dx$

$$= \int x^2\, dx - 2\int 1\, dx + \int x^{-2}\, dx$$

$$= \dfrac{x^3}{3} - 2x + \left(\dfrac{x^{-1}}{-1}\right) + c = \dfrac{x^3}{3} - 2x - \dfrac{1}{x} + c. \qquad \textbf{Ans.}$$

Example 6: Find the value of $\int (5x - 4)^3\, dx$.

Solution: $\int (5x - 4)^3\, dx = \int (125x^3 - 300x^2 + 240x - 64)\, dx$

$$= 125\int x^3 dx - 300\int x^2 dx + 240\int x\, dx - 64\int 1\, dx$$

$$= 125\left(\dfrac{1}{4}x^4\right) - 300\left(\dfrac{1}{3}x^3\right) + 240\left(\dfrac{1}{2}x^2\right) - 64x + c$$

$$= \dfrac{125}{4}x^4 - 100x^3 + 120x^2 - 64x + c. \qquad \textbf{Ans.}$$

Example 7: Integrate $\sin^2 \dfrac{x}{2}$ **with respect to** x.

Solution: $\int \sin^2 \dfrac{x}{2} dx = \int \dfrac{1 - \cos x}{2} dx = \dfrac{1}{2}\int dx - \dfrac{1}{2}\int \cos x\, dx$

$$= \dfrac{1}{2}x - \dfrac{1}{2}\sin x + c = \dfrac{1}{2}(x - \sin x) + c. \qquad \textbf{Ans.}$$

Example 8: Find the value of $\int (\cos^4 x - \sin^4 x)\, dx$.

Solution: $\int (\cos^4 x - \sin^4 x)\, dx = \int (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)\, dx$

$$= \int 1 \cdot \cos 2x\, dx$$

$$= \int \cos 2x$$

$$= \dfrac{\sin 2x}{2} + c. \qquad \textbf{Ans.}$$

Note

Example 9: Find the value of $\int \dfrac{x^4}{x^2 + 1} dx$.

Solution:

$$\int \dfrac{x^4}{x^2 + 1} dx = \int \left(x^2 - 1 + \dfrac{1}{x^2 + 1} \right) dx$$

$$= \dfrac{x^3}{3} - x + \tan^{-1} x + c$$ 　　　　**Ans.**

Example 10: Find the value of $\int \tan^2 x\, dx$.

Solution:

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$$

$$= \tan x - x + c.$$ 　　　　**Ans.**

Example 11: Find the value of $\int \left(\dfrac{5x + 7}{x} + e^x \right) dx$.

Solution:

$$\int \left(\dfrac{5x + 7}{x} + e^x \right) dx = \int \left(5 + \dfrac{7}{x} + e^x \right) dx$$

$$= 5x + 7 \log |x| + e^x + c.$$ 　　　　**Ans.**

Example 12: Find the value of $\int \left(\dfrac{ax^4 + bx^2 + c}{x^4} \right) dx$.

Solution:

$$\int \left(\dfrac{ax^4 + bx^2 + c}{x^4} \right) dx = \int \left(a + \dfrac{b}{x^2} + \dfrac{c}{x^4} \right) dx$$

$$= \int (a + bx^{-2} + cx^{-4})\, dx$$

$$= \int a\, dx + b \int x^{-2}\, dx + c \int x^{-4} dx$$

$$= ax + b\,\dfrac{x^{-1}}{-1} + c \cdot \dfrac{x^{-3}}{-3} + d = ax - \dfrac{b}{x} - \dfrac{c}{3x^3} + d$$ 　　**Ans.**

Example 13: Find the value of $\int \dfrac{x^2}{x^2 + 1} dx$.

Solution:

$$\int \dfrac{x^2}{x^2 + 1} dx = \int \dfrac{(x^2 + 1) - 1}{x^2 + 1} dx$$

$$= \int \dfrac{x^2 + 1}{x^2 + 1} dx - \int \dfrac{1}{x^2 + 1} dx$$

$$= \int dx - \int \frac{dx}{x^2 + 1}$$

$$= x - \tan^{-1} x + c. \qquad \textbf{Ans.}$$

Example 14: Find the value of $\displaystyle\int \frac{1}{1 + \cos x}\, dx$ **.**

Solution: $\displaystyle\int \frac{1}{1 + \cos x}\, dx = \int \frac{1 - \cos x}{1 - \cos^2 x}\, dx$, multiplying both numerator and denominator by $1 - \cos x$

$$= \int \frac{1 - \cos x}{\sin^2 x}\, dx = \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int (\text{cosec}^2 x - \cot x \cdot \text{cosec}\, x)\, dx$$

$$= -\cot x + \text{cosec}\, x + c .$$

$$\textbf{Ans.}$$

Example 15: Find the integration of function $\dfrac{\sec x + \tan x}{\sec x - \tan x}$ **with respect to x.**

Solution: $\displaystyle\int \frac{\sec x + \tan x}{\sec x - \tan x}\, dx = \int \frac{(\sec x + \tan x)\,(\sec x + \tan x)}{(\sec x - \tan x)\,(\sec x + \tan x)}\, dx$

$$= \int \frac{(\sec x + \tan x)^2}{\sec^2 x - \tan^2 x}\, dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x)\, dx$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x)\, dx$$

$$= \int 2\sec^2 x\, dx - \int dx + \int 2\sec x \tan x\, dx$$

$$= 2\int \sec^2 x\, dx - \int dx + 2\int \sec x \tan x\, dx$$

$$= 2\tan x - x + 2\sec x + c.$$

$$\textbf{Ans.}$$

Example 16: Find the integration of function $\dfrac{(x + 2)\,(4x^2 - 5)}{x}$ **with respect to x.**

Solution:

$$\int \frac{(x + 2)\,(4x^2 - 5)}{x}\, dx = \int \frac{(4x^3 + 8x^2 - 5x - 10)}{x}\, dx$$

$$= 4\int x^2 dx + 8\int x\, dx - 5\int dx - 10\int \frac{1}{x} dx$$

Note

$$= \frac{4}{3}x^3 + \frac{8}{2}x^2 - 5x - 10 \log |x| + c$$

$$= \frac{4}{3}x^3 + 4x^2 - 5x - 10 \log |x| + c .$$ **Ans.**

Questionnaire 12.2

Find the values of the following:

1. $\int (e^x + 2\sin x - 3\cos x)dx$.

2. $\int (x^2 - \cos x + \sec x \tan x)\, dx$.

3. $\int (7x^6 - 8x^3 + 5)\, dx$.

4. $\int (x + 2)\,(2x + 6)\, dx$.

5. $\int \frac{ax^3 + bx + c}{x^2}\, dx$.

6, $\int \frac{(1+x)^3}{\sqrt{x}}\, dx$.

7. $\int \left(\frac{1}{\sqrt{x}} - \sqrt{x} \right) dx$.

8. $\int \left(\frac{3x^2 + 4x + 5}{\sqrt{x}} \right) dx$.

9. $\int \frac{x}{a+x}\, dx$.

10. $\int \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \right) dx$.

11. $\int (1 - x^2)\,\sqrt{x}dx$.

12. $\int \frac{e^{\log x}}{x} dx$.

13. $\int \left(\cos x - \frac{5}{x} + e^x \right) dx$.

14. $\int \frac{x^4 + 1}{x^2 + 1} dx$.

Answers

1. $e^x - 2\cos x - 3\sin x + c$

2. $\frac{x^3}{3} - \sin x + \sec x + c$

3. $x^7 - 2x^4 + 5x + c$

4. $\frac{2}{3}x^3 + 5x^2 + 12x + c$

5. $\frac{a}{2}x^2 + b\log_e |x|\frac{-c}{x} + c_1$

6. $\frac{2}{7}x^{7/2} + \frac{6x^{3/2}}{5} + 2x^{3/2} + 2\sqrt{x} + c$

7. $2x^{1/2} - 2/3\, x^{3/2} + c$

8. $2\sqrt{x}\left[\frac{3}{5}x^5 + \frac{4}{3}x + 5 \right] + c$

9. $x\, a \log_e |a + x| + c$

10. $x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + ...$

11. $\frac{2}{3}x^{3/2} - \frac{2}{7}x^{7/2} + c$

12. $x + c$

13. $\sin x - 5 \log |x| + e^x + c$

14. $-\frac{2}{3}\operatorname{cosec} x + \sec x + c$.

Self Assessment

2. State whether the following statements are True or False:

6. Integration of addition or subtraction of any functions is equal to the addition or subtraction of their integrations.

7. Integration of multiplication of a constant and function is equal to the addition of that constant and the integration of the function.

8. $\int af'(x)\, dx = a\int f'(x)\, dx$

9. $\dfrac{d}{dx}\{f_1(x) \pm f_2(x)\} = \dfrac{d}{dx} f_1(x) \pm \dfrac{d}{dx} f_2(x)$

12.6 Summary

● The Inverse process of finding the differentiation of a function is called Integration. In differential Math's we find the differential coefficient of a function. But in Integral Mathematics we have to find those functions whose differential coefficient is the given function.

● Sign "$\int$" is the symbol of integration which is called the sign of integration. It represents the integration of a function. X in dx shows that integration is done with respect to variable x. If integration has to be done relative to some other variable, then that variable is kept in place of x.

● If on any function both integration and differentiation would be done then that function remains unaltered.

● Therefore the integral of any function is not unique, since many integrals of $F(x)$ were obtained on giving different values to c, therefore if $f(x)$ is any integral of $F(x)$, then $f(x) + c$ will be its Comprehensive Integral.

12.7 Keywords

● *Inverse:* Antipod
● *Symbol:* Icons

12.8 Review Questions

1. Find the integration of $\int x^5\, dx$ with respect to x where $x \neq -1$.   (**Ans.:** $\dfrac{x^6}{6} + c$)

2. Find the value of the integral $\int e^x\, dx$ (**Ans.:** $e^x + c$)

3. Find the value of the integral $\int 6 \sin x\, dx$ (**Ans.:** $-6 \cos x + c$)

4. Find the value of the integral $\int x - \dfrac{1}{x}\, dx$ (**Ans.:** $\dfrac{x^3}{3} - 2x - \dfrac{1}{x} + c$)

5. Find the value of the integral $\int \tan^2 x\, dx$ (**Ans.:** $\tan x - x + c$)

Answers: Self Assessment

1. Integration
2. Integrand
3. $\dfrac{a^x}{\log_e a}$
4. $\sin x$
5. $n + 1.$
6. True
7. False
8. True
9. True

12.9   Further Readings

Books

Mathematical Economics – Michael Harrison, Patric Walderen.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics and Finance – Martin Norman.

Unit 13: Methods of Integration

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Objectives

After reading this unit, students will be able to :

- Understand Substitution.
- Understand Some results Gained from Substitution.
- Take out the Integration of Function of x^n.
- Aware of Integrand.
- Take out the Integration of some Standard Functions by Substitution.

Introduction

Below mentioned are the two methods for knowing the integration of any function:

1.   Integration by Substitution
2.   Integration by Parts

13.1   Substitution

In this operation integration is done by changing the given integrand in standard formula. For this any integration function of variable amount is put equal to another variable amount and substituting in integration, integration is changed in form of function of new variable amount in such a way that use of formula may be done easily.

13.2   Some Results Gain from Substitution

(i) For the function sin ($ax + b$), find relative integration relative of x.

Solution: assume $ax + b = t$

Note

$$\frac{d}{dx}(ax+b) = dt \quad \therefore \quad a\,dx = dt$$

Or

$$dx = \frac{1}{a}\,dt$$

$$\therefore \quad \int \sin(ax+b)\,dx = \int (\sin t)\frac{1}{a}\,dt = \frac{1}{a}\int \sin t\,dt$$

$$= -\frac{1}{a}\cos t + c = -\frac{1}{a}\cos(ax+b) + c$$

Notes $\quad \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

From the above given example, relative integration of x may be known for the below mentioned functions:

$$\int \cos ax\,dx = \frac{\sin ax}{a} + c$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a} + c$$

$$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$$

$$\int \operatorname{cosec}(ax+b)\tan(ax+b) = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$$

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}f(ax+b) + c}$$

(ii) Find relative integration of x for the function $\dfrac{1}{a^2 + x^2}$

Solution: Assume, $\qquad x = at \qquad \therefore \qquad dx = a\,dt$

$$\int \frac{1}{a^2 + x^2}\, dx = \int \frac{1}{a^2 + a^2 t^2} \cdot a\,dt$$

$$= \frac{1}{a}\ \int \frac{1}{1 + t^2}\, dt \ = \ \frac{1}{a}\ \tan^{-1} t + c$$

$$= \frac{1}{a}\ \tan^{-1} \left(\frac{x}{a}\right) + c$$

Hence,

$$\boxed{\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\ \tan^{-1} \left(\frac{x}{a}\right) + c}$$

Similarly,

$$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c}$$

$$\boxed{\int \frac{1}{x\,\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\ \sec^{-1} \frac{x}{a} + c}$$

Above mentioned results may be used in integration without any substitution.

$$\boxed{\textbf{EXAMPLES WITH SOLUTION}}$$

Very Short Answer Questions

Example 1: $\int (\sqrt{1 - 5x} + 9e^{3x})\ dx.$

Solution: $\qquad \int (\sqrt{1 - 5x} + 9e^{3x})\ dx = \int (1 - 5x)^{1/2}\ dx + 9\int e^{3x}\ dx$

$$= \left(-\frac{1}{5}\right) \frac{(1 - 5x)^{3/2}}{3/2} + 9\,\frac{e^{3x}}{3}\ + c$$

$$= \frac{-2}{15}\ (1 - 5x)^{3/2} + 3e^{3x} + c. \qquad\qquad \textbf{Ans.}$$

Example 2: Find the Value $\int \sin\ 4x \cos 6x\ dx.$

Solution: $\qquad \int \sin 4x \cos 6x\ dx = \frac{1}{2}\ \int (2 \sin 4x \cos 6x)\ dx$

$$= \frac{1}{2}\ \int \sin 10x - \sin 2x)\ dx$$

$$= \frac{1}{2} \int \sin 10x \, dx - \frac{1}{2} \int \sin 2x \, dx$$

$$= -\frac{1}{2} \frac{\cos 10x}{10} + \frac{1}{2} \frac{\cos 2x}{2} + c$$

$$= -\frac{1}{20} \cos 10x + \frac{1}{4} \cos 2x + c. \qquad \text{Ans.}$$

Example 3: Find the Value of $\int \sin^3 \theta \, d\theta$.

Solution: $\qquad \int \sin^3 \theta \, d\theta = \frac{1}{4} \int (3 \sin \theta - \sin 3\theta) \, d\theta$

$$= \frac{3}{4} \int \sin \theta \, d\theta - \frac{1}{4} \int \sin 3\theta \, d\theta$$

$$= \frac{3}{4} (-\cos \theta) - \frac{1}{4} \left(-\frac{1}{3} \right) \cos 3\theta + c$$

$$= -\frac{3}{4} \cos \theta + \frac{1}{12} \cos 3\theta + c. \qquad \text{Ans.}$$

Example 4: Find the Value of $\int \sqrt{1 - \sin x} \, dx$.

Solution: Here, $\qquad \sqrt{1 - \sin x} = \sqrt{\sin^2 \frac{1}{2}x + \cos^2 \frac{1}{2}x - 2 \sin \frac{1}{2}x \cos \frac{1}{2}x}$

$$= \sqrt{\left(\sin \frac{1}{2}x - \cos \frac{1}{2}x \right)^2} = \pm \left(\sin \frac{1}{2}x - \cos \frac{1}{2}x \right)$$

$\therefore \qquad \int \sqrt{1 - \sin x} \, dx = \pm \int \left(\sin \frac{1}{2}x - \cos \frac{1}{2}x \right) dx$

$$= \pm \left[-2 \cos \frac{1}{2}x - 2 \sin \frac{1}{2}x \right] + c$$

$$= \pm 2 \left(\cos \frac{1}{2}x + \sin \frac{1}{2}x \right) + c. \qquad \text{Ans.}$$

Example 5: Find the Value of $\int \left(7e^{2x} + \dfrac{5}{2x + 7} + \sec^2 3x - 2\sqrt{1 - 5x} \right) dx$.

Solution: $\int \left(7e^{2x} + \dfrac{5}{2x + 7} + \sec^2 3x - 2\sqrt{1 - 5x} \right) dx$

$$= 7 \int e^{2x} \, dx + 5 \int \frac{1}{2x+7} \, dx + \int \sec^2 3x \, dx - 2 \int (1-5x)^{1/2} \, dx$$

$$= \frac{7}{2} \, e^{2x} + \frac{5}{2} \log (2x+7) + \frac{1}{3} \tan 3x - 2 \frac{(1-5x)^{3/2}}{(-5)\left(\frac{3}{2}\right)} + c$$

$$= \frac{7}{2} \, e^{2x} + \frac{5}{2} \log (2x+7) + \frac{1}{3} \tan 3x + \frac{4}{15} (1-5x)^{3/2} + c. \qquad \textbf{Ans.}$$

Questionnaire 13.1

Find the value of the following integrations with respect to x:

1. (a) $(x+2)^3$ (b) $(7x-2)^3$
   (c) $(ax+b)^4$ (d) $(3+4x)^5$

2. (a) $\sqrt{4-5x}$ (b) $\sqrt{4x+3}$

   (c) $\sqrt{ax+b}$ (d) $\sqrt{2x+\dfrac{5}{3}}$

3. (a) $\dfrac{1}{(5x+4)^2}$ (b) $\dfrac{1}{(a+bx)^4}$ (c) $\dfrac{1}{\left(\dfrac{c}{2}+bx\right)^3}$

4. (a) $\dfrac{1}{3x+1}$ (b) $\dfrac{1}{a-bx}$

5. (a) $\dfrac{1}{\sqrt{25-x^2}}$ (b) $\dfrac{1}{\sqrt{1-(3x-2)^2}}$

6. (a) $\dfrac{1}{5+(2-3x)^2}$ (b) $\dfrac{1}{1+9x^2}$

7. (a) $\sin 3x$ (b) $\cos (4x+5)$

8. $\sqrt{1+\sin \dfrac{1}{2} x}$ 9. (a) e^{3x+4} (b) $e^{x/2}$

10. $\cos \left(\dfrac{2}{5} x - 2\right) + A^{3x+2}$.

Answers

1. (a) $\dfrac{1}{4}(x+2)^4 + c$ (b) $\dfrac{1}{28}(7x-2)^4 + c$ (c) $\dfrac{1}{5a}(ax+b)^5 + c$

   (d) $\dfrac{1}{24}(3+4x)^6 + c$

Note

2. (a) $-\dfrac{2}{15}(4-5x)^{3/2}+c$ (b) $\dfrac{1}{6}(4x+3)^{3/2}+c$ (c) $\dfrac{2}{3a}(ax+b)^{3/2}+c$

    (d) $\dfrac{1}{3}\left(2x+\dfrac{5}{3}\right)^{3/2}+c$

3. (a) $-\dfrac{1}{5(5x+4)}+c$ (b) $-\dfrac{1}{3b(a+bx)^3}+c$ (c) $-\dfrac{1}{2b(c/2+bx)^2}+c$

4. (a) $\dfrac{1}{3}\log 3x+1+c$ (b) $-\dfrac{1}{b}\log|9-bx|+c$

5. (a) $\sin^{-1}\dfrac{x}{5}+c$ (b) $\dfrac{1}{3}\sin^{-1}(3x-2)+c$

6. $-\dfrac{1}{3\sqrt{5}}\tan^{-1}\dfrac{(2-3x)}{\sqrt{5}}+c$ (b) $\dfrac{1}{3}\tan^{-1}3x+c$

7. $-\dfrac{1}{3}\cos 3x+c$ (b) $\dfrac{1}{4}\sin(4x+5)+c$

8. $\left(\sin\dfrac{1}{4}x-\cos\dfrac{1}{4}x\right)+c$ 9. (a) $\dfrac{1}{3}e^{3x+4}+c$ (b) $2e^{x/2}+c$

10. $\dfrac{5}{2}\sin\left(\dfrac{2}{5}x-2\right)+\dfrac{1}{3}\dfrac{a^{3x+2}}{\log_e a}+c$

13.3 Integration of x^n Function

If in the function of x^n product of x^{n-1} is given, then integration of that function may be done, assuming $x^n=t$.

We know that,

$$\int f\{\phi(x)\}\,\phi'(x)\,dx = \int f(t)dt, \text{ where, } \phi(x)=t$$

Now if in the same formula,

$$\phi(x)=x^n$$

∴ $\phi'(x)=nx^{n-1}$ is kept, then on dividing by n,

$$\int f(x^n)\,x^{n-1}\,dx = \dfrac{1}{n}\int f(t)\,dt$$

?

Did u know?   If integration is of the form $\int f(x^n)\,x^{n-1}\,dx$ then by putting value $x^n=t$ integration will become easy.

$$\boxed{\textbf{EXAMPLES WITH SOLUTION}}$$

Example 1: Find the relative integration of x for function $\cos^2 x \sin x$.

Solution: $\int \cos^2 x \sin x \, dx$ assumed $\cos x = t$

$$\sin x \, dx = -dt = \int -t^2 \, dt$$

$$= -\frac{t^3}{3} + c = -\frac{\cos^3 x}{3} + c.$$ **Ans.**

Example 2: Find the value of $\int \dfrac{x^8 \, dx}{(1 - x^3)^{1/3}}$.

Solution: Assumed $\qquad 1 - x^3 = t$

$\therefore \qquad\qquad\qquad -3x^2 \, dx = dt$

$$x^2 \, dx = -\frac{dt}{3}$$

$$\int \frac{x^8 \, dx}{(1 - x^3)^{1/3}} = \int \frac{(x^3)^2 \, x^2 \, dx}{(1 - x^3)^{1/3}}$$

$$= -\frac{1}{3} \int \frac{(1 - t)^2 \, dt}{t^{1/3}}$$

$$= -\frac{1}{3} \int \frac{(t^2 - 2t + 1) \, dt}{t^{1/3}}$$

$$= -\frac{1}{3} \int (t^{5/3} - 2t^{2/3} + t^{-1/3}) \, dt$$

$$= -\frac{1}{3} \left[\frac{t^{8/3}}{\frac{8}{3}} - \frac{2t^{5/3}}{\frac{5}{3}} + \frac{t^{2/3}}{\frac{2}{3}} \right] + c$$

$$= -\frac{1}{3} \left[\frac{3}{8} (1 - x^3)^{8/3} - \frac{6}{5} (1 - x^3)^{5/3} + \frac{3}{2} (1 - x^2)^{2/3} \right] + c$$

$$= -\frac{1}{8} (1 - x^3)^{8/3} + \frac{2}{5} (1 - x^3)^{5/3} - \frac{1}{2} (1 - x^2)^{2/3} + c.$$ **Ans.**

Example 3: Find the value of $\int \dfrac{dx}{x + \sqrt{x}}$.

Solution: Considered, $\qquad \sqrt{x} + 1 = t$

$\therefore \qquad\qquad\qquad \dfrac{1}{2\sqrt{x}} \, dx = dt$

Note

$$\therefore \qquad \frac{dx}{\sqrt{x}} = 2\, dt$$

$$\int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(\sqrt{x} + 1)}$$

$$= \int \frac{2\,dt}{t} = 2 \log t + c$$

$$= 2 \log |\ \sqrt{x}\ + 1\ | + c. \qquad\qquad \textbf{Ans.}$$

Example 4: Find the value of $\int \sin\left(2x + \dfrac{\pi}{2}\right)\, dx.$

Solution: $\qquad \int \sin\left(2x + \dfrac{\pi}{2}\right)\, dx = \int \cos 2x\, dx$

$$= \frac{\sin 2x}{2} + c. \qquad\qquad \textbf{Ans.}$$

Example 5: Find the value of $\int \dfrac{\sin x}{\sqrt{9 - \cos^2 x}}\, dx.$

Solution: Considered $\qquad \cos x = t$

$$\therefore \qquad - \sin x\, dx = dt$$

$$\therefore \qquad \sin x\, dx = - dt$$

$$\int \frac{\sin x\, dx}{\sqrt{9 - x^2}} = \int \frac{- dt}{\sqrt{9 - t^2}}$$

$$= \int \frac{- dt}{\sqrt{3^2 - t^2}}, \text{ formula } \int \frac{- dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$= - \sin^{-1} \frac{t}{3} + c$$

$$= - \sin^{-1}\left(\frac{\cos x}{3}\right) + c. \qquad\qquad \textbf{Ans.}$$

Example 6: Find the value of $\int \dfrac{\log x}{x}\, dx.$

Solution: Consider, $\qquad \int \dfrac{\log x}{x}\, dx$

$$\log x = t$$

$$\frac{1}{x}\, dx = dt \qquad\qquad\qquad\qquad \text{(by relative differentation of } x\text{)}$$

$$= \int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}\,(\log x)^2 + c. \qquad\qquad \textbf{Ans.}$$

Example 7: Find the value of $\int \dfrac{x^7}{1+x^{16}} \, dx.$

Solution: Considered that $x^8 = t$ to $8x^7 \, dx = dt$

$$\int \frac{x^7}{1+x^{16}} \, dx \;=\; \int \frac{x^7}{1+(x^8)^2} \, dx \;=\; \frac{1}{8} \int \frac{1}{1+t^2} \, dt$$

By the use of the formula, $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + c$

$$= \frac{1}{8} \tan^{-1} t + c = \frac{1}{8} \tan^{-1} x^8 + c. \hspace{2cm} \textbf{Ans.}$$

Example 8: Find the value of $\int x \sin^3 x^2 \cos x^2 \, dx.$

Solution: Assume that, $\quad \sin x^2 = t \;\Rightarrow\; (\cos x^2) \, 2x \, dx = dt$

Or $\hspace{3cm} x \cos x^2 \, dx = \dfrac{1}{2} \, dt$

$$\int x \sin^3 x^2 \cos x^2 \, dx \;=\; \int (\sin x^2)^3 \cdot x \cos x^2 \, dx = \frac{1}{2} \int t^3 \, dt$$

$$= \frac{1}{2} \frac{t^4}{4} + c = \frac{1}{8} (\sin x^2)^4 + c = \frac{1}{8} \sin^4 x^2 + c. \hspace{1.5cm} \textbf{Ans.}$$

Example 9: Find the value of $\int \dfrac{ax^2}{1-2x^3} \, dx.$

Solution: Considered $\int \dfrac{ax^2}{1-2x^3} \, dx,\ 1 - 2x^3 = t$

$$-6x^2 \, dx = dt$$

$$x^2 \, dx = -\frac{1}{6} \, dt = a \int \frac{-\dfrac{1}{6} dt}{t}$$

$$= -\frac{a}{6} \log t + c = -\frac{a}{6} \log (1 - 2x^3) + c. \hspace{1.5cm} \textbf{Ans.}$$

Example 10: Find the value of $\int \dfrac{(1+\log x)^2}{x} \, dx.$

Solution: Consider, $1 + \log x = t$ then $\dfrac{1}{x} \, dx = dt$

Note

$$\therefore \qquad \int \frac{(1 + \log x)^2}{x} \, dx = \int t^2 \, dt = \frac{t^{2+1}}{2+1} + c \implies \frac{t^3}{3} + c$$

$$= \frac{(1 + \log x)^3}{3} + c. \qquad \text{Ans.}$$

Task   Find the value of $\int \dfrac{1}{x \, (1 + \log x)^n}$. $\left[\textbf{Ans.} : \dfrac{1}{1-n} \, (1 + \log x)^{-(n-1)} + c.\right]$

Questionnaire 13.2

Find the value of the following integrations:

1. $\int x^3 \cos x^4 \, dx.$

2. $\int \dfrac{x^2}{\sqrt{1 + x^3}} \, dx.$

3. $\int \dfrac{3x^2}{9^2 + x^3} \, dx.$

4. $\int n x^{n-1} \cos x^n \, dx.$

5. $\int \dfrac{x^2}{1 + x^6} \, dx.$

6. $\int \dfrac{x}{\sqrt{1 - x^4}} \, dx.$

7. $\int \dfrac{x}{1 + x^4} \, dx.$

8. $\int \dfrac{x \, dx}{\sqrt{1 + x^2}} \, dx.$

9. $\int \dfrac{x^{n-1}}{a + bx^n} \, dx.$

10. $\int \dfrac{x \, dx}{(1 + x^2)^{3/2}} \, dx.$

11. $\int \dfrac{x^{m-1}}{\sqrt{1 - x^m}} \, dx.$

12. $\int \dfrac{2x^3}{4 + x^8} \, dx.$

13. $\int \dfrac{x^3 \, dx}{(4 - x^4)^2}.$

14. $\int x^{-1/2} \cosec^2 \sqrt{x} \, dx.$

15. $\int \cosec^2 x \sqrt{\cot x} \, dx.$

16. $\int \dfrac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} = dx.$

17. $\int \dfrac{x^2}{16 + 25x^6} \, dx.$

18. $\int e^x \, (1 + x) \cosec^2 (xe^x) \, dx.$

19. $\int \dfrac{x^4 + 1}{x^6 + 1} \, dx.$

20. $\int x \cos^3 x^2 \sin x^2 \, dx.$

$$\boxed{\textbf{Answers}}$$

1. $\frac{1}{4}\sin x^4 + c$

2. $\frac{2}{3}\sqrt{1+x^3} + c$

3. $\log|9^2 + x^3| + c$

4. $\sin x^n + c$

5. $\frac{1}{3}\tan^{-1} x^2 + c$

6. $\frac{1}{2}\sin^{-1} x^2 + c$

7. $\frac{1}{2}\tan^{-1} x^2 + c$

8. $\sqrt{1+x^2} + c$

9. $\frac{1}{nb}\log|a + bx^n| + c$

10. $-\frac{1}{\sqrt{1+x^2}} + c$

11. $-\frac{2}{m}\sqrt{1-x^m} + c$

12. $\frac{1}{4}\tan^{-1}\frac{1}{2}x^4 + c$

13. $\frac{1}{4}\cdot\frac{1}{4-x^4} + c$

14. $-2\cot\sqrt{x} + c$

15. $-\frac{2}{3}(\cot x)^{3/2} + c$

16. $2\sin e^{\sqrt{x}} + c$

17. $\frac{1}{60}\tan^{-1}\frac{5x^3}{4} + c$

18. $-\cot(xe^x) + c$

19. $\tan^{-1} x + \frac{1}{3}\tan^{-1} x^3 + c$

20. $-\frac{1}{8}\cos^4 x^2 + c.$

Self Assessment

1. Fill in the blanks:

1.   In substitution operation .... is done by changing the given integrand in standard formula.

2. $\int \sin(ax + b)\, dx = -\frac{1}{a}\cos(\text{.........}) + c$

3. $\int \sec^2 ax + dx = \frac{\text{.........}}{a} + c$

4.   If in the function of x^n product of .... is given, then integration of that function may be done by assuming $x^n = t$.

13.4 Integrand

If integrand is a form of $\phi[f(x)] f'(x)$ i.e., it is a function of any amount $f(x)$ and product of integral coefficient $f(x)$ of this same amount $f'(x)$ or may be written in this form, then we do integration considering this amount $f(x)$ equal to t.

Note

Notes   If $f(x) = t$ then $f'(x) \, dx = dt$

$\therefore$ $\int \phi \, [f(x)] \, f'(x) \, dx \; = \; \int \phi(t) \, dt = \psi(t)$ (Assume)

$= \psi \, [\phi(x)], \, t$ (on putting the value of t)

EXAMPLES WITH SOLUTION

Example 1: Find the Value of $\int \sin^4 x \cos x \, dx$.

Solution: Consider, $\sin x = t \Rightarrow \cos x \, dx = dt$.

Hence, $\int \sin^4 x \cdot \cos x \, dx \; = \; \int t^4 \, dt$

$$= \frac{t^5}{5} + c = \frac{1}{5} \sin^5 x + c.$$ **Ans.**

Example 2: Find the Value of $\int \cot^3 \theta \cdot \operatorname{cosec}^2 \theta \, d\theta$.

Solution: Consider, $\cot \theta = x$ then, $- \operatorname{cosec}^2 \theta \, d\theta = dx$

$\Rightarrow$ $\operatorname{cosec}^2 \theta \, d\theta = - \, dx$

$\therefore$ $\int \cot^3 \theta \operatorname{cosec}^2 \theta \, d\theta \; = \; - \int x^3 \, dx \; = - \dfrac{x^4}{4} + c$

$$= - \frac{\cot^4 \theta}{4} + c.$$ **Ans.**

Example 3: Find the Value of $\int \dfrac{4 \sin^{-1} x}{(1 - x^2)} \, dx$.

Solution: Consider, $\sin^{-1} x = t$, then, $\dfrac{dx}{\sqrt{(1 - x^2)}} = dt$

$\therefore$ $\int \dfrac{4 \sin^{-1} x \, dx}{\sqrt{(1 - x^2)}} \; = \; \int 4t \, dt = 4 \times \dfrac{1}{2} t^2 + c = 2 \; (\sin^{-1} x)^2 + c.$ **Ans.**

Example 4: Find the Value of $\int \dfrac{\cos^2 (\log x)}{x} \, dx$.

Solution: Consider, $\log x = t \;\; \Rightarrow \;\; \dfrac{1}{x} \, dx = dt$

Note

$\therefore \qquad \int \dfrac{\cos^2 (\log x)}{x} \, dx = \int \cos^2 t \, dt = \dfrac{1}{2} \int (1 + \cos 2t) \, dt$

$= \dfrac{1}{2} \left(t + \dfrac{\sin 2t}{2} \right) + c$

$= \dfrac{1}{2} \left[\log x + \dfrac{\sin 2 (\log x)}{2} \right] + c.$ **Ans.**

Questionnaire 13.3

Very Short Answer Questions:

Find the Value of the Following:

1. (a) $\int \sin x \, \cos x \, dx$
(b) $\int \sin^2 x \, \cos x \, dx$

(c) $\int \cos^2 x \, \sin x \, dx$
(d) $\int \cot^2 x \, \mathrm{cosec}^2 x \, dx$

(e) $\int \sec^p x \, \tan x \, dx$

2. (a) $\int \dfrac{(\log_e x)^2}{x} \, dx$
(b) $\int \dfrac{\cos (\log_e x)}{x} \, dx$

(c) $\int \dfrac{1 + \log_e x}{x} \, dx$
(d) $\int \dfrac{1}{x \cos^2 (\log_e x)} \, dx$

3. (a) $\int e^x \cos e^x \, dx$
(b) $\int e^x (a + be^x)^n \, dx$

(c) $\int \dfrac{e^{m \sin^{-1} x}}{\sqrt{1 - x^2}} \, dx$
(d) $\int e^{\tan x} \sec^2 x \, dx$

4. (a) $\int \dfrac{\tan^{-1} x}{(1 + x^2)^{3/2}} \, dx$
(b) $\int \dfrac{\tan^{-1} x}{1 + x^2} \, dx$

(c) $\int \dfrac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}} \, dx$

5. $\int x \sqrt{x^2 - 1} \, dx$
6. $\int \dfrac{x \tan^{-1} x^2}{1 + x^4} \, dx$

7. $\int \dfrac{1}{\sqrt{x} \, [\sqrt{x} + 1]} \, dx.$

Note

$\boxed{\textbf{Answers}}$

1. (a) $\dfrac{1}{2}\sin^2 x + c$

   (b) $\dfrac{1}{3}\sin^3 x + c$

   (c) $-\dfrac{1}{3}\cos^3 x + c$

   (d) $-\dfrac{1}{3}\cot^3 x + c$

   (e) $\dfrac{1}{p}\sec^p x + c$

2. (a) $\dfrac{1}{3}(\log_e x)^3 + c$

   (b) $\sin(\log_e x) + c$

   (c) $\dfrac{1}{2}(1 + \log_e x)^2 + c$

   (d) $\tan(\log_e x) + c$

3. (a) $\sin e^x + c$

   (b) $\dfrac{1}{b(n+1)}(a + be^x)^{n+1} + c$

   (c) $\dfrac{1}{m} e^{m\sin^{-1} x} + c$

   (d) $e^{\tan x} + c$

4. (a) $\dfrac{1 + x\tan^{-1} x}{\sqrt{1 + x^2}} + c$

   (b) $\dfrac{1}{2}(\tan^{-1} x)^2 + c$

   (c) $\dfrac{1}{3}(\sin^{-1} x)^3 + c$

5. $\dfrac{(x^2 - 1)^{3/2}}{3} + c$

6. $\dfrac{1}{4}\left(\tan^{-1} x^2\right)^2 + c$

7. $2\log|1 + \sqrt{x}| + c$

13.5 Integration of the Fraction whose Numerator is Integral Co-efficient of the Denominator

Consider that we have to find the value of $\displaystyle\int \dfrac{f'(x)}{f(x)}$ where, $f'(x)$, is integral co-efficient of $f(x)$

On putting $f(x) = t$, $f'(x)\,dx = dt$

$\therefore$ $$\int \dfrac{f'(x)}{f(x)}\,dx = \int \dfrac{dt}{t} = \log t = \log[f(x)]$$

Hence, $$\int \dfrac{f'(x)}{f(x)}\,dx = \log f(x) + c$$

Did u know?   If numerator (N^r) of any fraction is integral co-efficient of denominator (D^r), then integration of that fraction will be equal to Log of the denominator.

$\therefore$ $\displaystyle\int \dfrac{d(\text{denominator})}{(\text{denominator})}\,dx = \log(\text{denominator})$

EXAMPLES WITH SOLUTION

Example 1: Find the Value of $\int \dfrac{e^x}{1+e^x}\, dx.$

Solution: Consider, $1 + e^x = t$ then $e^x\, dx = dt$

∴ $$\int \frac{e^x}{1+e^x}\, dx = \int \frac{1}{t}\, dt = \log\, |t| + c$$

$$= \log\, |1 + e^x| + c. \qquad \textbf{Ans.}$$

Example 2: Find the Value of $\int \dfrac{x^4}{x^5+4}\, dx.$

Solution: Consider, $\qquad x^5 + 4 = t$

∴ $\qquad\qquad\qquad 5x^4\, dx = dt$

Or $\qquad\qquad\qquad x^4\, dx = \dfrac{1}{5}\, dt$

∴ $$\int \frac{x^4}{x^5+4}\, dx = \frac{1}{5} \int \frac{1}{t}\, dt = \frac{1}{5}\, \log\, |t| + c$$

$$= \frac{1}{5}\, \log\, |x^5 + 4| + c. \qquad \textbf{Ans.}$$

Example 3: Find the Value of $\int \dfrac{e^x - e^{-x}}{e^x + e^{-x}}\, dx.$

Solution: Consider, $\qquad e^x + e^{-x} = t \;\Rightarrow\; (e^x - e^{-x})\, dx = dt$

so that $$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int \frac{dt}{t} = \log\, |t| + c = \log\, |e^x + e^{-x}| + c. \qquad \textbf{Ans.}$$

Example 4: Find the Value of $\int \dfrac{dx}{e^x - 1}.$

Solution: $$\int \frac{1}{e^x - 1}\, dx = \int \frac{e^{-x}}{1 - e^{-x}}\, dx$$

[on multiplying e^{-x} with the numerator and denominator]

$$= \int \frac{1}{t}\, dt, \text{ Assumed, } 1 - e^{-x} = t \Rightarrow e^{-x}\, dx = dt$$

$$= \log\, |t| + c = \log\, |1 - e^{-x}| + c. \qquad \textbf{Ans.}$$

Task Find the Value of $\int \dfrac{a}{b + ce^x}\, dx.$ $\qquad\qquad$ (**Ans.:** $\dfrac{a}{b}\, \log\, |be^{-x} + c| + c'.$)

13.6   Integration of Some Standard Function By Substitution

(i) Integration of tan x and cot x

(1) $$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Consider cos $x = t$, on differentiation, $- \sin x \, dx = dt$

∴ $$\int \tan x \, dx = - \int \frac{dt}{t} = - \log |t| + c$$

$$= - \log |\cos x| + c = \log |\sec x| + c$$

∴ $$\boxed{\int \tan x \, dx = - \log |\cos x| + c \text{ or } \log |\sec x| + c}$$

(2) $$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$$

Consider sin $x = t$, on differentiation, $\cos x \, dx = dt$

∴ $$\int \cot x \, dx = \int \frac{dt}{t} = \log |t| + c = \log |\sin x| + c$$

∴ $$\boxed{\int \cot x \, dx = \log |\sin x| + c.}$$

***

| Questionnaire 13.4 |

Very Short Answer Questions.

Find the value of the following:

1. (a) $\displaystyle \int \frac{ax^{n-1}}{x^n + b} \, dx$ $\qquad\qquad$ (b) $\displaystyle \int \frac{3x^2}{(x^3 + 4)^5} \, dx$

(c) $\displaystyle \int \frac{x^2}{1 - 2x^3} \, dx$ $\qquad\qquad$ (d) $\displaystyle \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx.$

***

| Answers |

1. (a) $\dfrac{a}{n} \log |x^n + b| + c$ $\qquad\qquad$ (b) $-\dfrac{1}{4}(x^3 + 4)^{-4} + c$

(c) $-\dfrac{1}{6} \log_e |1 - 2x^3| + c$ $\qquad\qquad$ (d) $2e^{\sqrt{x}} + c$

Self Assessment

2. Multiple Choice Questions:

5.   Value of $\int \sin x \cos x \, dx$ is :

(a) $\dfrac{1}{2} \sin^2 x + c$ $\qquad\qquad\qquad\qquad$ (b) $\dfrac{1}{2} \cos^2 x + c$

(c) $\dfrac{1}{2} \operatorname{cosec}^2 x + c$ $\qquad\qquad\qquad\qquad$ (d) None of these.

6. Value of $\int \cot^3 \theta \cdot \cosec^2 \theta \, d\theta$ will be,

    (a) $\dfrac{\cot^4 \theta}{4} + c$

    (b) $-\dfrac{\cot^4 \theta}{4} + c$

    (c) $\dfrac{\cosec^4 \theta}{4} + c$

    (d) None of these.

7. $\sin (\log_e x) + c$ is solution for which of the following:

    (a) $\int \sin^2 x \cos^2 x \, dx$

    (b) $\int \cot^2 x \cosec^2 x \, dx$

    (c) $\int \dfrac{\cos (\log_e x)}{x} \, dx$

    (d) None of these.

8. $\log |1 + e^x| + c$ is solution for which of the following:

    (a) $\int \dfrac{e^x}{1 + e^x} \, dx$

    (b) $\int \dfrac{1 + e^x}{e^x} \, dx$

    (c) $\int e^x \, dx$

    (d) $\int \dfrac{-e^x}{1 + e^x} \, dx$.

13.7 Summary

- There are two main methods to find the integration of any function:
    (i) Integration by Substitution
    (ii) Integration by Parts
- In this operation integration is done by changing the given integrand in standard formula.
- If integrand is a form of $\phi [f(x)] f' (x)$ i.e., it is a function of any amount $f(x)$ and product of integral coefficient $f(x)$ of this same amount $f \phi (x)$ or may be written in this form, then we do integration considering this amount equal to t.

13.8 Keywords

- *Substitution:* Replacement
- *Methods:* Manner, process

13.9 Review Questions

1. Find the Value of $\int \sec^2 3x \, dx$ (**Ans.:** $\dfrac{\tan 3x}{3} + c$)

2. Calculate the value of $\int \sin^3 \theta \, d\theta$ (**Ans.:** $-\dfrac{3}{4} \cos \theta + \dfrac{1}{12} \cos 3\theta + c$)

3. Find the relative integration of x for the function $\cos^2 x \sin x$ (**Ans.:** $-\dfrac{\cos^3 x}{3} + c$)

Note

4. Find the value of $\int \sin\left(2x + \dfrac{\pi}{2}\right) dx$ (**Ans.:** $\dfrac{\sin 2x}{2} + c$)

5. Find the Value of $\int \dfrac{4 \sin^{-1} x}{(1 - x^2)} dx$ (**Ans.:** $2 (\sin^{-1} x)^2 + c$)

Answers: Self Assessment

1. Integration
2. $ax + b$
3. $\tan ax$
4. x^{n-1}
5. (a)
6. (b)
7. (c)
8. (a)

13.10 Further Readings

Books

Mathematical Economics – Michael Harrison, Patric Walderen.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics and Finance – Martin Norman.

Unit 14: Integration as a Summation

CONTENTS

Objectives

Introduction

14.1   Finding Integral of Multiplication of two Functions (Divided Integrals)

14.2   Integration by Partial Fraction

14.3   Two Standard Integrals

14.4   Summary

14.5   Keywords

14.6   Review Questions

14.7   Further Readings

Objectives

After reading this unit, students will be able to :

- Know the Integral of Multiplication of two Functions.
- Find out the Integration by Partial Fraction.
- Get the Information about two Standard Integrals.

Introduction

We should be careful before using the integration rule and we should ensure that by doing the integration operation once or twice it takes the form of regular integration or is modify such that it could be integrated. If one function of integral is known then this known function will always be considerd as first function. If integrals becomes undo form with negative sign in right hand side then equation should be solved with integration rule.

14.1   Finding Integral of Multiplication of two Functions (Divided Integrals)

if the multiplication of $f(x)$ and $\phi(x)$ is $f(x).\,\phi(x)$ the differentiation is–

$$\frac{d}{dx}\{f(x)\,.\,\phi(x)\} = \ f(x)\,\phi'(x) + f'(x)\,\phi(x)$$

Integration of both side with respect of x

$$f(x).\,\phi(x) = \ \int f(x)\,\phi'(x)\,dx + \int f'(x)\,\phi(x)\,dx$$

Or
$$\int f(x)\,\phi'(x)\,dx \ = \ f(x).\,\phi(x) - \int f'(x)\,\phi(x)\,dx$$

$$= \ f(x)\int \phi'(x)\,dx - \int\{f'(x)\int \phi'(x)\,dx\}\,dx$$

By the definition of $\phi(x)$, $\phi'(x)$ is the integral with respect to x, so $\int \phi'(x)\,dx$ is put at the place of $\phi'(x)$.

Note

For expressing the above conclusion in easier form, on putting $f_1(x)$ and $f_2(x)$ instead of $f(x)$ and $\phi'(x)$ respectively, $\because$

$$\int f_1(x).\, f_2(x)\, dx \;=\; f_1(x) \int f_2(x)\, dx - \int \left\{ \frac{d}{dx} f_1(x). \int f_2(x)\, dx \right\} dx$$

Did u know?   Integral of Multiplication of two functions = First function x integral of second function–[differentiation coefficient of first function × integral of integration of second function].

It should be seen before using the above rule that on using above process once or two times the integrant would take the form of any authentic integrant or come in such form which could be integrated. The success of this function much more depends on the study of first and second function. The first function is so selected that the second part of RHS could be easily and successfully integrated. Therefore, there should take care in the selection of first function.

In the use of this method, few important things are as following:

(1) If between two function there is a function whose integration we don't know, then that function must be considered as the first function.

In $\int x.(\log x)\, dx$, we don't know the integral of log x. Therefore log x should be considered as the first function.

(2) if both the functions were so that the integral of both is known then among those if there is any function of x^n form, then always consider it as the first function.

For example, in $\int \log x\, dx, \int \tan^{-1} x\, dx$, x² must be considered as the first function.

(3) To know the integral of $\int \log x\, dx, \int \tan^{-1} x\, dx$, etc. integration is done with considering '1' as the second function. As $\int \log x\, dx = \int (\log x).\, 1\, dx$ etc.

(4) The divided integration rule should be used more than once according to the use.

(5) If the integral in RHS comes in its initial form with the negative sign then questions should be solved with using the rules of solving the equations.

Notes   We do divided integration with considering the first coming and second coming function, in the word 'ILATE', as the first and second function respectively.

Where I – is for the Inverse trigonometrical functions (as ($\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.).

L – is for the Logarithmic functions (as log x, log ($x^2 \pm a^2$ etc.).

A – is for the algebraic functions (as $x, x + 1, 2x, \sqrt{x}$, etc.).

T – is for the Trigonometrical functions. (as sin x, cos x, tan x etc.).

E – is for the Exponential functions (as $a^x, e^x, 2^x, 10^x, 3^{-x}$, etc.).

As there is no any authentic method to select the functions during the divided integration, but the students can do the divided integration on selecting the function with the help of above mentioned 'ILATE'.

$$\boxed{\textbf{EXAMPLES WITH SOLUTION}}$$

Example 1: Find the Value $\int \log_e x \, dx$ **.**

Solution: Let's consider that here the second function is unit because if $\log_e x$ is considered as second function then it can't be integrated.

$$\int \log_e x \, dx = \int \log_e x \cdot 1 \, dx$$

$$= \log_e x \cdot \int 1 \, dx - \int \left[\frac{1}{x} \int 1 \, dx \right] \quad \text{(On doing divided inegration)}$$

$$= \log_e x \cdot x - \int \frac{1}{x} x \, dx = x \log_e x - \int 1 \, dx$$

$$= x \log_e x - x + c. \qquad\qquad \textbf{Ans.}$$

We can also write $x \log \dfrac{x}{e}$ of $x \log_e x - x$ if want because $\log_e e = 1$.

Example 2: Find the Value $\int x \sin x \, dx$ **.**

Solution: There are two factors of integrant, the integrant of both x and $\sin x$ is known but among these the function x is of x^n type. Therefore it should be considered as the first function.

Therefore, $\qquad \int x \sin x \, dx = x \left[\int \sin x \, dx \right] - \int \left[\left(\frac{d}{dx} x \right) \int \sin x \, dx \right] dx$

$$= x (- \cos x) - \int 1 . (- \cos x) \, dx$$

$$= - x \cos x + \int \cos x \, dx$$

$$= - x \cos x + \sin x + c. \qquad\qquad \textbf{Ans.}$$

Example 3: Find the Value $\int x^2 \log x \, dx$ **.**

Solution: $\qquad \int x^2 \log x \, dx = \int (\log x) \, x^2 \, dx$

$$= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} \, dx$$

$$= \frac{1}{3} x^3 \log x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{1}{3} x^3 \log x - \frac{1}{3} \left(\frac{1}{3} x^3 \right) + c$$

$$= \frac{1}{3} x^3 \log x - \frac{1}{9} x^3 + c. \qquad\qquad \textbf{Ans.}$$

Note

Example 4: Find the Value $\int x^n \log x$.

Solution: $\int x^n \log x \, dx = \int \log x \, x^n \, dx$

[log x, the integration of which is unknown, consider is as the first function]

$$= \log x \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} \, dx$$

(On doing divided inegration)

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^n \, dx = \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \frac{x^{n+1}}{n+1}$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} = \frac{x^{n+1}}{n+1} \left[\log x - \frac{1}{n+1} \right] + c.$$ **Ans.**

Example 5: Find the Value $\int x^2 \cos x \, dx$.

Solution: Let's assume that x^2 is the first function.

$\therefore$ $\int x^2 \cdot \cos x \, dx = x^2 \cdot \sin x - \int 2x \cdot \sin x \, dx.$

Take x as the first function in the second integral then integrate by factors.

$\therefore$ $\int x^2 \cos x \, dx = x^2 \sin x - 2[x \, (-\cos x) - \int 1. \, (-\cos x) \, dx]$

$$= x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$= (x^2 - 2) \sin x + 2x \cos x + c.$$ **Ans.**

Example 6: Find the Value $\int e^x \sin x \, dx$.

Solution: Assume that e^x is the first function.

$\therefore$ $\int e^x \sin x \, dx = e^x \, (- \cos x) - \int e^x \, (- \cos x) \, dx$

$$= -e^x \cos x + \int e^x \cos x \, dx.$$

Again by the integration by factors,

$$\int e^x \sin x \, dx = -e^x \cos x + [e^x \sin x - \int e^x \sin x \, dx]$$

Shift the last term towards left. Then on dividing by 2,

$$\int e^x \sin x \, dx = \frac{1}{2} e^x \, (\sin x - \cos x) + c.$$ **Ans.**

Example 7: Find the Value $\int \dfrac{1}{x^2} \log x \, dx$.

Solution:

$$\int \dfrac{1}{x^2} \log x \, dx = \int \log x \cdot \dfrac{1}{x^2} \, dx$$

$$= \log x \left(\dfrac{-1}{x}\right) - \int \left(\dfrac{1}{x}\right)\left(\dfrac{-1}{x}\right) dx$$

$$= \dfrac{-\log x}{x} + \int \dfrac{1}{x^2} \, dx$$

$$= \dfrac{-\log x}{x} - \dfrac{1}{x} + c = \dfrac{-(1 + \log x)}{x} + c.$$ **Ans.**

Example 8: Integrate the function $x^2 \, a^x$ **with respect to** x**.**

Solution: Assume that $= \int x^2 a^x \, dx$

Here, the integrant is the multiplication of two functions x^2 and a^x. where x^2 is an algebraic function and a^x is an exponential function and firstly A comes in **ILATE**. Therefore we'll do divided integration on taking x^2 and a^x as the first and second function respectively.

$$I = x \int a^x \, dx - \int \left\{ \dfrac{d}{dx}(x) \cdot \int a^x \, dx \right\} dx$$

$$= x \left(\dfrac{a^x}{\log a}\right) - \int 1 \cdot \left(\dfrac{a^x}{\log a}\right) dx$$

$$= x \dfrac{a^x}{\log a} - \dfrac{1}{\log a} \int a^x \, dx$$

$\therefore$ $$I = x \cdot \dfrac{a^x}{\log a} - \dfrac{a^x}{(\log a)^2} + c.$$ **Ans.**

Example 9: Find the Value $\int \log \{x + \sqrt{x^2 + a^2}\} \, dx$.

Solution: $\int \log \{x + \sqrt{x^2 + a^2}\} \, dx$

$$= \int \log \{x + \sqrt{x^2 + a^2}\} \cdot 1 \, dx$$

$$= \log \{x + \sqrt{x^2 + a^2}\} \, x - \int \dfrac{1}{(x + \sqrt{x^2 + a^2})} \cdot \left\{1 + \dfrac{1(2x)}{2\sqrt{x^2 + a^2}}\right\} x \, dx$$

$$= \log \{x + \sqrt{x^2 + a^2}\} \, x - \int \dfrac{1}{(x + \sqrt{x^2 + a^2})} \left\{\dfrac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right\} x \, dx$$

$$= \log \{x + \sqrt{x^2 + a^2}\} \, x - \int \dfrac{x \, dx}{\sqrt{x^2 + a^2}}$$

Note

$$= \log\{x + \sqrt{x^2 + a^2}\}\, x - \frac{1}{2}\int \frac{2x\, dx}{\sqrt{x^2 + a^2}}$$

$$= \log\{x + \sqrt{x^2 + a^2}\}\, x - \frac{1}{2}\{2\sqrt{x^2 + a^2}\} + c$$

$$= \log\{x + \sqrt{x^2 + a^2}\}\, x - \sqrt{x^2 + a^2} + c$$

$$= x\log\{x + \sqrt{x^2 + a^2}\} - \sqrt{x^2 + a^2} + c. \qquad\qquad \textbf{Ans.}$$

Example 10: Find the Value $\int \tan^{-1} x\, dx$.

Solution:

$$\int \tan^{-1} x\, dx = \int (\tan^{-1} x).\, 1\, dx$$

$$= (\tan^{-1} x)\, x - \int \frac{1}{1 + x^2}.\, x\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1 + x^2}\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{dt}{t}, \qquad\qquad \text{(on putting } 1 + x^2 = t \text{ and } 2x\, dx = dt)$$

$$= x\tan^{-1} x - \frac{1}{2}\log|t| + c$$

$$= x\tan^{-1} x - \frac{1}{2}\log|1 + x^2| + c. \text{ (On putting the value of } t) \qquad \textbf{Ans.}$$

Task   Find the value of $\int x^2 \cos x\, dx$ [**Ans:** $(x^2 - 2)\sin x + 2x\cos x + c$]

Example 11: Find the Value $\int x\tan^{-1} x\, dx$.

Solution:

$$\int x\tan^{-1} x\, dx = \int (\tan^{-1} x).\, x\, dx$$

$$= (\tan^{-1} x).\frac{x^2}{2} - \int \frac{1}{1 + x^2}.\frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{1 + x^2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + c$$

$$= \frac{1}{2} \tan^{-1} x + \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + c$$

$$= \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{x}{2} + c. \qquad \text{Ans.}$$

Example 12: Find the Value $\int \frac{x}{1 + \cos x} \, dx$.

Solution: $\qquad \int \frac{x}{1 + \cos x} \, dx = \int \frac{x(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} \, dx$

$$= \int \frac{x - x \cos x}{\sin^2 x} \, dx$$

$$= \int \frac{x}{\sin^2 x} \, dx - \int \frac{x \cos x}{\sin^2 x} \, dx$$

$$= \int x \, \text{cosec}^2 \, x \, dx - \int x \cot x \, \text{cosec} \, x \, dx$$

$$= x \, (- \cot x) - \int 1. \, (- \cot x) \, dx$$

$$\qquad - [x \, (- \text{cosec} \, x) - \int 1. \, (- \text{cosec} \, x) \, dx] + c)$$

$$= - x \cot x + \log |\sin x| + x \, \text{cosec} \, x - \log \left| \tan \frac{x}{2} \right| + c. \qquad \text{Ans.}$$

Example 13: Find the Value $\int \cos^{-1} \frac{1}{x} \, dx$.

Solution: Assuming that $\qquad I = \int \cos^{-1} \frac{1}{x} dx, \qquad \qquad \left[\because \cos^{-1} \frac{1}{x} = \sec^{-1} x \right]$

$$= \int \sec^{-1} x \, dx$$

$$= \int (\sec^{-1} x). \, 1 \, dx$$

Integration on taking $\sec^{-1} x$ and 1 as the first and second function respectively

$$= \sec^{-1} x \int dx - \int \left\{ \frac{d}{dx} (\sec^{-1} x). \int dx \right\} dx$$

$$= x \sec^{-1} x - \int \frac{1}{x \sqrt{x^2 - 1}} . \, x \, dx$$

$$= x \sec^{-1} x - \int \frac{1}{\sqrt{x^2 - 1}} \, dx$$

$$= x \sec^{-1} x - \log |x + \sqrt{x^2 - 1}| + c. \qquad \text{Ans.}$$

Note

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Questionnaire 14.1

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Long Answer Question.

Find out the value of following integrals:

1. (a) $\int x \log x \, dx$ (b) $\int (\log x)^2 \, dx$

   (c) $\int x^3 \log x \, dx$ (d) $\int \log (1 + x^2) \, dx$

   (e) $\int \sec x \log (\sec x + \tan x) \, dx$

2. (a) $\int x e^{ax} \, dx$ (b) $\int \dfrac{e^{1/x}}{x^3} \, dx$

3. (a) $\int x \, \text{cosec}^2 \, ax \, dx$ (b) $\int x \sec^2 x \, dx$

   (c) $\int x \sec^2 2x \, dx$ (d) $\int x^2 \sin x \, dx$

   (e) $\int \sec^3 x \, dx$ (f) $\int x^2 \sin 2x \, dx$

4. (a) $\int x \sin nx \, dx$ (b) $\int \sin^2 x \, dx$

   (c) $\int x \tan^2 x \, dx$ (d) $\int \sin \sqrt{x} \, dx$

5. (a) $\int \sin^{-1} x \, dx$ (b) $\int \cot^{-1} x \, dx$

6. (a) $\int x^3 \tan^{-1} x \, dx$ (b) $\int \dfrac{\log x}{(1 + x)^2} \, dx$

7. $\int \dfrac{x^2}{(x \sin x + \cos x)^2} \, dx$ 8. $\int \sqrt{x} \, (\log x)^2 \, dx$

9. (a) $\int x \log (1 + x) \, dx$ (b) $\int \log_{10} x \, dx$

10. $\int \cot^{-1} (1 + x^2 - x) \, dx$ 11. $\int x^3 \, e^{x^2} dx$

12. $\int \sin x \log (\sec x + \tan x) \, dx$.

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Answers

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1.   (a) $\dfrac{1}{2} x^2 \log x - \dfrac{x^2}{4} + c$ (b) $x (\log (x))^2 - 2x \log x + 2x + c$

  (c) $\dfrac{1}{4} x^4 \log x - \dfrac{x^2}{16} + c$ (d) $x \log (1 + x^2) - 2 (x - \tan^{-1} x) + c$

2.   (a) $\dfrac{1}{a} x e^{ax} - \dfrac{1}{a^2} e^{ax} + c$ (b) $e^{1/x} \left(1 - \dfrac{1}{x} \right) + c$

3. (a) $\dfrac{1}{a^2}[\log \sin ax - ax \cot x] + c$

   (b) $x \tan x + \log|\cos x| + c$

   (c) $\dfrac{1}{2} x \tan 2x - \dfrac{1}{4} \log|\sec 2x| + c$

   (d) $-x^2 \cos x + 2x \sin x + 2 \cos x + c$

   (e) $\dfrac{1}{2} \sec x \cdot \tan x + \dfrac{1}{4} \log|\sec x + \tan x| + c$

   (f) $-\dfrac{1}{2} x^2 \cos 2x + \dfrac{1}{2} x \sin 2x + \dfrac{1}{4} \cos 2x + c$

4. (a) $\dfrac{1}{n} x \cos nx + \dfrac{1}{n^2} \sin nx + c$

   (b) $\dfrac{x^2}{4} - \dfrac{1}{4} x \sin 2x - \dfrac{1}{8} \cos 2x + c$

   (c) $x \tan x + \log \cos x - \dfrac{x^2}{2} + c$

   (d) $2[-\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x}] + c$

5. (a) $x \sin^{-1} x + \sqrt{1 - x^2} + c$

   (b) $x \cot^{-1} x + \dfrac{1}{2} \log|1 + x^2| + c$

6. (a) $\dfrac{1}{4} x^4 \tan^{-1} x - \dfrac{1}{4}\left(\dfrac{x^3}{3} - x + \tan^{-1} x\right) + c$

   (b) $\dfrac{-\log x}{1 + x} + \log\left|\dfrac{x}{x + 1}\right| + c$

7. $\dfrac{-x}{\cos x\,(x \sin x + \cos x)} + \tan x + c$

8. $\dfrac{2}{3} x^{3/2}[(\log x)^2 - \dfrac{4}{3} \log x + \dfrac{8}{9}] + c$

9. (a) $\dfrac{x^2 - 1}{2} \log(1 + x) - \dfrac{x^2}{4} + \dfrac{x}{2} + c$

   (b) $x (\log_e x - 1) \log_{10} e + c$

10. $2[x \tan^{-1} x - \dfrac{1}{2} \log(1 + x^2)] + c$

11. $\dfrac{1}{2} e^{x^2} (x^2 - 1) + c$

12. $x - \cos x \log(\sec x + \tan x) + c$

14.2 Integration by Partial Fraction

We proved that the integral of difference between one or more functions is equals to the addition or difference of integrals of that function. So the important method of integration is used the above principle to break the additional and difference form of given function. So the methods of partial fraction are used when necessary.

You already have study about the method of break partial fraction in algebra book.

You know by algebra that exponential and quadratic factors of every polynomial can be possible- it may be possible some factors are repeated. So Partial fraction of rational fraction will be like that-

(i) The related partial fraction of non-reaped denominator of single power factor *x-a* is of $\dfrac{A}{x - a}$, where $A \neq a$.

(ii) *R* partial fraction are following with respect to $(x - a)$ factor of repeated.

$$\dfrac{B_1}{x - b} + \dfrac{B_2}{(x - b)^2} + \ldots\ldots + \dfrac{B_r}{(x - b)^r}$$

Note

Here $B \neq 0$.

(iii) The function of $x^2 + px + q$, which are not repeated then respected partial fraction is like that-

$$\frac{Cx + D}{x^2 + px + q}$$

(iv) The factors of $x^2 + px + q$ is repeated r times and the factors are- $(x^2 + px + q)^r$

Then, partial fraction is-

$$\frac{C_1 x + D_1}{x^2 + px + q} + \frac{C_2 x + D_2}{(x^2 + px + q)^2} + \ldots\ldots + \frac{C_r x + D_r}{(x^2 + px + q)^r}$$

Self Assessment

1. Fill in the blanks:

1. If two functions are same and the …………….. of both function known, then if there is any x^n type function, then it always considers first function.

2. If necessity then the formula of ……………………… integration used many times.

3. There is no any…………………… method for the selection of function during divided integration.

4. The integral of addition or differentiation of one function is ………………… to the addition or difference of that function.

5. Exponential an …………………. of every polynomial can be possible.

14.3 Two Standard Integrals

1. Find out the value of $\int \dfrac{1}{x^2 - a^2}\, dx$ **.**

When $x > a$,

Solution: $\because$ $\qquad\qquad \dfrac{1}{x^2 - a^2} = \dfrac{1}{(x + a)(x - a)}$

Let, $\qquad\qquad \dfrac{1}{x^2 - a^2} = \dfrac{A}{x + a} + \dfrac{B}{x - a} = \dfrac{A(x - a) + B(x + a)}{(x + a)(x - a)}$

$\therefore$ $\qquad\qquad A(x - a) + B(x + a) = 1$

$\Rightarrow$ $\qquad\qquad (A + B)x + (B - A)a = 1$

comparing the factors of both sides

$$A + B = 0 \text{ and } (B - A)a = 1$$

$\therefore$ $\qquad\qquad B = \dfrac{1}{2a} \text{ and } A = -\dfrac{1}{2a}$

$\therefore$ $\qquad\qquad \dfrac{1}{x^2 - a^2} = -\dfrac{1}{2a(x + a)} + \dfrac{1}{2a(x - a)} = \dfrac{1}{2a}\left\{\dfrac{1}{x - a} - \dfrac{1}{x + a}\right\}$

$\therefore$ $\int \dfrac{1}{x^2 - a^2}\,dx = \dfrac{1}{2a} \int \left\{ \dfrac{1}{x - a} - \dfrac{1}{x + a} \right\} dx$

$= \dfrac{1}{2a} \int \dfrac{1}{x - a}\,dx - \dfrac{1}{2a} \int \dfrac{1}{x + a}\,dx$

$= \dfrac{1}{2a} \log |x - a| - \dfrac{1}{2a} \log |x + a| + c$

$= \dfrac{1}{2a} \log \left| \dfrac{x - a}{x + a} \right| + c$

$$\boxed{\int \dfrac{1}{x^2 - a^2}\,dx = \dfrac{1}{2a} \log \left| \dfrac{x - a}{x + a} \right| + c, \text{ when } x > a}$$

2. Find out the value of $\int \dfrac{dx}{a^2 - x^2}$.

When $x < a$.

Solution: $\int \dfrac{dx}{a^2 - x^2} = \int \dfrac{dx}{(a + x)\,(a - x)} = \dfrac{1}{2a} \int \left\{ \dfrac{1}{a + x} + \dfrac{1}{a - x} \right\} dx$

$= \dfrac{1}{2a} \left[\log |a + x| - \log |a - x| \right] + c = \dfrac{1}{2a} \log \left| \dfrac{a + x}{a - x} \right| + c$

$\therefore$ $$\boxed{\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a} \log \left| \dfrac{a + x}{a - x} \right| + c}$$

EXAMPLES WITH SOLUTION

Example 1: Find the Value $\int \dfrac{dx}{a^2 x^2 - b^2}$.

Solution: $\int \dfrac{dx}{a^2 x^2 - b^2}\,dx = \dfrac{1}{a^2} \int \dfrac{1}{x^2 - (b/a)^2}\,dx$

$= \dfrac{1}{a^2} \cdot \dfrac{1}{2(b/a)} \log \left| \dfrac{x - (b/a)}{x + (b/a)} \right| + c = \dfrac{1}{2ab} \log \left| \dfrac{ax - b}{ax + b} \right| + c.$ **Ans.**

Example 2: Find the Value $\int \dfrac{dx}{24 - 6x^2}$.

Solution: $\int \dfrac{1}{24 - 6x^2}\,dx = \dfrac{1}{6} \int \dfrac{1}{4 - x^2}\,dx = \dfrac{1}{6} \int \dfrac{1}{2^2 - x^2}\,dx$

$= \dfrac{1}{6} \dfrac{1}{2.2} \log \left| \dfrac{2 + x}{2 - x} \right| + c = \dfrac{1}{24} \log \left| \dfrac{2 + x}{2 - x} \right| + c.$ **Ans.**

Note

Example 3: Find the Value $\int \dfrac{1}{5 - 2x - x^2}\, dx$.

Solution: $\int \dfrac{dx}{5 - 2x - x^2} = \int \dfrac{dx}{5 + 1 - 1 - 2x - x^2} = \int \dfrac{dx}{6 - (1 + 2x + x^2)}$

$$= \int \dfrac{dx}{6 - (x+1)^2} = \int \dfrac{dx}{(\sqrt{6})^2 - (x+1)^2}$$

∵ by formula $\quad \int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a} \log \left| \dfrac{a+x}{a-x} \right| + c$

∴ $\quad \int \dfrac{dx}{5 - 2x - x^2} = \dfrac{1}{2\sqrt{6}} \log \left| \dfrac{\sqrt{6} + (x+1)}{\sqrt{6} - (x+1)} \right| + c.$ **Ans.**

Example 4: Find the Value $\int \dfrac{3x}{(x-1)(x-2)(x-3)}\, dx$.

Solution: $\dfrac{3x}{(x-1)(x-2)(x-3)} = \dfrac{3}{2} \dfrac{1}{(x-1)} - 6 \cdot \dfrac{1}{(x-2)} + \dfrac{9}{2} \cdot \dfrac{1}{(x-3)}$ devided by partial fraction

$\int \dfrac{3x}{(x-1)(x-2)(x-3)}\, dx$

$$= \dfrac{3}{2} \int \dfrac{dx}{x-1} - 6 \int \dfrac{dx}{x-2} + \dfrac{9}{2} \int \dfrac{dx}{x-3}$$

$$= \dfrac{3}{2} \log |(x-1)| - 6 \log |(x-2)| + \dfrac{9}{2} \log |(x-3)| + c. \qquad \textbf{Ans.}$$

Example 5: Find the Value $\int \dfrac{dx}{2x^2 + x - 1}$.

Solution: ∵ $\dfrac{1}{2x^2 + x - 1} = \dfrac{1}{(x+1)(2x-1)}$

Let, $\quad \dfrac{1}{2x^2 + x - 1} = \dfrac{A}{x+1} + \dfrac{B}{2x-1} = \dfrac{A(2x-1) + B(x+1)}{(x+1)(2x-1)}$

∴ $\quad A(2x-1) + B(x+1) = 1$

Or $\quad (2A + B)x + (B - A) = 1$

on comparing the function of both side,

$$2A + B = 0 \text{ and } B - A = 1$$

∴ $A = -\dfrac{1}{3}$ and $B = \dfrac{2}{3}$

$$\dfrac{1}{2x^2 + x - 1} = \dfrac{2}{3} \dfrac{1}{2x-1} - \dfrac{1}{3} \dfrac{1}{x+1}$$

∴ $\quad \int \dfrac{1}{2x^2 + x - 1}\, dx = \dfrac{2}{3} \int \dfrac{1}{2x-1}\, dx - \dfrac{1}{3} \int \dfrac{1}{x+1}\, dx$

$$= \frac{2}{3} \cdot \frac{1}{2} \log|(2x-1)| - \frac{1}{3}\log|(x+1)| + C$$

$$= \frac{1}{3}\log\left|\frac{2x-1}{x+1}\right| + C. \qquad\qquad \textbf{Ans.}$$

Example 6: Find the Value $\int \frac{dx}{x - x^3}$.

Solution: $$\frac{1}{x - x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)}$$

Let, $$\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$\therefore$ $$1 = A(1-x^2) + Bx(1+x) + Cx(2-x) \qquad\qquad …(1)$$

Putting $x = 0$, 1 and –1 in equation (1)

$$A = 1, B = \frac{1}{2}, C = -\frac{1}{2}$$

$\therefore$ $$\int \frac{dx}{x - x^3} = \int \left[\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right] dx$$

$$= \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x| + c. \qquad\qquad \textbf{Ans.}$$

Example 7: Find the Value $\int \frac{dx}{4 - x^2}$.

Solution: $$\int \frac{dx}{4 - x^2} = \int \frac{dx}{(2)^2 - x^2}$$

$$= \frac{1}{2.2}\log\left|\frac{2+x}{2-x}\right| + c, \qquad\qquad \left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{4}\log\left|\frac{2+x}{2-x}\right| + c. \qquad\qquad \textbf{Ans.}$$

Example 8: Find the Value $\int \frac{1}{(x+b)(x^2+a^2)}\, dx.$

Solution: On adding the integrant in partial fraction

$$\frac{1}{(x+b)(x^2+a^2)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+a^2}$$

on solving, $$\frac{1}{(x+b)(x^2+a^2)} = \frac{1}{a^2+b^2}\left(\frac{1}{x+b} + \frac{b-x}{(a^2+b^2)(x^2+a^2)}\right)$$

Note

$$\therefore \quad I = \int \frac{1}{(x+b)(x^2+a^2)}\, dx$$

$$= \frac{1}{a^2+b^2} \int \frac{dx}{x+a} + \frac{1}{a^2+b^2} \int \frac{b-x}{x^2+a^2}\, dx$$

$$= \frac{1}{a^2+b^2} \log|x+b| + \frac{b}{a^2+b^2} \int \frac{dx}{x^2+a^2} - \frac{1}{2(a^2+b^2)} \int \frac{2x}{x^2+a^2}\, dx$$

$$= \frac{1}{a^2+b^2} \left[\log|x+b| + \frac{b}{a} \tan^{-1} \frac{x}{a} - \frac{1}{2} \log|x^2+a^2| \right] + c$$

$$= \frac{1}{a^2+b^2} \left[\log \left| \frac{x+b}{\sqrt{x^2+b^2}} \right| + \frac{b}{a} \tan^{-1} \frac{x}{a} \right] + c. \qquad \textbf{Ans.}$$

Example 9: Find the Value $\int \frac{dx}{1+x^3}$.

Solution:

$$\frac{1}{1+x^3} = \frac{1}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2};$$

on solving,

$$\frac{1}{1+x^3} = \frac{1}{3} \cdot \frac{1}{1+x} + \frac{2-x}{3(1-x+x^2)}$$

$$\therefore \quad \int \frac{1}{1+x^3}\, dx = \frac{1}{3} \int \frac{dx}{1+x} + \frac{1}{3} \int \frac{(2-x)\, dx}{1-x+x^2}$$

$$= \frac{1}{3} \log|1+x| + \frac{1}{2} \cdot \frac{1}{3} \int \frac{(4-2x)\, dx}{1-x+x^2}$$

$$= \frac{1}{3} \log|1+x| + \frac{1}{6} \int \frac{3-(2x-1)}{1-x+x^2}\, dx$$

$$= \frac{1}{3} \log|1+x| + \frac{1}{2} \int \frac{dx}{1-x+x^2} - \frac{1}{6} \int \frac{2x-1}{1-x+x^2}\, dx$$

$$= \frac{1}{3} \log|1+x| + \frac{1}{2} \int \frac{dx}{\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}} - \frac{1}{6} \int \frac{2x-1}{1-x+x^2}\, dx$$

$$= \frac{1}{3} \log|1+x| + \frac{1}{2} \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+(\sqrt{3}/2)^2} - \frac{1}{6} \log|1+x+x^2| + c$$

$$= \frac{1}{6} \log|1+x| + \frac{1}{2} \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{x-\frac{1}{2}}{\sqrt{3}/2} - \frac{1}{6} \log|1-x+x^2| + c$$

$$= \frac{1}{6} \log \left| \frac{(1+x)^2}{1-x+x^2} \right| + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + c. \qquad \textbf{Ans.}$$

Example 10: Find the Value $\int \dfrac{2x-3}{x^2+3x-18}\,dx$.

Solution: $\qquad \dfrac{2x-3}{x^2+3x-18} = \dfrac{2x-3}{(x-3)(x+6)} = \dfrac{A}{x-3} + \dfrac{B}{x+6}$

or $\qquad\qquad x-3 = A\,(x+6) + B(x-3)$

when, $\qquad\qquad x-3 = 0 \qquad\qquad\qquad$ or $\quad x = 3$

$\qquad\qquad\qquad 3 = 9A \qquad\qquad\qquad$ or $\quad A = \dfrac{1}{3}$

when, $\qquad\qquad x+6 = 0 \qquad\qquad\qquad$ or $\quad x = -6$

$\qquad\qquad -15 = B\,(-9) \qquad\qquad$ or $\quad B = \dfrac{5}{3}$

then, $\qquad \int \dfrac{2x-3}{x^2+3x-18}\,dx = \dfrac{1}{3}\int \dfrac{dx}{x-3} + \dfrac{5}{3}\int \dfrac{dx}{x+6}$

$\qquad\qquad\qquad\qquad = \dfrac{1}{3}\log|x-3| + \dfrac{5}{3}\log|x+6| + c.$ $\qquad$ **Ans.**

Example 11: Find the Value $\int \dfrac{dx}{e^x-1}$.

Solution: $\qquad \int \dfrac{dx}{e^x-1} = \int \dfrac{e^x\,dx}{e^x\,(e^x-1)} = \int \dfrac{dt}{t\,(t-1)}$ $\qquad$ (when $e^x = t,\ e^x\,dx = dt$)

$\qquad\qquad\qquad\qquad = \int \left(\dfrac{1}{t-1} - \dfrac{1}{t}\right) dt = \log\left|\dfrac{t-1}{t}\right| + c$

$\qquad\qquad\qquad\qquad = \log\left|\dfrac{e^x-1}{e^x}\right| + c = \log|1-e^{-x}| + c.$ $\qquad$ **Ans.**

Example 12: Find the Value $\int \dfrac{x}{(x-2)(x-1)^2}\,dx$.

Solution: Let, $\qquad \dfrac{x}{(x-2)(x-1)^2} = \dfrac{A}{x-2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$\qquad\qquad\qquad x = A\,(x-1)^2 + B(x-1)\,(x-2) + C\,(x-2)$

putting equally the factor of same power

$\qquad\qquad A + B = 0,\ -2A - 3B + C = 1,\ A + 2B - 2C = 0$

∴ $\qquad\qquad A = 2,\ B = -2,\ C = -1.$

∴ $\qquad \int \dfrac{x\,dx}{(x-2)(x-1)^2} = \int \dfrac{x\,dx}{(x-2)(x-1)^2}$

$\qquad\qquad\qquad\qquad = 2\log|x-2| - 2\log|x-1| - \dfrac{(x-1)^{-1}}{-1} + c$

Note

$$= 2\{\log|x-2|-\log|x-1|\}+\frac{1}{x-1}+c$$

$$= 2\log\left|\frac{x-2}{x-1}\right|+\frac{1}{x-1}+c. \qquad \textbf{Ans.}$$

Questionnaire 14.2

Short Answer Questions

Find out the value of following integrals:

1. (a) $\int \frac{dx}{x^2-4}$

(b) $\int \frac{x^2\,dx}{x^2-4^2}$

(c) $\int \frac{dx}{x^2-5}$

(d) $\int \frac{dx}{(x+1)^2-4}$

(e) $\int \frac{x^2}{x^6-a^6}\,dx$

2. (a) $\int \frac{3x\,dx}{(x-2)(x+1)}$

(b) $\int \frac{(x+5)}{(x-1)(x-4)}\,dx$

(c) $\int \frac{(2x+3)\,dx}{(x+2)(x-2)}$

(d) $\int \frac{dx}{(x-1)(x^2-4)}$

Answers

1. (a) $\frac{1}{4}\log\left|\frac{x-2}{x+2}\right|+c$

(b) $x+2\log\left|\frac{x-4}{x+4}\right|+c$

(c) $\frac{1}{2\sqrt5}\log\left|\frac{x-\sqrt5}{x+\sqrt5}\right|+c$

(d) $\frac{1}{4}\log\left|\frac{x-1}{x+3}\right|+c$

(e) $\frac{1}{6a^3}\log\left|\frac{x^3-a^3}{x^3+a^3}\right|+c$

2. (a) $\log 1|(x-2)^2(x+1)|+c$

(b) $-2\log|x-1|+3\log|x-4|+c$

(c) $\frac{1}{4}\log|x+2|+\frac{7}{4}\log|x-2|+c$

(d) $-\frac{1}{3}\log|x-1|+\frac{1}{4}\log|x-2|+\frac{1}{12}\log|x+2|$

Self Assessment

2. Multiple Choice Questions:

6. What will be value of $\int \dfrac{dx}{a^2 - x^2}$

    When $x < a$,

    (a) $\dfrac{1}{2a} \log \left| \dfrac{a+x}{a-x} \right| + c$

    (b) $\dfrac{1}{a} \log \left| \dfrac{a+x}{a-x} \right| + c$

    (c) $\dfrac{1}{2a} \left| \dfrac{a+x}{a-x} \right| + c$

    (d) $\dfrac{1}{2a} \log \left| \dfrac{a-x}{a+x} \right| + c$

7. What will be value of $\int \dfrac{dx}{4 - x^2}$

    (a) $\dfrac{1}{4} \log \left| \dfrac{2-x}{2+x} \right| + c$

    (b) $\dfrac{1}{4} \log \left| \dfrac{2+x}{2-x} \right| + c$

    (c) $\dfrac{1}{2} \log \left| \dfrac{4-x}{4+x} \right| + c$

    (d) $\dfrac{1}{2} \log \left| \dfrac{4+x}{4-x} \right| + c$

8. What will be value of $\int \dfrac{dx}{x^2 - 4}$

    (a) $\dfrac{1}{4} \log \left| \dfrac{x+2}{x-2} \right| + c$

    (b) $\dfrac{1}{2} \log \left| \dfrac{x-2}{x+2} \right| + c$

    (c) $\dfrac{1}{4} \log \left| \dfrac{x-2}{x+2} \right| + c$

    (d) None of these.

14.4 Summary

- If the multiplication of $f(x)$ and $\phi(x)$ is $f(x)$. $f(x)$, the differentiation is –

$$\frac{d}{dx}\{f(x) \cdot \phi(x)\} = f(x)\,\phi'(x) + f'(x)\,\phi(x)$$

- The first function is so selected that the second part of RHS could be easily and successfully integrated. Therefore, there should take care in the selection of first function.

- If between two functions there is a function whose integration we don't know, then that function must be considered as the first function.

- If two functions are same and the integral of both functions known, then if there is any x^n type function, then it always considers first function.

- There is no any authentic method to select the functions during the divided integration, but the students can do the divided integration on selecting the function with the help of above-mentioned 'ILATE'.

- The important method of integration is used the above principle to break the additional and difference form of given function. So the methods of partial fraction are used when necessary.

14.5 Keywords

- *Summation:* addition
- *Integral:* whole

14.6 Review Questions

1. Find out the value of- $\int x \sin x \, dx$ (**Ans.:** $- x \cos x + \sin x + c$)

2. Find out the value of- $\int x^2 \cos x \, dx$ (**Ans.:** $(x^2 - 2) \sin x + 2x \cos x + c$))

3. Find out the value of- $\int x \tan^{-1} x \, dx$ (**Ans.:** $\dfrac{1}{2}(x^2 + 1) \tan^{-1} x - \dfrac{x}{2} + c$)

4. Find out the value of- $\int \dfrac{dx}{2x^2 + x - 1}$ (**Ans.:** $\dfrac{1}{3} \log \left| \dfrac{2x - 1}{x + 1} \right| + c$)

5. Find out the value of- $\int \dfrac{dx}{x - x^3}$ (**Ans.:** $\log |x| - \dfrac{1}{2} \log |1 - x| - \dfrac{1}{2} \log |1 + x| + c$)

Answers: Self Assessment

1. Integral	2. Divided	3. Proved	4. Equals
5. Quardatic	6. (a)	7. (b)	8. (c)

14.7 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Blum-Viva Publications.

Mathematics for Economist – Mehta and Madnani-Sultan Chand and sons.

Mathematics Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman

Mathematics for Economist – Council for Economic Education.

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 15: Definite Integration

CONTENTS

Objectives

Introduction

Objectives

After reading this unit, students will be able to :

- Know the Meaning, Limitation and Methods of Finding Definite Integration.
- Understand the Substitution of Definite Integration.
- Know the General Properties of Definite Integrals.
- Get Information of Integrals of Infinity Limits.

Introduction

Let the integral of $f(x)$ relative to x is $F(x)$.

If two values of independent variables x, assume, for a and b, difference between integral $F(x)$ of function $f(x)$ is $F(a) - F(b)$, then its difference are known as Definite Integrals of $f(x)$ for interval $[a, b]$ and it will show below:

$$\int_a^b f(x)\ dx$$

The meaining of definite integration- when integral of any function is find out for any two certain limitations then it's called definite integration.

$$\text{If} \int f(x)\, dx\ = F(x)$$

$$\text{Then } \int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

The value of any definite integrals is unique, since,

$$\text{If} \qquad \int f(x)\, dx\ = F(x) + c, \text{ then}$$

$$\int_a^b f(x)\, dx\ = \left[F(x) + c\right]_a^b\ = \{F(b) + c\} - \{F(a) + c\}$$

$$= F(b) + c - F(a) - c = F(b) - F(a).$$

Note

15.1 Limitations of Integration

Digit a, that is written below in integration sign called lower limit and digit b that is written up side of integration sign is called the upper limit of integration.

15.2 Method of Finding Definite Integration

To find out the given integrals and keep it in big bracket and written the limitation of integration at right hand side of bracket. Now put the upper limit at the place of variable digit in integrals and keep lower limit in same integrals then both are differentiate, the desired integrals will find out.

EXAMPLES WITH SOLUTIONS

Example 1: Find out the Value of $\int_a^b \dfrac{\log x}{x}\ dx$.

Solution: $\qquad\qquad \int_a^b \dfrac{\log x}{x}\ dx \qquad$ Let $\quad \log x = t$

$$\frac{1}{x}\ dx = dt$$

$$= \int t\ dt$$

$$= \frac{t^2}{2} = \frac{1}{2}\Big[(\log x)^2\Big]_a^b$$

$$= \frac{1}{2}\big[(\log b)^2 - (\log a)^2\big]$$

$$= \frac{1}{2}\ [\log b + \log a]\ [\log b - \log a]$$

$$= \frac{1}{2}\ \log ab \times \log \frac{b}{a}\ . \qquad\qquad \textbf{Ans.}$$

Example 2: Find out the Value of $\int_0^{\pi/4} \tan x \sec x\ dx$.

Solution: $\qquad \int_0^{\pi/4} \tan x \sec x\ dx = \Big[\sec x\Big]_0^{\pi/4} = \Big[\sec \dfrac{\pi}{4} - \sec 0\Big]$

$$= \sqrt{2}\ - 1. \qquad\qquad \textbf{Ans.}$$

Example 3: Find out the Value of $\int_1^3 \dfrac{dx}{x}$.

Solution: $\qquad\qquad\qquad\qquad\qquad = \Big[\log x\Big]_1^3 = \log 3 - \log 1 = \log 3. \qquad \textbf{Ans.}$

Example 4: Find out the Value of $\int_0^{\pi/2} \cos^2 x \; dx$.

Solution:
$$\int_0^{\pi/2} \cos^2 x \; dx = \int_0^{\pi/2} \left\{ \frac{1 + \cos 2x}{2} \right\} \; dx$$

$$= \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} + \frac{\sin 2 \cdot \frac{\pi}{2}}{2} - 0 - \frac{\sin 2 \times 0}{2} \right) = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4} \, . \qquad \textbf{Ans.}$$

Example 5: Find out the Value of $\int_0^{\pi/4} \tan^2 x \; dx$.

Solution: $\int_0^{\pi/4} \tan^2 x \; dx = \int_0^{\pi/4} (\sec^2 x - 1) \; dx$

$$= \left[\tan x - x \right]_0^{\pi/4} = \left[\tan \frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0) \right]$$

$$= 1 - \frac{\pi}{4} \, . \qquad \textbf{Ans.}$$

Example 6: Find out the Value of $\int_0^a y^2 \, , x^2 + y^2 = a^2$.

Solution: $\because \qquad\qquad x^2 + y^2 = a^2 \quad \therefore \quad y^2 = a^2 - x^2$

$\therefore \qquad\qquad \int_0^a y^2 \; dx = \int_0^a (a^2 - x^2) \; dx$

$$= \left[a^2 x - \frac{x^3}{3} \right]_0^a = a^3 - \frac{a^3}{3} - 0 = \frac{2}{3} \, a^3. \qquad \textbf{Ans.}$$

Example 7: Find out the Value of $\int_0^a \dfrac{dx}{\sqrt{a^2 - x^2}}$.

Solution:
$$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \left[\sin^{-1} \left(\frac{x}{a} \right) \right]_0^a$$

$$= \sin^{-1} \left(\frac{a}{a} \right) - \sin^{-1} \left(\frac{0}{a} \right)$$

$$= \sin^{-1} (1) - \sin^{-1} (0) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \, . \qquad \textbf{Ans.}$$

Example 8: Find out the Value of $\int_0^{\pi/2} x \cos x \; dx$.

Solution:
$$\int_0^{\pi/2} x \cos x \; dx = \left[x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \; dx$$

Note

$$= \left[\frac{\pi}{2} \sin \frac{\pi}{2} - 0 \right] - \left[-\cos x \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \left[\cos \frac{\pi}{2} - \cos 0 \right] = \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1. \qquad \textbf{Ans.}$$

Example 9: Find out the Value of $\int_0^{\pi/4} \sqrt{1 - \sin 2x} \ dx$.

Solution: $\int_0^{\pi/4} \sqrt{1 - \sin 2x} \ dx = \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x - 2 \sin x \cos x} \ dx$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2} \ dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) \ dx$$

$$= \left[\sin x + \cos x \right]_0^{\pi/4}$$

$$= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0° - \cos 0°)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 - 1)$$

$$= \frac{2}{\sqrt{2}} + 1 = (\sqrt{2} + 1). \qquad \textbf{Ans.}$$

Example 10: Find out the Value of $\int_1^2 x \log x \ dx$.

Solution: $\qquad \int_1^2 x \log x \ dx = \left[\frac{x^2}{2} \log x \right]_1^2 - \int_1^2 \frac{1}{x} \cdot \frac{x^2}{2} \ dx$

$$= \left[\frac{x^2}{2} \log x \right]_1^2 - \frac{1}{2} \int_1^2 x \ dx$$

$$= \left[\frac{x^2}{2} \log x \right]_1^2 - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^2$$

$$= 2 \log 2 - \frac{1}{4} \left[x^2 \right]_1^2$$

$$= 2 \log 2 - \frac{1}{4} [4 - 1]$$

$$= 2 \log 2 - \frac{3}{4}. \qquad \textbf{Ans.}$$

Example 11: Find out the Value of $\int_0^{\pi/4} \tan x \ dx$

Solution: $\qquad \int_0^{\pi/4} \tan x \ dx = \left[\log \sec x \right]_0^{\pi/4}$

$$= \log \sec \frac{\pi}{4} \quad \log \sec 0$$

$$= \log \sqrt{2} - \log 1, \qquad\qquad [\because \quad \log 1 = 0]$$

$$= \log \sqrt{2} . \qquad\qquad\qquad \textbf{Ans.}$$

Example 12: Find out the Value of $\int_0^{\pi/6} \sqrt{1 - \sin 2x} \; dx$.

Solution: $\qquad\qquad \int_0^{\pi/6} \sqrt{1 - \sin 2x} \; dx = \int_0^{\pi/6} \sqrt{\cos^2 x + \sin^2 x - 2 \sin x \cos x} \; dx$

$$= \int_0^{\pi/6} \sqrt{(\cos x - \sin x)^2} \; dx$$

$$= \int_0^{\pi/6} (\cos x - \sin x) \, dx = \left[\sin x + \cos x \right]_0^{\pi/6}$$

$$= \left[\sin \frac{\pi}{6} + \cos \frac{\pi}{6} \right] - [\sin 0 + \cos 0]$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) - (0 + 1) = \frac{1}{2} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{1 + \sqrt{3} - 2}{2} = \frac{\sqrt{3} - 1}{2} .$$

Questionnaire 15.1

Find out the values of following integrals:

1. $\int_1^2 x^4 \; dx.$

2. $\int_1^4 \frac{dx}{\sqrt{x}} .$

3. $\int_0^3 e^{x/3} \; dx.$

4. $\int_0^{\pi/2} \cos x \; dx.$

5. $\int_1^2 \frac{dx}{x} .$

6. $\int_0^{\pi/4} \sec^2 x \; dx.$

7. $\int_0^{\pi/2} \sin x \; dx.$

8. $\int_0^{\pi} \sin 3x \; dx.$

9. $\int_{\pi/6}^{\pi/2} \cos x \; dx.$

10. $\int_4^9 \sqrt{x} \; dx.$

11. $\int_0^1 \frac{dx}{1 + x^2} .$

12. $\int_0^1 \frac{1}{\sqrt{1 - x^2}} \; dx.$

13. $\int_{-\pi/4}^{\pi/4} \cosec^2 x \; dx.$

Answers

1. $\dfrac{31}{5}$

2. 2

3. $3 \, (e - 1)$

4. 1

5. $\log 2$

6. 1

7. 1

8. $\dfrac{2}{3}$

9. $\dfrac{1}{2}$

10. $\dfrac{38}{3}$

11. $\dfrac{\pi}{4}$

12. $\dfrac{\pi}{2}$

13. -2

15.3 Substitution in Definite Integration

Some time it is necessary to change the variable in definite integration like indefinite integration. Limits changed with the substitution for making the method easy.

Example, let we put $\psi\,(x) = t$, then

$$\int f[\psi(x)\,\psi'\,(x)]\;dx = \int(t)\;dt.$$

Did u know?   If integrals limit are a to b for variables x, then limit relation for t is $t = \psi(x)$.

Since, When $x = a$, then $t = \psi(x)$ and when $x = b$, then $t = \psi(x)$,

So,

$\therefore$ $\displaystyle\int_{a}^{b} f[\phi(x)]\,\phi'\,(x)\,dx = \int_{\psi(a)}^{\psi(b)} f(t)\,dt.$

EXAMPLES WITH SOLUTIONS

Example 1: Find out the value of- $\displaystyle\int_{1}^{3} \dfrac{\cos(\log x)}{x}\;dx.$

Solution: $\displaystyle\int_{1}^{3} \dfrac{\cos(\log x)}{x}$ Let, $\log x = t$ then $(1/x)\,dx = dt$

When $x = 1, t = \log(1) = 0$

and when, $x = 3, t = \log 3$

$\therefore$ $\displaystyle\int_{1}^{3} \dfrac{\cos(\log x)}{x}\;dx = \int_{0}^{\log 3}\cos t\;dt = \Big[\sin t\Big]_{0}^{\log 3}$

$= \sin(\log 3) - \sin(0) = \sin(\log 3).$ **Ans.**

Example 2: Find out the value of- $\displaystyle\int_{0}^{1} \dfrac{x}{\sqrt{1+x^2}}\;dx.$

Solution: Let, $x = \tan\theta$; $\therefore$ $dx = \sec^2\theta\,d\theta$

$x = 0$, and $\tan\theta = 0$ so, $\theta = 0$

and when, $x = 1$, then $\tan\theta = 1$ so, $\theta = \dfrac{1}{4}\,\pi$

$\therefore$ $\displaystyle\int_{0}^{1} \dfrac{x}{\sqrt{1+x^2}}\;dx = \int_{0}^{\pi/4} \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}\,.\,\sec^2\theta\,d\theta$

$$= \int_0^{\pi/4} \frac{\tan \theta}{\sec \theta} \cdot \sec^2 \theta \, d\theta = \int_0^{\pi/4} \sec \theta \tan \theta \, d\theta$$

$$= \left[\sec \theta\right]_0^{\pi/4} = \sec \frac{\pi}{4} - \sec 0 = \sqrt{2} - 1. \qquad \textbf{Ans.}$$

Example 3: Find out the value of- $\int_0^a \frac{x}{\sqrt{a^2 - x^2}} \, dx.$

Solution: Let, $a^2 - x^2 = t$, then, $-2x \, dx = dt$ or $x \, dx = -\frac{1}{2} \, dt$

when, $\qquad\qquad$ tc $x = 0$, $t = a^2$ and when, $x = a$, $t = a^2 - a^2 = 0$

$$\int_0^a \frac{x}{\sqrt{a^2 - x^2}} \, dx = -\frac{1}{2} \int_{a^2}^0 \frac{dt}{\sqrt{t}} = -\frac{1}{2} \int_{a^2}^0 t^{-1/2} \, dt$$

$$= -\frac{1}{2}\left[2t^{1/2}\right]_{a^2}^0$$

$$= -[0 - (a^2)^{1/2}] = (a^2)^{1/2} = a. \qquad \textbf{Ans.}$$

Example 4: Find out the value of- $\int_0^{\pi/3} \frac{\cos x}{3 + 4 \sin x} \, dx.$

Solution: Let, $3 + 4 \sin x = t$, then, $4 \cos x \, dx = dt$

and when, $\qquad\qquad x = 0$, $t = 3 + 4 \sin (0) = 3 + 4 (0) = 3$, $\because \sin (0) = 0$

or when, $\qquad\qquad x = \frac{1}{3} \pi$, $t = 3 + 4 \sin \left(\frac{1}{3}\pi\right) = 3 + 4 (\sqrt{3}/2) = 3 + 2\sqrt{3}$

$$\therefore \qquad\qquad \int_0^{\pi/3} \frac{\cos x}{3 + 4 \sin x} \, dx = \frac{1}{4} \int_3^{(3 + 2\sqrt{3})} \frac{dt}{t}$$

$$= \frac{1}{4}\left[\log t\right]_3^{3 + 2\sqrt{3}} = \frac{1}{4} [\log (3 + 2\sqrt{3}) - \log 3]$$

$$= \frac{1}{4}\left[\log\left(\frac{3 + 2\sqrt{3}}{3}\right)\right]. \qquad \textbf{Ans.}$$

Example 5: Find out the value of- $\int_0^\pi x \sin^2 x \, dx.$

Solution: $\qquad\qquad \int x \sin^2 x \, dx = \frac{1}{2}\int x (2 \sin^2 x) \, dx$

$$= \frac{1}{2}\int x (1 - \cos 2x) \, dx$$

$$= \frac{1}{2}\int x \, dx - \frac{1}{2} \int x \cos 2x \, dx$$

Note

$$= \frac{1}{2} \cdot \frac{1}{2} x^2 - \frac{1}{2} \left[\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]$$

$$= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x,$$

$\therefore \qquad \int_0^\pi x \sin^2 x \, dx = \left[\frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x \right]_0^\pi$

$$= \frac{1}{4} [\pi^2] - \frac{1}{4} [\pi \sin 2\pi] - \frac{1}{8} [\cos 2\pi] - \left[0 - 0 - \frac{1}{8} \cos 0 \right]$$

$$= \frac{1}{4} \pi^2 - \frac{1}{4} [0] - \frac{1}{8} [1] + \frac{1}{8}$$

$$= \frac{1}{4} \pi^2. \qquad\qquad \textbf{Ans.}$$

Example 6: Find out the value of $\int_1^2 \frac{\log_e x}{x^2} \, dx$.

Solution: $\qquad \int \frac{\log_e x}{x^2} \, dx = \int (\log_e x)(x^{-2}) \, dx$

$$= (\log_e x)(-x^{-1}) - \int (1/x)(-x^{-1}) \, dx,$$

$$= -(\log_e x)(x^{-1}) + \int x^{-2} \, dx$$

$$= -\frac{\log_e x}{x} - \frac{1}{x} = -\left[\frac{\log_e x + 1}{x} \right]$$

$\therefore \qquad \int_1^2 \frac{\log_e x}{x^2} \, dx = -\left[\frac{\log_e x + 1}{x} \right]_1^2$

$$= -\left[\left(\frac{\log_e 2 + 1}{2} \right) - \left(\frac{\log_e 1 + 1}{1} \right) \right]$$

$$= -\left[\frac{1}{2} \log_e 2 + \frac{1}{2} - 1 \right] \qquad\qquad [\because \log_e 1 = 0]$$

$$= -\left[\frac{1}{2} \log_e 2 - \frac{1}{2} \right] = \frac{1}{2} [1 - \log_e 2]. \qquad\qquad \textbf{Ans.}$$

Example 7: Find out the value of $\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} \, dx$.

Solution: Put $\sin x = t$. then solve by partial method

$$\int_0^{\pi/2} \frac{\cos x \, dx}{(1+\sin x)(2+\sin x)} = \left[\log\left(\frac{1+\sin x}{2+\sin x}\right)\right]_0^{\pi/2}$$

$$= \log\left(\frac{1+\sin\frac{1}{2}\pi}{2+\sin\frac{1}{2}\pi}\right) - \log\left(\frac{1+\sin 0}{2+\sin 0}\right)$$

$$= \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right) = \log\left(\frac{\frac{2}{3}}{\frac{1}{2}}\right) = \log\frac{4}{3}. \qquad \text{Ans.}$$

Example 8: Find out the value of $\int_0^{\pi/2} x \sin x \, dx.$

Solution: $\qquad \int_0^{\pi/2} x \sin x \, dx = \left[x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos x) \, dx$

$$= \left[-x\cos x\right]_0^{\pi/2} + \left[\sin x\right]_0^{\pi/2}$$

$$= -\left[x\cos x\right]_0^{\pi/2} + \left[\sin x\right]_0^{\pi/2}$$

$$= -\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\right] + \left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= -0 + 1$$

$$= 1. \qquad \text{Ans.}$$

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Questionnaire 15.2

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Short Answer Questions.

Find out the value of following integrals:

1. $\int_1^3 \frac{\log x}{x} \, dx$

2. $\int_1^2 \frac{\cos(\log x)}{x} \, dx$

3. $\int_0^1 x\log\left(1+\frac{x}{2}\right) dx$

4. $\int_1^e \frac{e^x}{x}(1+x\log x)\, dx$

5. $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx$

6. $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2} \, dx$

7. $\int_0^1 \frac{x\sin^{-1} x}{\sqrt{1-x^2}} \, dx$

8. $\int_0^\infty \frac{\sin\tan^{-1} x}{1+x^2} \, dx$

9. $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} \, ; \beta > \alpha$

10. $\int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx$

Note

11. $\int_0^a \sqrt{a^2 - x^2} \; dx$

12. $\int_0^1 x \sqrt{\dfrac{1-x^2}{1+x^2}} \; dx$

13. $\int_0^1 \dfrac{x^2 \, dx}{1+x^6}$

14. $\int_0^1 x \sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}} \; dx$

15. $\int_0^1 \dfrac{dx}{3 + 2x + x^2}$

16. $\int_0^1 \dfrac{2x}{1+x^4} \; dx$

17. $\int_0^2 \sqrt{\dfrac{2+x}{2-x}} \; dx.$

Answers

1. $\dfrac{(\log 3)^2}{2}$

2. $\sin(\log 2)$

3. $\dfrac{3}{4} - \dfrac{3}{2} \log \dfrac{3}{2}$

4. e^e

5. $\dfrac{1}{8} \pi^2$

6. $\dfrac{\pi^3}{192}$

7. 1

8. 1

9. π

10. 1

11. $\dfrac{\pi a^2}{4}$

12. $\dfrac{\pi - 2}{4}$

13. $\dfrac{\pi}{12}$

14. $\dfrac{a^2 (\pi - 2)}{4}$

15. $\dfrac{1}{\sqrt{2}} \tan^{-1} \dfrac{1}{2\sqrt{2}}$

16. $\dfrac{\pi}{4}$

17. $\pi + 2$

Self Assessment

1. Fill in the blanks:

1. When integral of any function is find out for any two certain limitations then it's called ……………. …..

2. The value of any definite integrals is …………………..

3. Some time it is………….. ………….. to change the variable in definite integration like indefinite integration

4. The value of definite integration are not affected by the integration of……………………

5. Digit a that are written below of integration sign, called the……………………..of integration.

15.4 General Properties of Definite Integrals

Property: 1. $\int_a^b f(x) \; dx = - \int_a^b f(x) \; dx.$

Proof: Let, $\int f(x) \; dx = F(x)$

$\therefore$ $\quad$ Left hand side = $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Right hand side = $-\int_b^a f(x)\ dx = -\left[F(x)\right]_b^a$

$= -[F(a) - F(b)] = F(b) - F(a)$

$\therefore$ $\quad \int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$

Property: 2. $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt.$

Proof: $\quad$ Left hand side = $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Right hand side = $\int_a^b f(t)\ dt = \left[F(t)\right]_a^b = F(b) - F(a)$

$\therefore$ $\quad \int_a^b f(x)\ dx = \int_a^b f(t)\ dt.$

Property: 3. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx,$ **when** $a < c < b.$

Proof: Let, $\quad \int f(x)\ dx = F(x)$

Left hand side = $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Right hand side = $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

$= \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$

$[F(c) - F(a)] + [F(b) - F(c)] = F(b) - F(a)$

On detailed from

$\int_a^b f(x)\ dx = \int_a^{c_1} f(x)\ dx + \int_{c_1}^{c_2} f(x)\ dx + \ldots\ldots + \int_{c_{n-1}}^{c_n} f(x)\ dx + \int_{c_n}^b f(x)\ dx$

when, $\quad a < c_1 < c_2 < c_3 < \ldots\ldots c_n < b.$

Property: 4. $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx.$

Proof: Let, $a - x = t$ $\quad \therefore\ -dx = dt$;k $dx = -dt$

and when $x = 0$, then, $t = a$ and when, tc $x = a$, then, $t = 0$

$\therefore$ $\quad$ Right hand side = $\int_0^a f(a-x)\ dx = \int_a^0 f(t)\ (-dt) = -\int_a^0 f(t)\ dt$

$= \int_0^a f(t)\ dt$ $\qquad\qquad$ By property 1

$= \int_0^a f(x)\ dx$ $\qquad\qquad$ By property 2

= Right hand side

Note

Property: 5. $\int_{-a}^{a} f(x)\ dx = 0$, **if** $f(x)$, **is the odd function of** x

mean
$$f(-x) = -f(x)$$

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

if $f(x)$ is the even function of x mean $f(-x) = f(x)$.

Proof: $\int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_{0}^{a} f(x)\ dx$ ...(i)

$$\because -a < 0 < a$$

Now, $\int_{-a}^{0} f(x)\ dx = \int_{a}^{0} f(-t)\ (-dt),$ Put $x = -t \Rightarrow dx = -dt$

$$= -\int_{a}^{0} f(-t)\ dt$$

$$= \int_{0}^{a} f(-t)\ dt \qquad \text{(By property 1)}$$

$$= \int_{0}^{a} f(-x)\ dx \qquad \text{(By property 2)}$$

Condition I: If $f(x)$ is the odd function of x.

then, $f(-x) = -f(x)$ (by definition)

$$\int_{-a}^{0} f(x)\ dx = -\int_{0}^{a} f(x)\ dx$$

By equation (i),

$$\int_{-a}^{a} f(x)\ dx = -\int_{0}^{a} f(x)\ dx + \int_{0}^{a} f(x)\ dx = 0.$$

Condition II: If $f(x)$ is the even function of x.

then, $f(-x) = f(x)$ (By definition)

$\therefore$ $\int_{-a}^{0} f(x)\ dx = \int_{0}^{a} f(x)\ dx$

Bt equation (i)

$$\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} f(x)\ dx + \int_{0}^{a} f(x)\ dx$$

$$= 2\int_{0}^{a} f(x)\ dx.$$

EXAMPLES WITH SOLUTIONS

Example 1: Prove that-

$$\int_{0}^{\pi/2} \sin^2 x\ dx = \int_{0}^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right)\ dx.$$

Solution: Let hand side $= \int_{0}^{\pi/2} \frac{1 - \cos 2x}{2}\ dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2} \right]_{0}^{\pi/2}$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin \pi - \sin 0}{2}\right] = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \qquad \text{...(i)}$$

Right hand side $= \displaystyle\int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx$

$$= \int_0^{\pi/2} \cos^2 x \ = \int_0^{\pi/2} \frac{1 + \cos 2x}{2} \ dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) + \frac{\sin \pi - \sin 0}{2}\right]$$

$$= \frac{\pi}{4} \qquad \text{...(ii)}$$

$\therefore$ Left hand side = Right hand side

Example 2: Prove that

$$\int_{-\pi/2}^{\pi/2} x \sin x \ dx = 2 \int_0^{\pi/2} x \sin x \ dx$$

and, $\displaystyle\int_{-\pi/2}^{\pi/2} x \cos x \ dx = 0.$

Solution: Here function is, $f(x) = x \sin x$

$\therefore$ $f(-x) = (-x)\sin(-x) = x\sin x = f(x)$

Function is even

$\therefore$ $\displaystyle\int_{-\pi/2}^{\pi/2} x \sin x \ dx = 2 \int_0^{\pi/2} x \sin x \ dx$

If, $f(x) = x \cos x$

$\therefore$ $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$

function is odd

$\therefore$ $\displaystyle\int_{-\pi/2}^{\pi/2} x \cos x \ dx = 0.$

Example 3: Prove that

$$\int_0^{\pi/2} \sin 2x \ \log \tan x \ dx = 0.$$

Solution:

Let, $I = \displaystyle\int_0^{\pi/2} \sin 2x \ \log \tan x \ dx$ ...(i)

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx, \text{ (By property) (4) is}$$

Note

$$= \int_0^{\pi/2} \sin \ (\pi - 2x) \log \cot x \ dx$$

$$= \int_0^{\pi/2} \sin \ 2x \log \cot x \ dx \qquad \qquad ...(ii)$$

Add equation (i) and (ii),

$$2I = \int_0^{\pi/2} (\sin 2x \log \tan x + \sin 2x \log \cot x) \ dx$$

$$= \int_0^{\pi/2} \sin 2x \ (\log \tan x + \log \cot x) \ dx$$

$$= \int_0^{\pi/2} \sin 2x . \log \ (\tan x . \cot x) \ dx$$

$$= \int_0^{\pi/2} \sin 2x . \log \ 1 \ dx$$

$$= \int_0^{\pi/2} \sin 2x . 0 \ dx = 0$$

$$\therefore \qquad \qquad I = 0.$$

Example 4: Prove that-

$$\int_{\pi/8}^{3\pi/8} \frac{\tan^2 x}{\tan^2 x + \cot^2 x} \ dx = \frac{\pi}{8} \ .$$

Solution: Let, $I = \int_{\pi/8}^{3\pi/8} \frac{\tan^2 x}{\tan^2 x + \cot^2 x}$ \qquad \qquad ...(i)

$$x = \frac{\pi}{2} - t \ \text{ because; } \ a + b = \frac{\pi}{8} + \frac{3\pi}{8} = \frac{\pi}{2}$$

$$I = \int_{3\pi/8}^{\pi/8} \frac{\cot^2 t}{\cot^2 t + \tan^2 t} \ (-dt)$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\cot^2 t}{\cot^2 t + \tan^2 t} \ dt$$

or $\qquad \qquad I = \int_{\pi/8}^{3\pi/8} \frac{\cot^2 t}{\cot^2 x + \tan^2 x} \ dx$ \qquad \qquad ...(ii)

Adding equation (i) and (ii),

$$2I = \int_{\pi/8}^{3\pi/8} dx = [x]_{\pi/8}^{3\pi/8}$$

$$2I = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$$

or $\qquad \qquad I = \frac{\pi}{8} \ .$ \qquad \qquad **Ans.**

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Questionnaire 15.3

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Long Answer Questions:

1. Prove that $\int_0^{\pi/2} \log\ (\tan x)\ dx = 0$.

2. Prove that $\int_1^4 \dfrac{\sqrt{x}\ dx}{\sqrt{5-x} + \sqrt{x}} = \dfrac{3}{2}$.

3. Prove that $\int_0^{\pi/2} \dfrac{dx}{1 + \sqrt{\tan x}} = \dfrac{\pi}{4}$.

4. Prove that $\int_0^{\pi/2} \dfrac{\sin x}{\sin x + \cos x}\ dx = \dfrac{\pi}{4}$.

15.5   Integral of Infinite Limit

If high limit of any definite integrals is ∞, then

$$\int_a^{\infty} f(x)\ dx \text{ it means } \lim_{b \to \infty} \int_a^b f(x)\ dx,$$

Condition is that limit is any finite variable.

If the limit of any definite integration is (-∞)

$$\int_{-\infty}^b f(x)\ dx = \lim_{a \to -\infty} \int_a^b f(x)\ dx,$$

Condition is that limit is any finite variable.

<div style="text-align:center;">

EXAMPLES WITH SOLUTIONS

</div>

Example 1: Find the value of $\int_1^{\infty} \dfrac{1}{x^2}\ dx$.

Solution: $\qquad\qquad \int_1^{\infty} \dfrac{1}{x^2}\ dx = \lim_{b \to \infty} \left[-\dfrac{1}{x} \right]_1^b = \lim_{b \to \infty} \left(-\dfrac{1}{b} + 1 \right) = 1.$ **Ans.**

Example 2: Find the value of $\int_0^{\infty} \dfrac{1}{1 + x^2}\ dx$.

Solution: $\qquad\qquad \int_0^{\infty} \dfrac{1}{1 + x^2}\ dx = \lim_{b \to \infty} \left[\tan^{-1} x \right]_0^b$

$$= \lim_{b \to \infty}\ (\tan^{-1} b - \tan^{-1} 0) = \dfrac{1}{2}\ \pi. \qquad \textbf{Ans.}$$

Example 3: Find the Value of $\int_0^{\infty} \dfrac{dx}{(1 + x^2)^2}$.

Solution: Firstly we find the value of $I = \int_0^b \dfrac{dx}{(1 + x^2)^2}$

Note   Let $x = \tan\theta$ $\therefore$ $dx = \sec^2\theta\, d\theta$

Limits of θ are, 0, $\tan^{-1} b$

$$I = \int_0^{\tan^{-1} b} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int_0^{\tan^{-1} b} \cos^2\theta\, d\theta = \int_0^{\tan^{-1} b} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\tan^{-1} b} = \frac{\tan^{-1} b}{2} + \frac{2b}{4(1 + b^2)}.$$

$\therefore$ Require integration $= \lim_{b \to \infty} \left[\dfrac{\tan^{-1} b}{2} + \dfrac{\dfrac{1}{2b}}{1 + \dfrac{1}{b^2}}\right] = \dfrac{\pi}{4} + 0 = \dfrac{\pi}{4}.$ **Ans.**

Questionnaire 15.4

Short Answer Questions:

Find the values of following integration:

1. $\displaystyle\int_0^\infty \frac{dx}{a^2 + x^2}.$

2. $\displaystyle\int_0^\infty \frac{dx}{x^2\sqrt{a^2 + x^2}}.$

3. $\displaystyle\int_0^\infty e^{-x/2}\, dx.$

4. $\displaystyle\int_0^\infty x e^{-x}\, dx.$

5. $\displaystyle\int_0^\infty \frac{e^x\, dx}{1 + e^{2x}}.$

6. $\displaystyle\int_1^\infty \frac{(x^2 + 3)\, dx}{x^6\,(x^2 + 1)}.$

Answers

1. $\dfrac{\pi}{2a}$

2. $\left(\dfrac{\sqrt{2} - 1}{a^2}\right)$

3. 2

4. 1

5. $\dfrac{\pi}{4}$

6. $\dfrac{1}{30}(58 - 15\pi)$

Self Assessment

2. Multiple Choice Question:

6. $\displaystyle\int_a^\infty f(x)\, dx$ it means

   (a) $\displaystyle\lim_{b \to \infty} \int_a^b f(x)\, dx$

   (b) $\displaystyle\lim_{b \to \infty} \int_b^a f(x)\, dx$

   (c) $\displaystyle\lim_{b \to 0} \int_a^b f(x)\, dx$

   (d) $\displaystyle\lim_{a \to \infty} \int_a^b f(x)\, dx$

7. If lower limit of any definite integration is $(-\infty)$, then $\displaystyle\int_{-\infty}^b f(x)\, dx$

   (a) $\displaystyle\lim_{a \to \infty} \int_a^b f(x)\, dx$

   (b) $\displaystyle\lim_{a \to -\infty} \int_a^b f(x)\, dx$

   (c) $\displaystyle\lim_{a \to -\infty} \int_b^a f(x)\, dx$

   (d) $\displaystyle\lim_{a \to 0} \int_a^b f(x)\, dx$

8. Value of $\int_1^\infty \frac{1}{x^2}\,dx$

   (*a*) 0   (*b*) ∞

   (*c*) 1   (*d*) –1

15.6 Summary

- If two values of independent variables *x*, assume, for a and b, difference between integral $F(x)$ of function $f(x)$ is $F(a)-F(b)$, then its difference are known as Definite Integrals of $f(x)$ for interval $[a, b]$

- The value of any definite integrals is unique.

- To find out the given integrals and keep it in big bracket and written the limitation of integration at right hand side of bracket. Now put the upper limit at the place of variable digit in integrals and keep lower limit in same integrals then both are differentiate, the desired integrals will find out.

- Some time it is necessary to change the variable in definite integration like indefinite integration. Limits changed with the substitution for making the method easy.

15.7 Keywords

- *Function:* A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

- *Lower limit:* inferior limit

- *Upper limit:* higher limit.

15.8 Review Questions

1. Find out the value of- $\int_0^{\pi/4}\tan^2 x\,dx$ (**Ans.:** $1-\frac{\pi}{4}$)

2. Find out the value of- $\int_0^a y^2\,dx$ Where, $x^2 + y^2 = a^2$ (**Ans.:** $\frac{2}{3}a^2$)

3. Find out the value of- $\int_0^\pi x\sin^2 x\,dx$ (**Ans.:** $\frac{1}{4}\pi^2$)

4. Prove that- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. Prove that- $\int_0^{\pi/2}\sin 2x\log\tan x\,dx = 0$

Answers: Self Assessment

1. Definite Integration   2. Unique   3. Necessary   4. Non-variable

5. Lower limit   6. (a)   7. (b)   8. (c)

Note

15.9   Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics for Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman.

Mathematics for Economist – Council for Economic Education.

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 16: Economic Applications of Integration

Objectives

After reading this unit, students will be able to :

● Find out the Area under Plane Curves.

● Understand the Use of Integration in Economics.

Introduction

Now-a-day's use of integration in economics has increased fast. Definite integral is the range of addition. Use of integration in economics can be explained from the following facts -

1. Marginal Costs, Average Costs and Total Costs since Marginal Costs = $\dfrac{d(c)}{dq}$, therefore total

    costs $\int \dfrac{d(c)}{dq}. dq$

2. *Consumer Saving:* Consumer saving can also be found out from integration.

16.1   Area Under Plane Curves

Suppose *CPQD* represents curve $y = f(x)$, where $f(x)$ is a continuous function of *x in prant* [*a*, *b*] and suppose like the value of *x* increases from a to *b*, *y* also increases. Suppose *CA* and *DB* are the ordinates at $x = a$ and $x = b$.

Take any point $P(x, y)$ located on the curve and assume that *PM* is its grade. Take another point $Q(x + \delta x, y + \delta y)$ on the curve close at point *P* and suppose its ordinate is *QN*.
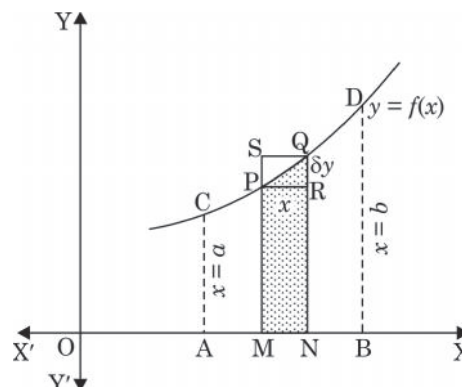
Draw perpendicular *PR*, *QS* on *QN* and *MP* then

$$OM = x, ON = x + \delta x;$$

∴ $MN = \delta x$

And $PM = y,$

$$QN = y + \delta y; \quad ∴ \quad QR = \delta y.$$

Note

If *AMPC* and *ANQC* are serially presented by $S + \delta S$ then

δS = Area *ANQC* – area *AMPC* = Area *MNQP*

Now *MNQP* in area result is located between the rectangles *MNRP* and *MNQS*, i.e.,

$$y\delta x < \ \delta S < (y + \delta y)\,\delta x$$

Or
$$y < \frac{\delta S}{\delta x} < (y + \delta y) \ \ \text{(dividing by } \delta x) \qquad\qquad\qquad ...(1)$$

Or
$$\frac{dS}{dx} = \ y = f(x), \qquad\qquad\qquad (\text{when } \delta x \to 0 \text{ and } dy \to 0^\circ)$$

Therefore
$$\int_a^b f(x)\,dx = \ \int_a^b \frac{dS}{dx}\cdot dx = \int_a^b dS = [S]_a^b$$

$$= \ (\text{Value of } S \text{ when } x = b) - (\text{Value of } S \text{ when } x = a)$$

$$= \ \text{Area ABDC}$$

Definite integral is a limit of addition. Suppose any function is continued in any interval $[a,b]$ its article is a curve in which *C* and *D* are two points. *CA* and *DA* are the perpendiculars on the *X*-axis. '*a*' and '<u>*b*</u>' are the abscissa of points *C* and *D*. Interval $[a,b]$ i.e. *AB* whose value is $(b-a)$, is divided into *n* equal small parts. Suppose the abscissa of points *P* and *Q* which are on the curve are $(a + rh)$ and $(a + r + 1h)$. *PM* and *QN* are their ordinates. This way under the marginal condition when $h \to 0$, *PRNM* is a very small rectangle. Area of this rectangle will be $f\ (a + rh) \times h$. This way definite Integral in the words of Addition,

$$\lim_{n \to \infty} h \cdot [f(a + h) + f(a + 2h) + ... + f(a + nh)]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} h \cdot f(a + rh), \text{ where } b - a = nh$$

$$= \int_a^b f(x)\,dx \text{ is as shown.}$$

Notes $\quad \int_a^b f(x)\,dx$ is the complete area which is formed within the limits of curve $y = f(x)$, *x*-axis, and $x = a$ to $x = b$.

Area of the space surrounded by the curve $y = f(x)$, *x*-axis, and $x = a$ to $x = b$ is calculated from the following formula:

$$A = \int_{x=a}^{x=b} y\,dx$$

Value of x $(x = a)$ from the left side at end *D* is considered as the lower limit of integration and value of x $(x = b)$ from the right side at end *C* is considered as the higher limit of integration.

In the same way, Area of the space surrounded by the curve $x = f(y)$, *y*-axis, and $y = a$ to $y = b$ s calculated from the following formula:

$$A = \int_{y=a}^{y=b} x\,dy$$

In this case limits of integration are taken from lower to upper side.
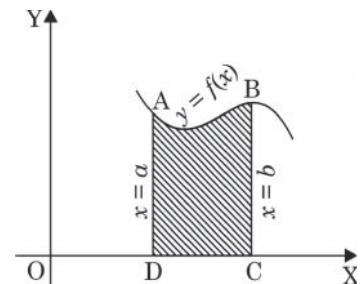
EXAMPLES WITH SOLUTION

Example 1: Find the area between $y = mx$, x-axis and ordinate $x = 2$.

Solution: Equation of the given curve is

$$Y = mx \qquad\qquad \dots (1)$$

On putting $y = 0$ in eq. 1, $x = 0$ ($\therefore$ at x-axis, $y = 0$)

$\therefore$ Desired area $= \int_0^2 y\,dx = \int_0^2 mx\,dx = m \int_0^2 x\,dx$

$$= m\left[\frac{x^2}{2}\right]_0^2 = \frac{m}{2}(4 - 0) = 2m \text{ square unit.} \qquad \textbf{Ans.}$$

Example 2: Find the area between Parabolic line $y^2 = 4x$ and simple line $x = 4$.

Solution: Equation of Parabola

$$y^2 = 4x$$

Line $x = 4$ meet the parabola at points A and B.

Desired area is *OBAO*.

Note   therefore $\qquad$ Area $= 2\int_0^4 y\,dx$

$$= 2\int_0^4 \sqrt{4x}\,dx = 4\int_0^4 x^{\frac{1}{2}}\,dx$$

$$= 4 \times \left[\frac{2}{3} x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{8}{3}\left[4^{\frac{3}{2}} - 0\right]$$

$$= \frac{8}{3}(8 - 0) = \frac{64}{3} \text{ square unit} \qquad \textbf{Ans.}$$
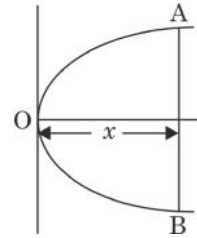
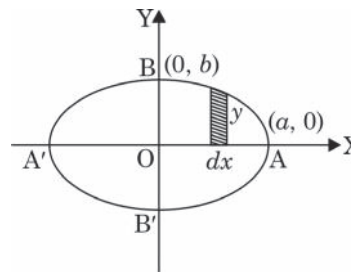**Example 3: Find the area of Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.**

Solution: Area of Ellipse *ABA'B'* has to calculate. First we will find the area of space *OAB* and on four folding it we will find the desired area.

$\because \qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore \qquad y^2 = b^2\left(1 - \dfrac{x^2}{a^2}\right)$

$\Rightarrow \qquad y = \dfrac{b}{a}\sqrt{a^2 - x^2}$

On putting $x = 0$ for point b and $y = 0$ for point A, coordinates of points B and A will be (a, b) and $(a, 0)$.

Note

$$\text{Desired Area} = 4\int_0^a y \, dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2} \, dx$$

$$= 4\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]_0^a = \pi ab \text{ square unit} \qquad \textbf{Ans.}$$

Example 4: Find the area of Parabola $y^2 = 4ax$ and the space surrounded by its Latus Rectum.
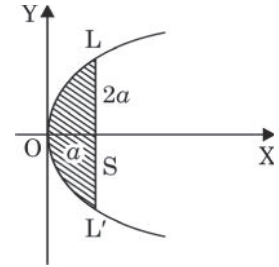
Solution: Equation of Parabola

$$y^2 = 4ax$$

Latus Rectum meets the parabola at the points L and L'.

Desired Area is $OL'SLO$

If Area is A, then

$$A = 2\int_0^a y \, dx$$

$$= 2\int_0^a \sqrt{4ax} \, dx$$

$$= 2\int_0^a 2a^{1/2}x^{1/2} \, dx = 2 \times 2a^{1/2}\int_0^a x^{1/2} \, dx$$

$$= 4a^{1/2}\left[\frac{x^{3/2}}{3/2}\right]_0^a = \frac{2.4}{3}a^{1/2}\left[x^{3/2}\right]_0^a$$

$$= \frac{8}{3}a^2 \text{ Square unit} \qquad \textbf{Ans.}$$

Task Find the area between the Parabola $y^2 = 4x$ and simple line $x = 4$

(**Ans.:** $\frac{8}{3}a^2$ square unit)

Questionnaire 16.1

Short Answer Questions:

Find the area surrounded by the following curves, X-axis and given Abscissa:

1. $y = e^x$ from $x = 0$ to $x = 2$

2. $y = x \sin x$, from $x = 0$ to 2π

    a.   [**Hint:** Desired Area $= \int_0^{2\pi} y \, dx = \int_0^{2\pi} x \sin x \, dx = \left[-x \cos x + \sin x\right]_0^{2\pi}$ on integrating]

3. $y = \left(1 + \dfrac{8}{x^2}\right)$, from $x = 2$ to $x = 10$

4. $y = 5 + \dfrac{1}{10}x^2$, from $x = 2$ to $x = 10$

5. $xy = b^2$ from $x = a$ to $x = b$

6. $y = x\, e^{x^2}$ from $x = 0$ to $x = b$

7. Find the area between simple line $y = mx$, x-axis, and abscissa $x = 3$

8. Find the ratio of areas of parts of curve $x^2 + y^2 = a^2$ divided by the simple line $x = \dfrac{1}{2}a$.

[**Hint:** let us assume the ratio of areas of the desired parts $x = a \sin \theta$].

$$= \int_{a/2}^{a} y\, dx : \int_{-a}^{a/2} y\, dx$$

$$= \int_{a/2}^{a} \sqrt{a^2 - x^2}\, dx : \int_{-a}^{a/2} \sqrt{a^2 - x^2}\, dx$$

Answers

1. $e^2 - 1$

2. 2π square unit

3. $\dfrac{56}{5}$ Square unit

4. $73\dfrac{1}{15}$ square unit

5. $b^2 \left[\log \dfrac{b}{a} \right]$

6. $\dfrac{1}{2} \left[e^{b^2} - 1 \right]$

7. $\dfrac{9}{2} m$ Square unit

8. $(4\pi - 3\sqrt{3}) : (8\pi + 3\sqrt{3})$

Self Assessment

1. Fill in the blanks:

1. Now a day's use integration in economics has increased ....
2. Definite integral is a limit of ....
3. Consumer saving can also be found out from ....

16.2 Use of Integration in Economics

Now-a-day's use integration in economics has increased very fast. This we can prove from the following facts–

1. **Marginal Costs, Average Costs and Total Costs**

   Since Marginal Costs = $\dfrac{d(c)}{dq}$, therefore

   Total Costs = $\int \dfrac{d(c)}{dq} \cdot dq$

Here, $\dfrac{d(c)}{dq}$ = Marginal Costs, C = Total Costs dq = Change in production.

Example 1: Marginal Costs = 3 + 150q – 9q^2 of any firm is given, then find the total costs when On manufacturing 3 units the total costs will be ₹ 300 .

Solution: We know that Total costs,

$$TC = \int (3 + 150q - 9q^2)\, dq$$

$$= 3q + 150\frac{q^2}{2} - \frac{q^3}{3} + c$$

$$= 3q + 75q^2 - 3q^3 + c$$

And Average costs

$$AC = \frac{TC}{Q} = 3 + 75q - 3q^2 + \frac{c}{Q}$$

Since $q = 3$ and $TC = 300$ given, therefore

$$300 = (3 \times 3) + 75\,(3)^2 - 3 \times 3 \times 3 + c$$

$$= 9 + (75 \times 9) - 27 + c$$

$$= 9 + 675 - 27 + c$$

Or

$$c = -675 + 300 = -357$$

$$TC = 3q + 75q^2 - 3q^3 - 357$$

And

$$AC = 3 + 75q - 3q^2 - \frac{357}{q}\,.$$

Example 2: If Constant costs is ₹ 100 and Marginal function is in the form ($6x$ + 3)Then find the Total costs function.

Solution: We know that Total Costs = $\int$ Marginal Costs

Therefore

$$TC = \int (6x + 3)\, dx$$

$$= \frac{6x^2}{2} + 3x + c$$

Here constant costs is ₹ 100 then in this condition if the production zero (i.e. $x = 0$) then also constant costs will remain ₹ 100 and Total costs will also be ₹ 100. In this condition

$$TC = (6 \times 0) + (3 \times 0) + c = 100$$

Therefore total costs function will be $3x^2 + 3x + 100$.

Consumer Saving

Consumer saving can also be found out from integration. This can be explained from the following.

In this diagram DD is a demand. Consumer give costs OP of anything and suppose he does OX purchase. Consumer is ready to give total costs of OX quantity of thing equal to the $OXRD$ part in the diagram, but actually he is ready to give $OXRP$ costs. Consumer Saving is the difference between the both.
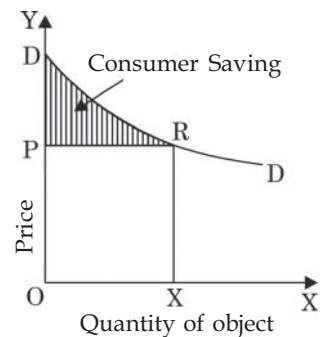
Did u know?   consumer saving = Area $OXRD$ - Area $OXRP$

$$\int_0^x (\text{Demand curve } dx - p \cdot x$$

Example 3: Find the consumer saving when demand curve is $p = 100 - 5x + 3x^2$ and $x = 5$ unit.   Note

Solution: Here demand curve is $p = 100 - 5x + 3x^2$ and quantity of demand is $x = 5$ then here

$$\text{Consumer Demand} = \int_0^x (\text{Demand Curve})\, dx - p.x$$

$$= \int_0^3 (100 - 5x + 3x^2)\, dx - (p \times 3)$$

$$= \left[100x - \frac{5x^2}{2} + \frac{3x^3}{3} \right]_0^3 - 30$$

$$= 100 \times 3 - \frac{5}{2}\,(3 \times 3) + (3 \times 3 \times 3) - 30$$

$$= 300 - 22.5 + 27 - 30 = 274.5.$$

Example 4: Find the consumer saving when demand function is $p = 50 - 5x - x^2$ and p = 0.

Solution: Demand Function; $p = 50 - 5x - x^2$

If $p = 0$ then

$0 = 50 - 5x - x^2$

or $x^2 + 5x - 50 = 0$

$x^2 + 10x - 5x - 50 = 0$

or $x(x - 10) - 5(x + 10) = 0$

or $(x - 5)(x + 10) = 0$

$x - 5 = 0$ or $x = 5$; in the same way, $x + 10 = 0$ or $x = -10$

If we ignore $x = -10$ since $x < 0$ therefore

$$\text{Consumer Saving} = \int_2^5 (50 - 5x + x^2)\, dx - p \times x$$

$$= \int_0^5 (50 - 5x + x^2)\, dx - 0 \times 5$$

$$= \left[50x - \frac{5x^2}{2} - \frac{x^3}{3} \right]_0^5$$

$$= \left[50 \times 5 - \frac{5 \times 5 \times 5}{2} - \frac{5 \times 5 \times 5}{3} \right]_0^5$$

$$= 250 - \frac{125}{2} - \frac{125}{3}$$

$$= \frac{1500 - 375 - 250}{6}$$

$$= \frac{825}{6} = \frac{275}{2} = 137.5$$ **Ans.**

Note

<div style="text-align:center; border:1px solid black; display:inline-block;">**Questionnaire 16.2**</div>

Solve Yourself :

1. Find : $\int x^2 \cos x \, dx$

2. Find the Integration : $\int (8 + 7x)^7 \, dx$

3. Find the values of the following: $\int_0^{\frac{1}{2}} \cos^2 x \, dx$

4. Find :

   (i) $\int \dfrac{dx}{a^2 + x^2}$ (ii) $\int \dfrac{\sin x}{\sin(x - \alpha)} \, dx$ –

5. Integrate:

   (i) $\int 4x^2 \sqrt{x^3 + 3}$ (ii) $\int \dfrac{dx}{4x^2 - 9}$

   (iii) $\int \dfrac{x^2 - 4}{x^2 + 3x + 2} \, dx$ (iv) $\int x^3 e^x \, dx$

6. Find the value of the followings:

   (i) $\dfrac{e^{3x}}{1 + e^{3x}}$ (ii) $\dfrac{3x}{(3x + 1)(x - 3)}$

   (iii) $x^3 \log x$,   (iv) $\sin(ax + b)$.

7. Find total costs function $f(x)$ if x = Total production.

   (i) ₹ 100 Total costs at $3x^3 - 4x + 5$, $x = 0$.

   (ii) ₹ 100 Total costs at $\dfrac{100}{\sqrt{x}}$, $x = 0$.

   (iii) $6.75 - 0.0006\, x$, $x = 0$, total costs = ₹ 10,485

8. Find:

   (i) $\int_1^5 x^3 x \, dx$ (ii) $\int_1^{14} \sqrt{x} \, dx$

   (iii) $\int_1^3 (1 + 5x + x^3) \, dx$ (iv) $\int_1^3 (e^{2x} + e^x) \, dx$

9. Find the consumer saving if Demand function curve is $p = 33 - 3x - 2x^2$ and $x = 3$ unit.

10. If p = ₹ 5 then How much will be the consumer saving on Demand Function $p = 100 - 2x^3$. If p = 0 then what will be the saving?

Self Assessment

2. Multiple Choice Questions:

4. Total Costs = ...?

   (a) $\int$ Marginal Costs   (b) $\int$ Total Costs   (c) $\int$ Total Profit   (d) $\int$ loss

5. Consumer Saving can be calculated by which method?

    (a) Differentiation   (b) Differential

    (c) Integration   (d) Cramer

6. Marginal Costs is ....

    (a) $\dfrac{d}{dq}$ (b) $\dfrac{dq}{d(c)}$

    (c) $\dfrac{d(c)}{dq}$ (d) None of these

16.3 Summary

- Suppose *CPQD* represents curve $y = f(x)$, where $f(x)$ is a continuous function of x *in prant* [*a*, *b*] and suppose like the value of *x* increases from a to *b*, *y* also increases. Suppose *CA* and *DB* are the ordinates at $x = a$ and $x = b$.

- Take any point *P* (*x*, *y*) located on the curve and assume that *PM* is its grade. Take another point $Q(x + \delta x, y + \delta y)$ on the curve close tom point *P* and suppose its ordinate is *QN*.

- Now a day's use integration in economics has increased very fast This we can prove from the following facts – Marginal Costs, Average Costs and Total Costs

    Since Marginal Costs = $\dfrac{d(c)}{dq}$, therefore

    Total Costs = $\int \dfrac{d(c)}{dq} \cdot dq$

- Here, $\dfrac{d(c)}{dq}$ = Marginal Costs, *C* = Total Costs, = Change in production.

- Consumer Saving can be calculated by Integration.

16.4 Keywords

- *Applications:* Experiments
- *Ordinate:* Grade

16.5 Review Questions

1. Find the area of Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.   (**Ans.:** $\pi\, ab$ square unit)

2. If a constant cost is ₹ 100 and Marginal Function is in the form $(6x + 3)$, then find the Total costs function.

    (**Ans.:** $3x^2 + 3x + 100$)

3. Find the area under simple line $Y = mx$, *X*-axis, and Ordinate $x = 3$.   (**Ans.:** square $\dfrac{9}{2} m$ unit)

Answers: Self Assessment

1. Fast   2. Addition   3. Integration

4. (a)   5. (c)   6. (c)

Note

16.6   Further Readings

Books

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

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Mathematics for Economics – Simon and Bloom – Viva Publlications.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications

Mathematics for Economics – Council for Economic Education.